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General Branching Processes with Local Dependencies

Peter Olofsson

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Key words: Branching process, multi-type, sibling dependencies, local dependencies, macro process, stable population.

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CHALMERS TEKNISKA HÖGSKOLA · GÖTEBORG ·

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Στην Αλκμήνη

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Peter Olofsson
April, 1994

Contents

1. Introduction

The theory of branching processes is, generally speaking, the study of idealized population dynamics. The concept "population" does not only refer to biological populations; it is used in a general sense to describe any set that may change as its members generate new set members. The origin of the theory is however the study of human populations, something mirrored in its language; we talk of individuals, mothers, daughters and so on. Apart from applications to human or biological populations, as in demography or cell kinetics, branching processes have been used for instance in the analysis of random trees (Aldous (1991)) and in the study of random Cantor sets (Larsson (1990)).

Branching process theory has provided answers to a multitude of questions about freely growing populations, where it is assumed that each individual reproduces independently of all other individuals. The three main issues have been extinction, growth and composition, the first of which can be treated quite exhaustively already in the simple *Galton-Watson process* counting only generation sizes. It does not last long before any student of probability theory learns how to compute the extinction probability in a Galton-Watson process and, further, that extinction is inevitable if and only if the average number of children of a mother is less than or equal to one.

If a population does not die out it grows exponentially, whether the population size is measured in discrete time as in the Galton-Watson process, or in continuous time as in the *general branching process* where individuals reproduce according to some point process on R_+, or, in the multi-type case, on $S \times R_+$, S being the *type space*. The well known explosion-extinction dichotomy, that any independently reproducing population either eventually becomes extinct or grows to infinity, holds quite generally if certain degenerate cases, such as all individuals always begetting exactly one child, are excluded.

The asymptotic composition of an exponentially growing population is best studied within the frame of general branching processes. It can be shown that, whereas population size is unstable, the composition stabilizes in the sense that proportions of individuals with various properties tend to constants as time tends to infinity. This consideration leads to the study of the so called *stable population*, a population centered round a typical individual, *Ego*. Her (this author sticks to the tradition to consider all individuals female) individual properties as well as her progeny and ancestry are described by the *stable population measure*, for single-type populations described in Nerman and Jagers (1984) and in the multi-type case in Jagers (1992) and Jagers and Nerman (1992).

It would be tempting to investigate how these results are affected if dependencies between siblings are allowed. Of course for theoretical reasons, as a first step away from independence, but also from a point of view of applications. Clearly some sort of dependencies could be expected between siblings in biological populations; they may compete or collaborate, thus giving rise to negative or positive correlations respectively.

One early treatment of sibling dependencies was performed in a paper by Crump and Mode (1969). In that reference a Bellman-Harris process, i.e. a process with splitting reproduction and random life-lenghts, allowing sibling dependencies is explored. Population-size-dependent Galton-Watson processes with dependent offspring were treated in Cohn and Klebaner (1986).

The effect of sibling dependencies on the extinction probability was investigated in the first of three papers in Broberg (1987). The third of these treats critical processes with sibling dependencies, whereas the second deals with general single-type branching processes with sibling dependencies. In that paper it is assumed that the population is *homogeneous* in the sense that all individuals have the same marginal reproduction, but that the dependence structure might vary from one group of siblings to another.

It turns out that leaving the homogeneity assumption actually leads to the study of general *multi-type* branching processes with sibling dependencies and hence the main objective of this thesis is to investigate such processes with respect to growth and stable population composition. Non-homogeneous single-type processes will be viewed as special cases.

The next chapter gives a short description of some general problems of branching processes, starting from Galton-Watson processes and aiming at the general multi-type branching process which is described in Section 2.2. In the following chapter it is argued why sibling dependencies should be studied and how this is done by embedding an independent process, the *macro process*, consisting of sibling groups, into the sibling dependent process. A series of simple examples are given to illustrate principal problems and general observations. The macro process is constructed in detail in Section 3.2. The multi-type structure calls for the use of Markov renewal theory, the definitions and results of which are given in Chapter 4, where it is also shown how Markov renewal theory and general multi-type processes relate to each other.

In Chapter 5 it is explored how the macro process and the individual process are connected. It is shown that they are actually equivalent with respect to Markov renewal properties and the fundamental convergence result for the expected population size is given. Chapter 6 deals with L^1-convergence of the actual population size and for that purpose it is also proved that the so called $x \log x$ *condition* is equivalent for the macro and the individual process.

Chapter 7 treats the stable population mentioned above. An explicit form of the stable population measure in a sibling dependent process is given, similarities to and differences from independent processes are pointed out. Non-homogeneous single-type processes are discussed in Chapter 8, where they are introduced as special cases of the multi-type process. The emphasis is on the special form their stable population measure takes. Some examples are also given. Section 8.3 deals with the stable population in another special case, Broberg's homogeneity.

In Chapter 9, L^2-convergence is proved under finite variance assumptions, and in Chapter 10 some concluding remarks are given to convince the reader that the theory actually covers more than it seems by a first glance. The theory is also extended to dependencies that go further than siblings, i.e. cousin dependencies, second cousin dependencies and so on.

Concepts such as "branching processes with sibling dependencies" will be used although the term "branching" strictly speaking means that individuals reproduce independently. However, as will soon be shown, in a sibling dependent population there is embedded branching also in the strict sense of the word.

2. Preliminaries

2.1. The Extinction and Growth of Populations

This section contains a short survey of the major problems of branching processes and how they are treated. The results are fundamental and may be found in any book on branching processes (and some of them indeed in most elementary probability books). We mention only two books which together cover the classical theory, Athreya and Ney (1972) and Jagers (1975).

The simplest type of a branching process is the *Galton-Watson process* (sometimes also called the Bienaymé-Galton-Watson process). In such a process an individual begets a random number, X say, of children, and individuals are assumed to reproduce independently. The number of individuals in the nth generation is then obtained by adding the number of children of all individuals in the $(n-1)$th generation. With z_n =the number of individuals in generation n and $X_{n-1,k}$ =the number of children of individual k in generation $n-1$,

$$z_n = \sum_{k=1}^{z_{n-1}} X_{n-1,k}.$$

The question of eventual extinction of such a process is answered with the aid of generating functions and therefore let

$$p_k = P(X = k), k = 0, 1, ...$$

$$f(s) = \sum_{k=0}^{\infty} s^k p_k$$

and

$$q = P(\exists n : z_n = 0),$$

the extinction probability. Then q is the smallest non-negative solution to the equation $s = f(s)$. Further, with $m = E[X]$ and from now on disregarding the degenerate case $p_1 = 1$,

$$q < 1 \Leftrightarrow m > 1.$$

Hence extinction is (almost) certain if and only if the expected number of children of an individual is less than or equal to one. The cases $m < 1, m = 1$ and $m > 1$ are called the subcritical, the critical and the supercritical case respectively. The reason for distinguishing between subcriticality and criticality is that, although they both lead to almost sure eventual extinction, they give rise to qualitatively different asymptotics for example when conditioning on non-extinction.

The next famous result concerns population growth and states that if a Galton-Watson population does not die out, its size will tend to infinity, i.e.

$P(z_n \to \infty) = 1 - q$. Hence, no independently reproducing population ever stabilises in size; it either dies out or grows beyond all bounds. The $x \log x$ *condition* is a crucial concept in the Galton-Watson theory, and indeed in branching process theory generally. In the Galton-Watson case it has the form $E[X \log^+ X] < \infty$ and it can be shown that there exists a non-degenarate random variable w, such that

$$E[X \log^+ X] < \infty \Leftrightarrow \frac{z_n}{m^n} \to w, \tag{2.1}$$

as $n \to \infty$, this convergence holding almost surely and in L^1, whereas $w \equiv 0$ if $E[X \log^+ X] = \infty$.

One extension is to allow individuals to be of different *types*, an individual's type affecting her reproduction. Thus, if the possible types are $\{1, 2, ..., r\}$, an individual of type k splits into k_1 children of type 1, k_2 of type 2,...,k_r of type r with probability $p_k(k_1, ..., k_r)$.

Let $z_n(k)$ be the number of individuals of type k in the nth generation. The asymptotic behaviour of $z_n(k)$ depends on the expected reproduction matrix, M, with entries M_{ij}=the expected number of type j children of an individual of type i. If M is *positive*, i.e. if for some integer $n > 0$ all the entries of M^n are positive, the growth rate is given by M's spectral radius, ρ, i.e. its largest eigenvalue. It is ρ that now plays the role m did previously. Without going into details, the fundamental convergence result is that there exist random variables $w(k)$ such that $z_n(k)/\rho^n \to w(k)$, $k = 1, 2, ..., r$, this again being equivalent to an $x \log x$ condition which will not be spelled out here.

A Galton-Watson process thus reveals some fundamental properties of populations of independently reproducing individuals, such as the extinction-explosion dichotomy and the exponential growth. However, it is clearly limited by its generation structure; it is not a model for real time evolution of populations. One way to obtain such a model is to assume that an individual lives for a random amount of time, L, with distribution function G, then dies and at the time of death gives birth to (or splits into) a random number of children, X, with expectation m. Individuals still reproduce independently and the Galton-Watson process can be viewed as the special case $L \equiv 1$. Rather than z_n, the number of individuals in the nth generation, the number of individuals alive at time t, z_t, is studied. If L and X are assumed independent, such a process is called a *Bellman-Harris process*, whereas we talk of a *Sevastyanov process* if dependencies between L and X are allowed.

In the latter process the growth is described by a real number α, called the *Malthusian parameter* and defined through

$$E[Xe^{-\alpha L}] = 1,$$

which for the Bellman-Harris process reduces to

$$m\hat{G}(\alpha) = 1,$$

where hat denotes Laplace transform i.e.

$$\hat{G}(\alpha) = \int_0^\infty e^{-\alpha t} G(dt).$$

The fundamental convergence result is

$$e^{-\alpha t} z_t \to w,$$

as $t \to \infty$.

These processes are special cases of the *general* or *Crump-Mode-Jagers process.* In the latter the pair (X, L) is replaced by a point process, ξ, the *reproduction process*, describing at which time-points an individual begets her children. All individuals are assumed to reproduce independently, an assumption referred to as the *branching property.* The Malthusian parameter is now defined through the relation

$$E[\hat{\xi}(\alpha)] = E[\int_0^\infty e^{-\alpha t} \xi(dt)] = 1,$$

and the $x \log x$ condition is $E[\hat{\xi}(\alpha) \log^+ \hat{\xi}(\alpha)] < \infty$.

The process can be made even more general by also allowing individuals to be of different types, in analogy with the multi-type Galton-Watson processes described above. Such multi-type general processes are the main objectives of this thesis and they will be described in the next section.

2.2. General Multi-type Branching Processes

The theory of general branching processes will be described without going too much into details. Some definitions are postponed to coming chapters where they will be stated as they appear in the construction of the macro process mentioned in the introduction. For a thorough description the reader is referred to Jagers (1989).

In a general branching process individuals are elements of the *Ulam-Harris space*

$$I = \{0\} \cup \bigcup_{k=1}^{\infty} N^k,$$

where 0 denotes the ancestor and $N = \{1, 2, ...\}$. An individual is thus of the form $(x_1, ..., x_n)$ with the interpretation that she is the x_nth child of the x_{n-1}th child of ... of the x_1th child of the ancestor.

If $x, y \in I$ then xy denotes the individual having first x's and then y's coordinates. In particular $x0 = 0x = x$. Hence $x1, x2, ...$ are x's children and if $x = (x_1, ..., x_n)$ then x's mother, $(x_1, ..., x_{n-1})$, is denoted by mx, with the convention that $m0 = 0$.

If $x = (x_1, ..., x_n)$ and, for some $k \leq n$, $y = (x_1, ..., x_k)$ we say that x *stems from* y and write $x \succ y$. If M is a subset of I such that there is a $y \in M$ with $x \succ y$ we write $x \succ M$. The set $\{xy : y \in M\}$ is denoted xM.

The set of possible life careers is called the *life space* and denoted by $(\Omega, \mathcal{A})$. Any property of an individual such as its life length or number of children is viewed as a random element on Ω. A few such variables of central interest will be given.

An individual's reproduction is described by the consecutive ages at childbearing, $0 \leq \tau(1) \leq \tau(2) \leq ... \leq \infty$, where $\tau(i)(\omega)$ is the age of an individual with life career ω when she begets her ith child, $\tau(i)$ being infinity if the individual begets less than i children.

At birth each individual gets a type in the *type space* S which can be quite general; it is only required to have a countably generated σ-algebra $\mathcal{S}$. The ith child of an individual with life career ω gets type $\sigma(i)(\omega)$. The birth times and types together form the *reproduction process*

$$\xi(\omega, A \times B) = \#\{i : \sigma(i)(\omega) \in A, \tau(i)(\omega) \in B\},$$

where $A \in \mathcal{S}$ and $B \in \mathcal{B}$, the Borel algebra on R_+. The notations $\xi(\omega)$ and $\xi(A \times B)$ will also be used depending on the context, and we will sometimes write $\xi(t)$ for $\xi(S \times [0, t])$.

From the life and type spaces the *population space* $S \times \Omega^I$ is constructed. An outcome of this consists of the ancestor's type and a life career for each individual in I. The projections $U_x : S \times \Omega^I \to \Omega$ single out the life career of the individual x, and enable us to lift entities defined on the life space into the population space. For instance, the reproduction process of an individual x is defined through

$$\xi_x = \xi \circ U_x,$$

other random elements defined similarily. An element of the population space is denoted (s, ω_I), and generally the notation ω_M is used for the set of lives of the individuals in $M \subseteq I$. In particular $\omega_x = U_x(s, \omega_I)$, the life of the individual x.

To count or measure the population *random characteristics* are introduced. These are random processes

$$\chi : S \times \Omega^I \times R \to R_+,$$

where $\chi(s, \omega_I, t)$ gives the contribution at age t of an s-type individual with life and progeny's lives described by ω_I. We assume that χ vanishes for negative times t. The simplest example of such a characteristic is $\chi(s, \omega_I, t) = 1_{R_+}(t)$, which is 1 if $t \geq 0$, i.e. if you are born, and 0 otherwise. Note that χ may depend on an individual's whole progeny, something which makes it possible to count individuals with a certain number of grandchildren or with children who are alive at certain timepoints etc. If we only allow χ to depend on an individual's own life and type, i.e. if

$$\chi(s, \omega_I, t) = f(s, \omega_0, t),$$

for some function $f : S \times \Omega \times R \to R_+$, it is called an *individual characteristic*.

Using the shift operator $S_x : S \times \Omega^I \to S \times \Omega^I$ defined through

$$S_x(s, \omega_I) = (\sigma_x, \omega_{xI}),$$

i.e. the operator that makes the individual x an ancestor, we write $\chi_x = \chi \circ S_x$ for the χ-value pertaining to x. The process starting from x is called the *daughter process* of x. The individual xi is born at time τ_{xi}, recursively defined through

$$\tau_0 = 0$$

and

$$\tau_{xi} = \tau_x + \tau(i) \circ U_x$$

and the *χ-counted population* is defined to be

$$z_t^\chi = \sum_{x \in I} \chi_x(t - \tau_x),$$

adding the contributions of all individuals born up to time t at their proper ages (τ_x is the birth time of the individual x and hence $t - \tau_x$ is her age at time t). Sometimes z_t^χ will simply be called the population size. With the characteristic 1_{R_+} mentioned above z_t^χ will be the total number of individuals born up to time t, usually denoted by y_t.

Assume that a set of probability kernels $\{P(r, \cdot) : r \in S\}$ on the life space is given. Choose an enumeration $x(0), x(1), \ldots$ of the set of individuals which is such that a mother always precede her daughters. Define transition probabilities from $S \times \Omega^{n+1}$ to Ω through

$$P(A | \sigma_0, \omega_{x(0)}, \ldots, \omega_{x(n)}) = P(\sigma_{x(n+1)}, A),$$

so that an individual chooses life according to a probability measure that only depends on her type. This is the *Markov property* for a multi-type process. It can be shown that this property implies the *branching property*, that daughter processes started from individual's who do not stem from each other are conditionally independent, see Jagers (1989) for a strict treatment of this. Note that the type $\sigma_{x(n+1)}$ is a function of $x(n+1)$'s mother's life which is one of the elements $\omega_{x(0)}, \ldots, \omega_{x(n)}$. The branching property for a single-type process is that all individuals choose lives independently and the multi-type analogue is thus that all individuals choose life independently *conditioned on their types.*

Under these assumptions the kernels $\{P(r, \cdot), r \in S\}$ define a unique probability measure P_s on $(S \times \Omega^I, \mathcal{S} \times \mathcal{A}^I)$.

With E_s denoting expectation with respect to P_s, both the asymptotics of $E_s[z_t^\chi]$, the expected size of the χ-counted population when the ancestor's type is s, and of z_t^χ itself are investigated.

The crucial part in this investigation is played by the *reproduction kernel* μ defined through

$$\mu(s, A \times [0, t]) = E_s[\xi(A \times [0, t])],$$

the expected number of children with types in A an s-type individual begets before age t. (Observe that the notation E_s is used also for expectation with respect to the measure $P(s,\cdot)$.) Under quite general conditions μ determines a real number α, called the *Malthusian parameter*, which is such that, roughly speaking, the population size grows as $e^{\alpha t}$.

Further, μ determines a function h on the type space, $h(s)$ being the reproductive value of the type s, and a measure π, also on the type space, which describes the distribution of the types in a large population.

For the time being nothing more is said about these entities. They will be properly defined and described in Chapter 4, where also the conditions required on μ are given.

The two fundamental convergence results are

$$E_s[e^{-\alpha t}z_t^{\chi}] \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}h(s),$$

and

$$e^{-\alpha t}z_t^{\chi} \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}w,$$

as $t \to \infty$. Here

$$\hat{\chi}(\alpha) = \int_0^\infty \alpha e^{-\alpha t}\chi(t)dt,$$

$E_\pi = \int_S E_s\pi(ds)$, i.e. expectation when the ancestor's type is not fixed, but distributed according to π, and β is a real number called the *stable age at childbearing*, a definition of which will be given in Section 4.1. The second convergence holds in $L^1(P_s)$ for π-almost all $s \in S$. The random variable required to have a finite $x\log x$ moment is now

$$\bar{\xi} = \int_{S\times R_+} e^{-\alpha t}h(r)\xi(dr \times dt),$$

and hence the $x\log x$ condition takes the form $E_\pi[\bar{\xi}\log^+\bar{\xi}] < \infty$. The entity $\bar{\xi}$ is interpreted as the relative contribution by the reproduction process ξ to the population. With this interpretation the equality

$$E_s[\bar{\xi}] = h(s)$$

is obvious, $h(s)$ being the reproductive value of the type s.

Note that in a single-type Galton-Watson process $\bar{\xi}$ reduces to $e^{-\alpha}X$, where X is the number of children, all born at time $t = 1$. Although the general definition of α has not yet been given, it is quite clear that a single-type Galton-Watson process should have $\alpha = \log m$ where $m = E[X]$, and hence the $x\log x$ condition is

$$E[\frac{X}{m}\log^+\frac{X}{m}] < \infty,$$

obviously equivalent to the previous formulation, $E[X\log^+ X] < \infty$.

3. Why Sibling Dependencies and How to Deal with Them?

3.1. Effects of Sibling Dependencies

As mentioned in the introduction, population models allowing sibling dependencies are of great interest for two reasons. First, they provide a natural extension of the theory of general branching processes, this theory being well suited to model dependencies that are local in the family tree. Secondly, it can of course be argued that such dependencies are most plausible in models for biological populations. Surely what first comes to mind is competition among siblings, thus giving rise to negative correlations.

In this section it will be argued why processes with sibling dependencies should be studied, how this was done in Broberg (1987) and how it can be extended and further generalised. This will be done by giving several examples of simple Galton-Watson processes. Hopefully they will exhibit the principal problems of sibling dependencies, and make it easier to understand the general theory of coming sections. Let us start with a very simple example, where the impact of sibling dependencies is substantial.

Example 3.1. Consider a critical binary splitting population, i.e. a Galton-Watson process where each individual can have either zero or two children with equal probability. The expected number of children is then one, and by classical results ultimate extinction has probability one.

Now consider a population where each individual can beget zero or two children such that, in a group of two siblings, one will always split and the other will not. The two are equally likely to be the splitting one. Clearly there are sibling dependencies; given that an individual begets two children her sister will not reproduce at all.

This second process is governed by a joint probability measure $P(2,\cdot)$ (the "2" is there to indicate that there are two individuals in the sibling group) on $\{0,2\}^2$ such that

$$P(2,(0,2)) = P(2,(2,0)) = 1/2.$$

This joint probability measure has marginals

$$p_k = P(\text{individual begets } k \text{ children}) = 1/2,$$

for $k = 0, 2$. The second process thus has the same individual marginal reproduction as the first, but the sibling dependencies give it a totally different behaviour; if it starts from a group of two siblings it never dies out (and will forever have generation size two), whereas the first process always dies out. □

So obviously sibling dependencies can have a crucial influence, at least as far as extinction is considered. Less drastic examples may be found in Broberg (1987) where it is also generally shown that negative correlations between sibling's reproductions decrease the extinction probability, whereas positive correlations increase it.

The basic idea for a theory that allows sibling dependencies is to embed another process, *the macro process*, into the sibling dependent process. This macro process, introduced in Broberg (1987), consists of sibling groups, to be called *macro individuals*. The point is of course that whilst individuals do not reproduce independently, macro individuals do, since the only dependencies are within the sibling groups. Of course the macro process corresponding to the sibling dependent process in Example 3.1 is degenerate, in the sense that each macro individual with probability one begets exactly one child. This case is excluded in classical Galton-Watson theory; it is the only situation that violates the extinction-explosion dichotomy.

Generally, if it is assumed that the reproduction and dependence structure of a sibling group of size k is described by a joint probability measure $P(k,\cdot)$ on N^k, the macro process may be considered as multi-type, the type of a macro individual being the number of siblings in that group.

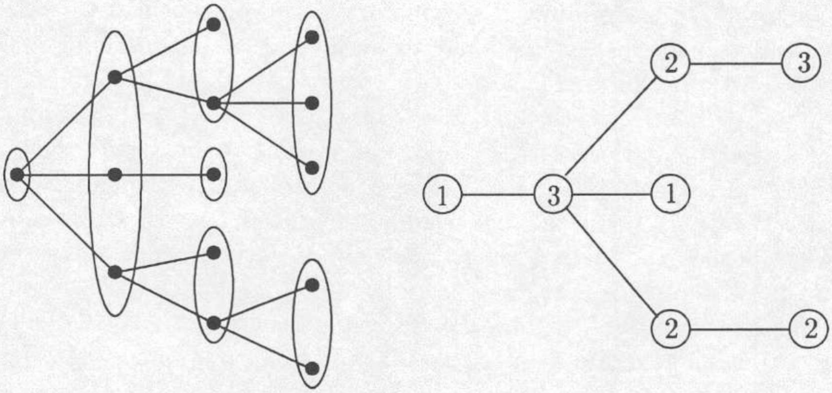

Figure 3.1: A Galton-Watson process with encircled sibling groups and the corresponding macro process with its types.

Hence classical theory for multi-type Galton-Watson processes applies to the macro process. Since what is interesting is still the individual process, this whole setting would be useless if it is not possible to count individuals by instead counting macro individuals. But indeed, with ζ_n=the number of individuals in the nth generation and $Z_n(k)$ =the number of macro individuals of type k of the nth generation, it holds that

$$\zeta_n = \sum_{k=1}^{\infty} k Z_n(k).$$

As mentioned, it is the individual process that is in focus; the macro process is merely a tool making the analysis simpler. Therefore it would be desirable to make all the assumptions on the individual level and a drawback if we had to invoke some extra conditions on the macro level. Indeed, the ideal situation would be if the only conditions needed were those from the ordinary theory of independent processes, now assumed for the individual's marginal reproduction, i.e. the marginals of $P(k,\cdot)$.

The theory for sibling dependent Galton-Watson processes will not be developed in detail, the emphasis of the thesis being general branching processes which clearly contain Galton-Watson processes as special cases. However, in order to provide the reader with a little more insight into the problems of sibling dependencies, a few more Galton-Watson illustrations will be given. The following example indicates that whereas dependencies might seriously affect the extinction probability, their influence in some sense disappears on the set of non-extinction.

Example 3.2. Consider a sibling dependent Galton-Watson process where an individual can beget zero or three children, and where the dependencies are such that, in a group of three siblings, two will always reproduce whilst the third never will. All individuals are equally likely to be among the reproducing ones. The joint probability measure $P(3,\cdot)$ on $\{0,3\}^3$ then gives equal probabilities to the points $(0,3,3),(3,0,3)$ and $(3,3,0)$ and has the marginals $p_0 = 1/3$ and $p_3 = 2/3$.

The independent process with these marginals has an expected number of children equalling two, and there is a positive probability of non-extinction, in which case the generation size tends to infinity. Indeed, the extinction probability may be computed explicitely as the smallest non-negative solution of the equation $s = 1/3 + (2/3)s^3$, which turns out to be $(\sqrt{3}-1)/2$.

The dependent process will have a deterministic generation size of $3 \cdot 2^n$ individuals in the nth generation (if it starts from a full group of siblings) and hence it always explodes.

The macro process is now such that each macro individual with probability one begets exactly two children. Of course it is a trivial and highly uninteresting process, but one observation can be made: its expected number of macro children is two, the same as the independent, individual process governed by the marginal reproduction. The growth rate is 2^n for both processes and if the independent one does not become extinct, it will tend to infinity at the same speed as the dependent one. From classical theory we know the asymptotics of the independent process:

$$\frac{z_n}{2^n} \to w$$

almost surely as $t \to \infty$, where w is a random variable having expectation one. For the macro process obviously

$$\frac{Z_n}{2^n} \equiv 1$$

and, with ζ_n=the number of individuals in the nth generation of the dependent process,

$$\frac{\zeta_n}{2^n} \equiv 3,$$

if the process starts from a full group of siblings. In the independent process, assume that we wish to count the number of individuals with some certain property A which appears independently with probability p. Let z_n^A be the number of individuals in the nth generation with property A. It can be shown that, almost surely,

$$\frac{z_n^A}{2^n} \to pw,$$

so that

$$\frac{z_n^A}{z_n} \to p,$$

almost surely on the set of non-extinction. The latter convergence still holds if we start from a group of three siblings since these will initiate three independent copies of the process.

Now do the same in the sibling dependent process. Let ζ_n^A be the number of A-individuals in the nth generation when starting from a group of three siblings. Then it is possible to define Z_n^A in the macro process such that $Z_n^A = \zeta_n^A$, namely by letting

$$Z_n^A = \sum_{i=1}^{2^n} Y_{n,i}^A,$$

where the $Y_{n,i}^A$ are binomial$(3,p)$, giving the numbers of A-individuals in each group of siblings. By the law of large numbers

$$\frac{Z_n^A}{2^n} \to 3p,$$

and hence also

$$\frac{\zeta_n^A}{2^n} \to 3p,$$

so that

$$\frac{\zeta_n^A}{\zeta_n} = \frac{\zeta_n^A}{2^n} \cdot \frac{2^n}{\zeta_n} \to p.$$

The asymptotic composition of A-individuals is thus the same in the independent process as in the sibling dependent one with the same marginals, of course under the condition that the independent process does not die out. □

We have seen an example where sibling dependencies affect the extinction probability, as compared with an independent process, but the growth rate and asymptotic composition are the same as in an independent non-extinct process. This is actually no coincidence; this thesis is devoted to the task of showing that quite generally, the growth rate or asymptotic composition of a branching process can never be affected by sibling dependencies. Hence, these two aspects are completely determined by the individual marginals of the joint probability measure describing reproduction and dependence structure.

Yet another example is given, now with the macro process being multi-type.

Example 3.3. Consider a sibling dependent process where an individual can beget zero, one or two children. In a group of two, one always splits into two, the other one either begets zero or one child with equal probabilities. The two siblings are equally likely to be the splitting one. A singleton, i.e. an individual with no sibling, begets two children with probability 1/2, one child with probability 1/4 and none with probability 1/4. The marginals p_k = the probability that an individual begets k children are

$$p_0 = 1/4,\ \ p_1 = 1/4,\ \ p_2 = 1/2.$$

The mean number of children in an independent process with this reproduction is thus 5/4 and the probability of eventual extinction turns out to be 1/2, whereas the dependent process never becomes extinct if it starts from a two-group. If it starts from a singleton it might become extinct only in case it avoids two-groups. Summing a geometric series shows that this probability is 1/3.

The macro process is now a multi-type Galton-Watson process with type space $\{1,2\}$. The matrix of expected reproduction is

$$\begin{bmatrix} 1/4 & 1/2 \\ 1/2 & 1 \end{bmatrix},$$

where the entry (i,j) is the expected number of j-type macro children of an i-type macro mother. The growth rate is determined by the spectral radius of M, i.e. M's largest eigenvalue denoted by ρ. Solving the characteristic equation for M yields $\rho = 5/4$, i.e. ρ equals the expected number of children in the independent process with the same marginals, and the growth rates again are the same, but the extinction probabilities differ. □

In the last example all individuals reproduce according to the same marginals, regardless of the sizes of their sibling groups. This is what is called *homogeneity* in Broberg (1987), a concept there also extended to general branching processes. Changing the probabilities for a singleton's reproduction in the example clearly implies that an individual's marginal reproduction might depend on the size of her sibling group, i.e. gives rise to a situation not covered by the homogengity concept. Two more examples are given, in order to argue why it is desirable to relax the condition that all marginals be the same.

Example 3.4. Consider a sibling dependent Galton-Watson process where individuals can beget zero, one or two children. Assume that the dependencies are described by the joint probabilities $P(k, \cdot)$ where k can be 1 or 2. The macro individuals can thus be of two types: 1 for singletons or 2 for twins. Let X_j be the number of children of the individual j and let the reproductive probabilities be given by

$$p_i = P(1, X_1 = i), \; p_0 + p_1 + p_2 = 1,$$

for singletons, and

$$p_{ij} = P(2, X_1 = i, X_2 = j), \; p_{00} + p_{01} + ... + p_{22} = 1,$$

for twins. Assume that the order between two individuals who are twins is irrelevant so that $p_{ij} = p_{ji}$ when $i \neq j$. This is essentially Example 1 in the second paper in Broberg (1987). The marginals for an individual in a twin group are

$$q_i = \sum_{j=0}^{2} P(2, X_1 = i, X_2 = j) = p_{0i} + p_{1i} + p_{2i},$$

where $i = 0, 1, 2$. Under the assumption of identical marginals it must then hold that

$$p_i = q_i = p_{0i} + p_{1i} + p_{2i}$$

for $i = 0, 1, 2$. Hence homogeneity in the sense mentioned above enforces restrictions on the marginals that obviously limit the scope of the theory. For instance, it does not cover cases where $p_{ii} > p_i$ for some i. □

A general observation is that homogeneity in sibling dependent Galton-Watson processes leads to the conclusion that the expected number of children of a sibling group is proportional to the size of that group. But, at least in biological applications, there might be strong reasons to suspect that this is not true. If a cat gives birth to ten kittens it is unlikely that they all survive, and hence their expected progeny should be less than ten times that of a single cat. It seems that homogeneity rules out such natural cases of negative correlations. Of course one can also think of situations where positive, or altering, correlations contradicts homogeneity. There might for instance be an ideal size of the sibling group, so that they are many enough to protect each other, but not so many that the competition between them gets too hard. We give another example.

Example 3.5. Consider a Galton-Watson process with sibling dependencies such that the probability of r survivors in a group of size k is $p_{r,k}$. Given that an individual survives she reproduces according to the probabilities $q(j) = P(j \text{ children})$. With $E_k[S]$ = the expected number of survivors in a group of size k, the marginals

$$p_k(j) = P(\text{individual in a group of size } k \text{ begets } j \text{ children}),$$

are

$$p_k(0) = 1 - (1 - q(0)) \sum_{r=1}^{k} \frac{r}{k} p_{r,k} = 1 - (1 - q(0)) \frac{E_k[S]}{k},$$

and

$$p_k(j) = q(j)\sum_{r=1}^{k}\frac{r}{k}p_{r,k} = q(j)\frac{E_k[S]}{k}.$$

Assuming homogeneity now implies that $E_k[S]/k$ (the probability of survival of an individual in a group of size k) is independent of k, which may be true in special cases, but certainly not generally. □

These considerations lead to the search for a richer theory of sibling dependencies. Both the examples above are cases which are not covered by homogenity, i.e. all individuals do not have the same marginals. However the marginals are the same *within* each sibling group, so this could be a first step away from homogeneity: to assume that the dependencies are such that individuals who are siblings have identical marginals. Since now an individual's reproduction depends on the size of her sibling group, not only the macro individuals but also the individuals constitute a multi-type process, the type of an individual being the number of siblings in her group. The type space will thus be the same for the individual process and the macro process. The expected reproduction matrices are then $M = (M_{ij})$ and $m = (m_{ij})$ for the macro and individual process respectively. Here M_{ij}=the expected number of macro children of type j of a macro mother of type i, which obviously equals $ip_i(j)$, where $p_i(j)$ is the marginal probability that an individual of type i begets j children. For the individual process the corresponding numbers are $m_{ij} = jp_i(j)$ and the relation

$$M_{ij} = \frac{i}{j}m_{ij}, \tag{3.1}$$

between the macro and individual process is obtained. Analysis of this shows that the two matrices are equivalent with respect to important properties such as positivity, and that in the finite case, i.e. if the number of children is bounded, the spectral radii coincide. These facts will later be proven in a more general context, and for now they are only illustrated by a simple special case.

