Approaches to New Gauge Theories

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APPROACHES TO NEW GAUGE THEORIES

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Abstract

All successful theories describing the fundamental particles and their interactions are gauge theories. In this thesis we make use of two quite different approaches to new gauge theories. In the first approach we construct relativistic particle models from representations of the Poincaré group. Specifically we consider Wigner's infinite spin particle representation and show that it gives rise to particle models described by simple and reparametrization invariant higher order Lagrangians. Possible external interactions are analyzed and a covariant quantization is made using a Gupta-Bleuler method.

In the second approach we consider the Batalin-Vilkovisky (BV) formalism from a reversed point of view, where the BV formalism is used as a framework for generating a class of consistent gauge field theories, rather than as a quantization procedure. Generated theories are obtained by means of a superfield algorithm. An analysis of four and six dimensional theories indicates that many master actions are (anti)canonically equivalent to much simpler master actions. It is shown how topological gauge field theories naturally fit into the framework set up by this superfield algorithm. A generalization of the algorithm is thereafter developed which allows for the construction of higher order gauge field theories.

Keywords: Gauge theory, constraints, superfield formulation, BV quantization, topological field theories.

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[2]This work was performed at the Department of Theoretical Physics, Göteborg University and Chalmers University of Technology, since 1 of April 2005 the Department of Fundamental Physics, Chalmers University of Technology.

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Thesis for the degree of Doctor of Philosophy in Physics

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This thesis consists of an introductory text and the following three appended research papers, henceforth referred to as Paper I, Paper II and Paper III:

I. Ludde Edgren, Robert Marnelius and Per Salomonson,
Infinite spin particles,
JHEP **05**(2005) 002 [hep-th/0503136].

II. Ludde Edgren and Niclas Sandström,
First order gauge field theories from a superfield formulation,
JHEP **09** (2002) 036 [hep-th/0205273].

III. Ludde Edgren and Niclas Sandström,
Superfield algorithm for higher order gauge field theories,
JHEP **01**(2004) 006 [hep-th/0306175].

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Ludde Edgren
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Contents

1

INTRODUCTION

Theoretical models of fundamental particles and their interactions are invaluable for our understanding of physical phenomena in Nature. Experimentally verified particle theories explaining more phenomena than others are vastly superior and a unification of different established theories is therefore a natural temptation for theoretical physicists.

One successful and important example of a unification was done in 1864 when J. C. Maxwell coupled electricity and magnetism in the well-known relations now carrying his name, i.e. Maxwell's equations[1]. Some years later, or more specifically one hundred years ago in 1905, A. Einstein wrote his famous article[2] and thereby founded the special theory of relativity. This theory implied a unified description of electromagnetism and mechanics. It is based on the invariance under Lorentz transformations, a symmetry property relating reference frames moving with constant velocities relative to each other and where the speed of light in vacuum is the same for all observers. The success of the special theory of relativity indicated that all future particle theories describing the laws of Nature should at least locally be invariant under the Lorentz group.

Even though it was Einstein who realized the profound meaning of the special theory of relativity also H. Poincaré should be mentioned in connection to the birth of this revolutionary theory. Poincaré also discussed possible principles of relativity[3] and showed that Maxwell's equations are invariant under the inhomogeneous Lorentz group, also named by others as the Poincaré

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[1] J. C. Maxwell, Phil. Trans. R. Soc., **155**, 459 (1864).

[2] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Ann. Phys., IV. Folge, **17**, 891 (1905).

[3] Presented at the world scientific congress of Saint Louis (Missouri), 1904.

group[4].

In 1916, Einstein probably made his greatest achievement in theoretical physics when he constructed a theory of gravity by a new remarkable geometrical view of space and time in his general theory of relativity[5]. Gravity was until then best described by Newton's theory.

The idea of particles with spin as an intrinsic property (like the particle's mass or charge) was introduced in the midst of the 1920's to explain the behavior of electrons in different experimental settings. This was a part of the development of the theory of quantum mechanics which has ever since occupied the minds of most theoretical physicists. Eventually one arrived at the Standard Model in the early seventies, which is the present relativistic quantum theory for fundamental particles describing the basic forces; electroweak and strong interactions, typically at the length scale of an atom or smaller. Einstein's theory of general relativity, on the other hand, is a classical theory for the gravitational interactions at length scales large compared to the scales of the quantum theory.

A natural goal in theoretical particle physics is of course to find a quantum theory of gravity. Still better is to find a theory unifying the Standard Model and the general theory relativity. In other words, we nourish the hope of finding a theory describing all fundamental forces, popularly called a *theory of everything*. The best candidate for such a theory is presently String/M-theory.

I would like to return to the importance of symmetry and invariance properties in fundamental particle theories since, in fact, all established theories as well as all proposed generalizations not only share the property of being relativistic but also gauge invariant. A gauge theory is a theory that has the peculiar feature to be formulated in terms of both unphysical and physical degrees of freedom. For a theory to be considered sensible it should be independent of these unphysical degrees of freedom and this is exactly what the gauge invariance takes care of. The gauge invariance means simply that the theory is invariant under gauge transformations. However, these symmetries are also responsible for the form of the different interactions. For instance, in the Standard Model the gauge group generating the electroweak and strong interactions is the semi-simple Lie group $U(1) \times SU(2) \times SU(3)$. The Standard Model is built from a class of gauge theories called Yang-Mills theories.

The Standard Model is a quantum theory describing interactions of elementary particles with spin $0, \frac{1}{2}, 1$, where the half-odd integer spin particles describe fermions (matter) and the even integer spins, bosons (force carriers and the spin zero Higgs particle). In a quantum theory of general relativity

[4]H. Poincaré, *Sur la dynamique de l'électron* Rendiconti del Circolo Matematico di Palermo, **21**, 129, (1906).

[5]A. Einstein, *Die Grundlage der allgemeinen Relativitätstheorie*, Ann. Phys. **49**, 769 (1916). All Einstein's Ann. Phys. papers can be found in Ann. Phys. **14**, Suppl. (2005).

there should exist a spin-2 particle, called the graviton, responsible for the gravitational interactions. Actually, in String/M-theory, even higher spins enter in a natural way. Even though there are no experimental evidence that fundamental particles with higher spins exist we cannot completely rule them out.

This Ph.D. thesis describes two quite different approaches to find new gauge theories. Both of them accounts for certain aspects of the construction of quantum gauge field theories. Roughly, a quantum theory is developed by starting from a classical theory and thereafter turning it into a quantum mechanical counterpart.

In the first approach we consider simple classical particle models derived from representations of the Poincaré group. Specifically we consider a model describing particles with infinite spin and quantize this covariantly in the sense of a first quantization. By first quantization is meant that it is a quantum theory for a particle or a dynamical system of particles just like in a first quantized string theory. To appropriately describe the laws of nature we should consider quantum gauge field theories in the sense of a second quantization. In the second approach, gauge fields theories are generated from an algorithm constructed within the framework of a general quantization procedure. We are then in a way considering second quantization directly.

The classical theory should before quantization be formulated in terms of an action which is a useful mathematical object enabling a compact description of a physical system. Standard classical theories allow for an action formalism where an action describing particles is written as

$$S = \int dt L(x, \dot{x}),$$

and where t is a time coordinate, $\dot{x}$ the time derivative of the length coordinate x and $L(x, \dot{x})$ is called the Lagrangian. For example, a classical particle model in Newtonian mechanics may be written as a Lagrangian of the form $L(x) = \frac{1}{2}m\dot{x}^2 - V(x)$, where $\frac{1}{2}m\dot{x}^2$ is the kinetic energy and $V(x)$ the potential energy related to a force as $F = -\frac{\partial V(x)}{\partial x}$. Thus, the Lagrangian consists of kinetic and potential energy terms of the considered particle. Requiring the action S to be stationary under small variations of the coordinates gives rise to the equations of motion (called the Euler-Lagrange equations) which in this case is Newton's equation in classical mechanics, $F = m\ddot{x}$. In this sense, the dynamics of particles can be derived from an action principle.

The implementation of the special theory of relativity into the action formalism is roughly done by replacing the coordinates $x^i = (x, y, z)$ (with $i = 1, 2, 3$) above by the spacetime variables $x^\mu = (ct, x, y, z)$, where c is the speed of light in vacuum and $\mu = 0, 1, 2, 3$. The Lorentz indices μ, ν etc. are raised and lowered by a constant Minkowski spacetime metric $\eta^{\mu\nu}$, and space and time is in a sense treated on equal footing. The metric is e.g. used to depict the Lorentz invariant interval ds^2 between two infinitesimal coordinate displacements dx^μ with $ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$ if $\eta^{\mu\nu}$ is chosen to be spacelike, i.e. with diagonal elements $(-1, +1, +1, +1)$.

In our first approach to new gauge theories in this thesis we consider a simple and elementary construction for relativistic and covariant particle models. This is done by starting from the irreducible representations of the Poincaré group. The Poincaré group consists of Lorentz transformations and translations in spacetime and is therefore an underlying symmetry of all theories of fundamental particles. A classification of these representations in terms of free spinning relativistic particles was first done in 1939 by E. Wigner [1]. In this seminal paper Wigner shows that there are four different types of irreducible representations that all are characterized by the particle's mass and spin. However, there is particularly one called the infinite spin particle representation (also named the continuous spin representation) that presently do not seem to be realized in Nature. Wigner showed how this representation contains particles with spins ranging from $-\infty$ to $+\infty$ and it was regarded as unphysical. On the other hand, it has not been sufficiently analyzed, least of all covariantly, and since higher spins have a natural role to play in String/M-theory the infinite spin particles merit further studies. In the first part of this thesis and in Paper I we consider relativistic particle models and specifically the infinite spin particles. It is shown how this basic model can be written in terms of a simple higher order Lagrangian. We discuss possible interactions e.g. with a gravitational background and make a covariant first-quantization using a Gupta-Bleuler procedure.

To describe the fundamental aspects of Nature appropriately we ought to consider field theories as well. Fields have physical reality in the sense that they carry energy and fill the space, like fluids. In contrast to particles, a field has infinitely many degrees of freedom. A particle is described by spacetime coordinates x^μ representing its position, while a field is a function of spacetime $\phi(x^\mu)$ $(\mu = 0, 1, 2, 3)$. Before we proceed to a quantum description of gauge field theories, let us emphasize an important difference between classical and quantum field theories. In a classical theory, fields and particles are considered as completely different objects. A field can interact with a particle or with other fields. For example, a charged particle is also responsible for its own radiation field. Whereas a classical theory distinguishes between fields and particles, a quantum field theory does not. It treats *only* quantum fields. Particles are seen as being created and annihilated by the fields alone.

For field theories the expression for the action above is given by an action functional

$$S = \int d^4x \mathcal{L}(\phi^i, \partial_\mu \phi^i),$$

where $\mathcal{L}(\phi^i, \partial_\mu \phi^i)$ is a Lagrange density and the measure d^4x is over spacetime, coordinatized by x^μ. $\phi^i(x^\mu)$ denotes a collective set of fields labeled by the spacetime coordinates x^μ and indices $i = 1, 2, \ldots, N$.

A central object in the path integral formulation of a quantum field theory is expressed as

$$\mathcal{I} = \int \mathcal{D}\phi e^{\frac{i}{\hbar}S(\phi)}.$$

This is a functional integral over the exponential of the action S, integrated over *all* possible paths for which $\mathcal{D}\phi$ is the measure. Hence, the theory is still described by an action S. A correlation function (amplitude of a function $\mathcal{O}$) can be found by considering a weighted integral $<\mathcal{O}> \sim \int \mathcal{D}\phi \, \mathcal{O} \, e^{\frac{i}{\hbar}S(\phi)}$.

Calculating the path integral above may create obstacles. The measure $\mathcal{D}\phi$ is not always well defined (not even from a physicist's point of view), which in turn gives rise to infinities. For instance, a gauge theory described by S has unphysical degrees of freedom and an integration over equivalent field configurations makes the path integral diverge. This is characteristic for all gauge theories. To render a well-defined path integral describing a finite and unitary theory one has to introduce gauge fixing terms and ghost fields. A unitary theory is necessary because a non-unitary theory would spoil the probability interpretation since it allows physical states with negative probability to propagate. The introduction of ghost fields and gauge fixing terms was first done by Faddeev and Popov for Yang-Mills theories [2]. The ghost fields were here introduced as some unphysical fields used to lift parts of the measure into the Lagrangian density. It was not until later when more complicated theories were considered, that one discovered that also the self-interaction of ghost fields could be necessary to account for in order to obtain a unitary theory.

A general formalism that takes these aspects into account and puts the fields and ghost fields on an equal basis is the BRST formalism [3–5]. It generalizes the Faddeev-Popov results of the Yang-Mills theories to a framework applicable to general gauge theories. The formalism is named after Bechi-Rouet-Stora-Tyutin [6, 7] who first treated Yang-Mills theories in this way. In the BRST formalism, the local gauge symmetry in gauge theories is replaced by a global fermionic BRST symmetry. The BRST invariance, which the BRST framework originate from, helps us distinguish between physical and unphysical observables in a precise and covariant way.

The BRST formalism may be formulated in a phase space with a Hamiltonian or in a configuration space with a Lagrangian. In the second part of this thesis we construct a class of gauge field theories by considering the Lagrangian version. This formalism is better known as the Batalin-Vilkovisky (BV) formalism[6] [8–11]. Even though the Faddeev-Popov method is very powerful,

[6]In the literature the BV formalism is also known as the field-antifield method.

it may not be applied on theories with more complicated algebras. This is for example the case for open algebras in supergravity theories [12]. One method developed to cope with these difficulties is the Batalin-Vilkovisky formalism. Even though the Hamiltonian version is equally applicable it has the drawback of not being covariantly formulated for field theories. However, the BV formalism is a covariant quantization procedure in the configuration space, naturally incorporating the BRST symmetry.

The main difference of the BV formalism compared to its Hamiltonian analogue is the introduction of so called **antifields**. One antifield is introduced to every field, ghost field, ghost for ghost etc, i.e. a doubling of the number of fields. The cornerstones of the BV formalism is a **master action**, formulated in terms of fields and antifields, and a **master equation** which carry the information of the gauge structure. In the final expression the antifields are given in terms of only fields through a gauge fixing procedure. Hence, the antifields may be seen as merely a mathematical construction. The BV formalism was e.g. used in the development of the bosonic open string field theory in [13, 14].

In the second approach to new gauge theories in this thesis we specifically consider a class of theories for which a **superfield formulation** of the BV method is possible. The superfield formulation is convenient since instead of writing down expressions for all fields, ghosts, ghost for ghosts etc, we consider all these fields as components of *one* superfield. To be able to do this we introduce fermionic coordinates and let the theory live on a supermanifold. This supermanifold is $2n$-dimensional, with n fermionic and n bosonic coordinates. A superfield can be described as an expansion in terms of the fermionic coordinates such that the fields, ghost fields and ghost for ghosts etc are components of this superfield. In that way one can reduce the field theory living in $2n$ dimensions to the original n-dimensional theory.

In [15, 16] Batalin and Marnelius introduced a **superfield algorithm** which naïvely can be seen as a machine for the construction of a class of consistent first order gauge field theories, once the basic field setup has been chosen. This is a reversed viewpoint, since usually the BV formalism is used as a quantization procedure, rather than as a method for generating theories.

Two of the appended papers, **Paper II** and **Paper III**, deals with superfield algorithms. In **Paper II** we use the superfield algorithm to investigate four and six dimensional theories. By using (anti)canonical transformations we are able to solve the master equation for general interacting six dimensional theories. In many cases these general six dimensional theories are canonically equivalent to much simpler ones. A puzzling feature of the superfield algorithm is that it only seems to generate topological gauge field theories (except in the one dimensional case, where all theories can be generated [16]). In **Paper III** a generalization of the superfield algorithm is proposed which allows for a treatment of higher order gauge field theories. This is achieved by introducing non-dynamical multiplier fields. By means of the generalized version developed in Paper III it is possible to generate higher order Chern-Simons theories. This is exemplified by constructing a five dimensional Chern-Simons theory.

1.1 Outline of the thesis

This thesis is naturally divided into two parts with two different approaches to new gauge theories. The first part includes ***chapter 2*** to ***chapter 4*** and is connected to Paper I and the second part includes ***chapter 5*** to ***chapter 7*** and is related to Paper II and Paper III.

In the first part we consider classical gauge theories of relativistic particles. From conditions on the physical subspace of the irreducible representations of the Poincaré group we show how to construct relativistic gauge theories. Especially, we construct a simple particle model constructed from the infinite spin particle representation. In the second part of the thesis we generate a class of consistent gauge field theories by means of the superfield algorithm. To be more precise, we find new gauge field theories from BV quantization.

From this introduction we continue in ***chapter 2*** by considering the gauge structure of particle models by the use of both the Lagrangian and the Hamiltonian formalism. The classification of the irreducible representations of the Poincaré group is thereafter described in ***chapter 3***. We also show how the Poincaré invariants yield a reparametrization invariant Hamiltonian and a corresponding Lagrangian. Two simple and well-known examples are considered, namely the massless relativistic particle and the relativistic spin-$\frac{1}{2}$ particle, with emphasis on the gauge structure discussed in the preceding chapter. ***Chapter 4*** is devoted to the infinite spin particle representation of the Poincaré group. It is shown how to construct a relativistic particle model for this representation in terms of a simple reparametrization invariant higher order Lagrangian. We describe a superversion including particles with half integer spins and perform a covariant quantization of these different infinite spin particle models using a Gupta-Bleuler procedure. We also discuss possible interactions with external fields.

In ***chapter 5*** we consider some aspects of the Batalin-Vilkovisky formalism with emphasis on the master equation. We will see how the BV formalism incorporates the BRST symmetry and how the quantum version of the master equation differs from the classical counterpart. The BV formalism is illustrated by constructing master actions for the massless, relativistic particle, the relativistic spin-$\frac{1}{2}$ particle and the infinite spin particle. The superfield algorithm and its generalized version is introduced in ***chapter 6***, it is shown how they are constructed and how one may find the original theories by a set of reduction rules. It is also shown how to derive the gauge transformations of the original theory. In ***chapter 7*** we discuss a class of theories generated by using the methods introduced in chapter 6, with focus on topological gauge field theories. This is an important class of interacting theories which naturally fit into the framework set up by the superfield algorithm. We also show how one can formulate topological Yang-Mills theories and higher dimensional Chern-Simons theories by means of the formalism discussed in chapter 5 and chapter 6.

2

PROPERTIES OF CLASSICAL GAUGE THEORIES

Gauge symmetries are fundamental properties that partly determine the form of the action in quantum field theories for particle physics. A particle theory which has inherent gauge symmetries is called a gauge theory and these theories all share the feature that not all degrees of freedom are physical ones. Examples of gauge theories are the Standard Model and the general theory of relativity where interactions and symmetries have a profound connection. That symmetries and conserved quantities are related was shown a long time ago by E. Noether [17]. To have a sensible theory in particle physics the properties of gauge theories should be treated properly within the action formalism, whether it is considered in a Lagrangian or a Hamiltonian framework. Actually, not all theories allow for an action formalism but most established theories do and especially the ones considered in this thesis.

The gauge transformations, corresponding to the gauge symmetries, are non-trivial transformations that do not change the physical states. This in turn implies that the classical solutions to the equations of motion are not all independent and the theory has some unphysical degrees of freedom. Specifically, in particle theories this means that we cannot uniquely describe the accelerations in terms of the velocities and coordinates and as a result there is an ambiguity in the classical solutions. At some point this ambiguity has to be taken care of, otherwise it gives rise to severe problems in the quantization procedure, like infinities in the path integral formalism. As was discussed in the introduction, different techniques have been developed to avoid these problems among which the Batalin-Vilkovisky formalism is the most efficient and general method used for gauge field theories.

Quantum gauge field theories are constructed starting from a classical counterpart. It is therefore important to get familiar with the general structure of classical gauge theories before we proceed to the quantum theory. In this chapter we describe the Lagrangian and Hamiltonian formalism for particle models. The Hamiltonian formalism in phase space is very efficient when considering particle theories which we will do in the next two chapters, regarding the irreducible representations of the Poincaré group and especially the infinite spin particles. In a later chapter we describe the Batalin-Vilkovisky formalism which is a powerful quantization method, especially for field theories, formulated in the Lagrangian framework

Below, we discuss how gauge transformations are naturally incorporated into the Lagrangian and Hamiltonian action formalism and how these transformations arise from the properties of the classical equations of motion. Following Noether we discuss the relation between symmetries and conserved quantities in both of these frameworks. This treatment of gauge theories is made for systems with finite degrees of freedom, the generalization to field theories (infinite degrees of freedom) is straightforward.

To get familiar with conventions and notations used throughout this thesis the following sections are quite elaborate on particularly some important aspects of the basic structure of gauge theories. Notice that, unless stated otherwise, we let $\hbar = c = 1$ and use Einstein's summation convention, i.e. a summation over repeated indices.

There are of course several good references on this topic, the ones relevant to this chapter and related to the subsequent chapters are e.g. [18–25].

2.1 Lagrangian formalism

Let us start with a theory given in a configuration space within the Lagrangian formalism. More presicely, consider a theory described by an action with a first order Lagrangian $L(x(t), \dot{x}(t))$ integrated over time t with generalized coordinates $x^i(t)$ (where $i = 1, \ldots, n$) describing the world-line trajectory,

$$S = \int_{t_1}^{t_2} dt L(x(t), \dot{x}(t)) \tag{2.1}$$

and where $\dot{x} = \frac{d}{dt}x$. The local variation $\delta x^i(t)$ is defined as an infinitesimal change in the trajectory from $x^i(t)$ to $\tilde{x}^i(t)$ such that

$$\tilde{x}^i(t) = x^i(t) + \delta x^i(t), \qquad \frac{d}{dt}\delta x^i = \delta \dot{x}^i. \tag{2.2}$$

By requiring the action S above to be stationary $\delta S = 0$ under a local variation δx we find the familiar equations of motion (i.e. the Euler-Lagrange equations) defined by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0. \tag{2.3}$$

We may write these equations of motion in the more explicit form

$$\ddot{x}^j W_{ij} - K_i = 0, \tag{2.4}$$

where $W_{ij}(x, \dot{x})$ and $K_i(x, \dot{x})$ are defined by

$$W_{ij}(x, \dot{x}) := \left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\right), \qquad K_i := \frac{\partial L}{\partial x^i} - \dot{x}^j \frac{\partial^2 L}{\partial x^j \partial \dot{x}^i}. \tag{2.5}$$

$W_{ij}(x, \dot{x})$ is called the Hessian matrix. By inspection of the equations of motion (2.3) the properties of the matrix W_{ij} now yield two different situations. Either this Hessian has zero modes or it does not and we call the Lagrangian representing these theories either singular or regular. If W_{ij} does have zero modes, this implies that the matrix is non-invertible and the accelerations $\ddot{x}^i$ can not be uniquely determined by (2.4). Hence, not all degrees of freedom enter in the dynamics. Let us study these two separate cases in more detail.

2.1.1 Regular Lagrangian

When the determinant of the Hessian is non-zero, $\det W_{ij} \neq 0$, the dynamics described by the equations of motion (2.3) is uniquely determined and the corresponding Lagrangian is of the regular type. Theories described by such Lagrangians do not have any superfluous degrees of freedom, i.e. no unphysical degrees of freedom exist. When quantizing a theory in a Lagrangian formalism, a regular form of the Lagrangian is necessary. The invertibility of the Hessian will then normally not produce any infinities in the path-integral measure and therefore a regular type of Lagrangian is what we eventually are looking for to quantize.

2.1.2 Singular Lagrangian

From the equations of motion (2.3) we see that when the Hessian matrix is non-invertible, the accelerations can not be uniquely determined. We then have

$$\det W_{ij} = 0 \tag{2.6}$$

and Lagrangians with this property are called singular. There exist $k = (n - m)$ independent null eigenvectors $R^i_\alpha(x, \dot{x})$ ($\alpha = 1, \ldots, k$) if the rank of the Hessian W_{ij} is m, such that

$$W_{ij} R^i_\alpha = 0. \tag{2.7}$$

This implies that $y_i R^i_\alpha = 0$, where $y_i(x, \dot{x}) := \ddot{x}^j W_{ij} - K_i$ which in turn gives us a number of constraints $\Upsilon_\alpha = K_i R^i_\alpha = 0$. The singularity of the Lagrangian and the corresponding constraints are characteristic properties of all gauge theories. As mentioned, quantization requires a regular Lagrangian and therefore there are techniques developed to transform the original singular Lagrangian to a corresponding regular one. Since all modern particle theories and their generalizations are gauge theories, the knowledge of the fundamental properties of classical gauge theories can hardly be overestimated. An analysis of the constraints in the Lagrangian framework has been made in e.g. [26].

Let us study the non-invertibility of the Hessian by defining a conjugate momentum to x^i by

$$p_i := \frac{\partial L}{\partial \dot{x}^i}. \tag{2.8}$$

Using this definition, the singular property (2.6) implies the non-invertibility of the velocities as functions of the momentum and coordinates. There will now be a number of relations $\chi_i(p, x) = 0$ (without the use of the equations of motion) where the momenta p_i are independent of the generalized velocities $\dot{x}^i$. These relations are called primary constraints.

The transformation between the phase space (x^i, p_i) and configuration space $(x^i, \dot{x}^i)$ is therefore not unique until one introduce these constraints χ_i via some associated variables called Lagrange multipliers λ_i. These multipliers also have conjugate momenta, but since they are not dynamical their momenta are zero, $p_{\lambda_i} = \frac{\partial L}{\partial \lambda_i} = 0$. Consistency of a gauge theory requires the constraints to be constants of motion. Hence, by considering the time evolution of a primary constraint we may find some new constraints, called secondary constraints and so forth. We will shortly come back to this when discussing the Hamiltonian formalism where a constraint analysis is most efficiently done. But let us first describe the possible symmetry and invariance properties of the action (2.1) in more detail.

