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Aspects of M-theory

supergravity, duality and noncommutativity

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Abstract

In this thesis we discuss various aspects of string/M-theory. After an introduction to some aspects of string/M-theory (perturbative string theory, T-duality, S-duality, M-theory, etc.), we discuss supergravity solutions corresponding to bound states of branes and noncommutative theories which are obtained as limits of string/M-theory. So called open brane theories are investigated, especially a six-dimensional non-gravitational theory containing light open membranes, called OM-theory. Decoupling limits for the different theories are derived using open brane data (open brane metric etc.). Moreover, the open membrane metric and generalized noncommutativity parameter are derived using a new method. After the main text there are seven appended research papers. In Paper I renormalization group flows in three and six dimensions are investigated using the AdS/CFT correspondence. Furthermore, in Papers II-VI non-gravitational theories with noncommutativity are investigated using supergravity duals. One result obtained is the notion of deformation independence and how this can be used to obtain open brane metrics and generalized noncommutativity parameters. Duality relations between various theories are also obtained. Finally, in Paper VII we discuss a new solution generating technique, which is used to generate supergravity solutions corresponding to M5-M2 and M5-M2-M2-MW bound states in 11 dimensions.

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This thesis consists of an introductory text and the following seven appended research papers, henceforth referred to as Paper I-VII:

I. V.L. Campos, G. Ferretti, H. Larsson, D. Martelli and B.E.W. Nilsson, "A study of holographic renormalization group flows in d=6 and d=3", JHEP **0006** (2000) 023, [hep-th/0003151].

II. D.S. Berman, V.L. Campos, M. Cederwall, U. Gran, H. Larsson, M. Nielsen, B.E.W. Nilsson and P. Sundell, "Holographic noncommutativity", JHEP **0105** (2001) 002, [hep-th/0011282].

III. H. Larsson and P. Sundell, "Open string/Open D–brane dualities: old and new", JHEP **0106** (2001) 008, [hep-th/0103188].

IV. H. Larsson, "A note on half-supersymmetric bound states in M-theory and type IIA string theory", Class. Quant. Grav. **19** (2002) 2689, [hep-th/0105083].

V. D.S. Berman, M. Cederwall, U. Gran, H. Larsson, M. Nielsen, B.E.W. Nilsson and P. Sundell, "Deformation independent open brane metrics and generalized theta parameters", JHEP **0202** (2002) 012, [hep-th/0109107].

VI. H. Larsson, "Light-like noncommutativity and duality from open strings/branes", JHEP **0203** (2002) 037, [hep-th/0201178].

VII. H. Larsson, "An M-theory solution generating technique and SL(2,$\mathbb{R}$)", JHEP **0302** (2003) 060, [hep-th/0212107].

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1

Introduction

This thesis deals with various aspects of string/M-theory, which is currently the main candidate for a unified theory of all forces and particles in the Universe. In string theory the fundamental particles (fermions) and the force mediating particles (bosons) are described as different kinds of vibration of a fundamental string. This is analogous to a piano string, where different notes correspond to different vibrational modes of the string. Pursuing this analogy further [1], we could say that the basic particles in the universe correspond to the musical notes of the superstring, and the laws of physics correspond to the harmonies that these notes obey. Finally, the universe itself corresponds to a symphony of superstrings (and perhaps also higher-dimensional objects).

String theory has now been around for more than 30 years. The history of string theory started at the end of the 1960's with bosonic string theory, which was first invented as a model to describe strong interactions in hadron physics. This, unfortunately, did not work. The main reasons for this were that it was found that bosonic string theory has a spin 2 particle in its spectrum and that it can only be a consistent theory in 26 dimensions. It was also found that QCD gave a better description of the strong force. However, this did not mean that string theory was entirely forgotten. The reason for this was that since having a spin 2 particle in the spectrum, it might be a candidate for a quantum theory of gravity. However, bosonic string theory can not be the correct quantum theory of gravity, because it has a tachyon in its spectrum and no fermions. Instead, a supersymmetric string theory might be a consistent quantum theory of gravity and of the other three fundamental forces of nature as well, since, as we will see below, a supersymmetric string theory has both bosons and fermions in the spectrum but no tachyon. That superstring theory is supersymmetric means that it has the same amount of bosonic (integer spin particles) and fermionic (half integer spin particles) degrees of freedom.

In the beginning of the 1970's superstring theories were constructed, which contain both bosons and fermions and no tachyons and are consistent theories in 10 space-time dimensions. Unfortunately, it was at this point not clear if there existed anomaly free superstring theories. In fact, it was not until 1984, starting the so called 'first superstring revolution', that Green and Schwarz showed that a special type of superstring theory, called the type I theory, is free from anomalies if the gauge group is uniquely $SO(32)$ [2, 3]. More precisely, in [2] they showed that supersymmetric Yang-Mills theory coupled to N=1 10-dimensional supergravity is anomaly free if the gauge group is SO(32) or $E_8 \times E_8$, where the former is the low energy limit of type I superstring theory, and the latter the low energy limit of heterotic string theory, which was later constructed in 1985 [4]. Note that there also exists a heterotic string theory with gauge group SO(32) [4]. Next, in [3], it was further shown that the full type I superstring theory is anomaly free.

Superstring theory is a theory in 10 space-time dimensions. However, we know that we live in four (large) dimensions, three space and one time dimensions, and not 10. Then a relevant question is, how can string theory be a serious candidate for a unified theory of 'everything', when it seems that it exists in the wrong number of dimensions? The answer to this question is that it does not necessarily mean that all these 10 dimensions are large. For instance, six of them could be curled up so that they are small enough to be invisible at 'our' length-scales[1]. Moreover, the idea is that some superstring theory compactified on a six-dimensional compact space might be a candidate as a description of our universe, since this effectively gives a theory in four dimensions. At present it is not known how to choose the correct six-dimensional compact space. This is a very important problem to solve.

Later, in 1994 the 'second superstring revolution' began with the important papers by Hull and Townsend [5] and Witten [6]. For example, it was conjectured that the five superstring theories are all connected to each other and a previously unknown 11-dimensional theory called M-theory, through various dualities (S- and T-dualities). This implies that we no longer have five inequivalent string theories but instead there should only exist one unique theory (M-theory), which in different limits gives the various string theories. Not much is known about this 11-dimensional theory except that it is not a string theory but a theory containing two-dimensional membranes. Moreover, at about the same time so called D-branes were discovered by Polchinski [7][2]. These are non-perturbative extended objects, which are very important when, e.g., investigating duality conjectures, see chapter 2.2. Furthermore, at the end of 1997 Maldacena conjectured [9] that superstring theories in certain backgrounds are dual (equivalent) to superconformal field theories, which led to many interesting results. Around the same time it was also found that taking various limits of string theory with certain non-zero constant background fields turned on, leads to field theories with noncommutative coordinates, see chapter four and, e.g., [10, 11, 12].

In this thesis we are mostly interested in two different but related areas, (1) dual string theory (or supergravity) descriptions (using the AdS/CFT correspondence [9]) of field theories, or open brane theories, with noncommutativity, (2) how to use dualities in order to deform supergravity solutions, which correspond to bound states of branes. In the first case, we are in particular interested in a six-dimensional theory containing light open

[1] The idea that there might exist extra dimensions is old and was first proposed by Kaluza and Klein around 1920 as a way to unify gravity and electro-magnetism.

[2] Note that the concept of D-branes was first discussed much earlier in [8]. However, it was not until [7] that D-branes were really shown to be the objects that have Ramond-Ramond (RR) charges.

membranes, which is called OM-theory. One reason why this theory is considered to be interesting is because further knowledge of this non-gravitational open membrane theory might in the future lead to a better picture of the role of the closed membrane in M-theory. At present this is far from being understood. Moreover, OM-theory seems to live in a new interesting geometry which exhibits some kind of generalized noncommutativity structure, which also makes a further study interesting. In the second case we are in particular interested in deformations of the M-theory M5-brane, using M-theory 'T-duality'. For example, in chapter 4.3 we discuss an intriguing connection between M-theory 'T-duality' and the open membrane metric and generalized noncommutativity parameter (which are relevant for OM-theory).

This thesis is divided into two parts. The first one contains five chapters. The aim of this part is to give an introduction to various aspects of string/M-theory in order to facilitate the reading of the second part, which consists of seven appended research papers. Of these, six are published in Journal of High Energy Physics (JHEP), while one is published in Classical and Quantum Gravity.

In chapter two we give an introduction to perturbative and non-perturbative string/M-theory. This is followed by chapter three where we introduce bound states and describe how supergravity solutions corresponding to bound states can be generated using dualities. Next, in chapter four it is shown how string/M-theory together with non-zero constant background fields, in various low energy limits, lead to non-gravitational theories. Some of these theories contain light open branes. The concept of deformation independence is also introduced, both for open strings and open membranes. We end with a discussion and some conclusions in chapter five.

As we have mentioned above, the first part of this thesis is an introduction, in order to facilitate the reading of appended papers. Note, however, that there are some new results in this introductory text, which were obtained after the papers were written. These results, which are discussed in section 3.2, 3.3, 4.2.2, 4.3.1 and 4.3.3, mostly concern Papers V and VII.

Note that although we in this thesis discuss various aspects of M-theory, it is by no means a complete description of the present knowledge of string/M-theory. There are many important and interesting areas that we have chosen to ignore altogether. For example, in the last few years many interesting papers have been produced concerning tachyon condensation and string field theory. For a review see, e.g., [13]. Other areas that we choose to leave out are, e.g., string cosmology, PP-waves and compactifications of string/M-theory.

Concerning the papers, we recommend the reader to read chapter two in the introductory text before reading any of the papers and also reading chapter three and four before reading Papers II-VII in sequence. However, most of Paper IV can be understood without reading chapter four of the introductory text.

2

Introduction to String/M-theory

In this chapter we will give an introduction to perturbative and non-perturbative string theory. For a more complete introduction to the many different aspects of string/M-theory we recommend the excellent books by Green, Schwarz and Witten [14], Polchinski [15] and Johnson [16].

We begin in the first section with an introduction to perturbative string theory, both the bosonic and the supersymmetric theory including a discussion of T-duality. There is also a brief account of supergravity in 10 and 11 dimensions. We continue in section 2.2 by discussing various important non-perturbative (S- and U-duality) dualities and introduce M-theory and branes. We conclude in section 2.3, with an introduction to the AdS/CFT correspondence.

2.1 Perturbative string theory

2.1.1 Bosonic string theory

Here, we will give an elementary introduction to bosonic string theory. This introduction is incomplete, e.g., we have entirely ignored to introduce string interactions. For this and other omissions we refer to the standard text books [14, 15, 16].

We begin by describing a classical string propagating in a d-dimensional manifold M. The two-dimensional world volume (world sheet) Σ of the propagating string is embedded in the target space M through the map $X : \Sigma \to M$. We note that an open string is represented by a world sheet which has boundaries, while the world sheet of a closed string has no boundaries.

For a classical string there are two equivalent actions, the Nambu and Goto action [17] and the Brink, Di Vecchia, Howe, Deser and Zumino action (BDHDZ) [18, 19]. The second action is sometimes called the Polyakov action [20]. The Nambu-Goto action is given by the area swept out by the world sheet

$$S_{\rm NG}[X^\mu] = -T \int_\Sigma dA = -T \int_\Sigma d^2\sigma \sqrt{-{\rm det} h_{ab}} = -T \int_\Sigma d^2\sigma \sqrt{-{\rm det}(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu})} \; , \quad (2.1)$$

where $h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$ is the induced metric on the world sheet, $T = (2\pi\alpha')^{-1}$ is the tension of the string and σ^a, $(a, b = 0, 1)$ are the world sheet coordinates, while X^μ $(\mu, \nu = 0, 1, \ldots, d-1)$ are the target space coordinates.

The second action, which is equivalent to the first, is given by

$$S_{\rm BDHDZ}[X^\mu, \gamma_{ab}] = -\frac{T}{2} \int_\Sigma d^2\sigma \sqrt{-\gamma}\gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = -\frac{T}{2} \int_\Sigma d^2\sigma \sqrt{-\gamma}\gamma^{ab} h_{ab} \; , \qquad (2.2)$$

where γ_{ab} is the auxiliary world sheet metric and $\gamma = {\rm det}\gamma_{ab}$.

In order to show that the two actions are classically equivalent we start by obtaining the algebraic equation of motion for the auxiliary world sheet metric γ_{ab}. Varying the action with respect of γ_{ab}, gives

$$\delta_\gamma S_{\rm BDHDZ}[X^\mu, \gamma_{ab}] = -\frac{T}{2} \int_\Sigma d^2\sigma \sqrt{-\gamma}\delta_\gamma \gamma^{ab} \Big(h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd} h_{cd} \Big) \; , \qquad (2.3)$$

where we have used that $\delta_\gamma \gamma = \gamma\gamma^{ab}\delta_\gamma \gamma_{ab} = -\gamma\gamma_{ab}\delta_\gamma\gamma^{ab}$. Demanding that the variation of the action is zero, implies that

$$h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd} h_{cd} = 0 \; , \qquad (2.4)$$

which in turn gives that

$$\gamma^{ab} h_{ab} = 2\frac{\sqrt{-h}}{\sqrt{-\gamma}} \; . \qquad (2.5)$$

Inserting this relation in the BDHDZ action (2.2) we easily obtain the Nambu-Goto action. Hence, the two actions are classically equivalent.

The BDHDZ action is invariant under the following symmetry transformations:

(1) target space Poincare transformations

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + A^\mu \; , \qquad (2.6)$$

where A^μ is a constant d-dimensional vector and $\Lambda^\mu{}_\nu$ is an SO$(1, d)$ Lorentz matrix.

(2) world sheet reparametrisations

$$\delta X^\mu = \xi^a \partial_a X^\mu \; , \qquad (2.7)$$

and

$$\delta\gamma^{ab} = \xi^c \partial_c \gamma^{ab} - \partial_c \xi^a \gamma^{cb} - \partial_c \xi^b \gamma^{ac} \; , \qquad (2.8)$$

with parameters $\xi^a(\sigma^0, \sigma^1)$.

(3) Weyl transformations

$$\gamma_{ab} \rightarrow \gamma'_{ab} = e^{2\varphi}\gamma_{ab} \ , \tag{2.9}$$

where φ is a function of σ^0 and σ^1. It is easy to see that the BDHDZ action is invariant under this transformation, since $\sqrt{-\gamma}\gamma^{ab}$ is invariant under Weyl transformations.

One very interesting consequence of the symmetries (2) and (3), is that locally on Σ, the three components of γ_{ab} can be completely specified (two from using reparametrisation invariance and the third from using Weyl invariance). Usually, one chooses $\gamma_{ab} = \eta_{ab}$. We note that this is something which is unique for string actions and does not exist for, e.g., membrane actions[1]. Choosing $\gamma_{ab} = \eta_{ab}$ we obtain the following action:

$$S_{\text{BDHDZ}}[X^\mu] = -\frac{T}{2}\int_\Sigma d^2\sigma\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \ . \tag{2.10}$$

From this action the equations of motion for the fields X^μ are easily obtained, and given by the wave equation

$$\eta^{ab}\partial_a\partial_b X^\mu(\tau,\sigma) = 0 \ , \tag{2.11}$$

where $\tau = \sigma^0$ and $\sigma = \sigma^1$. Any solution to this wave equation can be written as a combination of a left traveling wave and a right traveling wave. Hence,

$$X^\mu(\tau,\sigma) = X^\mu_{\text{L}}(\tau+\sigma) + X^\mu_{\text{R}}(\tau-\sigma) \ . \tag{2.12}$$

For closed strings the X^μ fields obey periodic boundary conditions

$$\begin{aligned} X^\mu(\tau,0) &= X^\mu(\tau,\pi) \ , \\ \partial_\sigma X^\mu(\tau,0) &= \partial_\sigma X^\mu(\tau,\pi) \ , \end{aligned} \tag{2.13}$$

while for open strings the X^μ fields instead obey Neumann boundary conditions

$$\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,\pi) = 0 \ . \tag{2.14}$$

Open strings might also obey Dirichlet boundary conditions ($X^\mu(\tau,0) = X^\mu(\tau,\pi) = b^\mu$) in some directions. This case will be discussed more in subsection 2.1.4.

Next, for the closed string a general solution (Taylor expanded) can be written as (remember that $X^\mu(\tau,\sigma) = X^\mu_{\text{L}}(\tau+\sigma) + X^\mu_{\text{R}}(\tau-\sigma)$)

$$\begin{aligned} X^\mu_{\text{R}}(\sigma^-) &= \frac{1}{2}q^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-2in\sigma^-} \ , \\ X^\mu_{\text{L}}(\sigma^+) &= \frac{1}{2}q^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-2in\sigma^+} \ , \end{aligned} \tag{2.15}$$

where q^μ and p^μ are the center of mass position and momentum, $\sigma^\pm = \tau \pm \sigma$, and α^μ_n are Fourier components. Since X^μ is real we require that q^μ and p^μ are real and that $\alpha^\mu_{-m} = (\alpha^\mu_m)^\dagger$ and $\tilde{\alpha}^\mu_{-m} = (\tilde{\alpha}^\mu_m)^\dagger$.

[1] Nor on higher genus surfaces in string theory.

For open strings we instead have the following general solution

$$X^\mu(\tau,\sigma) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos n\sigma\ . \tag{2.16}$$

Note that for the closed string there are both left and right moving oscillators, while for the open string there is only one type of oscillators.

The fields X^μ, together with the conjugate momentum $\Pi^\mu = T\partial_\tau X^\mu$, obey the following equal time Poisson brackets [14]:

$$\begin{aligned}\{X^\mu(\sigma),\Pi^\nu(\sigma')\} &= -\eta^{\mu\nu}\delta(\sigma-\sigma')\ ,\\ \{X^\mu(\sigma),X^\nu(\sigma')\} &= \{\Pi^\mu(\sigma),\Pi^\nu(\sigma')\} = 0\ .\end{aligned} \tag{2.17}$$

This implies that the oscillators have the following Poisson brackets:

$$\{\alpha_m^\mu,\alpha_n^\mu\} = im\eta^{\mu\nu}\delta_{m+n} \quad \{q^\mu,p^\nu\} = -\eta^{\mu\nu}\ . \tag{2.18}$$

The Poisson brackets for $\tilde{\alpha}_m^\mu$ is the same as above, while the Poisson brackets for one α_m^μ with one $\tilde{\alpha}_n^\nu$ is obviously zero. Note also that for the open string the zero oscillator (zero mode) is $\alpha_0^\mu = p^\mu\sqrt{2\alpha'}$, and $\alpha_0^\mu = p^\mu\sqrt{\frac{\alpha'}{2}}$ for the closed string, which is identified from (2.15) and (2.16).

Before we quantize the theory we turn to the constraints of the theory. Varying the action with respect to the world sheet metric should give zero, since this gives the equation of motion for γ_{ab}. This implies that the energy momentum tensor must be zero, because [14]

$$T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\mathrm{P}}}{\delta\gamma^{ab}}\bigg|_{\gamma=\eta}\ . \tag{2.19}$$

A short calculation leads to the following constraints

$$\begin{aligned}T_{++} &= \frac{1}{2}(T_{\tau\tau}+T_{\tau\sigma}) = \partial_+X^\mu\partial_+X^\nu\eta_{\mu\nu} = (\partial_\tau X_{\mathrm{L}}^\mu)^2 = 0\ ,\\ T_{--} &= \frac{1}{2}(T_{\tau\tau}-T_{\tau\sigma}) = \partial_-X^\mu\partial_-X^\nu\eta_{\mu\nu} = (\partial_\tau X_{\mathrm{R}}^\mu)^2 = 0\ ,\end{aligned} \tag{2.20}$$

where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. We also obtain that $T_{+-} = T_{-+} = 0$, because the energy momentum tensor is traceless (which should be obvious from (2.3)). Furthermore, the energy momentum tensor can be expanded in a Fourier series, with the following Fourier coefficients [14]:

$$L_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma}T_{--}d\sigma = \frac{1}{2}\sum_{-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n\ ,\quad \tilde{L}_m = \frac{T}{2}\int_0^\pi e^{2im\sigma}T_{++}d\sigma = \frac{1}{2}\sum_{-\infty}^{\infty}\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n\ , \tag{2.21}$$

for closed strings, and [14]

$$L_m = T\int_0^\pi (e^{-im\sigma}T_{--} + e^{im\sigma}T_{++})d\sigma = \frac{1}{2}\sum_{-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n\ , \tag{2.22}$$

for open strings. Here we have defined $\alpha_{m-n} \cdot \alpha_n = \alpha^\mu_{m-n}\alpha^\nu_n \eta_{\mu\nu}$. Next, using (2.21) and (2.18) we obtain the following Poisson brackets:

$$\{L_m, L_n\} = i(m-n)L_{(m+n)} \quad \{\tilde{L}_m, \tilde{L}_n\} = i(m-n)\tilde{L}_{(m+n)} \ , \quad \{L_m, \tilde{L}_n\} = 0 \ . \qquad (2.23)$$

This means that L_m satisfies the Witt algebra. From the constraint equation (2.20) we see that $L_m = 0$ and $\tilde{L}_m = 0$ for all values of m. In particular, this implies that $L_0 = 0$, i.e., for an open string

$$L_0 = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n = \alpha' p^\mu p^\nu \eta_{\mu\nu} + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = -\alpha' M^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = 0 \ . \qquad (2.24)$$

(Note that for an open string the Hamiltonian is $H = L_0$). Hence, for an open string the mass-squared for open string states is given by

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \ . \qquad (2.25)$$

Quantization

Next, we are going to quantize the string using the old covariant approach. For an introduction to light-cone quantization and the more modern BRST quantization we refer to [14, 15, 16], where also a more thorough introduction to covariant quantization can be found, for both open and closed strings.

To quantize the string we turn the X^μ etc into operators, which have non-trivial commutators given by letting the Poisson brackets (i.e., (2.17), (2.18) and (2.23)) become commutators, i.e., $\{A, B\} \to -i[A, B]$. This leads to the following commutation relations

$$[X^\mu(\sigma), \Pi^\nu(\sigma')] = i\eta^{\mu\nu}\delta(\sigma - \sigma') \ , \quad [X^\mu(\sigma), X^\nu(\sigma')] = [\Pi^\mu(\sigma), \Pi^\nu(\sigma')] = 0 \ , \qquad (2.26)$$

or equivalently

$$[\alpha^\mu_m, \alpha^\mu_n] = m\eta^{\mu\nu}\delta_{(m+n)} \quad [q^\mu, p^\nu] = i\eta^{\mu\nu} \ , \qquad (2.27)$$

with similar commutators for $\tilde{\alpha}^\mu_m$. Comparing with ordinary quantum mechanics it is quite clear that $\frac{1}{\sqrt{m}}\alpha^\mu_{-m}$ and $\frac{1}{\sqrt{m}}\alpha^\mu_m$ $(m > 0)$, respectively, behave as creation and annihilation operators.

In our Fock space we define the state $|0; k >= e^{ik\cdot q}|0>$, to be an eigenstate of the center of momentum operator p^μ with momentum k^μ. Note that $|0>$ is the ground state, which is annihilated by both p^μ and α^μ_m, $m > 0$. Note that also $|0; k>$ is annihilated by α^μ_m for $m > 0$. Using α^μ_{-m} any number of times gives the entire Fock space. This is, however, not the physical Fock space, because we have, so far, not included the quantum version of the classical constraints. The quantum constraints on a physical state are [14]

$$\begin{aligned} L_m|\text{phys}> &= 0 \ , \quad m > 0 \ , \\ (L_0 - a)|\text{phys}> &= 0 \ , \end{aligned} \qquad (2.28)$$

with similar constraints for $\tilde{L}_m$. For the constraint including L_0 we had to introduce a constant a because of an ambiguity in the normal ordering.

For the closed string the right and left moving parts can be treated separately except for the level matching condition $(L_0 - \tilde{L}_0)|\text{phys}>= 0$. The physical motivation of this constraint is that the combination $L_0 - \tilde{L}_0$ generates translations in σ. However, the physics should be invariant under σ translation, since there is no physical information in how the string is parametrized.

The L_m obeys the following Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{(m+n)} + \frac{c}{12}m(m^2-1)\delta_{m+n} \ . \tag{2.29}$$

Here the central charge is $c = d$.

So far, we have let the dimension d of space-time be arbitrary. However, it can be shown (see, e.g., [14, 15, 16]) that in order for negative norm states (coming from the fact that $\eta^{00} = -1$) to decouple from the physical spectrum, we must restrict the dimension to be $d = 26$ and $a = 1$, or $d < 26$ and $a < 1$. Here we choose to set $d = 26$ and $a = 1$ unless otherwise specified, because this is the only case where there are massless states in the spectrum. It is also only this case which is equivalent with light-cone quantization [15]. Moreover, for $d < 26$ and $a < 1$ there is no known consistent way to introduce interactions in the theory [15].

This leads to the following mass shell constraint ($a = 1$):

$$M^2 = \frac{1}{\alpha'}\Big(\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n - 1\Big) = \frac{1}{\alpha'}(N-1) \ , \tag{2.30}$$

for the open string, and

$$M^2 = \frac{2}{\alpha'}\Big(\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n - 2\Big) = \frac{2}{\alpha'}(N + \tilde{N} - 2) \ , \tag{2.31}$$

for the closed string, where $N = \tilde{N}$ due to the level matching condition.

Next, we investigate the spectrum of the closed string. The ground state is given by $N = \tilde{N} = 0$, i.e., by $|0;k>$. Hence, the mass squared is $M^2 = -\frac{4}{\alpha'}$, which means that the ground state is a tachyon and the theory is unstable. This is very disturbing and possibly implies that bosonic string theory is not a consistent theory. However, recently the tachyon has been interpreted as a signal that we are in the wrong vacuum of the theory [21], which means that bosonic string theory still might be a consistent theory. The bosonic string theory is also interesting as a toy model, since the full superstring theory, as we will see in the next subsection, is a consistent theory with no tachyon in its spectrum.

The next level is the massless level and contains a graviton $g_{\mu\nu}$, a dilaton ϕ and an antisymmetric two form $B_{\mu\nu}$. These are obtained by acting on the ground state $|0;k>$ in the following way:

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;k> \ , \tag{2.32}$$

where $i, j = 1, \ldots, 24$ and obviously $N = \tilde{N} = 1$. This gives a total of 24^2 massless states. From the symmetric and traceless part we obtain the graviton $g_{\mu\nu}$, while the trace gives a dilaton ϕ and the antisymmetric part gives an antisymmetric two form $B_{\mu\nu}$. Note that in (2.32) we have used creation operators which only have transverse indices, since this gives the correct physical spectrum. This is clearly an effect of the quantum constraints (2.28), and that this is true is obvious if one choose to quantize the string in light-cone gauge, see, e.g., [14, 15, 16]. Next, continuing by acting with more creation operators will give massive string states.

We see from the above investigation that bosonic string theory contains a graviton. Hence, it automatically includes gravity. We note that only closed bosonic string theory and not open string theory has a graviton in its spectrum, because in closed string theory we act with both α^i_{-1} and $\tilde{\alpha}^j_{-1}$ on $|0; k>$, while in open string theory we only act with $\xi \cdot \alpha_{-1}$ (where ξ^μ is a polarization vector), which gives a massless vector and not a graviton. In open string theory there is obviously also a tachyonic ground state.

Background fields

Next, we include non-trivial background fields. To be more specific: so far we have had a flat target space. However, it is also interesting to investigate what happens if we let the string propagate in a curved target space, as well as including other non-trivial background fields. For the closed bosonic string we found that the massless fields are given by a metric $g_{\mu\nu}$, an antisymmetric two form B and a dilaton ϕ. It is therefore interesting to couple these background fields to the closed bosonic string. This is achieved by the following non-linear sigma model [22]:

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\Big(\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) + \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \\ &\quad -\alpha'\sqrt{-\gamma}R^{(2)}\phi(X)\Big) , \end{aligned} \tag{2.33}$$

where $R^{(2)}$ is the two-dimensional Ricci scalar computed from the world sheet metric γ_{ab}. Note that the last term is higher order in α'.

At the classical level the first two terms of the action (2.33) are obviously Weyl (conformally) invariant, while the third is not. However, although the action is not classically Weyl invariant it is still possible to make it invariant at the quantum level. This is obtained by demanding that the three beta functions (functionals) for the three background fields are zero. Note that from the two-dimensional world sheet point of view of, the background fields can be seen as coupling constants (coupling functionals). A perturbative calculation, using α' as an expansion parameter, leads to the following beta functions [14]

$$\begin{aligned} \beta^{(g)}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + 2D_\mu D_\nu\phi + \mathcal{O}(\alpha') , \\ \beta^{(B)}_{\mu\nu} &= \frac{1}{2}D^\rho H_{\mu\nu\rho} - D^\rho\phi H_{\mu\nu\rho} + \mathcal{O}(\alpha') , \\ \beta^{(\phi)} &= -R + \frac{1}{12}H^2 - 4D_\mu\partial^\mu\phi + 4(\partial_\mu\phi)^2 + \mathcal{O}(\alpha') , \end{aligned} \tag{2.34}$$

where $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\mu\nu]}$ and $R_{\mu\nu}$ is the target space Ricci tensor, computed from the background metric $g_{\mu\nu}$. Next, demanding that all the beta functions are zero, implies that the right hand sides in (2.34) must be zero. This give the equations of motion for the background fields to lowest order in α'. Hence, for the action (2.33) to be Weyl invariant at the quantum level we get constraints on the background fields. Interestingly, these equations of motion can be obtained from the following 26-dimensional action (in the string frame):

$$S = -\frac{1}{2\kappa^2}\int d^{26}x\sqrt{-g}e^{-2\phi}\Big(R + 4(\partial_\mu\phi)^2 - \frac{1}{12}H^2\Big)\,. \tag{2.35}$$

In the Einstein frame this action is given by (using that $g_{\mu\nu} = e^{\frac{\phi}{6}}g^{\mathrm{E}}_{\mu\nu}$)

$$S = -\frac{1}{2\kappa^2}\int d^{26}x\sqrt{-g}\Big(R_{\mathrm{E}} - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{12}e^{-\frac{\phi}{3}}H^2\Big)\,. \tag{2.36}$$

Both of these actions are of course equivalent. However, depending on the situation one or the other might be more useful. For example, in T-duality calculations the string frame action (2.35) is easier to work with.

2.1.2 Superstring theory

We found in the last subsection that bosonic string theory has a serious drawback, namely the tachyon in its spectrum. Furthermore, there are also no fermions present in bosonic string theory. Hence, it can not be a candidate for a unified theory of nature. In this subsection we will solve both these problems by introducing the superstring, which includes both fermions in the spectrum and removes the tachyon. It is also both space-time supersymmetric and world sheet supersymmetric.

There are two different but equivalent formulations of superstring theory. (1) the Neveu-Schwarz-Ramond (NSR) [23] formulation, and (2) the Green-Schwarz formulation (GS) [24]. The NSR formulation has manifest world sheet supersymmetry and is covariant. It is also space-time supersymmetric. This, however, is difficult to show in this formulation. The GS formulation, on the other hand, has manifest space-time supersymmetry, while world sheet supersymmetry arises as a consequence of κ-symmetry. It is, however, not possible to use covariant quantization when quantizing the GS model. One instead must use light-cone quantization. This is a drawback, since covariance of the theory is not manifest in this approach.

In this thesis we will concentrate on the NSR formulation of the superstring. For an introduction to the GS model and for a more thorough introduction to the NRS model we refer to the standard text books [14, 15, 16]. Moreover, unless otherwise specified we deal with open superstrings.

We begin by generalizing the bosonic string action (2.10), in the following way [14]:

$$S = -\frac{1}{2\pi}\int_\Sigma d^2\sigma[\partial_a X^\mu\partial^a X_\nu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu]\,, \tag{2.37}$$

where we for simplicity have set $\alpha' = 1/2$. We are using the same conventions as in [14], as well as suppressing a two-dimensional index in ψ^μ. Here $\alpha = 0, 1$, $\rho^0 = \sigma^2$ and $\rho^1 = i\sigma^1$. Note that ρ^α obeys $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$. Furthermore, in this basis the world sheet spinor ψ which has two components ψ_- and ψ_+, is a real Majorana spinor. Note also that the anticommuting field ψ_μ transform as a vector of $SO(1, d-1)$.

When quantizing this theory we have the same commutators as before for the fields X^μ, see (2.26). For the fermion fields we instead have the usual equal time anticommutating relations

$$\{\psi^\mu_A(\sigma), \psi^\nu_B(\sigma')\} = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma - \sigma') \ . \tag{2.38}$$

As in the bosonic case we have negative norm states since $\eta^{00} = -1$. In the bosonic case these states were removed by imposing the constraints (2.28) and by choosing $d = 26$ and $a = 1$. To remove the un-physical states from the fermion fields we have to use a new symmetry. This new symmetry is (world sheet) supersymmetry, which is a symmetry between the (bosonic) X^μ and the fermionic ψ^μ_A fields that mixes them. For the action (2.37) the supersymmetry transformations, which leaves the action invariant, are given by

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \ , \quad \delta\psi^\mu = -i\rho^\alpha\epsilon\partial_\alpha X^\mu \ , \tag{2.39}$$

where ϵ is a constant infinitesimal anticommuting Majorana spinor.

Next, using the Noether method (see [14]), we obtain the following (conserved) supercurrent

$$J_{\alpha A} = \frac{1}{2}\rho^\beta\rho_\alpha\psi^\mu_A\partial_\beta X_\mu \ , \tag{2.40}$$

where we use light-cone index for the spinor index $A = +, -$. The classical constraint is that the supercurrent is zero.

From the action (2.37) we obtain that the equations of motion for the fermion fields are given by the two-dimensional Dirac equation $\rho^\alpha\partial_\alpha\psi^\mu = 0$. In the basis given above this implies that similar to the bosonic case, we obtain two decoupled equations

$$\partial_+\psi^\mu_- = 0 \ , \quad \partial_-\psi^\mu_+ = 0 \ , \tag{2.41}$$

which means that ψ^μ_- and ψ^μ_+ describe right- and left-moving modes, respectively. Next, rewriting the supercurrent as[2] $J_+ = \psi^\mu_+\partial_+X^\mu$, and $J_- = \psi^\mu_-\partial_-X^\mu$, we can summarize all the classical constraints as follows:

$$0 = J_+ = J_- = T_{++} = T_{--} \ . \tag{2.42}$$

The next step is to obtain the full quantum constraints. However, before doing this we take a look at the various boundary conditions that the fermion fields must obey. We note that for the X^μ fields we have the same boundary conditions, equations of motion

[2]Here we have renamed $J_{+A} \rightarrow J_+$ and similar for J_{-A}, since only the positive chirality spinor component of J_{+A} is non-zero, see [14].

and mode expansions as in the bosonic case[3]. For the fermion fields an investigation leads to the following possible boundary conditions [14]:

$$\psi^\mu_+(\tau,0) = \psi^\mu_-(\tau,0)\ , \quad \psi^\mu_+(\tau,\pi) = \pm\psi^\mu_-(\tau,\pi)\ . \tag{2.43}$$

The two cases are named Ramond (R) boundary conditions (for the plus case) and Neveu-Schwarz (NS) boundary conditions (for the minus case). In the (R) case the mode expansion of the fermion fields is given by

$$\psi^\mu_\pm(\sigma^\pm) = \frac{1}{\sqrt{2}}\sum_{n\in Z} d^\mu_n e^{-in\sigma\pm}\ , \tag{2.44}$$

where the sum runs over the integers. In the (NS) case we instead have

$$\psi^\mu_\pm(\sigma^\pm) = \frac{1}{\sqrt{2}}\sum_{n\in Z+1/2} b^\mu_r e^{-ir\sigma\pm}\ , \tag{2.45}$$

where the sum runs over the half integer numbers. This means that the theory has two different sectors, (R), which gives space-time fermions and (NS), which gives space-time bosons [14]. That the two sectors give fermions and bosons, respectively, will be obvious when we investigate their spectrum below.

The operators d^μ_m and b^μ_r obey the following anticommutating relations:

$$\begin{aligned}\{d^\mu_m, d^\nu_n\} &= \eta^{\mu\nu}\delta_{m+n}\ ,\\ \{b^\mu_r, b^\nu_s\} &= \eta^{\mu\nu}\delta_{r+s}\ .\end{aligned} \tag{2.46}$$

For the closed string we obtain similar expressions. Note, however, that in this case we have different mode operators for ψ^μ_- and ψ^μ_+ (i.e., we have d^μ_n and b^μ_r, and $\tilde{d}^\mu_n$ and $\tilde{b}^\mu_r$, respectively). This naturally leads to four sectors, since a closed string can be seen as built from different parings of left- and right-moving modes. These four sectors are called (RR), (R-NS), (NS-R) and (NS-NS), where the first and last give space-time bosons and the second and third space-time fermions.

Next, we obtain the super-Virasoro operators from the energy momentum tensor and the supercurrent. From the energy momentum tensor we get as in the bosonic case the infinite set of operators L_m (which will be different for (R) and (NS) boundary conditions), while from the supercurrent we get the infinite set of operators F_m for (R) boundary conditions and G_m for (NS) boundary conditions. The L_m has the following definition in terms of α^μ_m, d^μ_m and b^μ_r [14]:

$$\begin{aligned}L_m &= \frac{1}{2}\sum_{-\infty}^{\infty} :\alpha_{-n}\cdot\alpha_{m+n}: +\frac{1}{2}\sum_{-\infty}^{\infty}(n+\frac{1}{2}m) : d_{-n}\cdot d_{m+n}: \quad \text{(R)}\ ,\\ L_m &= \frac{1}{2}\sum_{-\infty}^{\infty} :\alpha_{-n}\cdot\alpha_{m+n}: +\frac{1}{2}\sum_{r\in Z+1/2}(r+\frac{1}{2}m) : b_{-r}\cdot b_{m+r}: \quad \text{(NS)}\ ,\end{aligned} \tag{2.47}$$

[3]Note, however, that the energy momentum tensor is not the same as in the bosonic case due to the fermion terms, see, e.g., [14]. This implies, as we will see below, that the expressions for L_m must be modified.

while for F_m and G_r we obtain

$$\begin{aligned} F_m &= \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{-n}\cdot d_{m+n} , \quad (\mathrm{R}) , \\ G_r &= \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{-n}\cdot b_{r+n} , \quad (\mathrm{NS}) . \end{aligned} \tag{2.48}$$

We note that for L_m it is only necessary to use normal ordering when $m = 0$.