Example 3.6. Consider a sibling dependent Galton-Watson process where each individual can have zero, one or two children. The marginals are $p_i(j), i = 1, 2, j = 0, 1, 2$ and the type space $\{1, 2\}$. The matrices for expected reproduction are $m = (m_{ij})$ and $M = (M_{ij})$ for the individual and the macro process respectively; the elements of which relate according to (3.1). Recall that a matrix A is called positive if for some n, all the elements of A^n are positive. Computation of the powers of M and m shows that

$$M_{ij}^n = \frac{i}{j}m_{ij}^n,$$

for all n, so positivity is equivalent for the two matrices.

The characteristic equation for m is

$$(m_{11} - \lambda)(m_{22} - \lambda) - m_{12}m_{21},$$

and for M

$$(M_{11}-\lambda)(M_{22}-\lambda)-M_{12}M_{21}=(m_{11}-\lambda)(m_{22}-\lambda)-m_{12}m_{21},$$

since $M_{11}=m_{11}, M_{22}=m_{22}, M_{12}=(1/2)m_{12}$ and $M_{21}=2m_{21}$. Hence the spectral radii coincide, and the growth of the sibling dependent process is determined through the marginals alone. □

We will now take a brief look at what happens in a single-type general branching process with sibling dependencies, thus slowly approaching the coming general theory.

In a general single-type branching process an individual reproduces according to some point process, ξ, on R_+. Assume that the reproduction and dependence structure of a sibling group born according to a particular realization s of ξ, are described by a joint probability measure $P(s,\cdot)$ on $\Omega^{s(\infty)}$, Ω being the life space and $s(\infty)$ the number of points in s. In obvious analogy with the Galton-Watson case, a macro individual (still a sibling group) will have type s if that sibling group is born according to the point process s. Hence the type space S will be $\mathcal{N}(R_+)$, the set of point processes on R_+, or some appropriate subset thereof. Since there is no point in dealing with empty sibling groups, it is assumed that $s(\infty)>0$. The possibility of infinitely many children is also ruled out by assuming that $s(\infty)<\infty$ for all $s\in S$.

Assume that the marginals of $P(s,\cdot)$ may depend on s, but not on the rank of an individual. Call these marginals $Q(s,\cdot)$ and for sake of convenience view them as measures on S, so that $Q(s,A)$ is the probability that an individual born in a sibling group with type s, reproduces according to a point process in the set $A\subseteq S$. If we define the birth time of a macro individual to be the moment the individual mother of that particular group of siblings is born, the macro process has reproduction measure

$$M(s,dr\times dt)=s(dt)Q(s,dr),$$

since its macro children are born at the points in s, each one having probability $Q(s,dr)$ to give rise to a macro individual with type in dr.

The individual process also becomes multi-type with the type space S, and its reproduction measure is

$$\mu(s,r\times dt)=r(dt)Q(s,dr).$$

It is possible to show that M and μ define the same Malthusian parameter α and with

$$\hat{s}(\alpha)=\int_0^\infty e^{-\alpha t}s(dt),$$

the Laplace transform of s, we will have

$$\hat{M}(s,dr)=\int_0^\infty e^{-\alpha t}M(s,dr\times dt)=\hat{s}(\alpha)Q(s,dr)$$

and

$$\hat{\mu}(s, dr) = \hat{r}(\alpha)Q(s, dr)$$

so that

$$\hat{M}(s, dr) = \frac{\hat{s}(\alpha)}{\hat{r}(\alpha)}\hat{\mu}(s, dr),$$

a relation reminding of (3.1) thus suggesting that properties of importance should be equivalent for $\hat{M}$ and $\hat{\mu}$, although we have not yet defined these properties. Again it seems that crucial properties of the sibling dependent process are determined through the marginals alone, or, if you wish, that we might just as well study an ordinary independent multi-type process with the same marginals as the sibling dependent one. At least this holds as long as growth and composition are regarded; it has from the beginning of this section been made clear that extinction is a phenomenon highly sensitive to dependencies.

If the marginals are also allowed to depend on an individual's rank they will be $Q_i(s, dr)$=the probability that individual i in a sibling group s will reproduce according to a point process in dr. The macro reproduction measure is then

$$M(s, dr \times dt) = \sum_{i=1}^{\infty} \delta_{\tau_i(s)}(dt)Q_i(s, dr),$$

where $\tau_i(s)$ is the time point for the ith point in s. Its Laplace transform is

$$\hat{M}(s, dr) = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(s)}Q_i(s, dr).$$

The type space for the macro process is obviously still S as above, but for the individual process the situation changes slightly. The type of an individual is now the pair (s, i), telling not only what point process she comes from, but also her rank. The type space is thus $S \times Z_+$ and the individual reproduction measure is

$$\mu((s, i), dr \times \{j\} \times dt) = \delta_{\tau_j(r)}(dt)Q_i(s, dr),$$

with Laplace transform

$$\hat{\mu}((s, i), dr \times \{j\}) = e^{-\alpha\tau_j(r)}Q_i(s, dr).$$

The relation between $\hat{M}$ and $\hat{\mu}$ is not so obvious this time, but a general observation to be made for both the cases above is that the individual process is multi-type. Of course the sibling dependencies are still there, but given an individual's type her *marginal* reproduction does not depend on the rest of her sibling group. Hence both these cases would be solved by investigating the general problem of *multi-type* sibling dependent general branching processes, such that an individual's marginal reproduction follows a law determined by her type only, and not dependent of the rest of her siblings. We thus start with a multi-type process from the beginning and since it has sibling dependencies this can be viewed as the multi-type analogue of Broberg's homogeneity. Applied to the different situations above it will obviously solve quite general situations of sibling dependecies in single-type processes.

In fact it will also cover the situation where an individual's marginal in a multi-type process *is* allowed to depend on the rest of the sibling group, but before showing tha, a lot of work has to be done building up the theory.

3.2. The Macro Process

Recall the life space $(\Omega, \mathcal{A})$, the set of all possible individual life careers, Ω, equipped with a σ-algebra $\mathcal{A}$. To describe the sibling dependencies assume that the individuals born in a particular realization, γ, of a point process choose their lives according to a joint probability kernel $\tilde{P}(\gamma, \cdot)$ on $(\Omega^{\gamma(S\times R_+)}, \mathcal{A}^{\gamma(S\times R_+)})$, independently of all other individuals. For the sake of convenience view $\tilde{P}(\gamma, \cdot)$ as a measure on $(\Omega^\infty, \mathcal{A}^\infty)$, simply by choosing the ω_k in some arbitrary way for $k > \gamma(S \times R_+)$. This can be done since the lives of never born individuals of course will not matter to the evolution of the population. The process of interest will be that of individuals starting from one group of siblings, with dependencies described by the $\tilde{P}(\gamma, \cdot)$. This process will be analyzed through the corresponding macro process where a macro individual as mentioned is a group of siblings. Since there are no dependencies between individuals that are not siblings, we can view the macro process as an ordinary multi-type process according to the following construction.

Let the life of a macro individual simply be the vector of the individual lives of that particular group of siblings. A macro individual's life is thus an element in $\bar{\Omega} = \Omega^\infty$ and of the form $\bar{\omega} = (\omega_k)_{k=1}^{\infty}$. In analogy with the discussion of single-type processes at the end of Section 3.1 the type of a macro individual is the point process that describes that group of siblings. Hence take the type space Γ to be

$$\{\gamma \in \mathcal{N}(S \times R_+) : 0 < \gamma(S \times R_+) < \infty\},$$

the set of point processes on $S \times R_+$ with a positive and finite number of points, or , since not all possible point processes necessarily appear (consider for example splitting populations) some appropriate subset thereof. This means that we rule out the possibility of infinitely many children which is a natural restriction, see for instance Theorem 7.2 in Jagers (1989). Of course the condition $\gamma(S \times R_+) > 0$ just means that we disregard empty groups of siblings. An element $\gamma \in \Gamma$ consists of the sequence $(\sigma_i(\gamma), \tau_i(\gamma))$ where $i = 1, ..., \gamma(S \times R_+)$, the types and time points where γ puts its masses. As a σ-algebra on Γ we take

$$\mathcal{G} = \sigma\{\{\gamma : \sigma_i(\gamma) \in A_i, \tau_i(\gamma) \in B_i, i = 1, ..., n\}, A_i \in \mathcal{S}, B_i \in \mathcal{B}, n = 1, 2, ...\},$$

the cylinder-generated σ-algebra of sets in G, $\mathcal{B}$ denoting the Borel σ-algebra. Now define the types $\sigma'(i) : \bar{\Omega} \to \Gamma$ of macro individuals such that the i:th macro child of a macro mother with life $\bar{\omega} = (\omega_k)_{k=1}^{\infty}$ will have the type

$$\sigma'(i)(\bar{\omega}) = \xi(\omega_i).$$

The *macro population space* is $(\Gamma \times \bar{\Omega}^I, \mathcal{G} \times \bar{\mathcal{A}}^I)$, an outcome of which is denoted $(\gamma, \bar{\omega}_I)$ and consists of the type of the macro ancestor and the lives of all conceivable macro individuals. The set $\{\tilde{P}(\gamma, \cdot), \gamma \in \Gamma\}$ defines a unique probability measure $\tilde{P}_\gamma(\cdot)$ on $(\Gamma \times \bar{\Omega}^I, \mathcal{G} \times \bar{\mathcal{A}}^I)$ for each starting type $\gamma \in \Gamma$. In order to view the types as functions on $\bar{\Omega}^I$ rather than on $\bar{\Omega}$, we use the coordinate projections $U_x : \Gamma \times \bar{\Omega}^I \to \bar{\Omega}$ defined through

$$U_x(\gamma, \bar{\omega}_I) = \bar{\omega}_x,$$

the life of the macro individual x. The type of macro individual xi is then defined as

$$\sigma'_{xi}(\gamma, \bar{\omega}_I) = \sigma'(i) \circ U_x(\gamma, \bar{\omega}_I) = \sigma'(i)(\bar{\omega}_x) = \xi(\omega_{xi}).$$

Henceforth we will write ξ_x rather than σ'_x for the macro individual x's type.

Returning to the individuals, starting from a group of siblings means that we consider the individual space $I_1 = I \setminus \{0\}$, where the ancestor has been removed. The population space will then be Ω^{I_1}. To relate this space to the macro population space we add to it the point process describing the initial group of siblings and introduce the mapping $\varphi : \Gamma \times \Omega^{I_1} \to \Gamma \times \bar{\Omega}^I$ such that

$$\varphi(\gamma, (\omega_x)_{x \in I_1}) = (\gamma, \bar{\omega}_x)_{x \in I}.$$

Hence φ groups lives of individuals that are siblings together to a macro life. Since it consists of concatenations and coordinate projections only, it is measurable. We will also be interested in the individual process starting from the individual x and thus introduce the projections $q_x : \Gamma \times \Omega^{I_1} \to S \times \Omega^I$ such that

$$q_x(\gamma, (\omega_y)_{y \in I_1}) = (\sigma_x, (\omega_{xy})_{y \in I}).$$

In a more compact notation we can write these as

$$\varphi(\gamma, \omega_{I_1}) = (\gamma, \bar{\omega}_I)$$

and

$$q_x(\gamma, \omega_{I_1}) = (\sigma_x, \omega_{xI}).$$

Composing q_x and φ enables us to pick individuals from the macro space and we define the projections $p_x : \Gamma \times \bar{\Omega}^I \to S \times \Omega^I$ through

$$p_x(\gamma, \bar{\omega}_I) = q_x(\varphi^{-1}(\gamma, \bar{\omega}_I)).$$

Hence p_x selects the type, life and progeny of the individual x from the macro space. The three spaces and their connections are illustrated in Figure 3.2.

The lives of the individuals in I_1 will then follow the law $P_\gamma(d\omega_{I_1}) = \tilde{P}_\gamma \varphi(d\omega_{I_1})$. Individuals thus choose lives according to the marginals of $\tilde{P}_\gamma$. Generally the ith individual in a group of siblings with type γ chooses her life according to the marginal

$$\tilde{Q}_i(\gamma, A) = \int_{\bar{\Omega}} 1_A(\omega_i) \tilde{P}(\gamma, d\bar{\omega})$$

$$\begin{tikzcd}
\Gamma \times \bar{\Omega}^I \arrow[rr, "\varphi^{-1}"] \arrow[dr, "p_x"'] & & \Gamma \times \Omega^{I_1} \arrow[dl, "q_x"] \\
& S \times \Omega^I &
\end{tikzcd}$$

Figure 3.2: The three population spaces.

for $A \in \mathcal{A}$. We will assume that an individual's marginal reproduction *only depends on her type*, so if the ith individual in a sibling group with type γ has type $\sigma_i(\gamma)$ we will have

$$\tilde{Q}_i(\gamma, \cdot) = \tilde{Q}(\sigma_i(\gamma), \cdot),$$

where $\tilde{Q}(s, \cdot)$ for each $s \in S$ is a probability measure on the life space. This means that given an individual's type, her *marginal reproduction* does not depend on anything that has to do with her siblings. There are still dependencies between the actual reproductions though, so that given some information about your siblings' number of children for instance, your marginals will change.

As will be seen later, it is convenient to transform $\tilde{Q}$ into a measure on Γ so we define

$$Q(s, A) = \tilde{Q}(s, \{\xi \in A\}), \quad A \in \mathcal{G}.$$

The population of individuals started from a single individual ancestor of type s has marginal defined through

$$\tilde{Q}_s(d\omega_I) = P_{\xi(\omega_0)}(d\omega_{I_1})\tilde{Q}(s, d\omega_0). \tag{3.2}$$

To define *birth times* for the individuals in I_1 we assume that the process starts from a particular group of siblings, γ. As for the macro process we need coordinate projections $u_x : \Gamma \times \Omega^{I_1} \to \Omega$ such that $u_x(\gamma, \omega_{I_1}) = \omega_x$. With $\tau_i(\gamma)$ as above we define

$$\tau_i(\gamma, \omega_{I_1}) = \tau_i(\gamma)$$

$$\tau_{xi}(\gamma, \omega_{I_1}) = \tau_x(\gamma, \omega_{I_1}) + \tau(i) \circ u_x(\gamma, \omega_{I_1})$$

A macro individual is considered born when the individual mother of that group of siblings is born, if she begets any children. This means that a macro individual of type γ begets her i:th macro child at age $\tau_i(\gamma)$. Then we can define birth times for the macro individuals recursively:

$$\tau'_0(\gamma, \bar{\omega}_I) = \begin{cases} 0 & \text{if } \gamma(S \times R_+) > 0 \\ \infty & \text{otherwise} \end{cases}$$

$$\tau'_{xi}(\gamma, \bar{\omega}_I) = \begin{cases} \tau'_x(\gamma, \bar{\omega}_I) + \tau_i(\xi_x(\gamma, \bar{\omega}_I)) & \text{if } \xi_{xi}(S \times R_+) > 0 \\ \infty & \text{otherwise.} \end{cases}$$

Note that by this we do not allow empty groups of siblings to be born in finite time. We also see that if the individual x begets any children then $\tau'_x = \tau_x \circ \varphi$. The macro reproduction process becomes

$$\eta((\gamma, (\omega_i)_{i=1}^{\infty}), A \times [0, t]) = \#\{i : \xi(\omega_i) \in A, \tau_i(\gamma) \leq t\}.$$

where $A \in \mathcal{S}$ and $t \geq 0$.

The macro reproduction kernel, i.e. the expectation of η with respect to the measure $\tilde{P}(\gamma, \cdot)$ is then

$$M(\gamma, A \times [0, t]) = E_\gamma[\eta(A \times [0, t])]$$

$$= \sum_{i=1}^{\infty} Q(\sigma_i(\gamma), A) 1_{\{\tau_i(\gamma) \leq t\}},$$

the number of macro individuals with types in A born before t. Hence, with δ_x denoting the unit point mass in x,

$$M(\gamma, d\gamma' \times dt) = \sum_{i=1}^{\infty} Q(\sigma_i(\gamma), d\gamma') \delta_{\tau_i(\gamma)}(dt).$$

The dependence structure could be made more general by letting the $\tilde{P}$ depend on the mother's whole life, ω, rather than just the reproduction process $\gamma = \xi(\omega)$. However, most of the results to come would then be modified basically by replacing γ by $\xi(\omega)$. The macro reproduction measure would for instance be

$$M(\omega, d\omega' \times dt) = \sum_{i=1}^{\infty} \tilde{Q}(\sigma_i(\xi(\omega)), d\omega') \delta_{\tau_i(\xi(\omega))}(dt),$$

so nothing substantial is lost by our special construction.

Let us rest here for a moment and sum up what has been done so far. We wish to study a branching process where there are dependencies between siblings. To get rid of these dependencies a new branching process consisting of sibling groups, called macro individuals, is constructed. This new process, the macro process, is an ordinary multi-type branching process with conditional independencies given types. It is natural to start from a full group of siblings and our primary objective is to study the sibling dependent process starting in that way. This process is related to the macro process through the mapping φ. Some coordinate projections are introduced to enable us to select individual lives and daughter processes. The type of a macro individual is the reproduction process that describes that group of siblings, and the birth time is the time when the individual mother of that sibling group is born. Empty sibling groups are simply disregarded.

So now a macro process has been constructed; it is an ordinary multi-type branching process but would not be of much use unless it could measure the same aspects of the population as the individual, sibling dependent, process does. The next objective is to show that the macro process is indeed useful in this way.

Recall the random characteristics $\chi : S \times \Omega^I \times R \to R_+$ introduced in Section 2.2, where $\chi(s, \omega_I, t)$ gives the contribution at time t of an s-type individual with life and progeny's lives described by ω_I. Also recall the χ-counted population

$$z_t^\chi = \sum_{x \in I} \chi_x(t - \tau_x),$$

measured by adding the contributions of all individuals born up to time t, at their proper ages.

In the sibling dependent case the process starts from a group of siblings and the population space is then $\Gamma \times \Omega^{I_1}$ so we define

$$Z_t^\chi = \sum_{x \in I_1} \chi_x(t - \tau_x), \tag{3.3}$$

the χ-counted population with the ancestor removed. Here χ_x should be interpretated as $\chi \circ q_x$ since the population space under consideration is $\Gamma \times \Omega^{I_1}$. Note that it is still the *individual* process that is of interest but it is started from a group of siblings with the intention to link it to the macro process in some fashion. We are interested in the asymptotics of $E_\gamma[Z_t^\chi]$, the expected population size starting from a sibling group born according to the point process $\gamma \in \Gamma$, as well as of Z_t^χ itself. To relate this individual χ-counted population with sibling dependencies to the macro process, which is an ordinary multi-type branching process where (much of) the known theory applies, a macro characteristic, χ', that measures exactly the same thing as the individual characteristic, χ, needs to be found. And indeed, such a macro characteristic exists, namely

$$\chi'(t) = \sum_{i=1}^{\infty} \chi_i(t - \tau_i), \tag{3.4}$$

where

$$\chi_i(t - \tau_i)(\gamma, \bar{\omega}_I) = \chi(p_i(\gamma, \bar{\omega}_I), t - \tau_i(\gamma)) = \chi(\sigma_i(\gamma), \omega_{iI}, t - \tau_i(\gamma)).$$

Lemma 3.1. *Let* χ' *be defined through* (3.4) *and*

$$Z_t^{\chi'} = \sum_{x \in I} \chi'(t - \tau'_x).$$

Then

$$Z_t^{\chi} = Z_t^{\chi'} \circ \varphi$$

Proof.

$$\begin{aligned}
Z_t^{\chi'} \circ \varphi(\gamma, \omega_{I_1}) &= Z_t^{\chi'}(\gamma, \bar{\omega}_I) \\
&= \sum_{x \in I} \chi'(S_x(\gamma, \bar{\omega}_I), t - \tau_x'(\gamma, \bar{\omega}_I)) \\
&= \sum_{x \in I} \chi'(\xi_x(\gamma, \bar{\omega}_I), \bar{\omega}_{xI}, t - \tau_x'(\gamma, \bar{\omega}_I)) \\
&= \sum_{x \in I} \sum_{i=1}^{\infty} \chi(p_i(\xi_x(\gamma, \bar{\omega}_I), \bar{\omega}_{xI}), t - \tau_x'(\gamma, \bar{\omega}_I) - \tau_i(\xi_x(\gamma, \bar{\omega}_I))) \\
&= \sum_{x \in I} \sum_{i=1}^{\infty} \chi(\sigma_{xi}(\gamma, \omega_{I_1}), \omega_{xiI}, t - \tau_{xi}(\gamma, \omega_{I_1})) \\
&= \sum_{y \in I_1} \chi(\sigma_y(\gamma, \omega_{I_1}), \omega_{yI}, t - \tau_y(\gamma, \omega_{I_1})) = Z_t^{\chi}(\gamma, \omega_{I_1}),
\end{aligned}$$

since

$$\tau_x'(\gamma, \bar{\omega}_I) + \tau_i(\xi_x(\gamma, \bar{\omega}_I)) = \tau_{xi} \circ \varphi(\gamma, \omega_{I_1}).$$

■

In words, the macro characteristic of course does nothing more complicated than summing the contributions of all individuals contained in each sibling group. This means that χ' does not maintain all the properties χ might have; if χ is an indicator χ' will not necessarily be an indicator, if χ is bounded then χ' is not necessarily bounded etc.

If the process starts from a single individual rather than a full sibling group, we can of course just add the contribution of the individual ancestor and start counting by χ' from the first generation. Thus

$$z_t^{\chi}(s, \omega_I) = \chi_0(s, \omega_I, t) + Z_t^{\chi'} \circ \varphi(\xi(s, \omega_I), \omega_{I_1}).$$

In order not to bother too much about the mapping φ the notation E_γ will be used for expectation both with resepct to $\tilde{P}_\gamma$ and P_γ. Hence when we for instance write

$$E_\gamma[Z_t^{\chi'}] = E_\gamma[Z_t^{\chi}],$$

the expectations actually refer to different measures, but what matters is that the subscript denotes the initial group of siblings. No ambiguities will arise from this.

4. Markov Renewal Theory and Its Connection to Branching Processes

4.1. Markov Renewal Theory...

The asymptotics of the expected population size are analyzed by means of Markov renewal theory, and we will give the definitions and results needed for this purpose, following Shurenkov (1992) which is an english version of Shurenkov (1989). Let $(E, \mathcal{E})$ be a measurable space and $K(x, dy \times dt)$ a non-negative kernel, i.e. $K(x, A \times [0,t])$ defines a measure on $(E \times R_+, \mathcal{E} \times \mathcal{B})$ for each $x \in E$ and an $\mathcal{E}$-measurable function for each $A \in \mathcal{E}$ and $t \geq 0$ ($\mathcal{B}$ the Borel algebra on R_+). The kernel U is defined as

$$U(x, dy \times dt) = \sum_{n \geq 0} K^n(x, dy \times dt)$$

where

$$K^n(x, dy \times dt) = \int_{E \times R_+} K^{n-1}(x, dz \times du) K(z, dy \times dt - u)$$

and

$$K^0(x, dy \times dt) = \delta_{(0,x)}(dy \times dt),$$

placing a unit point mass in $(0, x)$. Further, convolution between the kernel U and a non-negative function g is defined as

$$U * g(x,t) = \int_{E \times R_+} g(y, t-u) U(x, dy \times du).$$

The main objective of Markov renewal theory is the asymptotics of $U * g(x,t)$ as t tends to infinity. A crucial entity in this analysis is the kernel

$$\hat{K}(x, dy) = \int_0^\infty K(x, dy \times dt)$$

on $(E, \mathcal{E})$, called the *basis* of K.

Definition 4.1. *$\hat{K}$ is called σ-finite if there exists a strictly positive, $\mathcal{E} \times \mathcal{E}$-measurable function f such that*

$$\int_E f(x,y) \hat{K}(x, dy) < \infty$$

for each $x \in E$.

Now let

$$\hat{K}^n(x, dy) = \int_E \hat{K}^{n-1}(x, dz) \hat{K}(z, dy)$$

and

$$\hat{K}^0(x, dy) = \delta_x(dy),$$

the unit point mass in x.

Definition 4.2. *The Perron root of $\hat{K}$, denoted by $\rho(\hat{K})$, is defined through*

$$\frac{1}{\rho(\hat{K})} = \sup\{\lambda \geq 0 : \sum_{n\geq 0} \lambda^n \hat{K}^n \text{is } \sigma\text{-finite}\}$$

The Perron root might well be zero or infinity. From now on it is assumed that $\hat{K}$ has Perron root one.

Definition 4.3. *$\hat{K}$ is called (m-)positive if there exists a non-trivial σ-finite measure, m, on $(E, \mathcal{E})$ such that*

$$m(A) > 0 \Rightarrow \hat{K}(x, A) > 0$$

for all $x \in E$.

One more definition remains before we can give the first result.

Definition 4.4. *$\hat{K}$ is called conservative if the kernel $\sum_{n\geq 0} \hat{K}^n$ is positive and*

$$\sum_{n\geq 0} \int_{E\times E} g(y)\hat{K}^n(x, dy)m(dx) = \infty.$$

for all strictly positive $\mathcal{E}$-measurable functions g and all non-zero measures $m \in \mathcal{M}$, where $\mathcal{M} = \{m : \sum_{n\geq 0} \hat{K}^n$ is m-positive$\}$.

In the case where $\mathcal{E}$ is countably generated there is an equivalent definition of conservativity.

Lemma 4.5. *If $\mathcal{E}$ is countably generated then $\hat{K}$ is conservative if and only if there exists a σ-finite non-zero measure m on $(E, \mathcal{E})$ such that*

$$m(A) > 0 \Rightarrow \sum_{n\geq 0} \hat{K}^n(x, A) = \infty, \quad A \in \mathcal{E},$$

for all $x \in E$.

Theorem 4.6. *If $\hat{K}$ is conservative, then there exists a measure l and an l-almost everywhere finite, strictly positive function h, such that*

$$\int_E \hat{K}(x, A)l(dx) = l(A), \quad A \in \mathcal{E},$$

and

$$\int_E h(y)\hat{K}(x, dy) = h(x)$$

for l-almost every $x \in E$. Both l and h are unique up to multiplicative constants.

We are approaching the main result, the Key Markov Renewal Theorem. However a few more concepts will be needed. First the definition of latticeness.

Definition 4.7. *$K(x, dy \times dt)$ is called lattice if there exists a number $d > 0$ and a function $c : E \to [0, d]$ such that $K(x, dy \times dt)$ is concentrated on the lattice*

$$\{(t, y) \in R_+ \times E : t = c(y) - c(x) + nd, \textit{ for some } n = 0, 1, \ldots\}$$

for l-almost all $x \in E$.

We also denote

$$\beta = \int_{E \times E \times R_+} th(y) K(x, dy \times dt) l(dx) \tag{4.1}$$

and finally give the following definition.

Definition 4.8. *A function $g : E \times R_+ \to R_+$ is called directly Riemann integrable (l) if*

$$\sum_{n \geq 0} \int_E \sup_{n \leq t \leq n+1} g(x, t) l(dx) < \infty$$

and

$$\delta \sum_{n \geq 0} \int_E \Big(\sup_{n\delta \leq t \leq (n+1)\delta} g(x, t) - \inf_{n\delta \leq t \leq (n+1)\delta} g(x, t) \Big) l(dx) \to 0,$$

as $\delta \to 0+$.

Theorem 4.9. (Key Markov Renewal Theorem) *Assume that $K(x, dy \times dt)$ is non-lattice and that $\hat{K}(x, dy)$ is conservative. Also assume that $0 < \beta < \infty$, the function $g \geq 0$ is directly Riemann integrable and that the set*

$$\{x \in E : \sup_{t \geq 0} U * g(x, t) < \infty\}$$

has positive l-measure. Then, for l-almost all $x \in E$,

$$U * g(x, t) \to \frac{h(x)}{\beta} \int_{E \times R_+} g(x, u) l(dx) du$$

as $t \to \infty$.

There is also a lattice analogue of Theorem 4.9. It states that, if the kernel K is lattice with step d and shift c,

$$U * g(x, nd + c(x)) \to \frac{d}{\beta} h(x) \int_E \sum_{k \geq 0} g(y, kd + c(y)) l(dy),$$

for l-almost all $x \in E$. Since the lattice case is not treated in this thesis we do not give the conditions under which it holds; it is mentioned for the sake of completeness.

4.2. ...and Its Connection to Branching Processes

We will now see why Markov renewal theory is applicable to multi-type branching processes. For this purpose we follow Jagers (1989) and (1992). Forget about sibling dependencies for a while and look quite generally at a multi-type process with type space S and reproduction measure $\mu(s, dr \times dt)$. For any real λ define

$$\mu_\lambda(s, dr \times dt) = e^{-\lambda t}\mu(s, dr \times dt)$$

and

$$\hat{\mu}_\lambda(s, dr) = \int_0^\infty \mu_\lambda(s, dr \times dt).$$

The *Malthusian parameter*, α, is now defined so that the kernel $\hat{\mu}_\alpha(s, dr)$ has Perron root one. With

$$\nu_\alpha(s, dr \times dt) = \sum_{n \geq 0} \mu_\alpha^n(s, dr \times dt), \tag{4.2}$$

called the total population measure, we have

$$E_s[e^{-\alpha t} z_t^\chi] = \int_{S \times R_+} E_r[e^{-\alpha(t-u)}\chi(t-u)]\nu_\alpha(s, dr \times du). \tag{4.3}$$

Hence, with the notation from the previous section, we have

$$E_s[e^{-\alpha t} z_t^\chi] = U * g(s, t),$$

where $U = \nu_\alpha$ and $g(s, t) = E_s[e^{-\alpha t}\chi(t)]$. It is thus the kernel μ_α that plays the role of K and $\hat{\mu}_\alpha$ will correspond to $\hat{K}$. Once α has been fixed the subscript will be dropped, and for instance we write $\hat{\mu}$ rather than $\hat{\mu}_\alpha$. If the kernel $\hat{\mu}$ is conservative there exists a function h and a measure π such that

$$h(s) = \int_S h(r)\hat{\mu}(s, dr)$$

and

$$\pi(ds) = \int_S \hat{\mu}(r, ds)\pi(dr).$$

These entities now have interpretations. Thus $h(s)$ is the reproductive value of an individual of type s, in the sense that the larger the h-value, the greater the average contribution to the population in the long run. Assume that $h \in L^1[\pi]$, so that we can, and will, norm to $\int_S h(s)\pi(ds) = 1$. If further $\inf h > 0$, π can be normed to a probability measure. We will make this assumption and norming throughout this paper. Then π will be referred to as the stable type distribution, loosely meaning that a newborn individual picked at random from an old population will have its type distributed according to π, regardless of the initial conditions. We make the follwing definition:

Definition 4.10. *A branching process is called strictly Malthusian if* $\alpha > 0$, $0 < \beta < \infty$, $\hat{\mu}(s, dr)$ *is conservative and there exists an* $a > 0$ *such that*

$$\sup_s \mu(s, S \times [0, a]) < 1.$$

Note. If $\alpha > 0$, the process is called *supercritical* (whereas it is called critical or subcritical if $\alpha = 0$ or $\alpha < 0$ respectively). The number β is now

$$\beta = \int_{S \times S \times R_+} te^{-\alpha t} h(r)\mu(s, dr \times dt)\pi(ds),$$

to be called the *stable age at childbearing*; a precise interpretation of this is given in Jagers and Nerman (1992) and will also appear in Chapter 7. □

The condition $\sup_s \mu(s, S \times [0, a]) < 1$ is a natural regularity assumption which together with some condition on χ guarantees that the set $\{s \in S : \sup_{t \geq 0} E_s[e^{-\alpha t} z_t^\chi] < \infty\}$ has positive π-measure (actually even that this supremum is finite for *any* $s \in S$).

Under the appropriate conditions the Key Markov Renewal Theorem thus yields the convergence of $E_s[e^{-\alpha t} z_t^\chi]$ to some limit. Using the notation E_π for expectation when the ancestor's type is not fixed but distributed according to π, i.e. $E_\pi[X] = \int_S E_s[X]\pi(ds)$, we have

$$\begin{aligned} &\int_{S \times R_+} g(s,t)\pi(ds)dt \\ &= \int_{S \times R_+} E_s[e^{-\alpha t}\chi(t)]dt\pi(ds) = \int_S E_s[\hat{\chi}(\alpha)]\pi(ds) = E_\pi[\hat{\chi}(\alpha)] \end{aligned}$$

and hence the convergence is

$$E_s[e^{-\alpha t} z_t^\chi] \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta} h(s) \tag{4.4}$$

as $t \to \infty$, for π-almost all starting types $s \in S$.

