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Abstract

Title: Teaching systems of linear equations in Sweden and China: What is made possible to learn?

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A starting point for this study was an aim to understand better the relation between teaching and learning of mathematics. This interest was based on the assumption that *what* is possible for students to learn about mathematics must be related to *how* they experience the mathematical content, which in turn must be related to *how* the content is handled during the mathematics lesson. Many factors that may have impact on the teaching and learning of mathematics are beyond the influence of teachers. However, the handling of the content – for instance, what examples to use, what aspects of concepts and methods to emphasise, and what exercises students should work on – is something about which mathematics teachers must always make decisions. An intention with this study was to produce results that could inform practice on the classroom level, as well as teacher education, in this respect. Another intention was to contribute to the development of methods for analysing teaching that are sensitive to and focus on the specific content of instruction.

Sixteen lessons from six classes in Sweden and China – video recorded within the Learners' Perspective Study – were analysed and compared. The analysis was based on Variation theory and had its focus on differences in how the same mathematical content was taught. The concept 'object of learning' was used to denote what teachers try to teach and what students are supposed to learn. In this study the three objects of learning analysed were related to the mathematics taught in the six classrooms – *systems of linear equations in two unknowns*, *solutions to systems of linear equations in two unknowns* and *the method of substitution*. The analysis was done from the perspective of a student in the classroom and aimed at describing what aspects of the objects of learning were made possible to learn. An aspect was considered made possible to experience if the corresponding 'dimension of variation' was opened. The alternative was that the aspect in question was taken for granted during teaching and kept invariant.

The analytical approach employed made it possible to detect even subtle differences in how the teachers handled the content. The teachers thereby made different things possible to learn for the students. The description of these differences points out several aspects that probably are so familiar to many teachers that these aspects face the risk of being taken for granted in teaching. Further, findings regarding the use of s*ystematic and deliberate variation* in the Chinese classrooms are discussed.

Preface

Working with a doctoral thesis can easily be perceived as an individual endeavour. Of course, you will work by yourself for long periods of time. I know I have done that anyway. However, when looking back I realise that I have constantly received a lot of help and support from friends and colleagues. I am absolutely convinced that without all that backing this thesis would not be.

I wish to thank *all* my friends and colleagues at the Section of Mathematics Education and at the National Center for Mathematics Education at the University of Gothenburg. I especially would like to mention Wiggo Kilborn and Per-Olof Bentley who read and discussed with me the preliminary plan, Wang Wei Sönnerhed who helped me with the Chinese and John Mason who carefully read the first draft version and came all the way to Göteborg to discuss my work in a very constructive seminar. Berner Lindström also read the draft version and gave me a number of very useful suggestions. Anders Wallby and Karin Wallby have, with always so sharp eyes, scrutinized large parts of the thesis. Marj Horne have carefully read the thesis in a very late stage and helped me to improve it considerably.

The thesis would not have been possible without the all teachers and students that allowed researchers to enter their classrooms and to document their activities on videotape. The cooperation within the KULT project, the ILU project, the Variation theory group and the Learners' Perspective Study community have been, not only most valuable and enjoyable, but also a prerequisite for my study.

Finally I wish to thank my two excellent supervisors Ference Marton and Jonas Emanuelsson – *you are the best*!

Contents

Chapter 1
Introduction

My research interest concerns the relation between teaching and learning of mathematics in a broad sense. Now, the teaching and learning of mathematics are highly complex phenomena. To understand the relation between the two and find ways to develop the teaching in order to improve the learning of the students is difficult, to say the least.

The issue of improving mathematics education is not just of interest to mathematics educators. In many countries there is great concern regarding the outcomes of mathematics education. From a Swedish perspective there are several studies, for instance Nationella utvärderingen 2003 [National assessment] (Skolverket, 2004a), TIMSS (TIMSS, 2005), and PISA (OECD, 2005), that show unsatisfactory results in mathematics. The analysis of the TIMSS study of 2003 showed that Swedish grade 8 students performed much worse than in the study of 1995. In fact, the Swedish students show the largest drop in score of all countries and the result for the Swedish grade 8 students in 2003 is lower than the result for Swedish grade 7 students that participated in the study of 1995 (Skolverket, 2004c). Also the results for Sweden from the PISA study in 2003 are poor. Only about 80 % of the Swedish 15-year-olds in the study show the competencies in mathematics that are deemed necessary by PISA (Skolverket, 2004b). Neither are the results on the National test in grade 9 satisfactory. Since the first test in the current series, which started in 1998, between 10 and 13 % of the students constantly obtain the grade 'not pass'. The poor results in mathematics for Swedish students have given rise to a public debate and in 2003 the Ministry of Education and Research appointed a delegation, the Mathematics Delegation (Matematikdelegationen, 2004), to develop an action plan to improve attitudes to and increase interest in mathematics, as well as to improve the quality of teaching and learning of mathematics. The work of the Mathematics Delegation resulted in a report with proposals and concrete examples directed not just to the educational system, but also to other parts of society. Similar initiatives have been taken in many western countries. Another example is the National Advisory Panel that was established in USA in 2006 to advise the President regarding the best use of scientifically based research on teaching and learning of mathematics "to foster

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greater knowledge of and improved performance in mathematics among American students" (Bush, 2006).

Hand in hand with the national concern for the quality of mathematics education there are many large international projects which aim at evaluating and comparing the educational systems in different countries. The approach of these studies, for instance TIMSS and PISA, is generally to measure the 'outcome of education' by student tests. The tests can be constructed differently but still share the intention of being adequate and fair when it comes to comparing different countries in terms of students' performance. In addition to the student tests other data were collected. One intention was to analyse these 'background variables' in order to find possible explanations of the differences in student performance. Whatever possible impact these background variables may have on the outcomes, I would label them 'distal factors' (Pong & Morris, 2002). Some examples of such distal factors are "class size", "mathematics teacher education", "socio-economic situation of students", and "time spent on math every week". One characteristic of the distal factors is that none of them is controlled by teachers, and they cannot easily be changed by the teachers alone, at least not in the short term. Even if it can be shown that some of these factors are important and really influence student learning, I mean that, in most cases, the information is of little immediate use to the individual mathematics teacher. When planning a lesson or when entering a classroom the teacher has to accept the distal factors as they are. The actual number of students in the class, the teacher's own educational background, and the number of weekly math classes just have to be accepted. The distal factors are not among the ones that the teacher can change. The teacher can only teach the best way possible given the actual conditions. This line of reasoning does not propose that the distal factors necessarily are unimportant. They contribute to the external conditions within which teachers have to act. Distal factors, such as the number of students in a class, the number of mathematics lessons every week, the teaching materials and aids available no doubt can affect the practice of teaching.

Initiatives to improve mathematics education, like the ones described above, may benefit from the large international studies. These initiatives address the question of what can be done by political means on a national level. Political decisions regarding the funding to and the regulation of schools and teacher education, decisions of curricula, national testing and grading, just to mention a few areas, are of significance as they form the frames in which teaching and learning take place. However, there will always be a space in which teachers may act. This study has its focus on that space, the area where teachers still 'rule'.

Teaching mathematics

What factors then *can* the teacher control? One such factor concerns the handling of the mathematical content. Questions such as the following have to be answered by the teacher, explicitly or implicitly, ”What examples to use?”, ”In what order to use them?”, ”What are the key ideas that should be made clear to students?”, ”What exercises should the students to work on?”, and ”In what order?”. The teacher still, if not in absolute control, at least strongly influences how the mathematical content will be handled and presented to students during instruction. This closeness to the teaching practice was one circumstance that led me to focus the study on different ways of handling the mathematical content.

The approach taken in this study was to enter the classrooms and examine how the mathematics was handled. One reason, as already discussed above, was that this feature of mathematics teaching is accessible and possible for teachers to change. I also found support for this approach in the TIMSS 1999 video study. This study was an ambitious attempt to expand the ’regular’ TIMSS achievement studies by also including the teaching process. The first analysis of the video recorded lessons showed that the teaching in so-called high-achieving countries differed in many ways, and ”no single method of teaching appears to be necessary for high mathematics achievement” (Hiebert & Stigler, 2004). Only after a much closer look on how the mathematics was handled could a similar feature of the high-achieving countries be found, a feature that was different from the U.S.

> Although teachers in the United States presented problems of both types (practicing skills vs. ”making connection”), they did something different from their international colleagues when working on the conceptual problems with students. For these problems, they almost always stepped in and did the work for the students or ignored the conceptual aspect of the problem when discussing it [...] The significance of this finding cannot be overestimated.
>
> (Hiebert & Stigler, 2004)

The more general features of teaching, that were studied initially, were not helpful in distinguishing between teaching of different quality. The difference could not be found in the kind of problems used, but only in *how* the problems were handled. These findings strengthened me in the decision to keep a close focus on how teachers and students interact about the mathematical content.

One way of trying to learn about mathematics teaching, that struck me as sensible, was to study authentic teaching by experienced teachers. Within the international collaboration of the Learners’ Perspective

Study (LPS, 2003) a substantial number of mathematics lessons taught by 'competent teachers' had been documented. As a member of the Swedish research team I could gain access to data from the study. In the present study video-data from classrooms in Sweden and China (Hong Kong and Shanghai) were analysed. There were several reasons for this choice.

Students from some of the countries in South East Asia (e.g. Singapore, Hong Kong, China, Japan) perform at the top of almost every international achievement study. Quotations like "[...] the Chinese students in our study outperformed U.S. students in every respect [...]" (Cai & Lester, 2005, p. 234), "East Asian countries [...] consistently outperform western countries [...] in these international tests" (Leung, Graf & Lopez-Real, 2006, p. 7), and "Chinese students often outperform U.S. students on international tests in mathematics" (Wang & Lin, 2005, p. 3) could be found in virtually every study where some aspect of teaching and learning of mathematics in Asian and Western countries were compared. The consistency in the results of the students from some East Asian countries has generated a lot of interest from researchers in mathematics education. There were results from comparative studies indicating that at least some part of the explanation of the differences in performance could be found in the way mathematics is taught.

> As a research community, we are now beginning to develop a better understanding of why these differences exist. In particular, mathematics educators have observed that the superior results of Asian students may be due in part to what happens during classroom instruction [...] (Cai & Lester, 2005, p. 234)

It seemed to be quite natural then, if I was interested in finding out what characterises good teaching, to take a look at what was going on inside East Asian classrooms. Clearly, some things might be learnt from that.

Another reason to compare Sweden and China was that the way teaching was perceived, and the teaching traditions within those countries could be expected to differ considerably. The comparisons of such different classrooms may generate interesting and possibly fruitful contrasts. The contrasts might make some of the typical characteristics of both the Swedish and Chinese classrooms become visible. It is often hard to notice features that are deeply embedded in your own culture. Features that are taken for granted easily turn invisible. This was experienced in the TIMSS video study. "We concluded from our first study that teaching is a cultural activity: learned implicitly, hard to see from within the culture, and hard to change" (Stigler & Hiebert, 2004).

Finally, a study of differences in how the mathematical content was handled in the classroom would require the content to be the same. With this approach comparison of the teaching of different content, such as the teaching of geometry and the teaching of algebra, was not of interest. Any differences found regarding the teaching practice could in that case not be separated from the difference in content. The teaching of geometry might be different precisely because it is *geometry* and not algebra, but that would not be possible to decide. In other words, in this study the content taught must be kept invariant so that other aspects might stand out. In order to allow for this kind of comparisons the LPS research teams in Sweden and China agreed to choose classrooms where the same content was taught. The first part of the data was collected in Hong Kong and Shanghai and 'systems of equations' turned out to be a concept from school algebra that could be captured in all classrooms. Unfortunately, it was only possible to record lessons with precisely the same mathematics in one of the Swedish classrooms. The 'systems of equations' is not normally included in the syllabus for grade 8 in Sweden. However, the content in all recorded classrooms in Sweden were from the topic school algebra.

Beside the importance, to the design of the study, of having the *same* content in all classrooms, the choice of school algebra as the topic was in itself interesting. Firstly, school algebra is an area of mathematics where many students encounter difficulties. The topic has attracted at lot of attention from researchers in mathematics education (cf. Bednarz, Kieran & Lee, 1996; Kieran, 1992, 2007; Sutherland, Rojano, Bell & Lins, 2000). Secondly, from a Swedish perspective, school algebra was interesting as Swedish students performed especially poorly on algebra tasks in international tests (Adolfsson, 1997; Nyström, 2005).

Research questions

In conclusion, the main purpose of the study was to cast some light on the following questions:

> How is systems of equations taught differently?
>
> What are the crucial differences from a learners' perspective?

Another purpose of this study was to trial a method for analysing classroom data based on Variation theory (Marton & Booth, 1997; Marton & Morris, 2001; Runesson, 1999). How well can an analysis based on Variation theory capture the interesting differences in relation to the questions above? The attention to the 'what-aspect' of learning within this theory aligned well with my research interest and the clear-cut focus on the mathematical content in this study. Variation theory provided an elaborated framework for describing differences in the object of learning that was suitable for the kind of analysis I wished to do. Further it was possible to theoretically relate differences in how the content was handled during instruction to what learning was made possible – different ways of handling the mathematics may have put constraints on and/or offered opportunities for students to experience different features of the content. These circumstances suggested that Variation theory could effectively provide the theoretical support for the analysis. By doing this I also hoped the study would contribute to the further development of research methods that are sensitive to differences in how the mathematical content is handled and made possible for students to experience.

An overall aim of the study was to deepen the understanding of the relation between teaching and learning of mathematics and to cast some light on what could be crucial for 'good teaching'. This included the overarching objectives, that the research might contribute to the improvement of mathematics teaching and learning, and inform the education of mathematics teachers. My aim was that the results of the study would add to the knowledge base upon which both the teaching of mathematics and the education of mathematics teachers could be grounded. Studying teacher practice might also illustrate and concretise the notion of 'pedagogical content knowledge'. By comparing the teaching of the same content, important differences in teachers' pedagogical content knowledge 'in action' might be exposed.

When discussing what could be learned from the TIMSS video studies in terms of professional development of teachers and the improvement of teaching Hiebert and Stigler (2004) stated that, "most teachers learn to teach by growing up in a culture, watching their own teachers teach, and then adapting these methods for their own practice" and that "teaching can only change the way cultures change: gradually, steadily, over time as small changes are made in the daily and weekly routines of teaching" (ibid.). One of their suggestions in how to facilitate a positive development of teaching was that teachers must be given opportunities to experience different ways of teaching, "[...] teachers must be provided with vivid examples that illustrate alternative ways of teaching" (ibid.). I hope that this study will provide examples of different ways to teach the same mathematics that could make it possible to discern features of teaching that were previously taken for granted and open up a dimension of variation regarding how something could be taught differently.

Structure of the thesis

In the next chapter I review and discuss previous research relevant to this study. In the first section I review research on the teaching and learning of school algebra, with a particular focus on the teaching of systems of equations. In the following section I review mathematics classroom research, based on studies reported in some major mathematics education research journals. In the third section, international comparative research in mathematics education is discussed. Even though I have not placed this study in the field of international comparisons, studies in that field are not without relevance, as the classrooms I compared were from different countries.

In chapter 3 I discuss the research method and the theoretical basis of this study. The concepts and definitions from Variation theory that I use in the analysis are discussed.

In chapter 4 I describe the data, the design and methods used in the Learners' Perspective Study. In this chapter I also describe the implementation of the method of analysis.

The fifth chapter contains the results. First I describe, in detail, how the mathematical content, in terms of three enacted 'objects of learning', was handled in each of the six classrooms. This is followed by a description of the 'space of learning' for each object of learning. In this part the teaching in the different classrooms is compared.

The last chapter contains further discussion and conclusions based on the findings.

Chapter 2
Review of research

In this study I analyse differences in how the mathematical content is handled. School mathematics can be divided into a number of topics. In 'the first handbook' edited by Grouws (1992) there are nine chapters on research on teaching and learning specific mathematical topics, for example whole number addition and subtraction, multiplication and division, rational numbers, school algebra, geometry etc. In 'the second handbook' edited by Lester (2007) the corresponding section contains eleven chapters. The one chapter on the topic of school algebra from the first handbook is for instance divided into two chapters, "Early algebra" and "Learning and teaching of algebra at the middle school through college levels". There are certainly many ways to classify school mathematics. In this study I will use the term *topic* to denoted a certain area of the mathematics taught in school. One of these topics is *school algebra*. The topic of school algebra contains a number of different mathematical concepts and methods, such as the concepts *unknown*, *variable* and *equation*, and the methods *simplifying*, *substituting* and *solving*. The mathematics or mathematical *content* of the studied lessons are the same in the sense that the same concepts and methods are handled (This does, however, not mean that the *objects of learning* of the studied lessons necessarily are the same, see the next chapter). The contents of the lessons are concepts and methods related to *system of linear equations in two unknowns* from the topic of school algebra. Thus, in the first part of this chapter I review research related to the teaching and learning of school algebra with a special focus on 'teaching of systems of equations'. This review is conducted by taking some of the major handbooks as the point of departure.

In the study video-data from authentic classrooms is used. I use the term *authentic* as opposed to experimental or interventional. The field of 'mathematics classroom research' is in my opinion less well defined, at least compared to research on 'teaching of school algebra'. I therefore choose a different approach in the second part where I review previous research in this field. This review is done based on a systematic examination of research reported in some of the major research journals in mathematics education. This will provide an overall picture of studies where authentic classroom data are used. I then narrow the search, firstly to classroom studies where the mathematics taught in the lessons are so *central* to the study's actual research question that it cannot be exchanged, and secondly to studies where teaching in different classroom is *compared*.

As the design of the study also involves comparison of several lessons from *different countries* I will in the third part of this chapter discuss international comparative research in mathematics education. In this part I will pay special attention to research that makes comparisons involving East Asia, and especially China.

The overall purpose of this chapter is to present a picture of research to which the current study relates.

Systems of equations

The mathematical content of the lessons in this study contains concepts and methods related to *system of linear equations in two unknowns*. In this section I will start with a short orientation of this mathematical content and then review the research into the teaching and learning of school algebra.

Mathematical considerations – terminology

Throughout this thesis there will be frequent use of some mathematical terminology. The most central of the terms used will be explained in this section. The explanation is, to some extent, based on several sources. The sources are dictionaries such as Clapham and Nicholson (2005), Downing (1995), and Thompson, Martinsson, Thompson and Martinsson (1994), as well as some textbooks (Aufmann, Barker & Lockwood, 1999, 2000; Martin-Gay, 2003; Smith, 1991).

Equation

An equation is a statement of equality of two mathematical expressions. As the equation is a central mathematical concept it is used in most fields of mathematics and there are equations of many different kinds, for example differential equations, functional equations, and Diophantine equations. This study focuses only on elementary equations, where unknowns represent numbers. A distinction sometimes needs to be made between arithmetic equations, where all terms are specific numbers, and algebraic equations, which contain one or more unknown numbers with letters used to represent the unknown numbers.

An equation that is always true is called an *identity*, for example, the arithmetic equation $3 + 4 = 5 + 2$ and the algebraic equation $2x + 4 = 2(x + 2)$. An equation that is never true is called a *contradiction*, for example, $3 + 4 = 6 + 2$ and $x + 2 = x - 2$. An equation that is true for only some values of the unknowns is called a *conditional equation*, for example, as in $2x +4 = x + 6$. The values that make a conditional equation true and *satisfy the equation* are called solutions to the equation. For instance, the equation $2x + 4 = x + 6$ has one solution, $x = 2$, which satisfies the equation and makes the left-hand and right-hand side of the equation equal. If $x = 2$ is substituted into the equation (x is replaced by the numerical value of two) both sides will be eight. In many cases, not least in school, the term equation refers to conditional equations. It is more or less taken for granted that the equations are conditional equations.

Linear equations in two unknowns

A linear equation in two unknowns is an algebraic equation with two unknowns of the first degree. An unknown of the first degree is not raised to any power (other than one). A solution to an equation in two unknowns is an ordered pair of numbers that will satisfy the equation. A linear equation in two unknowns will have infinitely many solutions forming a line if illustrated graphically, hence the term *linear*.

System of linear equations in two unknowns

A system of equations is a group of equations that are considered together. It is also called *simultaneous equations*. A system of linear equations in two unknowns is a system of two (or sometimes more) linear equations that are considered simultaneously. The equations together contain two unknown numbers of the first degree. A solution to the system is an ordered pair of numbers that will satisfy both (or all) equations in the system. There are three possible cases – the system will have either none, one or infinitely many solutions.

In the actual school setting in this study the range of numbers (coefficients, constants and solutions) is in principle limited to the set of rational numbers.

Solving a system of linear equations in two unknowns

There are many methods for finding the number pairs that satisfy the equations simultaneously, for example the graphical method, the method of elimination by addition or subtraction, and the method of substitution. In the studied classroom sequences the method of substitution is introduced and it is also in some cases described by a number of steps. These descriptions vary in detail between different classrooms, but they all basically aim at describing the same method. Here I provide a description of the method of substitution from a U.S. college textbook just to give an indication of how it can be articulated.