The above given operators obey the following Virasoro algebra [14]:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{(m+n)} + \frac{d}{8}m^3\delta_{m+n} , \\ [L_m, F_n] &= (\frac{1}{2}m - n)F_{(m+n)} , \\ [F_m, F_n] &= 2L_{(m+n)} + \frac{d}{2}m^2\delta_{m+n} , \end{aligned} \tag{2.49}$$

for (R) boundary conditions (the fermionic sector), and

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{(m+n)} + \frac{d}{8}m(m^2-1)\delta_{m+n} , \\ [L_m, G_r] &= (\frac{1}{2}m - r)G_{(m+r)} , \\ [G_r, G_s] &= 2L_{(r+s)} + \frac{d}{2}(r^2 - \frac{1}{4})\delta_{m+n} , \end{aligned} \tag{2.50}$$

for (NS) boundary conditions (the bosonic sector).

Similar to the bosonic case in section 2.1.1, we have the following quantum constraints on physical states for the open superstring in the (R) sector[4]:

$$\begin{aligned} F_m|\mathrm{phys}> &= L_m|\mathrm{phys}>= 0 , \quad m > 0 , \\ L_0|\mathrm{phys}> &= 0 , \end{aligned} \tag{2.51}$$

and for the (NS) sector

$$\begin{aligned} G_r|\mathrm{phys}> &= L_m|\mathrm{phys}>= 0 , \quad m > 0 , \quad r > 0 , \\ (L_0 - \frac{1}{2})|\mathrm{phys}> &= 0 . \end{aligned} \tag{2.52}$$

It can also be shown that one has to choose the dimension of space-time to be $d = 10$, in order to decouple all negative norm states [14].

[4]See, e.g., [14] for a derivation of the non-trivial L_0 constraints, and that the dimension of space-time must be $d = 10$.

The above constraints imply that the mass spectrum in the (R) and (NS) sectors, respectively, is given by

$$\alpha' M_{\mathrm{R}}^2 = N_\alpha + N_d \ , \quad \alpha' M_{\mathrm{NS}}^2 = N_\alpha + N_b - \frac{1}{2} \ , \tag{2.53}$$

where we from (2.47) have obtained that

$$N_\alpha = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m \ , \quad N_d = \sum_{m=1}^{\infty} m d_{-m} \cdot d_m \ , \quad N_b = \sum_{r=1/2}^{\infty} r b_{-r} \cdot b_r \ . \tag{2.54}$$

We build our Fock space, in the (NS) sector, by using creation operators (i.e., using the eight transversal components of) α_{-m}^μ and b_{-r}^μ on the ground state $|0; k>_{\mathrm{NS}}$. As usual for a ground state, $\alpha_m^\mu |0; k>_{\mathrm{NS}} = 0$ and $b_r^\mu |0; k>_{\mathrm{NS}} = 0$, for $m > 0$ and $r > 0$. In the (NS) sector the ground state is a singlet with mass squared $-\frac{1}{2\alpha'}$, i.e., it is a tachyon. This might sound disturbing, however, as we will see below, this state is not included in the final spectrum, because it is removed by the GSO projection.

In the (R) sector the Fock space is obtained by using the creation operators (i.e., using the eight transversal components of) α_{-m}^μ and d_{-m}^μ on the ground state $|0; k>_{\mathrm{R}}$. Here the ground state is massless but not a singlet. The reason for this is that in the (R) sector there is a fermionic zero mode d_0^μ, which obeys $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$, which means that $\gamma^\mu = \sqrt{2} d_0^\mu$ obeys an SO(1,9) Clifford algebra. Moreover, d_0^μ commutes with L_0[5], which means that $|0; k>_{\mathrm{R}}$ and $d_0^\mu |0; k>_{\mathrm{R}}$ have the same mass. We therefore conclude that the ground state is a massless SO(1,9) spinor with 8 on-shell degrees of freedom. Note also that in the (R) sector there is no tachyon in the spectrum.

So far, there are a few drawbacks with our open superstring theory. (1) There is a tachyon in the (NS) spectrum. (2) It can be shown that the theory, at this point, is not space-time supersymmetric. These two problems are solved by the so called GSO projection [25]. The GSO projection also renders the theory modular invariant, which is essential in order for a superstring theory to be consisitent.

The GSO projection demands that states with odd fermion number should be removed from the spectrum. A further investigation shows that the fermion number F is given by

$$\begin{aligned} F &= n_b - 1 \ , \quad (\mathrm{NS}) \ , \\ F &= n_d + \delta \ , \quad (\mathrm{R}) \ . \end{aligned} \tag{2.55}$$

Here n_b and n_d is the number of creation operators b_{-r} and d_{-m}, respectively, that have been used on the ground state. Furthermore, $\delta = 0, 1$, for the (R) sector vacuum with positive and negative chirality, respectively[6]. From this condition we obtain that the

[5]This is easily obtained by using that $[d_{-n} \cdot d_n, d_0] = 0$, if $n \neq 0$.

[6]Note that here we could instead have set $\delta = 1, 0$, for the (R) sector vacuum with positive and negative chirality, respectively. The only difference from this would have been that the massless multiplet below would have been $\mathbf{8}_\mathrm{v} \oplus \mathbf{8}_\mathrm{c}$ instead. Physically there is no difference. Note, however, that this difference is important when constructing a closed string theory from two open strings, since we have two choices, (1) both open strings have the same chirality or (2) they have different chiralities, see below.

tachyon is removed, since $n_b = 0$. Hence, in the (NS) sector the true ground state is a massless vector with eight degrees of freedom $(b^i_{-1/2}|0; k>_{\rm NS})$. This means that the GSO projected open string has a ground state with eight bosonic and eight fermionic degrees of freedom. Hence, the massless spectrum is space-time supersymmetric. Next, since the little group for SO(1,9) is SO(8), the ground state multiplet can therefore be written as a $\mathbf{8}_{\rm v} \oplus \mathbf{8}_{\rm s}$ representation of SO(8).

The closed superstring can now be built from two open superstrings, which means that the spectrum is the direct product of two open string spectra. For example, the maximally supersymmetric type II string theories A and B, are obtained by using open strings with the same chirality (IIB) and different chirality (IIA). In the two cases we obtain the following massless spectrum:

$$\begin{aligned} &(\mathbf{8}_{\rm v} \oplus \mathbf{8}_{\rm s}) \otimes (\mathbf{8}_{\rm v} \oplus \mathbf{8}_{\rm c}) \,, \quad ({\rm IIA}) \,, \\ &(\mathbf{8}_{\rm v} \oplus \mathbf{8}_{\rm s}) \otimes (\mathbf{8}_{\rm v} \oplus \mathbf{8}_{\rm s}) \,, \quad ({\rm IIB}) \,. \end{aligned} \tag{2.56}$$

From this we obtain that both the type II theories have the same NS-NS sector, namely

$$\mathbf{8}_{\rm v} \otimes \mathbf{8}_{\rm v} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_{\rm v} \,, \tag{2.57}$$

which are identified with the dilaton ϕ, the NS-NS two form $B_{\mu\nu}$ and the metric $g_{\mu\nu}$. Note that this is the same spectrum as the closed bosonic string.

In the R-R sector we instead obtain different results for IIA/B, namely

$$\begin{aligned} \mathbf{8}_{\rm s} \otimes \mathbf{8}_{\rm c} &= \mathbf{8}_{\rm v} \oplus \mathbf{56}_{\rm t} \,, \quad ({\rm IIA}) \,, \\ \mathbf{8}_{\rm s} \otimes \mathbf{8}_{\rm s} &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_{\rm c} \,, \quad ({\rm IIB}) \,. \end{aligned} \tag{2.58}$$

These fields are identified with the vector $C_{(1)}$ and three form $C_{(3)}$ for type IIA and with the axion $C_{(0)}$, the two form $C_{(2)}$ and four form (with self-dual fields strength) $C^+_{(4)}$ for type IIB. The NS-NS and R-R sectors give a total of 128 bosonic degrees of freedom in both cases.

The fermionic R-NS and NS-R sectors are given by

$$\begin{aligned} \mathbf{8}_{\rm v} \otimes \mathbf{8}_{\rm c} &= \mathbf{8}_{\rm c} \oplus \mathbf{56}_{\rm s} \,, \\ \mathbf{8}_{\rm v} \otimes \mathbf{8}_{\rm s} &= \mathbf{8}_{\rm s} \oplus \mathbf{56}_{\rm c} \,, \end{aligned} \tag{2.59}$$

for type IIA, while for type IIB we have two copies of the second one in (2.59). This means that for IIA we have two spinors and gravitinos with different chiralities, while they have the same chiralities for IIB. The total number of fermionic degrees of freedom is 128. Hence, the massless spectrum for both type IIA/B superstring theory are space-time supersymmetric.

In the last subsection we saw that gravity coupled to a two form and a dilaton is the massless limit of closed bosonic string theory. Here, we have obtained that in the maximally supersymmetric case there are two different massless limits, IIA/B. This indicates that N=2 type IIA/B supergravity is obtained in the massless limit of type IIA/B string theory, since they have exactly the same spectrum as the massless part of the IIA/B spectrum.

Name	IIA	IIB	I	HetA	HetB
string type	closed oriented	closed oriented	open and closed non-oriented	closed oriented	closed oriented
10D SUSY	$N=2$ non-chiral	$N=2$ chiral	$N=1$	$N=1$	$N=1$
10D gauge group	none	none	$SO(32)$	$SO(32)$	$E_8 \times E_8$

Table 2.1: The five consistent string theories. Here SUSY means the number of supersymmetries. HetA is the heterotic string theory with gauge group $SO(32)$, while HetB is the one with $E_8 \times E_8$.

There are also three other interesting superstring theories, that we have not mentioned so far. These are the two heterotic theories and the type I theory. The massless part of these theories can be identified with the two heterotic supergravity theories and type I supergravity, which all have N=1 supersymmetry. For an introduction to these string theories and more on the type II theories we refer to the standard text books [14, 15, 16], see also the next subsection for a very short introduction to the type I and the heterotic string and supergravity theories.

In table 2.1 we summarizes the five superstring theories. From this we see, e.g., that all five string theories contain closed strings, while type I also contains open strings. Note however, that open strings can exist in the type II theories if they end on D-branes, see below.

2.1.3 Supergravity in 10 and 11 dimensions

We saw in the subsection 2.1.1 that the massless sector of bosonic string theory can be described by a 26-dimensional target space action. Similar, in the last subsection we identified the massless sector of type IIA/B string theory with type IIA/B supergravity. In this subsection we are going to give a brief introduction to these supergravity theories as well as the different N=1 supergravity theories and the unique supergravity theory in 11 dimensions.

(1) N=1 $d=11$

We begin by discussing the N=1 supergravity theory in 11 dimensions [27]. The massless on-shell multiplet consists of a graviton g_{MN}, a three form A_3 and a gravitino ψ^α_M. This multiplet is supersymmetric, since it has 128 bosonic degrees of freedom (44 + 84) and 128 fermionic degrees of freedom. The bosonic part of the action is given by

$$S = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\Big(R - \frac{1}{48}F_4^2\Big) - \frac{1}{12\kappa^2}\int F_4 \wedge F_4 \wedge A_3 \,, \qquad (2.60)$$

where $F_4 = dA_3$. This action if further supplemented with a Bianchi identity $dF_4 = 0$.

This theory is interesting for various reasons. In this thesis we will focus on the fact that the $d = 11$ supermembrane couples to the above 11-dimensional supergravity background, as well as the fact that there are solutions to the 11-dimensional supergravity equations of motion that correspond to solitonic membranes and 5-branes. Furthermore, and most importantly, as we will see in the next section, it has been conjectured that 11-dimensional supergravity is the low energy limit of a yet fairly unknown theory, called M-theory [6]. This non-perturbative theory is conjectured to give *all* perturbative string theories, in different perturbative limits. All these issues will be discussed in the next section.

(2) N=2 $d = 10$

In 10 dimensions there are two maximally supersymmetric supergravity theories, with 32 supercharges, called type IIA and IIB. From the last subsection we know that these theories are the low energy limit of type IIA and IIB string theory, respectively. The field content for these two theories is given in (2.57), (2.58) and (2.59). For type IIA theory the bosonic part of the action is given by (in the string frame) [16]

$$\begin{aligned} S_{\text{IIA}} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\Big(e^{-2\phi}[R+4(\partial\phi)^2-\frac{1}{12}H_3^2]-\frac{1}{4}F_2^2-\frac{1}{48}F_4^2\Big) \\ &- \frac{1}{4\kappa^2}\int F_4\wedge F_4\wedge B_2\,, \end{aligned} \tag{2.61}$$

where $F_4 = dC_3 + H_3 \wedge C_1$, $F_2 = dC_1$, $H_3 = dB_2$ and ϕ is the dilaton. For type IIB the bosonic part of the action is given by (in the string frame) [16]

$$\begin{aligned} S_{\text{IIB}} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\Big(e^{-2\phi}[R+4(\partial\phi)^2-\frac{1}{12}H_3^2]-\frac{1}{2}(\partial a)^2 \\ &- \frac{1}{12}[F_3+aH_3]^2-\frac{1}{480}F_5^2\Big)+\frac{1}{4\kappa^2}\int\Big(C_4+\frac{1}{2}B_2\wedge C_2\Big)\wedge F_3\wedge H_3\,, \end{aligned} \tag{2.62}$$

where $F_5 = dC_4 + H_3 \wedge C_2$, $F_3 = dC_2$, $H_3 = dB_2$, a is the RR axion and ϕ is the dilaton. In order to obtain the correct number of degrees of freedom we also have to demand that the four form C_4 has a self-dual field strength. This self-duality constraint is imposed, by hand, at the level of the equations of motion.

At this moment we will leave the IIA/B supergravity. However, in the next section we will return to them and see how the type IIA theory is related to 11 dimensional supergravity, and how the type IIB theory can be seen to be S-duality invariant.

(3) N=1 $d = 10$

Next, we take a look at the three supergravity theories, in 10 dimensions, which have N=1 supersymmetry. The first one is the type I theory. This is the low energy limit of type I string theory. The massless sector consists of one gravitino and one spinor, as well as a graviton, dilaton and an RR two form. These are massless modes from the closed unoriented type I string. However, there is also a massless gauge field (and its super partner), with gauge group SO(32), from the massless sector of an open superstring. This open string sector is added to the closed string, because the type I closed string is not consistent in it self, see, e.g., chapter 7.1 in [16]. This leads to a massless supermultiplet with N=1 supersymmetry. The bosonic part of the type I action is given by

$$S_{\text{I}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\Big(e^{-2\phi}[R+4(\partial\phi)^2] - \frac{1}{12}\tilde{F}_3^2 - \frac{1}{4}e^{-\phi}\text{Tr}F^2\Big)\,, \tag{2.63}$$

where $\tilde{F}_3$ is the modified field strength for the two form, see [16], ϕ is the dilaton and $F_2 = dA + A\wedge A$, where A is the gauge potential in the adjoint representation of SO(32).

For those familiar with D-branes (see next subsection and section 2.2) it might be interesting to know that the type I string theory can be obtained from type IIB string theory by adding a single space-filling O9-plane and 16 D9-branes. Here the O9-plane turns the IIB theory into type I unoriented string theory, while adding the 16 D9-branes on which open strings end, gives the open string sector. The number of D9-branes has to do with the fact that 16 is the rank of SO(32), see [16].

The heterotic string theories [4] were the last two string theories to be invented. These are a bit special, since the left-moving and right-moving sectors are not chosen to be the same as they are for type I and II theories. Instead, we choose to combine a left-moving bosonic string with a right-moving superstring. We note that although there is a bosonic part included in the heterotic string, there is no tachyon in the spectrum, which is a result of the level matching condition. Since the bosonic string lives in 26 dimensions and the superstring in 10, we have to compactify 16 dimensions on a torus T^{16}, for the left-moving bosonic string. This introduces a gauge field A. Furthermore, since we have compactified 16 dimensions the rank of the gauge group for the gauge field A has to be 16. Naively, this would lead to an abelian gauge group $\text{U}(1)^{16}$. However, if one choose the torus to be "self-dual"[7] (see, e.g., [16]) one obtains a non-abelin gauge group. The possible gauge groups have been shown to be SO(32) and $E_8\times E_8$ [4]. For a further discussion about heterotic string theory see, e.g., [4, 14, 15, 16].

The two heterotic supergravity theories are obtained as the low energy limit of heterotic string theory with gauge group SO(32) and $E_8\times E_8$, respectively. The bosonic part of the actions is given by

$$S_{\text{I}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\phi}\Big(R+4(\partial\phi)^2 - \frac{1}{12}\tilde{H}_3^2 - \frac{1}{4}\text{Tr}F^2\Big)\,, \tag{2.64}$$

where $\tilde{H}_3$ is the modified field strength for the two form, see [16], ϕ is the dilaton and $F_2 = dA + A\wedge A$, where A is the gauge potential with gauge group SO(32) or $E_8\times E_8$, respectively. Hence, the total field content is essentially the same as for the type I string, except that we here have two possible gauge groups SO(32) and $E_8\times E_8$. It is interesting to ask if this similarity between the heterotic theory with gauge group SO(32) the type I theory means that they are somehow related? The answer to this question seems to be yes, as we will see in the next section when we discuss S-duality.

[7] To be more specific: the torus $T^{16} = R^{16}/\Lambda$, where Λ must be an euclidian, even, self-dual lattice.

2.1.4 T-duality

In the final subsection on perturbative string theory we will discuss a duality between two different theories (or the same theory), compactified on a circle with radii R and $\frac{\alpha'}{R}$, respectively. This duality is called T-duality and does not exists for ordinary field theories, since only an extended object, like a string, can wound around a compact dimension, see below.

Closed strings

We start with a discussion about closed bosonic string theory compactified on a circle. Recall the mode expansion for the field X^μ for a closed bosonic string:

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) = \frac{x^\mu}{2} + \frac{\tilde{x}^\mu}{2} + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\tilde{\alpha}_0^\mu - \alpha_0^\mu)\sigma + (\text{oscillators}) \,. \tag{2.65}$$

We have chosen to only look at the zero modes since the oscillators are not important in the following discussion. In the above equation we have used, from the discussion in section 2.1.1, that $\alpha_0^\mu = \tilde{\alpha_0}^\mu = p^\mu\sqrt{\frac{\alpha'}{2}}$. (Hence, the forth term in (2.65) is really zero.) This is required for X^μ to be single valued when performing a periodic shift in σ.

Next, we investigate what will happen if we let, e.g., x^{25}, be periodic $x^{25} \sim x^{25} + 2\pi R$, where R is the radius of the circle. This has two consequences: (1) the momentum p^{25} is quantized as

$$p^{25} = \frac{n}{R} \,, \tag{2.66}$$

where n is an integer, and (2) we can no longer demand that $X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau,\sigma)$. Instead we relax this condition and instead demand that

$$X^{25}(\tau,\sigma + 2\pi) = X^{25}(\tau,\sigma) + 2\pi w R \,, \tag{2.67}$$

since the string can wind around the compact direction. Here w is an integer, the winding number. This means that for $w \neq 0$ we have $\alpha_0^{25} \neq \tilde{\alpha}_0^{25}$. Using equations (2.65)-(2.67) we obtain the following two coupled equations for the zero modes α_0^{25} and $\tilde{\alpha}_0^{25}$:

$$\begin{aligned}
\alpha_0^{25} + \tilde{\alpha}_0^{25} &= \frac{2n}{R}\sqrt{\frac{\alpha'}{2}} \,, \\
\tilde{\alpha}_0^{25} - \alpha_0^{25} &= wR\sqrt{\frac{2}{\alpha'}} \,.
\end{aligned} \tag{2.68}$$

Solving these equations give

$$\begin{aligned}
\alpha_0^{25} &= \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} \,, \\
\tilde{\alpha}_0^{25} &= \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} \,.
\end{aligned} \tag{2.69}$$

Next, considering the mass spectrum in the 25 uncompactified dimensions, we find [16]

$$\begin{aligned} M^2 &= -p^\mu p_\mu = \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(N-1) \\ &= \frac{2}{\alpha'}(\tilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\tilde{N}-1) , \end{aligned} \tag{2.70}$$

which can be rewritten in the following more symmetric way

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) , \tag{2.71}$$

where we have used that the level matching condition has been changed to $nw + \tilde{N} - N = 0$.

In (2.71) we have two different contributions from the compactification, (1) the usual Kaluza-Klein states with mass $\sim 1/R$ and (2) the stringy winding states with mass $\sim R$. With stringy we mean that these states are included due to the extended nature of strings. We also see that the Kaluza-Klein and the winding states have the opposite dependence of the radius R. This leads us to ask if there exists a symmetry between these states? The answer to this question is yes, and it is easy to see that the spectrum (2.71) is invariant under the following symmetry transformation:

$$n \to w , \quad w \to n , \quad R \to R' = \frac{\alpha'}{R} . \tag{2.72}$$

For the zero modes this implies that $\alpha_0^{25} \to -\alpha_0^{25}$ and $\tilde{\alpha}_0^{25} \to \tilde{\alpha}_0^{25}$, when we go from radius R to radius $R' = \frac{\alpha'}{R}$. This transformation between the two dual compactifications is called a T-duality transformation. For the field X^{25} this implies that

$$X'^{25} = X_{\text{L}}^{25}(\sigma^+) - X_{\text{R}}^{25}(\sigma^-) . \tag{2.73}$$

A consequence of this is that closed bosonic string theory compactified on a circle with radius R is equivalent to closed bosonic string theory compactified on a circle with radius $\frac{\alpha'}{R}$.

We have seen above that for closed bosonic string theory compactified on a circle there exists a T-duality symmetry, which maps a compactification on a circle with large radius to one with a small radius. This indicates that there exists a minimum radius, $R = \sqrt{\alpha'}$, where the theory is self-dual, since $R' = \frac{\alpha'}{R} = R$. However, although perturbative string theory seems to indicate a minimum radius, this does not mean that this is true when incorporating also non-perturbative effects. In fact it is believed that so called D-branes can probe smaller length scales, see, e.g., [16].

In the above discussion we note that for $n = w = 0$ and $N = \tilde{N} = 1$, we obtain the usual massless Kaluza-Klein states, which are obtained by using oscillators in the compact direction. This leads to two U(1) vector fields (from the off-diagonal components of the massless graviton and NS-NS two form)

$$A_\mu^{\text{R}} = \frac{1}{2}(g - B)_{\mu,25} , \quad A_\mu^{\text{L}} = \frac{1}{2}(g + B)_{\mu,25} , \tag{2.74}$$

where $\mu = 0, 1, \ldots, 24$. There is also a scalar which is identified from the graviton, as $\phi_{25} = \frac{1}{2}\log g_{25,25}$. We note that the vev of this scalar is proportional to the compactified radius R. As a summary: for closed bosonic string theory compactified on a circle with radius R, the massless spectrum is given by a graviton $g_{\mu\nu}$, dilaton, NS-NS two form B and from the fact that we have a compactified dimension, we have two vectors with $\mathrm{U}(1)_{\mathrm{L}}\times\mathrm{U}(1)_{\mathrm{R}}$ gauge symmetry and a scalar ϕ_{25}. Moreover, at the self-dual radius $R = \sqrt{\alpha'}$, it can be shown that there are extra degrees of freedom that enhance the gauge symmetry for the vectors to $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$. What happens is that we also have massless fields coming from the levels

$$n = w = \pm 1 \ , \quad N = 1 \ , \tilde{N} = 0 \ ; \quad n = -w = \pm 1 \ , \quad N = 0 \ , \tilde{N} = 1 \ , \tag{2.75}$$

since these values implies that $M^2 = 0$, which is obtained from (2.71), using $R = \sqrt{\alpha'}$. From (2.75) two vectors combine with $\mathrm{U}(1)_{\mathrm{L}}$ to form $\mathrm{SU}(2)_{\mathrm{L}}$, while the other two combine with $\mathrm{U}(1)_{\mathrm{R}}$ to form $\mathrm{SU}(2)_{\mathrm{R}}$. Hence, for the massless vectors, we have enhanced the gauge symmetry from $\mathrm{U}(1)_{\mathrm{L}}\times\mathrm{U}(1)_{\mathrm{R}}$ to $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$. Note that this gauge symmetry enhancement happens in a very similar way to how the gauge groups SO(32) and $E_8 \times E_8$ are obtained for heterotic string theory.

If one T-dualize in more than one dimensions, say n, then the T-duality group is $\mathrm{O}(n, n;\mathbb{Z})$. For supergravity the T-duality group is instead $\mathrm{O}(n, n;\mathbb{R})$, which is discussed, together with transformation properties for background fields, in chapter 3. For a further discussion on T-duality for closed bosonic string, see, e.g., [15, 16].

Open strings

Next, we give a brief introduction to T-duality for open bosonic strings. From (2.16) we obtain that compactifying in the x^{25} direction we have the following open string expansion in this direction:

$$X^{25}(\tau, \sigma) = x^{25} + 2\alpha' p^{25}\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^{25}}{n}e^{-in\tau}\cos n\sigma \ , \tag{2.76}$$

where $p^{25} = \frac{n}{R}$. To obtain the T-dual field X'^{25} we use (2.73), which for an open string gives

$$X'^{25}(\tau, \sigma) = x'^{25} + 2\alpha'\frac{n}{R}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^{25}}{n}e^{-in\tau}\sin n\sigma \ . \tag{2.77}$$

We note that compared to (2.76) we no longer have any τ dependence in the zero mode sector. Furthermore, at the endpoints $\sigma = 0, \pi$, the oscillator term vanishes. Hence, the endpoints of the open string does not move in the x^{25} direction. This means that instead of satisfying a Neumann boundary condition in the x^{25} direction, the open string satisfies a Dirichlet boundary condition

$$X'^{25}(\tau, \pi) - X'^{25}(\tau, 0) = \frac{2\pi\alpha' n}{R} = 2\pi R' n \ , \tag{2.78}$$

where we have used (2.77) and (2.72). This formula can be seen as defining an open string "winding" number $n \in Z$.

To conclude: we have started with an open string that obeys Neumann boundary conditions in all 26 dimensions, i.e., its endpoints can move freely in all 26 dimensions. After T-duality in one directions (x^{25}), the boundary condition in the T-dualized direction has changed to Dirichlet boundary condition. Hence, the endpoints of the open string can now only move freely in 25 dimensions. This means that in this case the open string can been seen as ending on a hyperplane with 24 spatial dimensions. Before the T-duality it is possible to view the string as ending on a space-filling 25-dimensional hyperplane. Furthermore, the winding number n tells us the "difference" between the two hyperplanes which the open string end on. For example, $n = 0$ implies that the two endpoints end on the same hyperplane. Note that the hyperplanes are identified periodically in the x^{25} direction, with period $2\pi R'$. Moreover, it should be obvious that T-duality in k dimensions implies that the T-dualized open string ends on a hyperplane with $25 - k$ spatial dimensions. This hyperplane has been named Dp-brane where $p = 25 - k$ and k is the number of dimensions that have Dirichlet boundary conditions. The number of dimensions with Neumann boundary conditions is obviously $p + 1$.

T-dual superstring theories

We end this subsection with a brief discussion about T-duality for superstring theories. For the five superstring theories it can be shown that type IIA and IIB are T-dual, when compactified on a circle with radius R and $R' = \frac{\alpha'}{R}$, respectively, see, e.g., [15, 16]. This implies that type IIA and IIB string theory are equivalent theories when compactified down to nine or less dimensions. For a type II string theory compactified on an n-torus, the T-duality group is $\mathrm{O}(n, n;\mathbb{Z})$.

A few indications that the IIA/B duality is correct are given by (note that the duality has been rigourously proven, see, e.g., [16]):

1. Using supersymmetry, we obtain from the bosonic case that when T-dualizing type IIA string theory compactified on a circle with radius R, there is a parity transformation for one of the fermions, which means that the fermions after T-duality have the same chirality (as in type IIB).

2. For the massless RR fields, C_1 and C_3 transform into C_0, C_2 and C_4^+, under T-duality. For the explicit transformation rules we refer to chapter 3 equation (3.5). That this happens can be motivated from the analysis of T-duality for open strings. There we found that under T-duality a hyperplane, i.e., a Dp-brane, on which open strings end transforms into a D$(p-1)$-brane or a D$(p+1)$-brane, depending on if the T-duality is performed in a direction parallel to the Dp-brane or transeverse, respectively. This happened because the open string boundary conditions are changed in the T-dualized directions. Moreover, in type II string theory it can be shown that the massless RR fields couple to D-branes, e.g., C_3 couples to a D2-brane. Hence, under T-duality, the

RR fields C_1 and C_3 transform into C_0, C_2 and C_4^+, since the corresponding branes D0 and D2 should transform into D(−1), D1 and D3-branes.

3. Both type IIA and IIB have the same NS-NS sector. Therefore using the result for closed bosonic string theory, indicates that the NS-NS sectors are T-dual.

It can also be shown that the two heterotic theories are T-dual to each other, when compactified on a circle with radius R and $R' = \frac{\alpha'}{R}$, respectively. In this case the T-duality group for a heterotic string theory compactified on an n-torus is $\mathrm{O}(n+16, n;\mathbb{Z})$ [28]. Here the extra 16 emerges because the left-moving modes are bosonic and already compactified in 16 directions.

To conclude: we have found that of the five perturbatively inequivalent superstring theories in 10 dimensions, there are only three perturbatively inequivalent superstring theories below 10 dimensions. This leads us to ask the following question: If we also are able to incorporate non-perturbative results, is it then possible to obtain that *all* string theories, in fact, in some sense, are equivalent, e.g., can they be shown to be different perturbative limits of the same non-perturbative theory? As we will see in the next section, the answer to this question is most likely yes.

2.2 M-theory and duality

In this section we will discuss non-perturbative aspects of string theory. The topics will be: (1) p-branes, which are non-perturbative extended objects (particle ($p = 0$), string ($p = 1$), membrane ($p = 2$), etc.), that correspond to solitonic solutions in the supergravity theories discussed so far in this thesis. (2) S-duality, which is a duality between theories at strong and weak coupling, respectively. (3) U-duality, which essentially is a combination of S- and T-dualities. (4) A new non-perturbative theory in 11 dimensions, called M-theory, which is not a string theory but a theory containing membranes and 5-branes, and has 11-dimensional supergravity as its low energy limit. For other introductions to non-perturbative string theory, see, e.g., [15, 16, 29].

2.2.1 Branes in 10 and 11 dimensions

A p-brane is an object that is extended in p space dimensions, e.g., a string is a 1-brane, a membrane is a 2-brane, etc. That these objects exist in string theory was hinted in subsection 2.1.4, when we discussed T-duality for open strings. They can also be argued to exist from using supergravity, since there exist solitonic solutions in supergravity, which have the interpretation that they have been generated from a stack of p-branes. Furthermore, all the p-branes that we discuss in this subsection preserve half of the target space supersymmetry, which means that they are BPS objects. This implies that these objects have the minimum value of the mass that is allowed by supersymmetry. That they are BPS is very important, since this implies that they receive no quantum corrections, perturbative or non-perturbative, to their mass [32], as long as supersymmetry remains unbroken.

For example, this means that p-branes are very useful for investigating strong-weak coupling conjectures. A concrete example is S-duality in type IIB string theory, where we can compare the tensions between a fundamental string and a D-string, and find that they transform into each other under S-duality. Here the BPS property of the two strings implies that their tensions are valid for any value of the closed string coupling constant.

p-branes in 11 dimensions

We begin in 11 dimensions, and claim that 11-dimensional supergravity indicates that there exists a membrane (2-brane) and a 5-brane in 11 dimensions. The reason for this is the following: in 11-dimensional supergravity there is a three form A_3, and similar to how a point particle couples to a one form and a string to a two form (i.e., the NS-NS two form), it should be possible for a membrane to couple to the three form A_3. Furthermore, if this is true it should be possible to obtain a solitonic supergravity solution, which interpolates between a vacuum with SO(1,2)×SO(8) symmetry (since a membrane breaks SO(1,10)→S(1,2)×SO(8)), for $r \sim 0$, and flat 11-dimensional space-time, for $r \to \infty$. Here r is the radius for the eight-dimensional space transverse to the membrane. Now the question is, can we obtain a solution of this type to the 11-dimensional supergravity equations of motion? The answer is affirmative and the solution is given by

$$\begin{aligned} ds^2 &= H^{-\frac{2}{3}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{1}{3}}\delta_{mn}dx^m dx^n \ , \quad \mu,\nu = 0,1,2 \ , \quad m,n = 3,\ldots,10 \ , \\ A_3 &= H^{-1}dx^0 \wedge dx^1 \wedge dx^2 \ , \quad H = 1 + \frac{R^6}{r^6} \ , \end{aligned} \tag{2.79}$$

where H is a harmonic function on the transverse space. A derivation of this solution can be found in [33, 34]. We note that it is possible to add an arbitrary constant to the three form, since this does not effect the solution. The above solution can be shown to preserve half of the supersymmetry. It is easy to see that this solution interpolates between $AdS_4 \times S^7$ ($r \sim 0$) and flat 11-dimensional space-time ($r \to \infty$). At $r = 0$ there is a coordinate singularity. However, in the interior there is a true time-like singularity [34]. The interpretation of this supergravity solution is that at $r = 0$ there is a membrane or a stack of N membranes on top of each other. Since the membranes are solitons they obey the no-force condition, which means that they do not attract or repulse each other. This is an effect of the gravitational attraction being exactly compensated by the non-zero three form. Note that all BPS branes obey the no-force condition.

The membrane couples electrically to the three form, which indicates that some other extended object couples magnetically to it. This must be a 5-brane, since a 5-brane in 11 dimensions couples magnetically to a (11-7)-form field strength, which appear in the dual version of the supergravity theory [35]. The corresponding supergravity solution should therefore interpolate between a vacuum with SO(1,5)×SO(5) symmetry, for $r \sim 0$, and flat 11-dimensional space-time, for $r \to \infty$. This solution looks very much like the membrane solution (2.79), and is given by [36]

$$\begin{aligned} ds^2 &= H^{-\frac{1}{3}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{2}{3}}\delta_{mn}dx^m dx^n \ , \quad \mu,\nu = 0,1,\ldots,5 \ , \quad m,n = 6,\ldots,10 \ , \\ A_6 &= H^{-1}dx^0\wedge\cdots\wedge dx^5 \ , \quad A_3 = \gamma_3 \ , \quad H = 1 + \frac{R^3}{r^3} \ , \end{aligned} \tag{2.80}$$

where H is a harmonic function on the transverse space and γ_3 is the three form dual to the six form, i.e., $\gamma_3 = 3R^3\epsilon_3$, where $d\epsilon_3$ is the volume form of the four-sphere. Again the solution is half supersymmetric and interpolates between $AdS_7 \times S^4$ ($r \sim 0$) and flat 11-dimensional space-time ($r \to \infty$). A difference between the 5-brane and membrane solutions is that the 5-brane solution is completely non-singular everywhere, see, e.g., [34].

Since the solitonic membrane and 5-brane are infinitely extended objects they naturally have an infinite mass. However, they still have a finite tension, which are $T_2 = \frac{1}{(2\pi)^2\ell_p^3}$ and $T_5 = \frac{1}{(2\pi)^5\ell_p^6}$, for the membrane and 5-brane, respectively. Here ℓ_p is the 11-dimensional Plank length, which is the only length scale in 11-dimensional supergravity.

Similar to the string we can also write down actions for the membrane and the 5-brane, that generalizes the BDHDZ action (2.2). These actions govern the dynamics of the membrane and the 5-brane. An important difference between the membrane action and the string action is that the latter is Weyl invariant and the former is not. The membrane action is also much harder to investigate, since it does not give a free theory, but a highly non-linear theory. In subsection 2.2.3 we continue to discuss the membrane action.

Concerning the 5-brane we would like to mention that it is described, at low energy, by a six-dimensional superconformal (2,0) theory, where the massless multiplet consists of five scalars, one two form with self-dual field strength and four chiral fermions, which gives a total of 8+8 degrees of freedom. For a single 5-brane the (2,0) theory is free, while for a stack of 5-branes on top of each other it is a very complicated interacting theory. For more on these fairly unexplored theories, see, e.g., [30, 16, 57], while for a new and interesting approach to the (2,0) theory, see [37].