Now, what happens in a sibling dependent process? Well, the equation (4.3) is derived by considering expectations of sums and the following property: Let $\mathcal{C}_n$ be the σ-algebra generated by the ancestor's type and the lives of all individuals up to the nth generation and let $x \in N^{n+1}$, the $(n+1)$th generation. Then

$$\tilde{Q}(s, A|\mathcal{C}_n) = \tilde{Q}(\sigma_x, A),$$

for $A \in \mathcal{A}$. Since this holds also in the sibling dependent case, (4.3) is still valid. Also note that the concept of strict Malthusianness is defined only through the marginal reproduction measures $\mu(s, dr \times dt)$ (recall that it has been assumed that marginals only depend on the individual's type so that μ makes sense). Therefore the convergence (4.4) holds also in the sibling dependent case. When we later turn our interest to convergence of the process $e^{-\alpha t} Z_t^\chi$ it will however be necessary to start from a full group of siblings so we must examine the convergence of $E_\gamma[e^{-\alpha t} Z_t^\chi]$, the normed expected χ-counted population starting from a group of siblings born according to the point process γ. The special form of χ' given in (3.4) yields

$$E_\gamma[e^{-\alpha t} Z_t^\chi] = E_\gamma[e^{-\alpha t} Z_t^{\chi'}] = \sum_{i=1}^{\infty} E_\gamma[e^{-\alpha t} z_{t-\tau_i(\gamma)}^\chi(i)]$$

$$= \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} E_{\sigma_i(\gamma)}[e^{-\alpha(t-\tau_i(\gamma))} z^{\chi}_{t-\tau_i(\gamma)}].$$

Note that this sum is actually finite, with $\gamma(S \times R_+)$ terms. Since all the $E_{\sigma_i(\gamma)}[e^{-\alpha(t-\tau_i(\gamma)} z^{\chi}_{t-\tau_i(\gamma)}]$ converge according to (4.4), the convergence

$$E_\gamma[e^{-\alpha t} Z^{\chi}_t] \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta} \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} h(\sigma_i(\gamma)) \qquad (4.5)$$

is established, at least for all $\sigma_i(\gamma)$ outside some set $S_0 \in S$ with π-measure zero. The question arises if there exists some measure on the macro type space Γ, telling for which starting types $\gamma \in \Gamma$ the macro process converges. If so, will this measure give mass zero to all point processes that contains the types in S_0?

Another question of importance arises from the fact that the macro process converges without any Markov renewal conditions; it is enough to assume these conditions at the individual level. This is of course in harmony with the intentions not to add any extra conditions on the macro level but the question poses itself: could the macro process possibly be conservative etc (and hence converge) even if the individual process is not?

These questions call for a more thorough examination of the relation between the individual and the macro process. How do the reproduction measures μ and M relate to each other? Do they determine the same Malthusian parameter? If $\hat{\mu}$ is conservative, is $\hat{M}$ and vice versa? If invariant functions and measures exist, how are they connected?

Don't miss the next section, you will get the answers there!

5. Properties of the Macro Process

5.1. A Preparatory Lemma

In the proofs to come it will be convenient to add to the macro type space some specific types, i.e. reproduction processes, that might not appear naturally in the population. The reason for this will be obvious, but the question is if it causes any trouble to manipulate the population like this. In the following lemmas it is shown that, in a certain sense, the answer is no. They are formulated in the general language of Section 4.1. Recall the concept of σ-finiteness and use the notation AB for convolution of two kernels on the same space E, i.e.

$$AB(x, dy) = \int_E A(x, dz)B(z, dy).$$

Lemma 5.1. *Let A and B be two non-negative kernels on some space $(E, \mathcal{E})$. If A and B are σ-finite then the two kernels AB and $A + B$ are both σ-finite.*

Proof. By assumption there exist measurable, positive functions f and g on $E \times E$ such that

$$\int_E f(x, y)A(x, dy) < \infty$$

and

$$\int_E g(x, y)B(x, dy) < \infty,$$

for all $x \in E$. Call these integrals $\varphi(x)$ and $\psi(x)$ respectively. Define

$$A_0(x, dy) = \frac{f(x, y)}{\varphi(x)} A(x, dy)$$

and

$$B_0(x, dy) = \frac{g(x, y)}{\psi(x)} B(x, dy).$$

Clearly $A \ll A_0$ and $B \ll B_0$ for all $x \in E$ and by the Radon-Nikodym theorem there exist functions a and b such that

$$A(x, dy) = a(x, y)A_0(x, dy)$$

and

$$B(x, dy) = b(x, y)B_0(x, dy).$$

(Since also $A_0 \ll A$, actually $a(x, y) = \varphi(x)/f(x, y)$ and similarily for b.) Now define $C = AB$ and $C_0 = A_0B_0$. Clearly $C \ll C_0$ and the Radon-Nikodym derivative c exists, satisfying $C(x, dy) = c(x, y)C_0(x, dy)$. Hence there exists a function h, namely $h(x, y) = 1/c(x, y)$, such that

$$\int_E h(x, y)C(x, dy) = \int_E C_0(x, dy) = C_0(x, E) = 1,$$

and C is σ-finite.

That $A + B$ is σ-finite follows more or less trivially by choosing instead the function $h(x, y) = \min(f(x, y), g(x, y))$. ■

Lemma 5.2. *Let $(E, \mathcal{E})$ and $(E', \mathcal{E}')$ be measurable spaces such that $E \subset E'$, the σ-algebras $\mathcal{E}$ and $\mathcal{E}'$ are countably generated and the restriction of $\mathcal{E}'$ to E is $\mathcal{E}$. Let K be a non-negative, finite kernel on $(E, \mathcal{E})$ and K' a non-negative finite kernel on $(E', \mathcal{E}')$ such that the restriction of K' to $(E, \mathcal{E})$ is K and $K'(x, E' \setminus E) = 0$ for all $x \in E'$. Then*

$$\rho(K') = \rho(K).$$

Further K' is conservative if and only if K is conservative.

Proof. Let

$$U_\lambda(x, dy) = \sum_{n \geq 0} \lambda^n K^n(x, dy)$$

and U'_λ defined for K'. Then, since K' puts all its mass on E, obviously

$$U'_\lambda = I + \lambda K' U_\lambda,$$

where $I(x, dy) = \delta_x(dy)$. Assume that U'_λ is σ-finite. Then there exists a measurable, positive function f on $E' \times E'$ such that

$$\int_{E'} f(x, y)U'_\lambda(x, dy) < \infty$$

for all $x \in E'$. But clearly g, the restriction of f to $E \times E$, satisfies

$$\int_E g(x, y)U_\lambda(x, dy) < \infty,$$

for all $x \in E$. Hence U_λ is σ-finite and by the definition of Perron root it is thus clear that $\rho(K) \leq \rho(K')$.

Now assume that U_λ is σ-finite. Clearly I is σ-finite and if K' is finite (i.e. $K(x, E') < \infty$ for all $x \in E'$) it is also σ-finite (the function $f \equiv 1$ will do). Thus also U'_λ is σ-finite and hence $\rho(K') \leq \rho(K)$.

Next assume that K' is conservative. By Lemma 4.5 there exists a measure m' on $(E', \mathcal{E}')$ such that

$$m'(A) > 0 \Rightarrow U'_\lambda(x, A) = \infty$$

for all $x \in E'$. Again since K' puts all its mass on E this implies the existence of a measure m on $(E, \mathcal{E})$ such that

$$m(A) > 0 \Rightarrow U_\lambda(x, A) = \infty,$$

namely the restriction of m', and K is conservative. Finally, if K is conservative for some measure m then the measure $m'(dx) = m(dx \cap E)$ will do for K', since K and K' coincide on E. ■

5.2. The Malthusian Parameter

Recall that the macro process follows the law $\tilde{P}_\gamma(\cdot)$ on $(\Gamma \times \bar{\Omega}^I, \mathcal{G} \times \mathcal{A}^I), \gamma \in \Gamma$ and that the ith individual of sibling group $\gamma \in \Gamma$ thus chooses life according to the marginal $\tilde{Q}(\sigma_i(\gamma), \cdot)$, only depending on her type $\sigma_i(\gamma)$. As mentioned in Section 3.2, it is convenient to work with the induced measures $Q(s, d\gamma') = \tilde{Q}(s, \xi^{-1}(d\gamma'))$ where $s \in S$. Recall also the reproduction measure

$$M(\gamma, d\gamma' \times dt) = \sum_{i=1}^{\infty} Q(\sigma_i(\gamma), d\gamma')\delta_{\tau_i(\gamma)}(dt),$$

the expected number of macro children with types in $d\gamma'$ born in the time interval dt, by a γ-type macro mother. Its Laplace transform is

$$\hat{M}_\alpha(\gamma, d\gamma') = \int_0^\infty e^{-\alpha t} M(\gamma, d\gamma' \times dt) = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} Q(\sigma_i(\gamma), d\gamma').$$

We wish to explore the connection between $\hat{\mu}^n$ and $\hat{M}^n$. First note that

$$\hat{\mu}_\alpha(s, dr) = \int_0^\infty e^{-\alpha t} E_s[\xi(dr \times dt)] = \sum_{i=1}^{\infty} \int_\Gamma e^{-\alpha\tau_i(\gamma')} \delta_{\sigma_i(\gamma')}(dr) Q(s, d\gamma'). \tag{5.1}$$

For $n = 2$ we obtain

$$\begin{aligned}
\hat{M}_\alpha^2(\gamma, d\gamma') &= \int_\Gamma \hat{M}_\alpha(\gamma, d\gamma'')\hat{M}_\alpha(\gamma'', d\gamma') \\
&= \int_\Gamma \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{-\alpha\tau_i(\gamma)} Q(\sigma_i(\gamma), d\gamma'') e^{-\alpha\tau_j(\gamma'')} Q(\sigma_j(\gamma''), d\gamma') \\
&= \int_{\Gamma \times S} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{-\alpha\tau_i(\gamma)} Q(\sigma_i(\gamma), d\gamma'') e^{-\alpha\tau_j(\gamma'')} \delta_{\sigma_j(\gamma'')}(ds) Q(s, d\gamma') \\
&= \sum_{i=1}^{\infty} \int_S e^{-\alpha\tau_i(\gamma)} \hat{\mu}_\alpha(\sigma_i(\gamma), ds) Q(s, d\gamma')
\end{aligned}$$

by (5.1). By induction

$$\hat{M}_\alpha^n(\gamma, d\gamma') = \sum_{i=1}^{\infty} \int_S e^{-\alpha\tau_i(\gamma)} \hat{\mu}_\alpha^{n-1}(\sigma_i(\gamma), ds) Q(s, d\gamma') \tag{5.2}$$

for $n \geq 1$ ($\hat{\mu}_\alpha^0(r, ds) = \delta_r(ds)$ and $\hat{M}_\alpha^0(\gamma, d\gamma') = \delta_\gamma(d\gamma')$). At this point α is any real number. However, it is possible to prove that μ and M define the same Malthusian parameter. Recall that this parameter is defined so that the Perron root of $\hat{\mu}_\alpha$, $\rho(\hat{\mu}_\alpha)$, equals one (and for the macro process so that $\rho(\hat{M}_\alpha) = 1$).

Lemma 5.3. *$\hat{\mu}_\alpha$ has Perron root one if and only if $\hat{M}_\alpha$ has Perron root one.*

Proof. Assume that $\rho(\hat{\mu}_\alpha) = 1$ (and from now on drop the subscript). Let first $\lambda < 1$. Then there exists a function $f > 0$ such that

$$\sum_{n\geq 0} \lambda^n \int_S f(s,r)\hat{\mu}^n(s,dr) < \infty, \tag{5.3}$$

for all $s \in S$ where, by (5.1),

$$\int_S f(s,r)\hat{\mu}^n(s,dr) = \sum_{j=1}^{\infty} \int_{S\times\Gamma} e^{-\alpha\tau_j(\gamma')} f(s,\sigma_j(\gamma'))\hat{\mu}^{n-1}(s,dr)Q(r,d\gamma'), \tag{5.4}$$

for $n \geq 1$ and

$$\int_S f(s,r)\hat{\mu}^0(s,dr) = f(s,s).$$

The question is if there exists a function $F > 0$ such that

$$\sum_{n\geq 0} \lambda^n \int_\Gamma F(\gamma,\gamma')\hat{M}^n(\gamma,d\gamma') < \infty,$$

for all $\gamma \in \Gamma$. Therefore take a $\gamma \in \Gamma$, fix an $i \leq \gamma(S \times R_+)$ and note that by (5.3) and (5.4)

$$\sum_{n\geq 0} \lambda^n \sum_{j=1}^{\infty} \int_{S\times\Gamma} e^{-\alpha\tau_j(\gamma')} f(\sigma_i(\gamma),\sigma_j(\gamma'))\hat{\mu}^n(\sigma_i(\gamma),dr)Q(r,d\gamma') < \infty$$

and hence also

$$\sum_{n\geq 0} \lambda^n \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{S\times\Gamma} e^{-\alpha\tau_i(\gamma)} e^{-\alpha\tau_j(\gamma')} f(\sigma_i(\gamma),\sigma_j(\gamma'))\hat{\mu}^n(\sigma_i(\gamma),dr)Q(r,d\gamma') < \infty.$$

Now define

$$F_i(\gamma,\gamma') = \sum_{j=1}^{\infty} e^{-\alpha\tau_j(\gamma')} f(\sigma_i(\gamma),\sigma_j(\gamma')), \tag{5.5}$$

so that

$$\sum_{n\geq 0} \lambda^n \sum_{i=1}^{\infty} \int_{S\times\Gamma} e^{-\alpha\tau_i(\gamma)} F_i(\gamma,\gamma')\hat{\mu}^n(\sigma_i(\gamma),dr)Q(r,d\gamma') < \infty.$$

Since there for each $\gamma \in \Gamma$ are a finite number of F_i $(i = 1,2,...,\gamma(S \times R+) < \infty)$ we can define

$$F(\gamma,\gamma') = \min_{i=1,2,\ldots,\gamma(S\times R_+)} F_i(\gamma,\gamma') \tag{5.6}$$

and, by (5.2), obtain that

$$\begin{aligned}
&\sum_{n\geq 0} \lambda^n \int_\Gamma F(\gamma,\gamma')\hat{M}^n(\gamma,d\gamma') \\
&= F(\gamma,\gamma) + \sum_{n\geq 1} \lambda^n \sum_{i=1}^{\infty} \int_{S\times\Gamma} e^{-\alpha\tau_i(\gamma)} F(\gamma,\gamma')\hat{\mu}^{n-1}(\sigma_i(\gamma),dr)Q(r,d\gamma') \\
&\leq F(\gamma,\gamma) + \sum_{n\geq 1} \lambda^n \sum_{i=1}^{\infty} \int_{S\times\Gamma} e^{-\alpha\tau_i(\gamma)} F_i(\gamma,\gamma')\hat{\mu}^{n-1}(\sigma_i(\gamma),dr)Q(r,d\gamma'),
\end{aligned}$$

where the second term is finite by the above. For the first term we note that, for each i,

$$F_i(\gamma,\gamma) = \sum_{j=1}^{\infty} e^{-\alpha\tau_j(\gamma)} f(\sigma_i(\gamma), \sigma_j(\gamma)) < \infty,$$

since if there exists an f satisfying (5.3) then there also exists an everywhere *finite* function satisfying (5.3) (for instance $g = \min(f, C)$ for any positive constant C). Hence $\sum_{n\geq 0} \lambda^n \hat{M}^n$ is σ-finite if $\lambda < 1$ and therefore $\rho(\hat{M}) \leq 1$. Next let $\lambda > 1$. For all $f > 0$ there exists an $s \in S$ such that

$$\sum_{n\geq 0} \lambda^n \int_S f(s,r)\hat{\mu}^n(s, dr) = \infty,$$

by the definition of α. The question is now if, given an $F > 0$, there exists a $\gamma \in \Gamma$ such that

$$\sum_{n\geq 0} \lambda^n \int_\Gamma F(\gamma,\gamma')\hat{M}^n(\gamma, d\gamma') = \infty$$

i.e if

$$F(\gamma,\gamma) + \sum_{n\geq 0} \lambda^n \sum_{i=1}^{\infty} \int_{S\times\Gamma} F(\gamma,\gamma') e^{-\alpha\tau_i(\gamma)} \hat{\mu}^n(\sigma_i(\gamma), dr) Q(r, d\gamma') = \infty.$$

For an arbitrary $F > 0$ let

$$g(\gamma, r) = \int_\Gamma F(\gamma,\gamma')Q(r, d\gamma'),$$

which is strictly positive if we assume that all the $Q(r,\Gamma) > 0$. This assumption means that every type has a positive probability to reproduce or, if you wish, that there are no sterile types. This assumption can be made since it will follow from the requirement $\inf h > 0$ mentioned in Section 4.2. Now let

$$\Gamma_1 = \{\gamma \in \mathcal{N}(S \times R_+) : \gamma(S \times R_+) = 1, \tau_1(\gamma) = 1\}. \tag{5.7}$$

Then every $s \in S$ corresponds to a unique $\gamma = \gamma(s)$ in Γ_1. This is where Lemma 5.2 is needed; the types in Γ_1 are added to the population in such a way that the conditions of that lemma are satisfied. Clearly $\hat{M}$ is finite since

$$\hat{M}(\gamma,\Gamma) = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} Q(\sigma_i(\gamma), \Gamma) \leq \gamma(S \times R_+) < \infty.$$

Hence, as far as Perron roots and conservativity conditions are concerned, the process does not change. Define

$$f(s,r) = g(\gamma(s), r). \tag{5.8}$$

Then there exists an $s_0 \in S$ such that

$$\sum_{n\geq 0} \lambda^n \int_S f(s_0, r)\hat{\mu}^n(s_0, dr) = \infty.$$

Choose $\gamma_0 = \gamma(s_0)$ to obtain

$$\sum_{n\geq 0} \lambda^n \int_\Gamma F(\gamma_0, \gamma')\hat{M}^n(\gamma_0, d\gamma')$$

$$= F(\gamma_0,\gamma_0) + \sum_{i=1}^{\infty}\sum_{n\geq 0} \lambda^n \int_S e^{-\alpha\tau_i(\gamma_0)} g(\gamma_0, r)\hat{\mu}^n(\sigma_i(\gamma_0), dr)$$

$$= F(\gamma_0,\gamma_0) + e^{-\alpha\tau_1(\gamma_0)} \sum_{n\geq 0} \lambda^n \int_S f(s_0, r)\hat{\mu}^n(s_0, dr) = \infty.$$

The question arises if f really is measurable with respect to the product σ-algebra $\mathcal{S}\times\mathcal{S}$. To show that it is, first assume that F is an indicator, i.e.

$$F(\gamma,\gamma') = 1_{A\times B}(\gamma,\gamma'),$$

where A and B belong to $\mathcal{G}$. Then

$$f(s,r) = \int_\Gamma 1_A(\gamma(s))1_B(\gamma')Q(r,d\gamma') = 1_A(\gamma(s))Q(r,B),$$

where $Q(r,B)$ is measurable by definition (it is a kernel). That also $1_A(\gamma(s))$ is measurable follows since

$$\{s\in S : \gamma(s)\in A\} = \{s : \gamma(s) \in A\cap\Gamma_1\} = \{s : \gamma(s)\in A_1\},$$

where A_1 is a set of the form

$$A_1 = \{\gamma\in\Gamma : \gamma(S\times R_+) = 1, \tau_1(\gamma) = 1, \sigma_1(\gamma)\in C, C\in\mathcal{S}\},$$

so that

$$\{s\in S : \gamma(s)\in A\} = C \in \mathcal{S}.$$

By Dynkin's $\pi - \lambda$ Theorem it follows that f is measurable if F is the indicator of any set in $\mathcal{G}\times\mathcal{G}$. Applying the standard procedure of approximating by simple functions then yields the measurability of f for a general positive F.

It has been shown that $\sum_{n\geq 0}\lambda^n\hat{M}^n$ is not σ-finite if $\lambda > 1$. Therefore $\rho(\hat{M}) \geq 1$ and hence equals one.

For the converse assume that $\hat{M}$ has Perron root one and let $\lambda < 1$. Then there exists an $F > 0$ such that

$$\sum_{n\geq 0}\lambda^n \int_\Gamma F(\gamma,\gamma')\hat{M}^n(\gamma, d\gamma') < \infty,$$

for all $\gamma\in\Gamma$. Define f through (5.8) and note that, for any $s\in S$,

$$\sum_{n\geq 0}\lambda^n \int_S f(s,r)\hat{\mu}^n(s,dr)$$

$$= e^{\alpha\tau_1(\gamma)} \sum_{n\geq 0}\lambda^n \int_{\Gamma\times S} e^{-\alpha\tau_1(\gamma)} F(\gamma(s),\gamma')\hat{\mu}^n(\sigma_1(\gamma(s)), dr)Q(r, d\gamma')$$

$$= e^{\alpha\tau_1(\gamma)} \sum_{n\geq 1} \lambda^n \int_\Gamma F(\gamma(s),\gamma')\hat{M}^n(\gamma(s),d\gamma') < \infty$$

(of course $\sigma_1(\gamma(s)) = s$). Hence $\rho(\hat{\mu}) \leq 1$.

Finally let $\lambda > 1$. Then for all $F > 0$ there exists a $\gamma \in \Gamma$ such that

$$\sum_{n\geq 0} \lambda^n \int_\Gamma F(\gamma,\gamma')\hat{M}^n(\gamma,d\gamma') = \infty.$$

Fix an arbitrary $f > 0$. Then, as before,

$$\begin{aligned} &\sum_{n\geq 0} \lambda^n \int_S f(s,r)\hat{\mu}^n(s,dr) \\ &= f(s,s) + \sum_{n\geq 0} \lambda^n \sum_{j=1}^{\infty} \int_{S\times\Gamma} e^{-\alpha\tau_j(\gamma')} f(s,\sigma_j(\gamma'))\hat{\mu}^n(s,dr)Q(r,d\gamma'). \end{aligned}$$

With F_i and F defined as in (5.5) and (5.6) we then obtain, for any $\gamma \in \Gamma$,

$$\begin{aligned} &\sum_{n\geq 0} \lambda^n \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{S\times\Gamma} e^{-\alpha\tau_i(\gamma)} e^{-\alpha\tau_j(\gamma')} f(\sigma_i(\gamma),\sigma_j(\gamma'))\hat{\mu}^n(\sigma_i(\gamma),dr)Q(r,d\gamma') \\ &= \sum_{n\geq 0} \lambda^n \sum_{i=1}^{\infty} \int_{S\times\Gamma} e^{-\alpha\tau_i(\gamma)} F_i(\gamma,\gamma')\hat{\mu}^n(\sigma_i(\gamma),dr)Q(r,d\gamma') \\ &\geq \sum_{n\geq 1} \lambda^n \int_\Gamma F(\gamma,\gamma')\hat{M}^n(\gamma,d\gamma'). \end{aligned}$$

Hence there exists a $\gamma \in \Gamma$ such that

$$\sum_{n\geq 0} \lambda^n \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} f(\sigma_i(\gamma),r)\hat{\mu}^n(\sigma_i(\gamma),dr) = \infty$$

and since there are only a finite number of terms in the inner sum ($\gamma(S \times R_+) < \infty$) there exists a k such that

$$\sum_{n\geq 0} \lambda^n \int_S f(\sigma_k(\gamma),r)\hat{\mu}^n(\sigma_k(\gamma),dr) = \infty,$$

so that $\rho(\hat{\mu}) \geq 1$ and hence equals one. ■

5.3. The Invariant Functions and Measures

First we treat the question of conservativity.

Lemma 5.4. *$\hat{\mu}$ is conservative if and only if $\hat{M}$ is conservative.*

Proof. Since $\mathcal{G}$ is countably generated we can use the second definition of conservativity, the one given in Lemma 4.5.

Assume first that $\hat{\mu}$ is conservative. Then there exists a measure m on $(S, \mathcal{S})$ such that

$$m(A) > 0 \Rightarrow \sum_{n \geq 0} \hat{\mu}^n(s, A) = \infty,$$

for each $s \in S$. We wish to find a measure l on $(\Gamma, \mathcal{G})$ such that

$$\sum_{n \geq 0} \hat{M}^n(\gamma, B) = \infty,$$

for all $\gamma \in \Gamma$ whenever $l(B) > 0$. Such a measure is

$$l(d\gamma) = \int_S Q(s, d\gamma) m(ds).$$

To see this, consider an $A \in \mathcal{G}$ with $l(A) > 0$. Then there exists a set $B \in \mathcal{S}$ and a number $c > 0$ such that $m(B) > 0$ and $Q(s, A) \geq c$ when $s \in B$. Therefore, for any $\gamma \in \Gamma$,

$$\sum_{n \geq 1} \hat{M}^n(\gamma, A) \geq c \sum_{i=1}^{\infty} \int_S e^{-\alpha \tau_i(\gamma)} \sum_{n \geq 0} \hat{\mu}^n(\sigma_i(\gamma), B) = \infty,$$

since $m(B) > 0$.

Now assume that $\hat{M}$ is conservative. Then there exists a measure l on $(\Gamma, \mathcal{G})$ such that

$$\sum_{n \geq 0} \hat{M}^n(\gamma, B) = \infty,$$

for all $\gamma \in \Gamma$ whenever $l(B) > 0$. A measure exhibiting $\hat{\mu}$ as conservative is

$$m(ds) = \sum_{i=1}^{\infty} \int_\Gamma e^{-\alpha \tau_i(\gamma)} \delta_{\sigma_i(\gamma)}(ds) l(d\gamma).$$

Indeed, if $m(B) > 0$ then there exists an $A \in \mathcal{G}$ and a number $c > 0$ such that $l(A) > 0$ and $\sum_{i=1}^{\infty} e^{-\alpha \tau_i(\gamma)} 1_B(\sigma_i(\gamma)) \geq c$ when $\gamma \in A$. Hence, by (5.1),

$$\begin{aligned} & \sum_{n \geq 1} \hat{\mu}^n(s, B) \\ & \quad = \sum_{n \geq 0} \sum_{i=1}^{\infty} \int_{S \times \Gamma} e^{-\alpha \tau_i(\gamma)} 1_B(\sigma_i(\gamma)) \hat{\mu}^n(s, dr) Q(r, d\gamma) \\ & \quad \geq c \sum_{n \geq 0} \int_S \hat{\mu}^n(s, dr) Q(r, A). \end{aligned}$$

But, by (5.2),

$$\sum_{n \geq 1} \hat{M}(\gamma, A) = \sum_{n \geq 0} \sum_{i=1}^{\infty} \int_S e^{-\alpha \tau_i(\gamma)} \hat{\mu}^n(\sigma_i(\gamma)), dr) Q(r, A) = \infty,$$

for every $\gamma \in \Gamma$ since $l(A) > 0$. In particular it holds for all γ in the set Γ_1 defined in (5.7). Therefore fix an $s \in S$, let $\gamma(s)$ be the corresponding element of Γ_1 so that $\gamma(S \times R_+) = 1$, $\tau_1(\gamma(s)) = 1$ and $\sigma_1(\gamma(s)) = s$. By the above,

$$\sum_{n \geq 0} \int_S \hat{\mu}^n(s, dr) Q(r, A) = \infty,$$

and hence

$$\sum_{n \geq 0} \hat{\mu}^n(s, B) = \infty,$$

for all $s \in S$ and $\hat{\mu}$ is conservative. ∎

This result is essential and fortunate. For instance it is clear that the macro individuals are always fewer than the individuals (because of the convention that empty sibling groups are disregarded), but could they be so few that their growth is slower, thus defining a smaller Malthusian parameter? Maybe the individual process could grow by some dependence structure that only allows a small number of the individuals in each sibling group to reproduce (in the extreme case only one) but that the size of their offspring would tend to increase as time went by. Of course this would be surprising and the lemma also excludes such obscurities.

From the reasoning at the end of the previous chapter it follows that the expectation of the normed macro process, $E_\gamma[e^{-\alpha t} Z_t^{\chi'}]$, converges (possibly with some care exercised about "almost all"-statements) without any other conditions than the usual ones for the individual process. This is good in the sense that we do not want to assume anything further; as mentioned the macro process is basically a tool for making statements about the individual process and should not be restricted by any extra conditions if possible. However we do not know what would happen if the individual process did *not* satisfy the Markov renewal conditions. Could the macro process perhaps, due to some intricate dependence structure, satisfy the conditions of the general theory and converge anyhow? The lemma gives the answer no; conditions on the individual level are equivalent to the corresponding conditions on the macro level.

We are now going to explore the connection between the two processes further. As before, the assumption that $\hat{\mu}$ is conservative yields the existence of h and π such that

$$h(s) = \int_S h(r) \hat{\mu}(s, dr)$$

and

$$\pi(ds) = \int_S \hat{\mu}(r, ds) \pi(dr).$$

Now it has been shown that this assumption implies that also $\hat{M}$ is conservative and there thus exists a function H and and a measure ψ such that

$$H(\gamma) = \int_\Gamma H(\gamma') \hat{M}(\gamma, d\gamma') \tag{5.9}$$

and

$$\psi(d\gamma) = \int_\Gamma \hat{M}(\gamma', d\gamma) \psi(d\gamma'). \tag{5.10}$$

One might ask if more can be done than just giving an existence statement; is it possible to make use of the relation between the individual and the macro process and give H and ψ explicitely in terms of h and π? The affirmative answer is given in the next lemma. First recall the notation

$$\bar{\gamma} = \int_{S \times R_+} e^{-\alpha t} h(s) \gamma(ds \times dt) = \sum_{i=1}^{\infty} e^{-\alpha \tau_i(\gamma)} h(\sigma_i(\gamma)).$$

Lemma 5.5. *If $\hat{\mu}$ has invariant measure π and invariant function h, then $\hat{M}$ has invariant measure ψ and invariant function H given by*

$$H(\gamma) = \bar{\gamma} \tag{5.11}$$

$$\psi(d\gamma) = \int_S Q(s, d\gamma) \pi(ds). \tag{5.12}$$

Proof. Just insert the candidates for H and ψ in the defining relations (5.9) and (5.10) and make use of the invariance properties of h and π. First part:

$$\int_\Gamma \bar{\gamma}' \hat{M}(\gamma, d\gamma') = \sum_{i=1}^{\infty} e^{-\alpha \tau_i(\gamma)} \int_\Gamma \bar{\gamma}' Q(\sigma_i(\gamma), d\gamma')$$

$$= \sum_{i=1}^{\infty} e^{-\alpha \tau_i(\gamma)} E_{\sigma_i(\gamma)}[\bar{\xi}] = \sum_{i=1}^{\infty} e^{-\alpha \tau_i(\gamma)} h(\sigma_i(\gamma)) = \bar{\gamma},$$

since $E_s[\bar{\xi}] = h(s)$. Second part:

$$\int_{\Gamma \times S} \hat{M}(\gamma', d\gamma) Q(s, d\gamma') \pi(ds)$$

$$= \sum_{i=1}^{\infty} \int_{\Gamma \times S} e^{-\alpha \tau_i(\gamma')} Q(\sigma_i(\gamma'), d\gamma) Q(s, d\gamma') \pi(ds)$$

$$= \sum_{i=1}^{\infty} \int_{\Gamma \times S \times S} e^{-\alpha \tau_i(\gamma')} \delta_{\sigma_i(\gamma')}(dr) Q(r, d\gamma) Q(s, d\gamma') \pi(ds)$$

$$= \int_{S \times S} \hat{\mu}(s, dr) Q(r, d\gamma) \pi(ds) = \int_S Q(r, d\gamma) \pi(dr),$$

where the last equality but one follows from (5.1). ■

The careful reader will object that although π can be normed to a probability measure on S, this norming will in general not render ψ a probability measure on Γ:

$$\psi(\Gamma) = \int_S Q(s, \Gamma) \pi(ds) \leq 1,$$

since $Q(s, \Gamma) = Q(s, \{\xi(S \times R_+) > 0\}) \leq 1$ and the inequality is of course strict on a set of positive π-measure if there is a possibility of extinction at all. Hence the normed measure, call it ψ', would be

$$\psi' = \frac{1}{c}\psi,$$

where

$$c = Q(\pi, \Gamma) = \int_S Q(s, \Gamma)\pi(ds).$$

Henceforth we will use the notation ψ both for the normed and the unnormed measure; it will be clear from the context which one that is considered. Also note that the norming of π to a probability measure was guaranteed by the request that $\inf_s h(s) > 0$. The corresponding condition on the macro level, that $\inf_\gamma H(\gamma) > 0$ does not necessarily hold. With $\kappa = \inf_s h(s)$ the best that can be done in general is

$$H(\gamma) = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} h(\sigma_i(\gamma)) \geq \kappa \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} = \kappa\hat{\gamma}(\alpha),$$

but this does not have to be bounded away from zero. However we have seen that ψ can be normed anyhow, and this agrees with our intentions not to invoke any extra conditions on the macro process.

Are there intuitive interpretations of these relations? Recall that $h(s)$ is the reproductive value of the type s in some sense measuring the average contribution to the population by an s-type individual. Hence $H(\gamma)$, the reproductive value of the sibling group γ, weighs together the reproductive values of the individuals in that group. The weights are exponentially decreasing functions of the birth times, taking into account that early born individuals on the average contribute more to the population the do later born ones (and that the population size grows exponentially). Also ψ is intuitively reasonable if we think of it as the distribution of macro types in a stable population (although it is not really a probability measure in its unnormed form). The stable population will be properly defined in Chapter 7.1; until then just think of it as a very old and large population. An individual of type s reproduces, and hence gives rise to a macro individual, according to the probability measure $Q(s, d\gamma)$. In a stable population this individual's type is chosen according to the stable distribution π and the expression in Lemma 5.5 follows.