> **Solving a system using substitution**
>
> Graphing the equations of a system by hand is often a good method for finding approximate solutions of a system, but it is not a reliable method for finding exact solutions. To find an exact solution, we need to use *algebra*. One such *algebraic* method is called the **substitution method.**
>
> **Solving a system of two equations using the substitution method**
>
> Step 1. Solve one of the equations for one of its variables.
>
> Step 2. Substitute the expression for the variable found in step 1 into the other equation.
>
> Step 3. Find the value of one variable by solving the equation from step 2.
>
> Step 4. Find the value of the other variable by substituting the value found in step 3 into the equation from step 1.
>
> Step 5. Check the ordered pair solution in *both* original equations.
>
> (Martin-Gay, 2003, p. 262)

I will use the term *source equation* to denote the equation that provides the expression (step 1) which will be inserted into the other equation, called the *target equation* (step 2). In order to avoid unnecessary computational complexity it is wise to be careful when choosing which unknown in which equation (the source equation) to use in step 1 in the description above. Some of the difficulties involved in the performance of the method of substitution will be discussed below.

Teaching and learning of school algebra

The concept 'system of linear equations in two unknowns' is part of the mathematical topic often referred to as 'school algebra'. The word 'algebra' comes from the Arabic word 'al-jabr', which is an operation used for solving equations described in *al-Kitab al-mukhtasar fi hisab al-jabr w'al-muqabala* [Approx. *A compendium on calculation by al-jabr and al-muqabala*] by the 9th century Persian mathematician al-Khwarizmi. It is basically a text about solving linear equations. Al-jabr is the process of adding equal quantities to both sides of an equation and al-muqabala is when you subtract equal quantities from both sides of an equation (Thompson et al., 1994).

There are some features of school algebra that distinguish it from arithmetic. To students, the most obvious feature is probably the presence of letters that denote numbers. In arithmetic you deal with problems where you can start with known numbers and work towards the unknown (answer). All operations are performed on particular numbers. In algebra you often operate on and handle relations between indeterminate numbers. The numbers can be specific but yet unknown, 'any number' or numbers that are allowed to vary within a given range (variables). Exactly which mathematical concepts, ideas and methods that are included in the curricula, courses and textbooks that deal with school algebra varies worldwide (Sutherland, 2002).

School algebra turns out to be difficult to master and thus has rendered a lot of research interest (Bednarz, Kieran & Lee, 1996; Kieran, 1992, 2007; Stacey, Chick & Kendal, 2004; Sutherland et al., 2000). A large part of the research in this field is directed towards the students – investigating learning and understanding of algebraic concepts, ideas and methods – and towards the 'beginning algebra' or the transition from arithmetic to algebra. Today there exist much knowledge about difficulties that beginning algebra students may face. Less research has been directed towards the teaching of algebra.

> We researchers have built a large base with respect to the learner of algebra and have developed an extensive understanding of the nature of algebra learning. We have yet to develop the same understanding of algebra teaching and of the kinds of practice that are effective in bringing about such learning in algebra students.
>
> (Kieran, 2007, p. 747)

'Systems of linear equations in two unknowns' is not among the early algebra concepts in any syllabus and, hence, there are quite few studies that focus on this area specifically. Two of the more extensive reviews of research on the teaching and learning of school algebra can be found in the two handbooks of 'research on mathematics teaching and learning' from NCTM (Grouws, 1992; Lester, 2007). Both reviews on school algebra are written by Carolyn Kieran. In the first review there is no research on 'systems of equations' present, and in the second just three studies specifically focused on the learning of this concept are reviewed.

> Up until quite recently, researchers have known very little about the ways in which students of this age range [grade 6 to 9, approx. 11 to 15 years of age] approach the solving of systems of equations and the manner in which they think about its underlying concepts.
>
> (Kieran, 2007, p. 723)

If there has been little research on the learning of this concept, there has been even less on the teaching. I have not been able to find any study on the teaching of 'systems of equations'.

The specific conditions for teaching and learning of systems of equations seem to be less researched than many other areas of school algebra. However, some of the findings from studies into other parts of school algebra might be relevant where others might be less so. The latter can be the case for many findings from the research on 'beginning algebra' or 'transition from arithmetic to algebra'. When students are introduced to 'systems of equations' they usually have experience of several other concepts and methods from school algebra, typically including 'linear equations in one unknown', 'simplifying expressions' etc., and may have begun to overcome some of the documented difficulties with beginning algebra. I will however give an overview of the research on the teaching and learning of school algebra regarding the findings I deem relevant to the teaching and learning of 'systems of equations'.

Research on learning of systems of equations

Much research attention is directed towards understanding students' obstacles when moving from arithmetic to algebra. I will briefly review the general research in this area and discuss some of the findings in relation to teaching and learning of 'systems of equations'. I will do so by attending to two key relations. The first is the relation between *form* and *meaning*, the other is the relation between *dynamic* aspects (procedure/operation) and *static* aspects (structure/object) of a mathematical entity.

In mathematics in general, and in algebra in particular, there is an interesting relation between the form and the meaning of mathematical symbols (see for example Bergsten, 1990). The attention can be directed to the symbols in themselves (the form) and/or towards what they might represent (the meaning). Looking at figure 2.1 while considering two different questions can provide a simple example of this. If the question is: *Which digit is the largest*? the attention will most likely be directed

5 8

Figure 2.1 *Relation between digit (form) and number (meaning)* towards the form of the symbols. If you are asked: *Which number is the largest*? the attention probably will be on the meaning of the symbols.

The use of letters is a key feature of school algebra and students' understanding of letters have been the focus of many studies (Kücheman, 1981; MacGregor & Stacey, 1997; Quinlan, 2001). The findings indicate that one major difficulty in mastering school algebra is related to the interpretation of the letters used in algebra. A letter (x for instance) can be used in different ways and will play different roles in different contexts. Some examples of the different use of a letter in algebraic expressions are shown in figure 2.2.

$$x + 3$$

$$x + 3 = 7$$

$$x + y = y + x$$

$$(x + 3) \cdot 2y$$

$$x + 3 = 2y$$

$$f(x) = x + 3$$

$$\begin{cases} x + 3 = 2y \\ 2x - y = 11 \end{cases}$$

Figure 2.2 *Different roles of the same symbol 'x'*

The meanings of x in the expressions in figure 2.2 are by large given by the context. In the context of equations the letters are used to represent specific (but yet) unknown numbers. This is the case in for example $x + 3 = 7$. The task when dealing with equations mostly concerns finding and revealing the identity of the unknown numbers. Letters can also be used to represent numbers that are *known* (given), *any number* or a *variable* number that can vary with a certain range. To an experienced person in algebra there is little difficulty in recognising and interpreting the meaning of x in these different expressions (figure 2.2). That is not the case for the novice.

One early study, often referred to, that describes different ways student understand letters in algebra was carried out by Kücheman (1981) as a part of the 'Concepts in Secondary Mathematics and Science Project' (Hart, 1981) that had the aim of providing a representative picture of English children's knowledge. A large number of students were tested and one focused aspect was the meaning students assigned to letters in mathematics. It was found that a large proportion of the students showed a mathematically insufficient understanding of the letters. Similar results can be found in other studies, (cf. MacGregor & Stacey, 1997; Quinlan, 2001). It seems that many students never try to understand the use of letters but rather aim at 'remembering the procedures'. Bergsten (1990) showed in his thesis that students (high school) can learn quite quickly to master a formal system (called VOX). In the study, VOX is introduced to students as a game where three symbols V, O and X can be manipulated according to four rules. Bergsten concludes "[...] students, after a very short introduction, without any larger difficulties can deduce simple theorems in a previously unknown formal system, [...]" (ibid., p. 139, my translation from Swedish). To the students, the symbols V, O and X have no specific meaning related to anything outside the 'game'. The symbols do not represent anything in particular. However, it is possible to interpret the VOX game as a way to represent and describe addition of even and odd integers. However, it turned out to be difficult to make the students see and use this interpretation.

> The interviewer tried, after having checked that addition of even and odd integers were known, to make the students realise that there was a possible interpretation as odd and even numbers. It turned out that such a connection between form and content didn't come automatically or naturally. The interviewer had to, with struggle, lead [pilot] the interviewees in the direction of at least a partial understanding of the actual interpretation.
>
> (Ibid., p. 138, my translation)

The students who first met VOX as a completely formal system where the symbols are void of 'external meaning' (they do not represent anything outside the game), seem to be reluctant to later assign such 'meaning' to the symbols and seem to be quite content with being able to 'mechanically' apply the rules. This finding can be an indication of what happens in many algebra classes; students are quite comfortable with just applying rules 'mechanically' and 'move around symbols', at least as long as they get the correct answers. In my opinion the relation between form and meaning is one key challenge in teaching algebra.

One of the major strengths of algebra when solving problems, be it problems from within mathematics or from other areas, is that when you have managed to represent the problem by algebraic symbols it is possible to 'disconnect' the symbols from any 'meaning' and apply the appropriate rules (in most cases this can be handed over to some technical device). An equation for instance can be solved by the same procedure no matter how it was generated and regardless of the specific problem context. The nature of the problem, the kinds of quantities etc. that are represented in the equation do not matter when it comes to the actual solving. When the solving part is finished, however, it is time to 'reconnect' the symbols to the 'meaning' and evaluate the solution in terms of the original problem. The possibility of working with algebra in this 'de-contextualised' mood is of course a powerful property. It makes it possible to apply and use algebra in many kinds of problems, but it is at the same time probably one of the reasons why so much difficulty is connected to the learning of this topic.

In school algebra there are basically three different ways to interpret and treat letters. 1) The letters are not interpreted and treated as numbers at all, 2) the letters are interpreted as representing one (unknown) specific number, and 3) the letters are interpreted as representing many numbers simultaneously. These ways parallel the three stages that often are used to describe the historical development of algebra (cf. Harper, 1987). The stages are called the rhetoric, the syncopated and the symbolic stages. In the first stage no symbols to represent unknowns were used. The second started with Diophantus (3rd century AD), who introduced the use of letters to represent unknown numbers. Up until the 16th century 'algebraists' developed the art of solving equations. The third stage, with letters to represent 'known' or 'given' numbers, began with the work of Vieta and Descartes, and opened up for the symbolic representation of concepts such as variable and function. This was called *the symbolic abstraction* by Thompson (1985, 1996). In the case of 'systems of linear equations in two unknowns' a letter in most cases denotes one particular but yet unknown number. The most common case is that a system of equations has one particular solution. However, when one of the equations is looked upon separately the letters represent infinitely many numbers, as there are infinitely many number pairs that solve a single linear equation in two unknowns. This variation in the number of solutions, and the shift between a letter standing for *one* number and *many* numbers, would indeed be an interesting focus for a study. The previously mentioned studies on students interpretation of letters (Kücheman, 1981; MacGregor & Stacey, 1997; Quinlan, 2001) all indicate that a larger proportion of students can cope with situations where letters represent one particular unknown than with situations where letters represent many numbers simultaneously.

In both Sweden and China *Latin letters* are used to denote numbers in algebra, for example x for an unknown number in an equation. Swedish students already know these letters and have to interpret them in a new context. Here we might have an interesting difference between students who use Latin letters in writing their native language and those, like the

Chinese students, who use other symbols for writing. To Chinese students the Latin letters might not be loaded with 'meaning' from another use, when they appear in mathematics.

As noted before, when students are introduced to 'systems of equations' they most probably have studied other areas of school algebra, such as 'linear equations in one unknown' and 'algebraic expressions'. The use of letters to represent numbers should be familiar to them when they start with 'systems of equations'. Letters standing for numbers will probably not be at the core of instruction. It is quite natural in a hierarchical subject as mathematics that teachers start from what is supposed to be known when new concepts are introduced. The teacher, however, should always be prepared for and sensitive to students who may not have a sufficient understanding of previous content and from other topics.

I have found no studies concerning the understanding of letters used in systems of equations. However, there is one interesting finding in a study by Vaiyavutjamai, Ellerton and Clements (2005). They studied students' attempts to solve quadratic equations.

> Many students (perhaps a majority of them) did not realise that if a variable (say x) appeared twice in an equation (for example, with $x^2 - 8x + 15 = 0$, or $(x - 3)(x - 5) = 0$), then it had the same value in the different "places" in which it appeared.
>
> (ibid., p. 736)

Most teachers will probably take for granted that students realise that the x:s in an equation represent the same number (or numbers). If there is insecurity whether x:s in one equation like $x^2 - 8x + 15 = 0$ are 'the same' or not, there might be similar uncertainty regarding x:s appearing in *two different* equations in a system of equations.

In discussing research on school algebra the second relation is between the *dynamic* and *static* aspects of student conceptions. Most mathematical entities can be interpreted in two significantly different ways. Sfard (Sfard, 1991, 1992; Sfard & Linchevski, 1994) has in a number of articles discussed the 'dual nature' of mathematical conceptions; the dynamic operational side on the one hand and the static, structural side on the other (see also Bergsten, 1990; Gray & Tall, 1994; Hiebert, 1986). Some of the difficulties that students have been found to meet in the area of school algebra can be interpreted in the frame of the dynamic-static duality. A simple example of the duality is provided by figure 2.3. Look at the symbols and think of their possible meaning to you.

One possible interpretation is that the symbols describe a procedure. Then the meaning is dynamic, 'two divided by three'. This interpretation almost makes you 'want to start to calculate' and get the answer. Another

$$\frac{2}{3}$$

Figure 2.3 *Example of the relation between dynamic and static aspects*

possible interpretation is static. The meaning is now structural, and the symbols describe an object, 'the number two thirds'. This time you do not get a feeling that something should be done. On the contrary, it is rather as if it has already been done.

Of course one could argue that it is just an unfortunate coincidence that two different concepts, division and fraction, happened to get the same symbolic representation. In fact, in this case and in many others it turns out to be convenient to use the same symbolic form (for example the use of '–'both for subtraction and for negative number). The two concepts in this symbolic form can technically be manipulated by the same rules. For example division is the inverse operation of multiplication, and if we view 2/3 as '2 divided by 3' this operation can be 'undone' by a multiplication by 3. Then we have: $2/3 \cdot 3 = 2$. If we instead view 2/3 as 'the fraction two thirds' and multiply this fraction by 3 we will also get 2 (again: $2/3 \cdot 3 = 2$). If we just work on the symbols (form) it does not matter if 2/3 is viewed as a division or as a fraction.

Some of the difficulties students encounter when they begin with algebra are related to the relation between dynamic and static conceptions. The conceptions of 'equality' have been shown to be important in relation to the understanding of equations (Kieran, 1981, 1992). The dynamic use of the equal sign is frequent in arithmetic and an expression as $2 + 3 = 5$ is then interpreted as "we have two, add three and then get 5". The expression is seen as a procedure going from left to right, where the left side is what we have in the start and the right side is what we get *after* the operation is preformed. This interpretation can also work on the most elementary equations that students typically meet first. In equations such as, $11 + x = 25$, the dynamic interpretation still makes sense – "What number (x) is added to 11 in order to get 25?" The shortcomings of this view become evident when students meet less simple equations where unknowns are present in both sides, for example $2x + 3 = 13 - x$. In this case the equal sign must be viewed statically, the left side and the right side have to exist simultaneously and to be equal. You can hardly start with the left side and end up with the right side. Also in order to apply the rules of cancellation when solving equations the whole equation must be regarded as an object (static) that can be operated on. These observations are still relevant when students move on from linear equations in one unknown to systems of equations in two unknowns.

Investigations of how students try to solve problems involving two unknowns (basically corresponding to two simultaneous linear equations in two unknowns) revealed another aspect regarding conceptions of the equal sign that might cause difficulties (Filloy, Rojano & Solares, 2003, 2004). The students had previously been introduced to solving linear equations in one unknown, but had not solved problems with two unknowns. Observations were made regarding difficulties in applying the ”transitivity of the equal sign”. For instance, students faced with the two equations $4x - 3 = y$ and $6x + 7 = y$ were found to be unable to use the transitivity to get for example $4x - 3 = 6x + 7$.

An alternative interpretation is to view it as an inability to substitute y in the first equation with an expression equal to y from the second equation. This can be interpreted as the student seeing the two y:s as different or, in the ’dynamic-static’ frame, difficulties with treating an expression as an object in its own right. In order to use the method of substitution and to replace an unknown (y) by an expression, the expression ($6x + 7$) must be seen and treated as a mathematical object, not as a procedure (”calculate six times an unknown number and then add seven”).

A similar obstacle was found in a study where students used CAS calculators when solving systems of equations by a method called ISS (Isolate-Substitute-Solve) (Drijvers & Herwaarden, 2000). ISS is basically the method of substitution performed with the aid of an advanced calculator. One of the unknowns in one of the equations can be ’isolated’ by the command *solve*, then substituted into the other equation by the command *with*. The resulting equation then contains only one unknown and is solved by a second use of *solve*. For example, the first step involves solving the equation $x + y = 31$ for y and thus getting $31 - x$ which can be substituted into the other equation. ”Students had conceptual difficulties here in using the command ’solve’. To them ’solve’ means finding a solution, whereas $31 - x$ is an expression, not a solution” (Drijvers & Herwaarden, 2000, 263). This indicates that the students interpreted the expression dynamically as an operation that should be executed and not as a static entity. Drijvers and Herwaarden found that the second step, the substitution, was the hardest point and explain this by students’ difficulties in seeing an expression as an ’object’ and not as a ’task’ or an action’. ”[...] it [$31 - x$] has to be seen as a static object, as an entity that can replace the variable y, [...]” (ibid., p. 270).

To summarize, not much research has been directed specifically at the learning of ’systems of equations’ but there are relevant findings from research into other areas of school algebra. Students' difficulties with handling the relation between form and meaning, and between dynamic and static aspects can be expected to prevail also regarding the concepts and symbols in relation to 'systems of equations'. More specific findings are the following.

- Students might not realise that the same letter represents the same number in both equations.
- Students might not realise that letters may stand for one number as well as many number simultaneously.
- Students might not be able to apply the transitivity of the equal sign.
- Students might not be able to see and operate on expressions as objects (in the method of substitution).

Research on teaching of systems of equations

As already mentioned I have not found any studies specifically directed to the teaching of 'systems of equations'. Nor is there much research on teaching of school algebra in general. Kieran concludes in her first review that "[...] the amount of research that has been carried out with algebra teachers is minimal" (Kieran, 1992, p. 413) and that

> [...] the literature on mathematics teaching does not describe the ways in which the teaching of algebra ought to be considered in a different light from, say, the teaching of geometry or arithmetic. This body of research tends to focus not on the distinctions to be made according to the various subject matters, but rather on the commonalities in the teaching of mathematics classes, such as time spent on whole-group instruction versus seat work, teaching for rote learning versus teaching for understanding, the role of reviewing, constructivist approaches to teaching, motivation, social dynamics of the classroom, and so on. In much of the research literature on teaching mathematics, the actual content to be delivered is generally treated as a variable.
>
> (Kieran, 1992, p. 394)

I found a similar lack of considration of the significance of the specific mathematical content in much 'mathematics classroom research'. This is discussed in the next section.

In spite of an observed increase in research on teaching and teachers of algebra during the 15-years period between the first and the second handbook Kieran still has to conclude that ”researchers still know relatively little about algebra teaching” (Kieran, 2007, p. 739). A reason for this can be that more research is directed to teachers than to teaching. There seem to be an increasing interest in ’teachers pedagogical content knowledge’ also regarding school algebra. In the report from the *Working group on teachers' knowledge and the teaching of algebra* in the 12th ICMI study (The future of the teaching and learning of algebra) (Doerr, 2004), the group presents their work ordered in three areas, none of which concerns ’the teaching of algebra’. The research on teachers’ knowledge and teaching practices is discussed in ”(a) teachers’ subject matter knowledge and pedagogical content knowledge, (b) teachers’ conceptualisations of algebra, (c) teachers learning to become teachers of algebra” (ibid., p. 270). The group observes that much of the research tend to focus on teachers’ shortcomings regarding both their own understanding of algebraic concepts and ideas, as well as their insight into students’ conceptions. Kieran also notes this last aspect of teachers’ knowledge.

> One of the current themes of the existing research into algebra teachers’ pedagogical content knowledge concerns teachers’ knowledge of students’ algebraic thinking. Recent research suggests discrepancies between teachers’ predictions of students’ difficulties and students’ actual difficulties.
>
> (Kieran, 2007, p. 740)

Approaches to algebra: perspectives for research and teaching (Bednarz et al., 1996) can be viewed as an ambitious initiative to examine the teaching of algebra. In this volume four different ways to introduce algebra are discussed. They are, introduction by *expressing generality*, via *problem solving*, via *modelling*, and a *functional* approach. Even so, much of the focus is still on the students. The discussions are more about how the different approaches can facilitate students’ development of algebraic thinking, rather than about the actual teaching of algebra. Of course, all discussions about findings from investigations in student thinking, student obstacles, students’ misunderstandings etc. can inform and be of importance to the teaching of a subject.

