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Boundary behaviour of eigenfunctions for the hyperbolic Laplacian

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Fakultetsopponent är professor Fausto Di Biase, Università 'G.d'Annunzio', Pescara, Italien.

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Abstract

Let $P(z, \varphi)$ denote the Poisson kernel in the unit disc. Poisson extensions of the type $P_\lambda f(z) = \int_{\mathbb{T}} P(z, \varphi)^{\lambda+1/2} f(\varphi)\, d\varphi$, where $f \in L^1(\mathbb{T})$ and $\lambda \in \mathbb{C}$, are then eigenfunctions to the hyperbolic Laplace operator in the unit disc. In the context of boundary behaviour, $P_0 f(z)$ exhibits unique properties.

We investigate the boundary convergence properties of the normalised operator, $P_0 f(z)/P_0 1(z)$, for boundary functions f in some function spaces. For each space, we characterise the so-called natural approach regions along which one has almost everywhere convergence to the boundary function, for any boundary function in that space. This is done, mostly, via estimates of the associated maximal function.

The function spaces we consider are $L^{p,\infty}$ (weak L^p) and Orlicz spaces which are either close to L^p or L^∞. We also give a new proof of known results for L^p, $1 \leq p \leq \infty$.

Finally, we deal with a problem on the lack of tangential convergence for bounded harmonic functions in the unit disc. We give a new proof of a result due to Aikawa.

Keywords: Square root of the Poisson kernel, approach regions, almost everywhere convergence, maximal functions, harmonic functions, Fatou theorem.

2000 AMS Subject classification: 42B25, 42A99, 43A85, 31A05, 31A20.

THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Boundary behaviour of eigenfunctions for the hyperbolic Laplacian

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The thesis consists of a summary and the following four papers:

[MB1] M. Brundin, *Approach regions for the square root of the Poisson kernel and weak L^p boundary functions*, Revised version of Preprint 1999:56, Göteborg University and Chalmers University of Technology, 1999.

[MB2] M. Brundin, *Approach regions for L^p potentials with respect to the square root of the Poisson kernel*, Revised version of Preprint 2001:55, Göteborg University and Chalmers University of Technology, 2001.

[MB3] M. Brundin, *Approach regions for the square root of the Poisson kernel and boundary functions in certain Orlicz spaces*, Revised version of Preprint 2001:59, Göteborg University and Chalmers University of Technology, 2001.

[MB4] M. Brundin, *On a theorem of Aikawa.*

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BOUNDARY BEHAVIOUR OF EIGENFUNCTIONS FOR THE HYPERBOLIC LAPLACIAN

MARTIN BRUNDIN

1. Introduction

This thesis deals mainly with the boundary behaviour of solutions to a specific partial differential equation. We shall content ourselves in this introductory paper with a discussion of the relevant harmonic analysis on the unit disc, where our differential equation is defined. In Section 5, however, we show how some of the concepts treated can be carried over to a different setting (the half space).

We shall be concerned with pointwise, almost everywhere, convergence. The solutions to our differential equation will be given by Poisson-like integral extensions of the boundary functions. More precisely, the integral kernel is given by the square root of the Poisson kernel and possesses unique properties relative to other powers. The associated extensions are eigenfunctions of the hyperbolic Laplacian, at the bottom of the positive spectrum. To recover the boundary values, the extensions must be normalised.

It is a well-known fact that solutions to boundary value problems behave more and more dramatically the closer one gets to the boundary. *A priori*, it is often not even clear in which sense the boundary conditions should be interpreted. Of course, in some sense, the solution should be "equal to the prescribed boundary values on the boundary", but that statement is not precise. It will be clear that if we approach the boundary, the unit circle, too close to the tangential direction, then almost everywhere convergence of the extension to the boundary function will fail. The question we wish to answer is, somewhat vaguely, the following:

Given a space A of integrable functions defined on the unit circle, how tangential can our approach to the boundary be in order to guarantee a.e.

convergence of the extension to the boundary function, for any boundary function in A?

A few comments are in order. The notion of tangency to the boundary will be measured by so-called approach regions, which will depend on the space A, beside the integral kernel. It is to my knowledge impossible to give an answer to the question above for all A. Instead, we shall consider more or less explicit examples of A. The examples we cover are $A = L^p$ for $1 \le p \le \infty$, $A = L^{p,\infty}$ (weak L^p) for $1 < p < \infty$ and $A = L^\Phi$ (Orlicz spaces) for certain classes of functions Φ. These results are covered in the papers [MB2], [MB1] and [MB3], respectively.