It is also possible to give h and π in terms of H and ψ:

Lemma 5.6. *If $\hat{M}$ has invariant measure ψ and invariant function H, then $\hat{\mu}$ has invariant measure π and invariant function h given by*

$$h(s) = \int_\Gamma H(\gamma)Q(s, d\gamma) \tag{5.13}$$

and

$$\pi(ds) = \sum_{i=1}^{\infty} \int_\Gamma e^{-\alpha\tau_i(\gamma)} \delta_{\sigma_i(\gamma)}(ds)\psi(d\gamma). \tag{5.14}$$

Proof. Again insert in the defining relations and use the invariance properties. By (5.1),

$$\begin{aligned}
&\int_{S\times\Gamma} H(\gamma)Q(r,d\gamma)\hat{\mu}(s,dr) \\
&= \sum_{i=1}^{\infty}\int_{S\times\Gamma\times\Gamma} H(\gamma)Q(r,d\gamma)e^{-\alpha\tau_i(\gamma')}\delta_{\sigma_i(\gamma')}(dr)Q(s,d\gamma') \\
&= \sum_{i=1}^{\infty}\int_{\Gamma\times\Gamma} H(\gamma)e^{-\alpha\tau_i(\gamma')}Q(\sigma_i(\gamma'),d\gamma)Q(s,d\gamma') \\
&= \int_{\Gamma\times\Gamma} H(\gamma)\hat{M}(\gamma',d\gamma)Q(s,d\gamma') \\
&= \int_{\Gamma} H(\gamma')Q(s,d\gamma').
\end{aligned}$$

Further

$$\begin{aligned}
&\sum_{i=1}^{\infty}\int_{S\times\Gamma} \hat{\mu}(r,ds)e^{-\alpha\tau_i(\gamma)}\delta_{\sigma_i(\gamma)}(dr)\psi(d\gamma) \\
&= \sum_{i=1}^{\infty}\int_{\Gamma} e^{-\alpha\tau_i(\gamma)}\hat{\mu}(\sigma_i(\gamma),ds)\psi(d\gamma) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{\Gamma\times\Gamma} e^{-\alpha\tau_i(\gamma)}e^{-\alpha\tau_j(\gamma')}\delta_{\sigma_j(\gamma')}(ds)Q(\sigma_i(\gamma),d\gamma')\psi(d\gamma) \\
&= \sum_{j=1}^{\infty}\int_{\Gamma\times\Gamma} e^{-\alpha\tau_j(\gamma')}\delta_{\sigma_j(\gamma')}(ds)\hat{M}(\gamma,d\gamma')\psi(d\gamma) \\
&= \sum_{j=1}^{\infty}\int_{\Gamma} e^{-\alpha\tau_j(\gamma')}\delta_{\sigma_j(\gamma')}(ds)\psi(d\gamma').
\end{aligned}$$

■

The interpretation of (5.13) is as follows: to obtain the reproductive value of an s-type individual, you average over the reproductive values of all possible macro individuals that this individual may give rise to, these macro individuals occurring according to the probability measure $Q(s,d\gamma)$. In fact this is generally the interpretation of the fact that $E_s[\bar{\xi}] = h(s)$ regardless of dependencies and macro individuals. And indeed, since $H(\gamma) = \bar{\gamma}$ this is exactly what (5.13) says. The intuition behind the relation (5.14) is slightly more complicated. If a macro individual is of type γ, then an individual in this sibling group will have type in the set $A \in S$ if any of the $\sigma_i(\gamma)$ are in A. The $e^{-\alpha\tau_i(\gamma)}$ again puts more weight on early born individuals and ψ averages over all possible macro individuals in a stable population.

5.4. The Mean Convergence Theorem

At the end of the previous section the convergence

$$E_\gamma[e^{-\alpha t}Z_t^\chi] = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} E_\gamma[e^{-\alpha(t-\tau_i)} z_{t-\tau_i}^\chi(i)]$$

$$\to \quad \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta} \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} h(\sigma_i(\gamma)) = \bar{\gamma}\frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}$$

was established, where

$$E_\gamma[e^{-\alpha(t-\tau_i)} z_{t-\tau_i}^\chi(i)] \to h(\sigma_i(\gamma))\frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta},$$

for π-*almost all* $\sigma_i(\gamma) \in S$. The question arises what the connection is between π-almost all $s \in S$ and ψ-almost all $\gamma \in \Gamma$. Could it possibly be so that removing all point processes containing *some* individual type for which the convergence fails, leaves us with a set whose complement is not a ψ-null set? The answer is, fortunately, no for the following reason. Let S_0 be any set with π-measure zero and define

$$\Gamma_0 = \{\gamma \in \Gamma : \sigma_i(\gamma) \in S_0 \text{ for some } i\}.$$

Then Γ_0 has ψ-measure zero since, by (5.14),

$$\pi(S_0) = \sum_{i=1}^{\infty} \int_\Gamma e^{-\alpha\tau_i(\gamma)} 1_{S_0}(\sigma_i(\gamma))\psi(d\gamma) = 0,$$

so that

$$1_{S_0}(\sigma_i(\gamma)) = 0,$$

for all i and ψ-almost all $\gamma \in \Gamma$. Hence ψ-almost all $\gamma \in \Gamma$ have none of their types in S_0 and therefore $\psi(\Gamma_0) = 0$. That the converse cannot be established is expected. If Γ_0 is a ψ-null set in Γ then, by (5.12),

$$\psi(\Gamma_0) = \int_S Q(s, \Gamma_0)\pi(ds) = 0,$$

from which it can be concluded that $Q(s, \Gamma_0) = 0$ for π-almost all $s \in S$. Hence the macro types, i.e. the reproduction processes in Γ_0, will never occur except possibly from individuals with types in a π-null set. Of course we can say nothing about the *individual* types that appear in the processes in Γ_0, other than that they will not appear in these specific combinations.

Thus it makes no substantial difference if the process is started from a full sibling group rather than from a single individual at least as far as convergence of means are considered.

Therefore the convergence of $E_\gamma[e^{-\alpha t}Z_t^\chi]$ can be established already at this stage. This can be done because of the special form of χ' given in (3.4). Since we will later work with macro characteristics not necessarily of this form, we will go on and investigate to what extent individual properties carry over to the macro process.

That the stable ages of childbearing, B and β, coincide is clear since

$$\begin{aligned} B &= \int_{\Gamma\times\Gamma\times R_+} te^{-\alpha t} H(\gamma')M(\gamma, d\gamma' \times dt)\psi(d\gamma) \\ &= \int_{\Gamma\times\Gamma\times R_+} \sum_{i=1}^{\infty} te^{-\alpha t} H(\gamma')Q(\sigma_i(\gamma), d\gamma')\delta_{\tau_i(\gamma)}(dt)\psi(d\gamma) \\ &= \int_{\Gamma\times R_+\times S} \sum_{i=1}^{\infty} te^{-\alpha t} h(\sigma_i(\gamma))\delta_{\tau_i(\gamma)}(dt)Q(s, d\gamma)\pi(ds) \\ &= \int_{\Gamma\times R_+\times S\times S} te^{-\alpha t} h(r)\delta_{\sigma_i(\gamma)}(dr)\delta_{\tau_i(\gamma)}(dt)Q(s, d\gamma)\pi(ds) \\ &= \int_{\Gamma\times R_+\times S\times S} te^{-\alpha t} h(r)\gamma(ds \times dt)Q(s, d\gamma)\pi(ds) \\ &= \int_{S\times S\times R_+} te^{-\alpha t} h(r)\mu(s, dr \times dt)\pi(ds) = \beta. \end{aligned}$$

Some care should be taken here when referring to B as the stable age of childbearing since ψ is not a probability measure. For the convergence results however, the only important thing is that they are positive and finite, and indeed the real stable age of childbearing for the macro process differs only by the postive multiplicative constant $Q(\pi, \Gamma)$. Another aspect to be treated is the question of latticeness. The individual process is assumed to be non-lattice, the question is if this implies non-latticeness of the macro process. The affirmative answer can be motivated shortly by the fact that individuals and macro individuals are born at the same timepoints, and by assumption these do not constitute a lattice. However there are also some almost everywhere qualifications that have to be considered and thus a little more care is needed.

Recall that M is lattice if there exists a number $D > 0$ and a function $C : \Gamma \to [0, D)$ such that $M(\gamma, d\gamma' \times dt)$ is concentrated on the lattice

$$\{(t, \gamma') \in R_+ \times \Gamma : t = C(\gamma') - C(\gamma) + nD, \text{ for some } n = 0, 1, ...\},$$

for ψ-almost all $\gamma \in \Gamma$. Clearly however, $M(\gamma, d\gamma' \times dt)$ is concentrated on the set $\{\tau_i(\gamma), i = 1, 2, ...\}$ so latticeness would imply that there is a subset $\Gamma_0 \subseteq \Gamma$ of ψ-measure zero such that, for any $\gamma \notin \Gamma_0$, the $\tau_i(\gamma)$ are concentrated on a lattice with step D.

But since $\mu(s, dr \times dt)$ is assumed non-lattice there is a set $A \subseteq S$ such that $\pi(A) > 0$ and $\mu(s, dr \times dt)$ is not concentrated on any lattice of the form $\{(r, t) \in S \times R_+ : t = c(r) - c(s) + nd, n = 0, 1, ...\}$ when $s \in A$. Since

$$\mu(s, dr \times dt) = \int_{\Gamma} \gamma(dr \times dt)Q(s, d\gamma),$$

non-latticeness means that there is a set $B \subseteq \Gamma_0$ such that $Q(s, B) > 0$ for $s \in A$. But then

$$\psi(B) = \int_S Q(s, B)\pi(ds) \geq \int_A Q(s, B)\pi(ds) > 0,$$

contradicting the latticeness of M. Hence, if the individual process is non-lattice then so is the macro process.

Therefore M inherits the properties of μ and so far the theory may be applied directly to the macro process by only assuming individual conditions. The next condition to examine is the boundedness of $E_\gamma[e^{-\alpha t} Z_t^{\chi'}]$ in the sense that the set

$$\{\gamma \in \Gamma : \sup_{t \geq 0} E_\gamma[e^{-\alpha t} Z_t^{\chi'}] < \infty\}$$

should be of positive ψ-measure. Under the usual assumptions of strict Malthusianness we know that the set

$$\{s \in S : \sup_{t \geq 0} E_s[e^{-\alpha t} z_t^{\chi}] < \infty\}$$

is of positive π-measure; in fact it holds *uniformly* in s that $\sup_{t \geq 0} E_s[e^{-\alpha t} z_t^{\chi}] < \infty$, see the proof of Theorem 3 in Jagers (1992). Since

$$E_\gamma[e^{-\alpha t} Z_t^{\chi'}] = \sum_{i=1}^{\infty} e^{-\alpha \tau_i(\gamma)} E_{\sigma_i(\gamma)}[e^{-\alpha(t-\tau_i)} z_{t-\tau_i}^{\chi}],$$

it also holds that

$$\sup_{t \geq 0} E_\gamma[e^{-\alpha t} Z_t^{\chi'}] < \infty, \tag{5.15}$$

for all $\gamma \in \Gamma$. On the individual level this condition is guaranteed by assuming χ bounded (see Jagers (1992)). Clearly, boundedness of χ does not imply that a χ' of the form (3.4) is bounded, but since the boundedness of $E_\gamma[e^{-\alpha t} Z_t^{\chi'}]$ is guaranteed anyway this is no limitation. What will be of future importance though, is that the direct Riemann integrability carries over from the individual to the macro process.

Lemma 5.7. *If $e^{-\alpha t} E_s[\chi(t)]$ is directly Riemann integrable (π) then $e^{-\alpha t} E_\gamma[\chi'(t)]$ is directly Riemann integrable (ψ).*

Proof. This is a consequence of Lemma 2 in Shurenkov (1992). With a slight modification it states the following. Assume that the function $g : E \times R_+ \rightarrow R_+$ is directly Riemann integrable (l) and that there exists a space E' and a kernel $K(x, dy \times dt)$ with basis $K(x, dy)$, now with the meaning that $K(x, dy)$ is a measure on $\mathcal{E}$ for each $x \in E'$ and that $K(x, A)$ is an $\mathcal{E}'$-measurable function for all $A \in \mathcal{E}$. If there exists a measure m on $(E', \mathcal{E}')$ such that

$$\int_{E'} K(x, A) m(dx) = l(A), \ \ A \in \mathcal{E},$$

then the function

$$K * g(x,t) = \int_E \int_0^t g(y,t-u) K(x, dy \times du)$$

is directly Riemann integrable (m). Since

$$\begin{aligned} E_\gamma[e^{-\alpha t}\chi'(t)] &= \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} E_\gamma[e^{-\alpha(t-\tau_i)}\chi_i(t-\tau_i)] \\ &= \sum_{i=1}^{\infty} e^{-\alpha(t-\tau_i(\gamma))} E_{\sigma_i(\gamma)}[e^{-\alpha(t-\tau_i)}\chi_i(t-\tau_i)] \\ &= \int_S \int_0^t E_r[e^{-\alpha(t-u)}\chi(t-u)] e^{-\alpha u}\gamma(dr \times du) \end{aligned}$$

and

$$\begin{aligned} &\int_\Gamma \int_0^\infty e^{-\alpha u}\gamma(dr \times du)\psi(d\gamma) \\ &= \sum_{i=1}^{\infty} \int_\Gamma e^{-\alpha\tau_i(\gamma)} \delta_{\sigma_i(\gamma)}(dr)\psi(d\gamma) = \pi(dr), \end{aligned}$$

$e^{-\alpha u}\gamma(dr \times du)$ plays the role of K above and hence $E_\gamma[e^{-\alpha t}\chi'(t)]$ is directly Riemann integrable (ψ). ■

It has now been shown that all the conditions needed for convergence of expectations on the individual level carry over nicely to the macro process and we state the first convergence theorem.

Theorem 5.8. *Consider a non-lattice, strictly Malthusian branching process with sibling dependencies. Let χ be bounded and the function $e^{-\alpha t}E_s[\chi(t)]$ directly Riemann integrable. Then*

$$E_s[e^{-\alpha t} z_t^{\chi}] \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta} h(s), \tag{5.16}$$

as $t \to \infty$ for π-almost all $s \in S$ and

$$E_\gamma[e^{-\alpha t} Z_t^{\chi'}] \to \frac{E_\psi[\widehat{\chi'}(\alpha)]}{\alpha B} \bar{\gamma},$$

as $t \to \infty$ for ψ-almost all $\gamma \in \Gamma$.

Proof. As for Theorem 3 in Jagers (1991). ■

Note. The discussion at the end of the previous chapter suggests that $E_\psi[\widehat{\chi'}(\alpha)] = E_\pi[\hat{\chi}(\alpha)]$ (Formula (4.5)). That can also be shown directly:

$$E_\gamma[\widehat{\chi'}(\alpha)] = E_\gamma[\alpha \int_0^\infty e^{-\alpha t}\chi'(t)dt]$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} E_\gamma[\alpha \int_0^\infty e^{-\alpha(t-\tau_i)} \chi_i(t-\tau_i)]dt] \\
&= \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} E_\gamma[\alpha \int_0^\infty e^{-\alpha t} \chi_i(t) dt] \\
&= \int_S \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} E_s[\hat{\chi}(\alpha)] \delta_{\sigma_i(\gamma)}(ds)
\end{aligned}
$$

and from (5.14) we can conclude

$$E_\psi[\hat{\chi'}(\alpha)] = E_\pi[\hat{\chi}(\alpha)].$$

Strictly speaking this equality is valid with ψ denoting the unnormed measure, but then ψ is not a probability measure and the notation E_ψ might not be appropriate. What actually should be stated is that

$$\frac{E_\psi[\hat{\chi'}(\alpha)]}{\alpha B} = \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta},$$

which always holds since any norming constant then appears multiplicatively both in the numerator and the denominator. □

Theorem 5.8 gives the answer to a fundamental question: under the conditions of Theorem 3 in Jagers (1991) (i.e. the same conditions as in Theorem 5.8 but without the sibling dependencies), is it possible to destroy, or substantially change, the asymptotics by inferring some dependence structure among siblings? The answer is no, the asymptotic behaviour of the population is determined through the behaviour of the marginals only, irrespectively of whether siblings are dependent or not. Always keep in mind that we talk about what happens on the set of non-extinction; the extinction probability might, as the examples of section 3.1 show, depend heavily on the dependence structure.

This also indicates, quite generally, that the Malthusianness is the crucial criterion for a growing population to behave regularily, and that the branching property on the individual level is of less importance. Of course there must be *some* branching structure; in the sibling dependent case it appears on the macro level.

Since the expectations dealt with in the theorem are expectations of sums, it might not be surprising that the dependencies do not influence the results. The next objective is to investigate how the actual population behaves asymptotically, and for that purpose the next chapter is devoted to L^1-convergence.

6. Convergence in L^1

6.1. The $x \log x$ Condition

The L^1-convergence of the macro process is explored by means of a certain martingale, introduced in Nerman (1984). Let mx denote x's mother and define

$$\mathcal{I}_t = \{x \in I : \tau'_{mx} \leq t < \tau'_x\},$$

i.e. the set of macro individuals whose macro mothers are born before t but who themselves are born after t. The *intrinsic martingale* is

$$W_t = \sum_{x \in \mathcal{I}_t} e^{-\alpha \tau'_x} H(\xi_x).$$

From Theorem 6.1 in Jagers (1989) we know that this martingale is uniformly integrable and hence L^1-convergent if it satisfies the $x \log x$ condition. Recall

$$\bar{\eta} = \int_{\Gamma \times R_+} e^{-\alpha t} H(\gamma) \eta(d\gamma \times dt)$$

and write E_ψ for $\int_\Gamma E_\gamma \psi(d\gamma)$, expectation with the macro ancestor's type $\xi_0 \sim \psi$. The $x \log x$ condition is

$$E_\psi[\bar{\eta} \log^+ \bar{\eta}] < \infty.$$

Recall the special birth time and type structure of the macro process: The reproduction process of a macro individual with life $(\gamma, \bar{\omega})$ places its point masses at $(\tau_i(\gamma), \xi_i(\bar{\omega})), i = 1, 2, \ldots$ and hence

$$\bar{\eta} = \sum_{i=1}^{\infty} e^{-\alpha \tau_i(\xi_0)} \bar{\xi}_i.$$

The following lemma states that the $x \log x$ condition will be guaranteed by the corresponding condition for the individual process; in fact there is an equivalence between individual $x \log x$ and macro $x \log x$.

Lemma 6.1. $E_\pi[\bar{\xi} \log^+ \bar{\xi}] < \infty \Leftrightarrow E_\psi[\bar{\eta} \log^+ \bar{\eta}] < \infty$.

Proof. The proof relies on the convexity of the function $x \log^+ x$. First assume that $E_\pi[\bar{\xi} \log^+ \bar{\xi}] < \infty$, denote

$$\hat{\gamma}(\alpha) = \sum_{i=1}^{\infty} e^{-\alpha \tau_i(\gamma)},$$

and note that

$$\begin{aligned}
E_\gamma[\bar\eta\log^+\bar\eta] &= E_\gamma\left[\bar\eta\log^+(\sum_{i=1}^{\infty}e^{-\alpha\tau_i(\gamma)}\bar\xi_i)\right] \\
&= E_\gamma[\bar\eta\log^+(\hat\gamma(\alpha)\sum_{i=1}^{\infty}\frac{e^{-\alpha\tau_i(\gamma)}}{\hat\gamma(\alpha)}\bar\xi_i)] \\
&\leq E_\gamma[\bar\eta\log^+\hat\gamma(\alpha)]+\hat\gamma(\alpha)E_\gamma\left[\sum_{i=1}^{\infty}\frac{e^{-\alpha\tau_i(\gamma)}}{\hat\gamma(\alpha)}\bar\xi_i\log^+(\sum_{i=1}^{\infty}\frac{e^{-\alpha\tau_i(\gamma)}}{\hat\gamma(\alpha)}\bar\xi_i)\right] \\
&= I_1(\gamma)+I_2(\gamma),
\end{aligned}$$

since $\log^+ ab \leq \log^+ a + \log^+ b$. Further

$$I_1(\gamma) = E_\gamma[\bar\eta]\log^+\hat\gamma(\alpha) = \bar\gamma\log^+\hat\gamma(\alpha).$$

Since, $\kappa = \inf h > 0$,

$$\bar\gamma = \sum_{i=1}^{\infty}e^{-\alpha\tau_i(\gamma)}h(\sigma_i(\gamma)) \geq \kappa\sum_{i=1}^{\infty}e^{-\alpha\tau_i(\gamma)} = \kappa\hat\gamma(\alpha),$$

we obtain

$$\bar\gamma\log^+\hat\gamma(\alpha) \leq \bar\gamma\log^+\frac{\bar\gamma}{\kappa},$$

and hence

$$\begin{aligned}
\int_\Gamma I_1(\gamma)\psi(d\gamma) &\leq \int_{S\times\Gamma}(\bar\gamma\log^+\frac{\bar\gamma}{\kappa})Q(s,d\gamma)\pi(ds) \\
&= \int_S E_s[\bar\xi\log^+\frac{\bar\xi}{\kappa}]\pi(ds) = E_\pi[\bar\xi\log^+\frac{\bar\xi}{\kappa}],
\end{aligned}$$

which is finite by assumption.

The function $x\log^+ x$ is convex and therefore

$$\sum_{i=1}^{n}\lambda_i x_i\log^+(\sum_{i=1}^{n}\lambda_i x_i) \leq \sum_{i=1}^{n}\lambda_i x_i\log^+ x_i,$$

if $\sum_{i=1}^{n}\lambda_i = 1, \lambda_i \geq 0$. For a fixed γ we take $\lambda_i = \dfrac{e^{-\alpha\tau_i(\gamma)}}{\hat\gamma(\alpha)}$ (and have $n = \gamma(S\times R_+)$) to obtain

$$\begin{aligned}
I_2(\gamma) &\leq \hat\gamma(\alpha)\sum_{i=1}^{\infty}\frac{e^{-\alpha\tau_i(\gamma)}}{\hat\gamma(\alpha)}E_\gamma[\bar\xi_i\log^+\bar\xi_i] \\
&= \sum_{i=1}^{\infty}e^{-\alpha\tau_i(\gamma)}E_\gamma[\bar\xi_i\log^+\bar\xi_i].
\end{aligned}$$

Hence

$$\int_\Gamma I_2(\gamma)\psi(d\gamma) \leq \sum_{i=1}^{\infty}\int_\Gamma e^{-\alpha\tau_i(\gamma)}E_\gamma[\bar\xi_i\log^+\bar\xi_i]\psi(d\gamma)$$

$$= \sum_{i=1}^{\infty} \int_{\Gamma} e^{-\alpha\tau_i(\gamma)} E_{\sigma_i(\gamma)}[\bar{\xi} \log^+ \bar{\xi}]\psi(d\gamma)$$

$$= \sum_{i=1}^{\infty} \int_{\Gamma\times S} e^{-\alpha\tau_i(\gamma)} \delta_{\sigma_i(\gamma)}(ds) E_s[\bar{\xi} \log^+ \bar{\xi}]\psi(d\gamma)$$

$$= \int_{\Gamma} E_s[\bar{\xi} \log^+ \bar{\xi}]\pi(ds) = E_\pi[\bar{\xi} \log^+ \bar{\xi}] < \infty,$$

and it is proved that $E_\psi[\bar{\eta} \log^+ \bar{\eta}] < \infty$. Now assume that $E_\psi[\bar{\eta} \log^+ \bar{\eta}] < \infty$. Because of the convexity we can apply Jensen's inequality which yields

$$E_\gamma[\bar{\eta} \log^+ \bar{\eta}] \geq E_\gamma[\bar{\eta}] \log^+ E_\gamma[\bar{\eta}] = H(\gamma) \log^+ H(\gamma).$$

Hence

$$\infty > E_\psi[\bar{\eta} \log^+ \bar{\eta}] \geq \int_{\Gamma} H(\gamma) \log^+ H(\gamma)\psi(d\gamma)$$

$$= \int_{\Gamma\times\Gamma} \bar{\gamma} \log^+ \bar{\gamma} Q(s, d\gamma)\pi(ds)$$

$$= \int_{\Gamma} E_s[\bar{\xi} \log^+ \bar{\xi}]\pi(ds) = E_\pi[\bar{\xi} \log^+ \bar{\xi}],$$

and the proof is complete.

■

6.2. The L^1-convergence Theorem

By the $x \log x$ result there exists a random variable $W \geq 0$ on the macro space $\Gamma \times \bar{\Omega}^I$, such that $W_t \to W$ in $L^1(P_\gamma)$ as $t \to \infty$, for ψ-almost all $\gamma \in \Gamma$. The process will now be analyzed through the normed population size

$$\zeta_t = \frac{e^{-\alpha t} Z_t^{\chi'}}{H(\xi_0)}.$$

With $\mathcal{I}_t$ as above we get, for any $t_0 < t$,

$$\zeta_t = \frac{e^{-\alpha t}}{H(\xi_0)} \sum_{x<\mathcal{I}_{t_0}} \chi'_x(t - \tau'_x) + \sum_{x\in\mathcal{I}_{t_0}} \zeta_{t-\tau'_x}(x) e^{-\alpha\tau'_x} \frac{H(\xi_x)}{H(\xi_0)}, \tag{6.1}$$

where $\zeta_{t-\tau'_x}(x) = \zeta_{t-\tau'_x} \circ S_x$ and $x < \mathcal{I}_{t_0}$ means that x strictly preceeds $\mathcal{I}_{t_0}$, i.e. has descendants in $\mathcal{I}_{t_0}$ but does not itself belong to it. With $U_x(\gamma, \bar{\omega}) = \bar{\omega}_x$, the projection on the macro individual x's life, we define the σ-algebras

$$\mathcal{F}_t = \mathcal{G} \times \sigma(U_x : x < \mathcal{I}_t),$$

i.e. the σ-algebras generated by the types and lives of the macro individuals strictly preceeding $\mathcal{I}_t$.

The individual branching process is called uniformly integrable if y_t=the total number of individuals born up to time t, is uniformly integrable over its starting type, i.e. if

$$\sup_{s\in S} E_s[y_t; y_t > c] \to 0,$$

as $c \to \infty$, for any fixed t.

We are ready for the main convergence result.

Theorem 6.2. *Consider a uniformly integrable strictly Malthusian branching process with sibling dependencies, such that* $E_\pi[\bar{\xi}\log^+\bar{\xi}] < \infty$. *Let* χ *be bounded and the function* $e^{-\alpha t}E_s[\chi(t)]$ *directly Riemann integrable. Then there exists a random variable* w *on* $\Gamma \times \Omega^{I_1}$ *such that*

$$e^{-\alpha t}Z_t^\chi \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}w$$

in $L^1(P_\gamma)$, *for* ψ*-almost all* $\gamma \in \Gamma$.

Note. We can not apply Theorem 7.2 in Jagers (1989) directly to the macro process since it does not satisfy all the conditions of that theorem. Thus it has already been shown the requirement $\inf h > 0$ does not necessarily yield $\inf H > 0$. Neither is Y_t, the total number of macro individuals at time t, uniformly integrable over the starting type. To see this, observe that

$$E_\gamma[Y_t] \geq \#\{i : i \leq \gamma(S\times[0,t]), \xi_i(S\times R_+) > 0\},$$

the number of individuals in the initial sibling group that ever beget children. Since this number might well be unbounded, $E_\gamma[Y_t]$ cannot be uniformly bounded in γ and hence not uniformly integrable. However, it will be possible to prove the theorem without imposing these extra conditions on the macro process. □

Proof. We will work with the macro process and show that $e^{-\alpha t}Z_t^{\chi'}$ converges to $(E_\pi[\hat{\chi}(\alpha)]/\alpha\beta)W$. Then we can define $w = W\circ\varphi^{-1}$ to complete the proof.

Let $\chi'(t) = \sum_{i=1}^\infty \chi_i(t-\tau_i)$ and assume that χ is bounded by n. Define $\chi^n(t) = \chi'(t)1_{[0,n)}(t)$ so that $\chi^n(t)$ vanishes for $t \geq n$. It is χ^n that will be the characteristic under consideration. With $t_0 = t-n$ in (6.1) we then have

$$\zeta_t = \sum_{x\in\mathcal{I}_{t-n}} \zeta_{t-\tau'_x}(x)e^{-\alpha\tau'_x}\frac{H(\xi_x)}{H(\xi_0)}, \tag{6.2}$$

since if $x < \mathcal{I}_{t-n}$ then $t-\tau'_x \geq n$.

We will show that the ζ_t are uniformly integrable. From (5.15) we know that the $E_\gamma[\zeta_t]$ are uniformly bounded and it remains to be shown that $\sup_t E_\gamma[\zeta_t; A] \to 0$ as $P_\gamma(A) \to 0$. With

$$y_t(x)(\gamma,\bar{\omega}_I) = y_t(p_x(\gamma,\bar{\omega}_I)),$$

the total number of individuals at time t stemming from the individual x, we first note that

$$\begin{aligned} Z_t^{\chi^n} &\leq Z_t^{\chi'} \\ &= \sum_{i=1}^{\xi_0(t)} z^{\chi}_{t-\tau_i(\xi_0)}(i) \leq n \sum_{i=1}^{\xi_0(t)} y_{t-\tau_i(\xi_0)}(i), \end{aligned}$$

and hence

$$\zeta_{t-\tau_x'}(x) \leq \frac{n}{H(\xi_x)} e^{-\alpha(t-\tau_x')} \sum_{i=1}^{\xi_x(t-\tau_x')} y_{t-\tau_x'-\tau_i(\xi_x)}(xi).$$

By (6.2) and the fact that

$$H(\xi) = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\xi)} h(\sigma_i(\xi)) \geq \kappa\hat{\xi}(\alpha),$$

it now holds that

$$\begin{aligned} \zeta_t &\leq \frac{n}{\kappa} \sum_{x\in\mathcal{I}_{t-n}} \frac{e^{-\alpha(t-\tau_x')}}{\hat{\xi}_x(\alpha)} \left(\sum_{i=1}^{\xi_x(t-\tau_x')} y_{t-\tau_x'-\tau_i(\xi_x)}(xi) \right) e^{-\alpha\tau_x'} \frac{H(\xi_x)}{H(\xi_0)} \\ &\leq \frac{n}{\kappa H(\xi_0)} \sum_{x\in\mathcal{I}_{t-n}} \frac{e^{-\alpha(t-\tau_x')}}{\hat{\xi}_x(\alpha)} \left(\sum_{i=1}^{\xi_x(t-\tau_x')} y_n(xi) \right) e^{-\alpha\tau_x'} H(\xi_x), \qquad (6.3) \end{aligned}$$

since if $x \in \mathcal{I}_{t-n}$ then $t - \tau_x' \leq n$ and hence $y_{t-\tau_x'-\tau_i(\xi_x)}(x) \leq y_n(x)$.

Now let $\epsilon > 0$ be given. There exists a $\delta > 0$ such that

$$P(A|\mathcal{F}_{t-n}) < \delta \Rightarrow \sup_{x\in\mathcal{I}_{t-n}} E[y_n(xi); A|\mathcal{F}_{t-n}] < \epsilon,$$

since

$$\begin{aligned} & E[y_n(xi); A|\mathcal{F}_{t-n}] \\ &\leq E[y_n(xi); y_n(xi) > a|\mathcal{F}_{t-n}] + aP(A|\mathcal{F}_{t-n}) \\ &= E_{\sigma_{xi}}[y_n; y_n > a] + aP(A|\mathcal{F}_{t-n}), \end{aligned}$$

which can be made small because of the uniform integrability of the y_n. The last equality follows from the fact that if $x \in \mathcal{I}_{t-n}$ then the σ_{xi} are measurable with respect to $\mathcal{F}_{t-n}$ (the individual types σ_{xi} are determined by the type ξ_x of the macro individual x, and this type is determined by the life of x's macro mother which belongs to $\mathcal{F}_{t-n}$). From this we see that

$$E\left[\frac{e^{-\alpha(t-\tau_x')}}{\hat{\xi}_x(\alpha)}\sum_{i=1}^{\xi_x(t-\tau_x')} y_n(xi); A \mid \mathcal{F}_{t-n}\right]$$

$$= \frac{e^{-\alpha(t-\tau_x')}}{\hat{\xi}_x(\alpha)}\sum_{i=1}^{\xi_x(t-\tau_x')} E[y_n(xi); A|\mathcal{F}_{t-n}] < \frac{e^{-\alpha(t-\tau_x')}\xi_x(t-\tau_x')}{\hat{\xi}_x(\alpha)}\epsilon \leq \epsilon,$$

if $P(A|\mathcal{F}_{t-n}) < \delta$, since, for any $\gamma \in \Gamma$ and $t \geq 0$,

$$\frac{e^{-\alpha t}\gamma(t)}{\hat{\gamma}(\alpha)} = \frac{1}{\hat{\gamma}(\alpha)}\int_0^t e^{-\alpha(t-u)}e^{-\alpha u}\gamma(du) \leq 1.$$

Now fix a δ_0 and consider an $A \in \mathcal{S} \times \mathcal{A}^I$ with $P_\gamma(A) < \delta_0$. With

$$B = \{P(A|\mathcal{F}_{t-n}) < \delta\},$$

we then get

$$\delta P_\gamma(B^c) \leq P_\gamma(A \cap B^c) \leq P_\gamma(A) < \delta_0,$$

and

$$\begin{aligned}
&\frac{\kappa H(\gamma)}{n}E_\gamma[\zeta_t; A] \\
\leq\ & E_\gamma\left[E\left[\sum_{x\in\mathcal{I}_{t-n}} e^{-\alpha\tau_x'}H(\xi_x)\frac{e^{-\alpha(t-\tau_x')}}{\hat{\xi}_x(\alpha)}\sum_{i=1}^{\xi_x(t-\tau_x')} y_n(xi); A \mid \mathcal{F}_{t-n}\right]; B\right] \\
& +E_\gamma\left[E\left[\sum_{x\in\mathcal{I}_{t-n}} e^{-\alpha\tau_x'}H(\xi_x)\frac{e^{-\alpha(t-\tau_x')}}{\hat{\xi}_x(\alpha)}\sum_{i=1}^{\xi_x(t-\tau_x')} y_n(xi); A \mid \mathcal{F}_{t-n}\right]; B^c\right] \\
\leq\ & E_\gamma\left[\sum_{x\in\mathcal{I}_{t-n}} e^{-\alpha\tau_x'}H(\xi_x)E\left[\frac{e^{-\alpha(t-\tau_x')}}{\hat{\xi}_x(\alpha)}\sum_{i=1}^{\xi_x(t-\tau_x')} y_n(xi); A \mid \mathcal{F}_{t-n}\right]; B\right] \\
& +E_\gamma\left[\sum_{x\in\mathcal{I}_{t-n}} e^{-\alpha\tau_x'}H(\xi_x)E\left[\frac{e^{-\alpha(t-\tau_x')}}{\hat{\xi}_x(\alpha)}\sum_{i=1}^{\xi_x(t-\tau_x')} y_n(xi) \mid \mathcal{F}_{t-n}\right]; B^c\right] \\
\leq\ & \epsilon E_\gamma[W_{t-n}] + \sup_{s\in S} E_s[y_n]E_\gamma[W_{t-n}; B^c],
\end{aligned}$$

where the first term can be made small since $E_\gamma[W_{t-n}] = E_\gamma[W] = H(\gamma)$ (W_t is a martingale), and by choosing δ_0 small enough also $P_\gamma(B^c)$ will be small and by the uniform integrability of W_t (this is where the $x \log x$ condition appears)

$E_\gamma[W_{t-n}; B^c]$ will be uniformly small in t. Hence the ζ_t and thereby also the $e^{-\alpha t} Z_t^{\chi'}$ are uniformly integrable.