The importance of teachers’ knowledge of students’ difficulties was, for instance, observed in a study by Tirosh, Even and Robinson (1998), in which two somewhat in-experienced teachers, unaware of a common student obstacle, did not manage to deal with it properly during instruction and even were unable to articulate what the students’ difficulties were in post-lessons interviews (there is a more extensive review of this study in the next section).

What can then be said about the teaching of 'systems of equations'? In conclusion, there is little research on the practices of algebra teaching in general and I was able to find none on the teaching of 'systems of equations' specifically. However, findings from the more numerous investigations into *learning* of school algebra, the characteristics of algebraic thinking, students' errors etc. may certainly have implications for teaching. It seems that one step for a teacher in developing efficient teaching is to be aware of in what ways students usually think regarding the content that are to be taught. More specifically concerning the teaching of 'systems of equations', the findings from the previous section may give some indications.

Following this line of reasoning, efficient teaching of 'systems of equations' in some ways has to deal with the problematic relation between form and meaning, and between the dynamic and static aspects, as well as, more specifically with the role of letters, the transitivity of the equal sign and operations on expression as objects.

Mathematics classroom research

This review is conducted in order to get a picture of current classroom research in mathematics education. The review is restricted to research papers published in some of the leading research journals in mathematics education during the last five to ten years (details below). Unfortunately only papers written in English and Scandinavian languages have been included in the review. There might of course be relevant research that is published in other ways, but I choose the systematic and thorough search in a restricted area in favour of a more wide-angled but less complete search. Whatever method, a review of this kind will anyhow produce just one of many possible pictures.

To 'get a grip' on the field of mathematics education research that is interested in studying 'authentic' mathematics classrooms (contrary to 'experimental' or 'interventional') my ambition was to keep the following questions actual and in the foreground of my attention throughout the review.

- What are the research interests – *what* is studied, in *what* classrooms?
- *How* is the research conducted?

– What is the significance of the *mathematics* taught in the studied classroom?

Review procedure

Mathematics classroom research is reported in many different ways, e.g. as articles in research journals, as presentations at conferences and in proceedings, as project reports, and as a chapter in or as complete books. Considering the vast amount of research that is published it soon became quite obvious that some sort of limitation was necessary. I decided to restrict the review to articles published in *mathematics education research journals*. The journals selected for the review would have to (1) publish articles reporting empirical research in mathematics education, (2) employ a referee-system for evaluating proposed articles and (3) publish articles in English or in Scandinavian languages. This will generate *one* picture of the research in question, a picture that, of course, might be different from a review where other sources also were to be included. As the attention of this study is on the mathematical content it seemed relevant to restrict the review to articles that are published in journals in *mathematics* education. The requirement of a referee procedure is the established way to ensure research quality and that reported findings are solid. The restriction to articles written in English or 'Scandinavian' is of course a weakness. The review, for instance, will not account for research that is only published in other major languages, such as Spanish.

Before deciding upon how to conduct the search for articles in more detail, a 'pilot-review' was done. The Journal for Research in Mathematics Education was chosen as the journal for the trial review (Häggström, 2006a). The reason for this was that the journal was the only research journal in mathematics education listed in the ISI web of knowledge database (www.isiwebofknowledge.com). Repeated searches, using search terms like *classroom* & *teaching*, *classroom* & *learning*, *classroom* & *interaction*, *classroom* & *observation*, *classroom* & *research* etc., both in ISI and at the journals' own website (http://my.nctm.org), generated a number of articles. The results from this search were compared to a "manual search" of the five most recent volumes. The manual search was done by means of reading abstracts, and when called for the whole article, looking for studies of authentic mathematics classrooms. The results from the manual search and from the search assisted by search-engines on the websites were then compared. The comparison found that more articles were generated by the manual search. There was even one article (Montis, 2000) found by the manual search that was completely missing from the ISI database. The decision was then made to use the manual search method in the main review, even though it was much more time-consuming. The manual search allowed for more informed decisions concerning which articles to include and which to exclude in the review. Studies of 'authentic' mathematics classrooms were not always easily identified. There are of course many cases where apparent elements of experiment or intervention are present in a study that by and large is 'naturalistic' in its approach. The data used in a study may also be of many different types. Field notes and video records etc. of classroom activities may just be employed to a smaller extent. The main contributions could be from other sources, such as interviews or student tests. The manual search was deemed to open for a possibility to be more inclusive when it came to this kind of decision, compared to a search by means of search-engines.

The library at NCM (National Centre of Mathematics Education, University of Gothenburg) has an ambition to subscribe to all major journals and magazines in the field of mathematics education. The library keeps more than seventy journals from all around the world. Of that large number of journals few are research journals that use a referee procedure before accepting articles for publication. Finally, eight journals that met the criteria stated above were included in the review and are listed in table 2.1 (Journals with specific interests, for example on the role of computers in mathematics education or on the education of mathematics teachers, were not included). At least the issues from the years 2001 to 2005 for each journal were reviewed. For some journals it was possible to expand the review further (se table 2.1).

Table 2.1 *Research Journals included in the review*

Journal	Issues	Time
Educational Studies in Mathematics, ESM	32 (1) – 60 (3)	1997–2005
For the Learning of Mathematics, FLM	21 (1) – 26 (2)	2001–2006
Journal for Research in Mathematics Education, JRME	28 (1) – 37 (2)	1997–2006
Mathematical Thinking and Learning, MTL	1 (1) – 8 (3)	1999–2006
Mathematics Education Research Journal, MERJ	13 (1) – 18 (1)	2001–2006
Nordic Studies in Mathematics Education, NOMAD	1 (1) – 10 (3/4)	1993–2005
The Journal of Mathematical Behavior, JMB	19 (1) – 24 (3/4)	2000–2005
Zentralblatt für Didaktik der Mathematik, ZDM	32 (1) – 38 (4)	2000–2006

As discussed earlier the review was conducted without the aid of search-engines and any kind of databases. All abstracts of the articles were read. When there was an indication that the study might involve observation, video- or audio-recordings of authentic mathematics teaching the entire article was read. The articles that reported studies of authentic teaching/learning situations were then selected. The first review of these articles, however, did not involve exhaustive reading and analysing in detail, but a reading where at least the following features in the reported studies were noted.

Object of the study

Main focus on Teacher/teaching or/and on Students/learning

Mathematical content (if any particular)

Age of the students/grades in the study

Number of classes in the study and number of lessons or time-span for observations

The articles were also categorised depending on how central the actual mathematical content was to the study. The categorisation was done by aid of the following questions. Can the mathematical content taught in the studied lessons/classrooms be exchanged for other mathematical content without altering the study? Or is the particular mathematical content so essential that it would be meaningless to study a classroom where something else was taught? For instance, if the research interest concerns students' ways of thinking when doing mental *subtraction* it is obviously pointless to observe classrooms where *multiplication* is taught. On the other hand, if the research interest concerns differences in how mathematics teachers ask questions to boys and girls, the particular mathematics is of less importance (or of no importance at all).

Some general points

Of the approximately 1260 articles looked at, a substantial part, 179 articles (14 %), made use of empirical data from 'authentic' mathematics classrooms (see table 2.2) The teaching and learning of mathematics described in these articles, may have elements of, but was judged not to be entirely of an experimental kind. In this section I will try to characterise the research presented in these articles. The first questions to answer is, what is studied when mathematics education researchers enter the classrooms? The simple answer is, almost everything.

The picture of mathematics classroom research that emerges from reviewing these articles is very diverse. There is a quite even distribution between studies in elementary and secondary classrooms, around 80 in

Table 2.2 *Mathematical topics focused in the reviewed articles*

Journal[1]	No. of articles	Studies of 'authentic' classrooms	Focus on math	Mathematical topic									
				Arithmetic	Geometry	Algebra	Fractions	Proof/reasoning	Functions/graphs	Statistics	Linear equations	Measurement	Other
ESM	370[2]	58	13	3	1	2	-	1	-	2	2	1	1
FLM	110[2]	8	3	1	1	1	-	-	-	-	-	-	-
JRME	200[2]	32	9	-	2	2	1	2	1	-	-	1	-
MTL	102	14	4	1	1	1	-	-	1	-	-	-	-
MERJ	70	14	2	-	-	-	-	-	-	1	1	-	-
Nomad	99	16	4	1	1	-	1	-	-	-	-	-	1
JMB	141	30	10	2	1	-	3	-	1	-	-	-	3
ZDM	167	7	3	1	1	-	-	1	-	-	-	-	-
Σ	1259	179	48	9	8	6	5	4	3	3	3	2	5

Notes. 1. Abbreviations see table 2.1. 2. Estimated number of articles.

both cases. The tertiary level is less represented with approximately 20 studies. My rather crude labelling of the research, into (a) studies that mostly focus the mathematics *teachers* or the *teaching* of mathematics (55 cases), (b) studies that mainly focus the *students* and/or *learning* (71) and (c) studies that are addressing both *teachers and students* (52), also demonstrated a fairly even distribution. This image of diversity is illstated here with short descriptions of some of the research interests addressed.

Role of teaching materials, textbooks, calculators (CAS) and computers
Math teaching in multi-age classrooms, and in multilingual classrooms
Math teaching and learning of low performing students, and of students with dyscalculia
Long-term development of student understanding (often of certain math concepts)
Role of student-student interaction, collaborative and small-group work
Interaction patterns
Cultural patterns
Implementation of reform curricula, and of "inquiry" teaching
Teachers' actions, mode of questioning and time management
Role of attitudes, emotions, self-concepts and student engagement
Relation between teacher beliefs and teaching practice
Gender issues, e.g. girls' experience of computers
Socio-economic factors (socio-political), and teaching ethnic/language minorities

Socio-mathematical norms in the classroom, and learning communities
Role (and use) of signs, and gestures in communicating mathematics
Role of mathematical modelling and type of problems (e.g. more or less challenging)
Teachers' understanding of, and implementation of reform practices

In 48 of the 179 articles with 'authentic classroom data', the research interest is tightly connected to the actual mathematics. In these studies the mathematics cannot easily be replaced by other content. It means that just about one out of four classroom studies, published in these mathematics education research journals, actually shows a research interest that places importance on the mathematical content. It might be considered somewhat surprising, or perhaps it just illustrates the complexity of mathematics teaching and learning. The complex processes involved can of course be studied from many different perspectives and with the research attention towards many different and equally important aspects.

The mathematics in these 48 studies is spread across the mathematical landscape. Table 2.2 gives an overview of the articles included in this review and the mathematics content. Arithmetic, geometry and algebra are the most frequent topics that attract the attention of the researchers. The main interest is directed to the actions of the students/learning in 22 of these articles and to both the teaching and learning aspects in 18. There is less interest towards the teacher/teaching of certain mathematical topics (8 articles).

Beside data collected inside the classrooms, in the form of observation notes, audio- and video-recordings, many studies also use other data-sources, such as teacher and student interviews, student material and assessments, teaching plans etc. However, the use of video technology seems to be increasing. This is a natural development considering the possibility of repeated watching, slow motion and the high quality of both vision and audio. This will of course contribute to more accurate and, thus improved, analysis compared to the reliance of "spur-of-the-moment" field notes.

A bit more surprising then is the rather low frequency of articles analysing data from large-scale classroom studies. Data from the TIMSS Video studies of 1995 and 1999 is found in just 6 articles (Huang & Leung, 2002; Jacobs & Morita, 2002; Kawanaka & Stigler, 1999; Leung, 2005; Lopez-Real & Leung, 2001; Santagata & Barbieri, 2005). Notice that three of these have the same author (Fredrick Leung, Hong Kong). Data from the Learner's Perspective Study is found in just 2 articles by the same authors (Amit & Fried, 2005; Fried & Amit, 2003). The fact that most of the studies are of a rather small scale is also reflected in that most studies are conducted in one country. There are only 7 articles that contain classroom data from more than one country.

Comparing classrooms

In a majority (35) of the 48 articles just one class/classroom is studied. In six of these the data comes from just one lesson (or part of one lesson) from the single class. More than one classroom is included in 13 articles – the most common is two, three or four classrooms (11 instances). These studies are of special interest to me, as comparisons between different classrooms would be possible. The contrasts, created by comparing two or more classrooms, could generate opportunities to discern features of the teaching and learning that might in other cases pass undetected. In most cases, however, the opportunity to make comparisons between different classrooms is not taken. In just 5 of these 13 articles, there are elements of comparison. In the following I will discuss them further.

Tirosh, Even and Robinson (1998) explicitly compare the teaching of four seventh-grade teachers – two ”novices” and two ”experienced” teachers. The two experienced teachers – both had more than 15 years of teaching experience – were regarded as ”excellent teachers” by students and colleagues. The novices had been teaching for less than 2 years. All four teachers were observed and audio-recorded in the three initial lessons when teaching the same content – ”equivalent algebraic expressions”. The teachers submitted their lesson plans before each lesson, and in post-lesson interviews were asked to reflect on the plan, expectations from their students, the objectives of the lesson and general feelings of the lesson. The research was concentrated around students' tendency to conjoin or ”finish” open expressions. This is a documented tendency by students to write expressions like $2x + 3$ as $5x$ or 5. All four teachers used a ”traditional” approach to teaching – ”[...] emphasis on formal language, procedures and algorithms. Their lessons were teacher-centred with no emphasis on students' investigations” (ibid., p. 53) – and the same textbook were used in all four classes. To observe lessons, where the same mathematics is taught, and to keep a quite narrow research focus on ”students' tendency to finish open expressions” proved to be fruitful in revealing differences between novice and experienced teachers. Differences in the teachers' awareness of the students' ways of handling expressions, as well as differences in how teachers handled the content in the lessons were found.

The two experienced teachers demonstrated apparent awareness of students' tendency to conjoin open expressions. None of the novice teacher showed the same knowledge. This difference was also reflected in the teaching, where the novice teachers seemed to lack strategies to cope with the phenomena. Gilah, one of the experienced teachers, used ”systematic variation” in an activity to help students distinguish between ’like’ and ’unlike’ terms.

> [Gilah in an interview] ”I think that differentiating between like and unlike terms should precede the issue of simplifying algebraic expressions. There is a need to work extensively on the topic of like and unlike terms.”
>
> Her introductory activity consisted of two main parts. In the first one, ’Identifying like terms’, students are told that ’like terms are terms that have an identical combination of variables’ and they receive a variety of examples of like and unlike terms (for example $2x^2$ and $4x^2$, $3ab$ and $6ab$, $5a$ and $6a^2$, $2bc$ and $3ac$, $3ab$ and $-2ba$). Then they practice and discuss identifying like and unlike terms with a great variety of examples. In the second part of this activity ’Collecting like terms’, students are told that ’in order to simplify algebraic expressions, one can collect like terms’. The students then receive a variety of examples that illustrate how to collect like terms, starting with $4a + 2a = 6a$ and gradually reaching more complicated expressions such as $2xy + 4x + 1.5y + 6xy + y = 8xy + 2.5y + 4x$.
>
> (ibid., p. 57)

The classroom observations indicate that this approach appears to have worked. ”In her class students seemed to have mastered this skill [simplifying expressions]” (ibid., p. 59). There are two important findings in this study. First, an awareness of common student ’difficulties’ seems to be vital for teachers in order to plan high-quality lessons and activities, and to be prepared for accurate responses to students. Second, by letting students compare expressions where important features were highlighted, necessary conditions for students to discern these features seemed to be produced.

This finding by Tirosh et al. (1998) – the importance of teacher knowledge of common student thinking – was also one findings in a study by Jacobson and Lehrer (2000). In their study four teachers were observed while teaching the same geometry ’unit’ in grade 2, where students engage in quilt design and transformations (for example rotations and reflections). All four teachers had long teaching experience (7–12 years) and had participated in professional development concerning understanding children’s thinking in arithmetic. In addition to that, two of the teachers were also participants in a professional development program on understanding students’ thinking in geometry and were regarded as having greater knowledge of student thinking in this particular area. Part of the geometry unit consisted of a 5-minute video. The video showed images of quilts and transformations, without any spoken narrative. Two different sequences in each classroom, when teachers and students discussed the videos, were audio taped. The analysis focused on the teachers' statements, and these were coded according to their functions. The students' learning was assessed one time immediately after the geometry unit and again one month after instruction. The differences in learning were found to be connected with differences in teacher knowledge.

> Teachers of all four classes attended to student thought as well as to students' actions during the course of the curriculum. Nevertheless, the trends in the data suggested differences in student achievement. In the two classes (A and B) in which teachers were more knowledgeable about students' thinking about space and geometry, not only did students learn more than did their counterparts, but this difference in learning was maintained over time. This finding suggests the benefits of teachers' having knowledge attuned to nuances of students thinking within a mathematical domain. (ibid., p. 86)

The greater knowledge of student thinking seemed to enable two of the teachers to discuss the mathematical concepts and to respond to students in a way that would clarify the meaning in a more thorough way. Jacobson and Lehrer found a significant difference between how the teachers were "revoicing" students' comments. The two more knowledgeable teachers used the revoicing techniques to reformulate students' informal language and to introduce new terminology to gain more precision, as well as to prompt students to consider why or how something changed in the quilt. The other two teachers also encouraged discussion but did not emphasize the concepts or transformations in the same way.

> Although all teachers elicited students' thinking, the two teachers who were more knowledgeable about students' thinking about space orchestrated classroom talk in ways that refined, elaborated and extended students' thinking, [...] (ibid., p. 86)

Even though the studies by Tirosh et al. (1998), and Jacobson and Lehrer (2000) just involve a small number of teachers they provide two cases showing that well informed teachers may act differently (regarding the mathematical content) in the classroom, and that teachers' actions may make a difference regarding student learning. Differences in how the mathematical content is handled have bearing on the students' experience of and learning of that content.

Saxe, Taylor, McIntosh and Gearhart (2005) also compared several classrooms where the same mathematics was taught, in this case fractions. This study can be seen as a taking an active part in the debate – or ”math war” (see e.g. Becker & Jacob, 1998) – between reform and traditional mathematics teaching in the U.S. The nineteen elementary school classrooms studied were categorised as High Inquiry (8), Low Inquiry (7) and Traditional (4). A coding scheme for the observations of whole-class lessons was employed for the categorisation. In the inquiry classrooms an ”inquiry unit” was used. This material was supposed to support student investigations whereas the textbooks used in the Traditional classrooms were oriented more towards skills and definitions. The distinction between High and Low Inquiry was based on the degree to which students were engaged in conceptually rich discussions. The classroom practice in the High Inquiry classrooms was more ”aligned with inquiry principles” than the practice in Low Inquiry classrooms, even though they used the same material. One of the research questions involved comparing the impact of different classroom practices.

> [...] for students who begin fractions instruction with the similar notational approaches, how are their trajectories affected by participation in classrooms that stress inquiry as contrasted with classrooms that stress skills? (ibid., p. 142)

The students' use of fraction notation and understanding of part-whole relations were assessed before and after the instruction unit on fractions. The analysis addressed students' trajectories of change from pre-test to post-test as a function of the different kinds of instructional conditions. Saxe et al. came to the conclusion that ”students in inquiry classrooms made greater progress than those in traditional classrooms [...] (ibid., p. 154)” and that ”[...] our findings support the value of inquiry instruction in students' learning of fractions notation and reference [...]” (ibid., p. 154).

Runesson and Mok (2005) are quite explicit about comparison in their article. They set out to compare the teaching of the same mathematical content – fractions – in different school cultures. ”Our aim was to capture features which may not be easily observed within one culture, but which might become more visible in the contrast” (ibid., p. 2). The intention of the study was clearly to inform mathematics teachers. The close attention to how the mathematical content is taught will more likely ”provide insights for improvement inside the classroom” (ibid., p. 2), than in the 'outside' organisation. One class in Hong Kong (grade 4) and one in Sweden (grade 6) were observed and audio taped during lessons where ”comparison of fractions with different denominators” was taught.

The comparison was made by an analysis of how aspects of the ”object of learning” (the mathematical concept) were varied (or not) in the interaction. The findings indicate that many aspects of comparing fractions were varied at the same time, simultaneously, in Hong Kong, whereas the pattern of variation in the Swedish classroom had a sequential character. Runesson and Mok conclude, ”the results of a study like this appeal to teachers' professional knowledge and could be inspiration and help for reflection on practice. Seeing what *could be* the case sheds light on what *is* done and *what is the case in our own practice*” (ibid., p. 12). In contrast to some of the other studies, where teaching in different classrooms is compared, there is no comparison of the possible ”effect” on students' learning. No student assessment is conducted.