2.1.3 Noether's theorem

Consider a Lagrangian $L(x, \dot{x})$ (which we for simplicity choose not to depend on time explicitly) in the action (2.1) we assume that it is invariant under a local symmetry variation $\delta_\epsilon L$ up to a total derivative

$$\delta_\epsilon L = \delta_\epsilon x^i \frac{\partial L}{\partial x^i} + \delta_\epsilon \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} = \frac{d}{dt} Z_\epsilon, \tag{2.9}$$

where Z_ϵ is dependent on the type of symmetry under consideration. In particular, a general local transformation is given by

$$\delta_\epsilon x^i = R^i[\epsilon] = R^i_{0\alpha}\epsilon^\alpha + R^i_{1\alpha}\dot{\epsilon}^\alpha + \cdots + R^i_{r\alpha}\overset{(r)}{\epsilon}{}^\alpha \tag{2.10}$$

where the path is parametrized by the infinitesimal parameters $\epsilon^\alpha(t)$ ($\alpha = 1,\ldots,k$). The parameters are chosen to depend on time up to some finite time-derivative r. An explicit calculation of this particular variation of the action (2.1) gives us the relation

$$y_i \delta_\epsilon x^i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\delta_\epsilon x^i - Z_\epsilon\right), \tag{2.11}$$

where again $y_i(x,\dot{x}) := \ddot{x}^j W_{ij} - K_i$. Hence, we can define a conserved quantity Q_ϵ, using (2.8), by

$$Q_\epsilon := p_i \delta_\epsilon x^i - Z_\epsilon. \tag{2.12}$$

When L obeys the equations of motion (2.3) it follows from (2.11) that Q_ϵ is a constant of motion, i.e. $\dot{Q}_\epsilon = 0$. Depending of what type of symmetries the action is invariant under, there are corresponding conserved quantities in the theory. This is Noether's theorem which will be described further below.

Let us parametrize the introduced quantities Z_ϵ and Q_ϵ in the same way as for the variation $\delta_\epsilon x^i$ above

$$\begin{aligned} Z_\epsilon &= Z_\epsilon[\epsilon] = Z_{0\alpha}\epsilon^\alpha + Z^i_{1\alpha}\dot{\epsilon}^\alpha + \cdots + Z^i_{r\alpha}\overset{(r)}{\epsilon}{}^\alpha, \\ Q_\epsilon &= Q_\epsilon[\epsilon] = Q_{0\alpha}\epsilon^\alpha + Q_{1\alpha}\dot{\epsilon}^\alpha + \cdots + Q_{r\alpha}\overset{(r)}{\epsilon}{}^\alpha, \end{aligned} \tag{2.13}$$

where $\epsilon^\alpha(t)$ and its time derivatives are independent. As stated before, Q_ϵ is a conserved quantity which implies that we have the following restrictions on the components

$$\begin{aligned} \dot{Q}_{0\alpha} &= 0, \\ \dot{Q}_{1\alpha} &= -Q_{0\alpha}, \\ \dot{Q}_{2\alpha} &= -Q_{1\alpha}, \quad \text{etc.} \end{aligned} \tag{2.14}$$

For a rigid (global) symmetry $\epsilon = const.$ it follows that Q_ϵ is a constant. If the infinitesimal parameters ϵ^α have independent non-zero finite order of derivatives up to r we see from (2.14) that $Q_\epsilon = 0$. Hence, this constant of motion originates from a gauge (local) symmetry with parameter ϵ, where the relations $Q_\epsilon = 0$ represent the constraints of the theory. Actually, since there are k numbers of Q_ϵ's, there are k independent primary constraints in the theory relating the coordinates and canonical momenta. It can also be shown that there are k secondary, k tertiary etc. constraints such that the remaining degrees of freedom are $n - kr$ [27].

The local invariance (2.10) can be shown to possess an infinitesimal group structure, i.e. a closed commutator algebra

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]x^i = \delta_{\epsilon_{12}}x^i, \tag{2.15}$$

where ϵ_{12} is an expression in ϵ_1 and ϵ_2.

More generally we may write the variation of an action depending on any higher order derivative $L(x^i, \dot{x}^i, \ddot{x}^i, \ldots, \overset{(n)}{x}{}^i)$, in terms of functional derivatives of the form

$$\delta_\epsilon S = \frac{\delta S}{\delta x^i} R^i, \tag{2.16}$$

where R^i denotes the transformation $\delta_\epsilon x^i$ as in (2.10). The equations of motion for such a higher order theory are given by

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^i} + \cdots + (-1)^n \frac{d^n}{dt^n}\frac{\partial L}{\partial \overset{(n)}{x}{}^i} = 0. \tag{2.17}$$

We will return to these kind of higher order theories ($n \geq 2$) when discussing the infinite spin particle in a later chapter.

So far we have found the basic properties that follows if the action is invariant under gauge transformations which are local symmetries parametrized by some gauge parameters $\epsilon^\alpha(t)$. These symmetries are of the nature that they do not change the physical states, i.e. they do not change the classical solutions to the equations of motion. A gauge invariance implies that there are too many degrees of freedom since it generates constraints. These constraints are most efficiently analyzed in a Hamiltonian framework which we now turn to.

2.2 Hamiltonian formalism

In the Hamiltonian formalism we write the action (2.1) in a phase space with coordinates p_i and x^i as a Legendre transformation

$$S = \int_{t_1}^{t_2} dt (p_i \dot{x}^i - H(p_i, x^i)), \tag{2.18}$$

where $H(p, x)$ is the Hamiltonian which we here assume not to have explicit time-dependence. The variation of this action yields Hamilton's equations of motion

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \tag{2.19}$$

The Poisson bracket for two phase space functions $A(p_i, x^i), B(p_i, x^i)$ is

$$\{A, B\} := \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x^i}, \tag{2.20}$$

such that $\{x^i, p_i\} = \delta^i_j$. In particular $\dot{F} = \{F, H\}$ for any function $F(p_i, q^i)$ (c.f. (2.19)). This implies that a phase space variable $Q_\epsilon(p_i, x^i)$ (not identical to the Q_ϵ in the previous section) is a constant of motion if $\dot{Q}_\epsilon = \{Q_\epsilon, H\} = 0$. From the definition of the Poisson bracket (2.20) we also define symmetry transformations by

$$\delta_\epsilon x^i := \{x^i, Q_\epsilon\}, \qquad \delta_\epsilon p^i := \{p_i, Q_\epsilon\}. \tag{2.21}$$

Notice that a symmetry transformation of the Hamiltonian yields $\delta_\epsilon H = \{H, Q_\epsilon\} = 0$. If we now calculate the variation $\delta_\epsilon S$ of the action (2.18), using the definitions in (2.21) and $\delta_\epsilon H = 0$, we see that

$$\delta_\epsilon S = \int_{t_1}^{t_2} dt \frac{d}{dt}\left[p_i \delta_\epsilon x^i - Q_\epsilon\right], \tag{2.22}$$

without the implication of the equations of motions (2.19). This means then that the symmetry transformations in (2.21) implies constants of motion. Since the Poisson bracket satisfy the Jacobi identity

$$\{\{A, B\}, C\} + \{\{C, A\}, B\} + \{\{B, C\}, A\} = 0 \tag{2.23}$$

we have for the symmetry generators Q_α, Q_β

$$\{\{Q_\alpha, Q_\beta\}, H\} = 0. \tag{2.24}$$

For the Q's to define a complete set of generators, the Poisson bracket with themselves should consist of polynomials in the generators such that

$$\{Q_\alpha, Q_\beta\} = c_{\alpha\beta}(p_i, x^i) + f_{\alpha\beta}{}^{\gamma}(p_i, x^i) Q_\gamma + \mathcal{O}(Q^2), \tag{2.25}$$

where $c_{\alpha\beta}(p_i, x^i)$ are called central charges. The Q's are the constraints of the theory when they represent local symmetries. When the central charge is zero these are classified by Dirac as first-class constraints [18–20], where $f_{\alpha\beta}{}^{\gamma}$ are first-class structure functions. Gauge theories consist of only first-class constraints. When $c_{\alpha\beta} \neq 0$ there are constraints of second-class and when the only non-vanishing coefficient is $f_{\alpha\beta}{}^{\gamma}$ and this is a constant, the gauge algebra is a Lie algebra.

There can also be situations where one has a mixture of both first- and second-class constraints. The second-class constraints may be eliminated by replacing the Poisson bracket by a Dirac bracket. When we for example have only second-class constraints the Dirac bracket is defined by

$$\{Q_\alpha, Q_\beta\}_{DB} := \{Q_\alpha, Q_\beta\} - \{Q_\alpha, Q_\gamma\} c^{\gamma\rho} \{Q_\rho, Q_\beta\} \approx 0. \tag{2.26}$$

This expression is weakly zero, i.e. zero on the constraint surface and here the central charge has an inverse, $c_{\rho\beta} c^{\gamma\rho} = \delta^\gamma_\beta$.

Given a gauge theory, the first-class constraints $\chi_\alpha(p_i, x^i)$ generate the gauge transformation, with $Q_\epsilon = \epsilon^\alpha \chi_\alpha$,

$$\delta_\epsilon F = \{F, \epsilon^\alpha \chi_\alpha\}, \tag{2.27}$$

where $F(p_i, x^i)$ is an arbitrary phase space function and $\epsilon^\alpha(t)$ the gauge parameters. In order for the action principle to generate the constraints $\chi_\alpha = 0$ we must introduce Lagrange multipliers λ^α. The action is the written as

$$S = \int_{t_1}^{t_2} dt(p_i \dot{x}^i - H(p_i, x^i) - \lambda^\alpha \chi_\alpha). \tag{2.28}$$

We have postulated that the constraints are of first-class which implies that we are dealing with a gauge theory with time-independent constraints that form a Lie algebra

$$\{H, \chi_\alpha\} = d_\alpha{}^\beta \chi_\beta, \qquad \{\chi_\alpha, \chi_\beta\} = f_{\alpha\beta}{}^\gamma \chi_\gamma, \tag{2.29}$$

where $d_\alpha{}^\beta$ and $f_{\alpha\beta}{}^\gamma$ are constants. It follows that the latter one is antisymmetric in the lower indices. Now choose the gauge transformation of the Lagrange multiplier to be

$$\delta_\epsilon \lambda^\alpha = \dot{\epsilon}^\alpha + \epsilon^\beta \lambda^\gamma f_{\beta\gamma}{}^\alpha - \epsilon^\beta d_\beta{}^\alpha \tag{2.30}$$

such that

$$\delta_\epsilon S = \int_{t_1}^{t_2} dt \frac{d}{dt}\left(p_i \delta_\epsilon x^i - \epsilon^\alpha \chi_\alpha\right) = \int_{t_1}^{t_2} dt \frac{d}{dt}\left(\epsilon^\alpha (p_i \frac{\partial \chi_\alpha}{\partial p^i} - \chi_\alpha)\right), \tag{2.31}$$

i.e. the variation is zero if we let the gauge parameters fall off appropriately at the boundaries, $\epsilon(t_1) = \epsilon(t_2) = 0$. Comparing this with (2.22) we find that the conserved quantity $Q_\epsilon = \epsilon^\alpha \chi_\alpha$ where ϵ^α is the infinitesimal gauge parameter associated with the first class constraint χ_α, in agreement with (2.27).

We may also include the transformation properties of the Lagrange multipliers λ^α in the expression (2.27) by choosing the gauge generator as [22]

$$G = \epsilon^\alpha \chi_\alpha + (\dot{\epsilon}^\alpha + \epsilon^\beta \lambda^\gamma f_{\beta\gamma}{}^\alpha - \epsilon^\beta d_\beta{}^\alpha) p_\alpha^{(\lambda)}, \tag{2.32}$$

where $p_\alpha^{(\lambda)}$ is the conjugate momenta to the Lagrange multipliers. However, this expression is just the general solution to the equation

$$\dot{G}|_{\chi_{\text{primary}}=0} = 0, \tag{2.33}$$

where the gauge generator G is defined as

$$G = \epsilon^\alpha \chi_\alpha + \gamma^\alpha p_\alpha^{(\lambda)}, \tag{2.34}$$

with gauge parameters $\epsilon^\alpha(t), \gamma^\alpha(t)$. This generator implies that a general gauge transformation is given by

$$\delta_\epsilon F = \{F, G\} \tag{2.35}$$

and where the relations between the gauge parameters are found by solving the equation in (2.33).

2.2.1 Constraint analysis

Let us now have a closer look on how to perform a constraint analysis for a gauge theory described by an action. The established way to do this is to follow the procedure developed by Dirac in the Hamiltonian formalism [18–20]. This is a constraint analysis that works for all gauge theories. Starting from a singular Lagrangian, as in (2.1), there are gauge invariances in the theory which means that primary constraints χ_i exist. As discussed briefly before, the definition of the canonical momenta $p_i := \frac{\partial L}{\partial \dot{x}^i}$ yields $k = (n - m)$ primary constraints corresponding to the independent nullvectors

$$\chi_\alpha(p_i, x^i) = 0, \qquad \alpha = 1, \ldots, k. \tag{2.36}$$

The transformation from the Lagrangian L to the Hamiltonian H is written as $L = p_i \dot{x}^i - H_{\text{tot}}$, where H_{tot} is the most general form of the phase space Hamiltonian in terms of k primary constraints

$$H_{\text{tot}} = H + \sum_{\alpha=1}^{k} \lambda_\alpha \chi_\alpha. \tag{2.37}$$

The λ_α's are the Lagrange multipliers which yield the constraints by their equations of motion. Notice that these constraints are defined without the use of the equations of motion for the phase space coordinates p_i, x^i. The time evolution of a phase space function $A(x^i, p_i)$ is given by the Poisson bracket relation $\dot{A} = \{A, H_{\text{tot}}\}$, where the Poisson bracket is defined in (2.20). Hence, for a primary constraint to be a constant of motion we must have

$$\dot{\chi}_\alpha = \{\chi_\alpha, H_{\text{tot}}\} \approx 0, \tag{2.38}$$

which is either identically zero, an already known constraint or gives rise to a new secondary constraint. There is also a possibility that it yields an inconsistent relation, which is an indication that the theory is not appropriately defined. The secondary constraint is a consequence of the use of the equations of motion together with the primary constraints. By performing the same consistency check on the secondary constraint there might occur further constraints which we accordingly call tertiary constraints etc. When all constraints (including primary, secondary, tertiary etc) are found we need to check the Poisson bracket relations between these secondary and tertiary constraints. If all these relations close, i.e. they yield either zero or an already known constraint, this algebra is called first-class or a gauge algebra. Otherwise some of the constraints are of second-class (how to handle these with the introduction of a Dirac bracket was discussed in the previous section). Thus the non-zero bracket relations has the property

$$\{\chi_\alpha, \chi_\beta\} = f_{\alpha\beta}{}^{\gamma} \chi_\gamma. \tag{2.39}$$

These constraints exist due to the gauge invariance inherent in the theory and give rise to the gauge generator and gauge transformations defined in (2.34) and (2.35).

In the next chapter we show how particle models derived from representations of the Poincaré group may be written as reparametrization invariant theories in terms of their constraints and the associated Lagrange multipliers. In such theories we have $H_{\text{tot}} = \Sigma\lambda_\alpha\chi_\alpha$. Many important theories can be formulated in this way, such as string theory, general relativity and the relativistic particle.

3

IRREDUCIBLE REPRESENTATIONS OF THE POINCARÉ GROUP

The underlying symmetry transformations in the special theory of relativity form a group which in its extended form is called the Poincaré group, also known as the inhomogenous Lorentz group. This fundamental symmetry group consists of translations in spacetime and the Lorentz transformations. Eventually, Wigner realized how to classify the irreducible representations of spinning relativistic quantum particles in the seminal paper [1]. By irreducible representations we mean that they cannot be decomposed into further representations which in turn means that the corresponding relativistic wave equations describe fundamental point particles with spin and mass.

To explain what Wigner did in his original work we consider a relativistic particle described by the coordinates x^μ and momenta p_μ, $\mu = 0, 1, 2, 3$, with the commutation relation

$$[x^\mu, p_\nu] = i\delta^\mu_\nu. \tag{3.1}$$

The corresponding Poincaré algebra is a Lie algebra with generators p_μ and $m_{\mu\nu}$ satisfying

$$\begin{aligned}
&[p^\mu, p^\nu] = 0, \\
&[m^{\mu\nu}, p^\sigma] = i(\eta^{\mu\sigma} p^\nu - \eta^{\nu\sigma} p^\mu), \\
&[m^{\mu\nu}, m^{\rho\sigma}] = i(\eta^{\mu\rho} m^{\nu\sigma} + \eta^{\rho\nu} m^{\sigma\mu} + \eta^{\nu\sigma} m^{\mu\rho} + \eta^{\sigma\mu} m^{\rho\nu}),
\end{aligned} \tag{3.2}$$

where $\eta^{\mu\nu}$ is chosen to be the flat Minkowski spacetime metric, with only the diagonal elements non-zero and given by $(-1, +1, +1, +1)$, used to raise and lower the Lorentz indices. Furthermore,

$$m^{\mu\nu} := l^{\mu\nu} + s^{\mu\nu}, \tag{3.3}$$

where the angular momentum operator $l^{\mu\nu}$ is defined by

$$l^{\mu\nu} := x^{\mu}p^{\nu} - x^{\nu}p^{\mu}. \tag{3.4}$$

In his first paper [1], Wigner did not specify the spin operator $s^{\mu\nu}$, he only stated that it obeys the same commutation relations as the generator $m^{\mu\nu}$ and commutes with the conjugate momentum and the angular momentum operators, i.e.

$$\begin{aligned}
&[s^{\mu\nu}, s^{\rho\sigma}] = i(\eta^{\mu\rho}s^{\nu\sigma} + \eta^{\rho\nu}s^{\sigma\mu} + \eta^{\nu\sigma}s^{\mu\rho} + \eta^{\sigma\mu}s^{\rho\nu}),\\
&[s^{\mu\nu}, p^{\mu}] = 0, \qquad [s^{\mu\nu}, l^{\mu\nu}] = 0.
\end{aligned} \tag{3.5}$$

He found that the irreducible representations are classified by the particle's mass and its intrinsic spin by considering a subgroup of the Lorentz group, called the little group, which leaves one particular momentum vector invariant. He showed how all irreducible representations of the Poincaré group was given by the representations of the little group (independent on the choice of momentum vector).

In this chapter we review Wigner's classification of the irreducible representations of the Poincaré group. We also consider two simple examples of the Poincaré algebra; the relativistic point particle and the spinning relativistic particle. This is done quite extensively since the same procedure will be considered in the next chapter when discussing the infinite spin particle. The gauge structure of these simple particle models will also be used later to exemplify the Batalin-Vilkovisky formalism.

3.1 Classification of the irreducible representations

We will here see how the irreducible representations of the Poincaré group are classified by their invariants. Consider the generators of the Poincaré group $p^{\mu}, m^{\mu\nu}$ given in (3.1)-(3.5). The Poincaré invariants are the Casimir operators, i.e. the operators that commute with the Poincaré generators p_{μ} and $m^{\mu\nu}$. In a four dimensional Minkowski spacetime there exist two Casimir operators which are the momentum squared $p_{\mu}p^{\mu}$ and the square of the Pauli-Lubanski operator $w_{\mu}w^{\mu}$ defined by

$$w^{\mu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}m_{\nu\rho}p_{\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}s_{\nu\rho}p_{\sigma}. \tag{3.6}$$

The equality holds, since $\epsilon^{\mu\nu\rho\sigma}$ is a totally antisymmetric tensor with $\epsilon^{0123} = 1$. The invariance is seen by calculating the commutator between the generators and the Casimir operators. That $p_\mu p^\mu$ is invariant is easily seen by the definitions above in (3.2) and (3.3). Notice that the Pauli-Lubanski vector (3.6) is orthogonal to the momentum, i.e. $w^\mu p_\mu = 0$ and it also follows that $[w^\mu, p^\nu] = 0$ such that w^2 commutes with p^2. It is also quite easily shown that w^2 is an invariant since

$$[w^\lambda, m^{\mu\nu}] = i(\eta^{\lambda\nu} w^\mu - \eta^{\lambda\mu} w^\nu), \qquad [w^2, m^{\mu\nu}] = 0. \tag{3.7}$$

We have found two Casimir operators as squares of the momentum and the Pauli-Lubanski operator which both should have constant values. The definition in (3.6) allow us to write the square of the Pauli-Lubanski operator more explicitely as

$$w^2 = -\frac{1}{2} s_{\mu\nu} s^{\mu\nu} p_\sigma p^\sigma - s_{\mu\nu} s^{\nu\lambda} p_\lambda p^\mu. \tag{3.8}$$

In [1] Wigner found four different kinds of irreducible and unitary representations:

I. $\mathcal{P}_s$ *Massive particles with discrete spin s.*

The possible physical subspace $|\text{phys}\rangle$ of the theory for a massless particle with $m^2 > 0$ and spin s, satisfies the conditions (in a Dirac quantization)

$$p^2|\text{phys}\rangle = -m^2|\text{phys}\rangle, \qquad w^2|\text{phys}\rangle = m^2 s(s+1)|\text{phys}\rangle, \tag{3.9}$$

where the first condition corresponds to the Klein-Gordon equation. A specific representation with a spin operator with eigenvalues $s = \frac{1}{2}$ yields the Dirac equation from the second equation.

II. $\mathcal{P}'_s$ *Tachyons with discrete spin s.* The conditions are here given as above in (3.9) but with $m^2 < 0$.

III. $\mathcal{O}_s$ *Massless particles with discrete spin s.*

Massless particles are described by a helicity operator λ with eigenvalues $\pm s$, such that the conditions on the possible physical subspace $|\text{phys}\rangle$ are

$$p^2|\text{phys}\rangle = 0, \qquad (w^\mu + \lambda p^\mu)|\text{phys}\rangle = 0. \tag{3.10}$$

The corresponding equations for the lowest spins are e.g. the scalar wave equation and Maxwell's source free equations. Detailed analysis of this representation may for example be found in [28–30].

IV. $\mathcal{O}(\Xi)$ *Massless particles with continuous spin.*

The representation is described by the state conditions

$$p^2|\text{phys}\rangle = 0, \qquad w^2|\text{phys}\rangle = \Xi^2|\text{phys}\rangle, \tag{3.11}$$

where Ξ is a real constant. This representation was first called the continuous spin representation [31] and later in [32] the infinite spin particle representation, since it contains helicities from $-\infty$ to $+\infty$. These infinite spin particles will be discussed in detail in the next chapter were we for example will show how a corresponding particle model is classically described by a simple reparametrization invariant higher order Lagrangian.

3.2 The massless relativistic point particle

To find relativistic particle models from the irreducible representations of the Poincaré group we consider the characteristic properties of a Dirac quantization. Here the constraints χ_i are turned into hermitian operators $\hat{\chi}_i$ satisfying a Lie algebra $[\hat{\chi}_i, \hat{\chi}_j] = i f_{ijk}\hat{\chi}_k$ where f_{ijk} is a real constant. The physical subspace satisfies the conditions (for all i)

$$\hat{\chi}_i|\text{phys}\rangle = 0. \tag{3.12}$$

The relativistic wave equations are thereafter found by using an appropriate wave function representation for the states.

Let us now specify the spin operator and find some simple models of the irreducible representations of the Poincaré group. The massive and massless (spinless) relativistic point particles are described by well-known relativistic particle model that are contained in the irreducible representations of the Poincaré group. They can both be described by the same methods used in this section but we choose here to only consider the massless case. In this theory there are no internal variables describing the spin (i.e. $s^{\mu\nu} = 0$) and the Pauli-Lubanski vector (3.6) is identically zero. This particle model is therefore characterized by the constraint $p^2 = 0$. Hence, the states of the physical subspace are such that they obey the Dirac condition

$$p^2|\text{phys}\rangle = 0, \tag{3.13}$$

which in a wave function representation is the Klein-Gordon equation. As we later will study much more involved theories, like infinite spin particles in the next chapter, it is instructive already at this point to be familiar with the procedure used to analyze this simple relativistic model. In a later chapter we will also show how the so called master action in the Batalin-Vilkovisky formalism is constructed for the relativistic point particle based on the gauge structure developed in this section. Since this is our first example of a gauge theory we will be quite explicit in our derivation. The relativistic particle model is well-known and there are several textbooks which discuss it in more detail.

A reparametrization invariant theory describing the massless relativistic particle within the Hamiltonian framework may be defined by

$$H = \frac{1}{2}vp^2, \qquad \chi_1 = \frac{1}{2}p^2, \tag{3.14}$$

where χ_1 is a constraint and v a Lagrange multiplier (here called the einbein). A reparametrization invariant theory is characterized by the condition that the Hamiltonian is zero on the constraint surface and may be expressed in terms of the constraints. When this is the case, the time evolution of a phase space variable (generated by the Hamiltonian) is not definitely determined since it yields a gauge transformation. To each quantity we introduce a canonical momenta with the Poisson bracket relations

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{v, p_v\} = 1. \tag{3.15}$$

By construction, the einbein v imposes the constraint $\chi_1 = 0$ by its equation of motion. Hamilton's equations of motion (2.19) also yield the relations $\dot{x}^\mu = \{x^\mu, H\} = vp^\mu$ and $\dot{v} = \{p^\mu, H\} = 0$ (where the dot indicates a derivative with respect to a time variable τ) and the corresponding Lagrangian is given by the Legendre transformation $L(x, \dot{x}, p) = p_\mu \dot{x}^\mu - H$. Hence the configuration space Lagrangian $L(x, \dot{x})$ is

$$L = \frac{1}{2v}\dot{x}^2. \tag{3.16}$$

Let us now start with this Lagrangian and show how the Dirac condition (3.13) is naturally found and also how the full gauge structure is derived in a simple way, based on the general formalism discussed in the previous chapter.