In 11-dimensional supergravity there are also two other solutions which preserve half supersymmetry. These are the plane wave and the Kaluza-Klein monopole, which are purely gravitational solutions, i.e., the three form is zero, see, e.g., [38].

p-branes in 10 dimensions

We have seen above that 11-dimensional supergravity admits half supersymmetric solutions, which correspond to a membrane and a 5-brane. This leads us to expect, together with the results obtained earlier, that in type IIA/B supergravity there should exists half supersymmetric solutions corresponding to other branes. In subsection 2.1.4, T-duality for open branes suggested that in type IIA/B there should exist D0, D2-branes in type IIA string theory, and D(-1), D1 and D3-branes in type IIB string theory. However, since these branes couple electrically to the relevant RR potentials, there should also exist various higher-dimensional D-branes that couple magnetically to the RR one and three forms in IIA and to the zero and two forms in IIB. Note that the D3-brane in type IIB couples both electrically and magnetically to the RR four form, since the five form field strength is self-dual. This leads us to conjecture that for type IIA supergravity there should exist solutions which correspond to Dp-branes for $p = 0, 2, 4, 6$, while for type IIB for $p = -1, 1, 3, 5, 7$. In fact, this is true and a generic Dp-brane solution of type IIA/B supergravity can be written as (in the string frame)

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2 + \cdots + (dx^p)^2) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_{8-p}^2)\ , \\ e^{2\phi} &= g^2 H^{\frac{3-p}{2}}\ , \quad H = 1 + \frac{gN(\alpha')^{\frac{7-p}{2}}}{r^{7-p}}\ , \\ C &= \frac{1}{gH} dx^0 \wedge \cdots \wedge dx^p + \frac{1}{g}(7-p) gN(\alpha')^{\frac{7-p}{2}} \epsilon_{7-p}\ . \end{aligned} \tag{2.81}$$

Here $d\epsilon_{7-p}$ is the volume element on the transverse $(8-p)$-sphere, H is a harmonic function on the transverse space and g is the closed string coupling constant. This supergravity solution corresponds to a stack of N Dp-branes $(0 \leq p \leq 6)$[8] on top of each other, aligned in the x^i, $i = 1, \ldots, p$, directions, with horizon at $r = 0$. Furthermore, the solitonic solution interpolates between $AdS_{p+2} \times S^{8-p}$ times some power of the dilaton ($r \to 0$) and flat 10-dimensional Minkowski space-time ($r \to \infty$). Note that only for $p = 3$ there is no dilaton factor, i.e., the D3-brane solution interpolates between $AdS_5 \times S^5$ and flat space-time. The tension of a Dp-brane is given by [16]

$$T_p = \frac{1}{(2\pi)^p g (\alpha')^{\frac{p+1}{2}}}\ , \tag{2.82}$$

where g is the closed string coupling constant. We see here that the tension is inversely related to the string coupling constant. This means that Dp-branes are solitonc objects.

For small fluctuations around a D-brane the world volume dynamics is described by the Dirac-Born-Infeld action together with a Wess-Zumino action

$$\mathcal{L}_{\text{DBI,WZ}} = -T_p \sqrt{-\det(g + \mathcal{F})} + T_p \int e^{\mathcal{F}} \wedge C\ , \tag{2.83}$$

where $\mathcal{F} = 2\pi\alpha' F + B$ is the gauge invariant world volume field strength and g and B are the pullbacks of the background metric and NS-NS two form. At low energy the dynamics of a Dp-brane can be described by maximally supersymmetric Yang-Mills theory in $(p+1)$ dimensions.

Now that we have obtained all the D-branes in type II supergravity we note that there must also exist half supersymmetric solutions, with non-zero NS-NS two form, which correspond to a string and a 5-brane, respectively. Indeed, these solutions exists, see, e.g., [16].

After the above discussion it should be obvious that for the type I and heterotic supergravities there also exist string and 5-brane solutions. For a much more thorough introduction to D-branes, the DBI action, the boundary state formalism etc, see, e.g., [16, 40].

[8] For the type IIB D7-brane solution, see, e.g., [39].

2.2.2 S- and U-duality

S-duality is a non-perturbative relation between two theories (or the same theory) at weak and strong coupling. For example, it has been conjectured that the maximally supersymmetric superYang-Mills theory in four dimensions is S-dual to itself. This conjecture has been fairly well tested, see, e.g, [41], but not rigorously proven. Under an S-duality transformation the elementary particles and the solitons are interchanged and the coupling constant is inverted.

Type IIB S-duality

Here it will argued that type IIB string theory is S-duality invariant. A first test of this conjecture is to check that the bosonic part of type IIB supergravity action is invariant under an S-duality transformation. In subsection 2.1.3 the type IIB supergravity action, in the string frame, is given in (2.62). In order to check S-duality it is more appropriate to work with the Einstein frame action, which is obtained by a rescaling of the metric $g_{\mu\nu} = e^{\frac{\phi}{2}} g^{\rm E}_{\mu\nu}$. This leads to the following action

$$\begin{aligned} S_{\rm IIB} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g_{\rm E}}\Big(R_{\rm E} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial a)^2 - \frac{1}{12}e^{-\phi}H_3^2 \\ &- \frac{1}{12}e^{\phi}[F_3 + aH_3]^2 - \frac{1}{480}F_5^2\Big) + \frac{1}{4\kappa^2}\int\Big(C_4 + \frac{1}{2}B_2\wedge C_2\Big)\wedge F_3\wedge H_3\ , \quad (2.84) \end{aligned}$$

where all metric dependence is in terms of $g^{\rm E}_{\mu\nu}$.

This action is invariant under the following transformation

$$\tau' = -\frac{1}{\tau}\ , \quad B_2' = C_2\ , \quad C_2' = -B_2\ , F_5' = F_5\ , \quad g'^{\rm E}_{\mu\nu} = g^{\rm E}_{\mu\nu}\ , \quad (2.85)$$

where $\tau = a + ie^{-\phi}$ is the complex "coupling constant". Note that the condition $F_5' = F_5$ implies that $C_4' = C_4 + B_2\wedge C_2$, since $F_5' = dC_4' + H_3'\wedge C_2' = dC_4' - dC_2\wedge B_2 = dC_4 + H_3\wedge C_2 = F_5$.

If we restrict to the special case where the RR axion $a = 0$, we find that $e^{\phi'} = e^{-\phi}$. Hence, we have a strong-weak coupling duality. Furthermore, note that $2\kappa^2 = (2\pi)^7 g^2\alpha'^4$ must be invariant under S-duality, which implies that the new length-scale $\tilde{\alpha}'$ is identified with

$$\tilde{\alpha}' = g\alpha'\ . \quad (2.86)$$

Here we have used that $2\kappa'^2 = (2\pi)^7 g'^2\tilde{\alpha}'^4 = (2\pi)^7 g^2\alpha'^4 = 2\kappa^2$, and $g' = 1/g$.

From the more general case, we see that the introduction of the complex parameter τ is appropriate. This leads us to suspect that the type IIB supergravity action is, in fact, invariant under a larger symmetry. We will now show that the action is actually SL(2,$\mathbb{R}$) invariant. To make this invariance manifest, we rewrite the action in the following way:

$$\begin{aligned} S_{\rm IIB} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g_{\rm E}}\Big(R_{\rm E} - \frac{1}{4}{\rm Tr}(\partial\mathcal{W}\mathcal{W}^{-1})^2 - \frac{1}{12}\tilde{H}^{\rm T}_{\mu\nu\rho}\mathcal{W}\tilde{H}^{\mu\nu\rho} \\ &- \frac{1}{480}F_5^2\Big) - \frac{1}{8\kappa^2}\int\tilde{C}_4\wedge H^r\wedge H^s\epsilon_{rs}\ , \quad (2.87) \end{aligned}$$

where $H^r = (H_3, F_3)$ for $r = 1, 2$, $\epsilon_{12} = 1$, $F_5 = d\tilde{C}_4 + \frac{1}{2}\epsilon_{rs} H^r \wedge B_2^s$ (where $\tilde{C}_4$ is the SL(2) invariant RR four form), and $\mathcal{W}$ is the following metric on the coset SL(2)/SO(2)

$$\mathcal{W} = \frac{1}{\mathrm{Im}(\tau)} \begin{pmatrix} |\tau|^2 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & 1 \end{pmatrix} = \begin{pmatrix} a^2 e^{\phi} + e^{-\phi} & a e^{\phi} \\ a e^{\phi} & e^{\phi} \end{pmatrix} , \quad \tilde{H} = \begin{pmatrix} H^1 \\ H^2 \end{pmatrix} . \tag{2.88}$$

The action (2.87) is invariant under a general SL(2,$\mathbb{R}$) transformation[9]

$$\mathcal{W}' = \Lambda \mathcal{W} \Lambda^{\mathrm{T}} , \quad \tilde{H}' = (\Lambda^{\mathrm{T}})^{-1} \tilde{H} , \quad g'^{\mathrm{E}}_{\mu\nu} = g^{\mathrm{E}}_{\mu\nu} , \tag{2.90}$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad ad - bc = 1 , \tag{2.91}$$

where a, b, c and d are integers. The invariance of the action (2.87) is trivial for all terms except the last two. However, checking these terms explicitly shows that also they are invariant under the above transformation (2.90). Note that the S-duality transformation (2.85), corresponds to the case $i = l = 0$ and $j = -k = 1$ in (2.90).

Since type IIB supergravity theory that is obtained in the low energy limit of type IIB superstring theory, has an SL(2,$\mathbb{R}$) invariance, it has been conjectured that the superstring must be SL(2,$\mathbb{Z}$) invariant [5, 6]. Note that the restriction to the subgroup SL(2,$\mathbb{Z}$) is motivated from the fact that strings with fractional charge do not exist, see, e.g., [5, 6].

Another non-trivial test of S-duality (we check the case (2.85) with RR axion $a = 0$) is to check that the tension for the various branes transform correctly. From (2.85) it is obvious that the fundamental string (F1-string), with tension $T_{\mathrm{F1}} = \frac{1}{2\pi\alpha'}$ transforms into a D1-brane. This means that the tension for a F1-string before S-duality should be the same as the tension for a D1-brane after S-duality. Checking this gives:

$$T'_{\mathrm{D1}} = \frac{1}{2\pi g' \tilde{\alpha}'} = \frac{1}{2\pi\alpha'} = T_{\mathrm{F1}} , \tag{2.92}$$

which shows that the tensions for a F1-string and D1-brane transform correctly. Next, the D3-brane should be invariant under S-duality, which means that its tension must also be invariant. This is easily seen to be true, since $T_{\mathrm{D3}} \sim 1/(g\alpha'^2) = 1/(g'\tilde{\alpha}'^2)$, where we have used (2.86). Finally, since the D5-brane and NS5-brane are the magnetic duals of the D1-brane and F1-string, respectively, they must transform into each other. Checking this claim for the tensions, gives

$$T'_{\mathrm{D5}} = \frac{1}{(2\pi)^5 g' \tilde{\alpha}'^3} = \frac{1}{(2\pi)^5 g^2 \alpha'^3} = T_{\mathrm{NS5}} , \tag{2.93}$$

which confirms that the NS5-brane transforms into a D5-brane, and vice versa.

[9] Note that the first transformation in (2.90) is equivalent to the following transformation of τ:

$$\tau' = \frac{a\tau + b}{c\tau + d} . \tag{2.89}$$

For more on S-duality for the type IIB string, see, e.g., [5, 6, 15, 16].

Type I-SO(32) heterotic S-duality

Next, we will argue that the type I string theory and the SO(32) heterotic string theory are S-dual to each other. Again we will show that this is true for the bosonic massless sector.

In subsection 2.1.3 we saw that the supergravity actions for the type I and SO(32) heterotic strings are very similar. This leads us to ask if they can be related in some way. Since they are not T-dual, we will check if they might be S-dual. Again we rewrite the actions in the Einstein frame, $g_{\mu\nu} = e^{\frac{\phi}{2}} g^{\mathrm{E}}_{\mu\nu}$. This gives the following type I and SO(32) heterotic supergravity actions:

$$S_{\mathrm{I}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g_{\mathrm{E}}}\Big(R_{\mathrm{E}} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{\phi}\tilde{F}_3^2 - \frac{1}{4}e^{\frac{\phi}{2}}\mathrm{Tr}F^2\Big) , \tag{2.94}$$

and

$$S_{\mathrm{H32}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g_{\mathrm{E}}}\Big(R_{\mathrm{E}} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-\phi}\tilde{H}_3^2 - \frac{1}{4}e^{-\frac{\phi}{2}}\mathrm{Tr}F^2\Big) , \tag{2.95}$$

respectively, which are easily seen to be S-dual under the following S-duality transformation:

$$\phi_{\mathrm{H}} = -\phi_{\mathrm{I}} , \quad \tilde{H}_3 = \tilde{F}_3 . \tag{2.96}$$

Note also that κ is invariant under the above transformation, which can be used to show that the tensions for the strings and 5-branes transform correctly under the above S-duality transformation.

Another evidence for this S-duality conjecture, is that the SO(32) heterotic string can be found to appear as a soliton in the type I theory, which is what one would expect since the solitonic and fundamental strings switch places under an S-duality transformation [42]. Moreover, another strong evidence is that the type I string has the same world sheet structure as the heterotic string, see [43] for details.

U-duality

U-duality [5, 6] is essentially a combination of S-duality and T-duality, e.g., U-duality implies that a theory A which is compactified on a space with large volume and has the closed string coupling constant g_{s}, can be dual to a theory B which is compactified on a space with small volume and has the inverse coupling constant $\tilde{g}_{\mathrm{s}} = 1/g_{\mathrm{s}}$. Note, however, that the T-duality and the S-duality groups together only form a subgroup of the larger U-duality group.

A more precise definition of U-duality than the above is given by [5]: The U-duality group for a compactified string theory is conjectured to be the full discrete subgroup of the non-compact symmetry group for the corresponding low energy supergravity theory.

For example, this means that the U-duality group for type II (A or B) compactified on a two-torus, is given by SL(2,$\mathbb{Z}$)$\times$ SL(3,$\mathbb{Z}$), since the symmetry group of 10-dimensional

Space-time dimension d	Supergravity duality group	String theory T-duality group	Conjectured U-duality group
10A	SO(1,1)/$\mathbb{Z}$	-	-
10B	SL(2,$\mathbb{R}$)	-	SL(2,$\mathbb{Z}$)
9	SL(2,$\mathbb{R}$) $\times O(1,1)$	$\mathbb{Z}_2$	SL(2,$\mathbb{Z}$) $\times \mathbb{Z}_2$
8	SL(2,$\mathbb{R}$)$\times$ SL(3,$\mathbb{R}$)	O(2,2;$\mathbb{Z}$)	SL(2,$\mathbb{Z}$)$\times$ SL(3,$\mathbb{Z}$)
7	SL(5,$\mathbb{R}$)	O(3,3;$\mathbb{Z}$)	SL(5,$\mathbb{Z}$)
6	O(5,5)	O(4,4;$\mathbb{Z}$)	O(5,5;$\mathbb{Z}$)
5	$E_{6(6)}$	O(5,5;$\mathbb{Z}$)	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}$	O(6,6;$\mathbb{Z}$)	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}$	O(7,7;$\mathbb{Z}$)	$E_{8(8)}(\mathbb{Z})$

Table 2.2: Duality symmetries for type II string theory compactified down to d dimensions. This table was obtained from [5].

type II supergravity compactified on a two-torus is SL(2,$\mathbb{R}$)$\times$ SL(3,$\mathbb{R}$). For type II string theory compactified on a n-torus the conjectured U-duality group is given in table 2.2[10].

A simple example of U-duality is compactification of type II (A or B) string theory on a six-torus, which is U-duality invariant [5]. In this case we obtain from table 2.2 that $E_{7(7)}(\mathbb{Z})$ is the U-duality group, which has a subgroup SL(2,$\mathbb{Z}$)$\times$O$(6,6;\mathbb{Z})$, that correspond to the S-duality group times the T-duality group.

Another U-duality conjecture is that heterotic string theory (any of the two) compactified on a four torus is dual to the type IIA string theory compactified on K_3 [5, 6]. This means that the moduli spaces are mapped into each other except that the strongly coupled type IIA theory maps to the weakly coupled heterotic string theory and vise versa. Note that compactifying on a torus breaks no supersymmetry, while compactifying on K_3 breaks half of the supersymmetry. Hence, the two compactifications preserve the same amount of supersymmetry, which of course is necessary for them to be dual. Furthermore, this conjecture is very interesting because it relates type IIA string theory and heterotic string theory, which can not be related using perturbative dualities.

A first test of this duality is to check that the low energy theories in six dimensions are the same. This was done in [6][11], with the conclusion that the two six-dimensional theories are the same if we identify $\phi_{\rm H} = -\phi_{\rm IIA}$ and $g_{6,\rm H} = e^{-2\phi_{\rm IIA}} g_{6,\rm IIA}$, where $g_{6,\rm H}$ and $g_{6,\rm IIA}$ are the string frame metrics. This confirms that we have a strong-weak coupling duality. It also means that the two theories have the same moduli space of vacua, see, e.g., [6]. Another consistency check is obtained from the result that when the K3 becomes singular, the heterotic string gets an enhanced symmetry group. Here both the singularities and the enhanced symmetries have an A-D-E classification [6].

[10] For a similar table for the heterotic string theories, see [5].

[11] Note that already in [44] it was found that 11-dimensional supergravity compactified on K3 and heterotic supergravity compactified on T^3 have the same moduli space. Obviously this means that type IIA supergravity on K3 and heterotic supergravity on T^4, also have the same moduli space.

2.2.3 M-theory and strong coupling

So far, we have explained why it has been conjectured that the type IIB string theory is S-dual to itself and that the type I string theory and the SO(32) heterotic string theory are S-dual to each other. This means that we have obtained some understanding of the strong coupling regions of these three superstring theories. However, we have not said anything about the strong coupling regions of the $E_8 \times E_8$ heterotic string theory or the type IIA theory. It is clear that if these two theories are compactified down to d dimensions ($d < 10$) the strong coupling regions can be understood, using U-duality results. However, the strong coupling limit of the uncompactified theories can not be understood in this way.

In this subsection we are going to argue that the strong coupling limits of the type IIA string theory and the $E_8 \times E_8$ heterotic string theory is the same theory. Namely, a non-perturbative theory in *11 dimensions*, which contains membranes and 5-branes, but *not* strings, and has 11-dimensional supergravity as its low energy limit. We call this theory M-theory, where M stands for "magic, mystery or membrane according to taste" [45]. To be more precise: one believes that M-theory compactified on a circle leads to the type IIA string theory and compactified on a line interval (which breaks half of the supersymmetry) it leads to the $E_8 \times E_8$ heterotic string theory. Furthermore, the radius of the 11'th dimension is $R \sim g$, which means that for strong coupling the radius is large and we have to use M-theory in 11 uncompactified dimensions, while for weak coupling the radius is small and one can use a perturbative string theory description.

The strong coupling limit of type IIA string theory

We begin with a discussion of the relation between M-theory and the type IIA string theory. We will give several convincing arguments why there must exist an 11-dimensional theory which is the strong coupling limit of IIA string theory.

Type IIA superstring theory has the following perturbative and non-perturbative objects: the fundamental string (F1-string), the solitonic NS5-brane and Dp-branes, for $p = 0, 2, 4, 6$. Above, we have given the tension (mass density) for these objects. For fixed α' the dependence on the closed string coupling constant is: $T_{\rm F1} \sim 1$ for the F1-string, $T_{\rm NS5} \sim 1/g^2$ for the NS5-brane and $T_{\rm Dp} \sim 1/g$ for the Dp-branes. Hence, at weak coupling the F1-string is the fundamental object, which we quantize and use in perturbation theory, while the other objects are heavy solitonic objects. On the other hand, if we let the coupling g be large, things change dramatically. In this case the D0-brane will be the lightest object, with a mass (tension)[12]

$$m_{\rm D0} = \frac{1}{g\sqrt{\alpha'}} \, . \tag{2.97}$$

Note that since the D0-brane is a BPS object this mass is correct for arbitrary value of the coupling constant.

[12]Note the even though the tension for the NS5-brane scale as $T_{\rm NS5} \sim 1/g^2$, it is still not the lightest object at strong coupling, since the effective lenght scale is $\ell_{\rm NS5} \sim (T_{\rm NS5})^{1/6} \sim g^{1/3}\sqrt{\alpha'}$, which is smaller than the D0-brane length scale $m_{\rm D0}^{-1}$.

Next, consider a bound state with an arbitrary number of D0-branes, say n. Then we expect there to be an ultrashort multiplet with mass given by [15, 6]

$$m_{n\text{D0}} = n m_{\text{D0}} = \frac{n}{g\sqrt{\alpha'}} \,. \tag{2.98}$$

Furthermore, in the limit of large coupling, $g \to \infty$, the spectrum become continuous. Interestingly this is very reminiscent of what happens when one compactifies a theory on a circle with radius R (see the subsection about T-duality), where the momentum in the compactified direction becomes quantized as

$$p = \frac{n}{R} \,. \tag{2.99}$$

Note that for $R \to \infty$, the momentum becomes continuous and we have de-compactified. This suggests that the D0-brane spectrum can be viewed from an 11-dimensional perspective as Kaluza-Klein states, where the 11-dimensional theory is compactified on a circle with radius R. Comparing the two relations (2.98) and (2.99), we find the following relation between the radius of the new 11'th dimension, and the string length and coupling to be [6]

$$R = g\sqrt{\alpha'} \,. \tag{2.100}$$

Hence, for small radius we effectively have type IIA string theory with small coupling constant, while for large radius (large coupling constant) we have a new 11-dimensional theory, which is named M-theory. This is a truly spectacular result, which from a perturbative string theory perspective was very unexpected.

Since M-theory is the strong coupling limit of the type IIA string theory, it must be an inherently non-perturbative theory, with no arbitrary coupling constant, but only a length scale ℓ_{p}. The relation between this length scale and the IIA length scale and coupling can be obtained by comparing the 11 and 10-dimensional gravitational constants κ_{11} and κ_{10}. These are related as:

$$\kappa_{11}^2 = 2\pi R \kappa_{10}^2 \,, \tag{2.101}$$

where

$$2\kappa_{11}^2 = (2\pi)^8 \ell_{\text{p}}^9 \,, \quad 2\kappa_{10}^2 = (2\pi)^7 g^2 \alpha'^4 \,. \tag{2.102}$$

Using (2.100)-(2.102), we obtain the following relation between ℓ_{p}, α' and g:

$$\ell_{\text{p}}^2 = g^{2/3}\alpha' \,. \tag{2.103}$$

Finally, by combining (2.100) and (2.103) one obtains that

$$\alpha' = \frac{\ell_{\text{p}}^3}{R} \,, \quad g = \left(\frac{R}{\ell_{\text{p}}}\right)^{3/2} \,. \tag{2.104}$$

Hence, when the 11-dimensional radius is much smaller than the 11-dimensional length scale we effectively have a 10-dimensional theory, which is type IIA string theory.

In subsection 2.1.2 we found that type IIA supergravity is the low energy limit of IIA string theory. This suggests that the low energy limit of the strong coupling limit of IIA string theory (which is M-theory) must be 11-dimensional supergravity. Hence, type IIA supergravity must be obtained from a dimensional reduction of 11-dimensional supergravity. To show this we start by recalling that the 11-dimensional supergravity action is given by (2.60). Reducing this action implies that we demand that no fields depend on the 10'th space direction x^{10}. We choose to split the metric and three form in the following convenient way:

$$ds^2_{11} = e^{-2\phi/3} ds^2_{\text{IIA}} + e^{4\phi/3}(dx^{10} + C_\mu dx^\mu)^2 \ , \quad A_{\mu\nu,10} = B_{\mu\nu} \ , \quad A_{\mu\nu\rho} = C_{\mu\nu\rho} \ . \qquad (2.105)$$

Next, inserting this relation into the 11-dimensional supergravity action (2.60), gives, after a straight forward calculation, the type IIA supergravity action (2.61), in the string frame.

A further test of the conjectured M-theory type IIA relation, is to show that all p-branes in IIA string theory have an M-theory interpretation [46]. However, before we can answer this question, we must discuss what possible branes can exists in M-theory. In subsection 2.2.1 we found that there are half supersymmetric solutions of 11-dimensional supergravity, which correspond to a membrane, 5-brane, wave and a KK6-brane. It therefore seems natural to expect that these branes exist in M-theory. They are known as the M2-brane, M5-brane, MW (M-wave) and KK6-brane. Note that only the M2-brane and M5-brane can be said to be completely 11-dimensional, since the MW and the KK6-brane always have a compact direction. Above, we have argued that the D0-brane is obtained from the MW. Furthermore, the F1-string and D2-branes are obtained from the M2-brane, where the M2-brane has one or zero directions in the 11:th direction (i.e., the 10'th space direction x^{10}), respectively. Next, the D4-brane and NS5-branes are obtained from the M5-brane, where the M5-brane has one or zero directions in the 11:th direction, respectively. Finally, we obtain the D6-brane from the KK6-brane. A further check of the type IIA M-theory relation is that the tensions for the 10-dimensional branes are correctly obtained from the 11-dimensional branes. That this is so, is easily checked and we have summarized this in table 2.3.

Yet another test of the type IIA string theory/M-theory conjecture is that the membrane action under double dimensional reduction reduces to the string action (2.33). This will be shown here. For more details on the bosonic case and the full supersymmetric case we refer to the original paper by Duff, Howe, Imani and Stelle [47].

The bosonic part of the supermembrane action is given by

$$\begin{aligned} S_{\text{M2}} &= -\frac{T_{\text{M2}}}{2}\int_\Sigma d^3\sigma \Big(\sqrt{-\gamma}\gamma^{\hat{a}\hat{b}}\partial_{\hat{a}}X^M \partial_{\hat{b}}X^M g_{MN} - \sqrt{-\gamma} \\ &\quad + \frac{1}{3}\epsilon^{\hat{a}\hat{b}\hat{c}}\partial_{\hat{a}}X^M\partial_{\hat{b}}X^N\partial_{\hat{c}}X^P A_{MNP}\Big) \ , \end{aligned} \qquad (2.106)$$

where $\gamma_{\hat{a}\hat{b}}$ is the auxiliary world sheet metric, $\gamma = \det\gamma_{\hat{a}\hat{b}}$, $\hat{a}, \hat{b}$ are the three-dimensional world volume indies and M, N are the 11-dimensional target-space indices.

M-brane	Tension	Type IIA-brane	Tension
MW	$\frac{1}{R}$	D0	$\frac{1}{R} = \frac{1}{g\sqrt{\alpha'}}$
M2	$\frac{1}{(2\pi)^2\ell_p^3}$	F1	$\frac{2\pi R}{(2\pi)^2\ell_p^3} = \frac{1}{2\pi\alpha'}$
M2		D2	$\frac{1}{(2\pi)^2\ell_p^3} = \frac{1}{(2\pi)^2 g\alpha'^{3/2}}$
M5	$\frac{1}{(2\pi)^5\ell_p^6}$	D4	$\frac{2\pi R}{(2\pi)^5\ell_p^6} = \frac{1}{2\pi g\alpha'^{5/2}}$
M5		NS5	$\frac{1}{(2\pi)^5\ell_p^6} = \frac{1}{(2\pi)^5 g^2\alpha'^3}$
KK6	$\frac{(2\pi R)^2}{(2\pi)^8\ell_p^9}$	D6	$\frac{(2\pi R)^2}{(2\pi)^8\ell_p^9} = \frac{1}{(2\pi)^5 g\alpha'^{7/2}}$

Table 2.3: Relations between branes in type IIA string theory and M-theory.

Next, we perform a double dimensional reduction of the 11-dimensional M2-brane action. This is done by splitting the background metric and three form as in (2.105), and the world volume metric as follows [47]:

$$ds_3^2 = e^{-2\varphi/3}ds_2^2 + e^{4\varphi/3}(dx^{10} + \tilde{C}_a dx^a)^2 \ , \tag{2.107}$$

and demand that all fields are independent of the target-space coordinate x^{10} and world volume coordinate σ^2. We also demand that $\sigma^2 = x^{10}$. Performing the reduction, using (2.105) and (2.107), gives the string action [47]

$$\begin{aligned} S_{\rm F1} &= -\frac{T_{\rm F1}}{2}\int_\Sigma d^2\sigma\Big(\sqrt{-\gamma}e^{2(\varphi-\phi)/3}[\gamma^{ab}\partial_a X^\mu\partial_b X^\nu g_{\mu\nu} + \gamma^{ab}(C_a - \tilde{C}_a)(C_b - \tilde{C}_b) \\ &+ e^{2(\phi-\varphi)}] - \sqrt{-\gamma} + \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}\Big) \ , \end{aligned} \tag{2.108}$$

where we have used that $T_{\rm M2} = 2\pi R T_{\rm F1}$ and $C_a = \partial_a X^\mu C_\mu$. From this action it is easy to obtain the algebraic equations of motions for φ and $\tilde{C}_a$. These are given by

$$\tilde{C}_a = C_a = \partial_a X^\mu C_\mu \ , \quad \varphi = \phi \ . \tag{2.109}$$

Hence, the world volume fields φ and $\tilde{C}_a$ are the pullbacks of the background fields ϕ and C_μ. Next, inserting (2.109) in (2.108) gives the following equivalent string action

$$S_{\rm F1} = -\frac{T_{\rm F1}}{2}\int_\Sigma d^2\sigma\Big(\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu g_{\mu\nu} + \epsilon^{ab}\partial_a X^\mu\partial_b X^\nu B_{\mu\nu}\Big) \ , \tag{2.110}$$

which is the string action discussed in (2.33), except for the higher order dilaton term. Hence, the string action coupled to background fields can be obtained from dimensional reduction of the 11-dimensional M2-brane action coupled to the background 11-dimensional metric and three form.

Although we have obtained a string action from a M2-brane action, we find it a bit disturbing that we did not obtain the dilaton term. As a matter of fact, so far, there is very little understanding of the 11-dimensional origin of the dilaton term at all. In our opinion this is an important problem to solve.

In this subsection it has been found that type IIA string theory can be obtained from M-theory by dimensional reduction. We also know that type IIA and IIB string theory are T-dual when compactified on a small and large circle, respectively. This means that it must be possible to also relate M-theory and type IIB string theory. To do this we begin by compactifying M-theory on a torus T^2, with radii R_{11} and R_{10}. This can be viewed as type IIA string theory compactified on a circle with radius R_{10}. As before, we have that $\alpha' = \frac{\ell_p^3}{R_{11}}$ and $g_A = \left(\frac{R_{11}}{\ell_p}\right)^{3/2}$, when comparing M-theory and type IIA parameters. Next, using T-duality, the above picture must be equivalent to type IIB string theory compactified on a circle with radius $R_B = \alpha'/R_{10}$. T-duality relates the type IIB coupling constant to the IIA coupling constant as follows [16]: $g_B = g_A \frac{\sqrt{\alpha'}}{R_{10}}$. Using this and the type IIA/M-theory relations, we obtain the following relations between M-theory and type IIB string theory parameters:

$$g_B = \frac{R_{11}}{R_{10}} , \quad R_B = \frac{\ell_p^3}{R_{10}R_{11}} . \tag{2.111}$$

From these we see that if we let both $R_{10} \to 0$ and $R_{11} \to 0$, then $R_B \to \infty$. Hence, we have obtained type IIB string theory in 10 dimensions, since we have de-compactified the type IIB theory. We also see from (2.111) that the type IIB coupling is given by the radius of the 11'th dimension divided by the type IIA radius. This is interesting, because it means that type IIB S-duality (i.e., $g_B \to 1/g_B$), from an M-theory perspective, is nothing but switching place between the two radii of the torus. It also implies that the entire SL(2,$\mathbb{Z}$) invariance of type IIB string theory is a consequence of the modular invariance of the complex structure of the torus, on which M-theory is compactified. This is an amazing result, which is totaly unexpected from a string theory point of view.

The strong coupling limit of $E_8 \times E_8$ heterotic string theory

Next, we continue by arguing that M-theory on a line interval gives the $E_8 \times E_8$ heterotic string theory, for small "radius" R. We begin by showing that the bosonic part of 11-dimensional supergravity reduced on a line interval (half circle), which breaks half of the supersymmetry, gives the bosonic part of $E_8 \times E_8$ heterotic supergravity. This reduction is similar to the reduction to type IIA supergravity, except for the following: Reducing on a line interval implies that the physics in 11 dimensions must be invariant under the following transformation:

$$\tilde{x}^{10} = x^{10} + 2\pi , \quad \tilde{x}^{10} = -x^{10} , \quad \tilde{x}^{\mu} = x^{\mu} . \tag{2.112}$$

If we demand that the action and ds^2 are invariant under this transformation, we find the following transformation rules for the 11-dimensional metric

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} , \quad \tilde{g}_{\mu,10} = -g_{\mu,10} , \quad \tilde{g}_{10,10} = g_{10,10} , \tag{2.113}$$

and three form

$$\tilde{A}_{\mu\nu\rho} = -A_{\mu\nu\rho} , \quad \tilde{A}_{\mu\nu,10} = A_{\mu\nu,10} . \tag{2.114}$$

Next, for the reduction to be consistent we demand that all fields must be the same in the two points x^{10} and $-x^{10}$. This means that only the parts of the metric and three form that are even under the transformation (2.112) can remain in the spectrum. Hence, the off-diagonal part of the metric (which would give a vector) and the part of the three form which reduces to a three form, must be discarded in the reduction. This leads to the following 10-dimensional bosonic action (use the type IIA action (2.61) with $C_1 = 0$ and $C_3 = 0$, which implies that the CS term is zero)

$$S_{\rm I} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\phi}\Big(R + 4(\partial\phi)^2 - \frac{1}{12}H_3^2\Big) . \tag{2.115}$$

This is however not the end of the story, because the supersymmetric version of this action has gravitational anomalies. These anomalies can be canceled by introducing into the 10-dimensional action a Yang-Mills term $\sim \int d^{10}x \mathrm{Tr}F^2$ [48], with the correct gauge group. These gauge fields can be seen as living on the two 10-dimensional boundaries of the line interval. In order to precisely cancel the gravitational anomalies one has to choose the gauge group to be $E_8 \times E_8$, where we have one E_8 on each 10-dimensional boundary. Moreover, including a gauge field with gauge group $E_8 \times E_8$ and the relevant higher order gravitational corrections (which implies that $H_3 \to \tilde{H}_3$, see [16]), gives the complete bosonic part of the $E_8 \times E_8$ heterotic supergravity action (2.64). A derivation of this anomaly cancelation from a 11-dimensional perspective can be found in [48, 49]. That this works is an important test of the conjecture relating M-theory on a line interval and $E_8 \times E_8$ heterotic string theory.

Similar to the type IIA case the "radius" $R \sim g$, which means that when compactifying M-theory on a line interval with a "small" radius ($g << 1$) we have a valid perturbative description using the $E_8 \times E_8$ heterotic string theory, while for large radius ($g >> 1$) we have to use M-theory, since the string theory is strongly coupled.

In this section, we have discussed the strong coupling limit of all the five consistent superstring theories. A unifying picture has emerged, where the new central theme is a new and entirely non-perturbative theory in 11 dimensions, named M-theory. All five string theories have been argued to be different perturbative limits of M-theory. This has of course not been rigorously proven. However, there are many convincing arguments why this conjecture must be true, of which we have discussed a few.

In figure 2.1 we have summarized all perturbative and non-perturbative dualities in nine and higher dimensions.

M-theory

As is clear from the discussion above, M-theory is a non-perturbative theory with a length scale $\ell_{\rm p}$, but no coupling constant. M-theory contains M2-branes and M5-branes, which are each others electro-magnetic duals. It is also known that at low energy (i.e., when the dimensionless "coupling" $E\ell_{\rm p} << 1$, where E is the energy scale), 11-dimensional

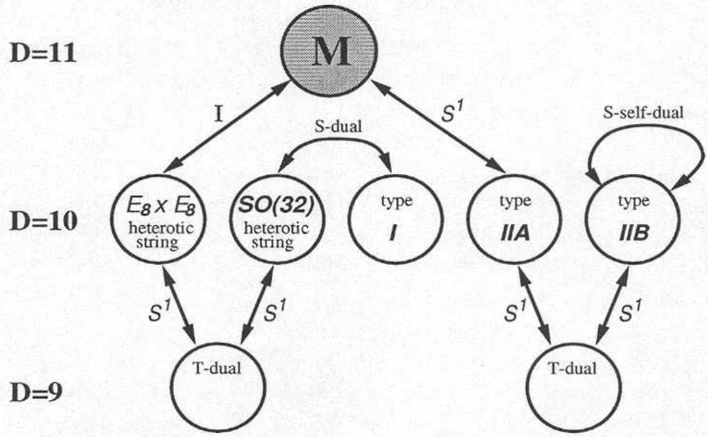

Figure 2.1: The net of string dualities in higher dimensions. As shown, the perturbatively different string theories in 10 dimensions are connected by duality transformations using certain compactifications. Moreover, in 11 dimensions a new theory arises, which is called M−theory and whose compactification on the circle S^1 or the finite line interval I gives rise to type IIA and $E_8 \times E_8$ heterotic string theory, respectively. This figure is taken from [50].

supergravity should give a good description of the physics. Apart from this we know very little about M-theory. For example, one of the most interesting questions is: What are the relevant (fundamental) degrees of freedom of M-theory? In perturbative string theory a similar question is easy to answer, since at week coupling ($g << 1$) we have one fundamental object, the F1-string and a bunch of solitonic p-branes. Hence, in this case the answer is that the F1-string gives the fundamental degrees of freedom (d.o.f.) of string theory. In the M-theory case this question is much harder to answer, since we have two objects, which have the *same* effective length scale, ℓ_p. At this stage we will leave this question unanswered and just assume that the M2-brane gives a fairly good description of M-theory. Note, however, that the M2-brane action in general is a non-renormalizable theory [51]. This is however not the only problem. It is also very difficult to quantize the M2-brane, since this is not a free theory, but an interacting one. Furthermore, it can also be shown that the spectrum of the super-membrane is continues [52]. We note that this does not mean that there is something wrong with the theory. It does, however, mean that the theory is very complicated.

In 1996 there was a proposal for a non-perturbative definition of M-theory, the so called matrix theory proposal [53][13]. Before discussing matrix theory we start by discussing the M2-brane action in light-cone coordinates, i.e., in the light-cone gauge. From this

[13] Note that we here only give a very sketchy description of matrix theory. For a much more complete introduction we refer to the original BFSS paper [53], or for a more up to date review, we suggest [54].

action we easily obtain the Hamiltonian in light-cone gauge. Unfortunately, this light-cone membrane theory is, contrary to the string, still very difficult to quantize due to non-linearities. One way to solve this is to use some clever regularization. If one assumes that the membrane world volume is $\Sigma \times \mathbb{R}$ (where Σ is a Riemann surface), it is consistent to use a regularization procedure in which functions are mapped to finite $N \times N$ matrices. After this regularization we have a classical theory with a finite number of d.o.f., which can be quantized. Quantization gives a 0+1-dimensional supersymmetric U(N) theory, where the matrices X^i ($i = 1, 2, \ldots, 9$) are in the adjoint representation. For larger and larger N one obtains a better and better description of the membrane, i.e., in the limit $N \to \infty$, this 0+1-dimensional supersymmetric U(N) theory gives a microscopic description of the supermembrane, in light-cone coordinates.