The last article that contains comparisons of different classroom practice is by Huang and Leung (2002). They compared how the same mathematical content, the Pythagoras' theorem, is taught in Czech Republic, Hong Kong and Shanghai. They also point to the importance of keeping the content the same while comparing teaching in different cultures. ”[...] a study on how the same topic is taught in different cultures will highlight the similarities and differences among different cultures” (ibid., p. 268). One 8^{th} grade lesson from each country – either the first lesson of the unit or the earliest possible – was analysed. The lessons from Czech Republic and Hong Kong were videotaped in the TIMSS-R Video Study and the lesson from Shanghai was taped following the same method. The analysis focused on how the Pythagoras' theorem was taught, including lesson structure and patterns of classroom interaction. Some of the differences found concerned the justification of the theorem, the types of problem used, variation in the exercises, and questioning.

> It seems that the Shanghai teacher stressed deductive mathematical reasoning, while the Hong Kong teacher and the Czech teacher preferred visual reasoning and verification. (ibid., p. 271)

> Overall, the Shanghai teacher provided more challenging problems than the others, and the Czech teacher offered the easiest problems to her students. (ibid., p. 272)

> According to this explicit-implicit distinction, it was found that both types of variation often appeared in the Shanghai lesson, but basically only explicit variation appeared in the Czech lesson and the Hong Kong lesson. By using implicit variation [when problems differ from the ”prototype” problem so that the method of solving

> cannot be applied directly], not only will the problems be more difficult, but they will also be more open-ended as well. (ibid., p. 273)

> [...] the Czech teacher and the Hong Kong teacher adopted a similar pattern of questioning, with more than 70 % of the questions requesting a simple yes or no response, [...] the Shanghai teacher only asked less than 5 % of the questions to request a simple yes or no response and asked around half of the questions to elicit students' explanations. (ibid., p. 273)

The findings suggest that there are similarities between the teaching in the lessons from the Czech republic and Hong Kong, and that The Shanghai lesson differ from the other two in all the studied aspects. The authors found that the students in the Shanghai classroom were quite involved in the "knowledge construction", which challenges the stereotype image of Chinese teaching. Huang and Leung found the methodology of comparing classrooms, where the *same mathematics* is taught, to have advantages, "[...] keeping the content invariant when comparing mathematics classrooms in different cultures will make a study more profitable" (p. 276).

To me it is surprising that the Czech classroom did not differ more from the other two, which can be regarded as culturally (as well as geographically) closer. Another comment concerns the possible application of a Variation theoretical approach to the data. Even without access to the actual data, I mean that it is possible to discern interesting features just from the descriptions in the article. It seems that in the Czech and Hong Kong classrooms it was *taken for granted* that the triangles discussed are right-angled, while the Shanghai lesson started with an exercise that highlighted the difference between *any* triangle and a *right-angled* triangle.

> The Shanghai teacher started the lesson by asking the questions "what is the relationship between the three sides of a triangle?" and "what is the relationship between the three sides of a right-angled triangle? (ibid., p. 270)

The different ways in which this feature of the Pythagoras' theorem is handled in the classrooms may no doubt have an impact on the students' learning and understanding of the theorem.

The five studies reviewed here share some common features. They use data from authentic teaching of mathematics, they compare the teaching in different classrooms and finally they keep the mathematical content invariant. All these features contribute to the findings considerably.

Concluding discussion

One finding in this review of mathematics classroom research, published in these eight research journals, is the huge variation concerning the ”object of study”. All of the studies could be regarded as dealing with the same ”object of knowledge”, namely the *teaching and learning of mathematics*, but still they show a tremendous diversity regarding the focus or *what* of the research. The studies are directed towards many different aspects of the common object of knowledge. I cannot say that I have found any clear trends in this diverse picture. Instead I am inclined to accept that there is a lack of larger clusters of research traditions or alliances. One possible explanation of this diversity is revealed when the low interest in the particular mathematics taught in the studied classrooms is considered. That is another finding of this review. Roughly one out of four studies of ”authentic” mathematics classrooms are dependent on the actual mathematics taught. For the other studies, could it be that the mathematics classroom is just a ”placeholder” for a more general interest in teaching and learning? The rationale behind placing the research inside the *mathematics* classrooms could be similar to the one behind the decision to use attainment in mathematics as a measurement of the effectiveness of educational systems in FIMS (The First International Mathematics Study, 1963–1967). The aim of FIMS was firstly to measure and compare the outcomes of school systems on a national level, only secondly were there an interest in mathematics education.

> The choice of mathematics for this first study was more a matter of convenience than interest in mathematics achievement per se. The organisers believed that it would be easier to make international comparisons in mathematics than in any other area, and they felt that mathematics achievement would serve as a surrogate for school achievement. (Travers & Weinzweig, 1999, p. 19)

The preferences for using mathematics in educational research generally have been noted and discussed by others. Kilpatrick (1993) raise the question of the value of this research to the field of mathematics education research.

> More often than one might predict, mathematics is used in a research study merely as a vehicle through which some aspect of teaching, learning, or schooling is explored. In many studies of problem solving, for example, mathematical problems could be replaced by problems in physics or poetry and the same psychological processes could still be studied. In studies of teachers' questions, the subject being taught might be mathematics, but it might also be history or

> French. In research on how children work in groups, the activities they are doing may be mathematical or they may be not, and the researcher may not care. In epistemological studies, the phenomena being discussed may be mathematical, but the work may do nothing to expand our understanding of what it means to know mathematics as opposed to biology or grammar.
> When mathematics acts as nothing more than a placeholder in a research study, one has to question whether such a study could make a strong contribution to the field. It may be useful research in some other domain, but if it does not treat mathematics in a serious way, it is unlikely to be of much use to mathematics educators.
>
> (ibid., p. 30)

My finding in this review indicates that the same situation is at hand considering classroom research and it raises the question whether these studies really embrace the same object of knowledge, as I initially assumed.

When it comes to large mathematics classroom research projects, such as the TIMSS video studies and the Learner's Perspective Study, one might also find less attention to the mathematics taught. In the TIMSS studies, with the ambition to gather nationally representative data, one would, of course, have to video-record a sample of mathematics lessons regardless of the topic. In the LPS project there is the possibility of paying more attention to the mathematics taught, even though the general idea is obtain data that exemplify teaching of many topics. There seems to be an idea that the data foremost should be suitable for cultural comparisons of mathematics teaching and learning in *general*, not for the study of teaching and learning of any *specific* mathematics. However, it is possible to find and compare lessons where the same mathematics is taught in these projects. The handling of the data from the LPS is still underway in many of the participating countries, which could explain the almost non-presence of this project in the reviewed journals (just 2 articles use LPS data). Much more remarkable is the fact that the data collected in 1995 and 1999 in the TIMSS video studies have not generated more than six articles, even though it has been around for a while. Some video taped lessons have also been published on the Internet.

Finally, there could be a political agenda regulating publishing in at least one of the journals. I found a number of articles in JRME that took a stand in favour of the reform movement with results supporting reform curricula and ”inquiry instruction”. One might speculate if an article with an opposite result (for example ”traditional instruction, focusing on skills and procedures, led to greater progress in student learning") would have any chance of being published regardless of the evidence.

My key finding from this review of mathematics clsssroom research is that the studies that combined a clear focus on the particular mathematics taught and carried out comparisons between different classrooms managed to come up with genuinely interesting results. This finding is also most encouraging in relation to my own research interest.

Comparative research

Large international comparative studies

Measuring and comparing student achievement is at the core of large international comparative studies like TIMSS and PISA. The existence of differences in student learning and performance is well known and obvious to every teacher of mathematics. Differences also exist between countries on an aggregated level as shown in TIMSS (IEA, 2005) and PISA (OECD, 2005). These studies have their focus on comparing the outcome or efficiency of education and use testing of student knowledge and skills as a means in this endeavour. The differences that were stimualted huge public interest. This is especially true of the 'ranking lists', where the participating countries are listed in order of performance.

In order to understand and explain differences in achievement found on a national level, a lot of other data are also collected. These additional data concern variables that possibly could influence student achievement. It includes variables ranging from curriculum content and class size to characterisation of teacher education and student attitudes to mathematics. A simple model that describes the framework that governs these studies is the input-output model shown in figure 2.4.

I use the term *distal factors* to denote the variables that constitute the frame for teaching. It comes from Pong and Morris (2002) who

Distal factors

$\Downarrow$

Teaching process

$\Downarrow$

Student achievement

Figure 2.4 *Framework governing many large comparative studies* distinguish between *distal* and *proximal variables* in their review of studies in this tradition. The distal variables are variables or factors that do not have a direct impact on teaching, such as most 'background' or 'policy' variables that are collected in order to explain differences in student achievement in the large international comparative studies. Proximal variables are the factors that more closely and directly impact teaching and learning. The effect of distal variables on learning outcomes is indirect. They may at best provide circumstances that create an environment for 'good teaching' rather than making it happen.

> One possible conclusion that can be drawn from across these meta-analysis [...] is that to improve school learning, we should focus on those variables that impact directly on the learning experiences of students such as teaching and feedback. Distal variables such as social, economic and political factors are indirect, and they must work through proximal variables to produce their effects. In short, they either support or interfere with the proximal factors rather than replacing them. (Pong & Morris, 2002, p. 11)

Class size is one distal variable that teachers often regard as important and many of them would like to reduce the number of students in their math classes. This view is supported by some studies where class size is shown to correlate to student achievement (Gustafsson, 2003). However, one should be careful not to interpret correlation as a causal relationship. In addition, it is quite simple to demonstrate that the effect of class size, per se, cannot determine the outcome of teaching. For instance, if a very rigid, teacher-centred lecturing style of instruction is used, it is hard to argue that the learning outcome will differ if there are 20, 45 or 100 students in the 'audience' (or if the students are shown an instruction movie). The potential effect of class size on achievement has to be indirect, in that it makes it *possible* to teach in a different way with a smaller number of students, not that it will *necessary* take place.

The large international studies, TIMSS and PISA, tend to neglect proximal factors. The focus on distal factors and student achievement leaves the 'teaching process' (Figure 2.4) as a 'black box'. It means that these studies are quite far away from the actual interaction in classrooms and that factors that mathematics teachers control are left outside of these studies. These studies may yield results that are of interest to school politicians and other policy makers, but of less interest to mathematics teachers.

Mathematics education in East Asia

The distal factors examined in TIMSS and PISA may be of little relevance to this study. However one very consistent result from these studies is the excellent performance of East Asian students. This has led to a considerable research interest in the mathematics education of these countries (see for example Chen, 1996; Fan, 2004; Huang & Leung, 2002; Leung, Graf & Lopez-Real, 2006). This is especially evident in the U.S. where a lot of attention is and has been directed towards comparing U.S. to countries in East Asia (see for example Ma, 1999; Stevenson & Stigler, 1992; Stigler & Hiebert, 1999).

Stevenson and Stigler (1992), who compared elementary school education in reading and mathematics, provided examples of the excellent performance of Asian students. Their rationale was to ”explore ways in which the United States might improve its educational system by learning from the successes of other cultures” (ibid., p. 9). In their first study in 1980 they involved 120 classrooms in the cities of Sendei (Japan), Taipei (Taiwan), and Minneapolis (U.S.). The second study, in 1987, included 204 classrooms in four cities: Sendei (Japan), Taipei (Taiwan), Beijing (China), and Chicago (U.S.). The motivation for choosing Asian countries was simply that they wanted to compare American education with the best.

> Rather than attempting to include a large number of cultures, we chose three for intensive study: Chinese, Japanese and American. Selecting the two Asian cultures was logical because they are among the world's most successful in producing students with high levels of achievement in mathematics. (ibid., p. 33)

Mathematics tests were given to 1st graders and 5th graders and the results show a clear difference between the American children and the Asian children.

> In tests of mathematics, for both grade levels and in both studies, the scores of American children were far lower than those of their Japanese and Chinese peers. (ibid., p. 33)

Even more serious was the finding that the differences in performance on the mathematics tests given seemed to increase over the grades.

> The scores of the American first-graders and the first-graders from the other cities overlap somewhat. By the fifth grade, however, the groups have diverged. The full range of the population was represented in all of the metropolitan areas sampled, yet even the best American schools were not competitive with their counterparts in

> Asia on mathematics achievement. The *highest*-scoring American school falls behind the *lowest*-scoring Asian school. These patterns also emerged in the second study, [...] (ibid., p. 34)

As no differences in the intellectual abilities of children from different countries could be established, explanations for the dramatic differences in achievement had to be found elsewhere. One obvious explanation put forward was that most American children spend less time studying mathematics, both in school and outside school, for example doing homework. But this circumstance could not completely explain the poor performance of the American students. The findings from studying *what* takes place inside the mathematics classroom provided additional explanations. In the two studies more than 800 hours of classroom observations were conducted and the quality of lessons in Japan and China were judged by the researchers as very high.

> If we were asked briefly to characterize classes in Japan and China, we would say that they consist of coherent lessons that are presented in a thoughtful, relaxed, and nonauthoritarian manner. Teachers frequently involve students as sources of information. Lessons are oriented toward problem-solving rather than rote mastery of facts and procedures, and make use of many different types of representational materials. (ibid., p. 176)

> We of course witnessed examples of excellent teaching in American classrooms. But what has impressed us in our personal observations and in our data is how remarkably well most Asian teachers teach. It is the widespread excellence of Asian class lessons that is so stunning. (ibid., p. 198)

The studies of Stevenson and Stigler described here share some common features with the larger studies of TIMSS and PISA in that they claim to compare countries with each other. They attempt to explain the differences in achievement found by means of cultural or national differences. On the other hand Stevenson and Stigler also investigated the classrooms in Japan, China and U.S. These observations made it possible for them to point to some differences in the actual teaching, even though mostly in general terms.

Mathematics teachers in East Asia

So, differences in student achievement could possibly be explained by differences in teaching. How can we understand the differences in teaching? Maybe the teachers are different?

Another study that also attracted much attention was the comparison of teacher knowledge in China and U.S. by Liping Ma (1999). Her study contained interviews with 23 American and 72 Chinese elementary school teachers. In the interviews the teachers were asked to solve some mathematical tasks and also to describe how they would teach the same content to children in school. One of the interview questions used was the following.

> People seem to have different approaches to solving problems involving divisions with fractions. How do you solve a problem like this one?
>
> $1\frac{3}{4} \div \frac{1}{2}$
>
> Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story for $1\,{}^3/_4 \div {}^1/_2 = ?$ (ibid., p. 55)

The Chinese teachers answered both parts of the questions in qualitatively better ways than their American colleagues, even though the American teachers had in total between 16 to 18 years of education and the Chinese teachers only between 11 and 12 years. The American teachers tended to try to remember how to do it and what rule to apply, in favour of trying to interpret any meaning into the problems. For the fhe first part of the question above only about half of the American teachers managed a correct solution, compared to all of the Chinese teachers. Many of the latter also provided alternative solutions. Regarding the second part of the question, which requred teachers to create a situation that could illustrate the mathematical operation, only *one* American teacher could formulate an acceptable answer. The others failed to produce a story or produced stories that contained serious misconceptions. Their stories involved expressions such as ”divide evenly by two” and ”take half of the total”. They displayed difficulties in distinguishing between ”dividing by a half” and ”dividing by two” and some of them confused ”dividing” and ”multiplying”. Most Chinese teachers (90 %) could produce accurate stories and they seemed to have a clear understanding of the meaning of division of fractions.

Ma comes to the conclusion that there was a huge difference in the understanding of fundamental mathematics and that ”the knowledge gap between the U.S. and Chinese teachers parallels the learning gap between U.S. and Chinese students revealed by other scholars" (ibid., p. 124). In order to explain the differences in knowledge Ma pointed to two factors. One is that Chinese teachers devoted a lot of time and effort to in-service training, and to discussing and planning lessons together with colleagues. Secondly, the pre-service teacher education is of higher quality and more relevant to the teaching profession which led to "that Chinese teachers begin their teaching careers with a better understanding of elementary mathematics than that of most U.S. elementary teachers" (ibid., p. xviii).

The findings of Ma were supported by Leung (2006) who in the first chapter in the report of the ICMI comparative study of East Asia and the West (Leung, Graf & Lopez-Real, 2006) reviewed the TIMSS-R 1999 study (and some other smaller studies).

> In this paper, we will look at a number of variables including societal resources, characteristics of the educational system, and teacher attitudes and attributes in order to whether there are any patterns that can be matched with the pattern of student achievement.
>
> (ibid., p. 22)

The variables considered were basically distal factors. Leung concluded that, beside differences in the factor 'teacher competence', there were no variables that could explain the differences in achievement.

> [...] we can see that other than the possible difference in competence of the teachers, none of the variables we reviewed at the levels of society, education system and teachers seemed to be able to explain the high achievement of East Asian students. (ibid., p. 40)

The teacher competence probably will manifest itself in the actual teaching and in factors 'proximal' to the teaching process inside the classrooms. The influence of distal factors on student learning is thus not supported by the finding of these studies.

Mathematics teaching in East Asia

So far we have that students in many East Asian countries performed very well in international mathematics achievement studies. This implies, of course, that they actually had learned mathematics better, which in turn might be a result of their teachers' apparently good understanding of mathematics and good knowledge in how to explain mathematical concepts to students. This line of reasoning leads to the interesting question of what difference the 'high teacher competence' really makes when it comes to the actual teaching. How does the Chinese teachers' good understanding of school mathematics that was documented by Ma manifest itself in their classrooms? What are the characteristics of the successful teaching in these Asian countries?

A very ambitious study to find out what features of mathematics teaching high-achieving countries have in common – features that may explain the differences in achievement – is the TIMSS 1999 Video Study (see for example Hiebert et al., 2003). They used a design called "video-survey", in which randomly selected 8th grade mathematics lessons were videotaped to generate a representative sample from each country. It is interesting to note that the comparison was done on a national level, between countries – as in the TIMSS achievement studies – even when the researchers stepped inside the classrooms and studied the actual teaching process.

In the TIMSS 1999 Video Study the video camera focused the teacher, who was carrying a microphone. In the analysis different aspects of mathematics teaching were coded in 50 to 140 lessons from the seven participating countries.

> The time and manpower we had allowed us to reliably code more than 60 distinct aspects of the lesson, from codes such as interaction pattern, mathematical content activity, and activity purpose, to myriad codes about each mathematical problem (e.g., evidence of real life connections, graphic representations, procedural complexity, and student choice in solution methods), to judgements of students engagement, lesson coherence, and overall quality.
>
> (Givvin, 2004, pp. 206–207)

The aspects studied and coded were of both a general kind (for example interaction pattern, judgements of students engagement), which were not unique to a mathematics classroom, and of a more specific kind (for example mathematical content activity, codes concerning mathematical problems), only relevant for mathematics education.

One aim of the study was to identify what was typical of mathematics teaching in countries that outperform the U.S. in international achievement tests. As a consequence, all countries in the study had scored significantly better than U.S. in the TIMSS 1995 achievement study. An interesting result is that, despite the huge effort, it turned out to be hard to find aspects of teaching that could be regarded as more important than the others in explaining differences in achievement. "[...] we had difficulty finding lesson features that correlate with differences in achievement" (Givvin, 2004. pp. 208). Not until a very close in-depth analysis of how the mathematical content was treated from a student perspective was it possible to find something that separated the teaching in U.S. from the higher-achieving countries.

> What, then, do the higher-achieving countries have in common? The answer does not lie in the organisation of classrooms, the kinds of technologies used, or even the types of problems presented to students, but in the way which teachers and students work on problems as the lesson unfolds. (Stigler & Hiebert, 2004, p. 14)

The mathematics problems used in the recorded lessons were coded as either ”using procedures” or ”making connections”. The first category was basically used for tasks that only require the students to perform a more-or-less ”routine type” operation. The second category was used for tasks that require students to use conceptual knowledge and where only applying a routine operation was not sufficient. The first analysis, of the extent to which the different types of problems were used, showed no pattern that could explain differences between U.S. and the higher achieving countries. A renewed analysis – that also took into account *in what way* teachers implemented the ”making connections” problems – made it possible to discern something interesting.

> Here is the most striking finding of all: In the United States, teachers implemented none of the making connections problems in the way in which they were intended. Instead, the U.S. teachers turned most of the problems into procedural exercises or just supplied students with the answers to the problems. (ibid., p. 15)

It is not a surprise that it is hard to identify distinct features that really matter and make a clear difference when it comes to complex phenomena, such as the mathematics classroom. What I find especially interesting is the fact that the more ’superficial factors’ that may be used to describe mathematics teaching, do not work when it comes to distinguishing teaching in higher achieving countries from teaching in U.S. The organisation of the teaching, the number of students in the class etc. seem to have much less importance than *how* the mathematics content is treated by the teacher and the students. Stigler and Hiebert came to the same conclusion.