From the Lagrangian (3.16) above we find the conjugate momenta

$$p^\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{v}\dot{x}^\mu, \qquad p_v = \frac{\partial L}{\partial \dot{v}} = 0, \tag{3.17}$$

which gives us back the Hamiltonian $H(x, v, p, p_v)$ in (3.14). In addition we also find the primary constraint $p_v = 0$. The consistency condition that the time evolution of a primary constraint should be zero on the constraint surface yields the secondary constraint $\chi_1 = \frac{1}{2}p^2$, since $\dot{p}_v = -\frac{1}{2}p^2$. There are no further constraint since $\dot{\chi}_1 = 0$ and χ_1 and p_v are first-class constraints. Hence, the massless relativistic point particle is a gauge theory described by an abelian

Lie algebra with Poisson bracket relation $\{\chi_1, p_v\} = 0$. We can now construct the gauge generator given in (2.34) which for this specific model is

$$G = \epsilon\chi_1 + \beta p_v, \tag{3.18}$$

where $\epsilon(t), \beta(t)$ are real and even, infinitesimal gauge parameters. The condition (2.33) yields a relation between these parameters, $\beta = \dot{\epsilon}$, and the gauge transformation (2.35) are therefore given here by

$$\delta_\epsilon x^\mu = \frac{\epsilon}{v}\dot{x}^\mu, \qquad \delta_\epsilon v = \dot{\epsilon}. \tag{3.19}$$

The action is invariant under this transformation by construction which is confirmed by a a quick calculation since the variation of the Lagrangian yields a boundary term $\delta_\epsilon L = \frac{d}{d\tau}(\frac{\epsilon\dot{x}^2}{2v^2})$ (compare with the expression in (2.31)). The theory is irreducible and abelian since there are no further gauge transformations due to the commutator relations of the gauge transformations $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]x^\mu = 0$ and $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]v = 0$.

We also notice that if we redefine the parameter $\epsilon \rightarrow v\epsilon$, the gauge transformations above transform into

$$\delta_\epsilon x^\mu = \epsilon\dot{x}^\mu, \qquad \delta_\epsilon v = \frac{d}{d\tau}(v\epsilon). \tag{3.20}$$

The action for the massless relativistic point particle is of course invariant under this transformation as well, with $\delta_\epsilon L = \frac{d}{d\tau}(\frac{\epsilon\dot{x}^2}{2v})$. But here we notice that the theory now is non-abelian since a short calculation yields the commutators $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]x^\mu = \delta_{\epsilon_{12}}x^\mu$ and $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]v = \delta_{\epsilon_{12}}v$ where the gauge parameter $\epsilon_{12} = (\dot{\epsilon}_1\epsilon_2 - \epsilon_1\dot{\epsilon}_2)$.

We may also calculate the degrees of freedom for this model. This is done by adding the total number of variables for the canonical pairs in phase space and subtracting the number of first-class constraints and half of the second-class constraints (since after gauge fixing there are only second-class constraints left) [21]. In this case we are therefore left with three degrees of freedom describing the motion of a free particle in space.

3.3 The massless spinning relativistic particle

Other established and well known particle models are found by specifying the spin operator $s^{\mu\nu}$ in (3.3) with the properties given in (3.5). For example, by adding anticommuting [28–30, 33–36] or commuting [30–32, 37] variables to the different irreduclble representation of the Poincaré group one can enlarge the space and find new free particle theories.

A famous model to describe these spinning particles is found from the irreducible representation of the Poincaré group by extending the theory to include some odd hermitian operators. Such a model is the one we consider in this section, the massless relativistic spin-$\frac{1}{2}$ particle described in e.g. [33–36]. Some of the gauge transformations will now turn out to be local supersymmetry transformations, relating even (bosonic) variable with odd (fermionic) ones.

The quantization of the massless relativistic spin-$\frac{1}{2}$ particle is known and yields the Dirac equation, i.e. a wave equation for the relativistic spin-$\frac{1}{2}$ particle.

Let us begin by adding to the previously discussed theory (i.e. the massless relativistic particle) an odd, hermitian operator ψ^μ satisfying the commutation relations

$$[\psi^\mu, \psi^\nu]_+ := \psi^\mu\psi^\nu + \psi^\nu\psi^\mu = \eta^{\mu\nu}, \tag{3.21}$$

where the index plus indicates an anticommutator. The Poincaré generators are now expressed as in (3.3) and (3.4) with the spin operator given by

$$s^{\mu\nu} := -\frac{i}{2}(\psi^\mu\psi^\nu - \psi^\nu\psi^\mu). \tag{3.22}$$

In order to construct the reparametrization invariant Hamiltonian for this massless spinning relativistic particle we need to add a constraint to the one above such that the Poincaré invariants p^2 and w^2 satisfy the state conditions

$$p^2|\text{phys}\rangle = 0, \qquad w^2|\text{phys}\rangle = 0, \tag{3.23}$$

where the square of the Pauli-Lubanski operator (3.8) now is given by

$$w^2 = p^2 - 2(p\cdot\psi)^2 + 2(\psi\cdot\psi)(p\cdot\psi)^2 - \frac{1}{2}(\psi\cdot\psi)^2p^2 \tag{3.24}$$

and where $p\cdot\psi = p_\mu\psi^\mu$. The state conditions (3.23) imply that the constraints are given by[1]

$$\chi_1 := \frac{1}{2}p^2, \qquad \chi_6 := p\cdot\psi. \tag{3.25}$$

Classically is ψ^μ an odd variable[2] which together with the variables x^μ, p_μ satisfy the Poisson bracket relations

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{\psi^\mu, \psi^\nu\} = -i\eta^{\mu\nu}. \tag{3.26}$$

The Poisson bracket relations (3.26) are such that the Lie algebra (the nonzero part) is

$$\{\chi_6, \chi_6\} = -2\chi_1. \tag{3.27}$$

[1]The numbering is chosen in analogy with the constraints introduced in the next chapter.

[2]Two odd (fermionic) classical variables ψ & Γ obey the relations $\psi\Gamma = -\Gamma\psi$ and $\psi^2 = 0$. This formalism will be streamlined later when introducing Grassmann parities.

The reparametrization invariant Hamiltonian is now given by

$$H = \frac{1}{2}vp^2 + i\lambda_6 p \cdot \psi, \tag{3.28}$$

where we have denoted the Lagrange multipliers to the corresponding constraints by the real einbein v and the real, odd variable λ_6. The equations of motion are given by $\dot{p}^\mu = 0$ and $\dot{x}^\mu = vp^\mu + \lambda_6\psi^\mu$ such that the corresponding Lagrangian is

$$L = \frac{1}{2v}(\dot{x}^\mu - i\lambda_6\psi^\mu)^2 + \frac{i}{2}\psi \cdot \dot{\psi}. \tag{3.29}$$

The careful reader probably notice that we have a term $\psi \cdot \dot{\psi}$ that we have not yet verified. This term is dictated by the Poisson bracket relation in (3.26) above. Due to the Legendre form of the Lagrangian we do not need to introduce a canonical momenta to ψ^μ, whereas it is needed for the x^μ variable. It can be seen by considering the general formalism below. A clarification will also explain the choice of the Poisson bracket relation in (3.26). So let us write a general Lagrangian as

$$L = \frac{1}{2}\rho^a \Omega_{ab} \dot{\rho}^b - H(\rho), \tag{3.30}$$

where $\rho = (p, x, \psi)$ is a collective variable describing all coordinates p_μ, x^μ, ψ^μ and Ω_{ab} (with inverse $(\Omega^{-1})^{ab}$) is a constant matrix to be calculated. Ω_{ab} is antisymmetric in a, b for even ρ^a and symmetric for odd ρ^a (otherwise the first term in the (3.30) is a total derivative and then it will not contribute to the equations of motion). The equations of motion are given by $\dot{\rho}^a = (\Omega^{-1})^{ab}\frac{\partial H}{\partial \rho^b}$ and the Poisson bracket is

$$\{A, B\} = A\frac{\overleftarrow{\partial}}{\partial \rho^a}(\Omega^{-1})^{ab}\frac{\overrightarrow{\partial}}{\partial \rho^b}B. \tag{3.31}$$

We may now write the Lagrangian on the form above as $L = p\dot{x} + \frac{i}{2}\psi \cdot \dot{\psi} - H$ where the Hamiltonian is given in (3.28). The form of the matrix in this case is now easily extracted

$$(\Omega^{-1})^{ab} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \eta^{\mu\nu}, \tag{3.32}$$

and from this expression and (3.31) the Poisson bracket relations (3.26) follows. Notice also that $\dot{\psi}^\mu = \frac{\lambda_6}{v}\dot{x}^\mu$. So let us now turn to the Lagrangian in (3.29) from which it follows that the canonical momenta

$$p^\mu = \frac{1}{v}(\dot{x}^\mu - i\lambda_6\psi^\mu), \quad p_v = 0, \quad p_{\lambda_6} = 0 \tag{3.33}$$

and the corresponding Hamiltonian is written as in (3.28). The primary constraints are given by $p_v = 0$ and $p_{\lambda_6} = 0$. A Dirac consistency analysis of these constraints yields the secondary constraints χ_1 and χ_6 in (3.25), since $\dot{p}_v = -\chi_1$, $\dot{p}_{\lambda_6} = -\chi_6$ and $\dot{\chi}_6 = 2\lambda_6\chi_1$. To find the gauge transformations we construct the gauge generator for this model. This generator consists of the four constraints together with the real, even gauge parameters a, b and the real, odd ones α, β

$$G = a\chi_1 + i\alpha\chi_6 + bp_v + i\beta p_{\lambda_6}. \tag{3.34}$$

The consistency condition (2.33) yields the relations between the gauge parameters, $\beta = \dot{\alpha}$ and $b = \dot{a} + 2i\alpha\lambda_6$. Hence, the gauge transformations (2.35) can here be written as

$$\begin{aligned}
\delta x^\mu &= \frac{a}{v}(\dot{x}^\mu - i\lambda_6\psi^\mu) + i\alpha\psi^\mu, \\
\delta\psi^\mu &= \frac{\alpha}{v}(\dot{x}^\mu - i\lambda_6\psi^\mu), \\
\delta v &= \dot{a} + 2i\alpha\lambda_6, \\
\delta\lambda_6 &= \dot{\alpha}.
\end{aligned} \tag{3.35}$$

As in the example of the massless relativistic particle we may redefine the parameters as $a \longrightarrow av$ and $\alpha \longrightarrow a\lambda_6 + v\alpha$ (many authors use $\alpha \longrightarrow a\lambda_6 + \alpha$, but the choices we make here slightly simplifies the calculation of the commutator algebra of the gauge transformations). Using these redefinitions we obtain the more familiar expressions of the gauge transformations, namely

$$\begin{aligned}
\delta x^\mu &= a\dot{x}^\mu + iv\alpha\psi^\mu, \\
\delta\psi^\mu &= a\dot{\psi}^\mu + \alpha(\dot{x}^\mu - i\lambda_6\psi^\mu), \\
\delta v &= \frac{d}{d\tau}(av) + 2iv\alpha\lambda_6, \\
\delta\lambda_6 &= \frac{d}{d\tau}(a\lambda_6 + v\alpha).
\end{aligned} \tag{3.36}$$

Notice that these local gauge tranformations are seperated in two parts with either an even or odd gauge parameter. The first term in each transformation above represents the so called worldline reparametrizations. The second terms are the local supersymmetry transformations which mixes the bosonic (even) variable with the fermionic (odd) ones. It should also be noted that the gauge transformations of the massless relativistic particle is obtained by setting ψ^μ and λ_6 to zero.

The commutator of the gauge transformations are now given by $[\delta_1, \delta_2]\rho = \delta_{12}\rho$, where ρ is either p_μ, x^μ or ψ^μ. The transformation δ_{12} corresponds to the parameters

$$\begin{aligned}
a_{12} &= a_2\dot{a}_1 - a_1\dot{a}_2 + 2iv\alpha_2\alpha_1, \\
\alpha_{12} &= a_2\dot{\alpha}_1 - a_1\dot{\alpha}_2 + \dot{a}_1\alpha_2 - \dot{a}_2\alpha_1 + 2i\lambda_6\alpha_2\alpha_1,
\end{aligned} \tag{3.37}$$

such that e.g. $[\delta_1, \delta_2]x^\mu = a_{12}\dot{x}^\mu + iv\alpha_{12}\psi^\mu$. Taking the limit when λ_6, α_1 and α_2 go to zero we find the commutator algebra of the gauge transformations of the free massless relativistic point particle in the previous section.

Calculating the degrees of freedom as described before we obtain four independent variables, describing the motion in space for the particle with spin up or down.

4

INFINITE SPIN PARTICLES

Not all irreducible representations of the Poincaré group described in the previous chapter have been exposed to an extensive study. In this chapter we focus on the massless representations of the Poincaré group with the conditions on the physical subspace given in (3.11). Except for the massless condition of these particles, $p^2 = 0$, the representations are also characterized by a real, constant Ξ which is the square of the Pauli-Lubanski vector (3.6) with $w^2 = \Xi^2$. This is one of those irreducible representations of the Poincaré group that we do not completely understand. Wigner dubbed it the **infinite spin particles** [32] or the **continuous spin representation** (CSR) [31]. We also call it Wigner's Ξ-representation.

In the communication by Bargmann and Wigner [31], Wigner's original work [1] was clarified and further developed. In their paper they also study models with the generators $p^\mu, m^{\mu\nu}$ given in (3.3), (3.4) and an explicit expression of the spin generator

$$s^{\mu\nu} = \xi^\mu \pi^\nu - \xi^\mu \pi^\mu \tag{4.1}$$

in terms of some internal variables ξ^μ and π_μ. These variables obey the commutation relation

$$[\xi^\mu, \pi_\nu] = i\delta^\mu_\nu. \tag{4.2}$$

This gives rise to a definite representation of the infinite spin particles when the conditions in (3.11) are fullfilled. This representation has since its birth been believed to not appear in any physical theories and one has not investigated it further. Wigner himself disregarded it since it gives rise to infinite heat capacity [32]. He showed that the state space is infinite dimensional and a summation over all possible states with positive energy yields infinite heat capacity. The continuous spin representation has also been found to have further obstacles, like negative norm states and non-locality [38, 39]. Recently the supersymmetric version has been considered in the light-cone gauge [40]. In [41] it was also shown that the infinite spin particle representation can be generated from the five dimensional Poincaré group by a combination of a group contraction and a Kaluza-Klein dimensional reduction.

The history of massless and massive higher spin fields has slowly been developed for a long period of time, beginning with the communications in [1, 42–44]. In recent years these theories have attracted a lot more attention, specially due to the possible connection to string theory. For good reviews on higher spin theories see [45–48] and references therein. The relevance of the infinite spin particles and other higher spin particles to be candidates for fundamental particles are of course doubtful. On the other hand we should not rule them out completely either since we know that string theories have an infinite tower of spins in its spectrum. Massless higher spin particles seem to be realized by a sector of string theories with zero tension [49–51]. This zero tension limit might be an unbroken phase of string theories [52]. However, this is not easily established since this is a singular limit.

In this chapter we start from Wigner's Ξ-representation and derive an infinite spin particle model described by a simple higher order Lagrangian. In this derivation we show how to find a phase space Hamiltonian build out of the constraints recovered from the infinite spin particle representation and thereafter take the theory into the simple higher order Lagrangian in configuration space. The introduction of fermionic variables is made in the same way as for the spinning relativistic particle model, i.e. by introducing the odd operators ψ^μ. It is shown how the classical higher order Lagrangian is written in a gravitational background with the only non-negative results in a (anti) de Sitter spacetime. This might have connections to the results found in other interacting higher spin theories [53–56]. We also make a Gupta-Bleuler quantization of the infinite spin particle and elaborate on possible extensions and generalizations to higher dimensions, such as a string theory generalization. This chapter is based on Paper I and all details are given there. Here we give supplementary comments and also some new results.

4.1 Wigner's Ξ-representation

As discussed in the previous chapter, the irreducible representations are recovered by finding the Poincaré invariants, i.e. the square of the momentum operator p^2 and the square of the Pauli-Lubanski operator w^2. Wigner's Ξ-

representation is a massless representation with $p^2 = 0$ and $w_\mu w^\mu = \Xi$. In a Dirac quantization we may obtain constraints in terms of operators acting on a physical state $|\text{phys}\rangle$ from the conditions

$$\begin{aligned} p^2|\text{phys}\rangle &= 0, \\ (w_\mu w^\mu - \Xi^2)|\text{phys}\rangle &= 0. \end{aligned} \tag{4.3}$$

These conditions are here written in a so called strong form $\chi_i|\text{phys}\rangle=0$, a weaker condition $\langle\text{phys}|\chi_i|\text{phys}\rangle = 0$ may in some cases be necessary to consider. An example of this will be seen later in this chapter when quantizing the infinite spin particles. The constraints above build up the reparametrization invariant Hamiltonian and constitute a gauge algebra by their mutual commutation relations.

Let the particle be described in terms of the coordinates x^μ with conjugate momenta p_μ. Also introduce an internal vector ξ^μ with conjugate momenta π_μ. These coordinates obey the commutation relations, the non-zero part,

$$[x^\mu, p_\nu] = i\delta^\mu_\nu, \qquad [\xi^\mu, \pi_\nu] = i\delta^\mu_\nu. \tag{4.4}$$

The conditions in (4.3) now yield two elementary sets of minimal constraints. With $\chi_i|\text{phys}\rangle = 0$ ($\forall i = 1, \dots, 4$) these are

$$\begin{aligned} \chi_1 &:= \frac{1}{2}p^2, \\ \chi_2 &:= \frac{1}{2}(\pi^2 - F^2), \\ \chi_3 &:= p \cdot \pi, \\ \chi_4 &:= p \cdot \xi - \frac{\Xi}{F}. \end{aligned} \tag{4.5}$$

With $\chi'_i|\text{phys}\rangle = 0$, an altenative description is given by the constraints

$$\begin{aligned} \chi'_1 &:= \frac{1}{2}p^2, \\ \chi'_2 &:= \frac{1}{2}(\xi^2 - F^2), \\ \chi'_3 &:= p \cdot \xi, \\ \chi'_4 &:= p \cdot \pi - \frac{\Xi}{F}. \end{aligned} \tag{4.6}$$

F is a non-zero constant or an operator commuting with $x^\mu, p_\mu, \xi^\mu, \pi_\mu$ (e.g. the inverse einbein introduced in the next section). These constraints are of first class and satisfy a closed Lie algebra with the nonzero relations

$$[\chi_4, \chi_2] = i\chi_3, \qquad [\chi_4, \chi_3] = 2i\chi_1. \tag{4.7}$$

The same relations with minus signs hold for the alternative representation with χ_i' above. The reparametrization invariant Hamiltonian (with $F = 1$) is given by

$$H := \lambda_1\chi_1 + \lambda_2\chi_2 + \lambda_3\chi_3 + \lambda_4\chi_4, \tag{4.8}$$

or explicitly

$$H := \lambda_1\frac{1}{2}p^2 + \lambda_2\frac{1}{2}(\pi^2 - 1) + \lambda_3 p\cdot\pi + \lambda_4(p\cdot\xi - \Xi). \tag{4.9}$$

The same Hamiltonian is given by the constraints χ_i' but with ξ^μ and π_μ interchanged.

4.2 Generating the higher order Lagrangian

Let us now turn to a classical description and write this theory in a configuration space represented by a Lagrangian. Hence, we need to transform the Hamiltonian above into a Lagrangian. The coordinates will now be treated as classical variables with the commutation relations

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{\xi^\mu, \pi_\nu\} = \delta^\mu_\nu. \tag{4.10}$$

The equations of motion (2.19) for the Hamiltonian (4.9) are given by

$$\begin{aligned} \dot{x}^\mu &= \lambda_1 p^\mu + \lambda_3\pi^\mu + \lambda_4\xi^\mu, \qquad & \dot{p}_\mu &= 0,\\ \dot{\xi}^\mu &= \lambda_2\pi^\mu + \lambda_3 p^\mu, & \dot{\pi}_\mu &= -\lambda_4 p_\mu, \end{aligned} \tag{4.11}$$

where the Lagrange multipliers $\lambda_i(t)$ are time-dependent. The corresponding configuration space Lagrangian is then

$$\begin{aligned} L &= \frac{\lambda_2}{2A}(\dot{x} - \lambda_4\xi)^2 + \frac{\lambda_1}{2A}\dot{\xi}^2 - \frac{\lambda_3}{A}(\dot{x} - \lambda_4\xi)\cdot\dot{\xi} + \frac{1}{2}\lambda_2 + \lambda_4\Xi,\\ A &:= \lambda_1\lambda_2 - \lambda_3^2 \neq 0. \end{aligned} \tag{4.12}$$

This expression can be reduced to a higher order Lagrangian in x^μ by choosing $\lambda_1 = 0$, $\lambda_3 = \alpha\lambda_4$, and $\lambda_2 \neq 0$ for any real $\alpha \neq 0$ and solving the equation of motion (2.3) for ξ^μ which yields

$$\xi^\mu = \frac{1}{\lambda_4}\dot{x}^\mu - \frac{\alpha}{\lambda_2}\frac{d}{d\tau}\left(\frac{1}{\lambda_4}\dot{x}^\mu\right). \tag{4.13}$$

Inserting this expression into (4.12) yields a simple unique higher order Lagrangian

$$L = \sqrt{\left(\frac{d}{d\tau}(\frac{1}{\lambda_4}\dot{x})\right)^2} + \lambda_4\Xi. \tag{4.14}$$

By unique we mean that exactly the same higher order Lagrangian is found if we instead start with the alternative reparametrization invariant Hamiltonian with the constraints given in (4.6). In this analysis we find the alternative Lagrangian

$$L' = -\frac{\lambda_1}{2\lambda_4^2}\dot{\xi}^2 + \frac{1}{\lambda_4}(\dot{x} - \lambda_3\xi)\cdot\dot{\xi} - \frac{1}{2}\lambda_2(\xi^2 - 1) + \lambda_4\Xi, \quad \lambda_4 \neq 0. \qquad (4.15)$$

Provided we again choose $\lambda_1 = 0$, $\lambda_3 = \alpha\lambda_4$, and $\lambda_2 \neq 0$ and insert the expression for ξ^μ, here given by

$$\xi^\mu = -\frac{1}{\lambda_2}\frac{d}{d\tau}\left(\frac{1}{\lambda_4}\dot{x}^\mu\right), \qquad (4.16)$$

we may write this model as a higher order Lagrangian which is exactly the one in (4.14).

The choice of Lagrange multipliers in both of these descriptions might seem strange but this is plausible since the associated constraints are recovered as secondary and tertiary constraint when performing a constraint analysis. This is studied below and shown in detail in Paper I.

To slightly simplify the calculations we write the Lagrangian (4.14) above in terms of the inverse einbein $e = \frac{1}{v}$, where $v = \lambda_4$ such that

$$L = \sqrt{(\dot{e}\dot{x} + e\ddot{x})^2} + \frac{1}{e}\Xi. \qquad (4.17)$$

4.3 The Ostrogradski method

Consider again the Lagrangian (4.17) and let us try to reproduce the constraints obtained in (4.5). In order to perform a constraint analysis of the Lagrangian we need to transform it into a Hamiltonian. However, the Lagrangian is not on the form we are used to since it consists of higher order time derivatives. To deal with these we make use of the method introduced by Ostrogradski [57] (see also [58] chapter X, or better [59] appendix I). This is nothing spectacular, it follows the standard procedure with the exception that we introduce some new variables which are equal to our higher order terms. Start with the Lagrangian $L(x, \dot{x}, \ddot{x}, \ldots, \overset{(n)}{x}; t)$ and let

$$\begin{aligned} x &= q_1, \\ \dot{q}_1 &= q_2, \\ \dot{q}_2 &= q_3, \\ &\ldots \\ \dot{q}_n &= q_{n+1}. \end{aligned} \qquad (4.18)$$

Write the Lagrangian $L(x, \dot{x}, \ddot{x}, \ldots, \overset{(n)}{x}; t)$ in terms of these new coordinates with Lagrange multipliers p_i (why we call them p will soon be obvious)

$$L'(q_1, q_2, \ldots, q_{n+1}; t) = L(q_1, q_2, \ldots, q_{n+1}; t) + \sum_{i=1}^{n} p_i(\dot{q}_i - q_{i+1}) \tag{4.19}$$

such that the Hamiltonian is written as

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L'(q_1, q_2, \ldots, q_{n+1}; t) = \sum_{i=1}^{n} p_i q_{i+1} - L(q_1, q_2, \ldots, q_{n+1}; t). \tag{4.20}$$

The only thing left is to eliminate the coordinate q_{n+1} by

$$p_n = \frac{\partial L(q_1, q_2, \ldots, q_{n+1})}{\partial q_{n+1}}. \tag{4.21}$$

4.4 Classical analysis of the infinite spin particles

Now we are ready to transform the Lagrangian description of the infinite spin particle into a Hamiltonian form and find all constraints. Redefine the coordinates as $x^\mu = q_1^\mu, \dot{q}_1^\mu = q_2^\mu, \dot{q}_2^\mu = q_3^\mu$ such that

$$L' = \sqrt{(\dot{e}q_2 + eq_3)^2} + \frac{1}{e}\Xi + p_{1\mu}(\dot{q}_1^\mu - q_2^\mu) + p_{2\mu}(\dot{q}_2^\mu - q_3^\mu). \tag{4.22}$$

The Hamiltonian $H(p_1, p_2, \omega, q_1, q_2, q_3)$ (where ω is the canonical momenta of the inverse einbein variable e) can now be written in a simple first order form by eliminating the coordinates q_3^μ, ω by

$$\begin{aligned} p_{2\mu} &= \frac{\partial \mathcal{L}(q_1, q_2, e, \dot{e})}{\partial q_3^\mu} = \frac{e(\dot{e}q_2^\mu + eq_3^\mu)}{\sqrt{(\dot{e}q_2^\mu + eq_3^\mu)^2}} \\ \omega &= \frac{\partial \mathcal{L}(q_1, q_2, e, \dot{e})}{\partial \dot{e}} = \frac{q_{2\mu}(\dot{e}q_2^\mu + eq_3^\mu)}{\sqrt{(\dot{e}q_2 + eq_3)^2}}. \end{aligned} \tag{4.23}$$

The Hamiltonian describing the infinite spin particles is now

$$H = p \cdot \xi - \frac{1}{e}\Xi \tag{4.24}$$

where we made the identifications $q_2^\mu = \xi^\mu$, i.e. $\dot{x}^\mu = \xi^\mu$, and $p_{1\mu} = p_\mu$, $p_{2\mu} = \pi_\mu$. As shown in Paper I, we in addition to the Hamiltonian (4.24) have the primary, secondary and tertiary constraints

$$\begin{aligned}\chi_1 &:= \frac{1}{2}p^2,\\ \chi_2 &:= \frac{1}{2}(\pi^2 - e^2),\\ \chi_3 &:= p\cdot\pi,\\ \chi_4 &:= p\cdot\xi - \frac{1}{e}\Xi,\\ \chi_5 &:= \pi\cdot\xi - \omega e.\end{aligned} \tag{4.25}$$

These constraints are all of first class and their nonzero Poisson brackets are

$$\begin{aligned}&\{\chi_4,\chi_2\} = \chi_3, \quad \{\chi_4,\chi_3\} = 2\chi_1, \quad \{\chi_5,\chi_2\} = 2\chi_2,\\ &\{\chi_5,\chi_4\} = -\chi_4, \quad \{\chi_5,\chi_3\} = \chi_3.\end{aligned} \tag{4.26}$$

A Lie algebra of this kind is called solvable. We may now express the total reparametrization invariant Hamiltonian as

$$H = \lambda_1\chi_1 + \lambda_2\chi_2 + \lambda_3\chi_3 + \lambda_4\chi_4 + \lambda_5\chi_5 \tag{4.27}$$

where λ_i is the Lagrange multiplier corresponding to the constraint χ_i. Thus we have shown how to generate the original Hamiltonian (4.9) from the higher order Lagrangian (4.17). Notice that we here have an extra constraint χ_5. This is due to the fact that the Lagrange multiplier became the dynamical einbein variable in the process discussed before, the constraints χ_5 removes this degree of freedom. By gauge fixing $e = 1$, this reduces to the familiar form of the Hamiltonian in (4.9) with the four constraints (4.5).