It can be proved that

$$\zeta_t - E[\zeta_t|\mathcal{F}_{t-n}] \to 0,$$

in P_γ-probability for ψ-almost all $\gamma \in \Gamma$. This relies on the uniform integrability of the y_n and is done exactly as in Jagers (1989) why it is omitted here. With

$$\rho_n = \frac{E_\psi[\hat{\chi}^n(\alpha)]}{\alpha B},$$

the next step is to show that

$$E_\gamma\left[|E[\zeta_t|\mathcal{F}_{t-n}] - \frac{\rho_n W}{H(\xi_0)}|\right] \to 0,$$

as $t \to \infty$, for ψ-almost all $\gamma \in \Gamma$ and thereby conclude that

$$\zeta_t \to \frac{\rho_n W}{H(\xi_0)}$$

in P_γ-probability for ψ-almost all $\gamma \in \Gamma$. With

$$W_t(A \times B) = \sum_{x \in \mathcal{I}_t} 1_{A \times B}(\xi_x, \tau'_x) e^{-\alpha \tau'_x} H(\xi_x),$$

we have that

$$\begin{aligned} &|E[\zeta_t|\mathcal{F}_{t-n}] - \frac{\rho_n W}{H(\xi_0)}| \\ \le\ & \frac{1}{H(\xi_0)} \int_{\Gamma \times R_+} |E_{\gamma'}[\zeta_{t-u}] - \rho_n| W_{t-n}(d\gamma' \times du) + \frac{\rho_n}{H(\xi_0)} |W_{t-n} - W|. \end{aligned}$$

We already know that $W_t \to W$ in $L^1(P_\gamma)$ for ψ-almost all $\gamma \in \Gamma$, which takes care of the second term. To deal with the first term introduce the special characteristic

$$\chi^*(t) = e^{\alpha t} \int_S \int_t^\infty 1_{A \times B}(\gamma', u) e^{-\alpha u} H(\gamma') \eta(d\gamma' \times du),$$

for which it can be shown that

$$W_t(A \times B) = e^{-\alpha t} Z_t^{\chi^*},$$

see Nerman (1984). To show that this converges, the conditions of the Key Markov Renewal Theorem will be investigated. The direct Riemann integrability relies on the general observation that, with notation from Section 4.1, if the function $g(x,t)$ is monotone in t and $\int_E g(x,t)\pi(dx)$ is bounded, then $g(x,t)$ is directly Riemann integrable (π). This observation relies on the connection between direct Riemann integrability in the sense of Definition (4.8) and one-dimensional direct Riemann integrability. The latter is defined for functions $f : R_+ \to R_+$ as

$$\sum_{n \ge 0} \sup_{n \le t \le n+1} f(t) < \infty$$

and

$$\delta \sum_{n \geq 0} \Big(\sup_{n\delta \leq t \leq (n+1)\delta} f(t) - \inf_{n\delta \leq t \leq (n+1)\delta} f(t) \Big) \to 0,$$

as $\delta \to 0+$. It is known that any bounded, monotone, integrable function is directly Riemann integrable in this sense. Now we see that if g is monotone in t then

$$\sup_{a \leq t \leq b} \int_E g(x,t)\pi(dx) = \int_E \sup_{a \leq t \leq b} g(x,t)\pi(dx),$$

with the obvious analogue for infimum. Hence direct Riemann integrability is equivalent to one-dimensional direct Riemann integrability of the function $f(t) = \int_E g(x,t)\pi(dx)$. But if g is monotone in t then also f is monotone and since f is assumed bounded it is also directly Riemann integrable. Since $e^{-\alpha t}E_\gamma[\chi^*(t)]$ is decreasing in t and

$$\int_\Gamma e^{-\alpha t}E_\gamma[\chi^*(t)]\psi(d\gamma) \leq \int_\Gamma H(\gamma)\psi(d\gamma) = 1,$$

$e^{-\alpha t}E_\gamma[\chi^*(t)]$ is directly Riemann integrable. The inequality is true since

$$e^{-\alpha t}E_\gamma[\chi^*(t)] \leq \int_{\Gamma \times R_+} e^{-\alpha u} H(r) M(\gamma, dr \times du) = H(\gamma),$$

and

$$\int_\Gamma H(\gamma)\psi(d\gamma) = \int_S \bar{\gamma} Q(s, d\gamma)\pi(ds) = \int_S E_s[\bar{\xi}]\pi(ds) = \int_S h(s)\pi(ds) = 1.$$

Clearly χ^* is not necessarily bounded so (5.16) is not directly applicable. However the boundedness was used only to guarantee that

$$\sup_{t \geq 0} E_\gamma[e^{-\alpha t} Z_t^{\chi^*}] < \infty,$$

and with our special χ^* this can be shown directly:

$$E_\gamma[e^{-\alpha t} Z_t^{\chi^*}] \leq E_\gamma[W_t] = H(\gamma) < \infty.$$

Hence all the conditions in the Key Markov Renewal Theorem are satisfied and it follows that

$$E_\gamma[W_t(A \times B)] \to E_\gamma[W(A \times B)] = \frac{E_\psi[\hat{\chi}^*(\alpha)]}{\alpha B} H(\gamma),$$

where

$$\begin{aligned} E_\psi[\hat{\chi}^*(\alpha)] &= \alpha \int_0^\infty \int_\Gamma \int_t^\infty 1_{A \times B}(\gamma', u) e^{-\alpha u} H(\gamma') M(\gamma, d\gamma' \times du)\psi(d\gamma)dt \\ &= \alpha \int_0^\infty \int_\Gamma 1_{A \times B}(\gamma', u) u e^{-\alpha u} H(\gamma') M(\gamma, d\gamma' \times du)\psi(d\gamma)dt. \end{aligned}$$

By this expression for $E_\gamma[W(A \times B)]$ and the invariance property of ψ, it is clear that $E_\gamma[W(\cdot \times B)]$ is absolutely continuous with respect to $\psi(\cdot)$ for any $B \in \mathcal{B}$

and ψ-almost all $\gamma \in \Gamma$. Therefore the convergence

$$|E_{\gamma'}[\zeta_{t-u}] - \rho_n| \to 0$$

holds for $E_\gamma[W(\cdot \times \cdot)]$-almost all $(u, \gamma') \in R_+ \times \Gamma$.

From the proof of Theorem 3 in Jagers (1992) we know that there exists a constant $C < \infty$ such that

$$\frac{E_s[e^{-\alpha t} z_t^\chi]}{h(s)} \leq C,$$

and hence

$$\begin{aligned} & \frac{E_\gamma[e^{-\alpha t} Z_t^{\chi'}]}{H(\gamma)} \\ = \ & \frac{1}{H(\gamma)} \sum_{i=1}^{\infty} e^{-\alpha \tau_i(\gamma)} h(\sigma_i(\gamma)) \frac{E_{\sigma_i(\gamma)}[e^{-\alpha(t-\tau_i(\gamma))} z_{t-\tau_i(\gamma)}]}{h(\sigma_i(\gamma))} \\ \leq \ & C \frac{1}{H(\gamma)} \sum_{i=1}^{\infty} e^{-\alpha \tau_i(\gamma)} h(\sigma_i(\gamma)) = C. \end{aligned} \tag{6.4}$$

Hence $|E_{\gamma'}[\zeta_{t-u}] - \rho_n|$ is bounded and we have, for any $G \in \mathcal{G} \times \mathcal{B}$,

$$\begin{aligned} & \int_{\Gamma \times R_+} |E_{\gamma'}[\zeta_{t-u}] - \rho_n| E_\gamma[W_{t-n}(d\gamma' \times du)] \\ \leq \ & \sup_{(\gamma', u) \in G} |E_{\gamma'}[\zeta_{t-u}] - \rho_n| E_\gamma[W_{t-n}(G)] + C E_\gamma[W_{t-n}(G^c)], \end{aligned}$$

where $W_t(G)$ and $W(G)$ are defined in the obvious way. By Egoroff's theorem, G can be chosen such that $|E_{\gamma'}[\zeta_{t-u}] - \rho_n|$ tends to zero uniformly on G and $E_\gamma[W(G^c)]$ is arbitrarily small, see for instance Folland (1984). Therefore, since $E_\gamma[W_{t-n}(G)] \to E_\gamma[W(G)] \leq H(\gamma)$ and $E_\gamma[W(G^c)] \to E_\gamma[W(G^c)]$,

$$\int_{\Gamma \times R_+} |E_{\gamma'}[\zeta_{t-u}] - \rho_n| E_\gamma[W_{t-n}(d\gamma' \times du)]$$

tends to zero as t tends to infinity and thus

$$\zeta_t \to \rho_n \frac{W}{H(\xi_0)}$$

in P_γ-probability for ψ-almost all $\gamma \in \Gamma$. Hence we have that

$$e^{-\alpha t} Z_t^{\chi^n} \to \rho_n W$$

in P_γ-probability, and by the uniform integrability also in $L^1(P_\gamma)$ for the characteristic χ^n under consideration (see for instance p.297 in Ash (1972)). Finally consider $\chi'(t) = \sum_{i=1}^{\infty} \chi_i(t - \tau_i)$. With

$$\rho = \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha \beta} = \frac{E_\psi[\hat{\chi'}(\alpha)]}{\alpha B},$$

we have that

$$\begin{aligned} E_\gamma\left[|e^{-\alpha t}Z_t^{\chi'} - \rho W|\right] &\le E_\gamma\left[|e^{-\alpha t}Z_t^{\chi'} - e^{-\alpha t}Z_t^{\chi^n}|\right] \\ &+ \; E_\gamma\left[|e^{-\alpha t}Z_t^{\chi^n} - \rho_n W|\right] + E_\gamma\left[|\rho_n W - \rho W|\right], \end{aligned}$$

where the first and third terms can be made small by choice of n large and monotone convergence, and the second term is the case just treated. ■

It has thus been shown that the process converges when it starts from a full group of siblings. One might ask what happens if it starts from one single individual. We would suspect the contribution of this individual to vanish as $t \to \infty$ so that the process might just as well start from her children. We already know from Theorem 5.8 that the expectations converge when starting from a single individual and would be surprised if the same was not true for the process itself. Indeed it holds that

$$e^{-\alpha t}z_t^\chi = e^{-\alpha t}\chi_0(t) + e^{-\alpha t}Z_t^\chi,$$

where Z_t^χ starts from the children of the ancestor. Since χ is bounded the first term tends to 0, and the second converges by the last theorem. However, writing down the above equality more carefully reveals where the problem could be. Hence

$$e^{-\alpha t}z_t^\chi(s,\omega_I) = e^{-\alpha t}\chi(s,\omega_I,t) + e^{-\alpha t}Z_t^\chi(\xi(\omega_0),\omega_{I_1}), \tag{6.5}$$

where the first term tends to 0 and the second does so in $L^1(P_\gamma)$ for *ψ-almost all* $\gamma \in \Gamma$. The question is if there is a set of full π-measure such that picking the ancestor's starting type from this set will guarantee that $\gamma = \xi(\omega_0)$ belongs to the set for which convergence holds. The answer is yes, and we state the following corollary. Recall the marginals $\tilde{Q}_s$ defined in (3.2).

Corollary 6.3. *Under the conditions of* Theorem 6.2 *there exists a random variable $\tilde{w}$ on $S \times \Omega^I$ such that*

$$e^{-\alpha t}z_t^\chi \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}\tilde{w},$$

as $t \to \infty$ in $L^1(\tilde{Q}_s)$ for π-almost all $s \in S$.

Proof. Recall the limit variable w on $\Gamma \times \Omega^{I_1}$ and define $\tilde{w}$ on $S \times \Omega^I$ through

$$\tilde{w}(s,\omega_I) = w \circ \xi_0(s,\omega_I) = w(\xi(\omega_0),\omega_{I_1}). \tag{6.6}$$

Let $Z_t^\chi(\xi(\omega_0),\cdot)$ be denoted by $Z_t^\chi \circ \xi_0$, let $\rho = E_\pi[\hat{\chi}(\alpha)]/\alpha\beta$ and note that, by (6.5),

$$\begin{aligned} &E_s\left[|e^{-\alpha t}z_t^\chi - \tilde{w}|\right] \\ \le\; &E_s[e^{-\alpha t}\chi_0(t)] + E_s\left[|e^{-\alpha t}Z_t^\chi \circ \xi_0 - \rho w \circ \xi_0|\right], \end{aligned}$$

where the first term tends to zero since χ is bounded and the second term is

$$E_s\left[\left|e^{-\alpha t}Z_t \circ \xi_0 - \rho w \circ \xi_0\right|\right] = \int_\Gamma E_\gamma\left[\left|e^{-\alpha t}Z_t^\chi - \rho w\right|\right] Q(s, d\gamma),$$

since an s-type ancestor chooses reproduction process ξ_0 according to the measure $Q(s,\cdot)$. By Theorem 6.2, $E_\gamma\left[\left|e^{-\alpha t}Z_t - \rho w\right|\right] \to 0$ as $t \to \infty$ for ψ-almost all $\gamma \in \Gamma$. Since

$$\psi(A) = \int_S Q(s, A)\pi(ds),$$

it is clear that $Q(s, d\gamma)$, and hence also $H(\gamma)Q(s, d\gamma)$, is absolutely continuous with respect to $\psi(d\gamma)$ for π-almost all $s \in S$. With C as in the previous proof, $E_\gamma\left[\left|e^{-\alpha t}Z_t^\chi - \rho w\right|\right] \leq (C+\rho)H(\gamma)$ and

$$\int_\Gamma E_\gamma\left[\left|e^{-\alpha t}Z_t^\chi - \rho w\right|\right] Q(s, d\gamma) \to 0,$$

for π-almost all $s \in S$ by dominated convergence. ■

Note. Of course the random variable $\tilde{w}$ has expectation $h(s)$ under $\tilde{Q}_s$. This is a direct consequence of the corollary together with Theorem 5.8 but can also be computed explicitly:

$$E_s[\tilde{w}] = \int_\Gamma E_\gamma[W]Q(s, d\gamma) = \int_\Gamma H(\gamma)Q(s, d\gamma) = E_s[\bar{\xi}] = h(s).$$

□

7. The Stable Population

7.1. The Stable Population Measure

We now turn to the problem of stable population composition. The objective here is to study the properties of a typical individual, loosely thought of as sampled from all those born in an old exponentially growing population. This individual, *Ego*, has siblings and progeny in the same manner as earlier, but she also has a mother, aunts, a grandmother and so on. Therefore the new individual space $J = Z_- \times I$ is introduced, where $Z_- = \{0, -1, -2, ...\}$. An element $(-n, x)$ of J is thus the individual x in the population stemming from Ego's nth grandmother. Here we note that since the individual $-n$ always has the child $-(n-1)$, $-nI$ is interpreted as the renumbered population that stems from $-n$ when $-(n-1)$ has been removed. Ego herself is of course $(0,0)$ or, for short, just 0.

To relate Ego to this nth grandmother we need information about the birth ranks. Hence we associate with Ego the vector $(i_n, ..., i_0)$ telling that she is the i_0th child of the i_1th child of...of the i_{n-1}th child of the individual $-n$ who herself has rank i_n. We also want to know Ego's age at sampling and we define the *doubly infinite population space* to be

$$\mathbf{\Omega} = R_+ \times N^\infty \times \Omega^J,$$

an element of which gives information about Ego's age, ancestry and the lives of all conceivable individuals in J. This space is endowed with the product σ-algebra denoted by $\mathcal{C}$. The objective is now to give a description of the *stable population measure* on $(\mathbf{\Omega}, \mathcal{C})$. First the ordinary independent multi-type case is described, following Jagers and Nerman (1992). In order to do this some random elements on $(\mathbf{\Omega}, \mathcal{C})$ are introduced.

Hence let T_0 denote Ego's age , S_0 her type, U_0 her life and R_0 her rank. T_1 is Ego's mothers age when she gave birth to Ego, S_1 her type, U_1 her life, R_1 her rank and so on backwards. The life and progeny of Ego is denoted by z_0, z_1 denotes the life and progeny of Ego's mother *except* the life and progeny of Ego, z_2 the life and progeny of Ego's grandmother except Ego's mother's life and progeny and so on. Thus, after renumbering the population to tell which one of $-k$'s children that has been removed in order to play the role of $-(k-1)$, z_k is the coordinate projection $\mathbf{\Omega} \to \Omega^{-kI}$ (see Jagers and Nerman (1992) for details).

Definition 7.1. *In an independent multi-type branching process, the stable population measure is determined by*

$$\begin{aligned}
&\mathbf{P}(z_k \in A_k, T_k \in dt_k, S_k \in ds_k, R_k = i_k, k = 0, ..., n) \\
&= E_\pi[e^{-\alpha\tau_{i_n}}; \sigma_{i_n} \in ds_n] E_{s_n}[e^{-\alpha t_n}; A_n \cap \{\sigma_{i_{n-1}} \in ds_{n-1}, \tau_{i_{n-1}} \in dt_n\}]
\end{aligned}$$

$$...E_{s_1}[e^{-\alpha t_1}; A_1 \cap \{\sigma_{i_0} \in ds_0, \tau_{i_0} \in dt_1\}]P_{s_0}(A_0)\alpha e^{-\alpha t_0}dt_0, \tag{7.1}$$

where $A_k \in \mathcal{A}^I$. This definition has a number of consequences described in the mentioned reference. For instance the stable population measure gives Ego an age which is exponentially distributed (α) and independent of everything else and a type distributed according to π, thus explaining its name. These and other interesting consequences will be more thoroughly described in the sequel when the independent case is compared with various forms of sibling dependent populations.

The main difference in a multi-type sibling dependent population is that even if an individual's marginal reproduction only depends on her type, there are still dependencies between siblings; the conditional independence structure in independent multi-type populations now only exists on the macro level. We want to study the individual stable population but for this it is necessary to invoke the macro process. Therefore any defining relation of the stable population measure should take into account the following two considerations: the whole reproduction processes should be used as information carriers between generations and lives and progenies of whole sibling groups should be considered rather than individuals. Hence let S_k, T_k and R_k be as above and define Γ_0 as the reproduction process according to which Ego was born, Γ_1 the process according to which Ego's mother was born and so on. Furter let Z_0 be the lives and progeny of Ego *and her siblings*, Z_1 the lives and progeny of Ego's mother and her siblings *except* the lives and progeny of Ego and her siblings and so on. In terms of coordinate projections, Z_k projects $\mathbf{\Omega}$ on $\Omega^{-(k+1)I_1}$.

Definition 7.2. *In a multi-type branching process with sibling dependecies, the stable population measure is determined by*

$$\begin{aligned}
&\mathbf{P}(Z_k \in B_k, T_k \in dt_k, \Gamma_k \in d\gamma_k, R_k = i_k, k = 0, ..., n) \\
&= e^{-\alpha\tau_{i_n}(\gamma_n)}\psi(d\gamma_n)E_{\gamma_n}[e^{-\alpha t_n}; B_n \cap \{\xi_{i_n} \in d\gamma_{n-1}, \tau_{i_{n-1}}(\xi_{i_n}) \in dt_n\}] \\
&\quad ...E_{\gamma_1}[e^{-\alpha t_1}; B_1 \cap \{\xi_{i_1} \in d\gamma_0, \tau_{i_0}(\xi_{i_1}) \in dt_1\}]P_{\gamma_0}(B_0)\alpha e^{-\alpha t_0}dt_0,
\end{aligned} \tag{7.2}$$

where $B_k \in \mathcal{A}^{I_1}$. Indeed this has basically the same form as in the independent case. The difference is, as mentioned above, that information between generation is passed over by the whole reproduction processes rather than just the individual types. With this in mind, also note that

$$\begin{aligned}
&\int_\Gamma e^{-\alpha\tau_{i_n}(\gamma)}1_C(\gamma)\psi(d\gamma) \\
&= \int_\Gamma e^{-\alpha\tau_{i_n}(\gamma)}1_C(\gamma)Q(\pi, d\gamma) = \int_\Omega e^{-\alpha\tau_{i_n}(\xi(\omega))}1_C(\xi(\omega))\tilde{Q}(\pi, d\omega) \\
&= \int_\Omega e^{-\alpha\tau(i_n)(\omega)}1_C(\xi(\omega)\tilde{Q}(\pi, d\omega) \\
&= E_\pi[e^{-\alpha\tau(i_n)}; \xi \in C],
\end{aligned}$$

so that

$$E_\pi[e^{-\alpha\tau(i_n)}; \xi \in d\gamma] = e^{-\alpha\tau_{i_n}(\gamma)}\psi(d\gamma). \tag{7.3}$$

7.2. Convergence towards the Stable Population Measure

In this section we will prove that the stable population measure indeed appears as a limit of averages in a growing population. We thus want to consider the process at time t, count the number of individuals that possess some certain property, E say, depending on age and ancestry and divide this by the total number of individuals born up to t. As we then let t tend to infinity this ratio will hopefully converge to the stable population probability of E.

To formulate this more accurately, let

$$J_n = \bigcup_{j=0}^{n} \{-j\} \times I,$$

the set of individuals stemming from $-n$. By considering $-n$ as ancestor we can identify the spaces J_n and I and hence also $(\Omega^{J_n}, \mathcal{A}^{J_n})$ and $(\Omega^I, \mathcal{A}^I)$. Now fix $n \in N$, $(i_0, ..., i_{n-1}) \in N^n$, $B \in \mathcal{A}^I$ and $a \in R_+$ and consider the set

$$E = [0, a] \times (i_0, ..., i_{n-1}) \times N^\infty \times B \times \Omega^{J \setminus J_n}.$$

Sets of this form generate a σ-algebra of sets in $\mathcal{C}$ which are such that they only depend n steps backwards, for fixed but arbitrary n. An individual is then counted if she has the property E at time t in the sense that her age is less than a, she is the i_0th child of the i_1th child of...of the i_{n-1}th child of some individual and that this individual has life and progeny in the set B. Since characteristics may depend on progeny but not on ancestry, the individuals with property E cannot be counted directly. However, if an individual x has property E, then she also has an ancestor, y say, $n+1$ generations back, whose life and progeny is in the set B and such that the individual $yi_{n-1}...i_0$ at time t has age less than a. Hence y might be counted just as well as x, a common trick in general branching processes. For that purpose put $\iota = (i_{n-1}, ..., i_0)$ and define

$$\chi_E(t) = 1_B 1_{[0,a]}(t - \tau_\iota),$$

Then $z_t^{\chi_E}$ is the number of individuals with property E born up to time t. Recalling y_t =the total number of individuals born up to t we first look at the *composition in expectation*

$$\frac{E_s[z_t^{\chi_E}]}{E_s[y_t]},$$

where $z_t^{\chi_E}$ is the process counted by χ_E when starting from a single individual. Under the assumptions of Theorem 5.8 the convergence

$$E_s[e^{-\alpha t} z_t^{\chi_E}] \to h(s)\frac{E_\pi[\hat{\chi}_E(\alpha)]}{\alpha\beta}$$

follows and since y_t is the process counted by $\chi(t) = 1_{R_+}(t)$, which has $E_\pi[\hat{\chi}(\alpha)] = 1$,

$$\frac{E_s[z_t^{\chi_E}]}{E_s[y_t]} \to E_\pi[\hat{\chi}_E(\alpha)].$$

And, indeed, this limit is $\mathbf{P}(E)$:

Lemma 7.3. *For sets E as above*

$$\mathbf{P}(E) = E_\pi[\hat{\chi}_E(\alpha)].$$

Proof. It suffices to consider sets E where B has the special form

$$B = \{\omega_I : \omega_{I_1} \in B_{n-1}, \omega_{i_{n-1}I_1} \in B_{n-2}, ..., \omega_{i_{n-1}...i_1 I_1} \in B_0\},$$

where B_k for $k \geq 1$ is such that it gives no restrictions on the lives of the individuals in the set $i_k I_1$ (because of the renumbering mentioned in the previous section).

$$\begin{aligned} E_\pi[\hat{\chi}_E(\alpha)] &= \int_{\Omega^I \times R_+} \alpha e^{-\alpha t} 1_{[0,a]}(t - \tau_\iota(\omega_I)) 1_B(\omega_I) dt \tilde{Q}_\pi(d\omega_I) \\ &= \int_0^a \alpha e^{-\alpha t} dt \int_{\Omega^I} e^{-\alpha \tau_\iota(\omega_I)} 1_B(\omega_I) \tilde{Q}_\pi(d\omega_I), \end{aligned}$$

where $\tilde{Q}_\pi(\cdot) = \int_S \tilde{Q}_s(\cdot)\pi(ds)$.

Now, because of the special form of B and the fact that

$$\tau_\iota(\omega_I) = \tau(i_{n-1})(\omega_0) + \tau_{i_{n-2}}(\xi_{i_{n-1}}(\omega_{I_1})) + ... + \tau_{i_0}(\xi_{i_1}(\omega_{i_{n-1}...i_2 I_1})),$$

where of course $\xi_{i_{n-1}}(\omega_{I_1}) = \xi(\omega_{i_{n-1}})$ and so on,

$$\begin{aligned} &\int_{\Omega^I} e^{-\alpha \tau_\iota(\omega_I)} 1_B(\omega_I) \tilde{Q}_\pi(d\omega_I) \\ &= \int_{\Omega^I} e^{-\alpha \tau_{i_{n-1}}(\omega_0)} e^{-\alpha \tau_{i_{n-2}}(\xi_{i_{n-1}}(\omega_{I_1}))} ... e^{-\alpha \tau_{i_0}(\xi_{i_1}(\omega_{i_{n-1}...i_2 I_1}))} \\ &\quad 1_{B_{n-1}}(\omega_{I_1}) 1_{B_{n-2}}(\omega_{i_{n-1}I_1}) ... 1_{B_0}(\omega_{i_{n-1}...i_1 I_1}) \tilde{Q}_\pi(d\omega_I) \\ &= \int_{\Omega \times \Gamma^{n-1}} e^{-\alpha \tau(i_{n-1})(\omega_0)} P_{\xi(\omega_0)}(B_{n-1} \cap \{\xi_{i_{n-1}} \in d\gamma_{n-2}\}) e^{-\alpha \tau_{i_{n-2}}(\gamma_{n-2})} \\ &\quad P_{\gamma_{n-2}}(B_{n-1} \cap \{\xi_{i_{n-2}} \in d\gamma_{n-3}\}) ... e^{-\alpha \tau_{i_1}(\gamma_1)} P_{\gamma_1}(B_1 \cap \{\xi_{i_1} \in d\gamma_0\}) \\ &\quad e^{-\alpha \tau_{i_0}(\gamma_0)} P_{\gamma_0}(B_0) \tilde{Q}(\pi, d\omega_0). \end{aligned}$$

For any $A \in \mathcal{G}$ and $k = 1, ..., n$,

$$\begin{aligned} &E_{\gamma_k}[e^{-\alpha \tau_{i_{k-1}}(\xi_{i_k})}; B_k \cap \{\xi_{i_k} \in A\}] \\ &= \int_{\Omega^{I_1}} e^{-\alpha \tau_{i_{k-1}}(\xi_{i_k}(\omega_{I_1}))} P_{\gamma_k}(B_k \cap \{\xi_{i_k} \in A\} \cap d\omega_{I_1}) \end{aligned}$$

$$\begin{aligned} &= \int_{\xi_{i_k}^{-1}(A)} e^{-\alpha\tau_{i_{k-1}}(\xi_{i_k}(\omega_{I_1}))} P_{\gamma_k}(B_k \cap d\omega_{I_1}) \\ &= \int_A e^{-\alpha\tau_{i_{k-1}}(\gamma_{k-1})} P_{\gamma_k}(B_k \cap \{\xi_{i_k} \in d\gamma_{k-1}\}), \end{aligned}$$

so that

$$E_{\gamma_k}[e^{-\alpha\tau_{i_{k-1}}(\xi_{i_k})}; B_k \cap \{\xi_{i_k} \in d\gamma_{k-1}\}] = e^{-\alpha\tau_{i_{k-1}}(\gamma_{k-1})} P_{\gamma_k}(B_k \cap \{\xi_{i_k} \in d\gamma_{k-1}\}).$$

Since $Q(\pi, \cdot) = \tilde{Q}(\pi, \{\xi \in \cdot\})$ we obtain

$$\int_\Omega e^{-\alpha\tau(i_{n-1})(\omega_0)} P_{\xi(\omega_0)}(\cdot)\tilde{Q}(\pi, d\omega_0) = \int_\Gamma e^{-\alpha\tau_{i_{n-1}}(\gamma_{n-1})} P_{\gamma_{n-1}}(\cdot) Q(\pi, d\gamma_{n-1}),$$

and hence

$$\begin{aligned} &E_\pi[\hat{\chi}_E(\alpha)] \\ &= \int_0^a \alpha e^{-\alpha t} dt \int_{\Gamma^n} e^{-\alpha\tau_{i_{n-1}}(\gamma_{n-1})} E_{\gamma_{n-1}}[e^{-\alpha\tau_{i_{n-2}}(\xi_{i_{n-1}})}; B_{n-1} \cap \{\xi_{i_{n-1}} \in d\gamma_{n-2}\}] \\ &\quad ...E_{\gamma_1}[e^{-\alpha\tau_{i_0}(\xi_{i_1})}; B_1 \cap \{\xi_{i_1} \in d\gamma_0\}] P_{\gamma_0}(B_0) Q(\pi, d\gamma_n) \\ &= \mathbf{P}(Z_k \in B_k, T_0 \le a, R_k = i_k, k = 0, ..., n-1) = \mathbf{P}(E). \end{aligned}$$

■

By Corollary 6.3,

$$e^{-\alpha t} z_t^\chi \to \frac{E_\pi[\hat{\chi}_E(\alpha)]}{\alpha\beta} \tilde{w},$$

in $L^1(\tilde{Q}_s)$ under the $x \log x$-condition. Since

$$e^{-\alpha t} y_t \to \frac{1}{\alpha\beta} \tilde{w}$$

is a special case of this, we should expect the *actual* composition

$$\frac{z_t^\chi}{y_t}$$

to converge to $E_\pi[\hat{\chi}_E(\alpha)] = \mathbf{P}(E)$ in $\tilde{Q}_s$-probability. To establish this we need some condition to guarantee that $\tilde{w} > 0 \Leftrightarrow y_t \to \infty$:

Lemma 7.4. *For a strictly Malthusian process with sibling dependencies, assume that* $\inf_s Q_s(\tilde{w} > 0) > 0$. *Then*

$$\tilde{w} > 0 \Leftrightarrow y_t \to \infty,$$

as $t \to \infty$, *almost surely* $\tilde{Q}_s, s \in S$.

Proof. From Theorem 6.2 it is clear that $\tilde{w} > 0 \Rightarrow y_t \to \infty$. For the converse enumerate macro individuals as they are born into the population: $X_0, X_1, ...$

and denote by $\mathcal{A}_n$ the σ-algebra generated by the ancestor's type and the lives of $X_0, ..., X_n$. Then $\tilde{Q}_s(\{\tilde{w} > 0\} \cup \{y_t \not\to \infty\} | \mathcal{A}_n)$ is a martingale and by Lévy's well known theorem, see for instance Ash (1972) p.300, it holds that

$$\tilde{Q}_s(\{\tilde{w} > 0\} \cup \{y_t \not\to \infty\} | \mathcal{A}_n) \to I_{\{\{\tilde{w}>0\}\cup\{y_t\not\to\infty\}\}},$$

as $n \to \infty$. Also it is easy to see that

$$\tilde{Q}_s(\{\tilde{w} > 0\} \cup \{y_t \not\to \infty\} | \mathcal{A}_n) \geq \inf_{r \in S} \tilde{Q}_r(\tilde{w} > 0) > 0$$

by assumption. Hence $I_{\{\{\tilde{w}>0\}\cup\{y_t\not\to\infty\}\}} = 1$ and the proof is complete. Not all the details are spelled out, they may be found in Jagers and Nerman (1992).