The paper [MB4] deals with a classical problem concerning the lack of convergence of bounded harmonic functions in the unit disc. We give a modified proof of a result by Aikawa, which in turn is a considerably sharpened version of a theorem of Littlewood (see below).

In the following sections we give an outline of the underlying theory and our results.

2. The Poisson kernel and harmonic functions in the unit disc

Let U denote the unit disc in $\mathbb{C}$, i.e.

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Then $\partial U \cong \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \cong (-\pi, \pi]$. Whenever convenient, we identify $\mathbb{T}$ with the interval $(-\pi, \pi]$.

The Dirichlet problem is the following: Given a function $f \in L^1(\mathbb{T})$, find a function u which is harmonic in U and such that $u = f$ on $\mathbb{T}$. As we shall see below, this question makes sense if $f \in C(\mathbb{T})$. If we only assume that $f \in L^1(\mathbb{T})$, this is a typical example where one has to be very careful with the meaning of the condition $u = f$ on $\mathbb{T}$ (see the results of Fatou and Littlewood below).

Let $P(z, \beta)$ be the Poisson kernel in U,

$$P(z,\beta) = \frac{1}{2\pi} \cdot \frac{1 - |z|^2}{|z - e^{i\beta}|^2},$$

where $z \in U$ and $\beta \in \mathbb{T}$. It is readily checked that $P(z, \beta)$ is the real part of the holomorphic function

$$u(z) = \frac{1}{2\pi} \cdot \frac{e^{i\beta} + z}{e^{i\beta} - z},$$

so that $P(\,\cdot\,, \beta)$ is harmonic in U.

The Poisson integral (or extension) Pf of $f \in L^1(\mathbb{T})$ is defined, for $z \in U$, by

$$Pf(z) = \int_{\mathbb{T}} P(z, \beta) f(\beta)\, d\beta.$$

Note that, if we write $z = (1-t)e^{i\theta}$, then

$$Pf(z) = K_t * f(\theta),$$

where the convolution is taken in $\mathbb{T}$ and

$$K_t(\varphi) = \frac{1}{2\pi} \cdot \frac{t(2-t)}{|(1-t)e^{i\varphi} - 1|^2}.$$

For positive functions f and g, we say that $f \lesssim g$ if $f \leq cg$ for some constant $c > 0$. If $f \lesssim g$ and $g \lesssim f$, we say that $f \sim g$. For later use, we note that

$$K_t(\varphi) \sim L_t(\varphi) = \frac{t}{(t + |\varphi|)^2}.$$

The Poisson extension Pf defines a harmonic function in U. Moreover, we have the following classical result (solution to the continuous Dirichlet problem):

Theorem (Schwarz, [10]). *If $f \in C(\mathbb{T})$, then $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in U$.*

A natural question is what happens when the boundary function f is less regular, e.g. when $f \in L^p(\mathbb{T})$. First of all, of course, the best thing one can hope for is convergence at, at most, almost every boundary point (i.e., convergence fails on at most a set of measure zero). However, it turns out that a.e. convergence may very well fail if the approach to the boundary is "too tangential". To guarantee a.e. convergence, one has to approach the boundary with some care, in the sense of staying within certain *approach regions.*

Definition 1. For any function $h : \mathbb{R}_+ \to \mathbb{R}_+$ we define the (natural) approach region, determined by h at $\theta \in \mathbb{T}$, by

$$\mathcal{A}_h(\theta) = \{z \in U : |\arg z - \theta| \leq h(1 - |z|)\}.$$

If $h(t) \sim t$, as $t \to 0$, we say that $\mathcal{A}_h(\theta)$ is a nontangential cone.

There are also other kinds of approach regions. Maybe the most interesting are those of so-called Nagel-Stein type, being given by means of a "cone condition" and a "cross-section condition". We shall not consider such approach regions, but will instead focus only on those given in Definition 1.

Theorem (Fatou, [6]). *Let $h(t) = O(t)$. Then, for all $f \in L^1(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$.*

In this case, to relate to what we said earlier, the condition $u = f$ on $\mathbb{T}$ should be interpreted as a nontangential limit.