4.5 Pseudoclassical model

As in the case of the spinning relativistic particle discussed in chapter 3 we may add an odd fermionic operator ψ^μ to the spin operator $s^{\mu\nu}$

$$s^{\mu\nu} = \xi^\mu\pi^\nu - \xi^\nu\pi^\mu - \frac{i}{2}\left(\psi^\mu\psi^\nu - \psi^\nu\psi^\mu\right). \tag{4.28}$$

The odd operators have the same anticommutation relation as considered before, i.e.

$$[\psi^\mu, \psi^\nu]_+ = \eta^{\mu\nu}. \tag{4.29}$$

To satisfy the relations in (4.3) we add the familiar state condition

$$p\cdot\psi|\text{phys}\rangle = 0. \tag{4.30}$$

In order to obtain a simple higher order Lagrangian we need to introduce another odd operator θ with

$$[\theta, \theta]_+ = -1, \tag{4.31}$$

and add the condition

$$(\pi \cdot \psi + \theta)|\text{phys}\rangle = 0. \tag{4.32}$$

At the pseudoclassical level these two odd operators ψ^μ and θ satisfy the Poisson bracket relations

$$\{\psi^\mu, \psi^\nu\} = -i\eta^{\mu\nu}, \quad \{\theta, \theta\} = i. \tag{4.33}$$

Hence the Hamiltonian consists of the constraints in (4.5) (with $F = 1$) and

$$\chi_6 := p \cdot \psi, \quad \chi_7 := \pi \cdot \psi + \theta. \tag{4.34}$$

The additional relations to the Lie algebra (4.26) are given by

$$\begin{aligned} \{\chi_6, \chi_6\} &= -2i\chi_1, \quad \{\chi_7, \chi_7\} = -2i\chi_2, \\ \{\chi_6, \chi_7\} &= -i\chi_3, \qquad \{\chi_7, \chi_4\} = -\chi_6. \end{aligned} \tag{4.35}$$

The Lagrangian is now obtained by a similar analysis as in the original case in the previous section, which is shown in detail in Paper I. The resulting expression is given by

$$\tilde{L} = \sqrt{(\dot{e}\dot{x} + e\ddot{x})^2 - 2i\lambda_7\psi \cdot (\dot{e}\dot{x} + e\ddot{x})} + \frac{1}{e}\Xi + \frac{i}{2}\psi \cdot \dot{\psi} - \frac{i}{2}\theta\dot{\theta} - i\lambda_7\theta. \tag{4.36}$$

By identifying $\xi^\mu = \dot{x}^\mu$ and following the Ostrogradski procedure we find the Hamiltonian (for details see Paper I)

$$\tilde{H} = p \cdot \xi - \frac{1}{e}\Xi. \tag{4.37}$$

This expression is in itself a constraint (which is seen by a constraint analysis and implies that the theory is reparametrization invariant) and together with the six other constraints in (4.25) and (4.34) it constitutes the Lie algebra described in (4.26), (4.35) and by

$$\{\chi_5, \chi_6\} = 0, \qquad \{\chi_5, \chi_7\} = \chi_7. \tag{4.38}$$

The constraint χ_5 is also here due to the fact that the inverse einbein e became dynamical in the process of obtaining the higher order Lagrangian.

4.6 Gravitational interaction

A natural way to introduce gravitational interaction in our model is to find the action that is independent on the choice of spacetime coordinates and reduces to the original model (4.17) in the flat case. The naïve idea is of course just to let $\sqrt{\ddot{x}^2} \rightarrow \sqrt{\ddot{x}^\mu g_{\mu\nu}\ddot{x}^\nu}$ where $g_{\mu\nu}(x)$ is the Riemannian metric. But it remains to check if this term is invariant under arbitrary coordinate transformations. So basically we need to find out how a term like $\sqrt{\ddot{x}^2}$ transforms in a curved Riemannian spacetime. We know that $\dot{x}^\mu$ transforms as a tensor, since $\dot{x}^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\dot{x}^\nu$ but what about $\ddot{x}^\mu$? With some algebra we find that we must add a term to $\ddot{x}^\mu$ such that it transforms adequately,

$$\ddot{x}'^\mu + \Gamma'^\mu_{\tau\nu}\dot{x}'^\tau\dot{x}'^\nu = \frac{\partial x'^\mu}{\partial x^\nu}(\ddot{x}^\nu + \Gamma^\nu_{\lambda\sigma}\dot{x}^\lambda\dot{x}^\sigma), \tag{4.39}$$

where $\Gamma^\nu_{\lambda\sigma}$ is the affine connection

$$\Gamma^\mu_{\alpha\beta} := \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}). \tag{4.40}$$

Hence, to introduce gravity we should replace all terms in the action by $\sqrt{\ddot{x}^2} \rightarrow \sqrt{(\ddot{x}^\mu + \Gamma^\mu_{\lambda\sigma}\dot{x}^\lambda\dot{x}^\sigma)g_{\mu\nu}(\ddot{x}^\nu + \Gamma^\nu_{\rho\tau}\dot{x}^\rho\dot{x}^\tau)}$. The Lagrangian corresponding to the infinite spin particle in a gravitational background is then

$$\begin{aligned} L_{\text{grav}} &= \sqrt{(\dot{e}\dot{x}^\mu + eD\dot{x}^\mu)g_{\mu\nu}(\dot{e}\dot{x}^\nu + eD\dot{x}^\nu)} + \frac{1}{e}\Xi, \\ D\dot{x}^\mu &:= \ddot{x}^\mu + \dot{x}^\nu\Gamma^\mu_{\nu\lambda}(x)\dot{x}^\lambda. \end{aligned} \tag{4.41}$$

Using the Ostrogradski method and identifying $\xi^\mu := \dot{x}^\mu$ the conjugate momenta to ξ^μ and e are given by

$$\begin{aligned} \pi_\mu &= \frac{eg_{\mu\nu}(\dot{e}\xi^\nu + eD\xi^\nu)}{\sqrt{(\dot{e}\xi^\lambda + eD\xi^\lambda)g_{\lambda\kappa}(\dot{e}\xi^\kappa + eD\xi^\kappa)}}, \\ \omega &= \frac{\xi^\mu g_{\mu\nu}(\dot{e}\xi^\nu + eD\xi^\nu)}{\sqrt{(\dot{e}\xi^\lambda + eD\xi^\lambda)g_{\lambda\kappa}(\dot{e}\xi^\kappa + eD\xi^\kappa)}}, \end{aligned} \tag{4.42}$$

such that the Hamiltonian

$$H_{\text{grav}} = \Lambda_\mu\xi^\mu - \frac{1}{e}\Xi. \tag{4.43}$$

Here, $\Lambda_\mu := p_\mu - \pi_\nu\Gamma^\nu_{\mu\lambda}(x)\xi^\lambda$ acts as a covariant derivative such that e.g.

$$\{\xi^\mu f^\nu_\mu(x)\pi_\nu, \Lambda_\sigma\} = \xi^\mu(\partial_\sigma f^\nu_\mu - f^\nu_\gamma\Gamma^\gamma_{\sigma\mu} + f^\gamma_\mu\Gamma^\nu_{\sigma\gamma})\pi_\nu = \xi^\mu f^\nu_{\mu;\sigma}\pi_\nu \tag{4.44}$$

for a general function $f^\nu_\mu(x)$. Λ_μ also has the useful Poisson bracket relation

$$\{\Lambda_\mu, \Lambda_\nu\} = \pi_\rho R^\rho{}_{\lambda\mu\nu}(x)\xi^\lambda, \tag{4.45}$$

where the Riemann tensor is defined by

$$R^{\rho}_{\ \alpha\gamma\beta} := \partial_\gamma \Gamma^{\rho}_{\alpha\beta} - \partial_\beta \Gamma^{\rho}_{\alpha\gamma} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\rho}_{\gamma\eta} - \Gamma^{\eta}_{\alpha\gamma}\Gamma^{\rho}_{\beta\eta}. \tag{4.46}$$

From the expression of the conjugate momenta above (4.42) we also find the primary constraints

$$\chi_2 := \frac{1}{2}(\pi_\mu g^{\mu\nu}(x)\pi_\nu - e^2), \qquad \chi_5 := \pi_\mu \xi^\mu - \omega e. \tag{4.47}$$

The constraint analysis of this model is rather straightforward but much more involved than before. The details are given in Paper I were we find that the only consistent solution to obtain a closed constraint algebra is given by some specially symmetric spacetimes with a constant curvature, such as the (anti) de Sitter background and with the additional condition $\Xi = 0$. The Riemann tensor is then given by

$$R_{\mu\alpha\beta\gamma} = K\big(g_{\alpha\beta}g_{\mu\gamma} - g_{\alpha\gamma}g_{\beta\mu}\big), \;\Rightarrow\; R_{\mu\alpha\beta\gamma;\nu} = 0, \tag{4.48}$$

where K is a real constant which is positive for a de Sitter space, and negative for an anti de Sitter space. By a consistent solution we mean that the degrees of freedom in the interacting model should be the same as in the free case.

4.6.1 The psudoclassical case

In the pseudoclassical case with the odd spin variables ψ^μ we need to introduce vierbeins $V^a_\mu(x)$ in our formalism. These are related to the metric as

$$g_{\mu\nu}(x) = V^a_\mu(x)V^b_\nu(x)\eta_{ab}, \tag{4.49}$$

where η_{ab} is the flat metric and $V^\mu_a(x)$ is the inverse. Except for the affine connection (4.40) we also need to introduce the spin connection $\omega^a_{\ b\gamma}$ such that the pseudoclassical Lagrangian is given by

$$\begin{aligned}\tilde{L}_{\text{grav}} = \;& \sqrt{\big(e\dot{x}^\mu + eD\dot{x}^\mu\big)g_{\mu\nu}\big(e\dot{x}^\nu + eD\dot{x}^\nu\big) - 2i\lambda_7\psi_a V^a_\alpha\big(e\dot{x}^\alpha + eD\dot{x}^\alpha\big)} + \\ & + \frac{1}{e}\Xi + \frac{i}{2}\psi_a D\psi^a - \frac{i}{2}\theta\dot{\theta} - i\lambda_7\theta \end{aligned} \tag{4.50}$$

where $D\psi^a \equiv \dot{\psi}^a + \omega^a_{\ b\gamma}\dot{x}^\gamma\psi^b$ and $\omega_{ab\mu}(x) = V^\nu_a(\partial_\mu V_{b\nu} - \Gamma^\rho_{\mu\nu}V_{b\rho})$. The corresponding Hamiltonian is again

$$\tilde{H}_{\text{grav}} = \tilde{\Lambda}_\mu \xi^\mu - \frac{1}{e}\Xi, \tag{4.51}$$

but here

$$\tilde{\Lambda}_\mu := p_\mu - \pi_\sigma \Gamma^\sigma_{\mu\nu}\xi^\nu - \frac{i}{2}\psi^a \omega_{ab\mu}\psi^b. \tag{4.52}$$

The Poisson bracket relation for this variable is

$$\{\tilde{\Lambda}_\mu, \tilde{\Lambda}_\nu\} = \pi_\rho R^\rho{}_{\lambda\mu\nu}\xi^\lambda + \frac{i}{2}\psi^a\psi^b R_{ab\mu\nu}, \quad R_{ab\mu\nu} = V_a^\sigma V_b^\rho R_{\sigma\rho\mu\nu}. \tag{4.53}$$

The (anti) de Sitter spacetime (with Riemann tensor given in (4.48)) yields a closed constraint algebra also in the pseudoclassical case but *only* if we introduce two new constraints

$$\chi_8 := \frac{1}{2}(\xi^\mu g_{\mu\nu}\xi^\nu - \omega^2), \tag{4.54}$$

$$\chi_9 := \theta\omega + \xi_\mu V_a^\mu \psi^a. \tag{4.55}$$

However, it does not form a Lie algebra. The detailed analysis is also in this case given in Paper I.

4.7 Quantization

We have showed how the infinite spin particle can be written in terms of a simple higher order Lagrangian and we have considered possible interactions with a gravitational background. The only non-negative results was found to be in (anti) de Sitter spacetimes. So far we have not mentioned how a covariant quantization of this infinite spin particle model might be done. To do a proper quantization we should use a BRST formulation, which we in this case expect to be inconsistent with the Dirac quantization conditions used before. Probably we need to consider a general framework of the BRST quantization such as the one considered by Batalin and Marnelius in [60] when quantization is made on inner product spaces. Another possibility is to use the Batalin-Vilkovisky formalism discussed in the next chapter. However, higher order theories are not easily described in the BV formalism. On the other hand, we do believe that a Gupta-Bleuler quantization is in the right direction and we will therefore discuss this further here.

4.7.1 Gupta-Bleuler quantization

The Gupta-Bleuler quantization procedure uses the weaker condition compared to the Dirac condition discussed in the previous sections

$$\langle \text{phys}|\hat{\chi}_i|\text{phys}\rangle = 0, \tag{4.56}$$

where the constraint may now act on either state. In a Gupta-Bleuler quantization this condition follows from some operators $\hat{Q}_r$ (not necessarily hermitian) with the properties

$$\hat{Q}_r|\text{phys}\rangle = 0, \qquad \langle\text{phys}|\hat{Q}_r^\dagger = 0. \tag{4.57}$$

The $\hat{Q}$'s are required to form a closed algebra

$$[\hat{Q}_r, \hat{Q}_s] = i f_{rst} \hat{Q}_t \tag{4.58}$$

where f_{rst} is a constant (not necessarily real).

Let us introduce the oscillators $\hat{a}^\mu$ expressed in terms of the internal variables ξ^μ and π_μ as

$$\hat{a}^\mu := \frac{1}{\sqrt{2}}(\xi^\mu + i\pi^\mu) \quad \Rightarrow \quad [\hat{a}^\mu, \hat{a}^{\nu\dagger}] = \eta^{\mu\nu}. \tag{4.59}$$

For the physical states we use a Fock ansatz

$$|\Psi\rangle = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k!} A^{(n)}_{\mu_1\mu_2\cdots\mu_k}(\hat{x})|0,n\rangle^{\mu_1\mu_2\cdots\mu_k} + \sum_{n=-\infty}^{\infty} \phi^{(n)}(\hat{x})|0,n\rangle, \tag{4.60}$$

where we have defined

$$\begin{aligned} &|0,n\rangle^{\mu_1\mu_2\cdots\mu_k} := \hat{a}^{\mu_1\dagger}\hat{a}^{\mu_2\dagger}\cdots\hat{a}^{\mu_k\dagger}|0,n\rangle, \quad |0,n\rangle := |0\rangle_p|0\rangle|n\rangle, \\ &|n\rangle := \hat{e}^n|0\rangle_\omega, \quad \hat{a}^\mu|0\rangle = 0, \quad \hat{p}_\mu|0\rangle_p = 0, \quad \hat{\omega}|0\rangle_\omega = 0 \end{aligned} \tag{4.61}$$

and $[e, \omega] = i$. We may also let the states be such that

$$\begin{aligned} \langle e| &= {}_e\langle 0|e^{ie\hat{\omega}}, & |p\rangle &= e^{ip\cdot\hat{x}}|0\rangle_p, \\ \langle x| &= {}_x\langle 0|e^{i\hat{p}\cdot x}, & |\omega\rangle &= e^{i\hat{e}\omega}|0\rangle_\omega \end{aligned} \tag{4.62}$$

where the exponentional should not be mistaken for the inverse einbein in (4.61). Notice that we have introduced a kind of Banach space by using different representations of the bra and ket states. The Fock ansatz in (4.60) with (4.61) and (4.62) now implies the wave function representation

$$\begin{aligned} \Psi(x,e) &:= \langle x,e|\Psi\rangle = \sum_{n=-\infty}^{\infty} \phi^{(n)}(x)e^n, \\ \Psi_{\mu_1\cdots\mu_k}(x,e) &:= {}_{\mu_1\cdots\mu_k}\langle x,e|\Psi\rangle = \sum_{n=-\infty}^{\infty} A^{(n)}_{\mu_1\cdots\mu_k}(x)e^n, \quad k \geq 1, \end{aligned} \tag{4.63}$$

with $e \neq 0$ and where we have defined

$${}_{\mu_1\cdots\mu_k}\langle x,e| := \langle x|\langle e|\langle 0|\hat{a}_{\mu_1}\cdots\hat{a}_{\mu_k} = {}_x\langle 0|_e\langle 0|\langle 0|e^{i\hat{p}\cdot x}e^{ie\hat{\omega}}\hat{a}_{\mu_1}\cdots\hat{a}_{\mu_k}. \tag{4.64}$$

To exemplify this Gupta-Bleuler quantization we analyze the free classical model considered in (4.17). The quantization of other related models are considered in Paper I as well.

4.7.2 Quantization of the free infinite spin particle model

In order to perform a Gupta-Bleuler quantization of the free classical model in (4.17) we define the constraints

$$\begin{aligned} \hat{Q}_0 &:= 2\hat{\chi}_1, \\ \hat{Q}_1 &:= \frac{1}{\sqrt{2}}(\hat{\chi}_4 + i\hat{\chi}_3), \\ \hat{Q}_2 &:= 2\hat{\chi}_2 - i\hat{\chi}_5, \end{aligned} \tag{4.65}$$

with the only non-zero commutation relation

$$[\hat{Q}_1, \hat{Q}_2] = \hat{Q}_1. \tag{4.66}$$

$\hat{\chi}_1, \ldots, \hat{\chi}_5$ are the operators corresponding to the constraints given in (4.25) and in terms of creation and annihilation operators

$$\begin{aligned} \hat{\chi}_1 &:= \frac{1}{2}\hat{p}^2, \\ \hat{\chi}_2 &:= -\frac{1}{2}\big((\hat{a} - \hat{a}^\dagger)^2 + 1\big), \\ \hat{\chi}_3 &:= \frac{i}{\sqrt{2}}\big(\hat{p}\cdot\hat{a}^\dagger - \hat{p}\cdot\hat{a}\big), \\ \hat{\chi}_4 &:= \frac{1}{\sqrt{2}}\big(\hat{p}\cdot\hat{a} + \hat{p}\cdot\hat{a}^\dagger\big) - \Xi, \\ \hat{\chi}_5 &:= \frac{i}{2}\big(\hat{a}^{\dagger 2} - \hat{a}^2\big) - \frac{1}{2}\big(\hat{\omega}\hat{e} + \hat{e}\hat{\omega}\big). \end{aligned} \tag{4.67}$$

Notice that the number of $\hat{Q}_r$'s and their hermitian conjugate are equal to the number of $\hat{\chi}_i$'s. Now we look for non-zero solutions for the ϕ- and A-fields to the conditions $\hat{Q}_{1,2,3}|\Psi\rangle = 0$, where $|\Psi\rangle$ is given by the Fock ansatz (4.60) and the constraint operators by (4.67) above. The condition $\hat{Q}_1|\Psi\rangle = 0$ yields the Klein-Gordon like equations

$$\partial^2\phi^{(n)}(x) = 0, \quad \partial^2 A^{(n)}_{\mu_1,\ldots,\mu_k} = 0 \tag{4.68}$$

and $\hat{Q}_2|\Psi\rangle = 0$;

$$\begin{aligned} &i\partial^\nu A^{(n)}_\nu(x) + \frac{\Xi}{\sqrt{2}}\phi^{(n+1)}(x) = 0, \\ &i\partial^\nu A^{(n)}_{\nu\mu_1\cdots\mu_k}(x) + \frac{\Xi}{\sqrt{2}}A^{(n+1)}_{\mu_1\cdots\mu_k}(x) = 0, \quad k \geq 1. \end{aligned} \tag{4.69}$$

Notice that if $\Xi = 0$ we find the Lorentz conditions from the equations above. The last condition $\hat{Q}_3|\Psi\rangle = 0$ gives us the expression

$$\left(n + \frac{5}{2}\right)\phi^{(n)}(x) - A_\nu^{(n)\nu}(x) - \phi^{(n-2)}(x) = 0,$$
$$\left(n + k + \frac{5}{2}\right)A^{(n)}_{\mu_1\cdots\mu_k}(x) - A^{(n)\nu}_{\nu\mu_1\cdots\mu_k}(x) - A^{(n-2)}_{\mu_1\cdots\mu_k}(x) = 0, \quad k \geq 1. \tag{4.70}$$

We have not yet investigated these relations in more detail but we notice that all spins are connected to one another which is specially clear in the last equation. Thus, one has an infinite tower of states with different spins connected to each other. This is quite different from other higher spin theories where the wave equations for different spins are uncoupled. For the infinite spin particles there is no way to covariantly separate a specific spin. One might analyze these equation and try to find solutions by for example imposing different conditions, but since we do not have a clear interpretation of the wave functions (4.63) we do not speculate on these issues further. Actually, the wave functions (4.63) are quite peculiar due to the presence of the inverse einbein e which in a way acts as an extra dimension. We believe that an interpretation of the dynamical einbein is important to understand in order to analyze the infinite spin particles further.

4.8 Remarks

Here we make some remarks on the behavior of the infinite spin particle and how it is related to other particle models. A description of the gauge structure in configuration space is obtained when rewritten in a first-order formulation. We also discuss how one may find this Ξ-representation from the rigid particle with curvature. In doing this we notice that two different kinds of parametrizations yield two completely separate models.

4.8.1 Infinite spin particles in a first order formulation

The higher order formulation of the infinite spin particle (4.20) is in itself very simple and elegant. However, to derive the gauge transformations and the complete gauge structure in a configuration space it is not as useful since the calculations easily get quite involved. We may instead start with a Lagrangian in a first-order formulation were we have both x^μ and ξ^μ variables. This is found from the Lagrangian given in (4.15) by requiring $\lambda_1 = \lambda_3 = 0$, $\lambda_2 = \lambda$,

$\lambda_4 = \frac{1}{e}$

$$L' = e\dot{x} \cdot \dot{\xi} + \frac{1}{e}\Xi - \frac{1}{2}\lambda(\xi^2 - 1). \tag{4.71}$$

This is exactly the classical model that Mourad starts from when he considers a string theory generalization in [61,62]. Since the action (4.71) is of first order, the procedure to obtain the gauge structure is straightforward as in the case of the massless relativistic point particle. From (4.71) we find the two primary constraints $p_e = \frac{\partial L'}{\partial \dot{e}} = 0$, $p_\lambda = \frac{\partial L'}{\partial \dot{\lambda}} = 0$ and also the conjugate momenta to x^μ, ξ^μ as $p^\mu = e\dot{\xi}^\mu, \pi^\mu = e\dot{x}^\mu$ respectively. The total Hamiltonian is then given by

$$H = \frac{1}{e}(p \cdot \pi - \Xi) + \frac{1}{2}\lambda(\xi^2 - 1) + p_e + p_\lambda. \tag{4.72}$$

The constraints in (4.6) (where $F = 1$) with Lie algebra $\{\chi_2', \chi_4'\} = \chi_3'$, $\{\chi_3', \chi_4'\} = 2\chi_1'$ are now found by a constraint analysis, since $\dot{p}_e = \frac{1}{e^2}\chi_4'$, $\dot{\chi}_4' = -\lambda\chi_3'$, $\dot{\chi}_3' = \frac{2}{e}\chi_1'$, $\dot{\chi}_1' = 0$ and $\dot{p}_\lambda = -\chi_2'$, $\dot{\chi}_2' = \frac{1}{e}\chi_3'$. Notice that we only have four constraints here, compared to the higher order version were we found five. Here the inverse einbein e does not have the same role as in the higher order version. This first order model is a gauge fixed version of the higher order theory considered in the previous sections.