> A focus on teaching must avoid the temptation to consider only the superficial aspects of teaching: the organisation, tools, curriculum content, and textbooks. The cultural activity of teaching – the ways in which the teacher and students interact about the subject – can be more powerful than the curriculum materials that teachers use.
>
> (ibid, p. 15)

The TIMSS video study design can be characterised by some features. First there is the idea of comparing mathematics teaching on a national or cultural level. This means that the lessons recorded must be selected in a manner that they can be regarded as representative of the participating countries. The design also involves coding lesson events in order to be able to make quantitative comparisons. Secondly, the research attention is on the teaching and the actions of the teacher. Then there is only need for one camera, which captures the activities of the teacher. Thirdly, there is a presumption that filming only one lesson is enough to capture what is characteristic of the teaching in the respective classes.

Further this study was clearly initiated and led by researchers from the U.S. This becomes obvious when looking at the choice of participating countries – they all performed better than U.S. on the achievement tests. The research agenda in TIMSS 1999 Video Study involved examining in what ways the teaching in these counties differed from the teaching in U.S.

Comparisons made on nationally representative sampling and comparison between 'cultures' have been criticised for example by Clarke et al. (2006) who claim that comparison made "between the Chinese/Confucian tradition on one side and the Greek/Latin/Christian tradition on the other", is at risk of oversimplifying.

> In the south-eastern suburb of Melbourne, a class of twenty-five children can include over twenty distinct ethnic backgrounds. [...] Asian-American students, participating in a school system that has been substantially maligned in the U.S. popular press, perform at a level comparable with their high-performing counterparts in schools in Asian countries. (ibid., p. 354)

Another problem with the comparison of mathematics teaching and learning on a general level was raised by Wang and Lin (2005) who conducted a systematic review of research that attempted to "identify the relationship between Chinese students' mathematics performance and the factors that contribute to their achievement" (ibid., p. 3). They found that the view of the performance of Chinese students turned out to be less clear when considering different mathematical competences and topics.

> First, although Chinese students perform better *in general*, their performance in the areas of mathematics competencies as envisioned by the U.S. curriculum standards is less well understood and, in some cases, is not substantially better than that of their U.S. counterparts.

> Second, although a limited number of studies show a positive relationship between students' performance, on the one hand, and curriculum materials, teachers' mathematics knowledge, organization of instruction, and representation of mathematics ideas in Chinese classrooms, on the other hand, these studies do not provide a satisfactory interpretation of the disparities in performance between the Chinese students and their U.S. peers. For example, although Chinese teachers possessed a deeper understanding of mathematical connections and required their students to develop flexible connections among mathematical concepts and to find multiple, divergent solutions to mathematics problems (Ma, 1999), Chinese students were not better than U.S. students at solving complex and open-process mathematics problems (Cai, 1995, 2000; Cai & Silver, 1995).
>
> Third, Chinese students' better *general* mathematics performance as compared with that of U.S. students cannot be attributed solely to the Chinese formal schooling or teaching process. As suggested in our literature review, several non-school-related factors very likely make important contributions to Chinese and Chinese Americans' mathematics performance. These factors may include the nature of Chinese language, students' self-concept and effort, and family values and processes. (Wang & Lin, 2005) p. 10)

There seems to be a problem when aggregated measures of achievement are used. The picture can be oversimplified. There is a need for a closer look into the teaching and learning of *particular* mathematical content.

> [...] the existing literature does not provide enough evidence to develop a complete picture of the network of factors and its adaptive transformation. To develop this picture, a different conception must be developed to guide comparative studies. It could include reconceptualizing research designs to view mathematics learning as influenced by adaptive rather than additive factors and by interactive rather than isolated variables. Moreover, rather than focusing on comparisons using *general* performances, future studies should examine the effects of influential factors on *specific* areas of mathematics competencies. (ibid., p. 10)

Neubrand (2006) came to similar conclusions in the report of a re-analysis of the TIMSS video studies. She found that despite a lot of coding and analysing not much can be said about eventual cultural differences and concluded that comparing without considering the actual mathematics taught is dubious. For instance, it is found that Japanese teachers use 'procedural' problems in algebra but 'conceptual' problems in geometry and hence it will not be productive to use general notions such as 'Japanese way of teaching'.

This review of international comparative studies shows that it is difficult to find 'distal factors' that can explain differences in student achievement. It further shows that research that considers mathematics teaching and learning in general will face difficulties. One conclusion is that it is important to consider the *specific* mathematical content.

Conclusion

In this chapter I have reviewed research from three areas within the field of mathematics education – research into the teaching and learning of school algebra, 'authentic' mathematics classrooms research and international comparative research.

Given the quite extensive amount of research into the area of school algebra I found surprisingly little research done on teaching of this topic. More specifically, there is a considerable lack of research into the teaching of 'systems of equations'. Research that is done with a clear focus on a specific mathematical content seems in most cases to have its interest directed towards the learners – their difficulties, errors and conceptions – rather than towards how the content is and can be taught. The review of the research that made use of authentic classroom data gave a similar picture – there are obviously many other things, beside the teaching of a specific content, that can be in focus when researchers enter mathematics classrooms. An exception from this 'general picture' is a recent study by Olteanu (2007). She analysed and compared how two teachers were teaching the same content – second degree equations and quadratic functions – in two different classes. She had a clear focus on the mathematics which enabled her to find and describe interesting differences in how the content was handled. Further, she could relate these differences to qualitative differences in how the students of the two classes had experienced second degree equations and quadratic functions.

A major finding from this review of mathematics education research is that studies combining a distinct focus on a specific mathematical content while comparing the teaching of this content in different classrooms can come up with interesting results. In my study all of these ingredients will be present.

Chapter 3
Theoretical base

Studying mathematics teaching

A starting point for this study is the acknowledgement of a relation between teaching and learning. A major purpose of the activities a mathematics teacher introduces during a lesson is to facilitate the development of students' competence in mathematics. What students may experience and learn is not only associated with mathematics. The many aspects experienced in a mathematics classroom, may be for example, social and emotional, as well as of mathematical nature. There are so many things taking place in a classroom that may affect student learning. This diversity also manifested itself in the research on mathematics classroom practice (see the review in the previous chapter).

However, *what* is possible for students to learn about mathematics must be related to *how* they experience the mathematical content. *How* students experience the content must in turn be related to *how* the content is handled during mathematics instruction. Different ways of handling the mathematics influence the possible learning of different aspects related to the content. This does not mean that there are a one-to-one relation between teaching and learning. Below I will discuss the relation between what the teacher intends, what takes place in the classroom and what students might experience – in relation to mathematics. Because of the many intricate processes taking place in the mathematics classroom, it will never be possible to predict or guarantee what learning will take place, especially not on the individual level. On the other hand, the absence of an absolute relation will not rule out the possibility of a complex of relationships.

However, the aim of this study is not to examine what students actually learn, but what possibilities there are to learn mathematics in the studied lesson. By extensive descriptions of what takes place during the lessons I hope to provide the reader with the opportunity to judge whether it is possible for students to to experience a particular aspect of the mathematics taught.

Häggström, J. (2008). *Teaching systems of linear equations in Sweden and China: What is made possible to learn?* Göteborg: Acta Universitatis Gothoburgensis.

Focus on the what aspect of teaching and learning

The approach of this study is to describe qualitative differences in how the same mathematical content is handled. The focus is not on general aspects of teaching and learning. Studies trying to find general characteristics of efficient teaching, where the specific content is not considered (see the previous chapter), seem to be less fruitful. Distal factors such as class-size, grouping or streaming, curricula, etc. seem not to discriminate between different qualities of education in the general case. This study has a much 'narrower' approach with a close focus on the content. Even with a broader research interest than this, there are limits to how much can be studied in one project. Quite substantial restrictions need to be made among the many factors that may influence the outcome of teaching. While this study necessarily focuses on the mathematics, it does not mean that other factors are not important. The interest in this study is restricted to teaching where the intention is that students develop their mathematical competence. An assumption is that the way teachers make the mathematical content available for, and handle it together with, the students will make a difference, and therefore be worthwhile studying.

Variation theory

I will base the analysis and comparison of the mathematics lessons in this study on Variation theory (Marton & Booth, 1997; Marton, Runesson & Tsui, 2004; Runesson & Marton, 2002). Variation theory provides a framework that should make it possible to discern and describe differences, even subtle ones, in how the mathematical content is handled. The theory has an explicit focus on the *what* aspect of teaching and learning as it recognises that teaching and learning is always the teaching and learning of *something*. There are comparative studies of mathematics teaching that have demonstrated the merit of Variation theory in this respect (cf. Runesson, 1999; Runesson & Mok, 2005). With this study I will follow in the tradition of 'content oriented' classroom studies at the University of Gothenburg (cf. Emanuelsson, 2001; Kilborn, 1979; Lybeck, 1981; Löwing, 2004; Runesson, 1999) and try to further develop the methodology of analysis by focusing on differences in how the same mathematics is taught.

Variation theory also offers a way to discuss potential implications of teaching for student learning. In Variation theory 'learning' is seen as a capability to see the 'object of learning' in new ways. Learning means to be able to discern features of the object of learning that were not discerned earlier. A cornerstone in the theory is that a requirement for discernment of an aspect of the object of learning is that variation in that respect is experienced. Features that are invariant – indicating that no variation is experienced – are not possible to discern. An analysis of a mathematics lesson that provides a description of what features are kept invariant and what features are varied may, according to the theory, have strong implications on what it is possible to learn concerning the actual object of learning in that particular lesson. The notion of *space of learning* is used to describe what it is possible to learn in relation to a certain object of learning (see below).

From Phenomenography to Variation theory

Difference runs through the development of Variation theory. The theory has its origins in the research tradition known as Phenomenography. The starting point was an interest in *differences* in learning and the observation that people learnt differently. This early interest in learning later evolved into what is sometimes referred to as *traditional* phenomenographic studies. The traditional phenomenographic research investigates and describes qualitatively *different* ways of understanding the same phenomena (Marton & Booth, 1997). The conceptions of – or ways of seeing – a certain phenomena are described in terms of what aspects of the phenomena are kept in focal awareness simultaneously.

> Traditional phenomenografic research aims to investigate the qualitatively different ways in which people understand a particular phenomenon or an aspect of the world around them. These 'different ways of understanding', or conceptions, are typically represented in the form of categories of description, which are further analysed with regard to their logical relations in forming an outcome space.
>
> (Marton & Pong, 2005, p. 335)

There are a large number of studies in this tradition (Marton & Booth, 1997), where the results are descriptions of a limited number of *different* ways to understand a particular phenomena. Most studies are done by interviews in what is called a 'second order perspective'. The researcher is investigating, not the phenomena in itself (first order perspective), but the interviewees' conception of the phenomena. Different conceptions are characterised in terms of which critical aspects of the phenomena are kept in focal awareness and which are not. An aspect of a phenomenon is *critical* if it distinguishes between two different conceptions. There are qualitative differences between the conceptions in an outcome space. Some ways to understand phenomena can be more powerful than others. Here powerful is used in the sense that it enables an individual to cope with situations or problems in a more powerful way. A less powerful conception often does not include all critical aspects of the phenomena simultaneously. A typical result from a study in the traditional phenomenographic traditions would be an outcome space where categories of understanding of the phenomenon in questions are ordered hierarchically. An example of such an outcome space is the conceptions of 'price' described in Marton and Pong (2005). The outcome space in this case consists of four different conceptions or ways of understanding the concept of price. The conceptions, a bit simplified, are the following. A – the price reflects the *value* of the object or commodity, B – the price reflects the *demand* for the object, C – the price reflects the *supply* of the object and D – the price reflects both the *demand* for and *supply* of the object simultaneously. The outcome space can be viewed as hierarchical where the conception D is the most complex and powerful and conception A is the least complex and powerful. Conception D is clearly better suited when it comes to dealing successfully with situations where the phenomena or concept of price is involved. In phenomenographic research 'critical aspect' is used for the specific aspects that have the quality of separating one category of understanding from the others. In the example of price, the aspects 'demand' and 'supply' are critical as they distinguish between the different conceptions. The aspects related to the mathematical concepts considered in the present study are not necessary *critical* in this respect. These aspects are not grounded in a phenomenographic outcome space.

Human awareness has a limited capability regarding the number of elements that can be focused simultaneously (cf. Araï, 1999). On the other hand, at every point in time there are basically an unlimited number of elements that could be focused upon. Due to the limitation of awareness most of them will be kept in the background, while others will be at focal awareness. The awareness is dynamic in nature, and the elements in focal awareness will change all the time. The descriptions of categories of understanding, 'the outcome space' are thus seen as descriptions of possible ways to understand the phenomenon in question on a collective level that cannot be ascribed to certain individuals. An individual may express different conceptions, different ways of understanding, in different situations and contexts, and at different points in time.

The meaning of 'learning' in this theoretical perspective is a change in the way a phenomenon is seen. This means, a change in what critical aspects are discerned and kept in focal awareness at the same time. Learning takes place when new critical aspects are discerned. The next step in the theoretical development was to start to ask questions like, how can such changes of conceptions be accomplished and how can a critical aspect of a phenomenon be discerned? The answer provided by

Variation theory is that discernment requires experience of variation in that respect. "[...] in order to discern a feature, a person must experience variation in that feature" (Marton, Runesson & Tsui, 2004, p. 17). Then we have come back to where we started. Variation is all about *differences*. Variation implies that an aspect can be experienced *differently* in one instance compared to how it is experienced in another. Aspects that are invariant cannot be discerned. Also the simultaneous variation of two aspects will make it impossible to separate and discern the individual aspects. This will be further discussed below.

Definition of concepts and terminology

In the analysis of how the mathematics is handled in the different lessons a number of concepts from Variation theory are used. These are presented here.

Object of learning

The 'object of learning' is used to denote the 'what' aspect of teaching and learning. In a school setting the object of learning is basically what teachers are trying to teach and what students are supposed to learn. This may be many different things and not necessary related to the subject matter content. For example, the teacher might have an objective in a lesson or with an activity that the students should develop an ability to work in groups or to speak in front of the whole class. In this study, however, the objects of learning considered are the ones related to students' conceptions of the mathematical content. The object of learning must not be confused with the mathematical content itself. The object of learning is an ability – in the present study, an ability related to the mathematical content – for instance the ability to understand the mathematics in a certain way or an ability to solve certain types of problem.

The object of learning can be viewed from three different perspectives. The *intended object of learning* is when it is viewed from the teachers' perspective. The teacher has an intention or a goal that the students will develop certain abilities.

> [...] teachers are trying to work toward an *object of learning*. This object may be more or less conscious for the teacher and it may be more or less elaborated. But, whatever the circumstances, what teachers are striving for is the intended object of learning, an object of the teachers awareness, that might change dynamically during the course of learning. (Marton, Runesson & Tsui, 2004, p. 4)

The *enacted object of learning* is when the object of learning is examined from a researchers' point of view.

> What the students encounter is the *enacted* object of learning, [...] The enacted object of learning is the researcher's description of whether, to what extent, and in what forms the necessary conditions of a particular object of learning appear in a certain setting.
>
> (Ibid., p. 4–5)

The *lived object of learning* is what a student actually experience of the object of learning.

> What is of decisive importance for the students, is what actually comes to the fore of their attention [...] What they actually learn is the *lived* object of learning, the object of learning as seen from the learner's point of view, that is, the outcome or result of learning.
>
> (Ibid., 2004, p. 5)

Figure 3.1 illustrates the relation between the three objects of learning. Each of the circles contains different features of the respective object of learning. The three circles only partly overlap which indicates that for instance some features of the intended object will be included in the enacted in the classroom, but possibly not all. Similarly, some features of the enacted object of learning might be experienced by a student (lived object of learning), and other features might not. The different parts of the figure will represent all possible cases. The central part is common to all three circles and shows the case when features of the intended object of learning also are enacted and experienced. As a teacher you strive to expand this area. The ultimate goal is that all students learn what was

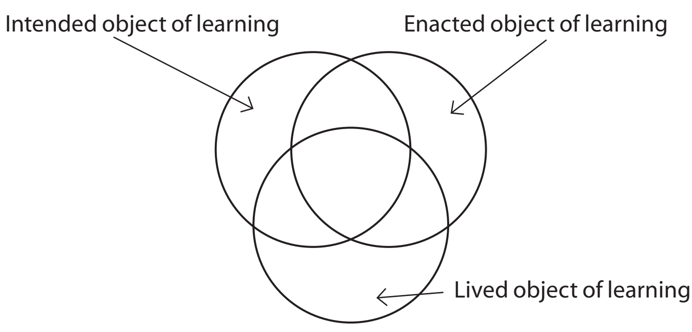


Figure 3.1 *The object of learning.*

intended – in the perfect teaching-learning situation the circle showing the lived object of learning will totally overlap the circle showing the intended object of learning. From the figure it will follow that there can be features common to two or just one of the three objects of learning. An aspect of the intended objected might for instance, be observed by an observer of a classroom but nevertheless not experienced by a certain student, or students may learn things that were neither intended nor enacted. Outside of the three circles are features that are not included in any of the three objects of learning. In this study it is the enacted object of learning that is examined.

Variation

Awareness is constituted by the elements that are simultaneously attended to. Aspects of a phenomenon that are invariant will never come to our focal awareness. An aspect being invariant means either that the aspect in question is not varied in the actual situation, or variation is not created by comparison to previous experience. The best strategy for a teacher to provide for the possible learning of the intended object of learning is to create a pattern of variation that will direct the students' attention to the critical aspects. However, as illustrated in figure 3.1 the intended object of learning might differ from the enacted object of learning. The possible variation created by individual students by use of their previous experience is out of reach for this study. Only the patterns of variations observable in the classrooms will be considered (enacted object of learning, see figure 3.1).

Marton, Runesson and Tsui (2004) discuss variation and could identify four different patterns of variation on a general level. In my analysis I will use variation in two of these respects.

Contrast – In order to discern a certain dimension of variation (or aspect) you need to experience a possible alternative in that respect. ”In order to understand what three' is, for instance, a person must experience something that is not three: 'two' or 'four', for example. This illustrates how a value (three, for instance) is experienced within a certain dimension of variation, which corresponds to an aspect (numerosity or 'manyness')” (ibid, p. 16).

Separation – ”In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant” (ibid, p. 16). Two aspects that vary at the same time cannot be separated.

The other two pattern of variations are Generalization, which may provide experience of an phenomenon in many different forms and situations in order to separate it from irrelevant aspects, and Fusion, which may provide opportunities to experience several critical aspects simultaneously (for instance the simultaneous experience of demand and supply in relation to the concept of price). The reason for not using these two are that they are closer connected to the act of deepening the understanding of a phenomenon rather than discerning new aspects. The pattern of variation named Fusion is also clearly connected to merging *critical* aspects of a phenomenon, but I cannot tell what the critical aspects of the studied mathematical objects of learning are. *Contrast* and *Separation* I judge as better suited for examining how the mathematics is handled when the critical aspects are not known or identified in advance.

The experience of variation in a mathematics lesson can be facilitated by the teacher, by students, or by the examples and tasks provided. The teacher can direct the students' attention to possible alternatives, for instance by posing questions and thus highlight a certain aspect. When solving a system of equations, such a question from the teacher can be, for instance "can we substitute x into the first equation or into the second equation?" By the question the teacher provides alternatives and 'opens the dimension of variation' regarding the target equation when using the method of substitution. Highlighting an aspect in this way means that variation in this respect is created and that it is made possible for students to discern. Another teacher, in the same situation, that takes the choice of equation for granted and just shows how to substitute x into one of the two equations without any comment will not open this dimension of variation. When something is taken for granted it stays invariant and will not possibly be discerned from that particular situation alone. In the same way can questions or statements from students direct the attention towards an aspect and thus open the dimension of variation. Below I will discuss and exemplify how a sequence of examples or tasks in a similar way may facilitate discernment of certain aspects of an object of learning.

Dimension of variation

Different ways of understanding the same thing can be categorized by the aspects of the phenomena that are discerned and focused simultaneously.

> Whenever people attend to something, they discern certain aspects of it and by doing so pay more attention to some things, and less attention or none to other things. [...] A particular way of seeing

> something can be defined by the aspects discerned, that is, the critical features of what is seen.
>
> (Marton, Runesson & Tsui, 2004, p. 9)

In this study *aspects* or *features* are considered as aspects of the particular enacted object of learning in the mathematics lessons. The attention of the analysis is directed towards the variation and invariance of aspects of the enacted object of learning. A certain aspect cannot be discerned without the experience of variation in that respect. As described earlier, the experience of variation can be accommodated in different ways. A student may experience variation of a certain aspect by comparing what takes place in a situation to his/her pervious experience. For example, a student may compare present systems of equations given with only natural numbers as coefficients to the experience of decimal numbers and fractions from previous situations and experience variation regarding the kind of numbers that can be used in an equation. The experience of variation in this case cannot be detected by analysing the enacted object of learning and is dependent on the particular experience of the student in question.