Let us sketch a proof of Fatou's result:

Proof. To keep the proof as simple as possible, assume that $h(t) = t$. As we shall see later, in Section 3, it is now sufficient to see that the maximal operator given by

$$Mf(z) = \sup_{|z|>1/2, |\arg z - \theta|<t} |Pf(z)|,$$

is of weak type $(1, 1)$. Note that

$$Mf(z) \lesssim \sup_{t<1/2, |\eta|<t} \tau_\eta L_t * |f|(\theta),$$

where τ_η denotes translation, i.e. $\tau_\eta F(\theta) = F(\theta - \eta)$ for any function F. Since $|\eta| < t$, it is easily seen that

$$\tau_\eta L_t(\varphi) \sim L_t(\varphi).$$

Now, since $\|L_t\|_1 \lesssim 1$ uniformly in t, it follows by standard results (see [14], §2.1) that

$$Mf(z) \lesssim M_{HL} f(\theta),$$

where M_{HL} denotes the ordinary Hardy-Littlewood maximal operator, and the weak type estimate follows, as desired. □

Littlewood [7] proved that Fatou's theorem, in a certain sense, is sharp:

Theorem (Littlewood, [7]). *Let $\gamma_0 \subset U \cup \{1\}$ be a simple closed curve, having a common tangent with the circle at the point 1. Let γ_θ be the rotation of γ_0 by the angle θ. Then there exists a bounded harmonic function f in U with the property that, for a.e. $\theta \in \mathbb{T}$, the limit of f along γ_θ does not exist.*

Littlewood's proof was not elementary. He used a result of Khintchine concerning the rapidity of the approximation of almost all numbers by rationals. Zygmund [15] gave two new proofs, one of which was elementary. The other, which was considerably shorter, used properties of Blaschke products.

Since then, Littlewood's result has been generalised in a number of directions. Aikawa [1] and [2] sharpened the result considerably. A discrete analogue was given by Di Biase, [5]. In the last paper [MB4], we present a new proof of Aikawa's result: If the function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\mathcal{A}_h(\theta)$ is a tangential approach region (i.e. $h(t)/t \to \infty$ as $t \to 0^+$), there exists a bounded harmonic function in U which fails to have a boundary limit along $\mathcal{A}_h(\theta)$ for *any* $\theta \in \mathbb{T}$.

For further results on Fatou type theorems and related topics, the book [4] by Di Biase is recommended.

3. Poisson extensions with respect to powers of the Poisson kernel

For $z = x + iy$ define the hyperbolic Laplacian by

$$L_z = \frac{1}{4}(1 - |z|^2)^2(\partial_x^2 + \partial_y^2).$$

Then the λ-Poisson integral

$$u(z) = P_\lambda f(z) = \int_{\mathbb{T}} P(z,\beta)^{\lambda+1/2} f(\beta)\, d\beta, \text{ for } \lambda \in \mathbb{C},$$

defines a solution of the equation

$$L_z u = (\lambda^2 - 1/4)u.$$

In representation theory of the group $SL(2,\mathbb{R})$, one uses the powers $P(z,\beta)^{i\alpha+1/2}$, $\alpha \in \mathbb{R}$, of the Poisson kernel.

From now on we shall deal only with real powers, greater than or equal to 1/2, of the Poisson kernel, i.e. $\lambda \geq 0$.

It is readily checked that

$$P_\lambda 1(z) \sim (1-|z|)^{1/2-\lambda}$$

as $|z| \to 1$ if $\lambda > 0$, and that

$$P_0 1(z) \sim (1-|z|)^{1/2} \log \frac{1}{1-|z|},$$

as $|z| \to 1$. To get boundary convergence we have to normalise P_λ, since $P_\lambda 1(z)$ does not converge to 1. If one considers normalised $\lambda-$Poisson integrals for $\lambda > 0$, i.e. $\mathcal{P}_\lambda f(z) = P_\lambda f(z)/P_\lambda 1(z)$, the convergence properties are the same as for the ordinary Poisson integral. This is because the kernels essentially behave in the same way. However, it turns out that the operator $\mathcal{P}_0$ has unique properties in the context of boundary behaviour of corresponding extensions. A somewhat vague explanation is that this is due to the logarithmic factor in $\mathcal{P}_0$, which is absent in $\mathcal{P}_\lambda$ for $\lambda > 0$.

If $f \in C(\mathbb{T})$ then $\mathcal{P}_0 f(z) \to f(\theta)$ unrestrictedly as $z \to e^{i\theta}$ for all $\theta \in \mathbb{T}$, just as in the case of the Poisson integral itself. This is because $\mathcal{P}_0$ is a convolution operator, behaving like an approximate identity.

Theorem (Sjögren, [11]). *Let $f \in L^1(\mathbb{T})$. For a.e. $\theta \in \mathbb{T}$ one has that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, where $h(t) = O(t \log 1/t)$ as $t \to 0$.*

This result was generalised to L^p, $1 \leq p < \infty$, by Rönning [9]:

Theorem (Rönning, [9]). *Let $1 \leq p < \infty$ be given and let $f \in L^p(\mathbb{T})$. For a.e. $\theta \in \mathbb{T}$ one has that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, where $h(t) = O(t(\log 1/t)^p)$ as $t \to 0$.*

Rönning also proved that Sjögren's result is the best possible, when the approach regions are given by Definition 1 and h is increasing, and that in his own theorem for L^p, the exponent p in $h(t) = O(t(\log 1/t)^p)$ cannot be improved.