The gauge generator (2.34) is given with real, even and time dependent parameters $\alpha, \beta, \gamma, \kappa, a, b$

$$G = \alpha\chi_1' + \beta\chi_2' + \gamma\chi_3' + \kappa\chi_4' + ap_e + bp_\lambda. \tag{4.73}$$

Solving the equation in (2.33) yields the conditions $\gamma = -\frac{1}{2}e\dot{\alpha}, b = \dot{\beta}, \beta = e(-\dot{\gamma} + \lambda\kappa), a = -e^2\dot{\kappa}$. Using these relations and the expression for the conjugate momenta above, the gauge transformations (2.35) for the infinite spin particles are given by

$$\begin{aligned}
\delta x^\mu &= \alpha e\dot{\xi}^\mu - \frac{1}{2}\dot{\alpha}e\xi^\mu + \kappa e\dot{x}^\mu, \\
\delta\xi^\mu &= \kappa e\dot{\xi}^\mu, \\
\delta e &= -e^2\dot{\kappa}, \\
\delta\lambda &= \frac{1}{2}\frac{d}{d\tau}(e(\dot{e}\dot{\alpha} + e\ddot{\alpha} + 2\lambda\kappa)).
\end{aligned} \tag{4.74}$$

Hence, the Lagrangian is invariant under these gauge transformations with gauge parameters α, κ (which may be verified by a long and tedious calculation, yielding a total derivative). The commutator of the gauge transformations are $[\delta_1, \delta_2]\rho = \delta_{12}\rho$ where ρ is either x^μ, ξ^μ, e, λ and

$$\alpha_{12} = e(\dot{\alpha}_1\kappa_2 - \dot{\alpha}_2\kappa_1), \qquad \kappa_{12} = 0, \tag{4.75}$$

such that e.g. $[\delta_1, \delta_2]x^\mu = \alpha_{12}e\dot{\xi}^\mu - \frac{1}{2}\dot{\alpha}_{12}e\xi^\mu$. Comparing the gauge transformations above in (4.74) with the ones for the massless relativistic point particle in (3.19) we notice that e has the role of the inverse einbein $v = \frac{1}{e}$. Actually, if we disregard the gauge parameter α and the multiplier λ we obtain the same gauge transformations, i.e. $\delta x^\mu = \kappa\dot{x}^\mu$, $\delta v = \dot{\kappa}$.

4.8.2 Tachyonic behavior

It is interesting to see what kind of behavior the higher order Lagrangian (4.17) represents. The constraints were found to be the ones given in (4.25). Let us focus on the first and the fourth constraint and write them in terms of vectors

$$\begin{aligned} |p_i| &= p_0, \\ |p_i||\xi^i|\cos\theta - p_0\xi_0 &= \frac{1}{e}\Xi \end{aligned} \tag{4.76}$$

where $\xi^\mu = (\xi^0, \xi^i)$, $p_\mu = (p_0, p_i)$ and $i = 1, 2, 3$. $|p_i|$ denotes the length of the vector p_i and θ is the angle between the two vectors p_i and ξ^i . Combining these two constraints we find an expression for the speed (remember that $\xi^\mu := \dot{x}^\mu$)

$$v = \frac{|\xi^i|}{\xi_0} = \frac{1}{\cos\theta}(1 + \frac{\Xi}{e|p_i|\xi_0}). \tag{4.77}$$

Assuming that e, ξ_0 are positive, the sign on Ξ determines if the particle has a tachyonic behavior or not. There are three different situations we should study;

$\Xi > 0$ *Tachyonic behavior*

If Ξ is positive the velocity of the particle is always larger than the speed of light, i.e. the particle has a tachyonic behavior.

$\Xi = 0$ *Tachyonic behavior*

If Ξ is zero, the velocity is either greater or equal to the speed of light $v \geq 1$. When the equality holds, the angle $\cos\theta = 1$, i.e. the vectors p_i and ξ^i are parallel. Since the momentum p_μ is lightlike this implies that ξ^μ is lightlike. But the second constraint in (4.25) implies that π_μ is spacelike which contradicts the fact that ξ^μ is lightlike, since π_μ is a linear combination of ξ^μ and $\dot{\xi}^\mu$. Hence, the only possibility is a tachyonic behavior.

Here a subtlety arise, since the minimal construction when $\Xi = 0$ consists only of the constraints χ_1, χ_3 and χ_4 these can be written in the form

$$\dot{\xi}^2 = 0, \qquad \dot{\xi}\cdot\xi = 0, \qquad \dot{\xi}\cdot\dot{x} = 0 \tag{4.78}$$

where now ξ^μ is not necessarily equal to $\dot{x}^\mu$. The velocity of the particle is either equal to or greater than the speed of light since $v = \frac{|\dot{x}_i|}{x_0} = \frac{1}{\cos\theta}$, where θ is the angle between the vectors $\dot{x}^i$ and ξ^i. When adding the constraint χ_2 (which is possible to do) the part where $v = 1$ is ruled out due to a contradiction. For $v = 1$, the vectors ξ^i and $\dot{x}^i$ are lightlike and parallel. This contradicts that π^μ is spacelike (which is inherent in the constraint χ_2) since it is a linear combination of $\dot{\xi}^\mu$, $\dot{x}^\mu$ and ξ^μ which all are lightlike. Thus the model with $\Xi = 0$ describes a tachyon.

$\Xi < 0$ *Possible non-tachyonic behavior*

If the constant Ξ is negative there might be situations were we have non-tachyonic behavior, i.e. velocities less than c, depending on the angle between the two vectors p_i and ξ^i.

4.8.3 The curvature action

In our construction we have found a higher order Lagrangian starting from Wigner's Ξ-representation in four spacetime dimensions. To accomplish this we had to do quite a lot of intricate calculations and one may ask whether there is a simpler way to obtain the same action starting from something that we already are familiar with. Let us consider a term that is proportional to the curvature of its world-line trajectory in analogy with the relativistic string with rigidity [63–65]

$$S = \int ds \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \Xi}. \tag{4.79}$$

Here $k^2 = (\frac{d^2x}{ds^2})^2$ is the curvature and Ξ is a constant. Now introduce the time τ and an einbein v with its inverse $e = v^{-1}$ such that $ds = vd\tau$ and the curvature $k^2 = e^2(e\ddot{x} + \dot{e}\dot{x})^2$. This implies the action

$$S = \int d\tau \sqrt{(e\ddot{x} + \dot{e}\dot{x})^2 + \frac{1}{e}\Xi} \tag{4.80}$$

which is exactly the one considered before in (4.17). Hence it reproduces all the constraints in (4.25) and Wigner's Ξ-representation, i.e. the infinite spin particles. Notice that by choosing $e = 1$ we obtain the particle model considered by Zoller [66].

It is interesting to see what happens if we instead parametrize the particle's world line trajectory with time τ and $x^\mu = x^\mu(\tau)$ such that $ds^2 = -\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu(d\tau)^2$. A simple calculation now shows that in this parametrization, the action (4.79) with the curvature of the world-line trajectory is given by

$$S = \int d\tau \frac{1}{\dot{x}^2}\sqrt{\ddot{x}^2\dot{x}^2 - (\ddot{x}\cdot\dot{x})^2} + \sqrt{-\dot{x}^2}\Xi. \tag{4.81}$$

This action was considered as a generalization of the point particle action in e.g. [67–70]. Let us now perform the Ostrogradski procedure on this action. We define $\xi^\mu = \dot{x}^\mu$ and eliminate the time derivatives by the introduction of the canonical momenta

$$\pi_\mu = \frac{\partial L}{\partial \dot{\xi}^\mu} = \frac{1}{\xi^2}\frac{\dot{\xi}_\mu \xi^2 - \xi_\mu(\dot{\xi}\cdot\xi)}{\sqrt{\dot{\xi}^2\xi^2 - (\dot{\xi}\cdot\xi)}}. \tag{4.82}$$

Hence, we find the Hamiltonian

$$H = p\cdot\xi - \sqrt{-\xi^2}\Xi. \tag{4.83}$$

In addition, from the definition of the canonical momenta (4.82) we also find the primary constraints (the numbering is chosen in analogy with the constraints in (4.25))

$$\begin{aligned}\chi_2 &= \frac{1}{2}(\pi^2 - \frac{1}{\xi^2}),\\ \chi_5 &= \pi\cdot\xi\end{aligned} \tag{4.84}$$

and the total Hamiltonian

$$H_{\text{tot}} = p\cdot\xi - \sqrt{-\xi^2}\Xi + \lambda_2\chi_2 + \lambda_5\chi_5. \tag{4.85}$$

The consistency conditions now give us the secondary constraints

$$\begin{aligned}\chi_3 &= \pi\cdot p,\\ \chi_4 &= p\cdot\xi + \sqrt{-\xi^2}\Xi\end{aligned} \tag{4.86}$$

and the tertiary constraint

$$\chi_1 = \frac{1}{2}(p^2 - \Xi^2). \tag{4.87}$$

So even though we started with a seemingly massless relativistic particle, the constraint χ_1 for this specific parametrization does suggest that it really is massive. On the other hand, since we have a term $\sqrt{-\dot{x}^2}\Xi$ in the action (4.81) it is not surprising, since this is just the action for a massive relativistic particle. Hence, this parametrization describes a relativistic particle with mass Ξ and with an extra curvature term k added to the action. Using the einbein as above we instead find the infinite spin particle.

Notice also that when Ξ is a non-zero constant, χ_2 and χ_3 are second class constraints with $[\chi_2, \chi_3] = \Xi\sqrt{-\xi^2}/\xi^4$ such that the degrees of freedom are four. However, if $\Xi = 0$ we only have three degrees of freedom since then all constraints are of first class.

So a natural question to ask now is whether we actually can parametrize the object by $x^\mu = x^\mu(\tau)$. It has been shown that such a parametrization of the curvature action (4.79) describe tachyons. Hence we should not choose this parametrization because time loses its meaning and the last term in (4.81) ceases to be real.

4.9 Generalizations

4.9.1 Infinite spin particles in higher dimensions

When classifying the irreducible representations of the Poincaré group we considered only four spacetime dimensions. A natural generalization is to allow for higher spacetime dimensions. In general dimensions the Poincaré invariants are the Casimir operators p^2 and $w_{\mu_1\dots\mu_n}w^{\mu_1\dots\mu_n}$ where the Pauli-Lubanski n-forms in d spacetime dimensions are defined as

$$w_{\mu_1\dots\mu_n} = \frac{\epsilon_{\mu_1\dots\mu_n\mu_{n+1}\dots\mu_d}p^{\mu_d}m^{\mu_{n+1}\mu_{n+2}}m^{\mu_{n+3}\mu_{n+4}}\dots m^{\mu_{d-2}\mu_{d-1}}}{\sqrt{n!\,2^{\frac{d-n+1}{2}}(\frac{d-n-1}{2})(\frac{d-n-1}{2})!}} \tag{4.88}$$

where $n = 1, 3, \dots, (d-3)$ for even dimensions and $n = 0, 2, \dots, (d-3)$ for odd dimensions. Hence, there are naturally more possible Poincaré invariants in higher spacetime dimensions. Notice for example that in four spacetime dimensions the only possible Casimir operators are p^2 and $w_\mu w^\mu$. In the case of $d = 5$ we obtain the invariants p^2, ww and $w_{\mu\nu}w^{\mu\nu}$ etc. The higher dimensional analogue of the infinite spin particles can now be constructed in the same way as described before.

4.9.2 String theory generalization

In Paper I we consider a string theory generalization of the infinite spin particle model. In d spacetime dimensions the action (4.17) can be written as

$$S = \int d^m\zeta\Big(\sqrt{h}\sqrt{(\triangle(h)X^\mu)^2} + \Xi\sqrt{h}\Big), \quad h = \det h_{ab}, \tag{4.89}$$

where ζ^a coordinatize the manifold with metric h_{ab}. $\triangle(h)$ is the Laplace-Beltrami operator

$$\triangle(h) = \frac{1}{\sqrt{h}}\partial_a\sqrt{h}h^{ab}\partial_b \tag{4.90}$$

where $\partial_a := \frac{\partial}{\partial\zeta^a}$ and $a, b = 1, 2, \dots, m$ are world sheet coordinates parametrized by the metric h^{ab}. X^μ ($\mu = 0, 1, \dots, d-1$) describes the location of the string in spacetime. For $\Xi = 0$ and $m = 2$ (4.89) is exactly the model B considered by Savvidy [71], see also [72–74].

In order to see that this really reduces to (4.17) we need to write the metric h_{ab} in terms of vielbeins. Remember that in four dimensions we may write the metric in terms of vierbeins as

$$h_{ab} = \eta_{\mu\nu}v_a^\mu v_b^\nu, \tag{4.91}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, such that for a point particle $(a, b = 1)$

$$\sqrt{h} = v, \quad h^{ab} = v^{-2}. \tag{4.92}$$

Since we use the inverse einbein $e = \frac{1}{v}$ the expression (4.89) reduces to (4.17) if $m = 1$.

The generalization of the action (4.71) to a two-dimensional world-sheet is proposed to be of the form [61, 62]

$$S = \tilde{\Xi} \int d^2\sigma \sqrt{h}(h^{ab}\partial_a X^\mu \partial_b Y^\nu \eta_{\mu\nu} + \lambda(Y^2 - 1)), \tag{4.93}$$

where Y^μ is introduced as an auxiliary field to reproduce the action (4.89) (with $\Xi = 0$). The action proposed by Savvidy is said to be found by expressing Y^μ in terms of X^μ with the use of the constraint relations and equations of motion. Mourad claims that in a BRST quantization, the ground level state carries an infinite spin particle representation and that the spectrum is ghost free [62], contrary to Savvidy's results.

It would be interesting to study the classical string action in more detail with the use of the Ostrogradski method. However, it seems like the constraint algebra for such a model easily gets quite involved and the procedure to obtain secondary, tertiary, etc. constraints does not seem to end.

4.10 Non-covariant treatments

There have been several non-covariant treatments of Wigner's Ξ-representation which indicates possible problems of the corresponding models. Firstly, we have the treatments done by Wigner and Bargmann themselves [31, 32] where they find that this representation contains infinite heat capacity. In [38, 39] it is shown that the infinite spin particles have non-causal behavior and allow negative norm states.

Wigner's Ξ-representation has also been studied in more detail in the light-cone gauge [40, 41, 75].

4.10.1 Infinite spin particles in the light-cone gauge

By introducing light-cone coordinates the Poincaré generators may be more conveniently expressed. Let the only non vanishing commutators be

$$\begin{aligned} &[x^i, p^j] = i\delta^{ij}, \quad [x^-, p^+] = [x^+, p^-] = -i, \\ &[\xi^i, \pi^j] = i\delta^{ij}, \quad [\xi^-, \pi^+] = [\xi^+, p^-] = -i, \end{aligned} \tag{4.94}$$

with $i, j = 1, 2$ and

$$\begin{aligned} x^{\pm} &= \frac{1}{\sqrt{2}}(x^0 \pm x^3), \qquad \xi^{\pm} = \frac{1}{\sqrt{2}}(\xi^0 \pm \xi^3), \\ p^{\pm} &= \frac{1}{\sqrt{2}}(p^0 \pm p^3), \qquad \pi^{\pm} = \frac{1}{\sqrt{2}}(\pi^0 \pm \pi^3), \end{aligned} \tag{4.95}$$

such that

$$x \cdot p = x_\mu p^\mu = x^1 p^1 + x^2 p^2 - x^+ p^- - x^- p^+. \tag{4.96}$$

With proper gauge choices we can now try to eliminate the constraint in (4.5). We have four different constraint so we need four pairs of canonical gauge choices. Consider first the constraints χ_1, χ_3, χ_4 in (4.5) (with $F = 1$) which makes the following gauge choices possible

$$\begin{aligned} \chi_1: &\qquad x^+ = \tau, \qquad p^- = \frac{p^i p^i}{2p^+}, \\ \chi_3: &\qquad \xi^+ = 0, \qquad \pi^- = \frac{p^i \pi^i}{p^+}, \\ \chi_4: &\qquad \pi^+ = 0, \qquad \xi^- = \frac{p^i \xi^i - \Xi}{p^+}. \end{aligned} \tag{4.97}$$

The generators of the Poincaré algebra are now $P^i = p^i$, $P^- = \frac{p^i \cdot p^i}{2p^+}$ and $P^+ = p^+$ such that $m^{\mu\nu}$ in (3.3) can be expressed in terms of

$$\begin{aligned} l^{ij} &= \epsilon^{ij}(x^1 p^2 - x^2 p^1) = \epsilon^{ij} l, & s^{ij} &= \epsilon^{ij}(\xi^1 \pi^2 - \xi^2 \pi^1) = \epsilon^{ij} s, \\ l^{i+} &= x^i p^+, & s^{i+} &= 0, \\ l^{-+} &= \frac{1}{2}(x^- p^+ + p^+ x^-), & s^{-+} &= 0, \\ l^{-i} &= x^- p^i - \frac{1}{4p^+}(x^i p^j p^j + p^j p^j x^i), & s^{-i} &= \frac{-1}{p^+}(\Xi \pi^i + \epsilon^{ij} p^j s), \end{aligned} \tag{4.98}$$

such that $m^{\mu\nu}$ has the non-vanishing components

$$m^{ij} = \epsilon^{ij} l + \epsilon^{ij} s. \tag{4.99}$$

But it remains to eliminate the second constraint in (4.5), $\chi_2 = \pi^i \pi^i - 1 = 0$ which is not as easy as the other ones since it is quadratic in π. But it does look like a particle moving on a circle. So let us choose

$$\begin{aligned} \pi^1 &= \cos\theta, \\ \pi^2 &= \sin\theta \end{aligned} \tag{4.100}$$

and

$$S = -i \frac{\partial}{\partial \theta}, \tag{4.101}$$

such that the Poincaré generators remain valid.

5

THE BATALIN-VILKOVISKY FORMALISM

So far in this thesis we have considered how to obtain relativistic particle models starting from different representations of the Poincaré group. We have then considered particle theories mainly in the Hamiltonian formalism. In this chapter, which can be seen as the beginning of the second part of this thesis, we have a different approach to obtain gauge theories. Starting from an established quantization procedure we generate possible consistent gauge field theories. Before we can show how this can be done we first present the formalism in more detail.

To construct a quantum theory one normally starts with a regular classical theory and turn it into a quantum mechanical counterpart. This quantization procedure is usually rather straightforward. However, this is not the case for gauge theories which are singular and therefore requires a gauge fixing procedure. This in turn breaks the gauge invariance and to compensate for this and preserve unitarity, ghost fields have to be introduced. What is left of the local gauge invariance after gauge fixing is the global BRST invariance [6,7].

The BRST method can be formulated within a path integral formalism both in a Lagrangian or a Hamiltonian framework. In the Lagrangian version the Batalin-Vilkovisky (BV) formalism [8–11,76–78] is the most general one formulated in a configuration space. A general method in the Hamiltonian framework is the Batalin-Fradkin-Vilkovisky (BFV) method [3–5]. The BV and BFV methods should give equivalent results and this have been shown for different simple models [79–82].

The purpose of the BV formalism is to quantize a general gauge theory in a simple and covariant way directly in the configuration space. The BFV method on the other hand is formulated in a phase space and has the disadvantage of not being covariantly formulated for field theories. Even though the BV formalism requires quite a lot of mathematical machinery, the procedure is straightforward. For lucid reviews of the BV formalism, see e.g. [21,23–25,83].

The BV formalism introduces some new conceptual ingredients such as **antifields** and is therefore often called the field-antifield formalism. One antifield is introduced to every field, ghost field and ghost for ghost etc, which doubles the number of fields. These antifields have opposite statistics (Grassmann parities) to the corresponding fields and can be seen as sources of BRST transformations. One also introduces an **antibracket** on the space of fields and antifields. This antibracket is analogous to the Poisson bracket in the Hamiltonian framework. In the BV formalism the theory is formulated in terms of a **master action**. The master action has a manifest BRST symmetry if it satisfies the **master equation**. By requiring the master action to satisfy the master equation the gauge structure of the theory is determined such that it represents a consistent gauge field theory. The only thing that remains is gauge fixing.

The treatments in Paper II and Paper III are based on the BV formalism. Even though this is done in a superfield formulation the main concepts are of course the same as the ones given in this chapter. Here we briefly introduce the BV formalism in analogy with the conventional Hamiltonian formalism. We consider the gauge structure of classical field theories and thereafter turn to the master equation and the properties of the antibracket. To connect with the two previous chapters and to exemplify the BV formalism we construct master actions for the massless relativistic point particle, the spinning relativistic particle and the infinite spin particle. This chapter ends by considering the quantum description of the BV formalism.

5.1 Gauge structure in classical field theories

Consider a bosonic gauge field theory that is invariant under the gauge transformations

$$\delta_\varepsilon \phi^i = R^i_\alpha \varepsilon^\alpha. \tag{5.1}$$

The parameters ε^α are depending on spacetime and represent local gauge symmetries. Here we use the condensed notation introduced by DeWitt [84]. This means that repeated indices should be summed over and integrated over spacetime, i.e. $\delta\phi^i(u) = \sum_\alpha \int du' R^i{}_\alpha(u,u')\varepsilon^\alpha(u')$. R^i is the generator of the transformation and ε^α a gauge parameter.

Varying the original action functional $S[\phi^i]$ we find that

$$\delta_\varepsilon S = \frac{\delta S}{\delta \phi^i}\delta_\varepsilon \phi^i = 0, \tag{5.2}$$

which gives rise to the Noether identities

$$\frac{\delta S}{\delta \phi^i} R^i_\alpha = 0. \tag{5.3}$$

Since there are gauge transformations that do not change the physical states, the solutions to the equations of motion are not uniquely defined. This was discussed earlier in the case of particle theories and the singular property of the Lagrangian. For field theories this means that the Hessian has zero modes on the stationary surface of solutions to the equations of motion

$$\frac{\delta^2 S}{\delta \phi^i \delta \phi^j} R^i_\alpha \varepsilon^\alpha = 0. \tag{5.4}$$

This equation now implies that there are redundancies in the theory, i.e. the gauge transformations map a classical solution to another equivalent classical solution.

Let us assume a complete set of gauge generators R^i for the Noether identity (5.3). A general solution to $\frac{\delta S}{\delta \phi^i} X^i = 0$ should be such that we may add a term proportional to the equations of motion. This is called the completeness relation and is a result of the theory being a regular theory, i.e. the non-invertibility of the Hessian comes solely from the gauge generators R^i. This is discussed in [23] and in detail in [11]. Let us write the solution to $\frac{\delta S}{\delta \phi^i} X^i$ as

$$X^i = R^i_\alpha \varepsilon^\alpha + \frac{\delta S}{\delta \phi^j} M^{ij}, \tag{5.5}$$

where M^{ij} is antisymmetric in the indices i, j, such that the last term vanishes when multiplied by $\frac{\delta S}{\delta \phi^i}$. Consider a commutator of two gauge transformations $[\delta_1, \delta_2]\phi$ which is also a gauge transformation. It therefore satisfies the Noether identity (where the gauge parameters are left out)

$$\frac{\delta S}{\delta \phi^i} \left[\frac{\delta R^i_\alpha}{\delta \phi^j} R^j_\beta - R^j_\alpha \frac{\delta R^i_\beta}{\delta \phi^j} \right] = 0. \tag{5.6}$$

In the same way as in (5.5), the general solution to this equation should be of the form

$$\frac{\delta R^i_\alpha}{\delta \phi^j} R^j_\beta - R^j_\alpha \frac{\delta R^i_\beta}{\delta \phi^j} = R^i_\gamma f^\gamma{}_{\alpha\beta} + \frac{\delta S}{\delta \phi^j} h^{ij}_{\alpha\beta}. \tag{5.7}$$

Depending on the properties of the quantities $f^\gamma{}_{\alpha\beta}(\phi)$ and $h^{ij}_{\alpha\beta}(\phi)$ these represent different kinds of algebras. First assume that the algebra is closed, i.e. $h^{ij}_{\alpha\beta} = 0$, then we have a Lie algebra with structure coefficients $f^\gamma{}_{\alpha\beta}$ if they are constants (if they are zero we are dealing with an abelian algebra). If $f^{\gamma}{}_{\alpha\beta}$ depends on the fields it is called a soft algebra. As soon as $h^{ij}_{\alpha\beta} \neq 0$ we have an open algebra, that only closes on-shell, i.e. on the level of the equations of motion.

An expression of the Jacobi identity $[[\delta_\alpha, \delta_\beta], \delta_\gamma]\phi + cyclic(\alpha, \beta, \gamma) = 0$ can be written in terms of the above quantities by an analogous construction. The details of this and how one may transform open algebras to closed ones can be found in [10, 11].

5.2 The master equation

Assume that we have a gauge field theory described by an action $\mathcal{S}_0[\phi^i]$ which is BRST invariant under the transformation

$$\delta_B \phi^i = R^i_\alpha c^\alpha. \tag{5.8}$$

Here we have introduced a ghost field c^α with ghost number one, to each generator R^i_α. This implies that the gauge parameters ε^α in (5.1) have been replaced by the odd ghost fields c^α. Hence, we have replaced the gauge transformation by a BRST transformation. If the considered theory is irreducible, i.e. the BRST transformations are independent, there is no need to introduce any further ghost fields. On the other hand, if the set of R^i_α's are not linearly independent there is an invariance of the ghost

$$\delta_B c^\alpha = Z^\alpha_\beta d^\beta, \tag{5.9}$$

where the term on the right hand side has ghost number two, e.g. given by $f^{\alpha}{}_{\beta\gamma} c^\beta c^\gamma$. When d^β carries ghost number two we have introduced a ghost for ghost . If the ghost field d^α possess a gauge invariance there is a further set of ghost for ghost fields to be implemented. Notice that we in this thesis mainly consider irreducible gauge theories and not reducible ones, i.e. no ghost for ghosts are needed.