Moreover, in 1996 that it was conjectured (the BFSS conjecture [53]) that the above 0+1-dimensional supersymmetric quantum mechanics theory, in the limit $N \to \infty$, could be a candidate for a microscopic description of M-theory in light-cone coordinates. However, in [53] the starting point was not a membrane in 11-dimensions, but instead they considered M-theory compactified on a circle with radius R. From the discussion above we know that the momentum modes in the compact direction can be interpreted as the type IIA string theory D0-branes. In [53] it was argued that in the "infinite momentum frame" (i.e., take both $p^{11} = (N/R) \to \infty$ and $R \to \infty$, in order to have 11 un-compactified dimensions), where the momentum is taken to be very large and the theory becomes non-relativistic, the dynamics should be described by the large N limit of a system of non-relativistic D0-branes. Such a system of D0-branes, in the non-relativistic limit, is given by the matrix quantum mechanics Lagrangian, which is obtained from dimensional reduction of 10-dimensional super Yang-Mills down to 0+1 dimensions, see, e.g., [15, 55].

What is very interesting is that both the regularization of the membrane, and the D0-brane argument lead to two theories, which are equivalent in the infinite momentum frame. Hence, it seems to be possible (at least in the infinite momentum frame) to somehow view D0-branes as the fundamental constituents of M-theory and the membrane as built from an infinite amount of D0-branes.

That M-theory in the infinite momentum frame can be described by supersymmetric matrix quantum mechanics has been tested in several ways, see, e.g., [53, 54, 51], but it is far from rigourously proven. There are, however, some problems with matrix theory: For example, it is not a theory where 11-dimensional Lorentz invariance is manifest. In fact, it has so far not been rigorously proven that matrix theory is Lorentz invariant at the quantum level [54, 51]. It has also not been possible to obtain the correct higher order corrections to 11-dimensional supergravity from matrix theory [54]. It is not entirely clear why this does not work, and is an important problem to solve.

To conclude: So far, there has been one really interesting non-perturbative proposal for a description of M-theory, namely Matrix theory. However, at present it is very unclear if matrix theory gives a complete and correct description of M-theory. In fact, that matrix theory does not give a complete description of M-theory seems clear, since it fails to describe M-theory compactified on a n-dimensional torus, when $n > 5$ [54]. This suggests that matrix theory at most gives a correct description of a sector of M-theory. It is therefore fair to say that the quest for a non-perturbative microscopic description of M-theory is far from over.

2.3 The AdS/CFT correspondence

In this section we will discuss the AdS/CFT correspondence [9] and try to motivate why it is believed to be correct. We begin by giving three versions of the correspondence. Next, we will give some convincing argument why at least the 'weakest' form of the correspondence seems to be valid.

The AdS_{d+1}/CFT_d correspondence[14] (sometimes called the Maldacena conjecture) was first proposed by Maldacena [9] and later more rigorously defined by Gubser, Klebanov and Polyakov [60] and Witten [61]. The correspondence describes a duality relation between a SCFT (superconformal field theory) living on the d-dimensional boundary (which is compactified d-dimensional Minkowski space-time) of AdS_{d+1}, and a compactified string theory or M-theory with vacuum $AdS_{d+1} \times M^{D-(d+1)}$, where $D = 10$ or 11.

The AdS/CFT correspondence in its strongest form is stated as follows (for AdS_5 and a four-dimensional SCFT): The following two theories are equivalent (or dual), see, e.g., [59]:

1. Type IIB superstring theory on $AdS_5 \times S^5$, with N units of five form flux, where the radius of AdS_5 and S^5 is R and g is the closed string coupling constant.

2. $\mathcal{N} = 4$ superconformal Yang-Mills in four dimensions, with Yang-Mills coupling constant $g^2_{\rm YM}$ and gauge group SU(N).

The closed string coupling constant g, string length $(\alpha')^{1/2}$, radius R and RR scalar expectation value C_0 in **1** and the Yang-Mills coupling constant $g^2_{\rm YM}$ and instanton angle θ_I in **2**, are related as follows:

$$2\pi g = g^2_{\rm YM} \ , \quad R = [4\pi g N(\alpha')^2]^{1/4} \ , \quad C_0 = \theta_I \ . \tag{2.116}$$

That these two theories are dual to each other implies that there exists a well defined map [60, 61] between the fields and correlation functions on the superstring side and the gauge invariant operators and correlation functions in the superconformal YM theory. The precise relation between a mode ϕ in string theory on $AdS_5 \times S^5$ and the corresponding operator $\mathcal{O}$ in the $\mathcal{N} = 4$ superconformal Yang-Mills theory, is given by [60, 61]

$$\left\langle \exp \int \phi_0 \mathcal{O} \right\rangle_{\rm CFT} = Z_{\rm S}(\phi_0) \ , \tag{2.117}$$

where $Z_{\rm S}(\phi_0) = \exp(-I_{\rm S}(\phi))$ is the string theory partition function, $I_{\rm S}(\phi)$ the string theory action and ϕ_0 is the boundary value of ϕ, see [61] for further clarifications.

In the above version, the correspondence is assumed to be valid for any values of the parameters in (2.116). The validity of this so called strong version of the correspondence is

[14] For reviews, see, e.g., [56, 57, 58, 59].

at the moment impossible to check. However, if we impose restrictions on the parameters, it is easier to obtain relevant results.

The first restriction that we impose is to take the integer N towards infinity and at the same time keep the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ fixed. Note that this implies that both the closed string coupling constant and Yang-Mills coupling constant are much less than one. With this restriction we have a correspondence between a classical string theory and $\mathcal{N} = 4$ superconformal Yang-Mills theory. Note that $1/N$ corrections to the YM theory corresponds to a loop expansion in powers of g on the string theory side, see, e.g., [59]. Although this version of the correspondence is weaker it is still very difficult to perform any tests. However, see [62] for some interesting results.

A further restriction is to take also the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ to be large. Note that in this limit g is still small, which is seen from (2.116). This implies that we now have a correspondence between type IIB classical supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ superconformal Yang-Mills theory with large 't Hooft coupling. It can be seen from (2.116) that an $\lambda^{-1/2}$ expansion on the YM side corresponds to an α'/R^2 expansion on the supergravity side [57], since $\lambda^{-1/2} \sim \alpha'/R^2$. This means that for large 't Hooft coupling, α'/R^2 is very small, i.e., string theory can be approximated with supergravity. This version of the correspondence is the weakest form and the most well understood and investigated. There is now a fair amount of evidence that at least this version of the AdS/CFT correspondence is correct. In the reminder we only consider this version of the correspondence. An interesting consequence of the above discussion is that when supergravity is a valid approximation of string theory (i.e., $\alpha'/R^2 << 1$), (2.116) implies that the dual field theory is strongly coupled, since $\lambda >> 1$. It is therefore possible to use perturbative supergravity calculations to obtain non-perturbative results in the dual field theory.

We continue by giving the following argument why the AdS/CFT correspondence is expected to be valid. Consider a stack of N D3-branes in 10-dimensional flat space. Here closed strings perturbatively describe the excitations of empty space, while open strings ending on the D3-branes perturbatively describe the excitations of the D3-branes. For low energy the physics of the D3-branes in four dimensions is described by a U(N) $\mathcal{N} = 4$ superconformal Yang-Mills theory, which is decoupled from the massless closed string states. The massless closed string states can be described by 10-dimensional type IIB supergravity.

Next, we describe the same set up but using a supergravity description. As been discussed in subsection 2.2.1, there exists a supergravity solution which corresponds to a stack of N D3-branes. The metric of this D3-brane solution is given by

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2 + \cdots + (dx^3)^2) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) \ , \\ H &= 1 + \frac{R^4}{r^4} \ , \quad R^4 = 4\pi g N (\alpha')^2 \ , \end{aligned} \tag{2.118}$$

where H is a harmonic function on the six-dimensional transverse space and N is the number of D3-branes on top of each other. Note that the factor in front of $(dt)^2 = (dx^0)^2$, implies that if an observer at a distance r from the stack of branes measures the energy to be E_r, then an observer at infinity measures $E_\infty = H^{-\frac{1}{4}}E_r$. This means that close to the horizon (i.e., $r \sim 0$) there is a very large red-shift. The consequences of this is that at low energy there are two decoupled theories. These are: (1) free supergravity in the bulk and (2) the near horizon region of the D3-branes. Excitations with long wavelength in the bulk are described by supergravity in flat space while excitations with any wavelength close to the horizon are described by the near horizon solution, which is $AdS_5 \times S^5$. The precise near horizon limit is given by taking $\alpha' \to 0$, while keeping the energy and [9]

$$x^\mu \ , \quad g_{\rm YM}^2 = 2\pi g \ , \quad u = \frac{r}{\alpha'} \ , \tag{2.119}$$

fixed. Inserting this limit in (2.118) gives

$$\frac{ds^2}{\alpha'} = \left[\frac{u^2}{\tilde{R}^2}(-(dx^0)^2 + \cdots + (dx^3)^2) + \frac{\tilde{R}^2}{u^2}(du^2)\right] + \tilde{R}^2 d\Omega_5^2 \ , \tag{2.120}$$

where $\tilde{R}^4 = 2g_{\rm YM}^2 N = 2\lambda$. This is an $AdS_5 \times S^5$ solution, where the AdS_5 and S^5 spaces have equal radius $\tilde{R}$ (in units of $\sqrt{\alpha'}$). It is important to note that ds^2/α' is fixed in the near horizon limit (2.119), which implies that the supergravity action is also fixed in the near horizon limit. This is easily seen from the gravity part of the type IIB supergravity action (in the string frame):

$$S_{IIBg} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\phi}R \ . \tag{2.121}$$

Now inserting into the action that the metric scales as $g_{MN} \sim \alpha'$ (see (2.120) above), which implies that $R \sim \alpha'^{-1}$, and using that $\kappa \sim \alpha'^2$, we obtain that S_{IIBg} is fixed in the near horizon limit[15].

If we now compare the two descriptions we see that there are two decoupled theories in both cases, where the bulk supergravity is common to both of them. This means that since both the open/closed string description and supergravity description are equivalent (dual) descriptions, we conclude that the $\mathrm{U}(N)$ $\mathcal{N} = 4$ superconformal Yang-Mills theory and type IIB supergravity on $AdS_5 \times S^5$ are dual to each other. We note here that above when we formulated the correspondence it was stated that it relates an $\mathrm{SU}(N)$ field theory to a supergravity theory, but in the discussion here we used a $\mathrm{U}(N)$ theory. This might seem to be conflicting, however, both these statements are true considering the following: The correspondence is between an $\mathrm{SU}(N)$ $\mathcal{N} = 4$ superconformal Yang-Mills theory and type IIB supergravity on $AdS_5 \times S^5$, if only the supergravity bulk modes are included. As a consequence the $\mathrm{U}(1)$ doubleton multiplet which is confined to the boundary of AdS_5 is not included in the supergravity spectrum. However, if this $\mathrm{U}(1)$ part which is decoupled from the $\mathrm{SU}(N)$ part is included in the spectrum, then the supergravity solution corresponds to

[15] If we also consider p-form fields, (e.g., the NS-NS two-form B_2) they must scale as $A_{M_1 \ldots M_p} \sim \alpha'^{\frac{p}{2}}$ in the near horizon limit, which is easily seen from the full supergravity action (2.62) and using how the metric scale.

a U(N) field theory. Note that U(N)=SU(N)×U(1). This means that the correspondence is valid for both U(N) and SU(N) field theories.

Next, we continue by giving further evidence for the correspondence. For the correspondence to be correct it is necessary that the global symmetries are the same for the two theories. The continuous global symmetries of SU(N) $\mathcal{N}=4$ superconformal Yang-Mills theory are given by the superconformal group $SU(2,2|4)$. The maximum bosonic subgroup is given by SU(2,2)×SU(4)$_{\rm R}$ ≃SO(2,4)×SO(6)$_{\rm R}$. Here SO(2,4) is the four-dimensional conformal group, while SO(6)$_{\rm R}$ is the R-symmetry group. On the supergravity side we identify the conformal group with the AdS_5 isometry group, while SO(6) is the isometry of S^5. The two sides also have the same amount of supersymmetry, which implies that all the global continuous symmetries $SU(2,2|4)$ can be found on both sides of the correspondence. Both sides are also believed to have a non-perturbative SL(2,$\mathbb{Z}$) duality symmetry, which acts on the complex coupling τ. Here τ is the following combination of the closed string coupling constant g and expectation value of the RR scalar C_0 [59]:

$$\tau = \frac{C_0}{2\pi} + \frac{i}{g} \, . \tag{2.122}$$

It has also been shown that additional discrete $\mathbb{Z}_N$ symmetries, which arise if the theories are compactified on non-simply connected manifolds, can be matched between the two sides [65].

As we mentioned above, it is possible to match operators in the field theory with supergravity fields in the bulk. The relation between the mass M of the supergravity fields and the dimension Δ of the operators, is given by [59]:

$$\begin{aligned}
&\text{scalars}: && \frac{M^2}{m^2} = \Delta(\Delta-4) \, , \\
&\text{spin}\frac{1}{2}, \frac{3}{2}: && \left|\frac{M}{m}\right| = \Delta - 2 \, , \\
&\text{p}-\text{forms}: && \frac{M^2}{m^2} = (\Delta-p)(\Delta+p-4) \, , \\
&\text{gravitons}: && \frac{M^2}{m^2} = \Delta(\Delta-4) \, ,
\end{aligned} \tag{2.123}$$

where m is the inverse of the AdS_5 radius R. For a further discussion about the duality between SU(N) $\mathcal{N}=4$ superconformal Yang-Mills theory and type IIB supergravity on $AdS_5 \times S^5$, see, e.g., [56, 57, 58, 59].

The AdS/CFT correspondence is also expected to be valid in certain cases for M-theory. In particular, M-theory compactified on $AdS_4 \times S^7$ is conjectured to be dual to a three-dimensional $\mathcal{N}=8$ superconformal field theory with supergroup $OSp(4|8)$ [9], while M-theory on $AdS_7 \times S^4$ is conjectured to be dual to a six-dimensional interacting $\mathcal{N}=(2,0)$ superconformal tensor theory with supergroup $OSp(6,2|4)$ [9]. These conjectures are not as well tested as the conjecture in AdS_5 but are believed to be correct. The reason that the conjecture in AdS_5 is better tested, is because the $\mathcal{N}=4$ SYM theory is much better understood than the field theories in three and six dimensions.

We note here that for all cases that we have discussed so far, the correspondence is between maximally supersymmetric theories. For less than maximum supersymmetry the AdS/CFT correspondence is still expected to be valid. However, in these cases it is harder to check the correspondence. The simplest examples with less supersymmetry are given by superconformal field theories which are AdS/CFT dual to type IIB supergravity on $AdS_5 \times M^5$. Here M^5 is a compact Einstein manifold, i.e., $R_{\mu\nu} = a g_{\mu\nu}$, where a is a positive constant. The number of Killing spinors on M^5 determines the number of supersymmetries in the dual SCFT. One type of M^5 space is $M^5 = S^5/\Gamma$, where Γ is a discrete subgroup of SO(6). Depending on Γ one obtains supergravity duals of $\mathcal{N} = 2, 1, 0$ SCFT. From a brane point of view $AdS_5 \times (S^5/\Gamma)$ is the near horizon region of N D3-branes placed at an orbifold singularity of R^6/Γ. This implies that the full D3-brane solution with an orbifold singularity at $r = 0$, interpolates between $AdS_5 \times (S^5/\Gamma)$ $(r \to 0)$ and Minkowski$_4 \times (R^6/\Gamma)$ $(r \to \infty)$. We note that in this case M^5 is locally equivalent to S^5.

It is also possible to choose an M^5 which is not locally equivalent to S^5. In this case the D3-branes are placed at a conifold singularity. As far as we know the only M^5 space of this type which preserves any supersymmetry is the compact Einstein space $T^{1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2))/U(1)$. This is the supergravity dual of a $\mathcal{N} = 1$ SCFT [66]. In this case the D3-brane solution interpolates between $AdS_5 \times T^{1,1}$ $(r \to 0)$ and Minkowski$_4 \times con_6$ $(r \to \infty)$, where the six-dimensional conifold metric can be written as

$$ds_6^2 = dr^2 + r^2 d\tilde{s}_5^2 \; . \qquad (2.124)$$

Here $d\tilde{s}_5^2$ is the metric on $T^{1,1}$. This six-dimensional space is called a cone and $T^{1,1}$ is its five-dimensional base. Note that the six-dimensional conifold is Ricci flat although $T^{1,1}$ is not.

Also for M-theory it is possible to obtain a correspondence in the case of less supersymmetry, by replacing S^7 (or S^4) with a compact manifold with a smaller number of Killing spinors. For $AdS_4 \times M^7$ there exists AdS/CFT correspondences for $\mathcal{N} = 8, 4, 3, 2, 1, 0$, while $AdS_7 \times M^4$ corresponds to SCFT's with $\mathcal{N} = 2, 1, 0$.

Another interesting class of solutions with less supersymmetry are warped compactifications. This means that there is a warp factor in front of the AdS metric, i.e., a function depending on the M^5 coordinates. In Paper II both 'commutative' and noncommutative versions of a warped solution called the Pilch-Warner solution [68] are discussed. The Pilch-Warner solution is dual to a four-dimensional SCFT with $\mathcal{N} = 1$ supersymmetry. This superconformal fix point is 'connected' with the $\mathcal{N} = 4$ fix point, which means that there exists a renormalization group flow between the two conformal fix points, see [69]. For an introduction to renormalization group (RG) flows, using supergravity duals and the AdS/CFT correspondence, see, e.g., [70, 57, 59] and Paper I.

3

Supergravity, duality and bound states of branes

In this chapter we will show how supergravity solutions corresponding to bound states of different branes can be obtained from a single brane solution (or a solution corresponding to a stack of N identical branes on top of each other). A bound state is a collection of strings, branes etc., which are bound together in the sense that they in general have a total energy that is lower than the total energy of the individual strings, branes etc. A bound state that has the same energy as the total individual energy of the parts, is called a threshold bound state. An example of a non-threshold bound state is a state containing m fundamental strings (F1-strings) and n D1-strings. The tension (energy) is given by [71]

$$T_{m,n} = T\sqrt{m^2 + \frac{n^2}{g_s^2}} \ , \tag{3.1}$$

where $T = 2\pi\alpha'$ is the tension for one F1-string and g_s is the closed string coupling constant. The tension for m F1-strings and n D1-strings is given by

$$T_{Fm} = Tm \ , \quad T_{Dn} = T\frac{n}{g_s} \ , \tag{3.2}$$

respectively. From (3.1) and (3.2) we see that the tension (energy) for a bound state containing m F1-strings and n D1-strings is smaller than the total tension for m free F1-strings and n free D1-strings, since $T_{m,n} \leq (T_{Fm} + T_{Dn})$. Note that for $T_{m,n}$ to be strictly smaller both m and n has to be non-zero. Hence, a bound state where e.g., $m = 0$ but $n \neq 0$ (i.e., a bound state consisting of n D1-strings) is a threshold bound state. Also a bound state containing N Dp-branes, for arbitrary p, is a threshold bound state. This means that two Dp-branes can be separated from each other without any change of energy. As was mentioned in chapter 2, the low energy world volume theory for a stack of N D3-branes on top of each other is U(N) SYM in four dimensions. If one of these D3-branes is separated from the stack by a distance a, the U(N) gauge symmetry is broken to U($N-1$)×U(1). This is a Higgs effect which gives rise to a mass for W bosons. The mass is $m \sim Ta$, where T is the tension for an open string which is stretched between the stack and the separated D3-brane. Naturally, if $a \to 0$ the W bosons become massless and the gauge symmetry is restored to U(N).

To be able to obtain supergravity solutions corresponding to bound states of different types of branes is very important in, e.g., investigations of non-gravitational theories which have noncommutative coordinates. For example, similar to how the D3-brane solution was used to get the supergravity dual of four-dimensional SYM in section 2.3, the bound state solution D3-D1 can be used to obtain the supergravity dual of four-dimensional noncommutative SYM (NCYM), see section 4.2. It is therefore important to know, as will be explained below, how this bound state solution can be found. There are two ways to derive supergravity solutions which correspond to bound states: (1) solve the supergravity equations of motion for the appropriate non-zero fluxes, or (2) start with a known brane solution and use some solution generating technique to obtain the bound state solution. Of these two, the second one is usually the most practical approach. To obtain a bound state solution with many different branes by solving the supergravity equations of motion can be very difficult, see [72]. Note that below when we use the name bound state, we mean bound states which contain several different types of branes, e.g., Dp-branes, D($p-2$)-branes, F-strings etc.

In the next section we show how the bound state D3-D1 (i.e., it contains both D3-branes and D1-branes) can be obtained from a D3-brane solution. This is done by using two different methods, which we also show give equivalent results. For examples on how bound states can be obtained by solving the supergravity equations of motion we refer to [72]. In section 3.2 and 3.3 we generalize some of the results to M-theory and in particular discuss the generalization of string theory T-duality to M-theory 'T-duality' and D-brane 'T-duality'.

3.1 T-duality as a solution generating technique

3.1.1 Different solution generating techniques

In this subsection we are going to discuss two different solution generating techniques, which are used to obtain supergravity solutions corresponding to bound states. The first one we call the rotation/boost method, while the second one is referred to as the O($p+1$,$p+1$) method. The latter method is further discussed in the subsection 3.1.2, see also Paper II-IV,VI-VII and [73]. We note that for both these solution generating techniques we have to start with a Dp-brane. To include NS5-branes in the bound state, we can use the O($p+1$,$p+1$) method together with S-duality (and possibly T-duality), or a more direct method, which is discussed in section 3.3, see also Paper VII section 5-6. An important feature of the various solution generating techniques is that none of them break supersymmetry. This implies that if we start with a Dp-brane solution which preserves n

supersymmetries, then the generated bound state also preserves n supersymmetries.

To obtain a bound state solution using the rotation/boost method, we start with a solution that only contains one type of branes. These branes are going to become the branes in the bound state with highest dimensionality. This means that if we, e.g., want to obtain a Dp-D$(p-2)$ bound state, we have to start with a Dp-brane solution. The next step is to T-dualize the solution in a direction x^p (parallel to the brane). This implies that if we have started with a Dp-brane solution, after the T-duality we have obtained a D$(p-1)$-brane solution smeared (see below) in the x^p direction. The reason for this is that T-duality in a direction x^p interchanges the Neumann and Dirichlet boundary conditions in this direction. Next, we rotate the D$(p-1)$-brane solution in the x^p and x^{p-1} directions. This implies that the Neumann and Dirichlet boundary conditions get mixed in the new x^p and x^{p-1} directions. Finally, we T-dualize back in the new x^p direction. From the open string boundary conditions we see that this has introduced a non-zero NS-NS $B_{p-1,p}$ field in the solution. As we will see below this implies that there is also a RR $C_{01\ldots(p-2)}$ field in the solution. This RR field implies that the new solution gives a non-zero D$(p-2)$-brane charge. The conclusion is therefore that we have obtained a supergravity solution corresponding to a Dp-D$(p-2)$ bound state. We also note that if we had performed a boost instead of a rotation above, we would have obtained a Dp-F1 bound state instead, since there is now a non-zero B_{0p} field in the solution, as well as a non-zero $C_{12\ldots(p-1)}$.

We continue by deriving the D3-D1 solution, using the rotation/boost method. This bound state solution was first derived in [74]. We start with the following type IIB D3-brane solution

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2+\cdots+(dx^3)^2)+H^{\frac{1}{2}}(dr^2+r^2d\Omega_5^2)\ , \\ e^{2\phi} &= g^2\ , \quad C_4=\frac{1}{gH}dx^0\wedge\cdots\wedge dx^3+\frac{4R^4}{g}\epsilon_4\ , \\ H &= 1+\frac{R^4}{r^4}\ , \end{aligned} \tag{3.3}$$

where H is a harmonic function on the transverse six-dimensional space, g is the closed string coupling constant and $d\epsilon_4$ is the volume form of the five-sphere. We start by T-dualizing in the x^3 direction. This gives a type IIA D2-brane solution, which is smeared in the x^3 direction (i.e., the harmonic function is the same as above). For T-duality between type IIA and type IIB solutions we use the Buscher rules [75, 76] (in the string frame)

$$\begin{aligned} \tilde{g}_{yy} &= \frac{1}{g_{yy}}\ , \quad \tilde{g}_{\mu\nu}=g_{\mu\nu}-\frac{g_{\mu y}g_{\nu y}-B_{\mu y}B_{\nu y}}{g_{yy}}\ , \\ \tilde{B}_{\mu y} &= \frac{g_{\mu y}}{g_{yy}}\ , \quad \tilde{B}_{\mu\nu}=B_{\mu\nu}-\frac{B_{\mu y}g_{\nu y}-g_{\mu y}B_{\nu y}}{g_{yy}}\ , \\ \tilde{g}_{\mu y} &= \frac{B_{\mu y}}{g_{yy}}\ , \quad e^{2\tilde{\phi}}=\frac{e^{2\phi}}{g_{yy}}\ , \end{aligned} \tag{3.4}$$

where y denotes the Killing coordinate with respect to which the T-dualization is applied, while μ,ν denote any coordinate direction other then y. The RR p-form fields transforms as [77]:

$$
\begin{aligned}
\tilde{C}_{(p)\mu\cdots\nu\rho y} &= C_{(p-1)\mu\cdots\nu\rho} - (p-1)\frac{C_{(p-1)[\mu\cdots\nu|y|}g_{\rho]y}}{g_{yy}} , \\
\tilde{C}_{(p)\mu\cdots\nu\rho\sigma} &= C_{(p+1)\mu\cdots\nu\rho\sigma y} + pC_{(p-1)[\mu\cdots\nu\rho}B_{\sigma]y} \\
&\quad + p(p-1)\frac{C_{(p-1)[\mu\cdots\nu|y|}B_{\rho|y|}g_{\sigma]y}}{g_{yy}} .
\end{aligned} \tag{3.5}
$$

Using the above T-duality rules we get

$$
\begin{aligned}
ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2) + H^{\frac{1}{2}}(dx^3)^2 + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) , \\
e^{2\phi} &= g^2 H^{\frac{1}{2}} , \quad C_3 = \frac{1}{gH} dx^0 \wedge dx^1 \wedge dx^2 , \quad C_5 = \frac{4R^4}{g}\epsilon_4 \wedge dx^3 .
\end{aligned} \tag{3.6}
$$

This is a D2-brane solution smeared in the x^3 direction, since the function H is harmonic on a six-dimensional space $(x^4 - x^9)$ and not on a seven-dimensional space $(x^3 - x^9)$, as it is for an 'ordinary' D2-brane solution. Next, we rotate the solution in the x^2 and x^3 directions, i.e., let

$$
\begin{aligned}
x^2 &= \cos\phi \hat{x}^2 + \sin\phi \hat{x}^3 , \\
x^3 &= -\sin\phi \hat{x}^2 + \cos\phi \hat{x}^3 .
\end{aligned} \tag{3.7}
$$

Inserting this coordinate transformation into (3.6) gives

$$
\begin{aligned}
ds^2 &= H^{-\frac{1}{2}}\{-(dx^0)^2 + (dx^1)^2 + [(\cos\phi)^2 + (\sin\phi)^2 H](d\hat{x}^2)^2\} \\
&\quad + 2H^{-\frac{1}{2}}\cos\phi\sin\phi[1-H]d\hat{x}^2 d\hat{x}^3 + H^{-\frac{1}{2}}[(\sin\phi)^2 + (\cos\phi)^2 H](d\hat{x}^3)^2 \\
&\quad + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) , \quad e^{2\phi} = g^2 H^{\frac{1}{2}} , \\
C_5 &= \frac{4R^4}{g}\epsilon_4 \wedge (-\sin\phi d\hat{x}^2 + \cos\phi d\hat{x}^3) , \\
C_3 &= \frac{\cos\phi}{gH} dx^0 \wedge dx^1 \wedge d\hat{x}^2 + \frac{\sin\phi}{gH} dx^0 \wedge dx^1 \wedge d\hat{x}^3 .
\end{aligned} \tag{3.8}
$$

This is followed by a T-dualization in the new $\hat{x}^3$ direction, which, using (3.4) and (3.5), gives the following D3-D1 bound state solution, where we have removed the hat symbol:

$$
\begin{aligned}
ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2 + (dx^1)^2 + \frac{1}{h'}[(dx^2)^2 + (dx^3)^2]) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) , \\
e^{2\phi} &= \frac{g^2}{h'} , \quad B = \frac{\tan\phi}{Hh'} dx^2 \wedge dx^3 , \\
C_4 &= \frac{\cos\phi}{gHh'} dx^0 \wedge \cdots \wedge dx^3 + \frac{4R^4}{g}\cos\phi\epsilon_4 , \\
C_2 &= \frac{\sin\phi}{gH} dx^0 \wedge dx^1 , \quad C_6 = -\frac{4R^4}{g}\cos\phi\epsilon_4 \wedge B , \\
&\text{where} \quad h' = (\cos\phi)^2 + (\sin\phi)^2 H^{-1} .
\end{aligned} \tag{3.9}
$$

After the T-duality we have also performed a gauge transformation $\delta B = \tan\phi dx^2 \wedge dx^3$, for the B-field. We do this to give the B-field a non-zero value when $r \to \infty$. We see from the above bound state solution that the metric, RR four form and dilaton have been affected by the transformations. There are also non-zero components of the NS-NS two form and the RR two and six forms. The non-zero values for the electric RR four and two forms imply that the supergravity solution corresponds to a D3-D1 bound state.

Next we continue by deriving the D3-D1 bound state with a different method. This solution generating technique is called the $O(p+1,p+1)$ method [73], since we only use elements in the T-duality group $O(p+1,p+1)$. To be more specific: we use T-duality and gauge transformations. This approach to obtaining supergravity solution corresponding to bound states, was first used in [78].

The $O(p+1,p+1)$ method works as follows: To get a Dp-D$(p-2)$ bound state solution we start with a Dp-brane solution. Next, we T-dualize in the x^p and the x^{p-1} directions. More generally one T-dualizes in those directions where one wants to have non-zero components of the NS-NS B-field in the final solution. We have now obtained a D$(p-2)$-brane solution smeared in the x^p and the x^{p-1} directions. Now since the x^p and the x^{p-1} directions are transverse to the D$(p-2)$-brane we perform a gauge transformation for the NS-NS B-field, i.e., $B = 0 \to B = -kdx^{p-1} \wedge dx^p$, where k is a dimensionless deformation parameter. This does not affect the solution, because a constant B-field transverse to the brane can be gauged away. Note however, that a non-zero B-field parallel to the brane can *not* be gauged away. Finally, we first T-dualize in the x^{p-1} direction followed by the x^p direction. Since there is a non-zero $B_{p-1,p}$-field this implies that after the first T-duality we obtain an off-diagonal component in the metric, i.e., $g_{p-1,p} \neq 0$, but $B_{p-1,p} = 0$. However, after the second T-duality we have a non-zero $B_{p-1,p}$-field and a diagonal metric as well as a non-zero RR $C_{01\ldots(p-2)}$. The reason for this is that after we T-dualize back, the x^p and the x^{p-1} directions are parallel to the Dp-brane. This implies that there is also a B-field parallel to the Dp-brane, which can *not* be gauged away.

As an example we now derive the D3-D1 bound state with this method. Below we will show that the obtained solution is equivalent to (3.9). Start with the D3-brane solution (3.3). Then T-dualize in the x^3 and the x^2 directions. Using (3.4) and (3.5) this gives the following D1-brane solution smeared in these directions:

$$
\begin{aligned}
ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2 + (dx^1)^2) + H^{\frac{1}{2}}((dx^2)^2 + (dx^3)^2) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) \, , \\
e^{2\phi} &= g^2 H \, , \quad C_6 = \frac{4R^4}{g} dx^2 \wedge dx^3 \wedge \epsilon_4 \, , \quad C_2 = \frac{1}{gH} dx^0 \wedge dx^1 \, , \\
B &= -k dx^2 \wedge dx^3 \, ,
\end{aligned}
\tag{3.10}
$$

where we after the T-dualities also have performed a gauge transformation for the B-field. Note that k is a dimensionless deformation parameter. Next we T-dualize in the x^3 direction followed by the x^2 direction, which gives the following D3-D1 bound state solution[1]:

$$
\begin{aligned}
ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2 + (dx^1)^2 + \frac{1}{h}[(dx^2)^2 + (dx^3)^2]) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) \ , \\
e^{2\phi} &= \frac{g^2}{h} \ , \quad B = -\frac{k}{Hh} dx^2 \wedge dx^3 \ , \\
C_4 &= \frac{1}{gHh} dx^0 \wedge \cdots \wedge dx^3 + \frac{4R^4}{g}\epsilon_4 \ , \quad C_2 = -\frac{k}{gH} dx^0 \wedge dx^1 \ , \\
C_6 &= -\frac{4R^4}{g}\epsilon_4 \wedge B \ , \\
&\text{where} \quad h = 1 + k^2 H^{-1} \ .
\end{aligned}
\tag{3.11}
$$

We note that if we instead of performing T-duality and a gauge transformation in two spatial directions, would have done this in one spatial direction and the time direction, we would have obtained a D3-F1 bound state solution, because this would lead to a non-zero electric B-field. This solution is 'similar' to the D3-D1 solution, except that $h = 1 - k^2 H^{-1}$. In section 4.2 we will see that this minus sign has very important consequences which differ from those of the plus sign. Next, we show that the two solutions (3.9) and (3.11) are equivalent D3-D1 solutions. It is easy to see that if we let

$$
\tan\phi = -k \ , \quad x^i \to x^i \cos\phi \ , i = 2, 3 \ , \quad g \to g\cos\phi \ , \tag{3.12}
$$

in the solution (3.9), we obtain the solution in (3.11).

3.1.2 The O($p+1$,$p+1$) method

In this subsection we will further discuss the O($p+1$,$p+1$) solution generating technique. For bound states like the D3-D1 one in the last subsection, it is easy to explicitly perform the T-duality transformations. However, to obtain bound states with many different branes, e.g., a D6-D4-D4-D4-D2-D2-D2-D0 bound state (see Paper IV for the supergravity solution corresponding to this case), it would be convenient to be able to simplify the computations. Especially when considering the many components of the different RR fields. In [73] (see also Paper II), one therefore derived formulas which facilitate the computation of supergravity solutions corresponding to bound states. To use these formulas one has to first specify the undeformed brane solution (a Dp-brane solution with maximum or less than maximum supersymmetry), and then choose which components of the NS-NS B-field that should be non-zero in the desired bound state. In general, a non-zero electric B_{01} implies that there are F1-strings in the bound state, while magnetic components of B (B_{12}, B_{34}, B_{56} etc.) imply that it contains D($p-2$)-branes, D($p-4$)-branes, D($p-6$)-branes etc. If we mix one electric and one magnetic component with equal strength (i.e., $B_{01} = \pm B_{21}$), then one finds also a gravitational wave in the bound state.

[1]In Paper II and III the constant $k = \theta/\alpha'$, while in Paper IV-VI it is $k = \theta$.

Let us start with the following generic Dp-brane solution[2]

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + g_{ij}dy^i dy^j \ , \quad e^{2\phi} = g^2 F \ , \\ gC &= \omega dx^0 \wedge \cdots \wedge dx^p + \gamma_{7-p} \ , \end{aligned} \tag{3.13}$$

where g is the closed string coupling constant, F is some function of the coordinates, x^μ, $\mu = 0, \ldots, p$, are coordinates in the brane directions while x^i, $i = p+1, \ldots, 9$, are coordinates in the transverse directions. Here γ_{7-p} is a transverse form, i.e., $i_\mu \gamma_{7-p} = 0$, where i_μ denotes the inner product with the vector field associated with x^μ. This is a rather general Dp-brane solution since we have not completely specified the metric, dilaton and $(p+1)$-form RR field.

Next, we would like to obtain a supergravity solution which corresponds to a general bound state, by deforming (3.13) with a B-field of arbitrary rank. We start by obtaining the deformed metric $\tilde{g}$ and two-form $\tilde{B}$. To obtain these we use that a deformation with constant parameter (and arbitrary rank) $\theta^{\mu\nu}$ is generated by the following $O(p+1, p+1)$ T-duality group element[3]

$$\Lambda = \Lambda_0 \ldots \Lambda_p \Lambda_{-\theta} \Lambda_p \cdots \Lambda_0 = J \Lambda_{-\theta} J = \Lambda_\theta^T = \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix} \ , \tag{3.14}$$

where $\theta^{\mu\nu}$ is dimensionless and carries indices upstairs since it starts life on the T-dual world volume [73]. In (3.14) above, Λ_i $(i = 0, \ldots, p)$ corresponds to a T-duality transformation in the i:th direction, while $\Lambda_{-\theta}$ corresponds to a constant shift in B (i.e., a gauge transformation).