■

Corollary 7.5. *Assume the conditions of* Theorem 6.2 *and add to these that* $\inf_s \tilde{Q}_s(\tilde{w} > 0) > 0$. *Then, for sets E as above*

$$\frac{z_t^{\chi_E}}{y_t} \to \mathbf{P}(E)$$

in $\tilde{Q}_s$-probability on $\{y_t \to \infty\}$ for π-almost all $s \in S$ as $t \to \infty$.

7.3. Properties of the Stable Population Measure

In this section the stable population measure is described through some of its consequences. The measure is compared with its independent analogue, that is the stable population measure stemming from an ordinary independent multi-type branching process with the same reproduction kernel $\mu(s, dr \times dt)$. For a description of the stable population in such a process, the reader is referred to Jagers and Nerman (1992). First look at Ego's individual properties.

Proposition 7.6. *For Ego's marginals it holds that T_0 is exponentially distributed (α) and independent of everything else and further*

$$\mathbf{P}(S_0 \in B) = \pi(B),$$

$$\mathbf{P}(R_0 = k) = E_\pi[e^{-\alpha\tau(k)}],$$

$$\mathbf{P}(U_0 \in A) = \tilde{Q}(\pi, A), \quad A \in \mathcal{A}.$$

Proof. The statement about T_0 follows immediately from the definition of $\mathbf{P}$. The other statements are also easily proved:

$$\mathbf{P}(S_0 \in B) = \sum_{i_0=1}^{\infty} \int_\Gamma e^{-\alpha\tau_{i_0}(\gamma_0)} \psi(d\gamma_0) 1_B(\sigma_{i_0}(\gamma_0)) = \pi(B),$$

by Lemma 5.14,

$$\mathbf{P}(R_0 = k) = \int_\Gamma e^{-\alpha\tau_k(\gamma_0)} \psi(d\gamma_0) = E_\pi[e^{-\alpha\tau(k)}],$$

and, letting $A_0 = \{\omega_{I_1} \in \Omega^{I_1} : \omega_{i_0} \in A\}$,

$$\begin{aligned}
\mathbf{P}(U_0 \in A) &= \mathbf{P}(Z_0 \in A_0) \\
&= \sum_{i_0=1}^{\infty} \int_{\Gamma} e^{-\alpha\tau_{i_0}(\gamma_0)} \psi(d\gamma_0) P_{\gamma_0}(A_0) \\
&= \sum_{i_0=1}^{\infty} \int_{\Gamma\times S} e^{-\alpha\tau_{i_0}(\gamma_0)} \psi(d\gamma_0) \tilde{Q}(s, A) \delta_{\sigma_{i_0}(\gamma_0)}(ds) \\
&= \int_S \tilde{Q}(s, A)\pi(ds) = \tilde{Q}(\pi, A).
\end{aligned}$$

■

Hence the sibling dependencies do not change the fact that Ego's age is exponential and independent of everything else. Neither do they affect the distributions of Ego's rank or life when looked upon separately. It is also interesting to investigate more general properties of **P** such as joint distributions and possible Markov structures in the family line backwards from Ego. It has just been shown that Ego's type, S_0, has distribution π but what happens if the sequence $S_0, S_1, \ldots$ is considered? The answer is the same as in the independent case:

Proposition 7.7. *The sequence $S_0,, S_1, \ldots$ of types backwards from Ego is a Markov chain with transition probabilities*

$$\mathbf{P}(S_{n+1} \in ds | S_n = r) = \pi(ds) \frac{\hat{\mu}(s, dr)}{\pi(dr)}.$$

The marginal distributions are

$$\mathbf{P}(S_0 \in ds) = \pi(ds)$$

and

$$\mathbf{P}(S_n \in ds) = \hat{\mu}^n(s, S)\pi(ds).$$

Proof.

$$\mathbf{P}(S_0 \in ds_0, \ldots, S_n \in ds_n)$$

$$= \sum_{i_0,\ldots,i_n} \int_{\Gamma^{n+1}} \mathbf{P}(\Gamma_0 \in d\gamma_0, R_0 = i_0, \ldots, \Gamma_n \in d\gamma_n, R_n = i_n)$$

$$\delta_{\sigma_{i_0}(\gamma_0)}(ds_0) \ldots \delta_{\sigma_{i_n}(\gamma_n)}(ds_n),$$

where

$$\mathbf{P}(\Gamma_0 \in d\gamma_0, R_0 = i_0, ..., \Gamma_n \in d\gamma_n, R_n = i_n)$$

$$= \int_{R_+^n} e^{-\alpha\tau_{i_n}(\gamma_n)}\psi(d\gamma_n)E_{\gamma_n}[e^{-\alpha t_n}; \xi_{i_n} \in d\gamma_{n-1}, \tau_{i_{n-1}}(\xi_{i_n}) \in dt_n]$$

$$...E_{\gamma_1}[e^{-\alpha t_1}; \xi_{i_1} \in d\gamma_0, \tau_{i_0}(\xi_{i_1}) \in dt_1]$$

$$= e^{-\alpha\tau_{i_n}(\gamma_n)}\psi(d\gamma_n)E_{\gamma_n}[e^{-\alpha\tau_{i_{n-1}}(\xi_{i_n})}; \xi_{i_n} \in d\gamma_{n-1}]$$

$$...E_{\gamma_1}[e^{-\alpha\tau_{i_0}(\xi_{i_1})}; \xi_{i_1} \in d\gamma_0].$$

Now, for any k,

$$E_{\gamma_k}[e^{-\alpha\tau_{i_{k-1}}(\xi_{i_k})}; \xi_{i_k} \in d\gamma_{k-1}]$$

$$= e^{-\alpha\tau_{i_{k-1}}(\gamma_{k-1})}P_{\gamma_k}(\xi_{i_k} \in d\gamma_{k-1}) = e^{-\alpha\tau_{i_{k-1}}(\gamma_{k-1})}Q(\sigma_{i_k}(\gamma_k), d\gamma_{k-1}),$$

and hence

$$\mathbf{P}(\Gamma_0 \in d\gamma_0, R_0 = i_0, ..., \Gamma_n \in d\gamma_n, R_n = i_n)$$

$$= e^{-\alpha\tau_{i_n}(\gamma_n)}\psi(d\gamma_n)e^{-\alpha\tau_{i_{n-1}}(\gamma_{n-1})}Q(\sigma_{i_n}(\gamma_n), d\gamma_{n-1})$$

$$...e^{-\alpha\tau_{i_0}(\gamma_0)}Q(\sigma_{i_1}(\gamma_1), d\gamma_0).$$

Further, for any k and γ,

$$Q(\sigma_k(\gamma), d\gamma')\delta_{\sigma_k(\gamma)}(ds) = Q(s, d\gamma')\delta_{\sigma_k(\gamma)}(ds),$$

so that

$$\mathbf{P}(S_0 \in ds_0, ..., S_n \in ds_n)$$

$$= \sum_{i_0,...,i_n} \int_{\Gamma^{n+1}} e^{-\alpha\tau_{i_n}(\gamma_n)}\psi(d\gamma_n)\delta_{\sigma_{i_n}(\gamma_n)}(ds_n)$$

$$e^{-\alpha\tau_{i_{n-1}}(\gamma_{n-1})}Q(s_n, d\gamma_{n-1})\delta_{\sigma_{i_{n-1}}(\gamma_{n-1})}(ds_{n-1})$$

$$...e^{-\alpha\tau_{i_0}(\gamma_0)}Q(s_1, d\gamma_0)\delta_{\sigma_{i_0}(\gamma_0)}(ds_0).$$

Since

$$\sum_{i=1}^{\infty}\int_{\Gamma} e^{-\alpha\tau_i(\gamma)}\delta_{\sigma_i(\gamma)}(ds)\psi(d\gamma) = \pi(ds)$$

and

$$\sum_{i=1}^{\infty}\int_{\Gamma} e^{-\alpha\tau_i(\gamma)}\delta_{\sigma_i(\gamma)}(dr)Q(s, d\gamma) = \hat{\mu}(s, dr),$$

summation and integration leads to

$$\mathbf{P}(S_0 \in ds_0, ..., S_n \in ds_n)$$

$$= \quad \pi(ds_n)\hat{\mu}(s_n, ds_{n-1})...\hat{\mu}(s_1, ds_0).$$

■

Hence the types backwards from Ego constitute a Markov chain and together with the interbirth times they should form a Markov renewal process. The next proposition states that also this Markovian structure remains the same as in the independent case.

Proposition 7.8. *The sequence of types and interbirth times backwards from Ego,* $(S_n, T_n)_{n=0}^{\infty}$, *is a Markov renewal process with transition kernel*

$$\mathbf{P}(S_{n+1} \in ds, T_{n+1} \in dt | S_n = r) = \pi(ds)\frac{e^{-\alpha t}\mu(s, dr \times dt)}{\pi(dr)}.$$

Proof.

$$\mathbf{P}(S_{n+1} \in ds, T_{n+1} \in dt, S_n \in ds_n, ..., S_0 \in ds_0)$$

$$= \quad \sum_{i_0,...,i_{n+1}} \int_{\Gamma^{n+2}} e^{-\alpha\tau_{i_{n+1}}(\gamma_{n+1})}\psi(d\gamma_{n+1})\delta_{\sigma_{i_{n+1}}(\gamma_{n+1})}(ds)$$

$$E_{\gamma_{n+1}}[e^{-\alpha t}; \xi_{i_{n+1}} \in d\gamma_n, \tau_{i_n}(\xi_{i_{n+1}}) \in dt]$$

$$e^{-\alpha\tau_{i_{n-1}}(\gamma_{n-1})}Q(s_n, d\gamma_{n-1})\delta_{\sigma_{i_{n-1}}(\gamma_{n-1})}(ds_{n-1})$$

$$...e^{-\alpha\tau_{i_0}(\gamma_0)}Q(s_1, d\gamma_0)\delta_{\sigma_{i_0}(\gamma_0)}(ds_0)$$

$$= \quad \sum_{i_n,i_{n+1}} \int_{\Gamma\times\Gamma} e^{-\alpha\tau_{i_{n+1}}(\gamma_{n+1})}\psi(d\gamma_{n+1}\delta_{\sigma_{i_{n+1}}(\gamma_{n+1})}(ds)$$

$$E_{\gamma_{n+1}}[e^{-\alpha t}; \xi_{i_{n+1}} \in d\gamma_n, \tau_{i_n}(\xi_{i_{n+1}}) \in dt]\delta_{\sigma_{i_n}(\gamma_n)}(ds_n)$$

$$\hat{\mu}(s_n, ds_{n-1})...\hat{\mu}(s_1, ds_0).$$

Since

$$\sum_{i=1}^{\infty} 1_{\{\tau_i(\xi_j)\in dt, \xi_j\in d\gamma'\}}\delta_{\sigma_i(\gamma')}(ds) = \gamma'(ds \times dt)1_{\{\xi_j\in d\gamma'\}}$$

and

$$E_\gamma[1_{\{\xi_j \in d\gamma'\}}] = Q(\sigma_j(\gamma), d\gamma'),$$

it holds for any j that

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_\Gamma E_\gamma[e^{-\alpha t}; \xi_j \in d\gamma', \tau_i(\xi_j) \in dt]\delta_{\sigma_i(\gamma')}(ds) \\
&= \; e^{-\alpha t} \int_\Gamma \gamma'(ds \times dt) Q(\sigma_j(\gamma), d\gamma') = e^{-\alpha t} \mu(\sigma_j(\gamma), ds \times dt).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbf{P}(S_{n+1} \in ds, T_{n+1} \in dt, S_n \in ds_n, ..., S_0 \in ds_0) \\
&= \sum_{i_{n+1}} \int_{\Gamma \times \Gamma} e^{-\alpha \tau_{i_{n+1}}(\gamma_{n+1})} \psi(d\gamma_{n+1}) \delta_{\sigma_{i_{n+1}}(\gamma_{n+1})}(ds) e^{-\alpha t} \mu(\sigma_{i_{n+1}}(\gamma_{n+1}), ds_n \times dt) \\
&\quad \hat{\mu}(s_n, ds_{n-1}) ... \hat{\mu}(s_1, ds_0) \\
&= \sum_{i_{n+1}} \int_{\Gamma \times \Gamma} e^{-\alpha \tau_{i_{n+1}}(\gamma_{n+1})} \psi(d\gamma_{n+1}) \delta_{\sigma_{i_{n+1}}(\gamma_{n+1})}(ds) e^{-\alpha t} \mu(s, ds_n \times dt) \\
&\quad \hat{\mu}(s_n, ds_{n-1}) ... \hat{\mu}(s_1, ds_0) \\
&= \pi(ds) e^{-\alpha t} \mu(s, ds_n \times dt) \hat{\mu}(s_n, ds_{n-1}) ... \hat{\mu}(s_1, ds_0).
\end{aligned}$$

■

As in Jagers and Nerman (1992) it can be shown that one consequence of this proposition is that $E[T_n] \to \beta$ as $t \to \infty$, i.e. the asymptotic expected age at childbearing backwards from Ego is β, thereby explaining its name, the stable age at childbearing.

The final proposition states that the Markov property also holds for the sequence of ranks, types and lives backwards from Ego. The transition probabilities are the same as in the independent case:

Proposition 7.9.

$$\begin{aligned}
&\mathbf{P}(R_{n+1} = j, S_{n+1} \in ds, U_{n+1} \in A | R_n = i, S_n = r, U_n, R_{n-1}, S_{n-1}, ...) \\
&= \; E_\pi[e^{-\alpha\tau(j)}; \sigma(j) \in ds] \frac{E_s[e^{-\alpha\tau(i)}; A \cap \{\sigma(i) \in dr\}]}{E_\pi[e^{-\alpha\tau(i)}; \sigma(i) \in dr]}.
\end{aligned}$$

Proof. Let $B_k = \{\omega_{I_1} \in \Omega^{I_1} : \omega_{i_k} \in A_k\}, B = B_{n+1}$. Then

$$P_{\gamma_k}(B_k)\delta_{i_k(\gamma_k)}(ds_k) = \tilde{Q}(s_k, A_k)\delta_{i_k(\gamma_k)}(ds_k)$$

and hence

$$\begin{aligned}
&\mathbf{P}(R_{n+1}=j, S_{n+1}\in ds, U_{n+1}\in A, R_n=i_n, S_n\in ds_n, U_n\in A_n \\
&\qquad ,\dots, R_0=i_0, S_0\in ds_0, U_0\in A_0) \\
&= e^{-\alpha\tau_j(\gamma_{n+1})}\psi(d\gamma_{n+1})\delta_{\sigma_j(\gamma_{n+1})}(ds)e^{-\alpha\tau_{i_n}(\gamma_n)}P_{\gamma_{n+1}}(\{\xi_j\in d\gamma_n\}\cap B) \\
&\qquad \dots e^{-\alpha\tau_{i_0}(\gamma_0)}P_{\gamma_1}(\{\xi_{i_1}\in d\gamma_0\}\cap B_1)\delta_{\sigma_{i_0}(\gamma_0)}(ds_0)P_{\gamma_0}(B_0) \\
&= e^{-\alpha\tau_j(\gamma_{n+1})}\psi(d\gamma_{n+1})\delta_{\sigma_j(\gamma_{n+1})}(ds)e^{-\alpha\tau_{i_n}(\gamma_n)}\tilde{Q}(s,\{\xi\in d\gamma_n\}\cap A) \\
&\qquad \dots e^{-\alpha\tau_{i_0}(\gamma_0)}\tilde{Q}(s_0,A_0)\delta_{\sigma_{i_0}(\gamma_0)}(ds_0).
\end{aligned}$$

But

$$\begin{aligned}
&\int_\Gamma e^{-\alpha\tau_{i_{k-1}}(\gamma_{k-1})}\tilde{Q}(s_k,\{\xi\in d\gamma_{k-1}\}\cap A_k)\delta_{\sigma_{i_{k-1}(\gamma_{k-1})}}(ds_{k-1}) \\
&\quad = \quad E_{s_k}[e^{-\alpha\tau(i_{k-1})}; A_k\cap\{\sigma(i_{k-1})\in ds_{k-1}\}],
\end{aligned}$$

and since also

$$\begin{aligned}
&\int_\Gamma e^{-\alpha\tau_j(\gamma)}\psi(d\gamma)\delta_{\sigma_j(\gamma)}(ds) \\
&\quad = \quad \int_{\Gamma\times S} e^{-\alpha\tau_j(\gamma)}Q(r,d\gamma)\pi(dr)\delta_{\sigma_j(\gamma)}(ds) \\
&\quad = \quad E_\pi[e^{-\alpha\tau(j)};\sigma(j)\in ds],
\end{aligned}$$

we arrive at

$$\begin{aligned}
&\mathbf{P}(R_{n+1}=j, S_{n+1}\in ds, U_{n+1}\in A, R_n=i_n, S_n\in ds_n, U_n\in A_n \\
&\qquad ,\dots, R_0=i_0, S_0\in ds_0, U_0\in A_0) \\
&= E_\pi[e^{-\alpha\tau(j)};\sigma(j)\in ds]E_s[e^{-\alpha\tau(i_n)};\{\sigma(i_n)\in ds_n\}\cap A] \\
&\qquad \dots E_{s_1}[e^{-\alpha\tau(i_0)};\{\sigma(i_0)\in ds_0\}\cap A_1]\tilde{Q}(s_0,A_0).
\end{aligned}$$

■

8. Applications to Single-type Processes

8.1. Single-type Processes with Siblings' Marginals the Same

In this section single-type processes with sibling dependencies will be investigated. Recall that our discussion on how to treat the problem with dependencies started with such processes, briefly described in Section 3.1. There it was also argued that the key to the problem is to investigate multi-type processes with sibling dependencies such that marginals depend on types only.

Now consider a single-type branching process with sibling dependencies described by the joint measure $\tilde{P}(s,\cdot)$ on Ω^∞, where s is a point process on R_+ with a positive and finite number of points. Assume that all individuals in a sibling group born according to s have the same marginals

$$\tilde{Q}(s,A)=\int_{\bar{\Omega}} 1_A(\omega_i)\tilde{P}(s,d\bar{\omega}),$$

where $A\in\mathcal{A}$. As mentioned, this is the natural first step away from Broberg's homogeneous case, but it is also complicated enough to illustrate basically all the problems that may arise from the dependencies. Since an individual's reproduction depends on the point process she is born in, and siblings have identical marginals the type space, S, is

$$\{s\in\mathcal{N}(R_+): 0<s(\infty)<\infty\},$$

the set of point processes with a positive and finite number of points, or some appropriate subset thereof. The individual types are defined through

$$\sigma(i)(\omega)=\xi(\omega),$$

for $i=1,2,\ldots$ The construction from Section 3.2 then gives the macro type space $\Gamma=\mathcal{N}(S\times R_+)$, the set of point processes on $S\times R_+$. There is obviously a lot of redundance here; since all the individual types within a group of siblings are the same and these types actually equal the process of birth times, each $\gamma\in\Gamma$ corresponds uniquely to an $s\in S$. Hence S can be viewed also as macro type space. By this very special structure the reproduction measures of interest are

$$\mu(s,dr\times dt)=r(dt)Q(s,dr)$$

and

$$M(s,dr\times dt)=s(dt)Q(s,dr),$$

for the individual and the macro process respectively. Their Laplace transforms are

$$\hat{\mu}(s,dr)=\hat{r}(\alpha)Q(s,dr)$$

and

$$\hat{M}(s, dr) = \hat{s}(\alpha)Q(s, dr).$$

From the general theory developed in the previous sections we know that μ and M are equivalent in the sense that they determine the same α and that $\hat{\mu}$ and $\hat{M}$ are both conservative at the same time. It can also easily be shown directly by making use of the relation

$$\hat{M}^n(s, dr) = \frac{\hat{s}(\alpha)}{\hat{r}(\alpha)}\hat{\mu}^n(s, dr).$$

The relations between invariant measures and functions are

$$H(s) = \hat{s}(\alpha)h(s)$$

and

$$\psi(ds) = \int_S Q(r, ds)\pi(dr).$$

The interpretation of H is quite obvious: all individuals in a sibling group born according to s have the same reproductive value $h(s)$. The factor $\hat{s}(\alpha)$ takes into account the size and structure of s ($\hat{s}(\alpha)$ increases with the number of siblings but also if individuals are born early since they then obviously will have more time to contribute to the population). Also the relation $\pi(ds) = \hat{s}(\alpha)\psi(ds)$ holds, and from this it is obvious that $\psi \ll \pi$ and $\pi \ll \psi$. Therefore π-almost everywhere and ψ-almost everywhere statements are equivalent, a situation somewhat simpler than in the general case. The convergence results are

$$E_s[e^{-\alpha t} z_t^{\chi}] \to h(s)\frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}$$

and

$$e^{-\alpha t} z_t^{\chi} \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}\tilde{w},$$

in $L^1(\tilde{Q}_s)$ under appropriate conditions.

Example 8.1. Consider a Bellman-Harris process, i.e. a process where each individual has a random lifelength L with distribution function G. By the time of death the individual begets a random number of children, this number assumed independent of the lifelength. We assume that the number of children is bounded by m and that the sibling dependencies arise only through the number of siblings so that we can take the type space $S = \{1, 2, ..., m\}$ and consider the kernels $Q(i, j)$. The reproduction measure is then

$$\mu(i, j \times dt) = jQ(i, j)G(dt)$$

and hence

$$\hat{\mu}(i, j) = jQ(i, j)\hat{G}(\alpha) = \mu(i, j)\hat{G}(\alpha).$$

If there exists an n such that all the $Q^n(i,j)$ are positive and ρ denotes the spectral radius (i.e. the largest eigenvalue) of the matrix with entries $\mu(i,j)$ then the Malthusian parameter, α, is defined through the relation

$$\hat{G}(\alpha) = \frac{1}{\rho},$$

and $\hat{\mu}$ is conservative. The proof relies on the Perron-Frobenius theorem which tells us that there exist numbers $k(i,j)$ such that

$$\frac{\mu^n(i,j)}{\rho^n} \to k(i,j), n \to \infty.$$

Now consider

$$\sum_{n \geq 0} \lambda^n \sum_{j=1}^{m} f(i,j)\hat{\mu}^n(i,j) = \sum_{n \geq 0} (\lambda \rho \hat{G}(\alpha))^n \sum_{j=1}^{m} f(i,j) \frac{\mu^n(i,j)}{\rho^n}.$$

Since $\mu^n(i,j) \sim \rho^n k(i,j)$ as n grows large, the second sum will tend to a constant. Thus if $\lambda > 1$ we must have $\rho\hat{G}(\alpha) \geq 1$ to ensure that for all f there is an i such that the sum over n is infinite. Conversely, if $\lambda < 1$ we must have $\rho\hat{G}(\alpha) \leq 1$ to ensure finiteness for some f and all i, and so $\rho\hat{G}(\alpha) = 1$.

The conservativity follows since

$$\sum_{n \geq 0} \sum_{i,j} g(j)\hat{\mu}^n(i,j)m(i) = \sum_{n \geq 0} \rho^n \sum_{i,j} g(j) \frac{\mu^n(i,j)}{\rho^n} m(i) = \infty,$$

if only $\rho > 1$ which is the supercritical case.

A trivial special case is the Galton-Watson process which has $\hat{G}(\alpha) = e^{-\alpha}$ and hence $\alpha = \log \rho$. To analyze this further in the general framework the results for lattice kernels must be invoked.

□

Of course splitting populations of this kind are the most natural examples when siblings' marginals are the same. Another example which could be of some relevance is given.

Example 8.2. Suppose the kernel $Q(s, dr)$ can be decomposed as

$$Q(s, dr) = q(s)Q(dr),$$

for some function $q > 0$ and some probability measure Q on S. Then obviously

$$q(s) = Q(s, S) = \tilde{Q}(s, \{\xi(\infty) > 0\}),$$

the probability that an s-type individual begets any children. Since $S = \{s \in \mathcal{N}(R_+) : s(\infty) > 0\}$,

$$\begin{aligned} Q(s, A) &= \tilde{Q}(s, \{\xi \in A\}) \\ &= \tilde{Q}(s, \{\xi \in A\} | \{\xi \in S\})\tilde{Q}(s, \{\xi \in S\}) \\ &= \tilde{Q}(s, \{\xi \in A\} | \{\xi(\infty) > 0\})q(s), \end{aligned}$$

$A \in \mathcal{S}$, we see that $Q(A) = \tilde{Q}(s, \{\xi \in A\} | \{\xi(\infty) > 0\})$ and hence the conditional probability measure $\tilde{Q}(s, \cdot\, | \{\xi(\infty) > 0\})$ does not depend on s. One possible interpretation of this is that the structure of your sibling group only matters to your survival probability, and that if you survive you are certain to beget children, all surviving individuals reproducing according to the same probability measure Q.

The Malthusian parameter, α, is defined through the relation

$$\int_S \hat{r}(\alpha) q(r) Q(dr) = 1, \tag{8.1}$$

and the kernel $\hat{\mu}_\alpha$ is conservative. To see this, let

$$I = \int_S \hat{r}(\alpha) q(r) Q(dr),$$

and note that

$$\hat{\mu}_\alpha(s, dr) = \hat{r}(\alpha) Q(s, dr) = q(s) \hat{r}(\alpha) Q(dr).$$

Hence we have

$$\hat{\mu}_\alpha^n(s, dr) = q(s) \hat{r}(\alpha) Q(dr) I^{n-1}.$$

Now, for $\lambda \geq 0$ we have

$$\sum_{n \geq 0} \lambda^n \int_S f(s,r) \hat{\mu}_\alpha^n(s, dr) = \lambda \sum_{n \geq 1} (\lambda I)^n \int_S f(s,r) q(s) \hat{r}(\alpha) Q(dr),$$

and we see that this will be infinite for all $\lambda > 1$ if and only if $I \geq 1$. Similarily, it will be finite for all $\lambda < 1$ if and only if $I \leq 1$. Thus we must have $I = 1$.

To check positivity we note that

$$\hat{\mu}^n(s, A) = q(s) \int_A \hat{r}(\alpha) Q(dr),$$

which is positive if

$$m(A) = \int_A \hat{r}(\alpha) Q(dr) > 0.$$

Also note that

$$\sum_{n \geq 0} \int_{S \times S} g(r) \hat{\mu}^n(s, dr) m(ds) = \sum_{n \geq 0} \int_{S \times S} g(r) q(s) \hat{r}(\alpha) Q(dr) m(ds) = \infty,$$

and $\hat{\mu}$ is conservative. To obtain expressions for h and π, note that $\hat{\mu}(s, dr) = q(s) \hat{r}(\alpha) Q(dr)$ and use (8.1) to see that

$$\int_S q(r) \hat{\mu}(s, dr) = q(s)$$

and

$$\int_S \hat{s}(\alpha) Q(ds) \hat{\mu}(s, dr) = \hat{r}(\alpha) Q(dr),$$

and hence $h(s) = q(s)$ and $\pi(ds) = \hat{s}(\alpha) Q(ds)$.

In the interpretation above, the form of h is clearly reasonable. Since the reproduction of a surviving individual does not depend on the type, the type reproductive value should equal the survival probability for that type. In the expression for π, the factor $\hat{s}(\alpha)$ mirrors the fact that a randomly sampled individual in some sense will be more likely to belong to a large sibling group (this will be further discussed after Proposition 8.1).

A special case is Broberg's homogeneity which corresponds to $q \equiv 1$. □

It has already been shown that the sequence $S_0, S_1, ...$ is a Markov chain. The special type structure that is now dealt with gives a special form of the transition probabilities.

Proposition 8.1. *The sequence of types backwards from Ego is a Markov chain with transition probabilities*

$$\mathbf{P}(S_{n+1} \in ds | S_n = r) = \pi(ds) \frac{Q(s, dr)}{Q(\pi, dr)}.$$

Proof. From Proposition 7.7 we know that

$$\mathbf{P}(S_{n+1} \in ds | S_n = r) = \pi(ds) \frac{\hat{\mu}(s, dr)}{\pi(dr)}.$$

Clearly the Radon-Nikodym derivatives

$$\frac{Q(s, dr)}{Q(\pi, dr)}$$

exist since

$$Q(\pi, B) = \int_S Q(s, B) \pi(ds),$$

so that $Q(s, \cdot)$ is absolutely continuous with respect to $Q(\pi, \cdot)$. But

$$\begin{aligned} \hat{\mu}(s, A) &= \int_A \hat{r}(\alpha) Q(s, dr) \\ &= \int_A \hat{r}(\alpha) \frac{Q(s, dr)}{Q(\pi, dr)} Q(\pi, dr) = \int_A \frac{Q(s, dr)}{Q(\pi, dr)} \pi(dr), \end{aligned}$$

where the second equality follows from a well known identity for Radon-Nikodym derivatives, see for instance p.68 in Folland (1984). Hence

$$\frac{\hat{\mu}(s, dr)}{\pi(dr)} = \frac{Q(s, dr)}{Q(\pi, dr)}.$$

■

The last proposition thus tells us that a typical individual has probability $\pi(ds)$ of belonging to a sibling group born according to the point process $s \in S$. It does not, however, say that the typical *sibling group* is distributed according to π. To clarify the difference, consider the following simple example. Suppose that only the number of individuals in each sibling group matters to the reproduction and that this number is either one or two so that we have the type space $S = \{1, 2\}$. Now sample an individual from a population consisting of equally many sibling groups of size one and two. Then this individual has probability 2/3 of being in a size two group. If we instead sample a sibling group we will pick a size two group with probability 1/2. Hence the typical sibling group will have an expected size of 3/2 individuals, whereas the typical individual's expected number of siblings will be 2/3.

If we want to explore what the typical sibling group looks like, we must invoke the macro process. Remember that this is an ordinary multi-type process without any dependencies and thus gives rise to stable population measure analogous to (7.1). The eigenmeasure in the macro process is $\psi(ds) = \int_S Q(r, ds)\pi(dr)$ and after the appropriate norming to a probability measure, this will be the stable type distribution for the typical sibling group or *macro Ego*. From the example we realize that the frequency of an individual's type depends on both the frequency of that particular sibling group and how many individuals there are in it. In the general case this is mirrored by the relation $\pi(ds) = \hat{s}(\alpha)\psi(ds)$ where large values of $\hat{s}(\alpha)$ correspond to many or early born children. Hence, denoting macro Ego's type by S_0' and her nth grandmother's type by S_n' we have

$$S_0' \sim \psi$$

and

$$S_n' \sim \hat{M}^n(s, S)\psi(ds).$$

From Corollary 1 in Jagers (1992) we know that

$$\hat{\mu}^n(s, S)\pi(ds) \to h(s)\pi(ds)$$

and

$$\hat{M}^n(s, S)\psi(ds) \to H(s)\psi(ds).$$

But $H(s)\psi(ds) = h(s)\pi(ds)$ so the limiting distribution backwards is the same for individuals and sibling groups.

Example 8.3. Recall Example 8.2 where $Q(s, dr) = q(s)Q(dr)$ for some function $q > 0$ and some probability measure Q on S. In this case

$$Q(\pi, dr) = \int_S Q(s, dr)\pi(ds) = \int_S q(s)Q(dr)\hat{s}(\alpha)Q(ds) = Q(dr),$$

and therefore Proposition 8.1 reduces to

$$\mathbf{P}(S_{n+1} \in ds | S_n = r) = q(s)\hat{s}(\alpha)Q(ds),$$

so that the types $S_0, S_1, \ldots$ are actually independent. The explanation is that the types only affect the probability of having or not having children and that given this, the distribution of the number of children does not depend on the type. But in the stable population, all individuals backwards from Ego of course have children and hence the types become independent. □

Since $\mu(s, dr \times dt)$ now equals $r(dt)Q(s, dr)$ the next proposition is trivial.