The Batalin-Vilkovisky (BV) and the classical Hamiltonian formalism have some similar properties. The Hamiltonian formalism is defined within phase space with coordinates x^i, p_i. In this framework, an important role is played by the Poisson bracket defined in (2.20) as

$$\{A, B\} := \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x^i} \qquad \text{and} \qquad \{x^i, p_j\} = \delta^i_j, \tag{5.10}$$

where $A(x^i, p_i)$ and $B(x^i, p_i)$ are phase space functions. The time evolution of a phase space function $A(x^i, p_i)$ is given by $\frac{dA}{dt} = \dot{A} = \{A, H\}$ such that the Hamiltonian can be seen as the generator of time translations. With a gauge generator $G(x^i, p_i)$ (introduced in (2.32)), the gauge transformations (2.35) are given by

$$\delta_\epsilon A = \{A, G\}. \tag{5.11}$$

The Poisson bracket has the property $\{A, B\} = -\{B, A\}$ which implies that we have the identity $\dot{H} = \{H, H\} = 0$ for a Hamiltonian without explicit time dependence.

Having this in mind, we consider now instead a field theory within the BV formalism. This is a field theory formulated in the Lagrangian formalism with a set of **fields** $\phi^P(u)$ and **antifields** $\phi^*_P(u)$, over the set of bosonic coordinates u. The latter are called antifields since they have the opposite statistics (Grassmann parities) to the fields. The fields $\phi^P(u)$ denote *all* fields, ghost fields, ghosts for ghosts etc. and $\phi^*_P(u)$ *all* corresponding antifields to these. To each field, ghost field, ghost for ghosts etc. and their respective antifields (ϕ^*_P), we assign **Grassmann parities** ($\epsilon(\phi^P)$, $\epsilon(\phi^*_P)$) and **ghost numbers** ($gh_\#\phi^P$, $gh_\#\phi^*_P$). The ghost number is a grading of the algebra where the physical fields have ghost number zero. Grassmann parity is a $\mathbb{Z}_2$ grading of the algebra with $FG = (-1)^{\epsilon(F)\epsilon(G)}GF$, where $\epsilon(X) = 0, 1(\text{mod } 2)$ denotes the Grassmann parity of X. For a composition of two terms we have $\epsilon(FG) = \epsilon(F) + \epsilon(G)$ and due to the $\mathbb{Z}_2$ grading we also have that $\epsilon(F)\epsilon(F) = \epsilon(F)$. The ghost number and Grassmann parity should be conserved. The field and antifield have the ghost number relation

$$gh_\#\phi^*_P + gh_\#\phi^P = -1 \tag{5.12}$$

and the Grassmann parity relation

$$\epsilon(\phi^*_P) + \epsilon(\phi^P) = 1. \tag{5.13}$$

Hence, the field and antifield have opposite statistics.

We also introduce an **antibracket**, here given for two functionals $\mathcal{A}[\phi^P, \phi^*_P]$, $\mathcal{B}[\phi^P, \phi^*_P]$ by

$$(\mathcal{A}, \mathcal{B}) = \mathcal{A}\frac{\overleftarrow{\delta}}{\delta\phi^P}\frac{\overrightarrow{\delta}}{\delta\phi^*_P}\mathcal{B} - (-1)^{(\epsilon(\mathcal{A})+1)(\epsilon(\mathcal{B})+1)}\mathcal{B}\frac{\overleftarrow{\delta}}{\delta\phi^P}\frac{\overrightarrow{\delta}}{\delta\phi^*_P}\mathcal{A}. \tag{5.14}$$

The functional derivatives are so called right and left derivatives, this distinction is necessary since the fields and antifields have different Grassmann parities, i.e. they are either bosonic (with Grassmann parity zero, i.e. Grassmann even) or fermionic (with Grassmann parity one, i.e. Grassmann odd). The left and right functional derivatives are related as

$$\frac{\overrightarrow{\delta}}{\delta\phi^i}\mathcal{A} = (-1)^{\epsilon(\phi)(\epsilon(\mathcal{A})+1)}\mathcal{A}\frac{\overleftarrow{\delta}}{\delta\phi^i}. \tag{5.15}$$

From the definition of the antibracket (5.14) it follows that

$$(\phi^P(u), \phi^*_K(u')) = \delta^P_K \delta(u-u'), \tag{5.16}$$

such that the antifields play the role of conjugates to the fields in analogy with the phase space coordinates in the Poisson bracket (5.10). Since we defined the fields and antifields to have opposite statistics, the antibracket will not have the same properties as the Poisson bracket. The antibracket has for instance the symmetry property $(\mathcal{E}, \mathcal{F}) = (\mathcal{E}, \mathcal{F})$ for bosonic functionals $\mathcal{E}$ and $\mathcal{F}$, which implies that $(\mathcal{E}, \mathcal{E}) \neq 0$ in general. We should already here stress the importance of this symmetry property of the antibracket in the BV formalism. More generally, we see from the relations (5.12), (5.13) and (5.16) that the antibracket $(\cdot,\cdot)$ carries ghost number one and has Grassmann parity one, i.e.

(i) Grassmann parity

$$\epsilon[(\mathcal{A}, \mathcal{B})] = \epsilon(\mathcal{A}) + \epsilon(\mathcal{B}) + 1, \tag{5.17}$$

(ii) Ghost number

$$gh_{\#}[(\mathcal{A}, \mathcal{B})] = gh_{\#}(\mathcal{A}) + gh_{\#}(\mathcal{B}) + 1. \tag{5.18}$$

The antibracket (5.14) is graded symmetric and obeys the graded Leibniz rule and the graded Jacobi identity

(iii) Graded symmetric

$$(\mathcal{A}, \mathcal{B}) = -(-1)^{(\epsilon(\mathcal{A})+1)(\epsilon(\mathcal{B})+1)} (\mathcal{B}, \mathcal{A}), \tag{5.19}$$

(iv) Graded Leibniz rule

$$\begin{aligned} (\mathcal{A}, \mathcal{B}\mathcal{C}) &= (\mathcal{A}, \mathcal{B})\mathcal{C} + \mathcal{B}(\mathcal{A}, \mathcal{C})(-1)^{\epsilon(\mathcal{B})\epsilon(\mathcal{C})}, \\ (\mathcal{A}\mathcal{B}, \mathcal{C}) &= (\mathcal{A}, \mathcal{C})\mathcal{B} + \mathcal{A}(\mathcal{B}, \mathcal{C})(-1)^{\epsilon(\mathcal{A})\epsilon(\mathcal{B})}, \end{aligned} \tag{5.20}$$

(v) Graded Jacobi identity

$$(\mathcal{A}, (\mathcal{B}, \mathcal{C}))(-1)^{(\epsilon(\mathcal{A})+1)(\epsilon(\mathcal{C})+1)} + \text{cycl}(\mathcal{A}, \mathcal{B}, \mathcal{C}) = 0. \tag{5.21}$$

Consider now a **master action** which is an even functional $\mathcal{S}[\phi^P, \phi^*_P]$ defined on the space of fields and antifields. Let this master action generate the gauge transformation of any functional $\mathcal{A}$

$$\delta\mathcal{A} = (\mathcal{S}, \mathcal{A}). \tag{5.22}$$

This may be compared with the expression for the gauge transformation in the Hamiltonian framework (5.11)[1]. The antibracket for the bosonic functional $\mathcal{S}$ is

$$\frac{1}{2}(\mathcal{S},\mathcal{S}) = \mathcal{S}\frac{\overleftarrow{\delta}}{\delta\phi^P}\frac{\overrightarrow{\delta}}{\delta\phi^*_P}\mathcal{S}. \tag{5.23}$$

From (5.22) it now follows that the master action $\mathcal{S}$ is gauge invariant if $\delta\mathcal{S} = (\mathcal{S},\mathcal{S}) = 0$, which will be required. Since $(\mathcal{A},\mathcal{A})$ in general is different from zero this relation is not trivially true. The equation

$$(\mathcal{S},\mathcal{S}) = 0 \tag{5.24}$$

is the **master equation**, the cornerstone of the BV formalism. The master action $\mathcal{S}$ is required to have the properties

$$\mathcal{S}[\phi,\phi^*]\big|_{\phi^*=0} = \mathcal{S}_0[\phi], \tag{5.25}$$

which implies the Grassmann parity and ghost number

$$\epsilon(\mathcal{S}) = 0 \qquad \text{and} \qquad gh_{\#}(\mathcal{S}) = 0, \tag{5.26}$$

i.e. the master action $\mathcal{S}[\phi,\phi^*]$ is Grassmann even which then means that the master equation (5.24) is not trivially satisfied.

Thus, the analogy between the classical Hamiltonian formalism and the BV formalism is the following: the Hamiltonian $H(q^i,p_i)$ and the Poisson bracket $\{\cdot,\cdot\}$ in the Hamiltonian formalism corresponds to the master action $\mathcal{S}[\phi^P,\phi^*_P]$ and the antibracket $(\cdot,\cdot)$ in the BV formalism.

5.2.1 The master action

Let us focus on the classical master equation (5.24) and demonstrate that there exists a master action $\mathcal{S}[\phi^P,\phi^*_P]$ that solves this equation. The solution to the master equation will be seen to reproduce the gauge structure previously discussed.

Consider again the set of fields ϕ^P,ϕ^*_P where P is a collective label for fields, ghosts and ghost for ghosts etc. An ansatz for the master action that solves the master equation (5.24) may be expanded in a power series in the antifields ϕ^*_P. The master action for an irreducible gauge theory is then

$$\begin{aligned}\mathcal{S}[\phi^P,\phi^*_P] \;&=\; \mathcal{S}_0[\phi^i] + \phi^*_i R^i_\alpha c^\alpha + \frac{1}{2}(-1)^{\epsilon(c^\gamma)} c^*_\alpha f^\alpha{}_{\gamma\beta} c^\beta c^\gamma \\ &\quad + \frac{1}{4}(-1)^{\epsilon(\phi^i)+\epsilon(c^\beta)} \phi^*_i \phi^*_j h^{ji}_{\beta\alpha} c^\alpha c^\beta + \ldots \end{aligned} \tag{5.27}$$

[1]Actually, equation (5.22) generates the BRST transformation (which will be shown later) and should be compared to the BRST transformation in the Hamiltonian formalism with $\delta_Q A = \{A, Q\}$ where Q is the nilpotent BRST charge $\{Q,Q\} = 0$.

where $\mathcal{S}_0$ is assumed to be a BRST invariant action. The factors $\frac{1}{2}$ and $\frac{1}{4}$ are inserted for convenience and ϕ^i, c^α are the fields and ghosts condsidered in (5.8). The dots indicate possible terms if we have a more complicated gauge theory. The general ansatz for the master action can be found in e.g. [23]. Notice that the terms quadratic in the antifields are needed only when we consider open gauge algebras. We also notice that the individual terms in the expansion are required to have vanishing ghost number and that the boundary condition (5.25) is fulfilled.

When solving the master equation we expect to find one equation for each coefficient of the antifields and ghost terms and it can be shown that all conditions on the structure tensors of the gauge theory is found. Actually, solving the master equation shows that the structure coefficients $f^\alpha{}_{\beta\gamma}$, $h^{ij}_{\alpha\beta}$ are exactly the ones considered in the previous section.

As a simple example we consider an irreducible theory with fields ϕ^i and ghosts c^α and a closed algebra with $h^{ij}_{\alpha\beta} = f^\alpha{}_{\beta\gamma} = 0$ in (5.27) such that

$$\mathcal{S}[\phi, \phi^*] = \mathcal{S}_0[\phi] + \phi^*_i R^i_\alpha c^\alpha. \tag{5.28}$$

The master equation (5.24) gives us the familiar equations

$$\begin{aligned} &\frac{\overrightarrow{\delta}\,\mathcal{S}}{\delta\phi^i} R^i_\alpha = 0, \\ &\frac{\overrightarrow{\delta}\,R^i_\alpha}{\delta\phi^j} R^j_\beta - (-1)^{\epsilon_\alpha\epsilon_\beta} R^j_\alpha \frac{\overrightarrow{\delta}\,R^i_\beta}{\delta\phi^j} = 0, \end{aligned} \tag{5.29}$$

where ϵ_α and ϵ_β are the Grassman parities of the ghost fields. Thus, the gauge structure encountered in (5.3) and (5.7) has been shown to be consistent with the master action. This implies that the master action generates the gauge algebra via the master equation.

5.2.2 BRST symmetry

It is also interesting to see how the form of the classical master equation can be understood by demanding that the master action should be invariant under a BRST transformation. Let us consider a BRST invariant even action functional $\mathcal{S}[\phi^P, \phi^*_P]$ of the simple form

$$\mathcal{S}[\phi^P, \phi^*_P] = \mathcal{S}_0[\phi] - \delta_B\phi^P\phi^*_P, \tag{5.30}$$

where $\delta_B\phi^i$ is a BRST transformation. It follows from (5.30) that the BRST variation $\delta_B\phi^P(u) = -\mathcal{S}\frac{\overleftarrow{\delta}}{\delta\phi^*_P(u)} = (-1)^{\epsilon(\phi_P)}\frac{\overrightarrow{\delta}}{\delta\phi^*_P(u)}\mathcal{S}$. This invariance can be written as

$$\delta_B\phi^P(u) = (\mathcal{S}, \phi^P(u)), \tag{5.31}$$

using the odd antibracket introduced in (5.14). Define in a similar way the variation $\delta_B\phi^*_P(u) = \mathcal{S}\frac{\overleftarrow{\delta}}{\delta\phi^P(u)} = (\mathcal{S}, \phi^*_P(u))$. Demanding BRST invariance of the action yields

$$0 = \delta_B\mathcal{S} = \delta_B\phi^P\frac{\overrightarrow{\delta}}{\delta\phi^P}\mathcal{S} + \delta_B\phi^*_P\frac{\overrightarrow{\delta}}{\delta\phi^*_P}\mathcal{S} = 2\mathcal{S}\frac{\overleftarrow{\delta}}{\delta\phi^P}\frac{\overrightarrow{\delta}}{\delta\phi^*_P}\mathcal{S}. \tag{5.32}$$

This is actually nothing but the master equation (compare with (5.23) and (5.24)), i.e. the variation is zero as a consequence of $(\mathcal{S},\mathcal{S}) = 0$. Hence, the master equation may be seen as a statement of invariance of the action $\mathcal{S}$ under BRST transformations, where the antifields are the sources of this transformation. In our simple example in (5.28) we see that $\delta_B\phi^i = R^i_\alpha c^\alpha$ as expected. Earlier we also noticed that the gauge structure is obtained by requiring the master action to satisfy the master equation.

From the Jacobi identity (5.21) we see that the variations

$$\delta_B\phi^P(u) = (\mathcal{S}, \phi^P(u)) \qquad \text{and} \qquad \delta_B\phi^*_i(u) = (\mathcal{S}, \phi^*_P(u)) \tag{5.33}$$

are nilpotent. This is clear since e.g.

$$\delta_B^2\phi^P(u) = (\mathcal{S}, (\mathcal{S}, \phi^P(u))) = -\frac{1}{2}(\phi^P(u), (\mathcal{S},\mathcal{S})) = 0, \tag{5.34}$$

with the use of the master equation (5.24). Actually, for any field $\xi[\phi, \phi^*]$,

$$\delta_B\xi = (\mathcal{S}, \xi) \tag{5.35}$$

which implies $0 = (\xi, (\mathcal{S},\mathcal{S}))$ off-shell. Notice that the master action is (classically) BRST symmetric due to the master equation (5.24). δ_B has also the graded derivation property (5.20) such that δ_B satisfies three important features of a BRST-operator; nilpotency, it is a graded derivation and it leaves $\mathcal{S}$ invariant. A functional $\mathcal{O}$ is an observable (classical) if $\delta_B\mathcal{O} = 0$. In this way the BV-formalism naturally incorporates the BRST symmetry.

However, we are still considering a theory defined on the space of fields and antifields. To obtain a theory described by a master action in terms of just fields we need to consider a gauge fixing procedure. Let us briefly discuss this by considering a gauge fixed action with a gauge fixing term $\delta_B\psi$ in analogy with a BRST construction

$$\mathcal{S}_G[\phi^P] = \mathcal{S}_0[\phi] - \delta_B\psi[\phi^P], \tag{5.36}$$

where ψ is a fermionic gauge fixing functional and we note that $\delta_B\psi = \delta_B\phi^P\frac{\overrightarrow{\delta}\psi}{\delta\phi^P}$. The master action $\mathcal{S}$ in (5.30) should thus be turned into a gauge fixed master action $\mathcal{S}_G$ in (5.36). Comparing (5.30) and (5.36) we find that

$$\mathcal{S}_G[\phi^P] = \mathcal{S}[\phi^P, \phi^*_P]\Big|_{\phi^*_P = \frac{\overrightarrow{\delta}\psi}{\delta\phi^P}} = \mathcal{S}_0[\phi] - \delta_B\phi^P\phi^*_P\Big|_{\phi^*_P = \frac{\overrightarrow{\delta}\psi}{\delta\phi^P}}, \tag{5.37}$$

where ψ makes the last term BRST exact. Thus, in a final path integral the theory should be represented by a gauge fixed master action which is obtained first after eliminating the antifield by the gauge fixing

$$\phi^*_P(u) = \frac{\overrightarrow{\delta}\,\psi}{\delta\phi^P(u)}. \tag{5.38}$$

The choice of the gauge fixing fermion ψ is restricted and must be investigated further, a discussion of this is found in e.g. [23].

5.3 Some examples of master actions

We have so far been rather formal in our description of the Batalin-Vilkovisky formalism. Let us therefore exemplify this procedure by considering the massless relativistic point particle and the spinning relativistic particle. These two models were considered before as examples of irreducible representations of the Poincaré group (see chapter 3). They have also been studied in the BV formalism in [23, 85]. In addition we construct the master action for the first-order formulation of the infinite spin particle model considered in chapter 4.

5.3.1 BV construction of the massless relativistic particle

To construct the master action for the relativistic (spinnless) point particle we introduce a ghost c and 'antifields' to all 'fields' and ghosts with ghost number ($gh_{\#}$) and Grassman parity (ϵ) such that they obey the relations given in (5.12) and (5.13)

	x^μ	x^*_μ	v	v^*	c	c^*
$gh_{\#}$	0	-1	0	-1	1	-2
ϵ	0	1	0	1	1	0

Consider the original Lagrangian in (3.16) with the gauge transformation given by (3.19), such that $R^\mu\varepsilon = \frac{\dot{x}^\mu}{v}\varepsilon$ and $R^v\varepsilon = \frac{d}{d\tau}\varepsilon$, or more specifically from the compact DeWitt notation $R^\mu(\tau,\sigma) = \frac{\dot{x}^\mu(\tau)}{v(\tau)}\delta(\tau-\sigma)$, $R^e(\tau,\sigma) = \frac{d}{d\tau}\delta(\tau-\sigma)$. By the expansion given in (5.27) we find that the master action in the Batalin-Vilkovisky formalism is

$$S = \int d\tau\big(\frac{1}{2v}\dot{x}^2 + x^*_\mu\frac{\dot{x}^\mu}{v}c + v^*\dot{c}\big). \tag{5.39}$$

No more terms are needed since the commutator of the two gauge transformations is zero. If we instead consider the gauge transformations (3.20) we find the master action

$$S' = \int d\tau\big(\frac{1}{2v}\dot{x}^2 + x^*_\mu\dot{x}^\mu c + v^*v\dot{c} + v^*\dot{v}c + c^*\dot{c}c\big). \tag{5.40}$$

The two master actions (5.39) and (5.40) are related by the redefinitions $c \to vc$, $v^* \to v^* - \frac{1}{v}c^*c$ and $c^* \to \frac{1}{v}c^*$. These master actions therefore describe the same theory and it is sufficient to consider only one of them. Let us therefore use the simpler expression in (5.39). The BRST transformations are given by the relation (5.35) such that

$$\begin{aligned} \delta_Q x^\mu &= \frac{\dot{x}^\mu c}{v}, & \delta_Q x^*_\mu &= \frac{d}{d\tau}\left(\frac{\dot{x}_\mu + x^*_\mu c}{v}\right), \\ \delta_Q v &= \dot{c}, & \delta_Q v^* &= \frac{\dot{x}^2 + 2x^*_\mu \dot{x}^\mu c}{2v^2}, \\ \delta_Q c &= 0, & \delta_Q c^* &= \frac{x^*_\mu \dot{x}^\mu}{v} - \dot{v}^*. \end{aligned} \tag{5.41}$$

Notice that these BRST-transformations are valid on the space of fields and antifields and should not be compared with the usual expression for the BRST-transformations. This can first be done when a proper gauge fixing has been performed.

5.3.2 BV construction of the spinning relativistic particle

In chapter 3 we also studied the massless relativistic spin-$\frac{1}{2}$ particle which was found from the Lorentz group by adding an odd hermitian operator ψ^μ. The Lagrangian describing this massless relativistic spinning particle is given by (3.29) (where we now let $\lambda_6 = \lambda$). To the variables $x^\mu, v, \psi^\mu, \lambda$ describing the spinning particle we add two ghosts c, Γ. We also add antifields to all these such that the ghost numbers and Grassman parities of the complete set of fields, ghosts, antifields and antighosts are:

	x^μ	x^*_μ	ψ^μ	ψ^*_μ	v	v^*	λ	λ^*	c	c^*	Γ	Γ^*
$gh_\#$	0	-1	0	-1	0	-1	1	-2	1	-2	1	-2
ϵ	0	1	1	0	0	1	1	0	1	0	0	1

From the derived expressions of the gauge transformations in (3.36) we can now construct the master action. The gauge parameters a, α are replaced by the ghosts c, Γ and due to the non-zero commutators between the gauge transformations (3.37) we need to add corresponding terms in the master action. The master action is given by

$$\begin{aligned} S &= \int d\tau \Big(\frac{1}{2v}(\dot{x}^\mu - i\lambda\psi^\mu)^2 + \frac{i}{2}\psi\cdot\dot{\psi} + \\ &+ x^*_\mu(\dot{x}^\mu c - iv\psi^\mu\Gamma) + \psi^*_\mu(\dot{\psi}^\mu c + (\dot{x}^\mu - i\lambda\psi^\mu)\Gamma) + \\ &+ v^*(\dot{v}c + v\dot{c} - 2iv\lambda\Gamma) + \lambda^*(\dot{v}\Gamma + v\dot{\Gamma} + \dot{\lambda}c + \lambda\dot{c}) + \\ &+ c^*(\dot{c}c + iv\Gamma\Gamma) + \Gamma^*(c\dot{\Gamma} - \dot{c}\Gamma + i\lambda\Gamma\Gamma)\Big) \end{aligned} \tag{5.42}$$

where the first two terms are the original ones in (3.29) and the last two are due to the commutator of the gauge transformations in (3.37). The BRST transformations in the space of fields and antifields are then found from (5.35) by an analogous construction as for the massless relativistic point particle. Notice that the master action above is reduced to the expression for the massless relativistic particle (5.40) when $\psi^\mu, \lambda, \Gamma$ are all zero.

5.3.3 BV construction of the infinite spin particle

Making a BV construction of the higher order Lagrangian (4.17) is difficult since e.g. the gauge structure is quite complicated to find in configuration space and it is not so obvious how to use the BV formalism on higher order theories. But in (4.71) these problems have been circumvented and the construction of the master action is therefore straightforward. Notice that the higher order Lagrangian can be seen as an extended version of this first-order expression (4.71). To the variables $x^\mu, \xi^\mu, e, \lambda$ we add two ghosts c, b and antifields to all these such that the complete set is given by

	x^μ	x^*_μ	ξ^μ	ξ^*_μ	e	e^*	λ	λ^*	c	c^*	b	b^*
$gh_\#$	0	-1	0	-1	0	-1	0	-1	1	-2	1	-2
ϵ	0	1	0	1	0	1	0	1	1	0	1	0

The ghost b, c can be seen as replacing the gauge parameters α, κ, respectively. From the gauge transformations in (4.74) and (4.75) it now follows that the master action for the infinite spin particle is given by

$$\begin{aligned} S &= \int d\tau \Big(e\dot{x}\cdot\xi + \frac{1}{e}\Xi - \frac{1}{2}\lambda(\xi^2-1) + x^*_\mu(e\dot{\xi}^\mu b - \frac{1}{2}e\xi^\mu\dot{b} + e\dot{x}^\mu c) \\ &+ \xi^*_\mu e\dot{\xi}^\mu c - e^* e^2 \dot{c} - \frac{1}{2}\dot{\lambda}^*(e\dot{e}\dot{b} + e^2\ddot{b} + 2e\lambda c) + b^* e\dot{b}c \Big). \end{aligned} \quad (5.43)$$

One of the things that is left is now to find a suitable gauge fixing fermion, but this will not be discussed in this thesis.

5.4 The quantum master equation

So far we have only been considering the classical part of the BV formalism. This analysis gave us a very important equation, namely the (classical) master equation for a (classical) master action. Now we briefly turn to a quantum description and look for a similar expression. Consider a path integral with a gauge fixing term

$$\mathcal{I} = \int \mathcal{D}\phi^P \mathcal{D}\phi^*_P \delta(\phi^*_P - \frac{\overrightarrow{\delta}\,\psi}{\delta\phi^P}) \exp(\frac{i}{\hbar}\mathcal{W}[\phi^P, \phi^*_P]). \quad (5.44)$$

The **quantum master action** $\mathcal{W}$ depends on the fields ϕ^P and antifields ϕ^*_P and its gauge fixed version is equal to $\mathcal{S}_G$ in (5.37) up to quantum corrections

$$\mathcal{W}[\phi^P, \phi^*_P]\Big|_{\phi^*_P = \frac{\vec{\delta}\psi}{\delta\phi^P}} = \mathcal{S}_G + \mathcal{O}(\hbar). \tag{5.45}$$

One can now show that requiring the path integral to be independent of the gauge fixing fermion ψ implies the **quantum master equation**

$$(\mathcal{W}, \mathcal{W}) = 2i\hbar\Delta\mathcal{W}, \tag{5.46}$$

where the operator Δ is defined as

$$\Delta := (-1)^{\epsilon(\phi^P)} \frac{\vec{\delta}}{\delta\phi^P} \frac{\vec{\delta}}{\delta\phi^*_P}. \tag{5.47}$$

Since $\epsilon(\Delta) = 1$, Δ is a nilpotent operator $\Delta^2 = 0$. This operator is the key object in the geometrical understanding of the BV formalism [86–89].