The experience of variation may be also more obviously connected to the enacted object of learning. For example, students who work with equations where coefficients are natural numbers, decimal numbers and fractions are provided with an opportunity to experience variation in the same respect without comparing to previous experiences. The possible experiences of variation in this case are directly related to the enacted object of learning and to what *dimensions of variation* are opened in the classroom. This establishes a possible link between the variation of an aspect of the object of learning during a mathematics lesson and the possible discernment of that aspect. Experiencing variation in a certain aspect means experiencing a difference in that *dimension of variation*. "[...] every feature discerned corresponds to a certain dimension of variation in which the object is compared with other objects" (Marton & Pong, 2005, p. 336). I will use the concept *dimension of variation – DoV* – to describe what aspects or features are made possible to discern.

Variation in a sequence of tasks

A pattern of variation that is possible for students to experience can be offered through tasks and examples. While working through a sequence of exercises some aspects will be the same and some will vary. The type of exercises and in what order they are presented to students will create different patterns of variation, highlighting different aspects of the object of learning. Thus, it will be made possible for students to discern different aspects. I will use the following fictitious examples to illustrate some patterns of variation that can be generated by a sequence of exercises.

In the first sequence there are three systems of equations to be solved. What dimensions of variation, DoVs, are possibly opened while working through this sequence?

Example 1.
Solve the following systems of equations

1) $\begin{cases} 3x - 2y = 5 \\ x + 2y = 7 \end{cases}$

2) $\begin{cases} 3t = 5 + 2s \\ t = 7 - 2s \end{cases}$

3) $\begin{cases} 2u = 3v - 5 \\ 7 - v = 2u \end{cases}$

At first sight the three systems of equations may look different but they are basically equivalent (mathematically) and, thus, have the same solutions. A closer analysis could, however, reveal a number of aspects that vary between the three examples. One aspect that can be discerned in this case is the use of different letters to denote the unknowns. In most cases the letters x and y are used, but a sequence like this opens up the DoV concerning what letters can be used for the unknowns and thus makes discernment of this aspect possible.

The next example also consists of three systems of equations to solve. What DoVs are possibly opened while working through this sequence?

Example 2.
Solve the following systems of equations

1) $\begin{cases} x + y = 5 \\ y + 4 = 2x \end{cases}$

2) $\begin{cases} x = 5 - y \\ 3x + 3y = 15 \end{cases}$

3) $\begin{cases} 2y = 10 - 2x \\ 3x = 12 - 3y \end{cases}$

Also in this case the three systems of equations may look different at first sight. The first of the equations in each system are equivalent while the second equations are different. The most striking difference may be, however, the solutions to the systems. The first system has a single solution ($x = 3$, $y = 3$). In the second example the two equations are equivalent and there are no unique solution. In fact, there are infinitely many number pairs that satisfy this system of equations. The third system contains two equations with no common solution (illustrated graphically in the coordinate plane they would be two parallel lines without any point of intersection). In this sequence of tasks the DoV regarding the number of solutions to a system of equations is opened. This aspect, thus, is made possible to discern. If you as a student only experience systems of equations with a single solution this aspect will not be possible to discern without relying on your previous experience.

In the third example there are four tasks. What DoVs are possibly opened by this sequence?

Example 3.
Solve the following equations and systems of equations

1) $7x = 30 - 3x$

2) $7x = 2y - 3x$

3) $\begin{cases} 3x - 2y = 5 \\ x + 2y = 7 \end{cases}$

4) $\begin{cases} 3x = 5 + 2y \\ x + z = 7 - 2y \end{cases}$

There are (at least) three aspects that vary in example 3 – the number of unknowns, the number of equations and the number of solutions. In the second item the number of unknowns has increased from one to two compared to the first task. At the same time the number of solutions increase from one single solution to infinitely many. When we move on to the third item the number of unknowns stays the same but the number of equations to consider simultaneously has increased to two. The number of solutions is reduced to a single solution. In the fourth item the number of unknowns is increased again, while the number of equations stay the same as in the third. Now the number of solutions again changes to infinitely many. This last example provides a more complex pattern of variation. The simultaneous variation of two or more aspects may make it hard to separate them. This last example is perhaps closer to what normally occurs in the mathematics classroom where the patterns of variation often can be multifaceted and hard to analyse.

Watson and Mason (2006) have studied patterns of variation that could be provided by mathematical tasks. They found that carefully structured exercises could give students opportunities to discern critical features.

> As already mentioned, there are no guarantees, no matter how carefully an exercise is structured, because of the many other relevant factors influencing the situation. That said, greater commonality can be achieved through careful structuring than through apparently random collections of questions treated as individual tasks by learners. (ibid., p. 100)

> [...] learners discern differences between and within objects through attending to variation. Teachers can therefore aim to constrain the number and nature of the differences they present to learners and thus increase the likelihood that attention will be focused on mathematically crucial variables. (ibid., p. 102)

Space of learning

Variation theory provides opportunities to systematically study links between how the mathematics it handled in a classroom and what students may possibly learn. Remember, learning is seen as the development of a capability to see or understand the object of learning in new ways. To see the object of learning in a new way requires the discernment of aspects that were not previously discerned. As discernment requires the experience of variation in that respect one may analyse teaching by examining what aspects vary and what aspects stay the same. "[...] it is necessary to pay close attention to what varies and what is invariant in a learning situation, in order to understand what it is possible to learn in that situation and what is not" (Marton, Runesson & Tsui, 2004, p. 16). The *space of learning* defines what is made possible to discern and is a result of the careful examination of the enacted object of learning. It is important to note once again that the space of learning does not describe what students necessarily will learn only what is *made possible* to learn.

All teaching naturally involves variation. It is not possible to teach anything without creating a pattern of variation. Some teachers might create a certain pattern of variation deliberately, to highlight certain features or aspects of the intended object of learning. In other cases the pattern of variation may be more or less accidental. The pattern of variation may also be initiated by students. For instance, a question from a student may open up a new dimension of variation, that otherwise possibly could have been taken for granted by the teacher, and kept invariant. Nevertheless, as a mathematics lesson unfolds a certain pattern of variation will form, that makes up the enacted object of learning. The space of learning is a description of the enacted object of learning, as perceived by the researcher. This study aims to take the perspective of a student in the classroom when describing the enacted object of learning.

Certain patterns of variation provide opportunities for learning the aspects that are varied, but cannot predict what learning will take place. On the other hand, the aspects of the object of learning that are kept invariant, taken for granted by the teacher and students, are not made possible for students to discern from the events in the actual lesson. The circumstance that some students actually may discern these aspects anyhow, can be explained by the creation of the necessary variation by help of their previous experience. Thus, the discernment is not caused by the variation in the actual situation alone. This might also be an indication of the importance of the previous experiences of students.

What is compared in this study is not what students will learn in the different lessons, but rather what has been made possible to learn and what has not been made possible to learn. The comparison is made through the space of learning constituted in each classroom. It is done in terms of what aspects of the object of learning have been opened for variation, DoV, and what aspects have been kept invariant.

> A teacher can never ensure that the intended learning will take actually take place, but a teacher should try to ensure that it is possible for students to learn what's intended. The teacher should ensure that the space of learning allows for the intended learning to take place. (Marton & Tsui, 2004, p. ix)

Variation theory on many levels

Variation in the classroom

I have discussed my intention to base the analysis of how the mathematical content is handled in the different classrooms in terms of Variation theory. The pattern of variation regarding aspects of the enacted object of learning will have implications for what aspects students may discern, the lived object of learning. The exact nature of these implications is not given. The lived object of learning cannot be deduced from the enacted object of learning. However, it is possible to discuss what student could *possibly* learn, rather than what they actually learn. It is also possible to discuss what has *not* been made possible for students to learn in a certain situation.

Variation in the data

Another application of Variation theory concerns what is possible for me, as a researcher, to discern. The comparison of teaching in different classrooms may create contrasts that might direct my attention to certain features. By keeping the mathematical content the same in all classrooms the comparison can create patterns of variation where differences will stand out and be more easily discerned for me as a researcher.

In the process of analysis I will use my previous experience in the field of mathematics education. "Attending to a certain aspect means comparing something we experience with other things that we have experienced earlier" (Marton, Runesson & Tsui, 2004, p. 9–10). During the analysing process I will contrast my direct experience of the data with my previous experience, enabling discernment of features in the data.

However, it is possible that aspects of the enacted object of learning that are taken for granted by me and at the same time kept invariant in all classrooms will not be detected. Comparing classrooms from different school cultures and traditions can increase the likelihood of discerning aspects of teaching that are taken for granted in your own culture. The difficulty with discerning what is familiar and regarded as natural was also experienced in the TIMMS video study. "We concluded from our first study that teaching is a cultural activity: learned implicitly, hard to see from within the culture, and hard to change" (Stigler & Hiebert, 2004, p. 13). The aspects taken for granted may disappear in the background and evade discernment.

Variation experienced by a reader

The framework of Variation theory will be used to describe the space of learning related to the same mathematics in different classrooms. By comparison of the spaces of learning, differences in how the mathematical content is handled can be exposed. The differences found can generate a discussion of the implications of differences in what is possible for student to learn.

A reader of such a description of how the mathematical content is handled in different classrooms may discern aspects against the background of her/his own experience of teaching the same content. Runesson and Mok in the conclusion of their comparison of teaching of fractions in Sweden and Hong Kong note "The results of a study like this appeal to teachers' professional knowledge and could be inspiration and help for reflection on practice. Seeing what *could be* the case sheds light on what *is* done and *what is the case in our own practice* (Runesson & Mok, 2005, p. 12). Similarly, a researcher may discern interesting aspects against a background of previous experience of research of mathematics classrooms.

Chapter 4
Documentation of mathematics classrooms

The Learner's Perspective Study

The empirical data available for this study came from 9 classes from three different regions in two countries – three classes from Sweden, three classes from Hong Kong and three classes from Shanghai. The data were collected within the Learners' Perspective Study, LPS (LPS, 2003). Unlike many other large international projects this project is not top-to-bottom organized and run. LPS is an ongoing international collaborative effort to collect and share empirical data coordinated by Professor David Clarke, Director of the International Centre for Classroom Research at the University of Melbourne (ICCR, 2007). The basic idea is to employ a common design when collecting data, to put the data together into a common format and share the data among the participating research groups.

Researchers from four countries, Australia, Germany, Japan and USA, participated from the beginning of the project. In September 2007 the number of research teams reached fifteen. A research group from Sweden joined the LPS community in 2001. The Swedish research team came from the Universities in Uppsala and Göteborg and collaborated in the KULT project [KULT projektet. Svensk skolkultur – klassrumspraktik i komparativ belysning. English: *Swedish school culture – classroom practice in comparative light*, (KULT, 2003)] with the aim of describing mathematics teaching in Sweden and making international comparisons. Preliminary results and analysis of the Swedish data can be found in for instance Emanuelsson et al. (2002, 2003), Emanuelsson and Sahlström (2006), Emanuelsson, Sahlström and Liljestrand (2003), Häggström (2004, 2006a, 2006b, 2007), and Liljestrand and Runesson (2006).

The LPS is an attempt to complement other international comparative studies of classroom practice, more particularly the TIMSS video studies, and the design aims at providing data suitable for in-depth analysis of different aspects of mathematics classrooms.

> The aim of our research is to document not just the obvious social events that might be recorded on a videotape, but also the participants' construal of those events, including their memories, feelings, and the mathematical and social meanings and practices which arose as a consequence of those events. (LPS, 2003)

Video documentation

The documentation was done with extensive video recording of (mostly) 8th grade mathematics classrooms employing a common design (Clarke, 2000). The LPS design differs from many other international classroom studies in some respects. One such is that classes are video recorded for at least 10 consecutive lessons. In many cases as many as 15 to 18 lessons in a row have been taped. A rational behind the decision to record a sequence of lessons was that teachers do not plan and execute one lesson at a time. The planning mostly stretches over a larger part of the mathematics syllabus. The plan for teaching a certain topic area can vary in time, from just a few lessons up to quite a substantial number. The lesson was regarded as just too short a unit to enable the capturing of a good description of the teaching in different classrooms.

The classrooms in the LPS were (and are) selected on the basis of the following criteria. The classrooms shall be from government schools in major towns with a demographic diversity, and non-typical student groups shall be avoided. The teachers shall be experienced and competent, as locally defined by the community. The classrooms in the LPS could be seen as examples of good mathematics education in each respective country. In contrast to the lessons recorded in the TIMSS Video Studies of 1995 and 1999, the classrooms documented in LPS did not form nationally representative samples. With only three classes taped from each country, representativity has obviously not been a matter of great concern in this design.

Another difference is the use of three cameras: the teacher camera, the whole-class camera and the student camera. The teacher camera (TC), which was operated by one of the researchers, was directed to record the teacher all the time. The teacher wore a small wireless microphone that captured the teacher's voice. This microphone, in most cases, also managed to capture the voices of the students the teacher was helping at their desks. The recording of the TC was similar to the method used in the TIMSS video studies, where one single camera was used. The whole class camera (WC) was placed in the front of the classroom and used a wide-angle lens that captured the whole classroom (or most of it). The third camera was directed at a smaller number of students (typically two, three or four students) sitting next to each other. The desks of these students were equipped with a couple of small wire-less microphones that captured their conversation.

After every lesson video-stimulated interviews were conducted with two of the focused students. Data from each sequence of lessons consisted of classroom videotapes, transcripts in original language and in English translations, teacher questionnaires, transcripts from student and teacher interviews, copies of students' notebooks and textbooks etc. In some cases the interviews were also video recorded.

Some comments on the data

The use of three cameras and the student interviews showed that this design recognised the importance of students' actions as well as students' interpretations of the interaction in the classroom. The basic idea was to capture interaction in the classrooms in a way that the data would be suitable to many different research interests and approaches. The same 'event' in a lesson could for instance be studied from different perspectives, and the data would allow also in-depth analysis in more ways than the data from the TIMSS Video Study. Further, the filming of long sequences acknowledged that teachers in most cases plan for sequences of lessons that cover a particular topic rather than just one lesson at the time. Therefore the filming of a series of consecutive lessons would provide data better equipped to support understanding of teaching practice. One thing, however, neglected in both the TIMSS video study and the Learner's Perspective Study was the actual mathematical content taught in the studied lessons. In TIMSS the sampling generated lessons that covered most grade 8^{th} mathematics topics. In the Learners' Perspective Study there was no general agreements regarding any specific mathematical content in the recorded lessons. Rather it was perceived as preferable to cover many different topics. "It was intended that the lesson sequences should be spread across the academic year in order to gain maximum diversity of local curricular content" (LPS, 2003). This lack of interest in the specific content may be a bit surprising since both studies no doubt consider themselves as research projects in the field of *mathematics* education. There seemed to be an underlying assumption in both studies that it is meaningful to analyse and compare mathematics classrooms and the teaching of mathematics in a general way, where the actual content was given less significance. The LPS research teams in China and Sweden, however, decided to document the teaching of the same topics. The opportunity of comparing classrooms where the mathematical content was kept invariant was an important part of the application to The Bank of Sweden Tercentenary Foundation that awarded the grants to the Swedish research group (KULT, 2003).

An aim of the LPS was (and is) to use a common design for generating data that would enable comparisons. However, there could not be a complete guarantee that lesson recordings, interviews, questionnaires etc. would generate equivalent data from all the participating classrooms. For instance, the notion of a "locally recognised competent teacher" might differ in many respects between different countries. Cultural, and other aspects might affect the students' behaviour and their willingness to be video-recorded differently. There were for instance students in the Swedish classes who were more than willing, both to be video recorded and interviewed. At the same time there was one student who declined any participation at all. There were differences in students' behaviour at interviews: some were quite reserved and shy whereas others showed no hesitation in expressing their opinions and feelings. The tasks in the common achievement test – IBT – were carefully explored and discussed in the international board before they were used in the TIMSS study of 1995. Nevertheless the tasks might not have been equally suitable with regards to the content of curricula used in the different schools.

Results from the Learner's Perspective Study are reported in a book series. The first two volumes are, *Mathematics classrooms in twelve countries: the insider's perspective* (Keitel, Clarke & Shimizu, 2006), and *Making connections: comparing mathematics classrooms around the world* (Clarke, Emanuelsson, Jablonka & Mok, 2006).

The Swedish data

As a member of the Swedish research team I now describe in more detail how the data from the three Swedish classrooms was collected.

Classroom recordings

In the fieldwork in Sweden portable recording and editing equipment was used and the major part of the recordings were digital and stored directly to hard drives. The mobile recording unit consisted of a computer with a monitor and hard drive, a digital video (DV) deck, a monitor for watching video and an audio mixer. During recording, the recording unit was kept in a room next to the classroom. Both the whole-class camera and the student camera were fixed on 'magic arms' attached to a bookshelf or on a wall-cupboard as high up as possible. Either the teacher or the student camera was attached to the recording unit, depending on the location of the side room. The recording was done both to a mini-DV tape in the camera as well as straight to the hard drive in the recording unit. In a similar way the other camera recorded both to a mini DV in the camera and to a separate hard drive. The third camera (the whole class camera, WC) recorded to the mini DV tape only. The recordings made directly on hard drives (TC and SC) enabled the showing of a mixed image at interviews immediately after each lesson. Individual interviews with two of the focused students were done as soon as practically possible after the lesson.

Throughout the recorded lessons two researchers were present in the classroom. One handled the teacher camera and one observed and took field notes. This last researcher would later conduct the student interviews. The two researchers kept in the back of the classroom and did not interact with the students or the teacher.

The data from the first Swedish classroom (SW1) was collected during the spring of 2002. Data from the other two classrooms (SW2 and SW3) were collected one year later in the spring of 2003.

The conversation from the recorded lessons was transcribed and translated to English. The transcription follows a common protocol in the Learners' Perspective Study. The excerpts from transcripts presented in this study use a somewhat 'simpler' style than the one used in LPS (see the next chapter).

Interviews

Most of the times the two interviews with focus students were done the same day as the lesson was recorded. In just a few cases the interviews were done the following day, for instance if the lesson was the last of the day and the student could not stay. Most interviews were done with a single student, but there were a few exceptions where two students were interviewed at the same time. The interviews were 'video stimulated'. This means that the student would watch the student camera video with a small screen in one corner showing the teacher camera video. The digital recordings made it possible to show a mixed and synchronised movie immediately after the lesson was recorded. During the interviews the students watched the lesson on the video and were asked to stop the video when there were events on which the student wanted to comment. In Sweden the mobile recording unit was used for watching the video from the lesson. At the same time it was also used for recording the interview. One of the cameras was attached to the top of the unit directed at the student and the interviewer. The typical student interview lasted between 40 and 80 minutes. The interviewer had in most cases been observing the lesson and could also stop the video and ask the student about certain events, for instance about the interaction with the teacher if the teacher had been at the students desk. The interviews basically followed a common Student Interview Protocol with the addition and adaptation of questions depending on the actual mathematical content. Every student was interviewed at least once.

Three video-stimulated interviews with each Swedish teacher were conducted. The teachers watched the TC video, and were asked to stop when they wanted to make a comment about a particular event. The interviews were done in the beginning, the middle and in the end of the recorded period. The interviews were conducted in line with a common Teacher Interview Protocol.

Other material

Beside the recorded lessons and interviews additional data was collected. Copies were made of the relevant pages in the textbooks that were used during the recorded period. For each lesson the focused students' notebooks and work sheets, any teacher material, such as overhead projector slides, and other relevant written material were collected and photocopied.

In order to get a measure of and to be able to compare the level of the mathematical ability of the classes in the study a common achievement test was used. It was called the International Benchmark Test. The IBT consisted of 50 multiple-choice items taken from the released part of the TIMSS 1995 test for grade 8. It covers large parts of the mathematics typically taught to students in this age group. Copies of the students' answers to the IBT were also collected. As these items were included in the TIMSS study there were international results available which made it possible to get an indication of the standard of the participating classes based on a large international background.

Students and teachers answered to questionnaires. The student questionnaire contained questions about their home situation – for example, parents' occupation, country of origin, language spoken at home – and about the conditions of the mathematics classroom, homework etc.

The teacher questionnaires contained questions about the teachers' teaching experience and education, the goals and expectations for the studies lessons and about the class in question. The teachers were also asked to grade how typical the recorded lessons were.

School package

In order to get exchangeable data sets there were quite detailed specifications, not just for the actual collection of data, but also for the further handling of the data collected in the LPS. The recorded lessons, both from the teacher and student cameras, were transcribed according to LPS protocols and translated into English. The transcribed statements were then time-coded in order to enable videos with subtitles in English. The data from each classroom was put into a *school package* that contained classroom videotapes with transcripts, transcribed video-stimulated student and teacher interviews, field notes from classroom observations, teacher and student questionnaires, copies of students' material, notebooks and textbooks, and other resources used in the teaching. Each recorded lesson was also described in a *lesson-table*, which gave a general outline of the mode of instruction, the mathematical content and in many cases quite detailed descriptions of the examples and tasks that were used.