Proposition 8.2. *The sequence of types and interbirth times backwards from Ego,* $(S_n, T_n)_{n=0}^{\infty}$*, is a Markov renewal process with transition kernel*

$$\mathbf{P}(S_{n+1} \in ds, T_{n+1} \in dt | S_n = r) = \pi(ds)\frac{e^{-\alpha t} r(dt) Q(s, dr)}{\pi(dr)}.$$

Proof. Follows immediately from Proposition 7.8. ■

Proposition 8.3. *The sequence of ranks, types and lives backwards from Ego has the Markov property*

$$\mathbf{P}(R_{n+1} = j, S_{n+1} \in ds, U_{n+1} \in A)|R_n = i, S_n = r, U_n, R_{n-1}, S_{n-1}, ...)$$
$$= e^{-\alpha\tau_i(s)} Q(\pi, ds)\frac{\tilde{Q}(s, A \cap \{\xi \in dr\})}{Q(\pi, dr)}.$$

Proof. From (7.9) it is clear that

$$\mathbf{P}(R_{n+1} = j, S_{n+1} \in ds, U_{n+1} \in A)|R_n = i, S_n = r, U_n, R_{n-1}, S_{n-1}, ...)$$
$$= E_\pi[e^{-\alpha\tau(j)}; \sigma(j) \in ds]\frac{E_s[e^{-\alpha\tau(i)}; A \cap \{\sigma(i) \in dr\}]}{E_\pi[e^{-\alpha\tau(i)}; \sigma(i) \in dr]}.$$

But

$$E_\pi[e^{-\alpha\tau(i)}; \sigma(i) \in B] = \int_\Omega e^{-\alpha\tau(i)(\omega)} 1_B(\sigma(i)(\omega))\tilde{Q}(\pi, d\omega)$$
$$= \int_\Omega e^{-\alpha\tau_i(\xi(\omega))} 1_B(\xi(\omega))\tilde{Q}(\pi, d\omega)$$
$$= \int_S e^{-\alpha\tau_i(s)} 1_A(s) Q(\pi, ds),$$

so that

$$E_\pi[e^{-\alpha\tau(i)}; \sigma(i) \in ds] = e^{-\alpha\tau_i(s)} Q(\pi, ds),$$

and obviously also

$$E_s[e^{-\alpha\tau(i)}; A \cap \{\sigma(i) \in dr\}] = e^{-\alpha\tau_i(r)}\tilde{Q}(s, A \cap \{\xi \in dr\}).$$

The Radon-Nikodym derivatives

$$\frac{\tilde{Q}(s, A \cap \{\xi \in dr\})}{Q(\pi, dr)}$$

exist since

$$Q(\pi, B) = \int_S Q(s, B)\pi(ds) = \int_S \tilde{Q}(s, \{\xi \in B\})\pi(ds),$$

so that $\tilde{Q}(s, A \cap \{\xi \in \cdot\}) \ll Q(\pi, \cdot)$. But

$$\begin{aligned}&\int_B E_s[e^{-\alpha\tau(i)}; A \cap \{\sigma(i) \in dr\}] \\ = \quad & \int_B e^{-\alpha\tau_i(r)}\tilde{Q}(s, A \cap \{\xi \in dr\}) = \int_B e^{-\alpha\tau_i(r)}\frac{\tilde{Q}(s, A \cap \{\xi \in dr\})}{Q(\pi, dr)}Q(\pi, dr),\end{aligned}$$

and therefore

$$\frac{E_s[e^{-\alpha\tau(i)}; A \cap \{\sigma(i) \in dr\}]}{E_\pi[e^{-\alpha\tau(i)}; \sigma(i) \in dr]} = \frac{\tilde{Q}(s, A \cap \{\xi \in dr\})}{Q(\pi, dr)}.$$

As the reader can see, some π-almost everywhere statements have been omitted above. ■

8.2. Single-type Processes with Siblings' Marginals Different

The assumption made in the previous section that siblings' marginals are identical could be justified in splitting populations, like those considered in the first example, and perhaps in some other special cases. It is however not satisfactory in more general situations. In order to be able to treat such situations one would want to examine the case when the marginals are different, i.e. when the ith individual in a sibling group s chooses life according to the marginal distribution

$$\tilde{Q}_i(s, A) = \int_{\bar{\Omega}} 1_A(\omega_i)\tilde{P}(s, d\bar{\omega}),$$

now depending on both s and i. That an individual's rank is allowed to influence her reproduction is of course more natural from a biological point of view; it might be an advantage (or maybe a disadvantage) to be born early in a group of siblings. It is also an interesting theoretical question; one might suspect that it becomes substantially more complicated than if all marginals are the same. Actually this is not so, and the reason is that this case can be brought back to the previous one by considering a population where all siblings in a group s choose lives according to the measure

$$\tilde{Q}(s, d\omega) = \frac{1}{\hat{s}(\alpha)}\sum_{i=1}^{\infty} e^{-\alpha\tau_i(s)}\tilde{Q}_i(s, d\omega), \tag{8.2}$$

an exponentially weighted average of the $\tilde{Q}_i$'s. This process defines the same α as the one with different marginals and it also has the same asymptotic composition. We will return to this but let us first show how the general theory can be applied without such considerations.

The individual type space is $S \times Z_+$, an individual's type being (s, i) if she is the ith child in the sibling group s. The macro space would then be $\Gamma =$

$\mathcal{N}(S \times R_+)$ but as in the previous section it can obviously be identified with S. The macro reproduction measure is

$$M(s, dr \times dt) = \sum_{i=1}^{\infty} \delta_{\tau_i(s)}(dt) Q_i(s, dr),$$

with Laplace transform

$$\begin{aligned}\hat{M}(s, dr) &= \int_0^{\infty} e^{-\alpha t} M(s, dr \times dt) \\ &= \sum_{i=1}^{\infty} Q_i(s, dr) \int_0^{\infty} e^{-\alpha t} \delta_{\tau_i(s)}(dt) = \sum_{i=1}^{\infty} e^{-\alpha \tau_i(s)} Q_i(s, dr).\end{aligned}$$

The individual reproduction measure is

$$\mu((s,i), dr \times \{j\} \times dt) = \delta_{\tau_j(r)}(dt) Q_i(s, dr)$$

with Laplace transform

$$\hat{\mu}((s,i), dr \times \{j\}) = e^{-\alpha \tau_j(r)} Q_i(s, dr).$$

It is easily checked that the invariant measures and functions satisfy

$$H(s) = \sum_{i=1}^{\infty} e^{-\alpha \tau_i(s)} h(s,i),$$

$$h(s,i) = \int_S H(r) Q_i(s, dr),$$

$$\psi(ds) = \sum_{i=1}^{\infty} \int_S Q_i(r, ds) \pi(dr \times \{i\})$$

and

$$\pi(ds \times \{i\}) = e^{-\alpha \tau_i(s)} \psi(ds).$$

The asymptotics and stable population properties follow from the general results derived previously.

These asymptotic results also follow if instead the process mentioned above is used, the one where all individuals in a sibling group s choose lives according to the probability measure given in (8.2). These individuals can be viewed as some kind of average individuals, representative for their sibling group. Let

$$\bar{Q}(s, dr) = \tilde{Q}(s, \xi^{-1}(dr)) = \frac{1}{\hat{s}(\alpha)} \sum_{i=1}^{\infty} e^{-\alpha \tau_i(s)} \tilde{Q}_i(s, \xi^{-1}(dr)),$$

and define

$$\bar{\mu}(s, dr) = \hat{r}(\alpha) \bar{Q}(s, dr).$$

It is then clear that

$$\hat{M}(s, dr) = \frac{\hat{s}(\alpha)}{\hat{r}(\alpha)} \bar{\mu}(s, dr),$$

so that $\bar{\mu}$ and $\hat{M}$ relate to each other exactly as $\hat{\mu}$ and $\hat{M}$ in the previous section. Hence $\bar{\mu}$ and $\hat{M}$ define the same α and are also conservative at the same time. Further $\bar{\mu}$ defines an invariant function $\bar{h}$ and an invariant measure $\bar{\pi}$ satisfying

$$\psi(ds) = \int_S \bar{Q}(r, ds)\bar{\pi}(dr),$$

$$\bar{\pi}(ds) = \hat{s}(\alpha)\psi(ds)$$

and

$$H(s) = \hat{s}(\alpha)\bar{h}(s).$$

The stable population measure given in Definition (7.2) (only now with S_k's instead of Γ_k's) is the same since it depends on individual behaviour only through the macro process. Therefore, in the limit, all individuals in some sense behave like the average individuals discussed above. For instance take a look at the distribution of Ego's life, first for the "average" population:

$$\mathbf{P}(U_0 \in A) = \tilde{Q}(\bar{\pi}, A),$$

by Proposition 7.6. For the original population the same proposition yields

$$\begin{aligned}
\mathbf{P}(U_0 \in A) &= \tilde{Q}(\pi, A) \\
&= \sum_{i=1}^{\infty} \int_S \tilde{Q}_i(s, A)\pi(ds \times \{i\}) = \sum_{i=1}^{\infty} \int_S \tilde{Q}_i(s, A)e^{-\alpha\tau_i(s)}\psi(ds) \\
&= \sum_{i=1}^{\infty} \int_S \tilde{Q}_i(s, A)\frac{e^{-\alpha\tau_i(s)}}{\hat{s}(\alpha)}\bar{\pi}(ds) = \tilde{Q}(\bar{\pi}, A).
\end{aligned}$$

8.3. The Homogeneous Case

We will now take a quick look at the stable population in the homogeneous case. Recall that, in the terminology from Broberg (1987), a branching process with sibling dependencies is called *homogeneous* if all individual lives are identically distributed, i.e if all the $\tilde{Q}(s, \cdot) \equiv \tilde{Q}(\cdot)$ for some $\tilde{Q}$. Of course this is a special case of what has been described in the previous sections, but nevertheless highlights the effects (or rather lack of effects) of sibling dependencies on population composition.

In the homogeneous case there are no types at the individual level and it is thus possible to compare the sibling dependent process and the "same" process without dependencies, i.e. the process where all individuals choose lives according to the law $\tilde{Q}(\cdot)$, independently of each other.

In the sibling dependent homogeneous case, the stable population measure is determined by

$$\begin{aligned}
&\mathbf{P}(Z_k \in A_k, T_k \in dt_k, R_k = i_k, \Gamma_k \in d\gamma_k k = 0, ..., n) \\
&= e^{-\alpha\tau_{i_n}(\gamma_n)}Q(d\gamma_n)E_{\gamma_n}[e^{-\alpha t_n}; A_n \cap \{\xi_{i_n} \in d\gamma_{n-1}, \tau_{i_{n-1}}(\xi_{i_n}) \in dt_n\}]
\end{aligned}$$

$$...E_{\gamma_1}[e^{-\alpha t_1}; A_1 \cap \{\xi_{i_1} \in d\gamma_0, \tau_{i_0}(\xi_{i_1}) \in dt_1\}]P_{s_0}(A_0)\alpha e^{-\alpha t_0} dt_0, \qquad (8.3)$$

where $A_k \in \mathcal{A}^{I_1}$. Now $\psi(\cdot) = Q(\cdot)$, but apart from this it does not seem to differ much from (7.2). However, when we start to investigate individual properties in the family line backwards from Ego, things will simplify substantially when all marginals are the same. The stable population measure in the independent, single-type case is determined by

$$\tilde{\mathbf{P}}(C) = \prod_{\substack{x \in J_n \\ x \neq -1, ... -n}} \tilde{Q}(C_x) \prod_{k=0}^{n} E[e^{-\alpha\tau(i_k)}; C_{-(k+1)}](1 - e^{-\alpha t}),$$

where C is a cylinder of the form

$$C = (\prod_{x \in J_n} C_x \times \Omega^{J \setminus J_n}) \times (\{i_n, ... i_0\} \times N^{\infty}) \times [0, t]. \qquad (8.4)$$

From Nerman and Jagers (1984) we know, apart from Ego's exponential age and other properties mentioned earlier, that events referring to Ego's kth ancestor alone, have probabilities which are independent of k. We also know that the ranks backwards from Ego are i.i.d. and that the same holds for the interbirth times. The marginals are given by

$$\tilde{\mathbf{P}}(U_k \in A) = E[\hat{\xi}(\alpha); A],$$

$$\tilde{\mathbf{P}}(R_k = j) = E[e^{-\alpha\tau(j)}]$$

and

$$\tilde{\mathbf{P}}(T_k \in dt) = e^{-\alpha t}\mu(dt),$$

where $k = 0, 1, ...$, except for T_0 which is exponential (α). That all this holds also under $\mathbf{P}$ we state in the following propositions.

Proposition 8.4. *The life of Ego's kth ancestor, U_k, has distribution given by*

$$\mathbf{P}(U_k \in A) = E[\hat{\xi}(\alpha); A],$$

for $A \in \mathcal{A}$.

Proof. Just replace $Q(\pi, ds)$ by $Q(ds)$ and $\hat{\mu}(s, \cdot)$ by $\hat{\mu}(\cdot)$ in Proposition 8.3 to obtain

$$\mathbf{P}(U_k \in A) = \sum_{i=1}^{\infty} \int_{S \times S} e^{-\alpha\tau_i(s)} Q(ds) \hat{\mu}^{k-2}(S) \hat{\mu}_A(dr)$$

$$= \int_{S \times S} \hat{s}(\alpha) Q(ds) \hat{\mu}_A(dr) = E[\hat{\xi}(\alpha); A],$$

since

$$\int_S \hat{s}(\alpha) Q(ds) = E[\hat{\xi}(\alpha)] = 1$$

and

$$\hat{\mu}^{k-2}(S) = E[\hat{\xi}(\alpha)]^{k-2} = 1.$$

■

Proposition 8.5. *The ranks backwards from Ego, $R_0, R_1, \ldots$, are independent and have the marginal distributions*

$$\mathbf{P}(R_k = j) = E[e^{-\alpha\tau(j)}].$$

Proof. Since we have no types, the Radon-Nikodym derivatives in Proposition 8.3 are identically one and it follows that

$$\mathbf{P}(R_{n+1} = j | R_n = i) = \int_S e^{-\alpha\tau_j(s)} Q(ds) = E[e^{-\alpha\tau(j)}].$$

■

Proposition 8.6. *The interbirth times backwards from Ego, $T_1, T_2, \ldots$ are independent and have marginal distributions*

$$\mathbf{P}(T_k \in dt) = e^{-\alpha t}\mu(dt).$$

Proof. Summation in (8.3) yields

$$\begin{aligned}
&\sum_{i_0,\ldots,i_n} \int_{\Gamma^{n+1}} e^{-\alpha\tau_{i_n}(\gamma_n)} Q(d\gamma_n) E_{\gamma_n}[e^{-\alpha t_n}; \{\xi_{i_n} \in d\gamma_{n-1}, \tau_{i_{n-1}}(\xi_{i_n}) \in dt_n\}] \\
&\ldots \quad E_{\gamma_1}[e^{-\alpha t_1}; \tau_{i_0}(\xi_{i_1}) \in dt_1] \alpha e^{-\alpha t_0} dt_0 \\
&= \quad \sum_{i_0,\ldots,i_n} e^{-\alpha t_n} \tilde{Q}(\{\tau_{i_{n-1}}(\xi_{i_n} \in dt_n\}) \ldots e^{-\alpha t_1} \tilde{Q}(\{\tau_{i_0}(\xi_{i_1}) \in dt_1\}) \alpha e^{-\alpha t_0} dt_0 \\
&= \quad e^{-\alpha t_n} E[\xi(dt_n)] \ldots e^{-\alpha t_1} E[\xi(dt_1)] \alpha e^{-\alpha t_0} dt_0,
\end{aligned}$$

since

$$\sum_{i_0=1}^{\infty} \tilde{Q}(\{\tau_{i_0}(\xi_{i_1}) \in dt_1\}) = E[\xi_{i_1}(dt_1)] = E[\xi(dt_1)] = \mu(dt_1),$$

and so on. ■

9. L^2-convergence

9.1. The Asymptotic Variance

This chapter is devoted to L^2-convergence under second moment conditions. Such convergence results for ordinary multi-type processes can be found in Nerman (1984). The homogeneous sibling dependent case was investigated in Broberg (1986).

First the asymptotic variance of $e^{-\alpha t}Z_t^{\chi'}$ is explored. The method is, basically, that of Nerman (1984) and the conditions assumed in that reference are adopted.

Throughout this chapter we will therefore allow only individual characteristics, i.e. characteristics that depend on your type and life only and not on your progeny. Of course if χ is individual then also χ' is so. Recall

$$\bar{\eta} = \int_{\Gamma\times R_+} e^{-\alpha t} H(\gamma')\eta(d\gamma' \times dt) = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\xi_0)}\bar{\xi}_i.$$

With

$$g(\gamma,t) = \mathrm{Var}_\gamma\left[\chi'(t) + \int_{\Gamma\times R_+} E_{\gamma'}[Z_{t-u}^{\chi'}]\eta(d\gamma' \times du)\right],$$

we can then write

$$\mathrm{Var}_\gamma[Z_t^{\chi'}] = \int_{\Gamma\times R_+} g(\gamma', t-u)V(\gamma, d\gamma' \times du),$$

see Lemma 3.3 in Nerman (1984). Here V denotes the total population measure, i.e.

$$V(\gamma, d\gamma' \times dt) = \sum_{n\geq 0} M^n(\gamma, d\gamma' \times dt).$$

Hence

$$\mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\chi'}] = \int_{\Gamma\times R_+} e^{-2\alpha(t-u)}\frac{g(\gamma', t-u)}{H(\gamma')}e^{-2\alpha u}H(\gamma')V(\gamma, d\gamma' \times du). \tag{9.1}$$

The limiting behaviour of this is given in the following lemma.

Lemma 9.1. *Add to the conditions of* Theorem 5.8 *that χ is individual and that, for some constant c_1,*

$$\sup_s \frac{E_s[\bar{\xi}^2]}{h(s)} = c_1 < \infty. \tag{9.2}$$

Then, for ψ-almost all $\gamma \in \Gamma$,

$$Var_\gamma[e^{-\alpha t}Z_t^{\chi'}] \to \left(\frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}\right)^2 \int_{\Gamma\times R_+} e^{-2\alpha u}\,Var_{\gamma'}[\bar{\eta}]V(\gamma, d\gamma' \times du).$$

Proof. If we can show that

$$\frac{e^{-2\alpha t}g(\gamma,t)}{H(\gamma)} \leq a + b\hat{\gamma}(\alpha),$$

for some constants a and b, and that the integrals

$$\int_{\Gamma\times R_+} e^{-2\alpha u}H(\gamma')V(\gamma, d\gamma' \times du) \tag{9.3}$$

and

$$\int_{\Gamma\times R_+} e^{-2\alpha u}\hat{\gamma}'(\alpha)H(\gamma')V(\gamma, d\gamma' \times du) \tag{9.4}$$

are finite, dominated convergence can be used to replace $e^{-2\alpha(t-u)}g(\gamma', t-u)$ by its limit in (9.1). Recalling that $\mathrm{Var}[X+Y] \leq 2(\mathrm{Var}[X] + \mathrm{Var}[Y])$ we obtain

$$\frac{e^{-2\alpha t}g(\gamma,t)}{H(\gamma)} \leq 2\frac{e^{-2\alpha t}}{H(\gamma)}\left(\mathrm{Var}_\gamma[\chi'(t)] + \mathrm{Var}_\gamma\left[\int_{\Gamma\times R_+} E_{\gamma'}[Z^{\chi'}_{t-u}]\eta(d\gamma' \times du)\right]\right),$$

so we look at these two terms separately. Since χ is bounded, $\chi \leq m$ say, it is clear that

$$e^{-2\alpha t}\mathrm{Var}_s[\chi(t)] \leq m^2 < \infty.$$

Bearing in mind that $\mathrm{Cov}[X,Y] \leq \mathrm{Var}[X] + \mathrm{Var}[Y]$, we first see that

$$\begin{aligned}
&\frac{e^{-2\alpha t}}{H(\gamma)}\mathrm{Var}_\gamma[\chi'(t)] \\
&= \frac{1}{H(\gamma)}\left(\sum_{i=1}^{\infty} e^{-2\alpha\tau_i(\gamma)}\mathrm{Var}_\gamma[e^{-\alpha(t-\tau_i(\gamma))}\chi_i(t-\tau_i(\gamma))]\right. \\
&\qquad \left. + \sum_{i\neq j} e^{-\alpha(\tau_i(\gamma)+\tau_j(\gamma))}\mathrm{Cov}_\gamma[e^{-\alpha(t-\tau_i(\gamma))}\chi_i(t-\tau_i(\gamma)), e^{-\alpha(t-\tau_j(\gamma))}\chi_j(t-\tau_j(\gamma))]\right) \\
&= \frac{1}{H(\gamma)}\left(\sum_{i=1}^{\infty} e^{-2\alpha\tau_i(\gamma)}\mathrm{Var}_{\sigma_i(\gamma)}[e^{-\alpha(t-\tau_i(\gamma))}\chi(t-\tau_i(\gamma))]\right. \\
&\qquad + \sum_{i\neq j} e^{-\alpha(\tau_i(\gamma)+\tau_j(\gamma))}(\mathrm{Var}_{\sigma_i(\gamma)}[e^{-\alpha(t-\tau_i(\gamma))}\chi(t-\tau_i(\gamma))] \\
&\qquad \left. + \mathrm{Var}_{\sigma_j(\gamma)}[e^{-\alpha(t-\tau_j(\gamma))}\chi(t-\tau_j(\gamma))])\right) \\
&\leq \frac{1}{H(\gamma)}(m^2\hat{\gamma}(2\alpha) + m^4\hat{\gamma}(\alpha)^2) \\
&\leq \frac{m^2}{\kappa} + \frac{m^4}{\kappa}\hat{\gamma}(\alpha),
\end{aligned}$$

since

$$H(\gamma) = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} h(\sigma_i(\gamma)) \geq \kappa\hat{\gamma}(\alpha),$$

and clearly $\hat{\gamma}(2\alpha) \leq \hat{\gamma}(\alpha)$. For the second term,

$$\begin{aligned}
&\frac{e^{-2\alpha t}}{H(\gamma)} \mathrm{Var}_\gamma \left[\int_{\Gamma \times R_+} E_{\gamma'}[Z_{t-u}^{\chi'}] \eta(d\gamma' \times du) \right] \\
&= \frac{1}{H(\gamma)} \mathrm{Var}_\gamma \left[\int_{\Gamma \times R_+} \frac{e^{-\alpha(t-u)} E_{\gamma'}[Z_{t-u}^{\chi'}]}{H(\gamma')} H(\gamma') e^{-\alpha u} \eta(d\gamma' \times du) \right] \\
&\leq c_2^2 \frac{E_\gamma[\bar{\eta}^2]}{H(\gamma)},
\end{aligned}$$

for some constant c_2, by (6.4). Now, by (9.2) and the inequality

$$E[XY] \leq E[X^2] + E[Y^2] + E[X]E[Y],$$

we obtain

$$\begin{aligned}
E_\gamma[\bar{\eta}^2] &= E_\gamma \left[\sum_{i=1}^{\infty} e^{-2\alpha\tau_i(\gamma)} \bar{\xi}_i^2 + \sum_{i \neq j} e^{-(\tau_i(\gamma)+\tau_j(\gamma))} \bar{\xi}_i \bar{\xi}_j \right] \\
&\leq c_1 \sum_{i=1}^{\infty} e^{-2\alpha\tau_i(\gamma)} h(\sigma_i(\gamma)) \\
&\quad + \sum_{i \neq j} e^{-\alpha(\tau_i(\gamma)+\tau_j(\gamma))} (c_1 h(\sigma_i(\gamma)) + c_1 h(\sigma_j(\gamma)) + h(\sigma_i(\gamma)) h(\sigma_j(\gamma))).
\end{aligned}$$

Again by (9.2) it holds that

$$h(s) = E_s[\bar{\xi}] = \frac{E_s[\bar{\xi}]^2}{E_s[\bar{\xi}]} \leq \frac{E_s[\bar{\xi}^2]}{h(s)} \leq c_1,$$

and therefore, since $\hat{\gamma}(\alpha) \leq H(\gamma)/\kappa$,

$$\frac{E_\gamma[\bar{\eta}^2]}{H(\gamma)} \leq \frac{1}{\kappa}(c_1^2 + 3c_1^2 \hat{\gamma}(\alpha)). \tag{9.5}$$

Hence we have

$$\frac{e^{-2\alpha t} g(\gamma, t)}{H(\gamma)} \leq \frac{2}{\kappa}(m^2 + m^4 \hat{\gamma}(\alpha) + c_1^2 + 3c_1^2 \hat{\gamma}(\alpha)) = a + b\hat{\gamma}(\alpha).$$

To show that the integrals (9.3) and (9.4) are finite we first show that

$$\int_{S \times R_+} e^{-2\alpha u} h(r) \nu(s, dr \times du) < \infty, \tag{9.6}$$

which will hold if

$$\sup_n \int_n^{n+1} \int_S e^{-\alpha u} h(r) \nu(s, dr \times du) < \infty. \tag{9.7}$$

Here ν is the individual counterpart of V, defined in (4.2). With

$$\chi(t) = e^{\alpha t} 1_{[0,1]}(t) h(\sigma_0),$$

we have

$$\begin{aligned} & E_s[e^{-\alpha t} z_t^\chi] \\ = & \int_{S \times R_+} e^{-\alpha(t-u)} E_r[\chi(t-u)] e^{-\alpha u} \nu(s, dr \times du) \\ = & \int_{t-1}^{t} \int_S h(r) e^{-\alpha u} \nu(s, dr \times du), \end{aligned}$$

and since χ fulfills the conditions of Theorem 3 in Jagers (1992) this is uniformly bounded so that (9.7), and thereby also (9.6) holds.

Next we shall establish the connection between ν and V. Since

$$M(\gamma, d\gamma' \times du) = \sum_{i=1}^{\infty} Q(\sigma_i(\gamma), d\gamma') \delta_{\tau_i(\gamma)}(du)$$

and

$$\mu(s, dr \times du) = E_s[\xi(dr \times du)] = \sum_{i=1}^{\infty} \int_\Gamma \delta_{\sigma_i(\gamma)}(dr) \delta_{\tau_i(\gamma)}(du) Q(s, d\gamma),$$

it is clear that

$$\begin{aligned} M^2(\gamma, d\gamma' \times du) &= \int_{\Gamma \times R_+} M(\gamma, d\gamma'' \times dv) M(\gamma'', d\gamma' \times du - v) \\ = & \sum_{i,j=1}^{\infty} \int_{\Gamma \times R_+} \delta_{\tau_i(\gamma)}(dv) Q(\sigma_i(\gamma), d\gamma'') \delta_{\tau_j(\gamma'')}(du - v) Q(\sigma_j(\gamma''), d\gamma') \\ = & \sum_{i,j=1}^{\infty} \int_{\Gamma \times R_+ \times S} \delta_{\tau_i(\gamma)}(dv) Q(\sigma_i(\gamma), d\gamma'') \delta_{\tau_j(\gamma'')}(du - v) Q(r, d\gamma') \delta_{\sigma_j(\gamma'')}(dr) \\ = & \sum_{i=1}^{\infty} \int_{S \times R_+} \mu(\sigma_i(\gamma), dr \times du - v) \delta_{\tau_i(\gamma)}(dv) Q(r, d\gamma'), \end{aligned}$$

and induction yields

$$M^n(\gamma, d\gamma' \times du) = \sum_{i=1}^{\infty} \int_{S \times R_+} \mu^{n-1}(\sigma_i(\gamma), dr \times du - v) \delta_{\tau_i(\gamma)}(dv) Q(r, d\gamma').$$

Therefore

$$\begin{aligned} & V(\gamma, d\gamma' \times du) \\ = & M^0(\gamma, d\gamma' \times du) + \sum_{n \geq 1} \sum_{i=1}^{\infty} \int_{S \times R_+} \mu^{n-1}(\sigma_i(\gamma), dr \times du - v) \delta_{\tau_i(\gamma)}(dv) Q(r, d\gamma') \\ = & \delta_{(\gamma,0)}(d\gamma' \times du) + \sum_{i=1}^{\infty} \int_{S \times R_+} \nu(\sigma_i(\gamma), dr \times du - v) Q(r, d\gamma') \delta_{\tau_i(\gamma)}(dv). \end{aligned}$$

Hence, recalling that

$$\int_\Gamma H(\gamma')Q(r, d\gamma') = h(r),$$

$$\begin{aligned}
&\int_{\Gamma\times R_+} e^{-2\alpha u} H(\gamma')V(\gamma, d\gamma' \times du) \\
&= H(\gamma) + \sum_{i=1}^{\infty} \int_{\Gamma\times S\times R_+^2} e^{-2\alpha u} H(\gamma')\nu(\sigma_i(\gamma), dr \times du - v)Q(r, d\gamma')\delta_{\tau_i(\gamma)}(dv) \\
&= H(\gamma) + \sum_{i=1}^{\infty} \int_{S\times R_+^2} e^{-2\alpha u} h(r)\nu(\sigma_i(\gamma), dr \times du - v)\delta_{\tau_i(\gamma)}(dv),
\end{aligned}$$

which is finite by (9.6) and the fact that there are $\gamma(S \times R_+) < \infty$ terms in the sum. Having shown this it is easy to see that

$$\begin{aligned}
&\int_{\Gamma\times R_+} e^{-2\alpha u} H(\gamma')\hat{\gamma}'(\alpha)V(\gamma, d\gamma' \times du) \\
&\leq \frac{1}{\kappa}\sum_{i=1}^{\infty} \int_{\Gamma\times S\times R_+^2} e^{-2\alpha u} H(\gamma')^2 Q(r, d\gamma')\nu(\sigma_i(\gamma), dr \times du - v)\delta_{\tau_i(\gamma)}(dv) \\
&\leq \frac{c_1}{\kappa}\sum_{i=1}^{\infty} \int_{S\times R_+^2} e^{-2\alpha u} h(r)\nu(\sigma_i(\gamma), dr \times du - v)\delta_{\tau_i(\gamma)}(dv),
\end{aligned}$$

which is finite by the calculations above.

We now turn to computing the limit of $e^{-2\alpha t}g(\gamma, t)$. By the definition of g we can write

$$e^{-2\alpha t}g(\gamma, t) = \mathrm{Var}_\gamma\left[e^{-\alpha t}\chi'(t) + \int_{\Gamma\times R_+} e^{-\alpha(t-u)}E_{\gamma'}[Z_{t-u}^{\chi'}]e^{-\alpha u}\eta(d\gamma' \times du)\right].$$

Since $\chi \leq m$,

$$\mathrm{Var}_\gamma[e^{-\alpha t}\chi(t)] \leq e^{-2\alpha t}\frac{m^2}{4} \to 0.$$

From this it follows that also $\mathrm{Var}_\gamma[e^{-\alpha t}\chi'(t)] \to 0$ since $\chi'(t) = \sum_{i=1}^{\infty}\chi_i(t - \tau_i)$, which for each starting type γ is a finite sum. Hence

$$\begin{aligned}
&\lim_{t\to\infty} e^{-2\alpha t}g(\gamma, t) \\
&= \lim_{t\to\infty} \mathrm{Var}_\gamma\left[\int_{\Gamma\times R_+} e^{-\alpha(t-u)}E_{\gamma'}[Z_{t-u}^{\chi'}]e^{-\alpha u}\eta(d\gamma' \times du)\right] \\
&= \left(\frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}\right)^2 \mathrm{Var}_\gamma\left[\int_{\Gamma\times R_+} e^{-\alpha u}H(\gamma')\eta(d\gamma' \times du)\right] \\
&= \left(\frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}\right)^2 \mathrm{Var}_\gamma[\bar{\eta}],
\end{aligned}$$

by dominated convergence, since

$$\int_{\Gamma\times R_+} e^{-\alpha(t-u)} E_{\gamma'}[Z_{t-u}^{\chi'}] e^{-\alpha u}\eta(d\gamma'\times du)$$

$$= \int_{\Gamma\times R_+} \frac{e^{-\alpha(t-u)} E_{\gamma'}[Z_{t-u}^{\chi'}]}{H(\gamma')} H(\gamma') e^{-\alpha u}\eta(d\gamma'\times du) \le c_2\bar{\eta}.$$

Replacing $e^{-\alpha t}g(\gamma,t)$ by its limit in (9.1) yields

$$\mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\chi'}] \to \left(\frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}\right)^2 \int_{\Gamma\times R_+} \mathrm{Var}_{\gamma'}[\bar{\eta}] e^{-2\alpha u} V(\gamma, d\gamma'\times du),$$

and the lemma is proved. ■

9.2. The L^2-convergence Theorem

As for L^1-convergence, also here the intrinsic martingale is crucial in the analysis. Recall

$$W_t = \sum_{x\in\mathcal{I}_t} e^{-\alpha\tau'_x} H(\xi_x),$$

from Section 6.1. It can be shown that with the special characteristic

$$\chi'_1(t) = e^{\alpha t}\int_\Gamma\int_t^\infty e^{-\alpha u}H(\gamma')\eta(d\gamma'\times du),$$

we can write

$$W_t = e^{-\alpha t}Z_t^{\chi'_1},$$

see Nerman (1984). With g as above we then see that

$$\mathrm{Var}_\gamma[W_t] = \int_{\Gamma\times R_+} e^{-2\alpha(t-u)} g(\gamma', t-u) e^{-2\alpha u} V(\gamma, d\gamma'\times du)$$

$$= \int_\Gamma\int_0^t e^{-2\alpha u}\mathrm{Var}_{\gamma'}[\bar{\eta}] V(\gamma, d\gamma'\times du).$$

Letting $t\to\infty$ we obtain

$$\mathrm{Var}_\gamma[W_t] \to \int_{\Gamma\times R_+} e^{-2\alpha u}\mathrm{Var}_{\gamma'}[\bar{\eta}] V(\gamma, d\gamma'\times du), \tag{9.8}$$

which is finite since by (9.5) we see that

$$\mathrm{Var}_{\gamma'}[\bar{\eta}] \le E_{\gamma'}[\bar{\eta}^2] \le H(\gamma')\frac{1}{\kappa}(c_1^2 + 3c_1^2\hat{\gamma}'(\alpha)),$$

which has already have shown to be integrable with respect to $e^{-2\alpha u}V(\gamma, d\gamma'\times du)$. Hence W_t converges in L^2 to some random variable W such that $E_\gamma[W] = H(\gamma)$. The following theorem can now be stated.

Theorem 9.2. *Assume the conditions of* Lemma 9.1. *Then, with* $\tilde{w}$ *as in* (6.6),

$$e^{-\alpha t}z_t^\chi \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}\tilde{w},$$

as $t \to \infty$ *in* $L^2(\tilde{Q}_s)$ *for* π*-almost all* $s \in S$.