For the metric $g_{\mu\nu}$ and two-form $B_{\mu\nu}$, the transformations in (3.14) imply that the tensor $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$ transforms by the following projective transformation [73] (Note that in (3.13) $B_{\mu\nu} = 0$)

$$\tilde{E}_{\mu\nu} = \Big(\frac{E}{\theta E + 1}\Big)_{\mu\nu} = \Big(\frac{g(1-\theta g)}{(1+\theta g)(1-\theta g)}\Big)_{\mu\nu} \ . \tag{3.15}$$

To obtain this result we have used that under T-duality the tensor E_{MN} $(M, N = 0, 1, \ldots, 9)$ transforms as follows (see, e.g., [16]):

$$E'_{\mu\nu} = (E^{-1})^{\mu\nu} \ , \tag{3.16}$$

$$E'_{ij} = E_{ij} - E_{i\mu}(E^{-1})^{\mu\nu} E_{j\nu} \ , \tag{3.17}$$

where we have T-dualized in all the x^μ directions. Note that in our case $E_{i\mu} = 0$. From (3.15) and (3.17) we obtain the following deformed metric and NS-NS two form

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\rho}\Big[(1-(\theta)^2)^{-1}\Big]^\rho{}_\nu \ , \quad \tilde{g}_{ij} = g_{ij} \ , \\ \tilde{B}_{\mu\nu} &= -g_{\mu\rho}\theta^{\rho\sigma} g_{\sigma\lambda}\Big[(1-(\theta)^2)^{-1}\Big]^\lambda{}_\nu \ , \end{aligned} \tag{3.18}$$

[2] We are using multi-form notation such that C is a sum of forms while B (see below) has fixed rank 2.

[3] See [73] and Paper II for conventions and definitions of the various elements of $O(p+1, p+1)$ appearing in the following discussion.

where

$$(\theta^2)^\mu{}_\nu = \theta^{\mu\nu'} g_{\nu'\rho} \theta^{\rho\sigma} g_{\sigma\nu} \ . \tag{3.19}$$

To obtain the deformed dilaton we use that under T-duality the dilaton transforms as follows [16]:

$$e^{2\phi'} = e^{2\phi} \det(E^{-1})^{\mu\nu} \ , \tag{3.20}$$

which implies that the dilaton, after performing the transformation (3.14), is given by

$$e^{2\tilde{\phi}} = e^{2\phi}\Big(\frac{\det\tilde{E}_{\mu\nu}}{\det E_{\mu\nu}}\Big) = e^{2\phi}\Big(\frac{\det(\tilde{g}+\tilde{B})_{\mu\nu}}{\det g_{\mu\nu}}\Big) = \frac{e^{2\phi}}{\sqrt{\det(1-\theta^2)}} \ . \tag{3.21}$$

To obtain the transformation of the RR-fields it was used in [73] and Paper II that the RR-fields transform in a chiral Spin$(p+1,p+1)$ representation, under a general T-duality transformation. Using this fact gives, after a short calculation, the following transformation rule for the RR-fields[4]:

$$\tilde{C} = e^{-\frac{1}{2}\tilde{B}_{\mu\nu}dx^\mu\wedge dx^\nu}\big(e^{\frac{1}{2}\theta^{\mu\nu}i_\mu i_\nu}C\big) \ , \tag{3.22}$$

where C is given in (3.13).

Finally, combining the results in (3.18), (3.21) and (3.22), we find that the most general deformation of the Dp-brane (3.13) is given by

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= g_{\mu\rho}\Big[(1-(\theta)^2)^{-1}\Big]^\rho{}_\nu \ , \quad \tilde{g}_{ij} = g_{ij} \ , \\
\tilde{B}_{\mu\nu} &= -g_{\mu\rho}\theta^{\rho\sigma}g_{\sigma\lambda}\Big[(1-(\theta)^2)^{-1}\Big]^\lambda{}_\nu \ , \\
e^{2\tilde{\phi}} &= \frac{e^{2\phi}}{\sqrt{\det(1-(\theta)^2)}} = e^{2\phi}\Big(\frac{\det\tilde{g}}{\det g}\Big)^{\frac{1}{2}} \ , \\
g\tilde{C} &= e^{-\frac{1}{2}\tilde{B}_{\mu\nu}dx^\mu\wedge dx^\nu}\big(\omega e^{\frac{1}{2}\theta^{\mu\nu}i_\mu i_\nu}dx^0\wedge\cdots\wedge dx^p + \gamma_{7-p}\big) \ .
\end{aligned} \tag{3.23}$$

Note that the deformed solution (i.e., the bound state) is completely specified by the constant deformation tensor $\theta^{\mu\nu}$. From the formula for $\tilde{B}_{\mu\nu}$, we see that non-zero values of $\theta^{\mu\nu}$ gives non-zero values for $\tilde{B}_{\mu\nu}$. When the components of the constant deformation tensor $\theta^{\mu\nu}$ have been specified one first computes the new metric $\tilde{g}_{\mu\nu}$, B-field $\tilde{B}_{\mu\nu}$ and dilaton, followed by the various RR forms. For several examples where these formulas have been used, see Paper II-VI.

Recalling the discussion above and using (3.23) we see that two types of deformations are possible: θ^{0i} and θ^{ij}, where $i,j = 1,2,\ldots,p$. The first one is called electric since we mix the time direction with a spatial direction, while the second is called magnetic since the time direction is not included. Since an electric deformation implies a non-zero electric component of the B-field, electric deformations are used to include F1–strings in the bound state. Similarly, magnetic deformations are used to include Dq–branes, where $q = p-2, p-4, \ldots, p-2n$, and n is the number of independent magnetic deformations.

[4] We skip the details here, see [73] for the complete derivation.

Finally, to include waves, one has to mix an electric and a magnetic deformation with equal 'strength', i.e., turn on e.g. θ^{01} and θ^{12} where $\theta^{01} = \pm\theta^{12}$. For examples on this last (light-like) case see Paper IV and VI.

If the Dp-brane solution we start with obeys the no force condition, it is important that the deformed solution also obeys this condition. Here we will show that this is always the case. We check this in the special case of a rank 2 deformation. For the general case, see Paper II. Note, however, that if it is shown for the rank 2 case, the arbitrary rank case follows if $\theta^{\mu\nu}$ is restricted to be block diagonal. Furthermore, since an arbitrary rank deformation always can be made block diagonal, using Lorentz transformations, to show the rank 2 case therefore implies that the general case also obeys the no force condition.

Next, we use the action for a Dp-brane to show that (3.23) obeys the no force condition, under the assumption that the undeformed solution (3.13) obeys the no force condition. The action for a Dp-brane is given by two terms, the Dirac-Born-Infeld (DBI) term and the Wess-Zumino term (WZ), $S_{\mathrm{D}p} = S_{\mathrm{DBI}} + S_{\mathrm{CS}}$. From subsection 2.2.1 we recall that the explicit form is given by (where we have written the action using the dilaton instead of g)

$$S_{\mathrm{D}_p} = \frac{1}{\alpha'^{\frac{p+1}{2}}}\Big[- \int d^{p+1}x e^{-\phi}\sqrt{-\det(g+\mathcal{F})} - \int e^{\mathcal{F}} \wedge C\Big] , \tag{3.24}$$

where $\mathcal{F} = F + B$ and $C = C_{p+1} + C_{p-1} + \ldots$. Here B and C are the pullback of the background NS-NS two form and RR forms. From the DBI action we obtain the following effective static potential [79] (we only consider rank 2 B-field):

$$\begin{aligned}\tilde{V} &= -e^{-\tilde{\phi}}\sqrt{-\det(\tilde{g}+\tilde{B})} - \frac{1}{(p+1)!}\epsilon^{\mu_1\ldots\mu_{p+1}}\tilde{C}_{\mu_1\ldots\mu_{p+1}} \\ &- \frac{1}{2}\epsilon^{\alpha_1\alpha_2}\tilde{B}_{\alpha_1\alpha_2}\frac{1}{(p-1)!}\epsilon^{a_1\ldots a_{p-1}}\tilde{C}_{a_1\ldots a_{p-1}} ,\end{aligned} \tag{3.25}$$

where $\epsilon^{012\ldots} = -1$, $\epsilon^{12\ldots} = 1$, $\alpha = 0,1$ or $p-1,p$, and $a = 2,3,\ldots,p$ or $0,1,\ldots,p-2$, for electric and magnetic deformations, respectively. Note that in (3.25) $\tilde{g}$ and $\tilde{B}$ are the deformed background fields (and not the pullback of the background fields). Next, inserting the supergravity solution (3.23) in (3.25), for rank 2 $\tilde{B}$-field, we get that

$$\tilde{V} = V = -e^{-\phi}\sqrt{-\det g_{\mu\nu}} + g^{-1}\omega , \tag{3.26}$$

where V is potential that we obtained for the undeformed solution (3.13). Hence, if $V = 0$, then $\tilde{V} = 0$. Note also that demanding that $V = 0$ for (3.13) implies that $\omega = ge^{-\phi}\sqrt{-\det g_{\mu\nu}}$, in (3.13). For a light-like deformation (3.25) is trivially zero if the undeformed solution has $V = 0$. Here a comment is in order: we note that it is enough if the undeformed potential is equal to an arbitrary constant, in order to obey the no force condition [79]. For simplicity we choose to set this constant to zero.

3.2 M-theory 'T-duality' and bound states

In this section we are going to discuss generalization of string theory T-duality to M-theory. We will further use the M-theory 'T-duality' rules in order to derive the solution generating technique first discussed in Paper VII. As we will see below, this M-theory solution generating technique is a generalization of the $O(p+1, p+1)$ solution generating technique to M-theory.

3.2.1 M-theory 'T-duality'

We know that type IIA string theory compactified on a two torus is invariant under string theory T-duality. This suggests that M-theory on a three torus should be invariant under some transformation, which is the generalization of string theory T-duality to some kind of M-theory 'T-duality'.

In this subsection we will derive (tensor) transformation rules for the 11-dimensional supergravity background fields (i.e., the metric and the three form A), under an M-theory 'T-duality' transformation. The notion of M-theory 'T-duality' was first discussed by Sen [80] (see also [81] and Paper VII). It was assumed that M-theory on a three torus is invariant under exchange of the Kaluza-Klein modes and the wrapping modes for the M2-brane.

Here we do not intend to try to derive the transformation rules for the supergravity fields from a duality on the M2-brane world volume (compactified on a three torus down to eight dimensions), since this is most likely very difficult. Instead we will derive transformation rules for the supergravity fields, which are consistent with the 11-dimensional supergravity equations of motion and U-duality. Moreover, we conjecture that these are the correct M-theory 'T-duality' transformation rules. It would be very interesting and important to verify this from a duality transformation on the M2-brane world volume, similar to how string T-duality is derived, see also the end of this subsection for some further comments.

Under M-theory 'T-duality' it is conjectured [80] that the complex Kähler parameter of the three torus

$$E = A_{123} + i\sqrt{\det g_{ab}}\ , \quad a, b = 1, 2, 3\ , \tag{3.27}$$

transforms as

$$E' = -\frac{1}{E}\ , \tag{3.28}$$

where we have 'T-dualized' in the x^1, x^2 and x^3 directions and g_{ab} is assumed to be diagonal. The above transformation is in the SL(2) part of the U-duality group SL(3)×SL(2) for M-theory compactified on a three torus. To be more specific, in (3.28) we have used an S transformation, whereas a general SL(2) transformation would transform E as follows:

$$E' = \frac{aE + b}{cE + d}\ , \tag{3.29}$$

with $ad - bc = 1$. The S transformation corresponds to $a = d = 0$ and $b = -c = 1$.

Under 'T-duality', we obtain from (3.28) together with the knowledge that $g'_{ab} = g^{ab}$ if

$A = 0$, the following transformation rules for g_{ab} and A_{123}, setting $A_{123} = A$:

$$\begin{aligned} g'_{ab} &= \frac{g^{ab}}{(1 + A^2 \mathrm{det} g^{-1}_{ab})^{2/3}} , \\ A'_{123} &= -\frac{A}{|\tau|^2} = -\frac{A \mathrm{det} g^{-1}_{ab}}{1 + A^2 \mathrm{det} g^{-1}_{ab}} , \\ g'_{ij} &= |\tau|^{2/3} g_{ij} , \end{aligned} \tag{3.30}$$

where $\mathrm{det} g^{-1}_{ab} = (\mathrm{det} g_{ab})^{-1}$. We have also given the transformation rule for the 8-dimensional transverse metric. The transformation of the transverse metric can be obtained by first reducing 11 dimensional supergravity on a three torus (see [82, 83]), followed by using that the SL(2)×SL(3) invariant 8-dimensional metric g^E_{ij} and the 8-dimensional transverse part of the 11 dimensional supergravity metric are related as $g_{ij} = e^{\gamma/3} g^E_{ij}$. Here $e^{-\gamma} = \sqrt{\mathrm{det} g_{ab}}$ transforms to $e'^{-\gamma} = e^{-\gamma}/|\tau|^2$ under an S transformation, which is easily seen from (3.28).

Since the metric g_{ab} was assumed to be diagonal we only had to use the fact that $g'_{ab} = g^{ab}$ if $A = 0$ under M-theory 'T-duality', in order to completely determine the first expression in (3.30). Note that the last condition implies that in order to obtain (3.30) we have to perform a certain SL(3) transformation together with the above SL(2) transformation[5]. Hence, an M-theory 'T-duality' transformation must include both an SL(2) and an SL(3) transformation. However, we see from the example with diagonal metric above, that it is the SL(2) transformation which gives the 'important' information. Moreover, this suggests that the M-theory 'T-duality' group (for M-theory compactified on a three torus) is SL(2)×SL(3), which is equivalent with the U-duality group. To conclude: The transformation rules in (3.30) are correct since we have used the SL(2)×SL(3) invariance of the eight-dimensional action, which is obtained from 11-dimensional supergravity dimensionally reduced on a three torus [82, 83]. Note, however, that this means that no fields can have any dependence on the coordinates in the three 'T-dualized' directions.

The above 'T-duality' can be generalized to 'T-duality' with the time direction included. However, in this case it is important to note that 'T-duality' with a time-like component is *not* a symmetry of M-theory, but transforms M-theory to M*-theory [85] with signature (2,9). In this case E is real and is given by [86]

$$E = A_{012} + \sqrt{\mathrm{det} g_{ab}} , \quad a, b = 0, 1, 2 . \tag{3.31}$$

A 'T-duality' transformation now gives (set $A_{012} = A$) that $E \to -1/E$, which gives the same transformation rules as in (3.30).

Our next objective is to obtain the transformation rules given in (3.30), in a tensor form. We conjecture that the tensor form of (3.30) is given by (under the condition that

[5]This is analogous to type II string theory compactified on a two torus where a T-duality transformation is equivalent to S transformations in both of the SL(2)'s. Note that in this case the T-duality group is essentially SL(2)×SL(2) (see, e.g., [84]). Here the first SL(2) has to do with transformations of the Kähler parameter $E = B_{12} + i\sqrt{\mathrm{det} g_{ab}}$ and the second with transformations of the complex structure $\tau = g_{12}/g_{22} + i\sqrt{\mathrm{det} g_{ab}}/g_{22}$.

$g_{ai} = A_{abi} = A_{aij} = 0$):[6]

$$
\begin{aligned}
g'_{ab} &= (\det G^a{}_b)^{1/9}(G^{-1})^{ab} , \\
g'_{ij} &= (\det G^a{}_b)^{1/9}(\det g_{ab})^{1/3} g_{ij} , \\
A'_{abc} &= -A_{def} g^{da} g^{eb} (G^{-1})^{fc} ,
\end{aligned}
\tag{3.32}
$$

where

$$
G_{ab} = g_{ab} + \frac{1}{2} A^2_{ab} , \quad A^2_{ab} = g^{cc'} g^{dd'} A_{acd} A_{bc'd'} . \tag{3.33}
$$

In (3.32) $G^a{}_b = g^{ac} G_{cb}$ and $(G^{-1})^{ab}$ is the inverse of G_{ab}. Note how similar (3.32) is compared to the string case (3.16) (i.e., if we extract g' and B' from (3.16)), except for the conformal factor in the metric.

Next, if we insert that $A_{012} = A$ or $A_{123} = A$ into (3.32) and assume that the metric is diagonal, we easily obtain (3.30) in both cases. Another check of the metric in (3.32) is to relate it, by dimensional reduction, to T-duality rules for type IIA string theory on a two torus. By using how the dilaton transform under T-duality (3.20) we find that the transverse metric g'_{ij} reduces to $g'^{(s)}_{ij} = g^{(s)}_{ij}$, which is the correct transformation if $g^{(s)}_{ai} = B_{ai} = 0$, i.e., $E_{ai} = 0$, see (3.17). To check that g'_{ab} is correct we use that in 10 dimensions T-duality demands that (see, e.g., [87])[7]

$$
e^{-2\phi'} \sqrt{-g'_{(s)}} = e^{-2\phi} \sqrt{-g_{(s)}} , \tag{3.34}
$$

where $g_{(s)}$ is the determinant of the string metric before T-duality and $g'_{(s)}$ after T-duality. Lifting this relation to M-theory gives the following condition

$$
\sqrt{-g'} = \sqrt{-g} (\det G^a{}_b)^{1/9} (\det g_{ab})^{1/3} , \tag{3.35}
$$

where g is the determinant of the metric before 'T-duality' and g' after 'T-duality'. Next, if we calculate $\sqrt{-g'}$ using (3.32), we obtain that (3.35) is satisfied. This implies that g'_{ab}, given in (3.32) is correct.

We end this subsection with some additional comments on M-theory 'T-duality'. Above we have derived transformation rules for the background supergravity fields, under an M-theory 'T-duality' transformation. Although we have not proved them from duality on the M2-brane world volume, we still expect them to hold, since they respect U-duality and the 11-dimensional supergravity equations of motion. Hence, the 11-dimensional supergravity Lagrangian is invariant under an M-theory 'T-duality' transformation. However, it does not mean that a generic solution to the supergravity equations of motion is invariant. For example, start with an M2-brane solution, which is smeared in the x^a, $a = 3, 4, 5$ directions. Applying a 'T-duality' transformation, using (3.30), we obtain an M5-brane solution.

[6]Note that if the 'T-duality' involves the time direction, one should take the absolute value of $\det g_{ab}$ in line two in (3.32).

[7]Using the path integral approach, (3.34) originates from using a different measure in the dual sigma model. The change of the measure introduces a Jacobian which leads to the condition (3.34), see, e.g., [87].

Similarly, starting with an M5-brane solution, gives, after a 'T-duality' transformation in the x^a, $a = 3, 4, 5$ directions, an M2-brane solution smeared in these directions. This is very different from T-duality in string theory, where a supergravity solution corresponding to a string transforms into itself under T-duality. This seems to imply that there is a fundamental difference between string theory T-duality and M-theory 'T-duality', which perhaps is not very surprising.

Furthermore, this has the following interpretation for M2-branes and M5-branes: If we take a membrane and compactify on a three torus, an M-theory 'T-duality' transformation implies that we are dualizing three of the scalars[8] to three vectors, since the membrane world sheet is three-dimensional. However, this can not be entirely true, since de-compactification (i.e., lift to 11 dimensions) can not give a membrane in 11 dimensions with three vectors and five scalars, since a membrane of this kind does not exist. Instead what must happen is the following[9]: Under an M-theory 'T-duality' the (electric) membrane in eight dimensions is transformed to a magnetic membrane. With magnetic membrane we mean that it couples magnetically to the background three form which the membrane before 'T-duality' coupled electrically to. This is similar to the fact that in 11 dimensions the M2-brane couples electrically to the three form, while the M5-brane couples magnetically to it. Thus, we have a membrane that couples magnetically to the background three form, which means that if we now de-compactify, then this magnetic membrane becomes an M5-brane in 11 dimensions, i.e., M-theory 'T-duality' changes an M2-brane into an M5-brane. Conversely, starting with an M5-brane compactified on a three torus gives a membrane, which couples magnetically to the background three form (which is obtained from the 11-dimensional three form). From the M5-brane the five transverse scalars stay as five scalars, while the world volume two form with self-dual three form field strength gives three scalars and three vectors, which obey a duality relation. Hence, the three scalars and three vectors together only give three d.o.f. Next, under an M-theory 'T-duality' transformation we get an electric membrane, since from an eight dimensional perspective we have performed an electro-magnetic transformation (i.e., magnetic membrane→electric membrane). Note that the vectors are now completely dualized to scalars. Finally, de-compactifying gives an 11-dimensional M2-brane.

We have seen above that an M-theory 'T-duality' transformation changes an M2-brane to an M5-brane, and vice versa. This is very different compared to string theory T-duality. Moreover, in string theory we only have to deal with fundamental strings, when discussing string theory T-duality. However, in M-theory it does not seem to be sufficient to only

[8]In string theory a T-duality transformation is obtained by dualizing a scalar, which gives a new scalar, since the string world sheet is two-dimensional.

[9]Some of the motivation to the following discussion can be found in [81, 83]. In particular, it is shown in [83] that, for a membrane in eight dimensions, it is not possible to dualize three world volume scalars to three vectors, because of the non-trivial Bianchi identity for the vectors. It is, however possible to have a description using both three scalars and three vectors, where the vectors and scalars are dual to each other. This is similar to how, in 11-dimensional supergravity, it is possible to use a description with only a three form or a description using a three form together with the dual six form potential, but *not* a description with only a six form.

include M2-branes, when discussing M-theory 'T-duality'. Instead we also have to include M5-branes. This further suggest that when dealing with M-theory 'T-duality', the M2-brane and M5-brane are on equal footing. Hence, perhaps it is not correct to view the M2-brane as any more fundamental than the M5-brane. This something that must be investigated further.

3.2.2 An M-theory solution generating method

In this subsection we are going to use the results about M-theory 'T-duality', obtained in the last subsection, to derive the solution generating method which was first discussed in Paper VII. The purpose of this solution generating technique is to be able to deform M5-branes by turning on a three form A_3, in the M5-brane directions. This will generate supergravity solutions corresponding to M5-M2 or M5-M2-M2-MW bound states with various amounts of supersymmetry depending on the M5-brane solution which we start from. Note that, similar to the string case (see section 3.1.2), also in 11 dimensions the deformed and undeformed solutions will have the same amount of supersymmetry. We will see below that, using M-theory 'T-duality', this solution generating technique can be constructed in a similar way to how the string case was constructed.

Consider the following general M5-brane solution

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + g_{mn}dx^m dx^n \ , \quad \mu,\nu = 0,1,\ldots,5 \ , \quad m,n = 6,\ldots,10 \ , \\ A_6 &= \omega dx^0 \wedge \cdots \wedge dx^5 \ , \quad A_3 = \gamma_3 \ , \end{aligned} \tag{3.36}$$

where γ_3 is a transverse three form dual to the six form (i.e., $*d\gamma_3 = dA_6$), while the function $\omega = \sqrt{-\det g_{\mu\nu}}$ (up to an arbitrary constant), due to the no force condition.

Next, we are going to deform this M5-brane by turning on a three form A_3 in some of the M5-brane directions. By comparing to the string case in section 3.1.2, we claim that this is done by first performing an M-theory 'T-duality' transformation in three directions parallel to the M5-brane, e.g., in the x^a, $a = 3,4,5$, directions, which gives a smeared M2-brane. Next, we perform a gauge transformation in the smeared directions, followed by a new M-theory 'T-duality' transformation in the same directions as before. Performing these three steps should give us the desired deformed solution (bound state).

Following the above three steps using the M-theory 'T-duality' rules (3.32), we obtain the following complete metric and three form (in the deformed directions):

$$\begin{aligned} \tilde{g}_{ab} &= \Big[\det\Big(1+\frac{1}{2}(\theta)^2\Big)\Big]^{1/9} g_{ac}\Big[\Big(1+\frac{1}{2}(\theta)^2\Big)^{-1}\Big]^c{}_b \ , \quad \tilde{g}_{\alpha\beta} = \Big[\det\Big(1+\frac{1}{2}(\theta)^2\Big)\Big]^{1/9} g_{\alpha\beta} \ , \\ \tilde{g}_{mn} &= \Big[\det\Big(1+\frac{1}{2}(\theta)^2\Big)\Big]^{1/9} g_{mn} \ , \\ \tilde{A}_{3a} &= \frac{1}{6}\tilde{A}^{3a}_{abc}dx^a \wedge dx^b \wedge dx^c \ , \quad \tilde{A}^{3a}_{abc} = -g_{ad}g_{eb}\theta^{def}g_{fg}\Big[\Big(1+\frac{1}{2}(\theta)^2\Big)^{-1}\Big]^g{}_c \ . \end{aligned} \tag{3.37}$$

Here θ^{abc} is a constant dimensionless anti-symmetric deformation tensor, and $(\theta^2)^a{}_b$ is defined as follows:

$$(\theta^2)^a{}_b = \theta^{acd}g_{ce}g_{df}\theta^{efg}g_{gb} \ . \tag{3.38}$$

There are three possible deformations: (1) $\theta^{012} = \theta$, (2) $\theta^{345} = \theta$, (3) $\theta^{-12} = \theta$, (where $x^{\pm} = \frac{1}{\sqrt{2}}(x^5 \pm x^0)$). Similar to the string case the first one is called an electric, the second a magnetic and the third a light-like deformation. Note also that the part of A_3 which is transverse to the M5-brane is unchanged (i.e., it is still equal to γ_3).

Next, we note an important difference compared to the string case. On the M5-brane the gauge invariant three form field strength $\mathcal{H} = dB + A_3'$ (where A_3' is the pullback of the background three form to the M5-brane world volume) satisfies a non-linear self-duality (NLSD) condition. This implies that also the background three form $\tilde{A}_3$, in the M5-brane directions, must satisfy the same NLSD condition, using the background metric. Hence, e.g., if we turn on an electric component of A_3 we automatically get a magnetic component of A (i.e., turning on $\tilde{A}_{012}$ implies that also $\tilde{A}_{345}$ is non zero). This means that $\tilde{A}_{3a}$ obtained in (3.37), must be supplemented by a new component, in order for the three form $\tilde{A}_3$ to satisfy the NLSD condition in the M5-brane directions. Therefore, using the NLSD condition (see equation (3.19) in Paper VII) we obtain that the new component is given by

$$\tilde{A}_{3b} = \frac{1}{6}\tilde{A}^{3b}_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \ , \quad \tilde{A}^{3b}_{\alpha\beta\gamma} = (-1)^k \theta\omega\epsilon_{\alpha\beta\gamma} \ , \tag{3.39}$$

where $\epsilon_{012} = \epsilon_{345} = 1$ and $k = 0$ for $\theta^{012} = \theta$, while $k = 1$ for $\theta^{345} = \theta$ and $\theta^{-12} = \theta$. Hence, including (3.39) to (3.37), implies that the NLSD condition is satisfied. Moreover, the NLSD condition implies that an electric and a magnetic deformation must be equivalent. Below this is explicitly shown for a deformation of the maximally supersymmetric M5-brane.

So far, we have obtained the deformed metric $\tilde{G}_{MN}$ and three form $\tilde{A}_3$. However, sometimes it can be convenient to also have the dual six form. Using that $*\tilde{H}_4 = \tilde{H}_7$ (with $\epsilon^{01\ldots 9,10} = -1$), where

$$\tilde{H}_4 = d\tilde{A}_3 \ , \quad \tilde{H}_7 = d\tilde{A}_6 - \frac{1}{2}\tilde{A}_3 \wedge d\tilde{A}_3 \ , \tag{3.40}$$

we obtain the following deformed six form:

$$\tilde{A}_6 = A_6 + \frac{1}{2}\tilde{A}_{3a} \wedge \tilde{A}_{3b} + \frac{1}{2}(\tilde{A}_{3a} + \tilde{A}_{3b}) \wedge \gamma_3 \ . \tag{3.41}$$

Combining the results in (3.37), (3.39), (3.41), and simplifying the notation, we get the following deformed M5-brane supergravity solution:

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= K^{\frac{1}{9}} g_{\mu\rho}\Big[\Big(1 + \frac{1}{2}(\theta)^2\Big)^{-1}\Big]^{\rho}{}_{\nu} \ , \quad \tilde{g}_{mn} = K^{\frac{1}{9}} g_{mn} \ , \\
\tilde{A}_3 &= \tilde{A}_{3a} + \tilde{A}_{3b} + \gamma_3 \ , \\
\tilde{A}_6 &= A_6 + \frac{1}{2}\tilde{A}_{3a} \wedge \tilde{A}_{3b} + \frac{1}{2}(\tilde{A}_{3a} + \tilde{A}_{3b}) \wedge \gamma_3 \ ,
\end{aligned} \tag{3.42}$$

where

$$\begin{aligned}
\tilde{A}_{3a} &= \frac{1}{6}\tilde{A}^{3a}_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \ , \quad \tilde{A}^{3a}_{\mu\nu\rho} = -g_{\mu\rho'} g_{\sigma\nu} \theta^{\rho'\sigma\sigma'} g_{\sigma'\lambda}\Big[\Big(1 + \frac{1}{2}(\theta)^2\Big)^{-1}\Big]^{\lambda}{}_{\rho} \ , \\
\tilde{A}_{3b} &= -\frac{1}{6}\omega\theta^{\mu\nu\rho} i_\mu i_\nu i_\rho dx^0 \wedge \cdots \wedge dx^5 \ , \quad K = \Big[\det\Big(1 + \frac{1}{2}(\theta)^2\Big)\Big] \ ,
\end{aligned}$$

where $i_\mu dx^\nu = \delta^\nu_\mu$, $\theta^{\mu\nu\rho}$ is a constant dimensionless anti-symmetric deformation tensor, and $(\theta^2)^\mu{}_\nu$ is defined as follows:

$$(\theta^2)^\mu{}_\nu = \theta^{\mu\nu'\rho} g_{\nu'\sigma} g_{\rho\sigma'} \theta^{\sigma\sigma'\lambda} g_{\lambda\nu} \,. \tag{3.43}$$

Here $\theta^{\mu\nu\rho}$ has only 'one' non-zero component, e.g., $\theta^{\alpha\beta\gamma} = \theta\epsilon^{\alpha\beta\gamma}$ ($\epsilon^{012} = 1$), where $\alpha = 0, 1, 2$, while $\theta^{abc} = 0$, $a = 3, 4, 5$. Note this is not a restriction on the possible bound states that we can obtain, because it has been shown in [72, 101] that one parameter is enough to parameterize *all* deformations of an M5-brane with a non-linearly self-dual three form A_3 (in the M5-brane directions), up to Lorentz transformations. An important difference between the M5-brane and D4-brane is that from an M5-brane point of view the difference between rank 2 and rank 4 B-field on the D4-brane is a Lorentz transformation (see, e.g., [110]).

The solution generating technique (3.42) must be correct, because we have used the M-theory 'T-duality' rules, which have been rigorously proven[10]. To be more specific: We have proven that (3.42) is correct if all functions are independent of the three directions in which we deformed. It would be very interesting to investigate if it is possible to relax this constraint, and see if (3.42), or some generalization of it, is valid for more general deformations. One reason to believe that this might be possible is that it has been conjectured that 11-dimensional supergravity has an E_{11} symmetry [88]. Moreover, since E_{11} has SL(2)×SL(3) as a subgroup, it should be possible to use SL(2)×SL(3) transformations like the ones we use in (3.42), directly in 11 dimensions without compactifying on a three torus. At this stage we unfortunately have no solid evidence that (3.42) is correct for more general deformations. We plan to investigate this further, since it is an important question to answer.

Next, we show how to use the method above to obtain an M5-M2 bound state. We will obtain the maximally supersymmetric M5-M2 bound state in two ways and show that the two solutions are equivalent[11]

We begin by giving the half-supersymmetric M5-brane solution [36]:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{3}} \eta_{\mu\nu} dx^\mu dx^\nu + H^{\frac{2}{3}} \delta_{mn} dx^m dx^n \,, \quad \mu,\nu = 0,1,\ldots,5 \,, \quad m,n = 6,\ldots,10 \,, \\ A_6 &= H^{-1} dx^0 \wedge \cdots \wedge dx^5 \,, \quad A_3 = \gamma_3 \,, \end{aligned} \tag{3.44}$$

where H is a harmonic function on the transverse space and γ_3 is the three form dual to the six form, i.e., $\gamma_3 = 3R^3\epsilon_3$, where $d\epsilon_3$ is the volume form of the four-sphere.

[10] More specifically, this means that magnetic deformations must be correct. However, since the NLSD condition implies that electric and magnetic deformations are equivalent, it also proves that electric deformations are correct.

[11] We have chosen to show this equivalence here, since this is not shown explicitly in Paper VII.

Deforming this M5-brane solution with $\theta^{012}=\theta$, using (3.42), gives

$$\begin{aligned}
d\tilde{s}^2 &= (Hh^{-1})^{-\frac{1}{3}}\Big[\frac{1}{h}\Big((dx^0)^2+(dx^1)^2+(dx^2)^2\Big)+(dx^3)^2+(dx^4)^2+(dx^5)^2\Big] \\
&\quad +h^{\frac{1}{3}}H^{\frac{2}{3}}\delta_{mn}dx^m dx^n\ , \\
\tilde{A}_3 &= \frac{\theta}{Hh}dx^0\wedge dx^1\wedge dx^2+\frac{\theta}{H}dx^3\wedge dx^4\wedge dx^5+\gamma_3\ , \\
\tilde{A}_6 &= (Hh)^{-1}\Big(\frac{h+1}{2}\Big)dx^0\wedge\cdots\wedge dx^5 \\
&\quad + \frac{1}{2}\theta H^{-1}(h^{-1}dx^0\wedge dx^1\wedge dx^2+dx^3\wedge dx^4\wedge dx^5)\wedge\gamma_3\ , \\
h &= 1-\theta^2H^{-1}\ ,\quad H=a+\frac{R^3}{r^3}\ ,
\end{aligned} \tag{3.45}$$

while a deformation with $\theta^{345}=-\theta'$, gives

$$\begin{aligned}
d\tilde{s}^2 &= (H'h'^{-1})^{-\frac{1}{3}}\Big[(dx^0)^2+(dx^1)^2+(dx^2)^2+\frac{1}{h'}\Big((dx^3)^2+(dx^4)^2+(dx^5)^2\Big)\Big] \\
&\quad +h'^{\frac{1}{3}}H'^{\frac{2}{3}}\delta_{mn}dx^m dx^n\ , \\
\tilde{A}_3 &= \frac{\theta'}{H'}dx^0\wedge dx^1\wedge dx^2+\frac{\theta'}{H'h'}dx^3\wedge dx^4\wedge dx^5+\gamma_3\ , \\
\tilde{A}_6 &= (H'h')^{-1}\Big(\frac{h'+1}{2}\Big)dx^0\wedge\cdots\wedge dx^5 \\
&\quad + \frac{1}{2}\theta' H'^{-1}(dx^0\wedge dx^1\wedge dx^2+h'^{-1}dx^3\wedge dx^4\wedge dx^5)\wedge\gamma_3\ , \\
h' &= 1+\theta'^2H'^{-1}\ ,\quad H'=b+\frac{R^3}{r^3}\ ,
\end{aligned} \tag{3.46}$$

where a and b are constants. It is not obvious that these two solutions are equivalent. However, if we use the following identification

$$b=a-\theta'^2\ ,\quad \theta'=\theta\ , \tag{3.47}$$

which implies that

$$h'=\frac{1}{h}\ ,\quad H'=Hh\ . \tag{3.48}$$

It is fairly straight forward to show that they indeed are equivalent.

3.3 D-brane 'T-duality' and bound states

In this short section we will generalize the results in section 3.1 and 3.2 to Dp-brane 'T-duality' (we restrict ourselves to $p=1,2,3$), which can be used to derive a solution generating technique, which in turn can be used to deforms NS5-branes and D$(p+2)$-branes by turning on a non zero RR $(p+1)$-form[12].

For type II string theory compactified on a $(p+1)$-torus, 'T-duality' for Dp-branes implies (similar to the membrane case in last section) that the complex Kähler parameter of the $(p+1)$-torus

$$E = C_{12\ldots(p+1)} + i\sqrt{\det g^{\mathrm{D}p}_{ab}}\ , \quad a,b = 1,2,\ldots,(p+1)\ , \tag{3.49}$$

transform as [80]

$$E' = -\frac{1}{E}\ , \tag{3.50}$$

where $g^{\mathrm{D}p}_{ab}$ is the closed Dp-brane metric and assumed to be diagonal, and we have 'T-dualized' in the x^a, $a = 1,\ldots,(p+1)$, directions. For simplicity we have set the closed string coupling constant $g = 1$. Equation (3.50) leads to the following transformation rules for the Dp-brane metric and RR $(p+1)$-form (let us set $C_{12\ldots(p+1)} = C$), under Dp-brane 'T-duality':

$$\begin{aligned} g'^{\mathrm{D}p}_{ab} &= \frac{g^{ab}_{\mathrm{D}}}{(1 + C^2 \det g^{-1}_{ab})^{\frac{2}{p+1}}}\ , \\ C'_{12\ldots(p+1)} &= \frac{-C \det g^{-1}_{ab}}{1 + C^2 \det g^{-1}_{ab}}\ , \end{aligned} \tag{3.51}$$

where $\det g^{-1}_{ab} = (\det g^{\mathrm{D}p}_{ab})^{-1}$.

Similar to subsection 3.2.1 there are also 'T-duality' rules when the time coordinate is included. These give transformation rules similar to (3.51).

Next, the generalization to tensor expressions for the 'T-duality' rules is straightforward and given by (we also give the transformation rules for the transverse metric $g^{\mathrm{D}p}_{ij}$ with the condition that $g^{\mathrm{D}p}_{ai} = C_{ab\ldots ci} = \ldots = C_{ai\ldots j} = 0$):

$$\begin{aligned} g'^{\mathrm{D}p}_{ab} &= (\det G^a{}_b)^{\frac{p-1}{(p+1)^2}} (G^{-1})^{ab}\ , \\ g'^{\mathrm{D}p}_{ij} &= (\det G^a{}_b)^{\frac{p-1}{(p+1)^2}} (\det g_{ab})^{\frac{p-1}{p+1}} g^{\mathrm{D}p}_{ij}\ , \\ C'_{a_1\ldots a_{p+1}} &= -C_{b_1\ldots b_{p+1}} g^{a_1 b_1}_{\mathrm{D}p} \cdots g^{a_p b_p}_{\mathrm{D}p} (G^{-1})^{a_{p+1} b_{p+1}}\ , \end{aligned} \tag{3.52}$$

where

$$G_{ab} = g^{\mathrm{D}p}_{ab} + \frac{1}{p!} C^2_{ab}\ , \quad C^2_{ab} = g^{a_1 b_1} \cdots g^{a_p b_p} C_{aa_1\ldots a_p} C_{bb_1\ldots b_p}\ . \tag{3.53}$$

Looking at a specific electric or magnetic case, we find that the first and third expressions in (3.52) reduce to (3.51). For $p = 1$ (3.52) follow from S-duality of the fundamental string

[12] The reason we will use this Dp-brane 'T-duality' to deform NS5-branes and D$(p+2)$-branes, by turning on a non zero RR $(p+1)$-form, is because open Dp-branes can end on NS5-branes or D$(p+2)$-branes [89]. This is similar to how open strings can end on Dp-branes but not on NS5-branes, which implies that ordinary T-duality can be used to deform Dp-branes with non zero B-field but not NS5-branes. To deform NS5-branes we instead have to turn on a non zero RR $(p+1)$-form, see below.

case, see Paper VII, while for $p = 2$ (3.52) can be obtained from (3.32) using dimensional reduction. Furthermore, for $p = 3$ the exponent in $(\det g_{ab})^{1/2}$ is fixed by demanding that a magnetic D3-brane 'T-duality' of a 1/2-SUSY D5-brane gives a 1/2-SUSY D1-brane.