The method used to prove these theorems was weak type estimates for the corresponding maximal operators. The continuous functions, for which convergence is known to hold, are dense in L^p, so the results follow by approximation.

The case of $f \in L^\infty$ was (thought to be, see below) a deeper question, basically because the continuous functions do not form a dense subset. However, using a result by Bellow and Jones [3], Sjögren [12] managed to determine the approach regions:

Theorem (Sjögren, [12]). *The following conditions are equivalent for any increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^\infty(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $h(t) = O(t^{1-\varepsilon})$ *as $t \to 0$ for any $\varepsilon > 0$.*

The content of paper [MB1] is the following result for $L^{p,\infty}$ (weak L^p):

Theorem. (Brundin, [MB1]). *Let $1 < p < \infty$ be given. Then the following conditions are equivalent for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^{p,\infty}(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $\sum_{k=0}^{\infty} \sigma_k < \infty$, *where* $\sigma_k = \sup_{2^{-2^k} \leq s \leq 2^{-2^{k-1}}} \frac{h(s)}{s(\log(1/s))^p}$.

Clearly, (*ii*) is slightly stronger than the condition $h(t) = O(t(\log 1/t)^p)$ appearing in Rönning's L^p result. The proof of the $L^{p,\infty}$ result above follows the same lines as Sjögren's proof for L^∞, in the sense that it relies on a "Banach principle for $L^{p,\infty}$" which is established in the paper.

In paper [MB2] we give a new proof for the L^p case, $1 \leq p \leq \infty$. It is considerably shorter and more straightforward than the earlier proofs. Also, the L^∞ case is proved without using the Banach principle. The key observation is that one part of the kernel, which previously was thought to be "hard", actually is more or less trivial. In the last section of paper [MB2], the L^∞ case is generalised to higher dimensions (polydiscs).

Paper [MB3], which contains what should be considered our main results, deals with specific classes of Orlicz spaces. The point is to get an insight in the difference between the approach regions for L^p (finite p) and L^∞ (note that the approach regions for L^p are optimal, whereas no optimal approach region exists for L^∞).

Orlicz spaces generalise L^p spaces. One simply replaces the condition $\int |f|^p < \infty$ by

$$\int \Phi(|f|) < \infty,$$

for some "reasonable" function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. Here Φ should be increasing and convex, and $\Phi(0) = \Phi'(0) = 0$. The space defined depends only on the behaviour of $\Phi(x)$ for large x.

In general, the integral condition defining our Orlicz space does not give a linear space of functions. But with a few modifications, which we omit here, we actually get a linear space.

The first class of Orlicz functions Φ treated is denoted by ∇. It consists basically of functions Φ for which $M(x) = \log(\Phi'(x))$ grows at least polynomially as $x \to \infty$. The precise growth condition imposed is given by

$$\liminf_{x\to\infty} \frac{M(2x)}{M(x)} = m_0 > 1.$$

This implies that Φ itself grows at least exponentially at infinity, i.e. we are in some sense closer to L^∞ than to L^p. A typical example is $\Phi(x) \sim e^x$ for large x.

The other class we consider is denoted Δ. It consists basically of functions Φ whose growth at infinity is controlled above and below by power functions (polynomials). Here, the precise growth condition is given by

$$\frac{x\Phi''(x)}{\Phi'(x)} \sim 1,$$

uniformly for $x > x_0$ (some $x_0 \geq 0$). Δ contains, for example, functions of growth $\Phi(x) \sim x^p(\log(1 + |x|))^\alpha$ at infinity, for any $p > 1$ and $\alpha \geq 0$. The Orlicz spaces related to Δ are closer to L^p than to L^∞.

The following two theorems are proved:

Theorem. (Brundin, [MB3]). *Let $\Phi \in \nabla$ be given. Then the following conditions are equivalent for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any* $f \in L^{\Phi}$ *one has for almost all* $\theta \in \mathbb{T}$ *that* $\mathcal{P}_0 f(z) \to f(\theta)$ *a.e. as* $z \to e^{i\theta}$ *and* $z \in \mathcal{A}_h(\theta)$.

(*ii*) $\dfrac{M\left(C\frac{\log 1/t}{\log g(t)}\right)}{\log g(t)} \to \infty$ *as* $t \to 0$ *for all* $C > 0$, *where* $g(t) = h(t)/t$.