Proof. Let $\rho = E_\pi[\hat{\chi}(\alpha)]/\alpha\beta$. Then

$$E_\gamma[(e^{-\alpha t}Z_t^{\chi'} - \rho W)^2] \leq 2(E_\gamma[(e^{-\alpha t}Z_t^{\chi'} - \rho W_t)^2] + E_\gamma[(\rho W_t - \rho W)^2]),$$

where the second term converges to 0 by the L^2-convergence of W_t. For the first term note that

$$E_\gamma[(e^{-\alpha t}Z_t^{\chi'} - \rho W_t)^2] = \mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\chi'} - \rho W_t] + E_\gamma^2[e^{-\alpha t}Z_t^{\chi'} - \rho W_t]$$

where

$$E_\gamma[e^{-\alpha t}Z_t^{\chi'} - \rho W_t] = e^{-\alpha t}E_\gamma[Z_t^{\chi'}] - \rho H(\gamma) \to 0, \text{ as } t \to \infty.$$

With χ_1' as above we now see that

$$\begin{aligned}\mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\chi'} - \rho W_t] &= 2\mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\chi'}] \\ &+ \ 2\mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\rho\chi_1'}] - \mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\chi'+\rho\chi_1'}].\end{aligned}$$

It has already been shown that

$$\mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\chi'}] \to \rho^2 \int_{\Gamma\times R_+} e^{-2\alpha u}\mathrm{Var}_{\gamma'}[\bar{\eta}]V(\gamma, d\gamma' \times du) = \rho^2 I,$$

and

$$\mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\rho\chi_1'}] = \rho^2\mathrm{Var}_\gamma[W_t] \to \rho^2 I,$$

so if

$$\mathrm{Var}_\gamma[e^{-\alpha t}\chi_1'(t)] \to 0,$$

$$\frac{\mathrm{Var}_\gamma[e^{-\alpha t}\chi_1'(t)]}{H(\gamma)} \leq a + b\hat{\gamma}(\alpha)$$

and

$$\frac{1}{H(\gamma)}\mathrm{Var}_\gamma\left[\int_{\Gamma\times R_+} e^{-\alpha(t-u)}E_{\gamma'}[Z_{t-u}^{\chi_1'}]e^{-\alpha u}\eta(d\gamma' \times du)\right] \leq c + d\hat{\gamma}(\alpha),$$

for some constants a, b, c and d, then the methods in the proof of Lemma 9.1 show that also $\mathrm{Var}_\gamma[e^{-\alpha t}Z_t^{\chi'+\rho\chi_1'}]$ converge. But

$$\mathrm{Var}_\gamma[e^{-\alpha t}\chi_1'(t)] = \mathrm{Var}_\gamma\left[\int_\Gamma\int_t^\infty e^{-\alpha u}H(\gamma')\eta(d\gamma' \times du)\right] \to 0,$$

$$\frac{\mathrm{Var}_\gamma[e^{-\alpha t}\chi_1'(t)]}{H(\gamma)} \leq \frac{E_\gamma[\bar{\eta}^2]}{H(\gamma)} \leq \frac{1}{\kappa}(c_1^2 + 3c_1^2\hat{\gamma}(\alpha))$$

and

$$\begin{aligned}&\frac{1}{H(\gamma)}\mathrm{Var}_\gamma\left[\int_{\Gamma\times R_+} e^{-\alpha(t-u)}E_{\gamma'}[Z_{t-u}^{\chi_1'}]e^{-\alpha u}\eta(d\gamma' \times du)\right] \\ &= \ \frac{1}{H(\gamma)}\mathrm{Var}_\gamma\left[\int_{\Gamma\times R_+} H(\gamma')e^{-\alpha u}\eta(d\gamma' \times du)\right] \leq \frac{E_\gamma[\bar{\eta}^2]}{H(\gamma)} \leq \frac{1}{\kappa}(c_1^2 + 3c_1^2\hat{\gamma}(\alpha)),\end{aligned}$$

so it can be concluded that

$$\mathrm{Var}_\gamma[e^{-\alpha t} Z_t^{\chi' + \rho\chi_1'}] \to (\rho + \rho)^2 I = 4\rho^2 I,$$

and hence

$$\mathrm{Var}_\gamma[e^{-\alpha t} Z_t^{\chi'} - \rho W_t] \to 0.$$

Finally recall (6.5) to obtain

$$\begin{aligned} & E_s[(e^{-\alpha t} z_t^{\chi} - \rho\tilde{w})^2] \\ \leq \quad & E_s[(e^{-\alpha t}\chi_0(t))^2] + E_s[(e^{-\alpha t} Z_t^{\chi} \circ \xi_0 - \rho w \circ \xi_0)^2], \end{aligned}$$

where the first term tends to zero since χ is bounded and the second term is

$$\int_\Gamma E_\gamma[(e^{-\alpha t} Z_t^{\chi} - \rho w)^2] Q(s, d\gamma).$$

It can be shown that the integrand is dominated by $a\bar{\gamma}^2 + b\bar{\gamma}$ for some constants a and b, so by (9.2) and dominated convergence, the integral tends to zero as $t \to \infty$ for π-almost all $s \in S$.

■

As pointed out in Nerman (1984) it is a rather serious restriction to assume that the characteristic is individual.

10. More General Dependencies

10.1. A More General Type Structure

The theory built up so far relies on the assumption that an individual's marginal reproduction only depends on her type. Hence an individual of type s has the same marginal reproduction whatever the rest of her sibling group looks like. This has shown to be a rich model which, amongst other things, can be applied to quite general forms of sibling dependencies in single-type populations. It is not totally satisfactory though; for reasons similar to those that motivated us to leave the homogeneous case for more general dependence structures, we would want to treat the most general case also in the multi-type setting, i.e. when the marginals $Q_i(\gamma, \cdot)$ depend on i and γ not only through the type $\sigma_i(\gamma)$.

This step is not by any means as big as it is for single-type populations. In fact, the only thing that has to be done is to redefine the type $\sigma_i(\gamma)$ to consist of the pair (γ, i). Then everything is brought back to what has been done; given your type, your marginal reproduction is independent of the rest of your sibling group and all the theory developed previously applies.

It could however be of interest to continue to keep track of the original types. These might still be interesting even though they no longer alone determine individual marginal reproduction. Therefore keep the notation $\sigma_i(\gamma)$ for the original types and simply use the notation (γ, i) for the entities that determine marginal reproduction (they might for instance be called *actual types*). This leads to the notation

$$\mu((\gamma, i), d\gamma' \times \{j\} \times dt) = Q_i(\gamma, d\gamma')\delta_{\tau_j(\gamma')}(dt)$$

and

$$\hat{\mu}((\gamma, i), d\gamma' \times \{j\}) = \sum_{j=1}^{\infty} e^{-\alpha\tau_j(\gamma')} Q_i(\gamma, d\gamma').$$

Under conservativity assumptions there will exist an invariant measure π on $\Gamma \times Z_+$ such that

$$\pi(d\gamma' \times \{j\}) = \sum_{j=1}^{\infty} \int_\Gamma \hat{\mu}((\gamma, i), d\gamma' \times \{j\})\pi(d\gamma \times \{i\}),$$

this π also being the distribution of the (γ, i) in the stable population. The original types will then be distributed according to the measure

$$\phi(ds) = \sum_{i=1}^{\infty} \int_\Gamma \delta_{\sigma_i(\gamma)}(ds)\pi(d\gamma \times \{i\}),$$

but besides this not much of interest can be said about them. There will be no Markov structure backwards in the stable population tree for the original types, this structure being closely related to the fact that types should determine marginal reproduction (a role now played by the actual types). Neither is ϕ invariant for any reproduction kernel; in fact an expression like $\hat{\mu}(s, dr)$ no longer makes sense since it matters also in what context s appears. Of course all the established theory holds for the actual types.

10.2. How to Deal with Cousins

A natural generalization is to extend the dependencies beyond sibling groups, a natural first step being to assume that also cousins are dependent. This can be subsumed in the old theory by noting that a group of (first) cousins can be described as a sibling group of macro individuals, where the macro individuals as before consist of sibling groups of individuals, these latter groups now possibly being dependent. Observe that by a group of cousins is meant a group of individuals such that any pair are either siblings or cousins. For a Galton-Watson population this can be neatly illustrated in a picture. Recall Figure 3.1 where individuals were lumped together to form sibling groups and go one step further:

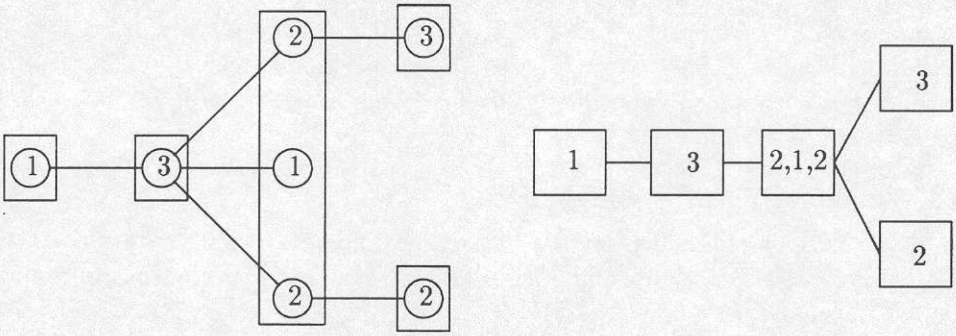

Figure 10.1: The macroprocess from Figure 3.1 with cousin groups in rectangles and the corresponding cousin macro process.

Note that the type of a group of cousins tells how many sibling groups it consists of and how many individuals there are in each sibling group.

In a general branching process, assume that a group of cousins born according to the point process γ' on $\Gamma \times R_+$ reproduces according to the joint probability measure $\tilde{\bar{P}}(\gamma', \cdot)$ on $\bar{\bar{\Omega}} = \bar{\Omega}^{\infty}$. This construction is exactly the same as in Section 3.2, only that it starts one step "higher", with sibling groups and that these in their turn are lumped together to form new macro individuals, cousin groups. The type γ' of a cousin group is thus an element of

$$\mathcal{N}(\Gamma \times R_+) = \mathcal{N}(\mathcal{N}(S \times R_+) \times R_+),$$

the set of point processes on $\Gamma \times R_+$. The ith sibling group in γ' reproduces according to the marginal

$$\tilde{P}(\sigma_i(\gamma'),\cdot)$$

on $\bar{\Omega}$. Here $\sigma_i(\gamma')$ is an element of Γ, i.e. a point process on $S \times R_+$. The jth individual in this sibling group in its turn reproduces according to the marginal $\tilde{Q}(\sigma_j(\sigma_i(\gamma')),\cdot)$, where $\sigma_j(\sigma_i(\gamma'))$ is an individual type, an element of S.

By this, there will be three reproduction kernels, μ,M and M', to be called the individual, the sibling macro and the cousin macro reproduction kernel respectively. It is clear that M' relates to M in the same way as M relates to μ and hence the three kernels define the same Malthusian parameter. Further, $\hat{\mu}$, $\hat{M}$ and $\hat{M}'$ are either all conservative or all not. The corresponding invariant measures and functions are π, ψ, ψ' and h, H, H', relating to each other in the obvious way.

The mean convergence follows immediately so the only concern is how well the proof of L^1-convergence applies. This is not obvious since the convergence of the sibling macro process was established through certain properties on the individual level (e.g. uniform integrability) which does not hold for the sibling macro process. Therefore the proof of Theorem 6.2 can not simply be copied. It can however be modified to extend also to cousin dependencies, and indeed to dependencies between nth cousins for each n, something that will be done in the next section.

10.3. General Local Dependencies

Let us start by introducing some terminology and convenient notation. Call a branching process *n-dependent* if individuals who are kth cousins reproduce independently if $k \geq n$ but possibly dependently if $k < n$. By convention siblings are considered as 0th cousins. Then 0-dependence means independence, 1-dependence means sibling dependences, 2-dependence means that individuals who are either siblings or first cousins reproduce dependently and so forth. There are no dependencies between generations other than through the types. The term "local" in the title obviously means local in the family tree.

There will be a hierarchy of macro individuals and we define a *k-macro individual* to consist of a group of individuals each two of whom are jth cousins for some $j = 0, 1, ..., k$ where $0 \leq k \leq n$.

There is now a sequence of life spaces $\Omega_0, \Omega_1, ...$ where

$$\Omega_{k+1} = \Omega_k^\infty,$$

Ω_0 being the individual life space. Likewise there will be a sequence of type spaces, $\Gamma_0, \Gamma_1, ...$ where

$$\Gamma_{k+1} = \mathcal{N}(\Gamma_k \times R_+),$$

(or again some appropriate subset) and, in the previous notation, $\Gamma_0 = S$ and $\Gamma_1 = \Gamma$. Let $\sigma_i : \Gamma_{k+1} \rightarrow \Gamma_k$, $k = 1, 2, ...$, so that $\sigma_i(\gamma_{k+1})$ is the type of the ith

k-macro individual in a $(k+1)$-macro individual. The sequence of probability measures $P_0, P_1, \ldots$ on $\Omega_0, \Omega_1, \ldots$ are then such that the ith k-macro individual in a $(k+1)$-macro individual reproduces according to the marginal

$$P_k(\sigma_i(\gamma_{k+1}), A) = \int_{\Omega_{k+1}} 1_A(p_i(\omega_{k+1}))P_{k+1}(d\omega_{k+1}),$$

the projections p_i defined in the obvious way. Again it will be convenient to consider the measures induced by the reproduction processes. Therefore define

$$Q_k(\gamma_k, d\gamma_{k+1}) = P_k(\gamma_k, \{\eta^k \in d\gamma_{k+1}\}),$$

with the obvious notation $\eta^0, \eta^1, \ldots$, where $\eta^0 = \xi$. These measures give rise to the sequence of reproduction kernels $M_0, M_1, \ldots$, where $M_0 = \mu$ and $M_1 = M$, and the next lemma is obvious.

Lemma 10.1. *In an n-dependent branching process the kernels $M_0, M_1, \ldots, M_n$ all define the same Malthusian parameter. Further, the kernels $\hat{M}_0, \hat{M}_1, \ldots, \hat{M}_n$, are either all conservative or all not.*

Now let $\tau_i : \Gamma_k \to R_+$ so that $\tau_i(\gamma_k)$ is the time point for the ith point in γ_k. Under conservativity assumptions, there will also be a sequence of functions $H_0, H_1, \ldots, H_n$ and a sequence of measures $\psi_0, \psi_1, \ldots \psi_n$ such that H_k and ψ_k are invariant for M_k,

$$\psi_k(d\gamma_k) = \sum_{i=1}^{\infty} \int_{\Gamma_{k+1}} e^{-\alpha\tau_i(\gamma_{k+1})}\delta_{\sigma_i(\gamma_{k+1})}(d\gamma_k)\psi_{k+1}(d\gamma_k), \tag{10.1}$$

$$\psi_{k+1}(d\gamma_{k+1}) = \int_{\Gamma_k} Q_k(\gamma_k, d\gamma_{k+1})\psi_k(d\gamma_k)$$

and

$$H_{k+1}(\gamma_{k+1}) = \bar{\gamma}_{k+1} = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma_{k+1})}H_k(\sigma_i(\gamma_{k+1})),$$

where, of course, $H_0 = h, H_1 = H, \psi_0 = \pi$ and $\psi_1 = \psi$. The mean convergence is obvious, the problems of L^1-convergence are two: only the individual process is assumed uniformly integrable and only the function $h = H_0$ is assumed to be bounded away from zero.

We start by proving that all the $x \log x$ conditions are equivalent. For that purpose note that

$$\bar{\eta}^{k+1} = \sum_{i=1}^{\infty} e^{-\alpha\tau_i}\bar{\eta}_i^k. \tag{10.2}$$

Here η_i^k is the point process stemming from the ith k-macro individual and $\bar{\eta}^k$ defined from η^k as was $\bar{\eta}$ was from η. The $x \log x$ conditions are

$$E_{\psi_k}[\bar{\eta}^k \log^+ \bar{\eta}^k] < \infty,$$

for $k = 0, 1, 2, \ldots, n$, and we state

Lemma 10.2. *Either*

$$E_{\psi_k}[\bar{\eta}^k \log^+ \bar{\eta}^k] < \infty,$$

for $k = 0, 1, 2, ..., n$, *or*

$$E_{\psi_k}[\bar{\eta}^k \log^+ \bar{\eta}^k] = \infty,$$

for $k = 0, 1, 2, ..., n$.

Proof. Let

$$f(\gamma_k) = \sum_{i_{k-1},...,i_0} e^{-\alpha\tau_{i_{k-1}}(\gamma_k)} e^{-\alpha\tau_{i_{k-2}}(\sigma_{i_{k-1}}(\gamma_k))} ... e^{-\alpha\tau_{i_0}(\sigma_{i_1}(...\sigma_{i_{k-1}}(\gamma_k))...)},$$

and note that

$$\begin{aligned}
\bar{\gamma}_k &= \sum_{i_{k-1}=1}^{\infty} e^{-\alpha\tau_{i_{k-1}}(\gamma_k)} H_{k-1}(\sigma_{i_{k-1}}(\gamma_k)) \\
&= \sum_{i_{k-1},i_{k-2}} e^{-\alpha\tau_{i_{k-1}}(\gamma_k)} e^{-\alpha\tau_{i_{k-2}}(\sigma_{i_{k-1}}(\gamma_k))} H_{k-2}(\sigma_{i_{k-2}}(\sigma_{i_{k-1}}(\gamma_k))) \\
&= ... = \sum_{i_{k-1},...,i_0} e^{-\alpha\tau_{i_{k-1}}(\gamma_k)} e^{-\alpha\tau_{i_{k-2}}(\sigma_{i_{k-1}}(\gamma_k))} ... e^{-\alpha\tau_{i_0}(\sigma_{i_1}(...\sigma_{i_{k-1}}(\gamma_k))...)} \\
&\quad\; h(\sigma_{i_0}(...\sigma_{i_{k-1}}(\gamma_k))...)) \\
&\geq \kappa f(\gamma_k). \qquad (10.3)
\end{aligned}$$

By (10.2),

$$\begin{aligned}
E_{\gamma_k}[\bar{\eta}^k \log^+ \bar{\eta}^k] &= E_{\gamma_k}\left[\bar{\eta}^k \log^+ \left(f(\gamma_k) \sum_{i_{k-1}=1}^{\infty} \frac{e^{-\alpha\tau_{i_{k-1}}(\gamma_k)}}{f(\gamma_k)} \bar{\eta}^{k-1}_{i_{k-1}} \right)\right] \\
&\leq E_{\gamma_k}[\bar{\eta}^k \log^+(f(\gamma_k))] \\
&\quad + f(\gamma_k) E_{\gamma_k}\left[\sum_{i_{k-1}=1}^{\infty} \frac{e^{-\alpha\tau_{i_{k-1}}(\gamma_k)}}{f(\gamma_k)} \bar{\eta}^{k-1}_{i_{k-1}} \log^+ \left(\sum_{i_{k-1}=1}^{\infty} \frac{e^{-\alpha\tau_{i_{k-1}}(\gamma_k)}}{f(\gamma_k)} \bar{\eta}^{k-1}_{i_{k-1}} \right)\right] \\
&= I_1(\gamma_k) + I_2(\gamma_k).
\end{aligned}$$

By (10.3),

$$I_1(\gamma_k) = \log^+(f(\gamma_k)) E_{\gamma_k}[\bar{\eta}^k] \leq \log^+(\frac{\bar{\gamma}_k}{\kappa}) H_k(\gamma_k) = \bar{\gamma}_k \log^+ \frac{\bar{\gamma}_k}{\kappa},$$

and hence

$$\begin{aligned}
\int_{\Gamma_k} I_1(\gamma_k)\psi_k(d\gamma_k) &\leq \int_{\Gamma_k} \log^+ \bar{\gamma}_k \frac{\bar{\gamma}_k}{\kappa} \psi_k(d\gamma_k) \\
&= \int_{\Gamma_k \times \Gamma_{k-1}} \bar{\gamma}_k \log^+ \frac{\bar{\gamma}_k}{\kappa} Q_{k-1}(\gamma_{k-1}, d\gamma_k)\psi_{k-1}(d\gamma_{k-1})
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Gamma_{k-1}} E_{\gamma_{k-1}}[\bar{\eta}^{k-1} \log^+ \frac{\bar{\eta}^{k-1}}{\kappa}]\psi_{k-1}(d\gamma_{k-1}) \\
&= E_{\psi_{k-1}}[\bar{\eta}^{k-1} \log^+ \frac{\bar{\eta}^{k-1}}{\kappa}].
\end{aligned}
$$

Iterate (10.2) down to $\eta^0 = \xi$ and use the convexity of $x \log^+ x$ to obtain

$$
\begin{aligned}
I_2(\gamma_k) = f(\gamma_k) E_{\gamma_k} \Bigg[&\sum_{i_{k-1},\ldots,i_0} \frac{e^{-\alpha\tau_{i_{k-1}}(\gamma_k)} \ldots e^{-\alpha\tau_{i_0}(\sigma_{i_1}(\ldots\sigma_{i_{k-1}}(\gamma_k))\ldots)}}{f(\gamma_k)} \xi_{i_{k-1}\ldots i_0} \\
&\log^+ \left(\sum_{i_{k-1},\ldots,i_0} \frac{e^{-\alpha\tau_{i_{k-1}}(\gamma_k)} \ldots e^{-\alpha\tau_{i_0}(\sigma_{i_1}(\ldots\sigma_{i_{k-1}}(\gamma_k))\ldots)}}{f(\gamma_k)} \xi_{i_{k-1}\ldots i_0} \right) \Bigg] \\
\leq\ & f(\gamma_k) \sum_{i_{k-1},\ldots,i_0} \frac{e^{-\alpha\tau_{i_{k-1}}(\gamma_k)} \ldots e^{-\alpha\tau_{i_0}(\sigma_{i_1}(\ldots\sigma_{i_{k-1}}(\gamma_k))\ldots)}}{f(\gamma_k)} E_{\gamma_k}[\bar{\xi}_{i_{k-1}\ldots i_0} \log^+ \bar{\xi}_{i_{k-1}\ldots i_0}] \\
=\ & \sum_{i_{k-1},\ldots,i_0} e^{-\alpha\tau_{i_{k-1}}(\gamma_k)} \ldots e^{-\alpha\tau_{i_0}(\sigma_{i_1}(\ldots\sigma_{i_{k-1}}(\gamma_k))\ldots)} E_{\sigma_{i_0}(\ldots\sigma_{i_{k-1}}(\gamma_k)\ldots)}[\bar{\xi} \log^+ \bar{\xi}].
\end{aligned}
$$

Now note that

$$
\begin{aligned}
&E_{\sigma_{i_0}(\ldots\sigma_{i_{k-1}}(\gamma_k)\ldots)}[\bar{\xi} \log^+ \bar{\xi}] \\
&\qquad = \int_{\Gamma_{k-1}} E_{\sigma_{i_0}(\ldots\sigma_{i_{k-2}}(\gamma_{k-1})\ldots)}[\bar{\xi} \log^+ \bar{\xi}] \delta_{\sigma_{i_{k-1}}(\gamma_k)}(d\gamma_{k-1}) \\
&\qquad = \ldots = \int_{\Gamma_{k-1} \times \ldots \times \Gamma_0} E_{\gamma_0}[\bar{\xi} \log^+ \bar{\xi}] \delta_{\sigma_{i_{k-1}}(\gamma_k)}(d\gamma_{k-1}) \ldots \delta_{\sigma_{i_0}(\gamma_1)}(d\gamma_0),
\end{aligned}
$$

and therefore, by iterating (10.1),

$$
\begin{aligned}
&\int_{\Gamma_k} I_2(\gamma_k)\psi_k(d\gamma_k) \\
&\leq \int_{\Gamma_k} \sum_{i_{k-1},\ldots,i_0} e^{-\alpha\tau_{i_{k-1}}(\gamma_k)} \ldots e^{-\alpha\tau_{i_0}(\sigma_{i_1}(\ldots\sigma_{i_{k-1}}(\gamma_k))\ldots)} E_{\sigma_{i_0}(\ldots\sigma_{i_{k-1}}(\gamma_k)\ldots)}[\bar{\xi} \log^+ \bar{\xi}]\psi_k(d\gamma_k) \\
&= \int_{\Gamma_0} E_{\gamma_0}[\bar{\xi} \log^+ \bar{\xi}]\psi_0(d\gamma_0) = \int_S E_s[\bar{\xi} \log^+ \bar{\xi}]\pi(ds) = E_\pi[\bar{\xi} \log^+ \bar{\xi}].
\end{aligned}
$$

Hence

$$
E_{\psi_k}[\bar{\eta}^k \log^+ \bar{\eta}^k] \leq E_{\psi_{k-1}}[\bar{\eta}^{k-1} \log^+ \bar{\eta}^{k-1}] + E_\pi[\bar{\xi} \log^+ \bar{\xi}].
$$

For the converse, apply Jensen's inequality to obtain

$$
E_{\gamma_k}[\bar{\eta}^k \log^+ \bar{\eta}^k] \geq E_{\gamma_k}[\bar{\eta}^k] \log^+ E_{\gamma_k}[\bar{\eta}^k] = \bar{\gamma}_k \log^+ \bar{\gamma}_k,
$$

and hence

$$
E_{\psi_k}[\bar{\eta}^k \log^+ \bar{\eta}^k] = \int_{\Gamma_k} E_{\gamma_k}[\bar{\eta}^k \log^+ \bar{\eta}^k]\psi_k(d\gamma_k)
$$

$$\geq \int_{\Gamma_k} \bar{\gamma}_k \log^+ \bar{\gamma}_k \psi_k(d\gamma_k) = \int_{\Gamma_k \times \Gamma_{k-1}} \bar{\gamma}_k \log^+ \bar{\gamma}_k Q_{k-1}(\gamma_{k-1}, d\gamma_k)\psi_{k-1}(d\gamma_{k-1})$$

$$= \int_{\Gamma_{k-1}} E_{\gamma_{k-1}}[\bar{\eta}^{k-1} \log^+ \bar{\eta}^{k-1}]\psi_{k-1}(d\gamma_{k-1}) = E_{\psi_{k-1}}[\bar{\eta}^{k-1} \log^+ \bar{\eta}^{k-1}].$$

Therefore, for any k,

$$E_{\psi_{k-1}}[\bar{\eta}^{k-1} \log^+ \bar{\eta}^{k-1}] \leq E_{\psi_k}[\bar{\eta}^{k} \log^+ \bar{\eta}^{k}]$$

$$\leq \; E_{\psi_{k-1}}[\bar{\eta}^{k-1} \log^+ \frac{\bar{\eta}^{k-1}}{\kappa}] + E_\pi[\bar{\xi} \log^+ \bar{\xi}],$$

and the proof is complete. ■

The population of individuals started from an s-type individual will follow the marginal law $\tilde{Q}_s(\cdot)$ on $S \times \Omega^I$, $\tilde{Q}_s$ defined in obvious analogy with (3.2). The analogue of Theorem 6.2 and Corollary 6.3 is

Theorem 10.3. *Consider an n-dependent non-lattice, uniformly integrable and strictly Malthusian branching process, such that $E_\pi[\bar{\xi} \log^+ \bar{\xi}] < \infty$. Let χ be bounded and the function $e^{-\alpha t} E_s[\chi(t)]$ directly Riemann integrable. Then there exists a random variable $\tilde{w}$ such that*

$$e^{-\alpha t} z_t^\chi \to \frac{E_\pi[\hat{\chi}(\alpha)]}{\alpha\beta}\tilde{w},$$

as $t \to \infty$ in $L^1(\tilde{Q}_s)$ for π-almost all $s \in S$.

Proof. The proof is essentially the same as that of Theorem 6.2 and only the differences will be pointed out. The major novelty is of course that we now deal with a hierarchy of characteristics, macro processes, reproduction measures and so on. With

$$\tau_{i_{n-1}\dots i_0}(\eta_0^{n-1}) = \tau_{i_{n-1}}(\eta_0^{n-1}) + \dots + \tau_{i_0}(\sigma_{i_1}(\dots\sigma_{i_{n-1}}(\eta_0^{n-1}))\dots),$$

the characteristic of interest is

$$\chi^n(t) = \sum_{i_{n-1}=1}^{\eta_0^{n-1}(t)} \quad \sum_{i_{n-2}=1}^{\eta_{i_{n-1}}^{n-2}(t-\tau_{i_{n-1}}(\eta_0^{n-1}))}$$

$$\dots \sum_{i_0=1}^{\xi_{i_{n-1}\dots i_1}(t-\tau_{i_{n-1}\dots i_0}(\eta_0^{n-1}))} \quad \chi_{i_{n-1}\dots i_0}(t - \tau_{i_{n-1}\dots i_0}(\eta_0^{n-1})),$$

and, in analogy with the proof of Theorem 6.2, we assume that $\chi \leq m$ and truncate $\chi^n(t)$ so that it vanishes for $t > m$. The key to the proof of Theorem 6.2 is the inequality (6.3) and we will try to generalise it to this setting. By (10.3) we have that $H_n(\eta^{n-1}) \geq \kappa f(\eta^{n-1})$ and therefore $\hat{\xi}_x(\alpha)$ is replaced by $f(\eta_x^{n-1})$. The sum

$$\sum_{i=1}^{\xi_0(t)} y_n(i)$$

is replaced by

$$\sum_{i_{n-1}=1}^{\eta_0^{n-1}(t)} \quad \sum_{i_{n-2}=1}^{\eta_{i_{n-1}}^{n-2}(t-\tau_{i_{n-1}}(\eta_0^{n-1}))} \quad \dots \quad \sum_{i_0=1}^{\xi_{i_{n-1}\cdots i_1}(t-\tau_{i_{n-1}\cdots i_0}(\eta_0^{n-1}))} \quad y_n(xi_{n-1}...i_0),$$

where, as before, the y_n are uniformly integrable over their starting types. The next step is therefore to establish a relation analogous to the previous fact that

$$\frac{e^{-\alpha(t-\tau'_x)}\xi_x(t-\tau'_x)}{\hat{\xi}_x(\alpha)} \leq 1.$$

For convenience reasons the notation τ rather than τ^n is used for birth times for n-macro individuals. Hence τ_x is the birth time of the n-macro individual x, τ_{xk} the birth time of the $(n-1)$-macro individual xk and so forth. Thus it is the subscript that tells which k that is under consideration. Also note that $\tau_{xk} = \tau_x + \tau_k(\eta_x^k)$. What we need to show is that

$$\frac{e^{-\alpha(t-\tau_x)}}{f(\eta_x^{n-1})} \sum_{i_{n-1}=1}^{\eta_x^{n-1}(t-\tau_x)} \quad \sum_{i_{n-2}=1}^{\eta_{xi_{n-1}}^{n-2}(t-\tau_{xi_{n-1}})}$$

$$\dots \sum_{i_1=1}^{\eta^1_{xi_{n-1}\cdots i_1}(t-\tau_{xi_{n-1}\cdots i_1})} \quad \xi_{i_{n-1}...i_1}(t-\tau_{xi_{n-1}...i_1}) \leq 1.$$

Since

$$\sigma_{i_1}(...\sigma_{i_{n-1}}(\eta_x^{n-1})...) = \xi_{xi_{n-1}...i_1},$$

we have

$$f(\eta_x^{n-1}) = \sum_{i_0,\dots,i_{n-1}} e^{-\alpha\tau_{i_{n-1}}(\eta_x^{n-1})} e^{-\alpha\tau_{i_{n-2}}(\sigma_{i_{n-1}}(\eta_x^{n-1}))} ... e^{-\alpha\tau_{i_0}(\sigma_{i_1}(...\sigma_{i_{n-1}}(\eta_x^{n-1}))...)}$$

$$= \sum_{i_1,\dots,i_{n-1}} e^{-\alpha\tau_{i_{n-1}}(\eta_x^{n-1})} ... e^{-\alpha\tau_{i_1}(\sigma_{i_2}(...\sigma_{i_{n-1}}(\eta_x^{n-1}))...)} \hat{\xi}_{xi_{n-1}...i_1}(\alpha).$$

By the proof of Theorem 6.2 we know that $e^{-\alpha t}\gamma_1(t) \leq \hat{\gamma}_1(\alpha)$ and hence

$$e^{-\alpha(t-\tau_x)} \sum_{i_{n-1}=1}^{\eta_x^{n-1}(t-\tau_x)} \quad \sum_{i_{n-2}=1}^{\eta_{xi_{n-1}}^{n-2}(t-\tau_{xi_{n-1}})}$$

$$\dots \sum_{i_1=1}^{\eta^1_{xi_{n-1}\cdots i_1}(t-\tau_{xi_{n-1}\cdots i_1})} \quad \xi_{i_{n-1}...i_1}(t-\tau_{xi_{n-1}...i_1})$$

$$
\begin{aligned}
&= \sum_{i_{n-1}=1}^{\eta_x^{n-1}(t-\tau_x)} \cdots \sum_{i_1=1}^{\eta^1_{xi_{n-1}\cdots i_1}(t-\tau_{xi_{n-1}\cdots i_1})} e^{-\alpha(\tau_{xi_{n-1}\cdots i_1}-\tau_x)} \\
&\quad e^{-\alpha(t-\tau_{xi_{n-1}\cdots i_1})}\xi_{i_{n-1}\dots i_1}(t-\tau_{xi_{n-1}\dots i_1}) \\
&\le \sum_{i_{n-1}=1}^{\eta_x^{n-1}(t-\tau_x)} \cdots \sum_{i_1=1}^{\eta^1_{xi_{n-1}\cdots i_1}(t-\tau_{xi_{n-1}\cdots i_1})} e^{-\alpha(\tau_{xi_{n-1}\cdots i_1}-\tau_x)}\hat{\xi}_{i_{n-1}\dots i_1}(\alpha) \le f(\eta_x^{n-1}),
\end{aligned}
$$

where the last inequality follows since

$$\tau_{xi_{n-1}\dots i_1} = \tau_x + \tau_{i_{n-1}}(\eta_x^{n-1}) + \dots + \tau_{i_1}(\sigma_{i_2}(\dots\sigma_{i_{n-1}}(\eta_x^{n-1}))\dots).$$

The rest of the proof is carried through by carefully copying the proofs of Theorem 6.2 and Corollary 6.3.

■

11. References

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