In Sweden a total of 43 lessons from three classrooms were recorded. This means that there were 129 lesson videos available. Further there were a total of 75 video-recorded student interviews and 12 interviews with the three teachers.

Video data, pros and cons

This study was based on the analysis of mainly video recorded mathematics lessons and the transcripts of classroom conversation. The transcripts used were in Swedish (original language of instruction) and English (translations of original conversation in Cantonese and Mandarin Chinese). In this section I discuss the pros and cons of video data.

Advantages of video data

There is an increasing use of video recording in classroom research. Of the 48 classroom studies with a certain focus on the mathematical content earlier described (chapter 2) no more than 27 used video in recording the events in the classrooms. This is no doubt due to the many advantages of using video recording for research purposes as well as the ease in handling modern video equipment.

A video camera captures the complex patterns of interaction that take place in mathematics classrooms, and allows the researcher to examine and re-examine the events unlimited many times. There are also opportunities to watch certain events in slow motion. In contrast to classroom observation and the taking of field notes the focus of attention can be shifted between repeated viewings. When doing observations it is hard to attend to the many things that may take place simultaneously. The taking of notes while observing to some degree always involves instant interpretation of what is occuring. In their review of literature on videotape methodology Powell, Francisco and Maher (2003) point to some important issues.

- Video is an important, flexible instrument for collecting aural and visual information.
- Video can capture rich behavior and complex interactions.
- Video allows investigators to re-examine data again and again.
- Video extends and enhances the possibilities of observational research by capturing moment-by-moment unfolding, subtle nuances in speech and non-verbal behaviour.
- Video overcomes human limitations of observing.
- Video is better than observer notes since it does not involve automatic editing.
- Video recordings capture two streams of data – audio and visual – in real time. An observer, even with access to all that a camera sees, has difficulty monitoring different, simultaneous details of ongoing behaviour.

Difficulties with video data

Regardless of the many advantages, the recording by video is in the end also dependent on the human factor. The researcher will have to decide issues such as where to place the camera, where to direct it, and when to start and stop taping. The extensive design in LPS with three cameras aims, to some degree at least, to aviod the pre-recording editing and decision making regarding focus and perspective, while at the same time establishing a protocol to avoid a camera operator making inappropriate decisions based on the moment. Three cameras increase the opportunities of capturing most of the classroom and allow viewing the same event from different perspectives and angles.

The recording of many classrooms can be made quite easily with modern video technology. Compared to the time and effort it takes to analyse video data there is a risk that ”videotaping ironically can produce too much data” (Powell, Francisco & Maher, 2003, p. 411). Powell, Francisco and Maher also point to another problem connected to abundant data, namely the temptation to researchers to use the ’best case' rather than ’typical cases'.

When you enter a classroom with video cameras there is always the danger that students and teacher do not ’act normally'. In LPS the classes were allowed to become familiar with the situation of being filmed a couple of lessons before the recording of the 'official' ten consecutive lessons began. Hence, the number of lessons in the nine classes in the available data stretches from 12 to 18 lessons.

In this study the analysis was based on video recordings and, regarding the Hong Kong and Shanghai classes, also on the translated transcription of the dialogue. This involved two conversions, first from the video to the transcript and then secondly a translation into English. Both involved issues of validity and were subject to selections and interpretations. In some cases I have had assistance from a native speaker of Chinese, who watched and listened to the video of certain key episodes. This extra check of the translations was also done regarding some formulations on worksheets and slides with examples and exercises that were used in the Hong Kong and Shanghai classrooms.

Ethical considerations

The Swedish Research Council has published 'Ethical principles for research' [Forskningsetiska principer inom humanistisk-samhällsvetenskaplig forskning] (Vetenskapsrådet, 2002) with the aim of regulating the relations between researchers and persons in a study. The ethical principles are basically expressed by the formulation of four 'requirements' on research projects in order to protect individuals. In short the principles are the following.

1. *Information requirement*. Individuals shall be completely informed regarding the study and the conditions of their participation.

2. *Consent requirement*. Individuals have the right to decide about their participation, the length and termination etc. For participants under the age of 15 the consent of parents or guardians is also required.

3. *Confidentiality requirement*. Personal data shall be kept safely and away from not authorised personal. Possible identification of individuals in sensitive circumstances shall be avoided

4. *Utility requirement*. Data regarding individuals shall not be used for non-scientific purposes, e.g. in commercial or business related activities.

The Swedish research team made sure to follow all requirements. The researchers from Uppsala University have extensive experience of doing video recordings in schools and kindergartens and have developed certain routines that were followed. A special consent form was used for both students and their parents/guardians. The recordings made in Hong Kong and Shanghai followed the appropriate regulations in their respective legislation.

The third point concerns the handling of data. In Sweden videotapes showing individuals are regarded as a "register of individuals" and regulated in the Personal Data Act [Personuppgiftslagen, PUL] (Justitiedepartementet, 1998) which aims at preventing the violation of personal integrity in the processing of personal data. This meant that storage of videotapes and hard drives containing video recordings was given special care. To make identification difficult all students and teachers in the LPS data were given pseudonyms, which were used in all transcripts.

Four steps of analysis

The complete data set consisted of nine video-recorded classes from the Learner's Perspective Study, three classes from Sweden (SW1-3), three from Hong Kong, (HK1-3) and three from Shanghai, (SH1-3). Twelve to eighteen consecutive lessons from each class were documented by the three-camera arrangement.

The lesson tables that were part of the LPS school package gave an overview of the type of activity, what the content was, and what examples and exercises were used for each recoded lesson. They were used together with the lesson videos to determine what mathematics was taught in the nine classrooms.

Step 1 – Identifying the content possible to compare

An overview of the mathematics taught in these lessons is displayed in table 4.1. The research teams in Sweden and China aimed to capture the teaching of the same content in all nine classes. This was not possible in all cases. In the available material I identified four different mathematical content areas which would be possible to compare and where at least one of the Swedish classes was included.

- The coordinate plane/system of coordinates
- The system of linear equations in two unknowns
- The solution to a system of linear equations in two unknowns
- The method of substitution for solving a system of linear equations in two unknowns

There were three lessons in three different classrooms where the coordinate plane was being introduced, one in Sweden and two in Shanghai ('framed' in table 4.1).

There were seven classrooms where 'system of equations', including its solutions and the method of substitution, were taught. This

Table 4.1 *Mathematics content of the video-recorded classes*

Lesson #	SW1	SW2	SW3	HK1	HK2	HK3	SH1	SH2	SH3
1	(–)	Volume of cylinders	Negative numbers - add/subtr	Factorisation of expressions	Revision algebra + System of linear eq.	Slope of parallel lines in coord-inate plane	Linear equations in 2 unknowns	Linear inequalities in 1 unknown	Linear equations in 2 unknowns
2	Linear equations in 1 unknown	(–)	Formulas, algebraic expressions	Factorisation of expressions	Method of substitution	Problem solving with parallel lines	The coordinate plane point -> (x,y)	Linear equations in 2 unknowns	The coordinate plane point -> (x,y)
3	Linear equations in 1 unknown	Volume of spheres (cylinder + cone)	Formulas, algebraic expressions	Factorisation of expressions	Method of substitution	Product of slope of perpendicular lines	The coordinate plane (x,y) -> point	System of linear eq. + Method of substitution	The coordinate plane (x,y) -> point
4	(–)	Linear equations in 1 unknown	Formulas, simplify algebraic expressions	Factorisation of expressions	Method of elimination (add/subtr)	Problem solving with perpendicular lines	The graph of a linear equation	Method of substitution	The graph of a linear equation
5	(–)	Problem solving w linear eq. 1 unknown	Simplify algebraic expressions	Factorisation of expressions	Method of elimination (addition)	Graph of a linear equation	System of linear equations	Method of elimination (add/subtr)	System of linear equations
6	The coordinate plane point -> (x,y) + (x,y) -> point	Problem solving w linear eq. 1 unknown	Revision + simplfy algebraic expressions	System of linear equations	Graphical method	System of linear eq. graphical method	Method of substitution	Method of elimination (add/subtr)	Method of substitution
7	Proportionality - line in coordinate plane (1/2 class)	Problem solving w linear eq. 1 unknown	Algebraic expressions w. brackets (var ex +/-)	Method of substitution	Graphical method	System of linear eq. - graph. 3 diff cases: //, /, X	Method of elimination (add/subtr)	Substitution or elimination (choice)	Method of elimination (add/subtr)
8	Proportionality - line in coordinate plane (1/2 class)	Problem solving w linear eq. 1 unknown	Multiplicat. w brackets	Method of substitution	Problem solving w syst. of eq.	Method of substitution	Method of elimination (add/subtr)	3 unknowns	Method of elimination (add/subtr)
9	Test (alg. expressions, linear equations)	Problem solving w linear eq. 1 unknown	Multiplicat. w brackets	Method of substitution	Problem solving w syst. of eq.	Method of substitution	Graphical method	3 unknowns	Substitution or elimination (choice)
10	Lines in coordinate plane (stud. work ind.)	Problem solving w linear eq. 1 unknown	(–)	Method of elimination (add/subtr)	Problem solving w syst. of eq.	Method of elimination (add/subtr)	Revision of system of equations	Problem solving w system of eq	Graphical method
11	Slope and gradient	(–)	Test (alg expressions)	Method of elimination (add/subtr)	Rev: meth of substitution / elimination	Problem solving w syst. of eq.	3 unknowns	Problem solving w system of eq	3 unknowns
12	Slope and y-intercept	System of linear eq. + method of substitution	Rev of some test items	Method of elimination (add/subtr)	Rev: Probl. solving w syst. of eq.	Problem solving w syst. of eq.	3 unknowns	Problem solving w system of eq	3 unknowns
13	Graphs of linear eq. (stud. work ind.)	Method of substitution		Graphical method	Rev: mixed algebra tasks	Problem solving w syst. of eq.	Problem solving w system of eq	Revision of system of equations	Problem solving w system of eq
14	Math quiz + Diagrams (stud. work ind.)	Method of substitution		Graphical method	Rev: mixed algebra tasks	Problem solving w syst. of eq.	Problem solving w system of eq	System of linear inequalities	Problem solving w system of eq
15	Reading diagrams (1/2 class)			Graphical method			Problem solving w system of eq		
16	Reading diagrams (1/2 class)			Problem solving w syst. of eq.					
17	IBT test			Problem solving w syst. of eq.					
18				Problem solving w syst. of eq.					

mathematical content was taught in one of the Swedish classrooms and in all classrooms from Hong Kong and Shanghai (table 4.1). However, in HK3, the introduction of the systems of equations was done with a graphical approach, which was quite different from the other six classrooms. The graphical representation and method of solving systems of equations were introduced at a later stage in the other classrooms.

I made the decision to concentrate the study on how systems of equations – together with its solutions and the method of substitution – were handled and included six of the classrooms. By doing this I gained the greatest possible number of lessons where the same mathematics was taught, as well as representation from all three geographical locations.

Step 2 – Analysing the handling of the mathematics

This step involved a much closer examination of the video recordings, the transcripts (in English and Swedish), the copies of the textbooks, worksheets, the notebooks of the focused students, and overhead slides or power point presentations shown on the screen. The analysis was mainly based on the video data and the transcribed conversation. This step involved a repeated re-examination of the data. The first examination was done lesson by lesson in the following order,

SW2-L12, SW2-L13, SW2-L14, HK1-L06, HK1-L07, HK1-L08, HK1-L09, HK2-L01, HK2-L02, HK2-L03, HK3-L06, HK3-L07, HK3-L08, HK3-L09, SH1-L05, SH1-L06, SH2-L03, SH2-L04, SH3-L05, SH3-L06.

The results of the analysis are also presented in this order. The next rounds of analysis followed the lessons in the same order but differed in the starting point. For instance, I could start with HK3-L06 (lesson 6 from Hong Kong class 3). After SH3-L06 I would continue with SW2-L12 and end with HK2-L03.

In the first stage I looked for variations of specific features regarding the objects of learning within a lesson sequence and for differences between classrooms in the handling of the content. The analysis aimed to take the perspective of a student in the classroom. For the student in the classroom what aspects of the object of learning were made possible to discern from what occered?

An aspect is either taken for granted during teaching, or the corresponding DoV is opened. A certain DoV can be opened more or less explicitly by the tasks and examples used, or by the contribution of possible or hypothetical alternatives. An example of the first case involved a number of tasks where the unknowns are represented by x and y in the first, by r and s in the second, and by u and v in the third system of equations. Variation among the letters that represent the unknowns could provide students with the opportunity to discern the aspect 'different letters may be used to represent unknowns'. An example of the second situation involved a sequence where the teacher pointed to x and y in a system of equations on the board and said "of course, we may also use other letters, a and b, or s and t". In this way the teacher introduced or provided alternatives to the letters used which opened this dimension for variation and made it possible to discern. The alternatives that open up a dimension of variation could differ in 'range'. As an example, consider the DoV 'number of equations' regarding the concept of system of equations. The alternatives that would open this DoV were of two types in the studied lessons. One was a variation between one and two equations while the other one was a variation between one and more than one equation. The 'values' or the 'range' of the variation was slightly different but the variation was still within the same DoV.

The first stage in the analysis generated a preliminary list of potentially interesting instances of variation. The dimensions of variation regarding the enacted object of learning that was noted in this stage were at first quite specific and tied to certain events. The DoVs first discerned were probably from the most obvious patterns of variations in each classroom.

Successively I started to look for a more general structure concerning the DoVs of the objects of learning found in the first stage. I noticed that for each round my sensitivity to the data increased and successively the, at first somewhat disparate, DoVs in the list could be grouped together. My experiences from careful examination of one classroom made it easier to analyse the pattern of variation in another classroom. One might argue that it also could make me biased towards discerning certain aspects in favour of others. However, in my judgement, the increased experience was beneficial for the analysis.

In the end of this first stage of systematic analysis of the sixteen lessons a set of DoVs that were opened in at least one of the classrooms were generated. These dimensions of variation corresponded to different aspects of the objects of learning that were made possible to discern. This set of DoVs had been subject to several alterations as the lessons were repeatedly analysed. The DoVs were grouped in three main groups, one for each object of learning. The position of a DoV or an aspect turned out to be far from straightforward in some of the cases. For instance, the DoV 'the unknowns represent the same number in both equations' corresponded to an important feature of the concept of systems of equations and, at the same time, was a necessary condition for the performance of the method of substitution. Thus, it was not possible to completely separate the three objects of learning. However, in the next step I tried to focus on each of the three objects of learning, one at a time.

Step 3 – Focusing one object of learning at a time

In this stage I once again repeatedly re-examined the video data with varying starting points. This time I focused on one of the objects of learning at a time. Each lesson was divided into episodes of varying length based on the content and activity. For each such episode the dimensions of variations that were opened were noted and finally summed up in tables (see chapter 5). It turned out to be quite necessary to narrow the analytical focus and concentrate on one object of learning at a time. At this point the total number of DoVs had exceeded the limit of what was practically possible to attend to simultaneously.

Step 4 – Focusing the aspects 'taken for granted'

The result from the previous steps in the analysis showed that some of the DoVs were not opened in all classrooms. This would indicate that the corresponding aspects of the three objects of learning were taken for granted and thus kept invariant. In an attempt to verify these results I posed questions such as, "Is it really the case that 'DoV X' never is opened in 'classroom A'?" or, put more specifically, "Will the aspect 'the unknowns represent the same number in both equations' be taken for granted in all three lessons in SW2?" With questions like that I once again returned to the data. In this last step of the analysis I just focused one aspect at a time – the 'missing' ones.

Chapter 5
Differences in teaching the same content

The same mathematics taught in six classes

In this chapter the result of the analysis is presented. The same mathematics was taught in the analysed lessons. The contents were the concept of the system of linear equations in two unknowns, the concept of a solution to a system of linear equations in two unknowns, and the method of substitution for solving a system of linear equations in two unknowns.

There were in total sixteen lessons where this content was handled. The time spent on this content in the six classes differed, both in the number of lessons devoted to it, as well in the total time spent (table 5.1). In the Shanghai classrooms two lessons were used to cover the content. In the other classes (SW and HK) three or four lessons were spent on the same content. The total 'time on topic' was about the same in the three Shanghai classes and SW2, even though three lessons were spent in SW2. An explanation for this was that during some parts of the lessons in the SW2 classroom the students were engaged with tasks from other mathematical topics.

Table 5.1 *Lessons and time used for teaching the mathematics in the study*

Classroom	Lesson no.	Time on topic
Sweden (SW2)	12, 13, 14	90' 05"
Hong Kong (HK1)	6, 7, 8, 9	122' 30"
Hong Kong (HK2)	1, 2, 3	108' 50"
Shanghai (SH1)	5, 6	85' 05"
Shanghai (SH2)	3, 4	91' 10"
Shanghai (SH3)	5, 6	83' 50"

In the lessons that followed the ones included in the study all classes moved on to deal with the method of elimination by addition and subtraction. This was the case in SW2, too, and if SW2-L14 had not been the

Häggström, J. (2008). *Teaching systems of linear equations in Sweden and China: What is made possible to learn?* Göteborg: Acta Universitatis Gothoburgensis.

last recorded lesson in this classroom a fourth content, the method of elimination, probably could have been added to the study.

The three enacted objects of learning in focus for this analysis, were more or less intertwined in the six lesson sequences. Aspects related to one of the objects of learning might 'come up' in the middle of an episode where the main focus was on another. Therefore I present the results from the analysis one classroom at a time. Firstly, by giving a short general description of the class and the usual activities of mathematics teaching and learning. Secondly, the examination of each classroom is described in detail and the handling of the mathematics is followed through the lessons. In this latter part, each lesson is divided into sections or episodes. The episodes are formed on the basis of the type of activity and content handled. I aim to describe and present the data, on which the analysis is based, in the most complete way possible, to ensure that an informed judgement concerning the analysis can be made. At the end of each lesson episode the DoVs that were opened are summed up in a table. In the tables the DoVs are labelled according to the different categories described in the last section of this chapter.

The last section contains a discussion of each of the three objects of learning, one at a time, across the six classrooms. In this part I compare the handling of the mathematical content in terms of the enacted object of learning. Depending on how the content was handled in the classrooms, aspects of the object of learning would either be subject to variation or taken for granted and kept invariant. The classrooms are compared regarding which dimensions of variation were opened and which were not.

Some comments on the transcripts

The transcripts in English for all classes are from the translations of the original conversation and texts in Swedish, Cantonese and Mandarin Chinese present in the lesson-transcripts and lesson-tables in the LPS data. The translations have been carefully checked where deemed necessary. In places where the translations are significantly changed from the original LPS transcript in English it will be indicated in the text. The author has done the translation from Swedish to English. All names of students and teachers in the transcripts are pseudonyms. The descriptions of the classrooms and excerpts from transcripts presented in this chapter will follow the style below.

06:20 time on the video (6 minutes : 20 seconds)

T: teacher

S:	student (in some cases students)
...	pause
(...)	speech impossible to detect
[...]	speech omitted
[text]	description of action or events outside detectable speech, and author comments

Textbook or worksheet

Text on the board or screen

Sweden 2

Students from four classes form this ninth grade mathematics class. In the teacher questionnaire no. 1 (TQ1) the teacher states that this is a high-ability group. The math class was formed in the beginning of grade seven. This particular grouping of students is used only for mathematics instruction. Other subjects are taught in the regular classes or in other formations. The mathematics class consists of 24 students.

The mathematical content in the studied lessons cannot be considered compulsory in the syllabus for the Swedish comprehensive school (grundskolan). Most Swedish pupils will not study systems of equations in their first nine years of schooling. This streamed mathematics class, however, would cover more topics than what is normally done. The opportunity to be able to compare teaching of the same mathematics as in the Chinese lessons was one reason for including a grade nine class in the study.

The activities from the students' perspective are basically of three kinds. The students often take part in shorter teacher-led whole-class instruction and discussions. The work is mostly with tasks from the textbook, individually or students sitting next to each other together. The work on textbook tasks makes up the larger part of the lessons. All students work with tasks from the same mathematical topic, but not necessary precisely the same tasks at the same time. The students are somewhat 'spread out' in the textbook, but basically kept within the same chapter. There are also occasions when students show their solutions on the board.

A lesson will typically start with a whole-class activity. This introduction is followed by students working with textbook tasks while the teacher walks around and talks to students at their desks. The teacher occasionally interrupts the seatwork and calls for the whole class' attention. He will then put particular problems, for instance tasks that he notices many students ask questions about, on the board and they will be discussed in whole class.

Overview of the lessons in Sweden 2

The three lessons analysed are no. 12, 13 and 14.

SW2-L12

In this lesson the concept of *system of linear equations in two unknowns* is introduced. The first example of a system of equations is generated as a way to represent (and solve) a problem.