An example would be $\Phi(x) \sim e^{x^\alpha}$, for $\alpha > 0$, i.e. $M(x) \sim x^\alpha$. It is easily seen that here condition (*ii*) is equivalent with

$$\log g(t) = o\left((\log 1/t)^{\alpha/(\alpha+1)}\right),$$

so that, expressed in a somewhat unorthodox way,

$$h(t) = t \exp\left(o\left((\log 1/t)^{\alpha/(\alpha+1)}\right)\right).$$

Clearly, no optimal approach region exists.

Theorem. (Brundin, [MB3]). *Let* $\Phi \in \Delta$ *be given. Then the following conditions are equivalent for any function* $h : \mathbb{R}_+ \to \mathbb{R}_+$*:*

(*i*) *For any* $f \in L^{\Phi}$ *one has for almost all* $\theta \in \mathbb{T}$ *that* $\mathcal{P}_0 f(z) \to f(\theta)$ *a.e. as* $z \to e^{i\theta}$ *and* $z \in \mathcal{A}_h(\theta)$.

(*ii*) $h(t) = O(t\Phi(\log 1/t))$, *as* $t \to 0$.

The natural example here is $\Phi(x) = x^p$, $p > 1$. Condition (*ii*) is then equivalent with Rönning's L^p condition.

The key proposition to prove these results could be thought of as an Orlicz space substitute for Hölder's inequality. It is formulated and proved in [MB3].

It is worth noting that optimal approach regions exist in the case the boundary functions are in L^p, $1 \le p < \infty$, and in L^{Φ}, where $\Phi \in \Delta$. For $L^{p,\infty}$, L^∞ and L^{Φ}, where $\Phi \in \nabla$, the conditions on h for a.e. convergence do not yield an optimal h. Given an admissible approach region, in these cases, one can always find an essentially larger region which is also admissible. Why is there a difference? It is reasonable to believe that the difference has to do with the fact that the "norms" in the latter spaces are *not* given by simple integral conditions.

4. Almost everywhere convergence and maximal operators

In this section we shall discuss the concept of almost everywhere convergence and how it is related to maximal operators.

Let $M = M(\mathbb{T})$ denote the set of Lebesgue measurable functions on $\mathbb{T}$. Assume that we are given a sequence of sublinear operators $S_n : A(\mathbb{T}) \to M$, where $A(\mathbb{T})$ is some normed subspace of $L^1(\mathbb{T})$ (e.g. $A(\mathbb{T}) = L^p(\mathbb{T})$). We say that $S_n f$ converges almost everywhere (w.r.t. Lebesgue measure m) if $S_n f(\theta)$ converges for a.e. $\theta \in \mathbb{T}$. This is equivalent to

$$m\left(E_\lambda\right) = 0,$$

for all $\lambda > 0$, where

$$E_\lambda(f) = \left\{\theta \in \mathbb{T} : \limsup_{n,m\to\infty} |S_n f(\theta) - S_m f(\theta)| > \lambda\right\}.$$

Define

$$S^* f(\theta) = \sup_{n\geq 1} |S_n f(\theta)|$$

and let

$$E^*_\lambda(f) = \{\theta \in \mathbb{T} : (S^* f)(\theta) > \lambda\}.$$

S^* is referred to as a *maximal operator*. Somewhat vaguely, one could say that maximal operators are obtained by replacing limits by suprema of the modulus.

Note that $E_\lambda(f) \subset E^*_{\lambda/2}(f)$. Now, assume that $g \in A(\mathbb{T})$ is some function for which $S_n g \to g$ a.e. as $n \to \infty$. Then $E_\lambda(f) = E_\lambda(f-g) \subset E^*_{\lambda/2}(f-g)$. Thus, it follows that

$$m(E_\lambda(f)) \leq m(E^*_{\lambda/2}(f-g)).$$

We are interested in proving a.e. convergence for all functions $f \in A(\mathbb{T})$, where $A(\mathbb{T})$ is equipped with a norm which we denote by $\|\cdot\|_A$.

In order to deduce that $m\left(E_\lambda(f)\right) = 0$ for all $\lambda > 0$, when $f \in A(\mathbb{T})$, it now suffices to have some weak continuity of $S^* : A(\mathbb{T}) \to M$ at 0, and to be able to approximate any f in the norm $\|\cdot\|_A$ with a "good" function g. We sum up this discussion in a theorem:

Theorem. *Let $A(\mathbb{T}) \subset L^1(\mathbb{T})$ be a function space, equipped with a norm $\|\cdot\|_A$. Assume that the following two conditions hold:*

(*i*) $S^* : A(\mathbb{T}) \to M(\mathbb{T})$ *is weakly continuous at* 0, *i.e.* $m(E^*_\lambda(f)) = C(\lambda)o(1)$ *as* $\|f\|_A \to 0$ *for all* $f \in A(\mathbb{T})$ *and some function* $C : \mathbb{R}_+ \to \mathbb{R}_+$.