A general quantum master action $\mathcal{W}$ is assumed to be expandable in powers of $\hbar$

$$\mathcal{W} = \mathcal{S} + \sum_{k=0}^{\infty} \hbar^k \mathcal{M}_k. \tag{5.48}$$

The quantum master equation (5.46) should then be satisfied order by order in $\hbar$.

Thus the classical master action $\mathcal{S}$ that was studied before should for quantum theories be replaced by a quantum master action $\mathcal{W}$, which coincide with $\mathcal{S}$ when $\hbar \to 0$. Hence, the classical master equation (5.24) should be replaced by the quantum version (5.46).

For many models, a master action satisfying the classical master equation also satisfies the quantum version assuming an appropriate regularization [90]. Regularization, renormalization, unitarity and locality are important aspects which should be considered in order to obtain a sensible quantum theory. This will not be considered here, but can be found in e.g. [91, 92].

The generalization of the BRST operator δ_B in the classical analysis, is for a quantum model

$$\hat{\delta}_B\mathcal{A} = (\mathcal{W}, \mathcal{A}) - i\hbar\Delta\mathcal{A}. \tag{5.49}$$

Violations of the quantum master equation implies that there are gauge anomalies [93].

6

GAUGE FIELD THEORIES FROM A SUPERFIELD FORMULATION

In the previous chapter we glanced at the structure of the BV formalism. We learned that the master equation was required to be satisfied by the master action, defined in terms of fields, antifields and ghost fields etc. The master equation contains all information about the gauge structure of the theory. Unfortunately, the BV formalism might sometimes seem unnecessary complicated. A more transparent form is found in some cases using a superfield formulation [15, 16, 88, 90, 94–96]. Due to the superfields it is fairly easy to cope with all the different fields, ghost fields and ghosts for ghosts etc since these then are components of one superfield.

In Paper II and Paper III we have chosen to investigate the superfield algorithm introduced by Batalin and Marnelius [15, 16]. They found a way to construct consistent quantum gauge field theories by means of a superfield algorithm which applies to a class of first order gauge field theories. It should be noted that in the superfield algorithm the BV formalism is used as a framework for generating gauge field theories, rather than as a technique for quantizing gauge theories. Possible theories are determined by a ghost number prescription and a simple local master equation. The resulting theories are represented by a master action living on a supermanifold. The original theories before quantization are obtained by a simple reduction procedure and by gauge fixing the master action the quantum theory may be found.

In Paper II we investigate four and six dimensional theories obtained by the superfield algorithm. We use (anti)canonical transformations to solve the master equation and as a tool for investigating canonically equivalent theories.

In Paper II we also discuss some general features concerning theories in various dimensions.

In Paper III we generalize the superfield algorithm to include higher order terms in the interaction part of the master action, since the superfield algorithm in [15,16] only generates first order gauge field theories which all seem to be of a topological nature. We also consider non-dynamical multiplier fields in the models. This helps us generate more general theories, such as a five dimensional Chern-Simons theory. This theory has previously been shown to include local degrees of freedom according to [97,98] which implies that this generalized version of the superfield algorithm might also include non-topological theories.

In this chapter we give an introduction to the superfield algorithm and state some of the results of Paper II and Paper III.

6.1 The superfield algorithm

The master action in n-dimensions may for a class of models be written as a field theory living on a $2n$-dimensional supermanifold $\mathcal{M}$

$$\Sigma[K^P, K^*_P] = \int_{\mathcal{M}} d^n u d^n \tau \mathcal{L}_n(u,\tau), \tag{6.1}$$

where $K^P(u,\tau)$ is a superfield and $K^*_P(u,\tau)$ is an associated superfield. (u^a, τ^a) are coordinates on the supermanifold $\mathcal{M}$, where $a = \{1, ..., n\}$ and u^a denotes the Grassmann even and τ^a the Grassmann odd coordinates. This implies that we have a $\mathbb{Z}_2$-grading on the algebra of the superfields, and where $\epsilon(u) = 0$ and $\epsilon(\tau) = 1$. The next thing to do is to choose a Lagrangian density $\mathcal{L}_n(u,\tau)$ with a kinetic and an interacting part:

$$\mathcal{L}_n(u,\tau) = K^*_P D K^P (-1)^{\epsilon_P + n} - S(K^*_P, K^P). \tag{6.2}$$

This choice of Lagrangian is of the same structure as the ones studied in e.g. [88,96]. The de Rham differential

$$D := \tau^a \frac{\partial}{\partial u^a}, \tag{6.3}$$

is odd and nilpotent, $D^2 = 0$. The nilpotency of this operator allows for a BRST interpretation. It was noticed in [15] that the expression in (6.2) has a similar structure to the string field theory considered by Zwiebach [99]. We will look into this BRST interpretation later on.

The fact that the master action Σ in (6.1) should be physical, $gh_\#(\Sigma) = 0$ and even $\epsilon(\Sigma) = 0$, gives us restrictions for the superfields. With $gh_\#(\tau) = 1$

and $gh_{\#}(u) = 0$ we see that $gh_{\#}(D) = 1$. The Berezin integral $\int d\tau\tau = 1$ implies that the measure $d^n\tau$ in (6.1) carries ghost number $-n$. This together with equations (6.1) and (6.2) now give us the relation

$$gh_{\#}K^P + gh_{\#}K^*_P = n - 1, \tag{6.4}$$

between superfields and associated superfields, with the Grassmann parities

$$\epsilon(K^*_P) = \epsilon_P + n + 1, \qquad \text{where} \qquad \epsilon(K^P) := \epsilon_P. \tag{6.5}$$

As a convention we choose $gh_{\#}K^*_P \geq gh_{\#}K^P$. Note that depending on the dimension n, the superfields K^P and associated superfields K^*_P do not necessarily have opposite statistics which is the case for the fields and antifields discussed in the conventional BV framework. It also follows that for $S(K^*_P, K^P)$,

$$gh_{\#}S = n \qquad \text{and} \qquad \epsilon(S) = n. \tag{6.6}$$

Self-consistency of the BV-formalism requires the master action Σ in (6.1) to obey the master equation[1]

$$(\Sigma, \Sigma) = 0. \tag{6.7}$$

The indicated bracket is the BV-bracket (antibracket) defined for two functionals $\mathcal{A}$ and $\mathcal{B}$ by

$$(\mathcal{A},\mathcal{B}) := \int_{\mathcal{M}} \mathcal{A}\frac{\overleftarrow{\delta}}{\delta K^P(u,\tau)}(-1)^{(\epsilon_P n)} d^n u d^n \tau \frac{\overrightarrow{\delta}}{\delta K^*_P(u,\tau)}\mathcal{B} - (\mathcal{A} \leftrightarrow \mathcal{B})(-1)^{(\epsilon(\mathcal{A})+1)(\epsilon(\mathcal{B})+1)} \tag{6.8}$$

A summation over the index P is assumed. This bracket has the usual properties, i.e. the graded versions of the antisymmetry property, the Leibniz rule and the Jacobi identity for functionals $\mathcal{A}, \mathcal{B}, \mathcal{C}$:

$$(\mathcal{A}, \mathcal{B}) = -(-1)^{(\epsilon(\mathcal{B})+1)(\epsilon(\mathcal{A})+1)}(\mathcal{A}, \mathcal{B}), \tag{6.9}$$

$$(\mathcal{A}, \mathcal{B}\mathcal{C}) = (\mathcal{A}, \mathcal{B})\mathcal{C} + \mathcal{B}(\mathcal{A}, \mathcal{C})(-1)^{\epsilon(\mathcal{B})\epsilon(\mathcal{C})}, \tag{6.10}$$

$$((\mathcal{A}, \mathcal{B}), \mathcal{C})(-1)^{(\epsilon(\mathcal{A})+1)(\epsilon(\mathcal{C})+1)} + \text{cycl}(\mathcal{A}, \mathcal{B}, \mathcal{C}) = 0. \tag{6.11}$$

6.1.1 The n-bracket and the local master equation

By now we have found a way to choose our superfields according to (6.4) and (6.5), but the beauty of the superfield algorithm lies ahead. Solving the master equation (6.7) for the specific choice of Lagrangian density (6.2) in the master action (6.1) yields the local expression

$$(S, S)_n = 0 \tag{6.12}$$

[1]Actually for quantum theories $(\Sigma, \Sigma) = 2i\hbar\Delta\Sigma$, but for actions local in the superfields we have $\Delta\Sigma = 0$. This was shown in [15].

and the boundary condition

$$\int_{\mathcal{M}} d^n u d^n \tau D\mathcal{L} = 0. \tag{6.13}$$

For the general functions $F(K^P(u,\tau), K^*_P(u,\tau))$ and $G(K^P(u,\tau), K^*_P(u,\tau))$, the n-bracket in (6.12) is defined by

$$(F,G)_n = F\frac{\overleftarrow{\partial}}{\partial K^P}\frac{\overrightarrow{\partial}}{\partial K^*_P}G - (F \leftrightarrow G)(-1)^{(\epsilon(F)+n+1)(\epsilon(G)+n+1)}. \tag{6.14}$$

Note that for the superfields and the associated superfields we have in particular

$$(K^{P_1}, K^*_{P_2})_n = \delta^{P_1}{}_{P_2}. \tag{6.15}$$

K^P and K^*_P may be seen as an analogue to the conjugate pairs in the Hamiltonian formalism. Due to (6.15), the n-bracket carries 1-n units of ghost number

$$gh_{\#}(F,G)_n = gh_{\#}F + gh_{\#}G + 1 - n, \tag{6.16}$$

and n+1 units of parity

$$\epsilon((F,G)_n) = \epsilon(F) + \epsilon(G) + n + 1. \tag{6.17}$$

The n-bracket possesses the graded symmetry property, the Leibniz rule and the Jacobi identity:

$$(F,G)_n = -(-1)^{(\epsilon(F)+n+1)(\epsilon(G)+n+1)}(G,F)_n, \tag{6.18}$$

$$(FG,H)_n = F(G,H)_n + (F,H)_n G(-1)^{\epsilon(G)(\epsilon(H)+n+1)}, \tag{6.19}$$

$$((F,G)_n,H)_n(-1)^{(\epsilon(F)+n+1)(\epsilon(G)+n+1)} + \text{cycle}(F,G,H) = 0. \tag{6.20}$$

Due to (6.20) and (6.18), $(\ ,\ )_n$ is an "ordinary" antibracket in even dimensions and a super Poisson bracket in odd dimensions.

By introducing the n-bracket (6.14) we have found a simple way of expressing the master equation (6.7) in terms of a local master equation (6.12), demanding the boundary condition (6.13).

6.1.2 BRST interpretation of the de Rham differential

The equations of motion for the action (6.1) given in terms of the n-bracket is

$$DK^P = (S, K^P)_n \qquad \text{and} \qquad DK^*_P = (S, K^*_P)_n. \tag{6.21}$$

For the local master action $S(K^P, K^*_P)$ we see that at the level of the equations of motion DS can be written as

$$DS = DK^P\frac{\overrightarrow{\partial} S}{\partial K^P} + DK^*_P\frac{\overrightarrow{\partial} S}{\partial K^*_P} = (S,S)_n. \tag{6.22}$$

The de Rham differential D defined in (6.3) is nilpotent which implies that

$$0 = D^2 K^P = (S, (S, K^P)_n)_n, \qquad 0 = D^2 K^*_P = (S, (S, K^*_P)_n)_n. \tag{6.23}$$

It also follows that

$$0 = D^2 S = (S, (S, S)_n)_n. \tag{6.24}$$

Due to the properties of the n-bracket specified in (6.18),(6.19) and (6.20) consistency requires

$$(S, S)_n = 0. \tag{6.25}$$

This gives the de Rham differential a natural interpretation as a BRST-charge operator. Remember that we had similar arguments for the δ_B-operator in the previous chapter. In this sense the de Rham differential may be seen as the BRST-charge operator of an underlying theory.

6.1.3 Reduction rules and gauge transformations

One advantage of the superfield algorithm is that it is easy to find the gauge transformations of the original theory. This is done by performing a Σ-variation of the superfields, then a reduction to the original model and replacing each form field by a gauge parameter, one at a time. This procedure will be exemplified later on in this section.

The Σ-variations of the superfield K^P and associated superfield K^*_P are given by

$$\begin{aligned}
\delta_\Sigma K^P &= (\Sigma, K^P) = (-1)^n (DK^P - (S, K^P)_n), & (6.26)\\
\delta_\Sigma K^*_P &= (\Sigma, K^*_P) = (-1)^n (DK^*_P - (S, K^*_P)_n). & (6.27)
\end{aligned}$$

Note that the Σ-variation of the superfields measures the failure of the superfields and the associated superfields to be on-shell, from the point of view of the equations of motion (6.21). The equations above can be used to determine the gauge transformation of the underlying classical theory after a reduction procedure.

Since the original fields are the ghost number zero components of the superfields K^P and K^*_P, reduction rules can be found by expanding the superfields in the odd coordinates τ. The following reduction rules are found for the extraction of the n dimensional classical field theory corresponding to a given master action Σ of the form (6.1) [16]

$$\begin{aligned} d^n u d^n \tau &\rightarrow 1 \\ D &\rightarrow \text{exterior derivative } d \\ K^P : gh_{\#} K^P = k \geq 0 &\rightarrow \text{k-form field } k^P \text{where,} \\ & \quad\ \epsilon(k^P) = \epsilon_P + k \\ K^*_P : gh_{\#} K^*_P = (n-1-k) \geq 0 &\rightarrow (n\text{-}1\text{-}k) - \text{form field } k^*_P \text{ where,} \\ & \quad\ \epsilon(k^*_P) = \epsilon_P + k \\ \text{all other superfields} &\rightarrow 0 \\ \text{pointwise multiplication} &\rightarrow \text{wedge product.} \end{aligned} \tag{6.28}$$

This means that ghost numbers in our superfield formulation corresponds to form degrees in the original theory. This in turn implies that the only surviving terms are those with ghost numbers greater than zero. That is why we in the following only consider superfields carrying positive ghost numbers. The original theory, obtained by performing the reduction rules, are thus described by forms and wedge products.

As an example, consider the decomposition of a ghost number two associated superfield $\Phi^*(u,\tau)$ in two dimensions ($n = 2$)

$$\Phi^*(u,\tau) = \Phi^*_1(u) + \tau^a \epsilon_{ab} \Phi^{*b}_2(u) + \tau^a \tau^b \epsilon_{ab} \Phi^*_3(u), \tag{6.29}$$

where ϵ_{ab} is defined to yield $\int d^2\tau \tau^a \tau^b = \epsilon^{ab}$. Since $\Phi^*(u,\tau)$ carries ghost number two and τ ghost number one it follows that Φ^*_3 has ghost number zero and thus is a physical field of the original theory. For the original field Φ^*_3 to be even we see that the associated superfield $\Phi^*(u,\tau)$ need to be even, due to the two fermionic coordinates coupled to Φ^*_3. Note that for a superfield with $gh_{\#}(\Phi(u,\tau)) < 0$ there is no zero component field and hence no corresponding physical field. Actually the superfield $\Phi(u,\tau)$ (that $\Phi^*(u,\tau)$ is "associated" to) does not have a physical component in the fermionic expansion, since this has ghost number minus one according to (6.4).

The gauge transformations are now found by replacing each k-form field by an $(k-1)$-form gauge parameter, one at a time. If the field happens to be a zero form field, i.e. a scalar, the gauge parameter is zero.

The superfield algorithm is now clear:

(i) *Choose superfields and associated superfields entering in the master action (6.2) combined with coefficients in such a way that the expressions for the ghost numbers (6.4) and Grassmann parities (6.5) are satisfied and (6.6) is valid.*

(ii) *Solve the local master equation (6.12) and note that the boundary condition (6.13) has to be satisfied.*

(iii) *Perform the reduction to the original theory following the rules in (6.28).*

The gauge transformations may also be found by calculating the Σ-variation of the superfields (6.26) and replacing the superfields by the original fields as discussed before.

Now the question remains what kind of theories can be generated using the superfield algorithm. From expression (6.2) we see that at least BF-theories [100–102] may be generated by means of the superfield algorithm. A BF-theory corresponds to the kinetic part of the Lagrangian density (6.2). In fact, all theories formulated in this way can be seen to include consistent deformations of BF-theories, where the local expression S represents the deformation [103, 104]. A quite peculiar feature of the superfield algorithm is that it seems to only generate topological gauge theories except in the one dimensional case where all theories can be generated [16], e.g. the relativistic particle theories considered in the beginning of this thesis. In the next section we consider a generalization of the superfield algorithm, henceforth referred to as the generalized superfield algorithm. This generalized version allows for the construction of higher order gauge field theories, such as five dimensional Chern-Simons theories. In the next chapter we discuss the class of theories that are possible to generate by means of the superfield algorithms. Let us first have a closer look at how the ordinary superfield algorithm works.

6.1.4 A simple example in four dimensions

As an example we consider a master action in four dimensions (i.e. $n = 4$)

$$\Sigma = \int d^4u d^4\tau \big(- T^*_E D T^E - S(T^E, T^*_E)\big). \tag{6.30}$$

Choose S to have the most general form with only one pair of superfields (T^*_E, T^E)

$$S = \frac{1}{2} T^*_{E_1} T^*_{E_2} \omega^{E_1 E_2} + \frac{1}{2} T^*_{E_1} \omega^{E_1}{}_{E_2 E_3} T^{E_2} T^{E_3} + \frac{1}{24} \omega_{E_1 E_2 E_3 E_4} T^{E_1} T^{E_2} T^{E_3} T^{E_4}, \tag{6.31}$$

where T^E is an odd superfield with ghost number one and T^*_E an even associated superfield with ghost number two

$$\begin{aligned} \epsilon(T^E) &= 1, & \epsilon(T^*_E) &= 0, \qquad (6.32)\\ gh_\# T^E &= 1, & gh_\# T^*_E &= 2. \qquad (6.33) \end{aligned}$$

From (6.4) and (6.5) it follows that the coefficients are all even, carrying ghost number zero. This model was also considered in [16] but there the master action was solved in a slightly different manner. Following the superfield algorithm, the next step is to solve the local master equation (6.12) to get restrictions on the coefficients, which in general can be quite difficult. The difficulties often occur for theories involving a lot of terms. This in turn usually happens in higher dimensions, since more fields with $gh_{\#}K^* \geq gh_{\#}K \geq 0$ are available to construct interaction terms in the master action. To solve the master equation we instead use (anti)canonical transformations[2] such that the local action S_0 is canonically transformed to

$$S_\Gamma = S_0 + \gamma(S_0,\Gamma)_n + \frac{\gamma^2}{2!}((S_0,\Gamma)_n,\Gamma)_n + \dots, \tag{6.34}$$

where Γ is a canonical generator and γ a real, even parameter of the canonical transformation. The expansion in the n-bracket comes from the definition of a canonical transformation: $S_\Gamma = e^{\mathrm{ad}\Gamma}S_0$, with the adjoint action $\mathrm{ad}\Gamma = (\cdot,\Gamma)_n$. The strategy is now to start with a simple term S_0 that trivially satisfies the local master equation $(S_0,S_0)_n = 0$. In this particular case we consider

$$S_0 = \frac{1}{2}T^*_{E_1}T^*_{E_2}\omega^{E_1E_2} \tag{6.35}$$

and the canonical generator

$$\Gamma = \frac{1}{3}\gamma\gamma_{E_1E_2E_3}T^{E_1}T^{E_2}T^{E_3}. \tag{6.36}$$

From these expressions we see that $\omega^{E_1E_2}$ is symmetric since T^*_E is even and $\gamma_{E_1E_2E_3}$ is totally antisymmetric since T^E is odd. Now we can perform the canonical transformation (6.34) and identify the coefficients in the transformed action S_Γ with the ones in (6.31). Due to the choice of the canonical generator, the expansion (6.34) will always truncate[3]. This is shown in detail in Paper II. In this way we have found the solution to the master equation to be, modulo a canonical transformation,

$$\omega^{E_1}{}_{E_2E_3} = -2\gamma\omega^{E_1E}\gamma_{EE_2E_3}, \tag{6.37}$$

$$\omega_{E_1E_2E_3E_4} = 12\gamma^2\gamma_{E_1E_2E}\omega^{EE'}\gamma_{E'E_3E_4}. \tag{6.38}$$

The master equation for the master action (6.31) actually yields a third equation which implies the Jacobi identity for the coefficients $\omega^{E_1}{}_{E_2E_3}$. If we let $\omega^{E_1E_2}$ be invertible we may interpret it as a group metric which lowers and raises indices. Since the Jacobi identity is required and $\gamma_{E_1E_2E_3}$ is totally antisymmetric it follows that $\omega^{E_1}{}_{E_2E_3}$ is the structure coefficient of a semi-simple Lie algebra.

[2]Note that we mean the transformations that preserves the local n-bracket, which are either canonical or anticanonical depending on n, therefore we will in the sequel denote this just as a canonical transformation.

[3]The truncation will always occur if the canonical generator is chosen not to involve both superfields and associated superfields in the same term.

The canonical transformations imply that we have made a trivial deformation of the theory described by (6.35). Hence, this theory is canonically equivalent to (6.31). Canonical equivalence is however a sufficient but *not* necessary condition for two actions to describe the same gauge theory.

To exemplify the reduction procedure and show how to obtain the gauge transformations we can again study the untransformed master action

$$\Sigma_0 = -\int d^4u d^4\tau \big(T^*_E DT^E + \frac{1}{2} T^*_{E_1} T^*_{E_2} \omega^{E_1 E_2}\big). \tag{6.39}$$

The reduction rules (6.28) implies that T^E reduces to an even one form t^E and T^*_E to an even two form t^*_E. Hence the master action (6.39) reduces to the bosonic action

$$\Sigma_{Cl.} = -\int t^*_E \wedge dt^E + \frac{1}{2} t^*_{E_1} \wedge t^*_{E_2} \omega^{E_1 E_2}. \tag{6.40}$$

The Σ-variation (6.26) and (6.27) of the superfields and associated superfields are

$$\begin{aligned} \delta_\Sigma T^*_E &= DT^*_E, & (6.41)\\ \delta_\Sigma T^E &= DT^E + T^*_{E_1} \omega^{E E_1}. & (6.42) \end{aligned}$$

This gives us the gauge transformations

$$\begin{aligned} \delta t^*_E &= d\tilde{t}^*_E, & (6.43)\\ \delta t^E &= d\tilde{t}^E + \tilde{t}^*_{E_1} \omega^{E E_1}, & (6.44) \end{aligned}$$

where $\tilde{t}^*_E$ and $\tilde{t}^E$ are one and zero form gauge parameters, respectively. Since $\tilde{t}^*_E$ is a one form ghost field this implies that we also have another zero form gauge parameter, namely the ghost for ghost $\tilde{\tilde{t}}^*_E$ with

$$\delta \tilde{t}^*_E = \tilde{\tilde{t}}^*_E. \tag{6.45}$$

It is easily checked that the action (6.40) is invariant under these transformations.

6.2 The generalized superfield algorithm

A natural generalization of the superfield algorithm would be to include higher order terms in the interaction part S in (6.2) and also allow for non-dynamical fields such as Lagrange multipliers. Such a type of generalization was developed in Paper III and will briefly be discussed below.

As in the superfield algorithm, discussed in the previous section, we have a master action on a $2n$-dimensional supermanifold with n Grassmann odd and n Grassmann even dimensions

$$\Sigma[K^P, K^*_P] = \int_{\mathcal{M}} d^n u d^n \tau \mathcal{L}_n(u,\tau). \tag{6.46}$$

The only difference in the setup in comparison with the superfield algorithm is the entering of higher order terms and Lagrange multiplier fields in the Lagrangian density, now given by

$$\mathcal{L}_n(u,\tau) = K^*_P(u,\tau) DK^P(u,\tau)(-1)^{\epsilon_P+n} - S(K^*_P, K^P, DK^*_P, DK^P). \tag{6.47}$$

Thus, DK^P- and DK^*_P-terms are here allowed to enter in the interaction part S. K^P and K^*_P denotes superfields and associated superfields as well as non-dynamical multiplier fields Λ^P and Λ^*_P. These multiplier fields are restricted by $D\Lambda^P = 0$ and $D\Lambda^*_P = 0$ such that they do not enter in the kinetic part of the Lagrange density (6.47). D is still the de Rham differential defined in (6.3).

It follows that the expression for the ghost numbers (6.4) and the Grassmann parities (6.5) are left unchanged

$$gh_{\#}K^P + gh_{\#}K^*_P = n - 1 \tag{6.48}$$

and

$$\epsilon(K^*_P) = \epsilon_P + n + 1, \qquad \text{where} \qquad \epsilon(K^P) := \epsilon_P. \tag{6.49}$$

It also follows that

$$gh_{\#}S = n \qquad \text{and} \qquad \epsilon(S) = n. \tag{6.50}$$

Since we still are in the BV formalism the master action (6.46) has to obey the master equation (6.7)

$$(\Sigma, \Sigma) = 0, \tag{6.51}$$

where the functional bracket $(\cdot\,,\cdot)$ is the usual BV-bracket (6.8).