Continuing in a similar way to section 3.2, we find that it is fairly simple to use the tensor 'T-duality' rules in (3.52), in order to derive deformations of NS5-branes and D$(p+2)$-branes by turning on a non-zero RR $(p+1)$-form, which would lead to bound states of the form NS5-Dp and D$(p+2)$-Dp, respectively. In the two cases $p = 1, 2$, we obtain the same result as in Paper VII. For $p = 3$ we have checked that one also obtain correct results, e.g., the maximally supersymmetric NS5-D3 bound state can be derived.

Although we have here restricted ourselves to the cases with $p = 1, 2, 3$, we will comment on the generalization of (3.49) and (3.52) for $p = -1$. In this case (3.49) and (3.52) generalize to (note that we use τ instead of E)

$$\tau = C_{(0)} + ie^{-\phi} , \tag{3.54}$$

where a D(-1)-brane 'T-duality' implies that $\tau \to -1/\tau$. Hence,

$$\tau' = -\frac{1}{\tau} = \frac{-C_{(0)} + ie^{-\phi}}{C_{(0)}^2 + e^{-2\phi}} . \tag{3.55}$$

As a conclusion we see from (3.54) and (3.55), that D(-1)-brane 'T-duality' seems to be *equivalent* to ordinary type IIB S-duality (see also [80]). This interesting result might imply that *all* U-duality transformations can be viewed as various (generalized) T-duality transformations.

4

Noncommutative theories, supergravity duals and open brane data

In this chapter we will discuss various maximally supersymmetric non-gravitational theories (see, e.g., [10]-[12], [73, 78], [90]-[122] and Papers II-VI), which live in flat space-time with noncommutative coordinates (or some kind of 'generalized' noncommutativity structure, see section 4.3). In section 4.1 and 4.3 we follow the more common 'flat-space scaling' approach, while in section 4.2 we will give a short introduction to the 'supergravity dual' approach (see Paper II-VI). We show that these two approaches give equivalent results. However, in many cases the latter is more convenient to use. Moreover, in subsection 4.2.2 we discuss the concept of deformation independence, while in 4.3.1 we give an alternative derivation of the open membrane metric and generalized noncommutativity parameter.

4.1 Dp-branes with background B-field

Here we give an introduction to maximally supersymmetric noncommutative super-Yang-Mills (NCYM) [10]-[12], with space-space or light-like noncommutativity, and noncommutative open string theory (NCOS) [93]-[98]. In particular we explain why these theories have noncommutative coordinates and why the former is a field theory and the latter an open string theory. We will obtain these theories by taking various decoupling limits (low energy limits) of Dp-branes with a background anti-symmetric NS-NS $B_{\mu\nu}$-field turned on in flat space ($g_{\mu\nu} = \eta_{\mu\nu}$, $\mu, \nu = 0, 1, \ldots, p$). Note that these theories are $(p+1)$-dimensional since they are world volume theories of a Dp-brane with a B-field turned on. The background B-field is equivalent to a constant magnetic or electric field on the brane. In this section it will be implicit that we only deal with noncommutative U(1) theories.

4.1.1 The open string metric, noncommutativity parameter and coupling constant

We start in this subsection by obtaining the two-point function for an open string ending on a Dp-brane with a constant NS-NS B-field turned on in flat space-time. The B-field has arbitrary even rank $r \leq p+1$. We will only include the $p+1$ directions parallel to the brane in the discussion, since a constant B-field transverse to the Dp-brane can be gauged away. The two-point function is important since it gives information about the open string metric $G_{\mu\nu}$ and noncommutativity parameter $\Theta^{\mu\nu}$. Here $G_{\mu\nu}$ is called the open string metric, since it governs the mass-shell condition for open string states propagating on the Dp-brane, while $\Theta^{\mu\nu}$ is the parameter of noncommutativity between Dp-brane coordinates, as will be explained below.

First, the two-point function for a closed string is given by (written with complex coordinates $z = \tau + i\sigma$ and $\bar{z} = \tau - i\sigma$) [16]:

$$< X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') >= -\frac{\alpha'}{2} g^{\mu\nu} (\log(z-z') + \log(\bar{z}-\bar{z}')) \ . \tag{4.1}$$

To obtain the two-point function for an open string ending on a Dp-brane, with a constant NS-NS B-field turned on in flat space-time, we start with the following (bosonic) action [12]

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma (g_{\mu\nu} \partial_a X^\mu \partial^a X^\nu - B_{\mu\nu} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu) \ , \tag{4.2}$$

where $X^\mu = X^\mu(z,\bar{z})$ and Σ is the two-dimensional open string world sheet (upper half of the complex plane, with boundary given by $z = \bar{z} = \tau$). Note that the closed string metric $g_{\mu\nu}$ and NS-NS field $B_{\mu\nu}$ are defined to be dimensionless. For the NS-NS field this is different compared to [12], where $B_{\mu\nu}$ has dimension (length)$^{-2}$. The relation between the two conventions is: $B_{\mu\nu} = 2\pi\alpha' B^{(\mathrm{sw})}_{\mu\nu}$, where $B^{(\mathrm{sw})}_{\mu\nu}$ is the B-field in [12]. Note also that the second term in (4.2) is a boundary term, i.e., it can be written as a total derivative term. This implies that a nonzero B-field does not effect the closed string two-point function, but, as we will see below, modifies the open string two-point function. The open string boundary conditions, which are obtained from (4.2), are given by

$$g_{\mu\nu} \partial_n X^\nu + B_{\mu\nu} \partial_t X^\nu \Big|_{z=\bar{z}} = 0 \ . \tag{4.3}$$

Here $\partial_n = \partial - \bar{\partial}$ is the normal derivative and $\partial_t = \partial + \bar{\partial}$ the tangential derivative with respect to the open string world sheet, where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$. For $B = 0$, the boundary conditions (4.3) are Neumann boundary conditions, while for $B \to \infty$, with rank $r = p+1$ and fixed metric $g_{\mu\nu}$, the boundary conditions become Dirichlet. This means that the boundary condition interpolates between Neumann ($B = 0$) and Dirichlet ($B \to \infty$). Below we will see that when we take a NCYM decoupling limit we obtain Dirichlet boundary conditions, while if we take a NCOS decoupling limit we obtain a mixture of Neumann and Dirichlet boundary conditions. This indicates that there is a distinct difference between these two decoupling limits.

To obtain the two-point function we use the "image" method (familiar from electrodynamics). This means that we make the following Ansatz:[1]

$$
\begin{aligned}
< X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') > &= a^{\mu\nu} \log(z-z') + b^{\mu\nu} \log(\bar{z}-\bar{z}') \\
&+ c^{\mu\nu} \log(z-\bar{z}') + d^{\mu\nu} \log(\bar{z}-z') \,.
\end{aligned} \tag{4.4}
$$

This Ansatz is motivated by the form of the closed string two-point function (4.1). The two extra terms (the last two in (4.4)) have to be included due to the boundary conditions (4.3). Close to the 'source' ($z \sim z'$), the first two terms dominate the two-point function and the last two terms can be ignored. This implies that for $z \sim z'$ the open string two-point function should be similar to the closed string two-point function (4.1), because the effect of the constant B-field and the boundary conditions are irrelevant when $z \sim z'$. Using this argument we obtain that the tensors $a^{\mu\nu}$ and $b^{\mu\nu}$ are given by

$$
a^{\mu\nu} = b^{\mu\nu} = -\frac{\alpha'}{2} g^{\mu\nu} \,. \tag{4.5}
$$

The other two unknown tensors can now be determined by using the boundary conditions (4.3). Using (4.3) in the Ansatz (4.4) for the coordinate z, at the boundary $z = \bar{z}$, gives

$$
\begin{aligned}
& g_{\rho\mu}\Big(a^{\mu\nu}\frac{1}{z-z'} - b^{\mu\nu}\frac{1}{\bar{z}-\bar{z}'} + c^{\mu\nu}\frac{1}{z-\bar{z}'} - d^{\mu\nu}\frac{1}{\bar{z}-z'} \Big) \\
+ \; & B_{\rho\mu}\Big(a^{\mu\nu}\frac{1}{z-z'} + b^{\mu\nu}\frac{1}{\bar{z}-\bar{z}'} + c^{\mu\nu}\frac{1}{z-\bar{z}'} + d^{\mu\nu}\frac{1}{\bar{z}-z'} \Big)\Big|_{z=\bar{z}} = 0 \,.
\end{aligned} \tag{4.6}
$$

Equating the terms with $1/(z-z')$ and $1/(z-\bar{z}')$, respectively, gives the following two equations:

$$
g_{\rho\mu}(a^{\mu\nu} - d^{\mu\nu}) + B_{\rho\mu}(a^{\mu\nu} + d^{\mu\nu}) = 0 \,, \tag{4.7}
$$

$$
g_{\rho\mu}(c^{\mu\nu} - b^{\mu\nu}) + B_{\rho\mu}(c^{\mu\nu} + b^{\mu\nu}) = 0 \,. \tag{4.8}
$$

From these two equations we can now obtain the two unknown tensors $c^{\mu\nu}$ and $d^{\mu\nu}$. However, we start by defining a new tensor

$$
C^{\mu\nu} = \frac{\alpha'}{2} g^{\mu\nu} - c^{\mu\nu} \,, \tag{4.9}
$$

since it simplifies the derivation of $b^{\mu\nu}$ and $c^{\mu\nu}$. Continuing by inserting (4.5) and (4.9) in (4.8), gives after some calculations

$$
C^{\mu\nu} = \alpha' \Big(\frac{1}{g+B} \Big)^{\mu\nu} \,. \tag{4.10}
$$

After a similar calculation we also obtain

$$
d^{\mu\nu} = \frac{\alpha'}{2} g^{\mu\nu} - C^{\nu\mu} \,. \tag{4.11}
$$

[1]The following derivation is based on [99].

Finally, inserting (4.9)-(4.11) in the Ansatz (4.4), gives the following open string two-point function

$$
\begin{aligned}
< X^{\mu}(z,\bar{z})X^{\nu}(z',\bar{z}') > &= -\alpha' g^{\mu\nu}(\log | z - z' | - \log | z - \bar{z}' |) \\
&- C_{\rm S}^{\mu\nu} \log | z - \bar{z}' |^2 - C_{\rm A}^{\mu\nu} \log \frac{z - \bar{z}'}{\bar{z} - z'} ,
\end{aligned} \tag{4.12}
$$

where the index S and A denotes the symmetric and anti-symmetric part of the tensor $C^{\mu\nu}$, respectively. In this thesis we are interested in world volume theories of deformed Dp-branes. It is therefore useful to evaluate the two-point function at the boundary points $z = \bar{z} = \tau$, $z' = \bar{z}' = \tau'$. This gives [99]

$$
< X^{\mu}(\tau)X^{\nu}(\tau') >= -\alpha' G^{\mu\nu} \log (\tau - \tau')^2 + \frac{i}{2}\Theta^{\mu\nu}\varepsilon(\tau - \tau') . \tag{4.13}
$$

Here $\varepsilon(\tau - \tau') = 1$ if $\tau > \tau'$ and -1 if $\tau' > \tau$, while

$$
G^{\mu\nu} = \frac{1}{\alpha'} C_{\rm S}^{\mu\nu} = \Big(\frac{1}{g + B}\Big)_{\rm S}^{\mu\nu} = \Big(\frac{1}{g + B} g \frac{1}{g - B}\Big)^{\mu\nu} , \tag{4.14}
$$

and[2]

$$
\Theta^{\mu\nu} = 2\pi C_{\rm A}^{\mu\nu} = 2\pi\alpha' \Big(\frac{1}{g + B}\Big)_{\rm A}^{\mu\nu} = -2\pi\alpha' \Big(\frac{1}{g + B} B \frac{1}{g - B}\Big)^{\mu\nu} . \tag{4.15}
$$

Here $G^{\mu\nu}$ is the inverse open string metric ($G_{\mu\nu}$ is called the open string metric) and $\Theta^{\mu\nu}$ is the noncommutativity parameter. $G_{\mu\nu}$ is called the open string metric since it is the effective metric seen by open strings. This can be understood from the following argument: In the closed string mass shell condition the relevant metric is $g_{\mu\nu}$ (called the closed string metric), while in the open string mass shell condition the relevant metric is $G_{\mu\nu}$.

It is sometimes convenient to write $G_{\mu\nu}$ and $\Theta^{\mu\nu}$ in the following form:

$$
\begin{aligned}
G_{\mu\nu} &= g_{\mu\nu} + B_{\rho\mu} g^{\rho\sigma} B_{\sigma\nu} = g_{\mu\nu} + B_{\mu\nu}^2 , \\
\Theta^{\mu\nu} &= -2\pi\alpha' g^{\mu\rho} B_{\rho\sigma} G^{\sigma\nu} .
\end{aligned} \tag{4.16}
$$

Note that $\Theta^{\mu\nu} = 0$ and $G_{\mu\nu} = g_{\mu\nu}$, if $B_{\mu\nu} = 0$. From the two-point function we also obtain that the commutator between $X^{\mu}(\tau)$ and $X^{\nu}(\tau)$ is nonzero as long as $B_{\mu\nu}$ is nonzero, since [12]

$$
[X^{\mu}(\tau), X^{\nu}(\tau)] = i\Theta^{\mu\nu} . \tag{4.17}
$$

This implies that the endpoints of the open string do not commute, i.e., the endpoints live in a noncommutative space. Also, as we will see below, this implies that if we take the appropriate low energy limit (in order to decouple gravity) of a Dp-brane with a nonzero B-field turned on, we obtain a theory with noncommutative coordinates.

We end this subsection by deriving an expression for the open string coupling constant $G_{\rm o}^2$. For $B = 0$ the open string coupling constant is $G_{\rm o}^2 = g_{\rm s}$, where $g_{\rm s}$ is the closed string coupling constant. Below we will see that this expression is modified by a nonzero B-field.

[2] Note that our conventions for $\Theta^{\mu\nu}$ differ with 2π from the conventions used in Paper II-VI.

To obtain the open string coupling constant, we first note that for slowly varying fields the effective Dp-brane Lagrangian is given by the Dirac-Born-Infeld Lagrangian [12]

$$\mathcal{L}_{\rm DBI} = \frac{1}{g_{\rm s}(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\sqrt{-\det(g+B+2\pi\alpha' F)}\;, \tag{4.18}$$

where $F = dA$ is the world volume field strength. Here the combination $B + 2\pi\alpha' F$ is invariant under the gauge transformation: $A \to A + \Lambda$ and $B \to B - 2\pi\alpha' d\Lambda$. Note that this Lagrangian was derived using Pauli-Villars regularization [12]. Instead of writing the Lagrangian in this form it is possible to write it in terms of the open string variables $G_{\mu\nu}$, $G_{\rm o}^2$ and $\Theta^{\mu\nu}$ [12], where the Θ dependence is entirely in the $*$ product (see below). In this new form the Lagrangian is given by (using point-splitting regularization) [12]

$$\mathcal{L}_{\rm NCDBI}(\hat{F}) = \frac{1}{G_{\rm o}^2(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\sqrt{-\det(G+2\pi\alpha' \hat{F})}\;, \tag{4.19}$$

where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\hat{A}_\mu * \hat{A}_\nu + i\hat{A}_\nu * \hat{A}_\mu\;, \tag{4.20}$$

is the noncommutative field strength. The $*$ product between two functions f and g is defined as

$$f(x) * g(x) = e^{\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\mu}} f(x+\xi)g(x+\eta)\Big|_{\xi=\eta=0} = fg + \frac{i}{2}\Theta^{\mu\nu}\partial_\mu f\partial_\nu g + \mathcal{O}(\Theta^2)\;. \tag{4.21}$$

The two Lagrangians (4.18) and (4.19) describe the same physics, and can therefore be related by a field redefinition [12] (the Seiberg-Witten map), which gives a relation between F and $\hat{F}$. This follows from the fact that the difference between the two Lagrangians is due to using different regularizations. Thus we have two different descriptions of the same physics: (1) we use the Lagrangian (4.18) on a commutative space-time or (2) we use the 'noncommutative' Lagrangian (4.19) and $[X^\mu(\tau), X^\nu(\tau)]_* = i\Theta^{\mu\nu}$, but instead of the 'ordinary' product we use the $*$ product (4.21) between functions.

Now, to obtain the open string coupling constant we simply compare the constant parts (i.e., set $F = \hat{F} = 0$) of the two Lagrangians (4.19) and (4.19), which must be equal. This gives the following expression for the open string coupling constant:

$$G_{\rm o}^2 = g_{\rm s}\Big(\frac{\det G}{\det(g+B)}\Big)^{\frac{1}{2}} = g_{\rm s}\Big(\frac{\det G}{\det g}\Big)^{\frac{1}{4}}\;, \tag{4.22}$$

where (4.16) was used to obtain the second expression.

In the next three subsections we will investigate what happens if we turn on a magnetic, a light-like or an electric B-field on a Dp-brane and take various decoupling limits. To obtain the appropriate decoupling limits we will use some of the results in this subsection.

4.1.2 Decoupling limits

The open string metric and noncommutativity parameter are useful for obtaining so called decoupling limits of string theory. With decoupling limit we mean a limit of string theory which leads to a theory *without* closed strings (i.e., without gravity). Sometimes, the limit is also taken in order to decouple massive open strings which means that only the massless open string modes are kept in the spectrum, i.e., the massive open string modes are infinitely massive. These massless open string modes can be described by a super-Yang-Mills theory, with or without noncommutativity, depending on the type of limit. In section 4.1.4 we give an example where the closed strings decouple but both the massless and massive open strings are kept, which leads to an open string theory with space-time noncommutativity (NCOS). As we will see below, to obtain these non-gravitational theories involves taking $\alpha' \to 0$, while at the same time keeping the energy scale fixed. Effectively this means that the $\alpha' \to 0$ limit is equivalent to a low energy limit. To be more precise: consider low energy, i.e., we introduce a cutoff at energy $E << 1/(\alpha')^{1/2}$, which implies that the massive open and closed string modes are irrelevant for the low energy theory, since there is not enough energy for them to be excited. Instead of restricting the energy to $E << 1/(\alpha')^{1/2}$, we can let $\alpha \to 0$ and at the same time keep the energy scale fixed, which implies that the massive open and closed string modes become irrelevant (i.e., infinitely massive).

As a simple example we consider first the decoupling limit which leads to super-Yang-Mills in $p+1$ dimensions. Start with a Dp-brane aligned in the x^i, $i = 1, \dots, p$, directions, in flat 10-dimensional space-time, with flat closed string metric g_{MN} $(M, N = 0, 1 \dots, 9)$ and an open string metric $G_{MN} = g_{MN}$. We have no B-field turned on which implies that the open and closed string metrics are the same.

First, to decouple massive open and closed strings we take $\alpha' \to 0$ and keep the closed string metric fixed in the brane directions, i.e., $g_{\mu\nu} = \eta_{\mu\nu}$ $(\mu, \nu = 0, 1, \dots, p)$. The closed string metric in the transverse direction scales as $g_{mn} \sim (\alpha')^2$, which effectively makes the metric $(p+1)$-dimensional. One also have to keep the Yang-Mills coupling constant $g^2_{\rm YM} = (2\pi)^{p-2} g_{\rm s} (\alpha')^{\frac{p-3}{2}}$ fixed in the $\alpha' \to 0$ limit. This naturally implies that massive $(N \geq 1)$ open and closed strings decouple, which is seen from the mass-shell condition $G^{\mu\nu} p_\mu p_\nu = N/\alpha'$. It also means that only the massless $(N = 0)$ open string modes are left in the spectrum. The massless closed strings decouple in this limit. In the generic case this is non-trivial to show, since $N = 0$ for the massless closed strings. However, it has been shown by calculating graviton absorption cross-sections that also the massless closed strings decouple in the decoupling limit, as long as $p \leq 5$, see, e.g., [116][3]. We have now obtained a super-Yang-Mills theory in $p+1$ dimensions with fixed Yang-Mills coupling constant $g^2_{\rm YM}$.

In the next subsection we show how to obtain a limit of string theory which leads to super-Yang-Mills theory with fixed noncommutativity parameter Θ. This case is a bit trickier, since the open and closed string metrics are not the same anymore and we also

[3]Note also that decupling is also suggested by the fact that the 10-dimensional Newton's constant is $\kappa \sim g_s \alpha'^2$, which approaches zero in the decoupling limit, for $p \leq 6$.

have to consider the noncommutativity parameter Θ which should be held fixed in the decoupling limit.

4.1.3 Noncommutative super-Yang-Mills theory

Here we will show how maximally supersymmetric noncommutative super-Yang-Mills theory (NCYM) can be obtained as a limit of string theory. We start by considering a Dp-brane (aligned in the x^i, $i = 1, \ldots, p$, directions with $2 \leq p \leq 5$) with a rank 2 magnetic (or light-like, see below) B-field turned on[4]. To obtain the correct decoupling limit, i.e., NCYM decoupled from gravity, we have to satisfy the following conditions:

1. We have to take $\alpha' \to 0$, while keeping the energy scale fixed, in order to decouple gravity.

2. We have to keep $G_{\mu\nu}$ fixed and flat ($G_{\mu\nu} = \eta_{\mu\nu}$) in order to decouple massive open string modes. We choose a flat metric since this is the simplest case. For cases with curved open string metric, see, e.g., [114].

3. We also have to keep the Yang-Mills coupling constant $g^2_{\rm YM}$ fixed, see (4.23) below.

4. Finally, we have to keep $\Theta^{\mu\nu}$ fixed in order to have fixed noncommutativity.

The Yang-Mills coupling is defined as [12]

$$\frac{1}{g^2_{\rm YM}} = \frac{(\alpha')^{\frac{3-p}{2}}}{(2\pi)^{p-2}G^2_{\rm o}} = \frac{(\alpha')^{\frac{3-p}{2}}}{(2\pi)^{p-2}g_{\rm s}}\Big(\frac{\det g}{\det G}\Big)^{\frac{1}{4}} . \tag{4.23}$$

This expression is obtained from the $\hat{F}^2$ term by expanding (4.19). Note that for 'ordinary' super-Yang-Mills ($\Theta = 0$) conditions 1-3 have to be satisfied, as we have seen above, while condition 4 is of course irrelevant.

As the first case we consider turning on a rank 2 magnetic B-field. To satisfy the above four conditions we propose the following decoupling limit [12]

$$\begin{aligned} \epsilon &\to 0 \,, \quad \alpha' = \theta\epsilon^{1/2} \,, \quad B_{p-1,p} = -\epsilon^{1/2} \,, \\ g_{\alpha\beta} &= \eta_{\alpha\beta} \,, \quad \alpha,\beta = 0,1,\ldots,p-2 \,, \\ g_{ab} &= \epsilon\delta_{ab} \,, \quad a,b = p-1,p \,, \\ g_{mn} &= \epsilon\delta_{mn} \,, \quad m,n = p+1,\ldots,9 \,, \\ g_{\rm s} &= \tilde{g}_{\rm s}\epsilon^{\frac{5-p}{4}} = (2\pi)^{2-p}g^2_{\rm YM}\theta^{\frac{3-p}{2}}\epsilon^{\frac{5-p}{4}} \,, \end{aligned} \tag{4.24}$$

where $g^2_{\rm YM} = (2\pi)^{p-2}\tilde{g}_{\rm s}\theta^{\frac{p-3}{2}}$, $\tilde{g}_{\rm s}$ and θ are kept fixed. Note that $g^2_{\rm YM}$ has dimension L^{p-3} (L=length), $\tilde{g}_{\rm s}$ is dimensionless and θ has dimension L^2. To derive this decoupling limit,

[4] We choose the rank 2 case for simplicity. The generalization to arbitrary even, magnetic, rank r is straight-forward.

we first used condition 1 to *define* $\alpha' = \theta\epsilon^{1/2}$, $\epsilon \to 0$, then condition 2 and 4 to obtain the correct expressions for $g_{\mu\nu}$ and $B_{p-1,p}$. Finally, we used condition 3 and (4.23) to obtain the expression for g_s.

As a check that we have found the correct decoupling limit, we calculate the open string metric and noncommutativity parameter, as well as the open string coupling constant. Inserting (4.24) into (4.16) and (4.22), gives

$$G_{\mu\nu} = \eta_{\mu\nu} \ , \quad \Theta^{p-1,p} = 2\pi\theta \ , \quad G_o^2 = \tilde{g}_s \epsilon^{\frac{3-p}{4}} = (2\pi)^{2-p} g_{\rm YM}^2 \theta^{\frac{3-p}{2}} \epsilon^{\frac{3-p}{4}} \ , \tag{4.25}$$

which satisfy the four conditions above. Next, if we insert the above decoupling limit into the open string mass shell condition we get

$$\eta^{\mu\nu} p_\mu p_\nu = \frac{N}{\alpha'} = \frac{N}{\theta}\epsilon^{-1/2} \ , \tag{4.26}$$

which implies that in the decoupling limit the massive open strings ($N > 0$) becomes infinitely heavy and decouple. Hence, only the massless open string modes are kept in the spectrum and we can therefore use NCYM theory to describe the physics of the Dp-brane in this limit. The Lagrangian for this theory is given by [12][5]

$$\mathcal{L}_{\rm NCYM} = \frac{1}{4g_{\rm YM}^2}\sqrt{-{\rm det}G}\, G^{\mu\mu_1} G^{\nu\nu_1} \hat{F}_{\mu\nu} * \hat{F}_{\mu_1\nu_1} \ , \tag{4.27}$$

where $\hat{F}_{\mu\nu}$ is the noncommutative field-strength (4.20), the $*$ product is given by (4.21) and $g_{\rm YM}^2$ is the Yang-Mills coupling constant (4.23). Note that for energies $E << \theta^{-1/2}$, the noncommutativity becomes unimportant and the physics can be described by 'ordinary' super-Yang-Mills.

For $p = 3$ we see that the coupling constant is dimensionless ($g_{\rm YM}^2 = 2\pi\tilde{g}_s$). This implies that NCYM in four dimensions might be renormalizable, i.e., ultraviolet complete (finite). In [109] it has been shown that maximally supersymmetric NCYM in four dimensions is indeed renormalizable to all orders. For $p > 3$ one on the other hand have a coupling constant that has dimension length to a positive power. This implies that in dimensions larger than four, NCYM is non-renormalizable, i.e., the theory breaks down for energies $E > 1/(g_{\rm YM}^2)^{\frac{1}{p-3}}$. This can be shown in the following way: For U(1) NCYM in $p+1$ dimensions the dimensionless effective coupling is given by

$$g_{\rm eff}^2 = g_{\rm YM}^2 E^{p-3} \ , \tag{4.28}$$

where E is the energy scale. To have a perturbatively well defined NCYM theory this effective coupling constant has to be $g_{\rm eff}^2 << 1$, i.e., $E << 1/(g_{\rm YM}^2)^{\frac{1}{p-3}}$. For energies $E \sim 1/(g_{\rm YM}^2)^{\frac{1}{p-3}}$ ($p > 3$) and above NCYM becomes strongly coupled and perturbation theory is useless. For large energies these theories therefore have to be completed by relating them to some other theory. By completed we mean that in the energy region

[5]For a U(N) theory there is a Tr=trace in front of $\hat{F}_{\mu\nu} * \hat{F}_{\mu_1\nu_1}$, since $\hat{F}_{\mu\nu}$ is a N $\times$ N matrix.

where perturbation theory is useless for the NCYM theory, one has to switch to a different description (new theory) of the world volume physics. This usually means that new degrees of freedom have to be introduced. A simple example is the 'ordinary' five-dimensional super-Yang-Mills theory which at high energies can be completed by the M5-brane (2,0) tensor theory compactified on a circle of radius $R = 1/g_{\rm YM}^2$, see, e.g., Paper II, III and V-VI for a further discussions on this topic. For $p = 2$ instead, the coupling constant has dimension length to a negative power, which means that in three dimensions NCYM is UV complete but breaks down in the infrared for energies $E < g_{\rm YM}^2$. In the IR this theory is completed by a theory containing light D0-branes ($\widetilde{\rm OD0}$), see Paper III.

Before we consider turning on an electric B-field, we turn to the case with a rank 2 light-like B-field turned on. Again we must obey the four conditions above. We propose the following decoupling limit[6]

$$\begin{aligned} \epsilon &\rightarrow 0\ , \quad \alpha' = \theta\epsilon^{1/2}\ , \quad B_{+1} = -\epsilon^{-1/2}\ , \\ g_{++} &= -\epsilon^{-1}\ , \quad g_{-+} = g_{+-} = 1\ , \\ g_{ab} &= \delta_{ab}\ , \quad a, b = 1, 3, 4, \ldots, p\ , \\ g_{mn} &= \epsilon\delta_{mn}\ , \quad m, n = p+1, \ldots, 9\ , \\ g_{\rm s} &= \tilde{g}_{\rm s}\epsilon^{\frac{3-p}{4}} = (2\pi)^{2-p} g_{\rm YM}^2 \theta^{\frac{3-p}{2}} \epsilon^{\frac{3-p}{4}}\ , \end{aligned} \tag{4.29}$$

where $x^{\pm} = \frac{1}{\sqrt{2}}(x^2 \pm x^0)$, and $g_{\rm YM}^2 = (2\pi)^{p-2}\tilde{g}_{\rm s}\theta^{\frac{p-3}{2}}$, $\tilde{g}_{\rm s}$ and θ are kept fixed. Calculating the open string metric, coupling constant and noncommutativity parameter, gives

$$G_{\mu\nu} = \eta_{\mu\nu}\ , \quad \Theta^{-1} = 2\pi\theta\ , \quad G_{\rm o}^2 = \tilde{g}_{\rm s}\epsilon^{\frac{3-p}{4}} = (2\pi)^{2-p} g_{\rm YM}^2 \theta^{\frac{3-p}{2}} \epsilon^{\frac{3-p}{4}}\ , \tag{4.30}$$

which is same as in (4.25) except that we in this case have light-like noncommutativity.

A relevant question is if this NCYM theory, with light-like noncommutativity, is a unitary theory. This question is relevant because turning on a light-like B-field means that part of the B-field is electric ($B_{01} \neq 0$). And as we will see in the next subsection, a purely electric B-field can not lead to a consistent (unitary) NCYM theory with space-time noncommutativity. To check unitarity for the light-like case, we have to check that the inner product $p \circ p$ (p external momentum) is never negative, where [120]

$$p \circ p = -p_\mu \Theta^{\mu\rho} G_{\rho\sigma} \Theta^{\sigma\nu} p_\nu\ . \tag{4.31}$$

In [120] this restriction is shown to be necessary in order for the one-loop Feynman integrals to be convergent. For $p \circ p < 0$ there are branch cuts. For example, in a field theory with space-time noncommutativity (i.e., $p \circ p < 0$) these branch cuts lead to problems with the so called cutting rules, which causes the theory to be non-unitary, see [120] for details. In the light-like case, inserting (4.30) into (4.31), gives

$$p \circ p = (2\pi\theta)^2 p_-^2 = \frac{1}{2}(2\pi\theta)^2 (p_2 - p_0)^2 \geq 0\ , \tag{4.32}$$

[6]This limit is slightly different from the limit taken in [119]. However, this has no consequence for the physics. We have chosen a different limit, since 'our' limit is easier to compare to the 'supergravity dual' approach. It also gives $G_{\mu\nu} = \eta_{\mu\nu}$.

which implies that the theory is unitary. Similar to the space-space noncommutativity case the light-like case is also not UV complete for dimensions $d > 4$, while it is a complete theory in four dimensions. It is interesting to note that in four dimensions, the S-dual of the above light-like NCYM is also a light-like NCYM theory [119, 121] but with (see Paper VI)

$$\Theta_{(s)}^{-3} = 2\pi\theta_{(s)} = -\Theta^{-1}G_{\rm o}^2 = -2\pi\theta G_{\rm o}^2\ ,\quad G_{{\rm o}(s)}^2 = \frac{1}{G_{\rm o}^2}\ , \tag{4.33}$$

where $G_{{\rm o}(s)}^2$ is the S-dual open string coupling constant. Note also that these theories are not conformal, due to the length scale Θ.

4.1.4 Noncommutative open string theory

In this subsection we will show that it is not possible to obtain a unitary NCYM theory with space-time noncommutativity, as a limit (similar to (4.24)) of string theory. However, by changing the decoupling limit, we show that it is possible to obtain an open string theory with space-time noncommutativity *without* gravity (i.e., without closed strings).

Consider a Dp-brane ($1 \leq p \leq 6$) with a B_{01}-field turned on. If we try to take a NCYM limit (4.24) (switching place between $g_{\alpha\beta}$ and g_{ab}, $\alpha, \beta = 0, 1$), then we get into trouble, since this gives a flat open string metric but with the wrong sign ($G_{\alpha\beta} = -\eta_{\alpha\beta}$), which is clearly unphysical. We can also insert this limit into (4.31), which gives

$$p \circ p = (2\pi\theta)^2(p_1^2 - p_0^2) \leq 0\ , \tag{4.34}$$

implying, according to [120], that the above limit gives a theory which is *not* unitary. The conclusion is that it is not possible to take a large B-field limit ($B/\sqrt{-g_{00}g_{11}} \to \infty$) with an electric B-field and obtain an unitary field theory. Now the natural question is, is it possible to take some other decoupling limit, which would give a well defined theory without gravity? If the answer to this question is yes, we expect it to be an open string theory with space-time noncommutativity, since it cannot be a field theory[7]. This open string theory has to satisfy the following four conditions:

1. We have to take $\alpha' \to 0$, while keeping the energy scale fixed, in order to decouple gravity.

2. We have to keep $\alpha' G^{\mu\nu}$ fixed and flat, in order to *keep* massive open string modes in the spectrum.

3. We also have to keep the open string coupling constant $G_{\rm o}^2$ fixed.

4. As in the previous cases we keep $\Theta^{\mu\nu}$ fixed in order to have fixed noncommutativity.

[7]It is obviously possibly to take a decoupling limit which gives an 'ordinary' super Yang-Mills theory, without noncommutativity. However, here we are only interested in noncommutative theories.

5. The dimension one (i.e., (length)$^{-1}$) 'transverse' scalar fields ϕ should be fixed (see below) and given by $\phi \sim \frac{X^m}{\alpha'_{\text{eff}}}$, where α'_{eff} is the effective length scale in the decoupled theory (NCOS).

6. The tension of open strings aligned in the x^1 direction must be fixed and finite.

To satisfy these conditions we propose the following decoupling limit [93, 94, 100]:

$$\begin{aligned}
\epsilon &\rightarrow 0\,, \quad \alpha' = \alpha'_{\text{eff}}\epsilon\,, \quad B_{01} = 1 - \frac{\epsilon}{2}\,,\\
g_{\alpha\beta} &= \eta_{\alpha\beta}\,, \quad \alpha,\beta = 0,1,\,,\\
g_{ab} &= \epsilon\delta_{ab}\,, \quad a,b = 2,\ldots,p\,,\\
g_{mn} &= \epsilon\delta_{mn}\,, \quad m,n = p+1,\ldots,9\,,\\
g_{\text{s}} &= \tilde{g}_{\text{s}}\epsilon^{-\frac{1}{2}}\,,
\end{aligned} \tag{4.35}$$

where we keep α'_{eff} and $\tilde{g}_{\text{s}}$ fixed. Note that [93] $B^{\text{c}}_{01} = \sqrt{-g_{00}g_{11}} = 1$ is the critical value of the electric field. If we take B_{01} to be larger than this critical value we obtain a non-unitary theory, since the electric field would then exceed the open string tension and tear the open string apart. This corresponds to open string modes becoming tachyonic. Next, calculating the open string co-metric, coupling constant, noncommutativity parameter and open string tension for an open string aligned in the x^1 direction, gives

$$\begin{aligned}
\alpha' G^{\mu\nu} &= \alpha'_{\text{eff}}\eta^{\mu\nu}\,, \quad G^2_{\text{o}} = \tilde{g}_{\text{s}}\,, \quad \Theta^{01} = 2\pi\alpha'_{\text{eff}}\,,\\
T_{\text{os}} &= \frac{1}{2\pi\alpha'}\Big(1 - \Big(1 - \frac{\epsilon}{2}\Big)\Big) = \frac{1}{4\pi\alpha'_{\text{eff}}}\,,
\end{aligned} \tag{4.36}$$

which satisfy condition 2-4 and 6. That condition 1 is satisfied, i.e., that gravity decouples, have been shown in [93] by analyzing one loop and higher loop diagrams.

We see from (4.36) that we have obtained a finite effective open string tension. This is an effect of that we turned on a near critical electric field. This electric field tries to pull apart the two open string endpoints, which compensates the tension which tries to pull the endpoints together, leaving a finite effective tension T_{os}. To see that condition 5 is satisfied, we note that the part of the string sigma model, which involves the transverse coordinates X^m, is given by [100]

$$S = \frac{1}{4\pi\alpha'}\int g_{mn}\partial X^m\bar{\partial}X^n = \Big/\text{decoupling} \quad \text{limit}\Big/ = \frac{1}{4\pi\alpha'_{\text{eff}}}\int \delta_{mn}\partial X^m\bar{\partial}X^n\,. \tag{4.37}$$

This implies that the dimension one scalar fields $\phi \sim \frac{X^m}{\alpha'_{\text{eff}}}$, and also that correlation functions of the X^m-fields are finite in the decoupling limit [100].