(*ii*) *There exists a set* $D(\mathbb{T}) \subset A(\mathbb{T})$, *dense in* $A(\mathbb{T})$, *such that, for all* $g \in D(\mathbb{T})$, $S_n g(\theta) \to g(\theta)$ *a.e. as* $n \to \infty$.

Then, for all $f \in A(\mathbb{T})$, $S_n f(\theta) \to f(\theta)$ *a.e. as* $n \to \infty$.

To be specific, if $A(\mathbb{T}) = L^p(\mathbb{T})$, part ($i$) follows if one for example establishes a weak type (p,p) estimate for S^*. In our case, later on, the continuous (or bounded) functions on $\mathbb{T}$ will serve as the set $D(\mathbb{T})$.

It should be pointed out that our results concern *families* of operators S_t, $t \in (0,1)$, and not sequences. However, the difference is slight and the above reasoning works just as well for families (as $t \to 0$) as for sequences (as $n \to \infty$).

A natural question is what one loses by studying the maximal operator instead of the sequence itself. Remarkably enough, as was proved by Stein [13] and by Nikishin [8], in a multitude of cases one does not lose anything. Continuity of the the maximal operator is quite simply often (without going into any details) equivalent with a.e. convergence.

5. An example

In this section, we prove a result for fractional Poisson extensions of L^p boundary functions in the half space. Thus, the setting but also the methods that we shall use are a bit different from those in the papers [MB1], [MB2] and [MB3]. I acknowledge the help received from Yoshihiro Mizuta, who came up with the idea and a brief sketch of the proof.

Let $P_t(x)$ denote the Poisson kernel in the half space

$$\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n \text{ and } t > 0\},$$

that is

$$P_t(x) = c_n \cdot \frac{t}{(t^2 + |x|^2)^{(n+1)/2}},$$

where c_n is the constant determined by

$$\int_{\mathbb{R}^n} P_t(x)\, dx = 1.$$

We fix an open, nonempty and bounded set $\Omega \subset \mathbb{R}^n$. In the unit disc we consider the square root of the Poisson kernel, but in higher dimensions it is the $\frac{n}{n+1}$:th power of the Poisson kernel that exhibits special properties. Therefore, for $f \in L^p(\mathbb{R}^n)$, let

$$(P_0 f)(x,t) = \int_{\mathbb{R}^n} P_t(x-y)^{\frac{n}{n+1}} f(y)\, dy.$$

We normalise the extension, with respect to Ω, by

$$(\mathcal{P}_0 f)(x,t) = \frac{(P_0 f)(x,t)}{(P_0 \chi_\Omega)(x,t)}.$$

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be given and let $x_0 \in \Omega$. We define the natural approach region at x_0, determined by h, to be

$$\mathcal{A}_h(x_0) = \{(x,t) \in \mathbb{R}^{n+1}_+ \ : \ \sqrt{|x-x_0|^2 + t^2} < h(t)\}.$$

We define

$$A_p(f,r,x) = \left(\frac{1}{r^n} \int_{B(x,r)} |f(y)|^p \, dy \right)^{1/p},$$

and

$$L_f^{(p)}(\Omega) = \{x \in \Omega : A_p(f - f(x), r, x) \to 0 \text{ as } r \to 0\}.$$

Note that, if $f \in L^p(\mathbb{R}^n)$, then $|\Omega \setminus L_f^{(p)}(\Omega)| = 0$ (a.e. point is a Lebesgue point).

Theorem. *Let $1 \leq p < \infty$ be given and assume that $h(t) = O(t(\log 1/t)^{p/n})$ as $t \to 0^+$. Furthermore, let $f \in L^p(\mathbb{R}^n)$ be given. Then, for any $x_0 \in L_f^{(p)}(\Omega)$ (in particular, for a.e. $x_0 \in \Omega$) one has that $(\mathcal{P}_0 f)(x,t) \to f(x_0)$ as $(x,t) \to (x_0, 0)$ along $\mathcal{A}_h(x_0)$.*

Proof. We shall prove the result directly, i.e. without using estimates of maximal operators.