6.2.1 Generalized n-bracket

Introducing higher order terms requires some new ingredients as is seen when computing the master equation $(\Sigma, \Sigma) = 0$ for the chosen functional (6.46). We are forced to do a rather long, but straightforward calculation of the functional BV-bracket (6.8) and see what comes out of it. One finds that

$$0 = (\Sigma, \Sigma) = \int_{\mathcal{M}} d^n u d^n \tau (\mathcal{L}_n, \mathcal{L}_n)_n, \tag{6.52}$$

which by inserting the expression for the Lagrangian density (6.47) implies

$$0=\int_{\mathcal{M}} d^n u d^n \tau\Big(\frac{1}{2}(S,S)_n + D\mathcal{L}_n(u,\tau) - D\Big(DK^P \frac{\vec{\partial}}{\partial DK^P}S + DK^*_P \frac{\vec{\partial}}{\partial DK^*_P}S\Big)\Big). \tag{6.53}$$

The expression $(\cdot\,,\cdot)_n$ is a generalized n-bracket here defined for the general local functions A and B by

$$\begin{aligned}(A,B)_n = &\Big(A\frac{\overleftarrow{\partial}}{\partial K^P} + (-1)^{\epsilon_P+\epsilon(A)} D\Big(A\frac{\overleftarrow{\partial}}{\partial DK^P}\Big)\Big)\Big(\frac{\vec{\partial}}{\partial K^*_P}B + (-1)^{\epsilon_P+n} D\Big(\frac{\vec{\partial}}{\partial DK^*_P}B\Big)\Big)\\ &-(A\leftrightarrow B)(-1)^{(\epsilon(A)+n+1)(\epsilon(B)+n+1)}.\end{aligned} \tag{6.54}$$

From the expressions of the master equation (6.52) and (6.53) we choose the requirement

$$(S,S)_n = 0, \tag{6.55}$$

which then also prescribes the allowed boundary conditions to be those satisfying

$$0=\int_{\mathcal{M}} d^n u d^n \tau\Big(D\mathcal{L}_n(u,\tau) - D\Big(DK^P \frac{\vec{\partial}}{\partial DK^P}S + DK^*_P \frac{\vec{\partial}}{\partial DK^*_P}S\Big)\Big). \tag{6.56}$$

It is possible to choose another way to establish $(\Sigma,\Sigma)=0$ if we choose another bracket and boundary conditions. This is clearly the case since one always can shift the expression by an exact term. On the other hand, when computing the master equation for Σ the choice above seems to be the most natural one.

The generalized n-bracket (6.54) has the graded symmetry property

$$(F,G)_n = -(-1)^{(\epsilon(F)+n+1)(\epsilon(G)+n+1)}(G,F)_n, \tag{6.57}$$

carries 1-n units of ghost number

$$gh_{\#}(F,G)_n = gh_{\#}F + gh_{\#}G + 1 - n, \tag{6.58}$$

and n+1 units of parity

$$\epsilon((F,G)_n) = \epsilon(F) + \epsilon(G) + n + 1. \tag{6.59}$$

The Jacobi identity and the Leibniz rule are not trivially satisfied, but since the BV-bracket (6.8) have these properties the generalized n-bracket will satisfy them up to a total derivative, i.e. modulo a D-term [105]. The exact expressions for the graded Jacobi identity and the graded Leibniz rule have not been calculated.

The gauge transformations can be found in the same way as was explained in the previous section. The reduction rules (6.28) and Σ-variations (6.26) and (6.27) are the same as in this generalized version, i.e.

$$\delta_\Sigma K^P = (\Sigma,K^P) = (-1)^n(DK^P - (S,K^P)_n), \tag{6.60}$$

$$\delta_\Sigma K^*_P = (\Sigma,K^*_P) = (-1)^n(DK^*_P - (S,K^*_P)_n). \tag{6.61}$$

6.2.2 BRST interpretation

In the framework of the ordinary superfield algorithm we studied the de Rham operator D and its interpretation as a BRST-charge operator. The natural question for this generalized version of the superfield algorithm is of course whether or not this BRST-interpretation is possible also here.

The equations of motion from the master action (6.46) are given by

$$DK^P = (S, K^P)_n, \quad DK^*_P = (S, K^*_P)_n. \tag{6.62}$$

Since $D^2 = 0$ it allows this operator to be interpreted as a BRST-charge operator, but if D is acting on a general function $A(K^P, DK^P, K^*_P, DK^*_P)$ this is not in general true. Define then instead another operator Q by

$$QA := DA + D(DK^P \frac{\vec{\partial}}{\partial DK^P} A + DK^*_P \frac{\vec{\partial}}{\partial DK^*_P} A), \tag{6.63}$$

with $\epsilon(Q) = 1$ and $gh_{\#}Q = 1$. Using the equations of motion above we find

$$QA = (S, A)_n. \tag{6.64}$$

Thus Q may here be interpreted as a BRST-charge operator. The equation above implies

$$QS = (S, S)_n \qquad \text{and} \qquad Q^2 S = (S, (S, S)_n)_n. \tag{6.65}$$

From the definition of Q in (6.63) one also finds that it is nilpotent on-shell, i.e. $Q^2 = 0$. This in turn implies that $(S, S)_n = 0$ due to the properties of the generalized n-bracket.

Note that the equations of motion for the multiplier fields Λ^P and Λ^*_P are

$$D\Lambda^P = (S, \Lambda^P)_n, \quad D\Lambda^*_P = (S, \Lambda^*_P)_n. \tag{6.66}$$

Since the multiplier fields are restricted by $D\Lambda^P = 0$, $D\Lambda^*_P = 0$ it follows from (6.66) that

$$0 = (S, \Lambda^P)_n, \quad 0 = (S, \Lambda^*_P)_n, \tag{6.67}$$

which are the constraint equations. The constraints are found by a variation of the action with respect to a multiplier field. Thus, the requirements $D\Lambda^P = 0$ and $D\Lambda^*_P = 0$ are quite natural since the equations of motion for the multiplier fields (6.67) then imply the constraints.

Examples of generated theories using the generalized superfield algorithm will be shown in the next chapter. The generalized version allows for the construction of higher order gauge field theories like the five dimensional Chern-Simons theories.

7

THE CLASS OF GENERATED THEORIES

So far we have briefly considered the BV formalism from a general point of view and we have seen how it is possible to construct consistent gauge field theories by means of a superfield algorithm. As previously discussed, the superfield algorithm only generates first order gauge field theories. An important class of theories that naturally fit into this framework is topological gauge field theories. Due to this feature, we will in this chapter consider topological gauge field theories in more detail and show examples of theories that can be constructed using a superfield algorithm. However, we will also consider non-topological theories generated by means of the generalized superfield algorithm developed in Paper III. Models treated are a topological Yang-Mills theory and a three and a five dimensional Chern-Simons theory. The latter is only possible to construct using the generalized superfield algorithm.

7.1 Topological gauge field theories

A topological theory is a theory without local degrees of freedom, where all observables are independent of the metric. Hence, there are no physical propagators. In a sense it is a *theory of nothing* and quite the opposite of String/M-theory which is thought of being a *theory of everything*. But it is for that sake not unimportant. A topological theory can be a fully interacting theory with the advantage of being exactly solvable. Topological theories have many local symmetries and can be seen as a subclass of non-topological theories. By considering the 'underlying' topological theories of two different models, Vafa and Witten where able to show the strong and weak coupling correspondence in supersymmetric Yang-Mills theories [106]. This was shown by using the necessary condition of duality of the 'underlying' topological theories. Topological theories have in the past years also been successfully used in connection to quantum chromodynamics (QCD) [107, 108].

A topological gauge field theory has a very large group of symmetries, one believes that non-topological gauge theories, e.g. string theories can be obtained by breaking some of these symmetries. How this is done is however not yet known. In a sense one may consider a topological theory as an embryo of a fully fledged theory.

In this chapter we only consider the necessary material in order to understand the topological nature of the generated theories. A more complete review of general topological theories can be found in [109].

A topological field theory is usually defined as a field theory living on a Riemannian manifold and where an observable is independent of the metric. This implies that the correlation function in a quantum field theory does not depend on the Riemannian metric structure. All topological field theories are either of Schwarz type [110] or Witten type [111] which both are lacking physical degrees of freedom. The Witten type is characterized by a BRST exact quantum action. In the Witten type, the energy-momentum tensor is also BRST exact. The energy-momentum tensor is found by a variation of the action with respect to the metric. Examples of topological field theories of Witten type are topological Yang-Mills theories [112, 113] and topological sigma models [114]. A Schwarz type topological theory on the other hand consists of an action which can be split into a BRST exact and a metric independent term. Chern-Simons theories in three dimensions [110, 115] and BF-theories [100–102] are examples of Schwarz type topological gauge field theories.

There have also been some progress in understanding the quantization of topological open membranes using the BV formalism [116–118].

7.2 Topological Yang-Mills theory

A topological Yang-Mills theory (TYM) in four dimensions [112,113] is defined by

$$S_{TYM} = \int tr(F_\wedge F) = \int tr(dA_\wedge dA + 2A_\wedge A_\wedge dA). \tag{7.1}$$

where the field strength F is related to the one form gauge connection A through $F = dA + A_\wedge A$. Since the $tr(A^4)$ vanishes, this is just a boundary term

$$S_{TYM} = \int tr(F_\wedge F) = \int d[tr(A_\wedge dA + \frac{2}{3}A_\wedge A_\wedge A)], \tag{7.2}$$

where $tr(A{\wedge}dA+\frac{2}{3}A{\wedge}A{\wedge}A)$ is the Chern-Simons Lagrangian. We let $A = g^a A_a$, where g^a satisfies a Lie algebra $[g^a, g^b] = f^{ab}{}_c g^c$. The expression (7.1) above may now be given in terms of a symmetric group metric $g^{ab} := tr(g^a g^b)$ and a totally antisymmetric structure coefficient $f^{abc} = f^{ab}{}_d g^{dc}$,

$$S_{TYM} = \int g^{ab} dA_a dA_b + f^{ab}{}_d g^{dc} A_a A_b dA_c. \tag{7.3}$$

We can write the corresponding master action within the framework of the superfield algorithm (or its generalized version) by introducing superfields T^*_E and T^E with ghost numbers

$$gh_{\#} T^*_E = 2, \qquad gh_{\#} T^E = 1, \tag{7.4}$$

and Grassmann parities

$$\epsilon(T^*_E) = 0, \qquad \epsilon(T^E) = 1. \tag{7.5}$$

The Grassmann parities of the fields are chosen such that the original fields are even after the limit (6.28) has been taken.

The master action of a topological Yang-Mills theory may now be written in at least two different ways. We will in the following show how this is possible using either the framework of the superfield algorithm or its generalized version.

7.2.1 Using the superfield algorithm

The local action corresponding to a topological Yang-Mills theory can in the superfield algorithm be written with auxiliary fields T^*_E as

$$S'_{TYM} = \frac{1}{4} T^*_{E_1} T^*_{E_2} \omega^{E_1 E_2} - T^*_{E_1} \omega^{E_1}{}_{E_2 E_3} T^{E_2} T^{E_3}, \tag{7.6}$$

where the coefficients are even and have ghost number zero. The ghost numbers and Grassmann parities of the superfields are the ones prescribed above. The master equation $(S, S)_4 = 0$ then yields

$$0 = \omega^{(E_1|}{}_{E_2 E} \omega^{E E_3)}, \tag{7.7}$$

$$0 = \omega^{E_1}{}_{(E_2 E} \omega^{E}{}_{E_3 E_4)}. \tag{7.8}$$

The brackets in the subscripts indicate antisymmetrization with respect to the enclosed indices. If we assume that $\omega^{E_1 E_2}$ is invertible it is quite natural to interpret this as a symmetric group metric. This implies that we can lower the indices in the first equation (7.7) in which case $\omega_{E_1 E_2 E_3}$ is a totally antisymmetric coefficient. The second equation (7.8) is a graded Jacobi identity, which means that $\omega^{E_1}{}_{E_2 E_3}$ can be seen as a structure coefficient of a semi-simple Lie algebra. A reduction to the original theory and eliminating the auxiliary fields t^*_E yields the desired topological Yang-Mills action (7.3). This is seen after an identification of the coefficients $\omega_{E_1 E_2 E_3}$ and $\omega^{E_1 E_2}$ with f_{abc} and g^{ab}.

7.2.2 Using the generalized superfield algorithm

The topological Yang-Mills theory can be written in an alternative way using the generalized superfield algorithm. This is shown in detail in Paper III. The corresponding local master action for the generalized superfield algorithm is given by

$$S''_{TYM} = T^*_{E_1}\omega^{E_1}{}_{E_2E_3}T^{E_2}T^{E_3} + \omega_{E_1E_2E_3E_4}T^{E_1}T^{E_2}T^{E_3}T^{E_4} + \Lambda^{E_1}(T^*_{E_1} - g_{E_1E_2}DT^{E_2}). \tag{7.9}$$

The ghost number and Grassmann parities of the superfield and associated superfields are the ones given by (7.4) and (7.5). The coefficients are Grassmann even and independent of the coordinates (u, τ). The super multiplier fields Λ^E are even, carrying ghost number two. When solving the master equation we should remember to use the generalized n-bracket (6.54). The constraint is found after varying the action Σ (6.46) with respect to Λ^E (i.e. applying (6.62), where $D\Lambda^E = 0$)

$$T^*_{E_1} = g_{E_1E_2}DT^{E_2}. \tag{7.10}$$

If we define

$$\phi_{E_1} := T^*_{E_1} - g_{E_1E_2}DT^{E_2}, \tag{7.11}$$

then we see that

$$(\phi_{E_1}, \phi_{E_1})_4 = 2D(g_{E_1E_2}) = 0. \tag{7.12}$$

Thus (7.11) is a first class constraint with respect to the antibracket $(\cdot,\cdot)_4$.

Actually, this classification is already included in the master equation. The first class constraint implies that this is a gauge theory and we can proceed by calculating the master equation. In the case of a second class constraint, we need to add extra terms to the action to convert them into first class constraints.

The constraint is inserted after the master equation has been calculated. To solve the master equation we also need to use the equation of motion given by (6.62)

$$DT^E = -\omega^E{}_{E_1E_2}T^{E_1}T^{E_2} - \Lambda^E. \tag{7.13}$$

This is needed in order to replace the expressions involving D-terms, produced by the generalized n-bracket.

The master equation $(S''_{TYM}, S''_{TYM})_4 = 0$ gives

$$0 = (2\omega_{E_1E_2E_3E}\omega^E{}_{E_4E_5} + \omega_{E'E_1E_2}\omega^{E'}{}_{E_3E}\omega^E{}_{E_4E_5})T^{E_1}T^{E_2}T^{E_3}T^{E_4}T^{E_5} \tag{7.14}$$

$$0 = (2\omega_{E_1E_2E_3E_4} + \omega_{E_4E_1E}\omega^E{}_{E_2E_3} + \omega_{EE_1E_2}\omega^E{}_{E_3E_4})T^{E_1}T^{E_2}T^{E_3}\Lambda^{E_4}, \tag{7.15}$$

$$0 = \omega_{E_1E_2E_3}\Lambda^{E_1}\Lambda^{E_3}T^{E_3}. \tag{7.16}$$

Here we have treated $g_{E_1E_2}$ as a group metric. Λ^E is even, such that equation (7.16) implies $\omega_{E_1E_2E_3} = -\omega_{E_3E_2E_1}$. From (7.9) we know that $\omega_{E_1E_2E_3} =$

$-\omega_{E_1E_3E_2}$, i.e. $\omega_{E_1E_2E_3}$ is totally antisymmetric. If the graded Jacobi identity is satisfied (which should be the case since we are dealing with a topological Yang-Mills theory) a solution to the remaining equations is $\omega_{E_1E_2E_3E} = \frac{1}{2}\omega_{E'E_1E_2}\omega^{E'}{}_{E_3E}$. Due to this last relation and the Jacobi identity, the T^4-term in (7.9) vanishes. Thus, the master equation implies that $\omega_{E_1E_2E_3}$ is a totally antisymmetric structure coefficient obeying the Jacobi identity.

A reduction to the original theory gives us the action

$$\Sigma_{TYM} = \int t^*_E{\wedge}dt^E + t^*_{E_1}\omega^{E_1}{}_{E_2E_3}{\wedge}t^{E_2}{\wedge}t^{E_3} + \lambda^{E_1}(t^*_{E_1} - g_{E_1E_2}dt^{E_2}). \tag{7.17}$$

Implementing the constraint,

$$t^*_{E_1} = g_{E_1E_2}t^{E_2}, \tag{7.18}$$

now yields

$$\Sigma_{TYM} = \int g_{E_1E_2}dt^{E_1}{\wedge}dt^{E_2} + g_{EE_1}\omega^{E}{}_{E_2E_3}dt^{E_1}{\wedge}t^{E_2}{\wedge}t^{E_3}. \tag{7.19}$$

Hence, we have generated the topological Yang-Mills action (7.3).

7.3 Higher dimensional Chern-Simons theories

Higher dimensional Chern-Simons theories are constructed in the same way as the familiar three dimensional Chern-Simons theory, i.e. from the characteristic classes in $2n+2$ dimensions [110, 115]. The field setup is a one-form gauge connection $A = t_a A^a = t_a A^{a\mu}{\wedge}dx_\mu$, where t_a is the generator of the gauge group G. From this we can construct a two form $F^a = dA^a + \frac{1}{2}f^a{}_{bc}A^b{\wedge}A^c$, where $f^a{}_{bc}$ is the structure coefficient of the gauge group and d the exterior derivative, $d = dx^\mu\partial_\mu$.

With a symmetric tensor $g_{a_1,a_2,\dots,a_{n+1}}$ the Chern-Simons Lagrangian in $2n+1$ dimensions is defined by

$$d\mathcal{L}^{2n+1}_{CS} = g_{a_1,a_2,\dots,a_{n+1}}F^{a_1}{\wedge}\dots{\wedge}F^{a_{n+1}}. \tag{7.20}$$

This implies that the Chern-Simons Lagrangian in 3-dimensions is defined by $d\mathcal{L}^3_{CS} = g_{ab}F^a{\wedge}F^b$. The tensor g_{ab} can here be viewed as a group metric. In five dimensions we have a tensor g_{abc} and in seven dimensions a g_{abcd} tensor. The latter may be chosen to be decomposed into various group metrics, but for the five dimensional case g_{abc} it is not likely that simple. This is investigated in more detail in [97, 98]. We will study the peculiarity of five dimensional Chern-Simons theories later on in this section.

One finds that the Chern-Simons Lagrangian in 3,5 and 7 dimensions are defined by

$$\mathcal{L}^3_{CS}=tr(A{\wedge}dA+\frac{2}{3}A{\wedge}A{\wedge}A), \tag{7.21}$$

$$\mathcal{L}^5_{CS}=tr(A{\wedge}dA{\wedge}dA+\frac{3}{2}dA{\wedge}A{\wedge}A{\wedge}A+\frac{3}{5}A{\wedge}A{\wedge}A{\wedge}A{\wedge}A), \tag{7.22}$$

$$\begin{aligned}\mathcal{L}^7_{CS}=tr(&A{\wedge}dA{\wedge}dA{\wedge}dA-\frac{8}{5}A{\wedge}A{\wedge}A{\wedge}dA{\wedge}dA-\frac{4}{5}dA{\wedge}A{\wedge}dA{\wedge}A{\wedge}A\\&+2dA{\wedge}A{\wedge}A{\wedge}A{\wedge}A{\wedge}A-\frac{4}{7}A{\wedge}A{\wedge}A{\wedge}A{\wedge}A{\wedge}A{\wedge}A).\end{aligned} \tag{7.23}$$

The gauge field A takes values in some representation of a Lie algebra, hence the trace (tr) is needed. Since $S[A]=\int_{\mathcal{M}}\mathcal{L}_{CS}$ we see that the integrand is a volume form, independent of the metric and without reference to the Hodge dual operator. The Chern-Simons theories are accordingly topological field theories of Schwarz type.

Chern-Simons theories are interesting since they can reconstruct gravitational theories [119–121]. The $d=26$ bosonic string field theory has also been found to resemble a Chern-Simons theory [122].

The 3-dimensional Chern-Simons theory is a topological theory. Hence it does not have any local degrees of freedom. One could naïvely expect that this will be the case also for all higher dimensional Chern-Simons theories since these are constructed in an analogous way. But surprisingly this is not necessarily true. Higher dimensional Chern-Simons theories with $d\geq 5$ have been shown to possess local degrees of freedom according to [97, 98] and are therefore non-topological theories. This is shown by turning to the Hamiltonian formalism and performing a Dirac analysis and thereafter count the number of degrees of freedom as explained in [21]. It seems like the local degrees of freedom increases faster as a function of dimension than the gauge symmetry does.

7.3.1 Five dimensional Chern-Simons theory

The ordinary superfield algorithm does not provide us with theories like the five dimensional Chern-Simons theory. But by means of its generalized version it is possible to generate such a theory.

Consider a classical Chern-Simons theory in five dimensions with the Lagrangian density

$$\mathcal{L}^5_{CS}=tr[(dA)^2A+\frac{3}{2}dAA^3+\frac{3}{5}A^5], \tag{7.24}$$

where A is a one form gauge connection. Let $A=g^aA_a=g^aA_{a\mu}dx^\mu$ where g^a satisfies a Lie algebra $[g^a,g^b]=f^{ab}{}_cg^c$. The trace may now be expressed in terms of the generators g^a and the structure coefficient $f^{ab}{}_c$ in such a way that

$$\mathcal{L}^5_{CS} = g^{abc} dA_a dA_b A_c + \frac{3}{4} f^{cd}{}_e g^{abe} dA_a A_b A_c A_d + \frac{3}{20} f^{ab}{}_f f^{cd}{}_h g^{fhe} A_a A_b A_c A_d A_e, \tag{7.25}$$

where $g^{abc} := tr(g^a g^b g^c)$.

This classical Chern-Simons action may be obtained from the generalized superfield formulation of the form (6.47) by introducing an odd ghost number one superfield S^D and an odd associated superfield S^*_D with ghost number three, together with multiplier fields Λ^D and Λ^*_D.

Superfield	ghost number	Grassmann parity
S^*_D	3	1
S^D	1	1
Λ^*_D	2	0
Λ^D	2	0

With these superfields and associated superfields, the master action may be written as

$$\Sigma^5_{CS} = \int d^5u d^5\tau \mathcal{L}^5_{CS} = \int d^5u d^5\tau (S^*_D D S^D - S^5_{CS}), \tag{7.26}$$

where the local master action is given by

$$\begin{aligned} -S^5_{CS} &= \frac{3}{5} \omega_{D_1 D_2 D_3 D_4 D_5} S^{D_1} S^{D_2} S^{D_3} S^{D_4} S^{D_5} + \frac{3}{2} S^*_{D_1} \omega^{D_1}{}_{D_2 D_3} S^{D_2} S^{D_3} \\ &\quad + \Lambda^{D_1} (S^*_{D_1} - g_{D_1 D_2 D_3} D S^{D_2} S^{D_3}). \end{aligned} \tag{7.27}$$

The coefficients $\omega_{D_1 D_2 D_3 D_4 D_5}, \omega^{D_1}{}_{D_2 D_3}, g_{D_1 D_2 D_3}$ are all even and carries ghost number zero. They are also chosen to be independent of the coordinates (u, τ). The constraint given by

$$\phi_{D_1} := S^*_{D_1} - g_{D_1 D_2 D_3} D S^{D_2} S^{D_3} \tag{7.28}$$

is of first class with respect to the super Poisson bracket $(\cdot,\cdot)_5$, since

$$(\phi_{D_1}, \phi_{D_2})_5 = -2D(g_{D_1 D D_2} S^D + g_{D_1 D_2 D} S^D) = 0 \tag{7.29}$$

due to boundary conditions. We choose to have a symmetric tensor $g_{D_1 D_2 D_3}$, such that the solution to the master equation, worked out in Paper III, is given by

$$0 = \omega^{D_1}{}_{(D_2 D} \omega^D{}_{D_3 D_4)}, \tag{7.30}$$

$$\omega_{D_1 D_2 D_3 D_4 D_5} = \frac{3}{2} g_{D D' D_2} \omega^D{}_{D_3 D_1} \omega^{D'}{}_{D_4 D_5}, \tag{7.31}$$

$$g_{D_1 D_2 D} \omega^D{}_{D_3 D_4} = g_{D D_1 D_3} \omega^D{}_{D_4 D_2}. \tag{7.32}$$

This is expected to reduce to the original Chern-Simons action (7.25) when taking the limit (6.28) and by an appropriate redefinition of the coefficients.

7.3.2 Three dimensional Chern-Simons theory

In Paper III we also discuss the three dimensional Chern-Simons theory. A way to obtain this theory, after taking the limit (6.28), is naïvely by implementing a constraint

$$\phi_{F_1} := U^*_{F_1} - g_{F_1F_2}U^{F_2} \tag{7.33}$$

such that the master action may be written as

$$\Sigma = \int U^*_F DU^F + U^*_{F_1}\omega^{F_1}{}_{F_2F_3} + \Lambda^{F_1}(U^*_{F_1} - g_{F_1F_2}U^{F_2}). \tag{7.34}$$

However, the constraint is second-class with respect to the bracket $(\cdot,\cdot)_3$ since

$$(\phi_{F_1}, \phi_{F_2})_3 = -g_{F_1F_2} - g_{F_2F_1}, \tag{7.35}$$

if we disregard the case when $g_{F_1F_2}$ is antisymmetric with respect to the subscripts F_1 and F_2 which not will lead to a Chern-Simons theory. Due to the second-class constraint we can not proceed in analogy with the construction of the five dimensional Chern-Simons theory. Instead one may introduce new terms to convert the constraint into first-class. Another way to approach the construction of a three-dimensional Chern-Simons theory by means of a superfield algorithm is considered in [16], and which may be viewed as a direct solution of the constraint (7.33).

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