Moreover, inserting the obtained open string metric into the open string mass shell condition, gives

$$\eta^{\mu\nu}p_\mu p_\nu = \frac{N}{\alpha'_{\text{eff}}}\,. \tag{4.38}$$

The implications of (4.36) and (4.38) is that in the above decoupling limit (4.35), we have obtained a noncommutative open string theory (NCOS), where *all* open string oscillator modes are part of the decoupled theory on the brane and their mass scale is set by α'_{eff}. As we have seen above, in order to decouple gravity and also keep massive open string modes we have to turn on a near critical electric field, which also leads to space-time noncommutativity, $[X^0, X^1] = i2\pi\alpha'_{\text{eff}}$. This implies that space-time noncommutativity and obtaining a decoupled open string theory are connected in the sense that it is not consistent to have one without the other. We also see from (4.36) that the open string coupling constant is fixed and dimensionless, which leads us to expect that NCOS theories are complete (renormalizable) in dimensions $2 \leq d \leq 6$ [122]. For $p > 6$ it does not seem to be possible to obtain a decoupled NCOS theory, see, e.g., [122].

For low energy, i.e., $E << (\alpha'_{\text{eff}})^{-1/2}$, NCOS theory reduces to 'ordinary' super-Yang-Mills with Lagrangian and Yang-Mills coupling constant given by [100]

$$\mathcal{L}_{\text{YM}} = \frac{1}{4g^2_{\text{YM}}}\eta^{\mu\mu_1}\eta^{\nu\nu_1}F_{\mu\nu}F_{\mu_1\nu_1}\,, \quad g^2_{\text{YM}} = (2\pi)^{p-2}G^2_{\text{o}}(\alpha'_{\text{eff}})^{\frac{p-3}{2}}\,, \tag{4.39}$$

where $F_{\mu\nu}$ is the Yang-Mills field strength. Note that this low energy limit theory has no noncommutativity, since the noncommutativity scale and the massive open string scale are the same for NCOS.

Finally, for a discussion about the strong coupling limit of NCOS (NCYM) theories, see section 4.2 and, e.g., [93, 100] and Papers II,III,V,VI.

4.2 Supergravity duals and deformation independence

In this section we are going to discuss supergravity duals of NCYM and NCOS in four dimensions, and show that these theories are S-dual to each other. We are also going to give a complete description of the concept of deformation independence, which is discussed and used in many of the Papers.

4.2.1 NCYM, NCOS and supergravity duals

As been explained in earlier sections, it is possible to use supergravity duals to obtain information about field theories, e.g., SYM in four dimensions. For theories with noncommutativity it is also possible to obtain supergravity duals, by taking the appropriate near horizon limit of a supergravity solution corresponding to a bound state, containing different types of branes. For example, the supergravity dual of four-dimensional NCYM is obtained by taking a 'magnetic' (see below) near horizon limit of the D3-D1 solution (this solution is given in (3.11), in section 3.1). This is the appropriate bound state because it contains a non-zero magnetic B-field. In Papers II-VI we have used the supergravity duals as follows: The supergravity dual gives the geometry of the background. This background is then probed by a probe brane. It is important that the background solution is obtained from a bound state with a large number of branes, since we do not want the probe brane to deform the background solution in any significant way. Next, by taking this probe brane infinitely far away from the stack (i.e., let the transverse radius $\tilde{r}/\tilde{R} \to \infty$, while keeping all other parameters fixed), the U(1) degrees of freedom on the probe brane will decouple from gravity, and the physics of the probe brane can be described by a non-gravitational theory with noncommutativity, see Paper II,V-VI for further clarifications. As we will see below, if, e.g., the probe brane is a D3-brane probing a supergravity dual obtained from a D3-D1 bound state, then open strings can end on it. Therefore, the open string metric, coupling constant and noncommutativity parameter (since $B_{23} \neq 0$) give important information about the physics on this probe brane. In this case the decoupled world volume theory is U(1) NCYM with non-zero Θ^{23}.

Below we will compare this approach with the method used in section 4.1.3 where NCYM was obtained as a limit of string theory. We show that with the appropriate identifications the two methods give equivalent results. However, using supergravity duals have other advantages over taking scaling limits in flat space. For example, scaling limits can easily be obtained from supergravity duals, but not the reverse, deformation independence (see below for the definition of this concept) is much more transparent and information regarding open brane metrics and generalized noncommutativity parameters can be obtained, see below and Paper II-VI. Moreover, thermodynamic quantities can also be obtained using supergravity duals, see, e.g., [96, 103].

Next, we take a closer look at the supergravity dual of NCYM with space-space noncommutativity in four dimensions. The supergravity dual is obtained from the D3-D1 bound state (3.11) by taking a 'magnetic' near horizon limit. This limit is called magnetic because the bound state has been obtained by turning on a non-zero magnetic B-field. Below we will see that an 'electric' limit is different.

To obtain the correct near horizon limit we demand that

$$\frac{ds^2}{\alpha'} , \quad e^{2\phi} , \quad \frac{B_2}{\alpha'} , \quad \frac{C_p}{(\alpha')^{p/2}} , \tag{4.40}$$

are held fixed in the $\alpha' \to 0$ limit, because this implies that the supergravity Lagrangian is finite, see section 2.3 and Paper II. Since we want to obtain the supergravity dual of NCYM in four dimensions we expect that the near horizon limit should be of 'field-theory type' [9], i.e., r/α' and the Yang-Mills coupling constant $g^2_{\rm YM}$ should be kept fixed when taking the near horizon limit. This means that in order to obtain the correct supergravity dual we should take the same limit as is taken to obtain the supergravity dual of four-dimensional SYM. Note, however, that there must be an extra condition which gives how the deformation parameter θ scales in the limit. From (4.40) we obtain that $\alpha'\theta$ should be kept fixed when taking the near horizon limit. This leads to the following 'magnetic' near horizon limit, where

$$x^\mu , \quad \tilde{r} = \frac{\ell^2}{\alpha'} r , \quad g^2_{\rm YM} = 2\pi g \ , \ell^2 = \alpha'\theta , \tag{4.41}$$

are kept fixed in the $\alpha' \to 0$ limit. Note that ℓ has dimension length. Inserting this limit in the D3-D1 bound state solution, gives

$$\begin{aligned}
\frac{ds^2}{\alpha'} &= \frac{1}{\ell^2}\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^2\Big(-(dx^0)^2+(dx^1)^2+\frac{1}{h}((dx^2)^2+(dx^3)^2)\Big) \\
&+ \frac{1}{\ell^2}\Big(\frac{\tilde{R}}{\tilde{r}}\Big)^2(d\tilde{r}^2+\tilde{r}^2 d\Omega_5^2)\ , \\
e^{2\phi} &= \frac{g^2}{h}\ ,\quad \frac{C_{0123}}{(\alpha')^2}=\frac{1}{g\ell^4 h}\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^4\ , \\
\frac{C_{01}}{\alpha'} &= -\frac{1}{g\ell^2}\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^4\ ,\quad \frac{B_{23}}{\alpha'}=-\frac{1}{\ell^2 h}\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^4\ , \\
\text{where}\quad & h=1+\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^4\ ,\quad \tilde{R}^4=g\ell^4 N\ .
\end{aligned} \tag{4.42}$$

This is the supergravity dual of four-dimensional NCYM.

Next, if we take the separation between the probe brane and the stack towards infinity (i.e., $\tilde{r}/\tilde{R}\to\infty$), in order to decouple gravity on the probe brane, then (4.42) reduces to

$$\begin{aligned}
\frac{ds^2}{\alpha'} &= \frac{\epsilon^{-1/2}}{\ell^2}(-(dx^0)^2+(dx^1)^2)+\frac{\epsilon^{1/2}}{\ell^2}((dx^2)^2+(dx^3)^2)+\frac{\epsilon^{1/2}}{\ell^2}(d\tilde{r}^2+\tilde{r}^2 d\Omega_5^2)\ , \\
e^{2\phi} &= g^2\epsilon\ ,\quad \frac{B_{23}}{\alpha'}=-\frac{1}{\ell^2}\ ,\quad \epsilon\to 0\ , \\
\text{where}\quad & \epsilon=\Big(\frac{\tilde{R}}{\tilde{r}}\Big)^4\ .
\end{aligned} \tag{4.43}$$

Note that we have only included the 'relevant' parts of the solution. Now let $\alpha'=\ell^2\epsilon^{1/2}\to 0$, and compare (4.43) with the decoupling limit for NCYM given in (4.24), $p=3$. This gives that (4.24) and (4.43) correspond to the same decoupling limit if we identify $\theta=\ell^2$ and $\tilde{g}_s=g$. Note also that the dilaton in (4.43) is identified with the closed string coupling constant g_s in (4.24). These results imply that the appropriate decoupling limit for NCYM can be obtained in two completely different ways: (1) As in section 4.1.3, where the decoupling limit was obtained using open string data and scaling, and (2) by first constructing the D3-D1 bound state, followed by taking a 'magnetic' near horizon limit and finally, separating the probe brane from the stack by an infinite distance (i.e., let $\tilde{r}/\tilde{R}\to\infty$).

Next, we calculate the open string metric, coupling constant and noncommutativity parameter for a probe brane probing the background given in (4.42), at a distance $\tilde{r}$ from the stack. This gives (using (4.16) and (4.22))

$$\frac{G_{\mu\nu}}{\alpha'}=\frac{1}{\ell^2}\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^2\eta_{\mu\nu}\ ,\quad G_o^2=g\ ,\quad \Theta^{23}=2\pi\ell^2\ . \tag{4.44}$$

The dimensionless NCYM coupling constant is given by $g^2_{\mathrm{YM}}=2\pi G_o^2$. The divergence of the open string metric, in units of α' in the $\tilde{r}/\tilde{R}\to\infty$ limit, implies that massive open string modes decouple.

Next, we show that NCYM with space-space noncommutativity is S-dual to NCOS in four dimensions. Under S-duality the supergravity fields (in the string frame and in the case of zero axion) transforms as follows:

$$\begin{aligned} \frac{ds_s^2}{\alpha'_s} &= e^{-\phi}\frac{ds^2}{\alpha'}\,, \quad e^{\phi} = e^{-\phi}\,, \\ \frac{B_s}{\alpha'_s} &= \frac{C_2}{\alpha'}\,, \quad \frac{C_2^s}{\alpha'_s} = -\frac{B}{\alpha'}\,, \\ \frac{C_4^s}{(\alpha'_s)^2} &= \frac{C_4}{(\alpha')^2} + \frac{B}{\alpha'}\wedge\frac{C_2}{\alpha'}\,. \end{aligned} \tag{4.45}$$

Using (4.45) on (4.42), gives

$$\begin{aligned} \frac{ds_s^2}{\alpha'_s} &= \frac{1}{\ell'^2}H^{\frac{1}{2}}\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^4(-(dx^0)^2+(dx^1)^2) + \frac{1}{\ell'^2}H^{-\frac{1}{2}}((dx^2)^2+(dx^3)^2) \\ &+ \frac{1}{\ell'^2}H^{\frac{1}{2}}(d\tilde{r}^2+\tilde{r}^2 d\Omega_5^2)\,, \\ e^{2\phi} &= g'^2 H\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^4\,, \quad g' = \frac{1}{g}\,, \quad \frac{C_{0123}^s}{(\alpha'_s)^2} = \frac{1}{g'\ell'^4}\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^4\,, \\ \frac{C_{23}^s}{\alpha'_s} &= \frac{1}{g'\ell'^2}H^{-1} \quad \frac{B_{01}^s}{\alpha'_s} = -\frac{1}{\ell'^2}\Big(\frac{\tilde{r}}{\tilde{R}}\Big)^4\,, \\ \text{where} \quad & H = 1+\Big(\frac{\tilde{R}}{\tilde{r}}\Big)^4\,, \quad \tilde{R}^4 = g'\ell'^4 N \quad \ell'^2 = g\ell^2\,, \quad \alpha'_s = g\alpha'\,. \end{aligned} \tag{4.46}$$

This is the supergravity dual of NCOS in four dimensions. This can easily be seen by comparing (4.46) with the NCOS supergravity dual given in, e.g., Paper II. In Paper II the supergravity dual of four-dimensional NCOS was obtained by first deforming the D3-brane with a non-zero electric component of the NS-NS B-field, which gives a D3-F1 bound state, followed by taking an 'electric' near horizon limit (see (4.57) below).

To really show that (4.46) is the NCOS supergravity dual, we calculate the open string metric, coupling constant, noncommutativity parameter as well as the tension for an open string aligned in the x^1 direction. This gives (using (4.16) and (4.22))

$$\frac{G_{\mu\nu}}{\alpha'_s} = \frac{1}{\ell'^2}H^{-\frac{1}{2}}\eta_{\mu\nu}\,, \quad G_o^2 = g'\,, \quad \Theta^{01} = -2\pi\ell'^2\,, \quad T_{os} = \frac{1}{4\pi\ell'^2}\,, \tag{4.47}$$

which is exactly the open string data we expect for NCOS in four dimensions. From (4.42), (4.46) and (4.47), we obtain the following relations between the parameters of the two S-dual theories:

$$G_{\rm NCOS}^2 = \frac{2\pi}{g_{\rm YM}^2}\,, \quad \Theta^{01} = -\frac{g_{\rm YM}^2}{2\pi}\Theta^{23}\,. \tag{4.48}$$

Note that $G_{\rm NCOS}^2 = G_o^2 = g'$. These relations imply that NCOS is weakly coupled when NCYM is strongly coupled and vice versa.

For a further discussion on NCOS and NCYM in four dimensions and possible dual descriptions using open D1-strings, see Paper III and V.

4.2.2 Deformation independence

A key concept in several of the included Papers, is deformation independence. Deformation independence of open string data has the following meaning: We start with a Dp-brane and deform it using the $\mathrm{O}(p+1,p+1)$ method. Next, if we calculate the open string metric and coupling constant, both before and after the deformation, we obtain the same result, i.e., the open string metric and coupling constant are independent of the dimensionless deformation parameter θ. It also implies that the noncommutativity parameter Θ is proportional to the deformation parameter θ, after the deformation (see below). We note that only the $\mathrm{O}(p+1,p+1)$ method gives manifestly deformation independent open string data. To obtain deformation independent result for solutions obtained with the rotation/boost method, one has to perform the coordinate transformation given in (3.12). The reason for this is that the rotation/boost method uses a coordinate transformation (i.e., a rotation or a boost).

We continue by showing explicitly that the $\mathrm{O}(p+1,p+1)$ method *always* gives manifestly deformation independent open string data. Begin with the observation from section 3.1.2 that the tensor $E_{\mu\nu}$ transform as follows, when using the $\mathrm{O}(p+1,p+1)$ method (note that we deform in all $p+1$ directions and do not explicitly write out indices):

$$\tilde{E} = (S^{-1}TS)E \ , \tag{4.49}$$

where $S^{-1} = S$, S is a T-duality transformation and T is a gauge transformation (i.e., a shift in $B \sim \theta$). Introducing

$$\tau^{\mu\nu} = G^{\mu\nu} + \frac{\Theta^{\mu\nu}}{2\pi\alpha'} \ , \tag{4.50}$$

it is easy to see from the results obtained in section 4.1 that

$$\tau^{\mu\nu} = (E^{-1})^{\mu\nu} \ . \tag{4.51}$$

Or written with a slightly different notation (and not explicitly writing out indices)[8]

$$\tau = (S)E \ . \tag{4.52}$$

Now using (4.49) and (4.52) we obtain that, using the $\mathrm{O}(p+1,p+1)$ method, τ changes to

$$\tilde{\tau} = (S)\tilde{E} = (T)\tau \ . \tag{4.53}$$

Hence, the new inverse open string metric and noncommutativity parameter are given by

$$\tilde{G}^{\mu\nu} = G^{\mu\nu} \ , \quad \tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \theta^{\mu\nu} \ . \tag{4.54}$$

Note that in the examples in this thesis we have always set $\Theta^{\mu\nu} = 0$, i.e., we have always started with an undeformed solution. However, from (4.54) we see that this is not necessary

[8] Note that in the relation $\tau = (S)E$, S of course does not mean T-duality but simply means that the tensor $E_{\mu\nu}$ is inverted. However, as we have seen earlier, this is just what T-duality does to $E_{\mu\nu}$.

in order to obtain deformation independence. The derivation of (4.53) implies that we have *proved* that the $\mathrm{O}(p+1, p+1)$ method always gives manifest deformation independent open string metric and noncommutativity parameter.

Moreover, that the open string coupling constant is manifestly deformation independent is easily obtained if one inserts the deformed dilation (3.23) and uses (4.54), in

$$\tilde{G}_{\mathrm{o}}^2 = e^{\tilde{\phi}}\Big(\frac{\det\tilde{G}}{\det\tilde{g}}\Big)^{\frac{1}{4}} , \qquad (4.55)$$

which gives that $\tilde{G}_{\mathrm{o}}^2 = G_{\mathrm{o}}^2 = e^{\phi}$.

Before we continue discussing deformation independence for a special case, the deformed D3-brane, we would like to make the following reflection: Above we saw that if the inverse open string metric and noncommutativity parameter is combined into the tensor $\tau^{\mu\nu}$, then $\tau^{\mu\nu}$ expressed in terms of the closed string metric and NS-NS two form, is given by the simple expression in (4.51) (or (4.52)). That $\tau^{\mu\nu}$ is given by this expression is interesting, since, as we have seen in subsection 3.1.2, under T-duality the new tensor $E'_{\mu\nu} = g'_{\mu\nu} + B'_{\mu\nu}$ (in the T-dualized directions) is given by *exactly* the same expression (see (3.16)). This is perhaps a bit strange considering that (3.16) and (4.51) are obtained in two completely different ways; (3.16) is derived from a duality between two dual sigma models, while (4.51) is derived by computing the open string two-point function on the boundary (for an open string ending on a Dp-brane). Also, (3.16) is a duality relation between two T-dual closed string theories, while (4.51) is just an expression of how the open string data (for open strings ending on a Dp-brane) depend on the closed string data in the same closed string theory. Furthermore, it should be obvious that deformation independence is related to this similarity. It would be interesting to further study this relation.

For the D3-brane the above proof has the consequence that if the D3-brane is deformed with a magnetic B-field (i.e., D3 $\rightarrow$ D3-D1) or an electric B-field (i.e., D3 $\rightarrow$ D3-F1)[9], then calculating the open string metric and coupling constant yields the same results *before* taking any near horizon limits. In both cases we obtain

$$G_{\mu\nu} = H^{-\frac{1}{2}}\eta_{\mu\nu} , \quad G_{\mathrm{o}}^2 = g . \qquad (4.56)$$

Note, however, that Θ has different non-zero components (i.e., $\Theta^{23} \neq 0$ or $\Theta^{01} \neq 0$) in the two cases. Next, if we take a magnetic near horizon limit (4.41), as we did for the D3-D1 bound state, we obtain the open string data given in (4.44). For the D3-F1 bound state we can not take a magnetic near horizon limit (field theory limit), because this would lead to an electric B-field which becomes larger than the critical value when the separation between the probe brane and the stack approach an infinite distance. This would give a non-unitary field theory with space-time noncommutativity as we have seen in section 4.1.4. Instead we have to take a different limit in order to obtain non-trivial results. This limit should imply that the electric field becomes near critical when the separation between

[9]Note that also a light-like deformation of course yields deformation independent open string data, see Paper VI.

the probe brane and the stack becomes large. We therefore consider the following 'electric' near horizon limit, where

$$\tilde{x}^{\mu} = \frac{\ell'}{\sqrt{\alpha'}} x^{\mu} \,, \quad \tilde{r} = \frac{\ell'}{\sqrt{\alpha'}} r \,, \quad g' \,, \theta = 1 \,, \tag{4.57}$$

are kept fixed in the $\alpha' \to 0$ limit. Inserting this limit into a D3-F1 bound state solution gives the NCOS supergravity dual (4.46). Note also that inserting it into (4.56) gives (4.47). The main result here is that although the open string data are deformation independent for any deformation, we are forced to take different limits in different cases, which naturally leads to different results.

For four-dimensional NCYM we draw the following conclusions: We know that NCYM and ordinary SYM (i.e., $\Theta^{\mu\nu} = 0$) have different supergravity duals. But both of these supergravity duals have been obtained by taking a magnetic near horizon limit (field theory limit). Deformation independence therefore implies that for both these field theories we have the same open string metric and coupling constant (both open string and Yang-Mills coupling constant). The only difference is that there is a constant shift in $\Theta^{\mu\nu}$ between them. This implies that if a SYM theory is deformed with a noncommutative deformation, i.e., we impose that, e.g., x^2 and x^3 no longer commute, then this deformation does *not* effect the Yang-Mills coupling constant and open string metric. It only induces a shift in the noncommutativity parameter. We therefore conclude, as a consequence of deformation independence, that both NCYM and 'ordinary' SYM are part of a continuous one parameter (assuming same value of the Yang-Mills coupling constant, otherwise two parameter) family of maximally supersymmetric field theories in four dimensions. The *only* difference between the different theories is that they have different values of the continuous noncommutativity parameter Θ. For the special value $\Theta = 0$, the theory is conformal (i.e., 'ordinary' SYM).

In Paper V deformation independence is conjectured to also be valid for M-theory open membrane data, as well as for open Dp-brane data. This means that open brane data is invariant (deformation independent) under the (U-duality) transformations which gives the solution generating techniques in Paper VII, see section 4.3.1, 4.3.3 and Paper VII for more details.

4.3 Open brane theories and open brane data

In this section we will obtain non-gravitational theories which contain light open branes in a geometry which have some kind of generalized noncommutativity structure. These theories are presently not as well understood as the NCYM and NCOS theories. In particular, there is no explicitly known microscopic formulation of them. Still it is possible to obtain some important information about these theories. We begin with the perhaps most interesting of these theories in subsection 4.3.2. This is the so called OM (open membrane) theory [100], which is the world volume theory of a deformed M5-brane in a certain limit. However, before we discuss OM-theory we will, in subsection 4.3.1, show how the open membrane metric and generalized noncommutativity parameter can be derived, using a new method which is very different from the methods used in Paper V and [108, 110]. In subsection 4.3.4 we discuss the ODp and $\widetilde{\text{OD}q}$ theories (see [100, 103, 105, 106] and Paper III), which are world volume theories of NS5 and D$(q+2)$-branes that contain light open Dq-branes, after we in subsection 4.3.3 have derived the open D-brane metric and generalized noncommutativity parameter, using the same method as in 4.3.1.

4.3.1 A derivation of open membrane data

Here we derive the open membrane metric and generalized noncommutativity parameter (theta parameter). These objects are conjectured to be the open membrane generalization of the open string metric and noncommutativity parameter (see below and next subsection for a further description of these objects). The main difference between the method used here and the one used in Paper V, is that here deformation independence is not assumed as a starting point, but easily seen to follow as a consequence of the method used.

We first generalize the string theory relation (4.51) to the M-theory membrane, i.e., we will *assume* that similar to how the open string metric and noncommutativity parameter can be 'obtained' from the string theory T-duality rules (by comparing (4.51) and (3.16)), the open membrane metric and theta parameter should be possible to obtain by using the M-theory 'T-duality' rules (3.32) in a clever way. The main difference compared to the string case is that the three form A_3 obeys a non-linear self-duality (NLSD) condition on the M5-brane. At first we will ignore this condition. However, as we will see below, in order to obtain the correct open membrane data we naturally have to include the NLSD condition into our calculations, which will lead to a modification of the first obtained results. Moreover, these new results fit perfectly with the open membrane data obtained earlier in Paper V and [108, 110].

We start by generalizing the string case. This, however, is not straightforward, since we have to generalize (4.51) for the open metric and noncommutativity parameter separately, because it does not seem to be possible to generalize the tensor $\tau^{\mu\nu}$. Instead we must first look at the generalization of the open string metric to an open membrane metric, using M-theory 'T-duality' rules for the metric. Then we generalize the open string noncommutativity parameter to a three form open membrane noncommutativity parameter, using M-theory 'T-duality' rules for the three form A_3.

Generalizing the results for the open string metric lead to that the open membrane (inverse) metric (in three directions) should be given by the right hand side of the first expression in (3.32), i.e.,

$$G^{ab}_{\text{OM}} = (\det G^a{}_b)^{1/9}(G^{-1})^{ab} , \quad G_{ab} = g_{ab} + \frac{1}{2}A^2_{ab} , \tag{4.58}$$

which implies that the open membrane metric is given by (so far only in three directions)

$$G^{\text{OM}}_{ab} = (\det G^a{}_b)^{-1/9}(g_{ab} + \frac{1}{2}A^2_{ab}) . \tag{4.59}$$

Here $a,b=0,1,2$, or $a,b=3,4,5$. Note that $G^a{}_b = g^{ac}G_{cb}$ and $(G^{-1})^{ab}$ is the inverse of G_{ab}.

Next, if we ignore the NLSD condition for the three form on the M5-brane world volume, the result in (4.59) indicates that the complete open membrane metric (in all six directions) is given by

$$G^{\rm OM}_{\mu\nu} = (\det[1+\frac{1}{2}A^2]^\mu{}_\nu)^{-1/9}(g_{\mu\nu}+\frac{1}{2}A^2_{\mu\nu})\ ,\quad \mu,\nu=0,1,\ldots,5\ . \tag{4.60}$$

However, this can not be corect, since we have ignored the NLSD condition. This means that we have to modify (4.60). We note that even the tensor structure of (4.60) has to be modified. This is clear, since the NLSD condition implies that $(\det[1+\frac{1}{2}A^2]^\mu{}_\nu)=1$.

Let us assume that (4.59) is still valid (in the three directions given by the indices a,b), adopt the following ansatz for the complete open membrane metric:

$$G^{\rm OM}_{\mu\nu} = R(K)(g_{\mu\nu}+\frac{1}{4}A^2_{\mu\nu})\ ,\quad \mu,\nu=0,1,\ldots,5\ , \tag{4.61}$$

where $R(K)$ is the yet to be determined conformal factor and

$$K = (\det[1+\frac{1}{4}A^2]^\mu{}_\nu)^{1/6} = \sqrt{1+\frac{1}{24}A^2}\ . \tag{4.62}$$

The tensor structure in the ansatz (4.61) is motivated from results obtained in [101], and it is also this tensor structure which is found in the NLSD condition, the M5-brane equations of motion and the energy momentum tensor on the M5-brane [123, 108]. One could also motivate the form of (4.61) from the fact that K should be given by a unique expression (as it also is in (4.62)). With this we mean that if we assume a more general ansatz of the tensor structure, e.g., $G^{\rm OM}_{\mu\nu} = R(K)(g_{\mu\nu}+\alpha\frac{1}{4}A^2_{\mu\nu})$, where α is an unknown constant, then K must be possible to express in the following two equivalent ways: (1) $K_1 = (\det[1+\alpha\frac{1}{4}A^2]^\mu{}_\nu)^{1/6}$, (2) $K_2=\sqrt{1+\beta\frac{1}{24}A^2}$, where β is a constant. Next, demanding that $K=K_1=K_2$ and that there should be a unique solution, gives that $\alpha=\beta=1$ (where we have used the 3+3 split below and the NLSD condition to obtain this result). Hence, we get the ansatz in (4.61).

The next step is to try to determine the conformal factor $R(K)$. To do this we demand that (4.61), under a 3+3 split of the coordinates, reduces to (4.59) (see below for further explanation). To simplify the calculations we use that if one goes to a frame (u^α_μ, v^a_μ), where $\alpha=0,1,2$ and $a=3,4,5$, parameterize the coset SO(1,5)/SO(1,2)×SO(3) as defined in [101] and Paper V, then (4.59) can be written as:

$$\begin{aligned} G^{\rm OM}_{rs} &= \left(1+\frac{1}{6}A^2_i\right)^{2/3} g_{rs}\ ,\quad A^2_i = g^{rs}(A^2_i)_{rs}\ ,\\ (A^2_i)_{rs} &= g^{r_1s_1}g^{r_2s_2}A_{rr_1r_2}A_{ss_1s_2}\ , \end{aligned} \tag{4.63}$$

where r,s, is either α,β, or a,b, depending on which case we have, while $i=1$ or 2, respectively. We note that the NLSD condition implies that

$$1+\frac{1}{6}A^2_2 = \left(1+\frac{1}{6}A^2_1\right)^{-1}\ . \tag{4.64}$$

Next, using the above parameterization we find that the ansatz (4.61) reduces to

$$G^{\rm OM}_{rs} = R(K)\Big(1 + \frac{1}{12}A_i^2\Big)g_{rs}\ . \tag{4.65}$$

Below, the following relations will be useful

$$K^2 = \frac{(1+\frac{1}{12}A_i^2)^2}{1+\frac{1}{6}A_i^2}\ , \quad i = 1,2\ , \tag{4.66}$$

and

$$K(1 \pm \sqrt{1-K^{-2}}) = \Big(1 + \frac{1}{6}A_i^2\Big)^{1/2}\ , \tag{4.67}$$

where the minus sign corresponds to $i = 1$ and the plus sign to $i = 2$ and we have used that $A_1^2 < 0$ and $A_2^2 > 0$.

As mentioned above, to determine $R(K)$ we demand that (4.63) and (4.65) are the same, which implies that

$$R(K) = \frac{\Big(1+\frac{1}{6}A_i^2\Big)^{2/3}}{1+\frac{1}{12}A_i^2}\ . \tag{4.68}$$

Next, using (4.66) and (4.67) in (4.68), the conformal factor can be written as

$$R(K) = \Big(\frac{1 \pm \sqrt{1-K^{-2}}}{K^2}\Big)^{1/3}\ . \tag{4.69}$$

We see here that we have obtained *two* results depending on if we have an electric (i.e., $r, s = 0, 1, 2$, in (4.63)) or a magnetic case (i.e., $r, s = 3, 4, 5$, in (4.63)). Since there should only be one unique open membrane metric, this means that only one of the results are correct. For various reasons (e.g., inserting the OM-limit should give the appropriate result [108] and the open membrane metric should reduce correctly to the open string metric Paper V), it has to be the electric case that is correct, which means that we should use the minus sign in (4.69). Thus we have derived the following covariant open membrane metric:

$$G^{\rm OM}_{\mu\nu} = \Big(\frac{1-\sqrt{1-K^{-2}}}{K^2}\Big)^{1/3}\Big(g_{\mu\nu} + \frac{1}{4}A^2_{\mu\nu}\Big)\ , \tag{4.70}$$

which is the same as the one derived in [108] and Paper V, using other methods.

It is also convenient to rewrite the open membrane metric as follows

$$G^{\rm OM}_{\mu\nu} = [K^4(1+\sqrt{1-K^{-2}})]^{-1/3}\Big(g_{\mu\nu} + \frac{1}{4}A^2_{\mu\nu}\Big)\ , \tag{4.71}$$

or

$$G^{\rm OM}_{\mu\nu} = \frac{[(2K^2-1) - 2K^2\sqrt{1-K^{-2}}]^{1/6}}{K}\Big(g_{\mu\nu} + \frac{1}{4}A^2_{\mu\nu}\Big)\ . \tag{4.72}$$

To obtain these expressions we have used that

$$\begin{aligned} K(1-\sqrt{1-K^{-2}}) &= [K(1+\sqrt{1-K^{-2}})]^{-1}\ , \\ (2K^2-1) - 2K^2\sqrt{1-K^{-2}} &= [K(1-\sqrt{1-K^{-2}})]^2\ . \end{aligned} \tag{4.73}$$

We continue by deriving the open membrane theta parameter. Generalizing the string case and ignoring the NLSD condition on the M5-brane, gives the following theta parameter (from identifying the (dimensionless) theta parameter with the right hand side of the third expression in (3.32))[10]:

$$\frac{\Theta_{\rm OM}^{rst}}{(2\pi)^2\ell_{\rm p}^3} = -A_{r's't'}g^{r'r}g^{s's}(G^{-1})^{t't} , \tag{4.74}$$

where G_{rs} is given in (3.33) above. The next step is to use this result together with the NLSD condition in order to obtain the complete open membrane theta parameter $\Theta_{\rm OM}^{\mu\nu\rho}$. To obtain $\Theta_{\rm OM}^{\mu\nu\rho}$ we demand that it, in the above 3+3 parameterization, should give the same result as (4.74) in the electric and magnetic cases together. From (4.74) we get

$$\Theta_{\rm OM}^{\alpha\beta\gamma} = -(2\pi)^2\ell_{\rm p}^3(1+\frac{1}{6}A_1^2)^{-1}A^{\alpha\beta\gamma} , \quad \Theta_{\rm OM}^{abc} = -(2\pi)^2\ell_{\rm p}^3(1+\frac{1}{6}A_2^2)^{-1}A^{abc} . \tag{4.75}$$

Next, using (4.75) and (4.77) below, we obtain the following tensor structure for $\Theta_{\rm OM}^{\mu\nu\rho}$:

$$\Theta_{\rm OM}^{\mu\nu\rho} = -(2\pi)^2\ell_{\rm p}^3 g^{\mu_1\mu}A_{\mu_1\nu_1\rho_1}(C^{-1})^{\nu_1\nu}(C^{-1})^{\rho_1\rho} , \tag{4.76}$$

where $(C^{-1})^{\mu\nu}$ is the inverse of

$$C_{\mu\nu} = S(K)\Big(g_{\mu\nu}+\frac{1}{4}A^2_{\mu\nu}\Big) , \tag{4.77}$$

and $S(K)$ is a conformal factor yet to be determined, and that $C_{\mu\nu}$ is conformal to the open membrane metric (4.70). In principle one could make a more general ansatz of the form $\Theta_{\rm OM} \sim g^{-1}\cdots g^{-1}AC^{-1}\cdots C^{-1}$, where the number of inverse metrics plus the number of C^{-1}'s, is three. However, it is not too difficult to see that it is only the combination '$g^{-1}C^{-1}C^{-1}$' that gives a unique solution. For the other three cases no solution exists. This means that the tensor structure is uniquely fixed by (4.75).

In the above parameterization (4.76) reduces to

$$\Theta_{\rm OM}^{\alpha\beta\gamma} = -(2\pi)^2\ell_{\rm p}^3S^{-2}(1+\frac{1}{12}A_1^2)^{-2}A^{\alpha\beta\gamma} , \quad \Theta_{\rm OM}^{abc} = -(2\pi)^2\ell_{\rm p}^3S^{-2}(1+\frac{1}{12}A_2^2)^{-2}A^{abc} . \tag{4.78}$$

Comparing (4.78) and (4.75) using (4.64), we obtain that

$$S = \frac{(1+\frac{1}{6}A_i^2)^{1/2}}{1+\frac{1}{12}A_i^2} = K^{-1} , \quad i=1,2 , \tag{4.79}$$

where we have used (4.66) in the last equality. Hence, the theta parameter is given by the covariant expression in (4.76), where[11]

$$C_{\mu\nu} = \frac{1}{K}\Big(g_{\mu\nu}+\frac{1}{4}A^2_{\mu\nu}\Big) , \quad (C^{-1})^{\mu\nu} = \frac{1}{K}\Big[\Big(1+\frac{1}{12}A^2\Big)g^{\mu\nu} - \frac{1}{4}(A^2)^{\mu\nu}\Big] . \tag{4.80}$$

[10]For $\Theta_{\rm OM}^{\mu\nu\rho}$, the conventions used in this thesis differ with a factor of $(2\pi)^2$ from the conventions used in Paper V.

[11]Note that $C_{\mu\nu}$ is exactly the so called M5-brane Boillat metric [123, 108, 110]. It is this metric that introduces the non-linearity in the M5-brane NLSD condition. Furthermore, the equations of motion and the energy momentum tensor on the M5-brane can be conveniently rewritten using $(C^{-1})^{\mu\nu}$ [123, 108].

In order to compare this result to Paper V and [110], we rewrite (4.76) in the following way, using (4.70) and (4.80):

$$\Theta_{\rm OM}^{\mu\nu\rho} = -(2\pi)^2 \ell_{\rm p}^3 [K(1-\sqrt{1-K^{-2}})]^{2/3} g^{\mu\mu_1} A_{\mu_1\nu_1\rho_1} G_{\rm OM}^{\nu_1\nu} G_{\rm OM}^{\rho_1\rho} , \tag{4.81}$$

where $G_{\rm OM}^{\mu\nu}$ is the inverse open membrane metric. This is exactly the same result as in Paper V and [110].

Above, we have obtained the open membrane metric and generalized noncommutativity parameter, from using the M-theory 'T-duality' rules and the NLSD condition only. However, although this method seems to work we still have no complete understanding why it works. It would be interesting to investigate this problem further.

We end this subsection with a short comment on deformation independence of open membrane data. Comparing to the discussion of deformation independence of open string data in subsection 4.2.2, it should be obvious, considering the results obtained in this subsection, that the open membrane data are deformation independent if we use deformed M5-brane solutions obtained using the solution generating technique derived in Paper VII. To be more specific: M5-branes which are deformed with an electric or light-like deformation are manifestly deformation independent. A magnetic deformation does not give manifest deformation independence since the NLSD condition implies that a magnetic deformation of an M5-brane is equivalent to an electric deformation, up to a coordinate transformation, see Paper VII and section 3.2.2 for an example. Hence, only one of the deformations (i.e., electric or magnetic) can be manifestly deformation independent, and since we had to chose the minus sign in (4.67) it must be the electric case.