As $(x,t) \to (x_0,0) \in \Omega \times \{0\}$, it is easy to see that

$$(P_0 \chi_\Omega)(x,t) \sim t^{\frac{n}{n+1}} \log 1/t.$$

Now, let $f \in L^p(\mathbb{R}^n)$ and $x_0 \in L_f^{(p)}(\Omega)$ be given. We may, without loss of generality, assume that $f(x_0) = 0$. Furthermore, we assume that $(x,t) \in \mathcal{A}_h(x_0)$. For short, let $r = \sqrt{|x-x_0|^2 + t^2}$. We write

$$
\begin{aligned}
(P_0 f)(x,t) &= \int_{B(x_0,2r)} P_t(x-y)^{\frac{n}{n+1}} f(y)\,dy \\
&\quad + \int_{B(x_0,2r)^c} P_t(x-y)^{\frac{n}{n+1}} f(y)\,dy \\
&= I_1(x,t) + I_2(x,t).
\end{aligned}
$$

By using Hölder's inequality, we obtain

$$
\begin{aligned}
|I_1(x,t)| &\lesssim t^{\frac{n}{n+1}} \left(\int_{|y-x_0|<2r} \frac{dy}{(t+|x-y|)^{nq}} \right)^{1/q} \cdot \left(\int_{|y-x_0|<2r} |f(y)|^p\,dy \right)^{1/p} \\
&\lesssim r^{n/p} \cdot t^{\frac{n}{n+1}} \left(\int_{|x-y|<3r} \frac{dy}{(t+|x-y|)^{nq}} \right)^{1/q} \cdot A_p(f,2r,x_0) \\
&\lesssim (r/t)^{n/p} \cdot t^{\frac{n}{n+1}} \cdot A_p(f,2r,x_0).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|I_2(x,t)| &\lesssim t^{\frac{n}{n+1}} \sum_{k=1}^{\infty} \int_{B(x_0,2^{k+1}r)\setminus B(x_0,2^k r)} \frac{1}{(t+|x_0-y|)^n} |f(y)|\,dy \\
&\lesssim t^{\frac{n}{n+1}} \sum_{k=1}^{\infty} (2^k r)^{-n} \int_{B(x_0,2^{k+1}r)} |f(y)|\,dy \\
&\lesssim t^{\frac{n}{n+1}} \sum_{k=1}^{\infty} A_1(f,2^{k+1}r,x_0).
\end{aligned}
$$

We now note that

$$
\begin{aligned}
A_1(f,2^{k+1}r,x_0) &\lesssim \frac{1}{2^k r} \int_{2^{k+1}r}^{2^{k+2}r} A_1(f,s,x_0)\,ds \\
&\lesssim \int_{2^{k+1}r}^{2^{k+2}r} \frac{A_1(f,s,x_0)}{s}\,ds.
\end{aligned}
$$

Invoking this in the estimate above, we obtain

$$\begin{aligned} |I_2(x,t)| &\lesssim t^{\frac{n}{n+1}} \sum_{k=1}^{\infty} \int_{2^{k+1}r}^{2^{k+2}r} \frac{A_1(f,s,x_0)}{s}\,ds \\ &\lesssim t^{\frac{n}{n+1}} \int_{r}^{\infty} \frac{A_1(f,s,x_0)}{s}\,ds \\ &\lesssim t^{\frac{n}{n+1}} \int_{t}^{\infty} \frac{A_1(f,s,x_0)}{s}\,ds. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} |(\mathcal{P}_0 f)(x,t)| &\lesssim \frac{1}{t^{n/(n+1)}\log 1/t}(|I_1(x,t)| + |I_2(x,t)|) \\ &\lesssim \frac{1}{\log 1/t} \cdot [(r/t)^{n/p} \cdot A_p(f,2r,x_0) + \int_t^{\infty} \frac{A_1(f,s,x_0)}{s}\,ds]. \end{aligned}$$

Now, using the fact that $r < h(t) \lesssim t(\log 1/t)^{p/n}$, we get

$$|(\mathcal{P}_0 f)(x,t)| \lesssim A_p(f,2r,x_0) + \frac{1}{\log 1/t}\int_t^{\infty} s^{-1} A_1(f,s,x_0)\,ds.$$

It is clear that

$$\int_t^{\infty} \frac{A_1(f,s,x_0)}{s}\,ds$$

is a convergent integral, since

$$\begin{aligned} \frac{A_1(f,s,x_0)}{s} &\lesssim s^{-1}s^{-n}s^{n/q}\|f\|_p \\ &\lesssim s^{-1-n/p}\|f\|_p, \end{aligned}$$

by Hölder's inequality.