4.3.2 Open membrane theory (OM-theory)

We have seen in subsection 4.1.4 that by deforming a Dp-brane with an electric B-field and taking the appropriate decoupling limit, we obtain a theory (NCOS) which contains light open strings. An interesting question is if this can be generalized to higher-dimensional objects, i.e., are there non-gravitational theories which contain light higher-dimensional branes? Before answering this question, we note that these higher-dimensional branes are not expected to be fundamental objects in the same sense that a string is fundamental. At the present not much is known about the role of these objects in a microscopic formulation of open brane theories.

The answer to the above question seems to be yes. For example, in M-theory there is an M5-brane, which has a superconformal (2,0) tensor theory as its world volume theory. Similar to how a Dp-brane can be deformed with a constant B-field this M5-brane can be deformed by turning on a constant background three form A_3. Therefore, since an open membrane couples to a three form potential, we expect that it is possible to take a decoupling limit, which leaves us with a six-dimensional theory containing light open membranes. Below we will show that this is indeed true. We also expect this theory to be very different from the (2,0) tensor theory, because it contains fully fluctuating open membranes and therefore contains many more d.o.f. than the (2,0) tensor theory. Similar to the

(2,0) theory this new open membrane theory (OM-theory) has no dimensionless coupling constant. A consequence of this is that it is not possible to use perturbative expansions for this theory, i.e., it is a non-perturbative theory. Since we have turned on a constant three form we also expect that there should be some generalization of noncommutativity, which is controlled by a generalized noncommutativity parameter $\Theta^{\mu\nu\rho}_{\rm OM}$, as seen in the previous subsection.

We start by considering an M5-brane in flat 11-dimensional space-time, where the M5-brane is aligned in the x^i, $i = 1, \ldots, 5$, directions. Next we deform this M5-brane by turning on a constant 'electric' three form A_{012}, which is dimensionless using our conventions. This three form obeys the same non-linear self-duality equation [117] as the gauge invariant M5-brane world volume three form $\mathcal{H}$. Note that the gauge invariant three form on the M5-brane is the combination $\mathcal{H} = H + A'_3$ [115] (where A'_3 is the pull-back of the background field A_3 and H is the self-dual field strength of the of the world volume two form B), which implies that a constant three form A_3 can always be transformed to a constant three form H (or vise versa), using a gauge transformation. Therefore, we will from here on only discuss a constant three form A_3.

Moreover, the NLSD condition implies that if one turns on an 'electric' three form A_{012} then there also has to be a 'magnetic' three form A_{345}, which is obtained using A_{012} and the NLSD equation. An interesting effect of the above mentioned self-duality is that for the M5-brane we can not have an electric deformation *or* a magnetic deformation, as is possible for the Dp-branes considering B-field deformations. Instead we have one case which has both magnetic and electric components of the three form turned on at the same time.

To obtain the correct decoupling limit in order to obtain a non-gravitational theory (OM-theory) which contain light open membranes, it must satisfy the following conditions (similar to the NCOS conditions):

1. We have to take the Planck length $\ell_{\rm p} \to 0$, while keeping the energy scale fixed, in order to decouple gravity.

2. We have to keep $\ell_{\rm p}^2 G^{\mu\nu}_{\rm OM}$ fixed and flat [101], in order to describe a genuine six-dimensional theory which has a fixed length scale $\ell_{\rm OM}$.

3. We also keep $\Theta^{\mu\nu\rho}_{\rm OM}$ fixed in order to have fixed generalized noncommutativity.

4. The tension of open membranes aligned in the x^1 and x^2 directions must be fixed and finite, in order to have light open membranes in the theory.

5. The dimension two (i.e., (length)$^{-2}$) 'transverse' scalar fields Φ^m should be fixed (see below) and given by $\Phi^m \sim \frac{X^m}{\ell^3_{\rm OM}}$ [100].

Here $G^{\rm OM}_{\mu\nu}$ is the open membrane metric, see (4.70) in subsection 4.3.1, and $\Theta^{\mu\nu\rho}_{\rm OM}$ is the generalized noncommutativity parameter (theta parameter), see (4.76). The latter object is a generalization of the open string noncommutativity parameter to the open membrane case. Moreover, a non-zero value of the theta parameter implies that the geometry is deformed. At the moment it is not quite clear in what way. It has been suggested that a non-zero theta parameter gives rise to some kind of noncommutative loop-space structure and/or non-associative geometry [101, 113, 125]. From here on we will not speculate any further on the meaning of this theta parameter. However, we expect it to be important in a microscopic formulation of the open membrane theory (OM-theory).

Next, to satisfy conditions 1-5 we propose the following decoupling limit:

$$\begin{aligned} \epsilon &\to 0\ , \quad \ell_{\rm p}^3 = \ell_{\rm OM}^3\epsilon\ , \quad A_{012} = 1 - \frac{\epsilon}{2}\ , \quad A_{345} = -\epsilon\ , \\ g_{\alpha\beta} &= \eta_{\alpha\beta}\ , \quad \alpha,\beta = 0,1,2\ , \\ g_{ab} &= \epsilon\delta_{ab}\ , \quad a,b = 3,4,5\ , \\ g_{mn} &= \epsilon\delta_{mn}\ , \quad m,n = 6,\ldots,10\ , \end{aligned} \tag{4.82}$$

where we keep $\ell_{\rm OM}$ fixed. Note that $A^{\rm c}_{012} = \sqrt{-g_{00}g_{11}g_{22}} = 1$ is the critical value of the electric field.

This decoupling limit satisfies condition 5 [100]. To check that it satisfies condition 1-4 above, we calculate the open membrane metric, theta parameter and the tension for an open membrane aligned in the x^1 and x^2 directions. We could calculate the former two by using (4.70) and (4.76), but instead we use some simpler expressions obtained in Paper V. In Paper V it is shown that if one goes to a frame (u_μ^α, v_μ^a), where $\alpha = 0,1,2$ and $a = 3,4,5$, parameterize the coset SO(1,5)/SO(1,2)×SO(3) as defined in [101] and Paper V, then the open membrane metric and theta parameter can be written as:

$$\begin{aligned} G^{\rm OM}_{\alpha\beta} &= \left(1 + \frac{1}{6}A_1^2\right)^{2/3} g_{\alpha\beta}\ , \quad A_1^2 = g^{\alpha\beta}(A_1^2)_{\alpha\beta}\ , \\ (A_1^2)_{\alpha\beta} &= g^{\alpha_1\beta_1}g^{\alpha_2\beta_2}A_{\alpha_1\alpha_2\alpha}A_{\beta_1\beta_2\beta}\ , \quad \alpha,\beta = 0,1,2\ , \\ G^{\rm OM}_{ab} &= \left(1 + \frac{1}{6}A_2^2\right)^{1/3} g_{ab}\ , \quad A_2^2 = g^{ab}(A_2^2)_{ab}\ , \\ (A_2^2)_{ab} &= g^{a_1b_1}g^{a_2b_2}A_{a_1a_2a}A_{b_1b_2b}\ , \quad a,b = 3,4,5\ , \end{aligned} \tag{4.83}$$

and

$$\Theta^{\alpha\beta\gamma}_{\rm OM} = -(2\pi)^2\ell_{\rm p}^3(1 + \frac{1}{6}A_2^2)A^{\alpha\beta\gamma}\ , \quad \Theta^{abc}_{\rm OM} = -(2\pi)^2\ell_{\rm p}^3(1 + \frac{1}{6}A_1^2)A^{abc}\ , \tag{4.84}$$

where we have used (4.64) and (4.67). Note that equation (4.84) implies that $\Theta^{\mu\nu\rho}_{\rm OM}$ is completely anti-symmetric. In Paper V it is further shown, using (4.84) and (4.83), that

$$*_G\Theta_{\rm OM} = \Theta_{\rm OM}\ , \quad (*_G\Theta_{\rm OM})^{\mu\nu\rho} = \frac{1}{6}\frac{1}{\sqrt{-G}}\epsilon^{\mu\nu\rho\iota\kappa\lambda}\Theta^{\rm OM}_{\iota\kappa\lambda}\ , \tag{4.85}$$

where G is the determinant of the open membrane metric and the indices on $\Theta_{\rm OM}$ are lowered with $G^{\rm OM}_{\mu\nu}$. From this relation we see that $\Theta^{\mu\nu\rho}_{\rm OM}$ is *linearly* self-dual with respect to the open membrane metric $G^{\rm OM}_{\mu\nu}$ [12]. That $\Theta^{\mu\nu\rho}_{\rm OM}$ is *linearly* self-dual might indicate that

[12] To be more precise: $\Theta^{\mu\nu\rho}_{\rm OM}$ is linearly self-dual with respect to *any* metric which is conformal to the open membrane metric, including the Boillat metric.

it would be more appropriate to define OM-theory using the open membrane metric and theta parameter, instead of the closed metric and the non-linearly self-dual three form. However, to clarify this issue requires further investigation.

Next, inserting the decoupling limit (4.82) into (4.83) and (4.84), as well as calculating the tension for an open membrane aligned in the x^1 and x^2 directions, gives

$$\begin{aligned} \frac{G^{\rm OM}_{\mu\nu}}{\ell_{\rm p}^2} &= \frac{\eta_{\mu\nu}}{\ell^2_{\rm OM}}\ , \quad \Theta^{012}_{\rm OM} = \Theta^{345}_{\rm OM} = \ell^3_{\rm OM}\ , \\ T_{\rm OM} &= \frac{1}{(2\pi)^2\ell_{\rm p}^3}\Big(1-\Big(1-\frac{\epsilon}{2}\Big)\Big) = \frac{1}{2(2\pi)^2\ell^3_{\rm OM}}\ , \end{aligned} \tag{4.86}$$

which clearly satisfy conditions 1-4. The fixed tension implies that OM-theory contains light open membranes. This is similar to how fixed open string tension implies that NCOS contains light open strings.

As we mentioned before, OM-theory has no dimensionless coupling constant. It has, however a length scale $\ell_{\rm OM}$ where both the noncommutative effects and the open membrane effects become important. This implies that for low energy, i.e., $E << 1/\ell_{\rm OM}$ we can use the (2,0) tensor theory as an effective description, since both the noncommutative effects and the open membrane effects are unimportant. At energies $E \sim 1/\ell_{\rm OM}$ and above we of course have to use the full OM-theory description. The introduction of this length scale in OM-theory implies that OM-theory is not a conformal theory. We also expect that the introduction of a generalized noncommutativity parameter implies that OM-theory is not a free theory for a single M5-brane. This is different from the (2,0) tensor theory which is free (a single tensor multiplet) for a single M5-brane. To check this claim we will next show that OM-theory compactified on a circle in the x^2 direction reduces to five-dimensional NCOS, which we know is not a trivial theory even for a single D4-brane.

The OM-theory decoupling limit (4.82) compactified on an 'electric' circle with radius R in the x^2 direction, gives the following type IIA solution (using the same conventions as in [100], except that our B-field is dimensionless):

$$\begin{aligned} \ell_{\rm p} &\to 0\ , \quad \alpha' = \frac{\ell_{\rm p}^3}{R}\ , \quad g_{\alpha\beta} = \eta_{\alpha\beta}\ , \quad g_{ab} = \frac{\ell_{\rm p}^3}{\ell^3_{\rm OM}}\delta_{ab}\ , \quad g_{mn} = \frac{\ell_{\rm p}^3}{\ell^3_{\rm OM}}\delta_{mn}\ , \\ g_{\rm s} &= \Big(\frac{R}{\ell_{\rm p}}\Big)^{3/2}\ , \quad B_{01} = 1 - \frac{\ell_{\rm p}^3}{2\ell^3_{\rm OM}}\ , \quad \alpha'_{\rm eff} = \frac{\ell^3_{\rm OM}}{R}\ , \\ G_{\rm o}^2 &= g_{\rm s}\Big(\frac{\ell_{\rm p}}{\ell_{\rm OM}}\Big)^{3/2} = \Big(\frac{R}{\ell_{\rm OM}}\Big)^{3/2}\ , \quad T_{\rm os} = \frac{2\pi R}{2(2\pi)^2\ell^3_{\rm OM}} = \frac{1}{4\pi\alpha'_{\rm eff}}\ , \end{aligned} \tag{4.87}$$

where $\alpha, \beta = 0, 1$, $a, b = 3, 4, 5$, and $m, n = 6, \ldots, 10$. Next, we define

$$\epsilon = \frac{\ell_{\rm p}^3}{\ell^3_{\rm OM}} = \frac{\alpha'}{\alpha'_{\rm eff}}\ , \tag{4.88}$$

using (4.87). Now, comparing (4.87) and (4.88) with (4.35) in section 4.1.4 (with $p = 4$), we obtain that (4.87) is the decoupling limit for NCOS in five dimensions. We therefore conclude that OM-theory compactified on an 'electric' circle with radius R gives five-dimensional NCOS theory when the radius is small ($R << \ell_{\rm OM}$). To be more precise: The relations between the OM-theory data and open string data are given by

$$G_{\rm o}^2 = \left(\frac{R}{\ell_{\rm OM}}\right)^{3/2} , \quad \Theta^{01} = \frac{\Theta^{012}_{\rm OM}}{2\pi R} . \tag{4.89}$$

A consequence of (4.89) is that NCOS is perturbatively well defined when $R << \ell_{\rm OM}$, since the open string coupling constant is $G_{\rm o}^2 << 1$ in this case.

Next we discuss the different phases which exists for OM-theory compactified on an 'electric' circle with radius R.

Case **1**. $R << \ell_{\rm OM}$. At very low energies $E << 1/(\alpha'_{\rm eff})^{1/2}$ we use five-dimensional SYM, since both noncommutativity and the Kaluza-Klein modes are irrelevant. At energies above this scale but $E << 1/R$ we use five-dimensional NCOS, since noncommutativity and massive open string states now are important but Kaluza-Klein modes are still irrelevant (i.e., the compactified dimension is too small to be seen at these energies). Finally, at energies $E \sim 1/R$, we have to use OM-theory compactified on an 'electric' circle, since the Kaluza-Klein modes are now relevant.

Case **2**. $R >> \ell_{\rm OM}$. In this case we can use five-dimensional SYM as long as $E << 1/R$. Above this but much less than $1/\ell_{\rm OM}$, we use the (2,0) tensor theory compactified on a circle with radius R. Finally, for energies $E \sim 1/\ell_{\rm OM}$, we use the full OM-theory compactified on a circle with radius R.

We see here that in case 2 there is no phase where NCOS is perturbatively well defined, since the open string coupling constant is $G_{\rm o}^2 >> 1$. This means that for $G_{\rm o}^2 >> 1$, when the NCOS theory is strongly coupled one can use OM-theory compactified on an 'electric' circle with radius R as an alternative description.

For a discussion about OM-theory compactified on a 'magnetic' circle with radius $R_{\rm m}$ we refer to [100] and Paper III. In Paper V and VI there are also discussions concerning the relation between open membrane and open string data in five and six dimensions (see also [110]).

4.3.3 A derivation of open D-brane data

In this short subsection we are going to show how the open D-brane data (i.e., the open Dp-brane metric and generalized noncommutativity parameter) can be derived in the same way as the open membrane data were derived in subsection 4.3.1. For the D-brane case we do not have to care about any NLSD conditions, which simplifies the derivation[13].

Following section 4.2.2 and 4.3.1 and using the D-brane 'T-duality' results from section 3.3, we see that the open Dp-brane (inverse) metric and generalized noncommutativity parameter (made dimensionless) can be identified with the expressions given by the first

[13]This means that we have not included the case with open D2-branes ending on the type IIA NS5-brane, since the RR three form obeys a NLSD condition. Note, however, that this case is essentially treated in the same way as the open membrane case in subsection 4.3.1.

and third right hand sides in (3.52). This gives (in $p+1$ directions)

$$\begin{aligned} G^{ab}_{\mathrm{OD}p} &= (\det G^a{}_b)^{\frac{p-1}{(p+1)^2}}(G^{-1})^{ab} \,, \\ \frac{\Theta^{a_1\ldots a_{p+1}}_{\mathrm{OD}p}}{(2\pi)^p(\alpha')^{\frac{p+1}{2}}} &= -C_{b_1\ldots b_{p+1}} g^{a_1b_1}_{\mathrm{D}p}\cdots g^{a_pb_p}_{\mathrm{D}p}(G^{-1})^{a_{p+1}b_{p+1}} \,. \end{aligned} \tag{4.90}$$

If we now constrain the open Dp-brane data to be valid only for RR (p+1)-forms with *one* non-zero component (as in Paper V), then (4.90) can be generalized to (note that we give the metric and not the inverse metric below)[14] :

$$\begin{aligned} G^{\mathrm{OD}p}_{\mu\nu} &= \left[1+\frac{1}{(p+1)!}C^2_{p+1}\right]^{\frac{1-p}{1+p}}\left(g^{\mathrm{D}p}_{\mu\nu}+\frac{1}{p!}(C^2_{p+1})_{\mu\nu}\right) \,, \\ \Theta^{\mu_1\ldots\mu_{p+1}}_{\mathrm{OD}p} &= -(2\pi)^p(\alpha')^{\frac{p+1}{2}}C_{\nu_1\ldots\nu_{p+1}}g^{\mu_1\nu_1}_{\mathrm{D}p}\cdots g^{\mu_p\nu_p}_{\mathrm{D}p}G^{\mu_{p+1}\nu_{p+1}} \,, \end{aligned} \tag{4.91}$$

where

$$G_{\mu\nu} = g^{\mathrm{D}p}_{\mu\nu} + \frac{1}{p!}(C^2_{p+1})_{\mu\nu} \,, \tag{4.92}$$

and we now have $\mu,\nu = 0,1,\ldots,5$ and $\mu,\nu = 0,1,\ldots,p+2$ for open Dp-branes ending on NS5-branes ($p \neq 2$ in the NS5-brane case) and D$(p+2)$-branes, respectively. Furthermore, to obtain (4.91) we have used that

$$(\det[1+\frac{1}{p!}C^2]^{\mu}{}_{\nu})^{\frac{1}{p+1}} = 1 + \frac{1}{(p+1)!}C^2 \,, \tag{4.93}$$

under the restriction that one only has one non-zero component of the RR (p+1)-form. Comparing (4.91) with Paper V we find that the results are identical, as expected.

Similar to the discussions in subsections 4.2.2 and 4.3.1, the results in this subsection implies that the open Dp-brane data are manifestly deformation independent for any one parameter deformation, with an RR $(p+1)$-form, of an NS5-brane or D$(p+2)$-brane, using the methods described in Paper VII.

4.3.4 Open D-brane theories (ODp/$\widetilde{\mathrm{OD}q}$-theory)

In this subsection we will generalize open string and open membrane theories to theories containing open Dp-branes. Similar to open strings and M-theory open membranes which can end on Dp-branes and the M-theory M5-brane, respectively, open Dp-branes can end on type IIA or IIB NS5-branes as well as on D$(p+2)$-branes. This suggest the possibility that there exist theories containing light open Dp-branes, which are obtained by taking the appropriate decoupling limit. This limit should involve turning on a near critical electric RR $(p+1)$-form. In this subsection we will first consider the case where a near critical electric RR $(p+1)$-form is turned on on an NS5-brane, which in the decoupling limit leads

[14]Note that for $\Theta^{\mu_1\mu_2\ldots\mu_p}_{\mathrm{OD}p}$ the conventions used in this thesis differ with a factor of $(2\pi)^p$ from the conventions used in Paper V.

to the so called ODp-theory [100]. Next, we continue with the case of a near critical electric RR $(q+1)$-form on a D$(q+2)$-brane, which in the decoupling limit leads to the $\widetilde{\text{OD}q}$-theory, see Paper III and [106]. In particular, we will discuss the OD3 and $\widetilde{\text{OD3}}$ theories and show that they are S-dual.

Start by considering an NS5-brane aligned in the x^i, $i = 1, \ldots, 5$, directions. To obtain the correct decoupling limit we demand that (similar to the membrane case in 4.3.2) the tension of open Dp-branes ($0 \leq p \leq 4$) aligned in the x^α, $\alpha = 1, \ldots, p$, directions is fixed. Furthermore, similar to the open string and open membrane cases we also keep $\alpha' G^{\mu\nu}_{\text{OD}p}$ and $\Theta^{\mu_1\mu_2\ldots\mu_p}_{\text{OD}p}$ fixed in the decoupling limit. Here $G^{\text{OD}p}_{\mu\nu}$ is the open Dp-brane metric and $\Theta^{\mu_1\mu_2\ldots\mu_p}_{\text{OD}p}$ is the open Dp-brane generalized noncommutativity parameter (theta parameter). For $p = 1$ this theta parameter implies 'ordinary' noncommutativity, while for $p > 1$ some generalization of noncommutativity. The open Dp-brane metric and theta parameter for a one parameter deformation, are given in (4.91) above (for $p \neq 2$), where $g^{\text{D}p}_{\mu\nu} = g_s^{-\frac{2}{q+1}} g_{\mu\nu}$ is the closed Dp-brane metric. For $p = 2$ these expressions are not valid. The reason for this is that for a one parameter deformation with an electric RR C_{012} of an NS5-brane, there is also a magnetic component (i.e., C_{345}) of the RR three form C_3. This is due to a non-linear self-duality equation for the RR three form on the NS5-brane. The origin of this self-duality equation is the non-linear self-duality equation which the M-theory three form A_3 obeys, see section 4.3.2. For $p = 2$ we instead have the following expressions:

$$
\begin{aligned}
G^{\text{OD2}}_{\mu\nu} &= \left(\frac{1-\sqrt{1-K^{-2}}}{K^2}\right)^{1/3} \left(g^{\text{D2}}_{\mu\nu} + \frac{1}{4}(C^2)_{\mu\nu}\right) , \\
\Theta^{\mu\nu\rho}_{\text{OD2}} &= -(2\pi)^2(\alpha')^{3/2}[K(1-\sqrt{1-K^{-2}})]^{2/3} g^{\mu\mu_1}_{\text{D2}} C_{\mu_1\nu_1\rho_1} G^{\nu_1\nu}_{\text{OD2}} G^{\rho_1\rho}_{\text{OD2}} , \qquad (4.94) \\
K &= \sqrt{1+\frac{1}{24}C^2} .
\end{aligned}
$$

Next, we consider the following decoupling limit [100]:

$$
\begin{aligned}
\epsilon &\to 0 , \quad \alpha' = \ell^2 \epsilon^{1/2} , \quad g_s = G^2_{\text{OD}p} \epsilon^{\frac{3-p}{4}} , \\
C_{01\ldots p} &= \frac{1}{G^2_{\text{OD}p}} \epsilon^{\frac{p-3}{4}} \left(1 - \frac{\epsilon}{2}\right) , \quad C_{(p+1)\ldots 5} = -\frac{1}{G^2_{\text{OD}p}} \epsilon^{\frac{5-p}{4}} , \\
g_{\alpha\beta} &= \eta_{\alpha\beta} , \quad g_{ab} = \epsilon \delta_{ab} , \quad g_{mn} = \epsilon \delta_{mn} , \qquad (4.95) \\
\alpha, \beta &= 0, 1, \ldots, p , \quad a, b = (p+1), \ldots, 5 , \quad m, n = 6, \ldots, 9 ,
\end{aligned}
$$

where the ODp-theory coupling constant $G^2_{\text{OD}p}$ and ℓ are kept fixed. Note that we have included the dual 'magnetic' $(5-p)$-form in the decoupling limit. To see that this is the appropriate decoupling limit we calculate the open Dp-brane metric, theta parameter and the tension for an open Dp-brane aligned in the x^α, $\alpha = 1, \ldots, p$, directions. This gives

$$
\begin{aligned}
\frac{G^{\text{OD}p}_{\mu\nu}}{\alpha'} &= \frac{\eta_{\mu\nu}}{\ell^2} , \quad \Theta^{01\ldots p}_{\text{OD}p} = G^2_{\text{OD}p} \ell^{p+1} , \qquad (4.96) \\
T_{\text{OD}p} &= \frac{1}{(2\pi)^p (\alpha')^{\frac{p+1}{2}}} \left(\sqrt{-\det g^{\text{D}p}_{\alpha\beta}} - \epsilon^{01\ldots p} C_{01\ldots p}\right) = \frac{1}{2(2\pi)^p G^2_{\text{OD}p} \ell^{p+1}} ,
\end{aligned}
$$

which satisfy the required conditions. Again, fixed tension implies that the ODp-theory contains light open Dp-branes.

As we have mentioned before these ODp-theories contain light open Dp-branes. But we also expect them to contain closed little strings. The reason for this is that for no deformation of an NS5-brane the decoupled world volume theory is Little String Theory (LST) at high energy [124]. LST is a non-perturbative theory, without gravity in six dimensions, which contains closed strings. In the type IIA case these closed little strings can be seen from an 11-dimensional point of view to come from cylindrical open membranes ending on M5-branes, where the 11'th (compactified) dimension is transverse to the M5-brane. When the radius is small we have a type IIA string theory description, and the little strings can be viewed as the boundary of the open membranes wrapped on a circle with small radius. After taking the decoupling limit (4.95) the tension of the little strings is given by [100] (see also Paper III)

$$T_{\mathrm{s}} = \frac{2\pi R}{(2\pi)^2 \ell_{\mathrm{OM}}^3} = \frac{1}{2\pi\ell^2} , \tag{4.97}$$

where R is the coordinate radius which is fixed in the decoupling limit, see below. For the type IIB cases the little string tension is also given by $T_{\mathrm{s}} = \frac{1}{2\pi\ell^2}$, see Paper III.

This means that the ODp-theories contain *both* light closed little strings and light open Dp-branes. Note that, at small ODp coupling constant (i.e., much less than one) the little strings are 'lighter' than the open Dp-branes[15]. This suggests that the little strings are more important than the open Dp-branes at weak ODp coupling. However, since there does not exists a complete microscopic description of the ODp-theories, the exact relevance of the little strings and open Dp-branes is not clear.

Next, we will look closer at the OD2 and OD3 theories. The OD2-theory is special since there are both a magnetic and an electric component of the three form C_3. This is a type IIA theory and can therefore be related to M-theory. The deformed NS5-brane can be seen as an deformed (with a three form A_3) M5-brane with a small compactified transverse direction with radius R_{T}. Furthermore, this means that the OD2-theory is related to OM-theory. If we compare the two decoupling limits (4.82) and (4.95), the parameters of the two theories are related as follows:

$$\ell_{\mathrm{p}}^3 = G_{\mathrm{OD2}}^2 \ell^3 \epsilon , \quad \ell_{\mathrm{OM}}^3 = G_{\mathrm{OD2}}^2 \ell^3 , \quad R_{\mathrm{T}} = G_{\mathrm{OD2}}^2 \ell \epsilon^{1/2} . \tag{4.98}$$

Hence, OD2-theory is OM-theory with a transverse circle with coordinate radius $R = G_{\mathrm{OD}p}^2 \ell$. Note also that $\Theta_{\mathrm{OD2}} = \Theta_{\mathrm{OM}}$ and that with our conventions the three forms are related as: $A_3 = g_{\mathrm{s}} C_3$, while the closed metrics are the same.

Next, we investigate the OD3-theory and argue (using S-duality) for the existence of a theory called $\widetilde{\mathrm{OD3}}$-theory, which is a theory containing light open D3-branes obtained from turning on a near critical electric RR four form on a D5-brane, followed by taking the appropriate decoupling limit.

[15] They are lighter in the sense that the little string effective length scale is larger than the open Dp-brane effective length scale.

The decoupling limit for the OD3-theory is given in (4.95) with $p = 3$. This theory contains light open D3-branes with tension given in (4.96) as well as closed little strings with tension given in (4.97). In the case with small coupling constant, i.e., $G^2_{\rm OD2} << 1$ we have to use the full OD3-theory at high energies ($E \sim 1/(\Theta_{\rm OD3})^{1/4}$ and above), while LST is adequate for lower energies. However, at even lower energies (i.e., $E << T_{\rm s}^{1/2} = 1/g_{\rm YM}$), we can switch to 'ordinary' six-dimensional SYM theory.

If we instead have a large OD3 coupling constant then things get trickier. Since the coupling constant is very large we can not have any perturbative OD3 description of the world volume physics. We therefore conclude that we must S-dualize to obtain a new (perturbatively well behaved) description. If we S-dualize the decoupling limit (4.95) for $p = 3$ we get the following decoupling limit (note that α' transform but the closed string metric does not)

$$
\begin{aligned}
\epsilon &\rightarrow 0\,, \quad \tilde{\alpha}' = \tilde{\ell}^2\epsilon^{1/2}\,, \quad \tilde{g}_{\rm s} = G^2_{\widetilde{\rm OD3}}\,, \\
C_{0123} &= \frac{1}{G^2_{\widetilde{\rm OD3}}}\Big(1-\frac{\epsilon}{2}\Big)\,, \quad B_{45} = -\epsilon^{\frac{1}{2}}\,, \\
g_{\alpha\beta} &= \eta_{\alpha\beta}\,, \quad g_{ab} = \epsilon\delta_{ab}\,, g_{mn} = \epsilon\delta_{mn}\,, \\
\alpha,\beta &= 0,1,2,3\,, \quad a,b = 4,5\,, \quad m,n = 6,\ldots,9\,,
\end{aligned}
\tag{4.99}
$$

where

$$
\tilde{g}_{\rm s} = \frac{1}{g_{\rm s}}\,, \quad \tilde{\alpha}' = \alpha' g_{\rm s}\,, \quad \tilde{\ell}^2 = \ell^2 G^2_{\rm OD3}\,, \quad G^2_{\widetilde{\rm OD3}} = \frac{1}{G^2_{\rm OD3}}\,. \tag{4.100}
$$

Note that we have included the dual magnetic B-field in the decoupling limit. We see here that we have obtained a decoupling limit which is very similar to the OD3 decoupling limit. For example, this decoupling limit implies that there are light open D3-branes whose tension is given by

$$
T_{\widetilde{\rm OD3}} = \frac{1}{2(2\pi)^3 G^2_{\widetilde{\rm OD3}}\tilde{\ell}^4} = \frac{1}{2(2\pi)^3 G^2_{\rm OD3}\ell^4} = T_{\rm OD3}\,, \tag{4.101}
$$

which is the same as the tension for open D3-branes in the OD3-theory. There are also closed little strings in this theory with tension

$$
\tilde{T}_{\rm s} = \frac{1}{2\pi G^2_{\widetilde{\rm OD3}}\tilde{\ell}^2} = \frac{1}{2\pi\ell^2} = T_{\rm s}\,, \tag{4.102}
$$

which means that also the little string tension is the same as in the OD3-theory. This leads us to propose that (4.99) gives the decoupling limit for a theory containing both light open D3-branes and closed little strings. We name this theory $\widetilde{\rm OD3}$. This theory is S-dual to the OD3-theory, since $G^2_{\widetilde{\rm OD3}} = \frac{1}{G^2_{\rm OD2}}$ (see (4.100) above). Furthermore, this has the consequence that when the OD3-theory is strongly coupled we can instead use the $\widetilde{\rm OD3}$-theory, which is weakly coupled.

Next, we note that the $\widetilde{\rm OD3}$-theory reduces to six-dimensional NCYM at low energy. This is clear from the fact that NCYM has the same decoupling limit as the $\widetilde{\rm OD3}$-theory, as seen by comparing (4.99) and (4.24) with $p = 5$ and set $\theta = \tilde{\ell}^2$ and $\tilde{g}_s = G^2_{\widetilde{\rm OD3}}$. The difference between the two cases is due to the fact that for the $\widetilde{\rm OD3}$-theory we have the four form electric RR field on the world volume, while in the NCYM case we use the dual two form B-field. These two different cases originate from the fact that both open F-strings and open D3-branes can end on a D5-brane. This means that using open F-strings leads to NCYM while open D3-branes leads to the $\widetilde{\rm OD3}$-theory. Since we know that NCYM is strongly coupled at large energies this is good news because we can now use the $\widetilde{\rm OD3}$-theory to complete NCYM at high energies, when $G^2_{\widetilde{\rm OD3}} << 1$. For $G^2_{\widetilde{\rm OD3}} >> 1$ we instead use the OD3-theory to complete NCYM at high energies.

We have seen above that the OD3 and $\widetilde{\rm OD3}$ theories essentially have the same decoupling limit, the same open D3-brane and little string tension, but that the coupling constants are inversely related. It has therefore been suggested that the OD3 and $\widetilde{\rm OD3}$ theories can be viewed as one self-dual theory [106]. We will here argue that this might not be true. As we have mentioned above, the two theories have light open D3-branes and little strings with tensions given in (4.101) and (4.102). There is, however, a crucial difference between the two theories: For OD3 at weak coupling the little strings are the lightest objects, while for the $\widetilde{\rm OD3}$-theory the open D3-branes are the lightest objects at weak $\widetilde{\rm OD3}$ coupling. We note that self-duality implies that a theory is identical at weak and strong coupling. But since the 'fundamental' objects (we here assume that the lightest object is 'more' fundamental) are different in the two theories we find it unlikely that they are identical. To obtain a definite answer we expect that one has to know the full microscopic descriptions of the two theories.

Above we have motivated the existence of a D5-brane world volume theory that contains light open D3-branes. Similar to how open D3-branes can end on D5-branes, we know that open Dq-branes can end on D$(q+2)$-branes. We therefore expect that it should be possible to obtain decoupled $(q+3)$-dimensional theories containing light open Dq-branes, for $0 \leq q \leq 3$. For $q = 4$ it is unclear if the theory is decoupled from gravity, see Paper IV. Note that there are only little strings included for $q = 3$. Also in the general case these theories are related to NCYM, in the sense that NCYM is the low energy limit of $\widetilde{{\rm OD}q}$-theory. For a further introduction to the ODp and $\widetilde{{\rm OD}q}$-theories we refer to [100, 103, 105, 106] and Paper III.

5

Summary and discussion

In this thesis we have discussed various aspects of string/M-theory. We began in chapter two with an introduction to perturbative string theory, dualities (T-, S- and U-duality), as well as a brief discussion about M-theory and its relation to type IIA and $E_8 \times E_8$ heterotic string theory. There was also an introduction to the AdS/CFT correspondence, which has given much new information about the relation between field theories and string/M-theory. The main conclusion in chapter two was that at a non-perturbative level much evidence points to the existence of *one* theory (M-theory), which in various limits give all the five consistent string theories in 10 dimensions. Note that in a low energy limit M-theory reduces to 11-dimensional supergravity. Furthermore, M-theory is not a string theory, but a theory containing membranes and 5-branes. A longstanding problem is to understand the exact role of the membrane and 5-brane in M-theory. It is also important to understand the nature of the fundamental degrees of freedom.

In chapter three we discussed bound states and how supergravity solutions corresponding to bound states can be derived using certain T-duality transformations. In particular, we derived M-theory 'T-duality' rules, which we then used in order to obtain a solutions generating method that can be used to deform M5-branes solutions (i.e., to obtain supergravity solutions corresponding to M5-M2 and M5-M2-M2-MW bound states). Moreover, we found indications that M-theory 'T-duality' has aspects that are very different compared to string theory T-duality. What is most striking is that under string theory T-duality transformations the string is always transformed into a string, while under M-theory 'T-duality' transformations (for M-theory compactified on a three torus) an M2-brane is transformed into an M5-brane and vice versa. Hopefully further study will lead to a microscopic derivation of the M-theory 'T-duality' rules (see chapter three) and a better understanding of the role of the M2-brane and M5-brane in M-theory. The results in chapter three and [80, 81, 83] indicate that both the M2-brane and M5-brane are important when discussing M-theory 'T-duality'.

In chapter four we showed how string theory with non zero background NS-NS two form turned on, in certain limits leads to non-gravitational theories with noncommutativity. In particular, taking the appropriate decoupling limit for a Dp-brane with non zero electric NS-NS two form turned on leads to noncommutative open string theory (NCOS). Here we found that we obtain an open string theory, because a field theory can not have space-time noncommutativity. Instead one has to keep all the massive open string states in order to have a unitary theory. Hence, a field theory can only have space-space or light-like noncommutativity.

Perhaps the most interesting theory discussed in chapter four is OM-theory, a six-dimensional theory that contains light open membranes and lives in some kind of non-trivial geometry. Hopefully, an improved understanding of OM-theory will lead to new information regarding the role of the membrane in M-theory. There is so far no satisfactory microscopic formulation of OM-theory. However, in Paper V (see also [108, 110]) we made some progress by deriving the open membrane metric and generalized noncommutativity parameter (theta parameter). These open membrane data are expected to be an important part in a microscopic formulation of OM-theory. This is motivated from the fact that these objects are the generalizations of the open string metric and noncommutativity parameter, which are derived from computing the two-point function for open strings ending on a Dp-brane. Furthermore, the theta parameter indicates that OM-theory lives in a geometry with some kind of generalized noncommutativity structure. It has been speculated that the geometry should include some kind of noncommutative loop-space structure and/or non-associativity, see [101, 113, 125] and Paper V. One approach to understanding this kind of deformed geometry could be to investigate an M5-brane with a non zero light-like three form turned on in the decoupling limit, since this leads to a (2,0) field theory with light-like generalized noncommutativity (see Paper VI). Furthermore, since this theory does not contain fully fluctuating membranes, as OM-theory does, it might be easier to investigate than OM-theory.

In chapter four we also discussed deformation independence of open string data. In Paper V this was generalized to deformation independence of open membrane data, and used to derive the open membrane metric and theta parameter. In Paper VII it was further shown that using the solution generating technique introduced in this paper, always leads to deformation independence of open membrane data, for electric and light-like deformations. Moreover, in chapter 4.3.1 an alternative derivation of the open membrane data was given. That this way of deriving the open membrane data worked, indicates that there might be some unknown connection between M-theory 'T-duality' and open membrane data, where deformation independence may play a role. Further investigation is necessary to fully understand this problem.

String theory and M-theory have so far led to much progress in obtaining a unified description of nature. However, there is still a long way to go before string theory can give any verifiable predictions about nature. For example, a long-standing problem is to select the correct vacuum when compactifying string theory down to four dimensions. Furthermore, although string theory is understood in great detail, M-theory is not. A breakthrough in the understanding of M-theory, especially concerning the role of the membrane and 5-brane, would be a very important next step in the search for a unified theory.

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