Now, as $t \to 0$ we also have $r \to 0$. Since $f(x_0) = 0$ and since we have assumed that $x_0 \in L_f^{(p)}(\Omega)$ (and thus that $x_0 \in L_f^{(1)}(\Omega)$), it follows that

$$(\mathcal{P}_0 f)(x,t) \to 0 = f(x_0),$$

as $(x,t) \to (x_0, 0)$ along $\mathcal{A}_h(x_0)$. This concludes the proof. $\square$

6. Open questions

6.1. **The unit disc.** A more complete picture of the convergence results for the "square root operator" in the unit disc would be desirable. The best one could hope for is a unified convergence theorem, for all function spaces (of some particular but general kind), where the convergence condition is given in terms of the norm on the space. This is probably a very hard problem, and most likely even impossible. However, more partial results would be interesting in their own right, to complete the picture. For instance, results for $\mathrm{BMO}(\mathbb{T})$ and for classes of Orlicz spaces, between ∇ and Δ, would be interesting. A typical example is given by the function $\Phi(x) \sim e^{(\log x)^p}$, where $p > 1$. Attempts have been made to characterise the approach regions for spaces related to such functions, but without success.

6.2. **Higher dimensions.** Results for polydiscs have been obtained by both Sjögren and Rönning, for L^p boundary functions. A natural questions is what happens for Orlicz spaces, weak L^p and so on. The results are of "restricted convergence" type, i.e. the speed with which one approaches the boundary should be approximately the same in all the discs. Whether or not this is necessary is not known. The Russian mathematicians Katkovskaya and Krotov claim that they have proved that a certain maximal operator is of strong type (p, p), which immediately would yield unrestricted convergence. However, the result has not been published. Another natural generalisation is to replace the unit disc with a symmetric space. Results have been obtained for rank 1 spaces, but higher rank generalisations are still a relatively unexplored field.

6.3. **Littlewood type theorems.** It would be nice to replace the negative results "not a.e. convergence" with "everywhere divergence". This is done in the paper [MB4] for the ordinary Poisson integral and bounded boundary functions. An attempt was made to transfer the same machinery to the square root case, but it failed. In this sense, the normalised square root operator behaves completely differently from the ordinary Poisson integral. A new approach is necessary.

6.4. **Weakly regular boundary functions.** One could increase the regularity of the boundary functions (e.g. by transforming L^p in some suitable way) and sharpen the convergence. The natural thing here is to replace Lebesgue measure with some capacity or Hausdorff measure, and obtain corresponding quasi everywhere results, which are stronger than almost everywhere.

References

[1] Hiroaki Aikawa. Harmonic functions having no tangential limits. *Proc. Amer. Math. Soc.*, 108(2):457–464, 1990.

[2] Hiroaki Aikawa. Harmonic functions and Green potentials having no tangential limits. *J. London Math. Soc. (2)*, 43(1):125–136, 1991.

[3] Alexandra Bellow and Roger L. Jones. A Banach principle for L^∞. *Adv. Math.*, 120(1):155–172, 1996.

[4] Fausto Di Biase. *Fatou type theorems: Maximal functions and approach regions.* Birkhäuser Boston Inc., Boston, MA, 1998.

[5] Fausto Di Biase. Tangential curves and Fatou's theorem on trees. *J. London Math. Soc. (2)*, 58(2):331–341, 1998.

[6] P. Fatou. Séries trigonométriques et séries de Taylor. *Acta Math. 30*, 1906.

[7] J.E. Littlewood. On a theorem of Fatou. *J. London Math. Soc. 2*, 1927.

[8] E.M. Nikishin. Resonance theorems and superlinear operators. *Russ. Math. Surveys*, 25:125–187, 1970.

[9] Jan-Olav Rönning. Convergence results for the square root of the Poisson kernel. *Math. Scand.*, 81(2):219–235, 1997.

[10] H.A. Schwarz. Zur Integration der partiellen Differentialgleichung $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. *J. Reine Angew. Math. 74*, 1872.

[11] Peter Sjögren. Une remarque sur la convergence des fonctions propres du laplacien à valeur propre critique. In *Théorie du potentiel (Orsay, 1983)*, pages 544–548. Springer, Berlin, 1984.

[12] Peter Sjögren. Approach regions for the square root of the Poisson kernel and bounded functions. *Bull. Austral. Math. Soc.*, 55(3):521–527, 1997.

[13] E. M. Stein. On limits of seqences of operators. *Ann. of Math. (2)*, 74:140–170, 1961.

[14] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[15] A. Zygmund. On a theorem of Littlewood. *Summa Brasil. Math.*, 2(5):51–57, 1949.

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