This system is then solved (almost) by the *method of substitution*. Two more systems of equations of the same kind are later solved on the board and the students also work with the same type of tasks in the textbook.

SW2-L13

A system of equations that the teacher and the students make up together is solved in the initial whole class episode. Then the teacher leaves this topic and brings up a question about the relation between sides and areas of papers in the A-format (A3, A4, A5 etc.). This is caused by a question from the national test in mathematics that the students took earlier this day. The lesson ends with some exercises of mental calculation.

SW2-L14

The lesson starts with a continued discussion about the problem concerning the A-format for papers.

Then one system of equations is solved by the method of substitution on the board. The students work with similar tasks in the textbook. This makes up about half the lesson.

A task (a linear equation in one unknown with rational expressions) from the homework is also solved on the board. The lesson ends with a whole class discussion about various algebra tasks.

Sweden 2 – Lesson 12

SW2-L12 [6:10-9:00]

The teacher returns to a problem that in the previous lesson has been handled by the formulation of an equation with one unknown. Some students had tried to use two unknowns. The teacher shows how it is possible to start with two unknowns and then how these can be combined – how one unknown can be expressed by means of the other.

06:10 T: We will continue with equations today, but we'll take one more step forward and do what some of you already tried ... to have not just one but two. Two unknowns ... er ... I thought of this dog-and-cat task, 'There are 660 cats and dogs in the town, of which some are out of their mind'. Some of you labelled the number of dogs ... D I think you wrote ... and cats C ... Then we have two variables.

Teacher writes "DOG = d [of them], CAT = c [of them], 660 all together" on the board.

07:25 T: ... we can move from two unknowns to one unknown ... right?

[...]

07:50 T: If we put D as dogs [writes "d = DOG" on the board] ... how many cats are there? ... If I exchange the variable, or the unknown C, to something else? [...] How can we express this? Move from two unknowns to one? ... How many cats are there?

[...]

08:40 T: 660 minus [writes "660 – d = CAT" on the board] ... That's the number of cats. You switch. You have two unknowns. You exchange one for the other. Let's do a task like that.

DOG = d [of them]
CAT = c [of them]
660 all together

d = DOG
660 - d = CAT

In this episode the problem is invariant. The number of unknowns used is varied. There is a contrast between the use of one unknown and two unknowns, both in the actual episode and by how the same problem was handled in the previous lesson. The DoV 'number of unknowns' is opened.

Opened dimensions of variation SW2-L12 [6:10-9:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
2a. Two unknowns not one	–	–

SW2-L12 [9:10-13:15]
The teacher and the students generate a problem together. A student is asked to ”think of a number” (x) and inform the teacher without telling anyone else. The teacher then ”thinks of another number” (y). The teacher is the only one (supposedly) who knows the two numbers.

09:10	T:	Tony, think of a number ... tell me, but no-one else. [writes on the board]
[...]		
10:10	T:	I think of another number ... I know that. [writes ”x + y = 60” on the board]
10:25	T:	Those two numbers together are 60 ... and then, the big question is – which are the numbers? [...] Which are the numbers? ... Any suggestions? ... Joel.
	S:	56 and 4.
	T:	Alright. Why is that?
	S:	It’s 60.

Teacher writes ”56 + 4 = 60” on the board.

11:10	T:	Are there any other possibilities? ... Yes, of course there are ... Now, I know this is correct because you have seen through me here, [Joel happened to see Tony’s number] but ... is it enough to know the sum of two numbers is 60 in order to find the answer? Michael?
	S:	No.
	T:	No it isn’t.
	S:	(...) it could be 40 and 20.

Teacher writes this and a third possible pair of numbers on the board.

Think of a number: x
Think of another number: y

$x + y = 60$

$56 + 4 = 60$
$40 + 20 = 60$
$35+25 = 60$
$\vdots \quad \vdots$

They reach the conclusion that there is not enough information to determine the two numbers.

12:10 T: Thus, it's not enough to know one condition.

[...]

12:35 T: Your [Tony's] number is 14 times my number.

The teacher then adds this second condition and gets the following on the board.

$$\begin{cases} x + y = 60 \\ x = 14 \cdot y \end{cases}$$

13:00 T: Now, we already know the answers, it's a little ... no good, but anyhow ... now I have two conditions and two unknowns. Now we can easily calculate ... the whole, so let's do it.

During this episode the two numbers 'thought of' are kept invariant. The student and the teacher 'keep thinking' of the same two specific numbers throughout this episode. At the same time the 'possible' or 'potential' values of *x* and *y* are varied when there is just one equation on the board. There is also variation with respect to the number of equations. With just one equation there is no single solution to be determined. However it is not shown, nor stated clearly by the teacher, that the addition of the second condition/equation will generate *one* unique solution to the problem. The fact that the numbers thought of are revealed do complicate the situation too.

Opened dimensions of variation SW2-L12 [9:10-13:15]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one	2a. One equation has many solutions/no unique solution	–

SW2-L12 [13:20-14:40]

The teacher demonstrates how to find the value of *y* with the method of substitution.

13:20 T: That is condition one and that is condition two [The teacher adds (1) and (2) to the equations] ... then we combine them, put them together, sort of ... and then you put in, exchanges [points to x in equation (2)] ... there [points to x in equation (1)] ... I know ... [points to (2)] one number is 14 times the other, then I exchange, suitably,

> there [points to x in (1)] ... you usually write one ... and two ... gives [writes ”(1) and (2) gives:”] ... I exchange that unknown [points to x in (1)] for 14y, it says here [points to (2)] x is 14 times the number ... 14y plus. [writes 14y + y = 60 ...]

The teacher finishes the solving for *y* on the board.

$$\begin{cases} x + y = 60 & (1) \\ x = 14 \cdot y & (2) \end{cases}$$

(1) and (2) gives: $14y + y = 60$

$$15y = 60$$

$$y = 4$$

14:35 T: And then we got ... what Joel recently have told us.

The teacher does not finish the process of solving – the value of *x* is not calculated. The two unknown numbers (the student's and the teacher's) had already been revealed by mistake. None of the numbers are really unknown to the students during this episode. Maybe this circumstance makes the teacher abort the solving when the value of the first unknown (*y*) is found.

The characteristic properties of a solution to a system of equations are left uncommented. The teacher seems to assume that the students know that a complete solution consists of the values of *both* unknowns. Also the procedure of how to find the second unknown *x* by use of the found value of *y* is taken for granted by the teacher and not shown at all. Instead the teacher returns to the original system of equations on the board as soon as the value of *y* is found.

The teacher takes for granted that the letters *x* and *y*, respectively, represent the same two unknown numbers in both equations. Maybe because it is an ’obvious consequence’ of how the equations were generated from the two numbers ’thought of’. It is, at the same time, quite necessary for the two *x*:es to be equal if one *x* is to be exchanged for 14*y*. However, the teacher never expresses this explicitly. This dimension of variation is not opened.

During the solving of *y*, the two equations with two unknowns, in the system, are transformed into one equation with one unknown. The main idea behind this method of solving a system of equations – generating a solvable equation in *one* unknown – is of course displayed by the teachers’ solution on the board but otherwise taken for granted.

The use of unknown (x) from equation (2) for the initial substitution is also taken for granted. There is no indication at all that there might be any alternatives.

Opened dimensions of variation SW2-L12 [13:20-14:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

SW2-L12 [14:50-30:20]

The teacher points to the two equations and use them to 'define' the system of equations.

14:50 T: What is this then? [points to the equations on the board]

[...]

T: Well, it's two equations. A system of equations, it is called.

The teacher writes "system of equations" beside the equations. Nothing else about the properties of this new concept is said.

The teacher puts a task from the textbook (920a, see below) on the board for the students to work on. When they have finished that one they will continue to work on similar tasks in the textbook. After a couple of minutes, while the students are working the teacher writes the solution (to 920a) on the board without saying anything.

After another couple of minutes the teacher puts another example (922b, see below) on the board and solves it, this time by a whole-class discussion of the steps in the solution. The teacher addresses the transformation from two unknowns to one unknown.

27:55 T: We have two equations ... two unknowns and then we can calculate their values. How do we do? Yes ... because it's a quite nicely formulated ... y in number one [points at (1)] equals five times an unknown number minus three... then in equation two [points at (2)] we transfer to one unknown. We replace y [points to y in (2)] by 5x–3 [points to "5x–3" in (1)] ... How do we do that then? ... Maybe it's not just to write directly? Is there anything we have to ... watch out for? ... Tony, do you know what I mean?

S: (...) use brackets.

The teacher writes "(1) and (2) gives: $7x - (5x - 3) = 6$". The necessary use of brackets $(5x - 3)$ is commented.

29:25 T: why, well, it wouldn't be correct if I just had [teacher erases the brackets] written like that ... exchange y for 5x–3 ... then something happens ... we shall take away [puts back the brackets] exactly the expression 5x–3.

The teacher continues to solve the system of equations. This time the teacher explains how to get the value of the second unknown, *y*.

30:10 T: Now, when I calculate the y-value I choose to take my x-value [points to x=1,5] and put it into this here [points to 5x in (1)]. Five times one and a half ... seven and a half ... minus three, four and a half.

The teacher puts the examples side by side on the board. The solutions can be compared directly.

920a)	922b)
$\begin{cases} y = x + 2 & (1) \\ 3x + y = 6 & (2) \end{cases}$	$\begin{cases} y = 5x - 3 & (1) \\ 7x - y = 6 & (2) \end{cases}$
(1) and (2) gives:	(1) and (2) gives:
$3x + x + 2 = 6$	$7x - (5x - 3) = 6$
$4x = 4$	$7x - 5x + 3 = 6$
$x = 1$	$2x = 3$
Answer: $x = 1$	Answer: $x = 1{,}5$
$y = 4$	$y = 4{,}5$

The same procedure is used for solving all three systems of equations put on the board so far. The students are supposed to solve tasks in the textbook in the same way. The method is kept invariant as the systems of equations vary. However the formats of the system of equations are not that different. In all cases one of the equations is in the format $y = f(x)$ or $x = f(y)$, suitable for substitution without any rearrangement. In the first example, *x* from equation (2) was substituted into (1). The latter two examples use *y* from equation (1) into (2).

The variation between the first and the latter two examples would make it possible to discern that both unknowns (*x* and *y*) can be used in the initial substitution, as well as that both equations (1 and 2) can give the expression to substitute into the other. This is, however, in each case taken for granted. In all examples so far (this is also the case for the tasks in the textbook) one of the equations is possible to use for substitution directly without having to rearrange it in any way. This fact makes the choice of unknown and equation for the initial substitution so unproblematic, that this aspect in a way seems to 'vanish'.

The direct comparing of the other two solutions, displayed on the board side by side in this episode, shows no difference regarding what unknown and equation to use for the initial substitution.

The teacher at first seems to take for granted how you find the value of the second unknown. In the very first example, in the previous episode, the value of the second unknown is not worked out at all. How the teacher finds the value of *y* in example 920a is neither revealed in any way, as the teacher just writes the solution on the board without saying anything. When the third example (922b) is solved, the teacher describes verbally how the value of the second unknown is found, but this is not noted on the board. Which equation to use for this second substitution is taken for granted.

Opened dimensions of variation SW2-L12 [14:50-30:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	3b. The unknowns can be rational numbers	2a. Getting an equation in one unknown is not taken for granted

SW2-L12 [30:25-31:20]

The solution to the task 922b is checked by substitution of $x = 1{,}5$ and $y = 4{,}5$ into the two equations.

30:25 T: Is this correct? ... Have we calculated correctly? ... Well, it is easy to check ... five times one and a half [points at 5x in (1)] ... minus three ... four and a half [points at y in (1)]. That's what we got just before ... what about here? [points at 7x in (2)] ... seven times one and a half ... what's that? ... seven times one and a half ... Tony?

S: Ten and a half.

T: Ten and a half ... minus [points at y in (2) and then to y=4,5] ... four point five ... ten point five minus four point five [writes 10,5–4,5=6] that's [points at 6 in (2)] ... six ... then these two values are valid for both equations ... a check up.

The values of *x* and *y* are kept invariant while they are substituted into both equations of the system. The teacher takes for granted that the unknowns (x and y) represent the same numbers in both equations and just performs the check.

Opened dimensions of variation SW2-L12 [30:25-31:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	1b. Substitution of solution will get equal sides in both equations	–

SW2-L12 [31:20-39:30]

The students work with similar systems of equations from the textbook. The teacher walks around talking to students at their desks. The degree of variation is quite low among the tasks in the textbook.

Textbook (page 207)

920a)	$\begin{cases} y = x+2 \\ 3x+y=6 \end{cases}$	b)	$\begin{cases} y = 2x–1 \\ x+y=5 \end{cases}$	c)	$\begin{cases} y = 4x–3 \\ 2x+y=6 \end{cases}$
921a)	$\begin{cases} y = 2x \\ 9x–2y=15 \end{cases}$	b)	$\begin{cases} y = x+1 \\ x+2y=11 \end{cases}$	c)	$\begin{cases} y = 3x–2 \\ 2y–5x=0 \end{cases}$
922a)	$\begin{cases} y = 3x \\ 4x–y=1 \end{cases}$	b)	$\begin{cases} y = 5x–3 \\ 7x–y=6 \end{cases}$	c)	$\begin{cases} y = 4–x \\ 5x–y=5 \end{cases}$
923a)	$\begin{cases} y = x+4 \\ 5x–2y=1 \end{cases}$	b)	$\begin{cases} y = 2–x \\ 4x–3y=8 \end{cases}$	c)	$\begin{cases} y = 2x+5 \\ 6x–2y=5 \end{cases}$
924a)	$\begin{cases} y = 7 \\ 8x–3y=-1 \end{cases}$	b)	$\begin{cases} y = 2,5x–4 \\ 6x–2y=12 \end{cases}$	c)	$\begin{cases} 16x–2y = 9 \\ y=7x–2 \end{cases}$
925a)	$\begin{cases} x = 4y–2 \\ 3x–10y=3 \end{cases}$	b)	$\begin{cases} x = 9–2y \\ 5y–3x=6 \end{cases}$	c)	$\begin{cases} x = 0,5y+10 \\ 2y–x=5 \end{cases}$

The 18 items (systems of equations) in the textbook have many similar properties. The authors do not use variation between items in a sequence to bring forward important features of the content.

- Only the letters x and y are used to denote the unknowns.
- One of the equations is either $y = f(x)$ or $x = f(y)$ which makes the application of the method of substitution quite straightforward. The first 14 items have the first equation in the format $y = f(x)$. No. 15 has the second equation as $y = f(x)$ and the last three have the first equation as $x = f(y)$.
- The other equation is either of the format $ax \pm by = c$ or $ay \pm bx = c$.
- In all but one task, both equations contains both unknowns. Equation (1) in task 924a is $y = 7$.
- Natural numbers are used for all coefficients, with two exceptions. (2,5x in 924b and 0,5y in 925c). There is one negative number, (-1) in 924a.

There are both natural and positive rational numbers as solutions and one task has a solution where x and y are the same number (925b: $x = 3$, $y = 3$).

The question of which unknown in which equation to choose for the initial substitution is never necessary to raise. Even though there is variation both regarding the unknowns and the equations used in the whole set of tasks there are in no case any real alternatives. Which unknown from which equation to use is basically given in every case. The lack of alternatives would make it difficult to discern that *any* unknown in *any* of the equations can be used in principle.

However small differences between the items in the textbook are, some DoVs will anyhow be opened. But here there probably will be differences regarding the individual student. The actual pattern of variation depends on which of the items in the textbook a student works on.

Opened dimensions of variation SW2-L12 [31:20-39:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5a. Rational numbers can be used as coefficients 5b. Negative numbers can be used as constants 7c. Both unknowns are not present in both equations	3b. The unknowns can be rational numbers 4a. Solution with two equal numbers possible	–

SW2-L12 [39:30-40:50]
The teacher explains the term ”substitutionsmetoden” [Swedish for ”method of substitution”. The word ”substitution” is not really everyday language in Swedish] by relating it to how the English word substitution is used in football, when one player is replaced by another.

40:40	T:	We replace one ... not player, but one variable by, sort of, something else.

Opened dimensions of variation SW2-L12 [39:30-40:50]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

Sweden 2 – Lesson 13

SW2-L13 [5:50-6:30]
The teacher asks the students to explain what a system of equations is. A student answers, but unfortunately the sound is not good at this part and it is difficult to hear everything she says. The teacher anyhow seems to accept the answer right away and the properties of a system of equations are not elaborated on further.

05:50	T:	A little about systems of equations, I said. What is that? Explain. Not just to be able to ... calculate and write and do, or solve a system of equations. What is it? ... Tell me ... nobody wants to tell? ... Erica?
06:15	S:	When you with two equations can solve (...).
	T:	That’s right.
	S:	Or, eh, with two equations can tell (...) and find out what x and y are.
	T:	Then I think, let’s do a, a system of equations ... we’ll make one up.

Opened dimensions of variation SW2-L13 [5:50-6:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

SW2-L13 [6:30-25:30]

The teacher 'constructs' a system of equations on the board together with the students. The first student contributes 3*x*, the next -2 and so on. Finally, a complete system of equations is on the board.

$$\begin{cases} 3x - 2 + y = 15 & (1) \\ 3 - 5x + 17 = 4{,}5y & (2) \end{cases}$$

10:35 T: From number one here, we can get a ... a description of how much y is? ... Can you do that? ... From one we can say that y equals [writes "y="] ... Think ... Think as you usually do when solving equations.

[...]

T: We can't calculate what y is, but we can rewrite this number one. [points at (1)]

[...]

11:50 S: Seventeen ... minus three x. [The teacher writes "$y = 17 - 3x$" on the board]

The choice of using the unknown *y* from equation (1) for the initial substitution is taken for granted. One of the students suggests that the expression (17 – 3*x*) be substituted into equation 1. The teacher frowns a little but follows the suggestion and writes on the board.

$$3x - 2 + 17 - 3x = 15$$

13:40 T: What will we get from this? [...] We will get 15 equals 15 ... it won't give us any more.

[...]

14:20 T: We sort of used one to get that [points at "17 – 3x"] ... then we insert it into two [points at (1) and (2)] ... Do it, try it yourself.

The students start to work on the examples in their notebooks. After a while the teacher writes on the board.

(1) and (2) give:

$$\begin{array}{ll} 3 - 5x + 17 = 4{,}5(17 - 3x) & \\ 20 - 5x = 76{,}5 - 13{,}5x & x = 56{,}5/8{,}5 \approx 6{,}647 \\ 8{,}5x = 56{,}5 & y \approx -2{,}941 \end{array}$$

This is the fourth system of equations put onto the board. The method for solving it is the same as before, but some new features might be possible to discern, if a comparison is done with the examples from the previous lesson.

The fact that the teacher invites the students to take part in the construction of the system that is solved on the board indicates that 'any system of equations' will do. This teacher seems not to use the variation a sequence of examples can create in any deliberate way. The rational numbers in the answer were not intended. The teacher explains to the students – almost makes an excuse – that the "strange answer" is a result of how the example was generated. There is no discussion regarding *how* the value of y is calculated, nor is it written on the board.

Opened dimensions of variation SW2-L13 [6:30-25:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5a. Rational numbers can be used as coefficients 7a. Format of individual equations not invariant	3a. The unknowns can be negative numbers 3b. The unknowns can be rational numbers	6b. Equation (1) $\Rightarrow$ (3) should not be substituted back into (1)

SW2-L13 [25:30-45:20]
The topic is changed. The class deal with a problem about the A-format for papers.

Opened dimensions of variation SW2-L13 [25:30-45:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

Sweden 2 – Lesson 14

SW2-L14 [0:00-11:15]
The class is dealing with other content (A-format problem continued).

Opened dimensions of variation SW2-L14 [0:00-11:15]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

SW2-L14 [11:15-22:30]
After the first 11 minutes, when the class was dealing with other topics, they continue with systems of equations. The teacher writes a task (925a) from the textbook on the board and the students are asked to solve it.

12.25 T: I thought, let's do a system of equations, anyway, to start with.

The teacher walks around while the students work on the problem. After a couple a minutes one of the students is sent to the board to write the solution. The teacher calls for attention and discusses some steps in the solution on the board.

$$\begin{cases} x = 4y - 2 & (1) \\ 3x - 10y = 3 & (2) \end{cases} \qquad \text{Answer: } \begin{cases} x = 10 \\ y = 4{,}5 \end{cases}$$

$$3(4y - 2) - 10y = 3$$
$$12y - 6 - 10y = 3$$
$$2y - 6 = 3$$
$$2y = 9 \qquad \frac{2y - 9}{2} \qquad y = 4{,}5$$
$$x = 4(4{,}5 - 2)$$
$$x = 18 - 8$$
$$x = 10$$

20.00 T: Here we exchange ... x is this much [points to (1)] ... we exchange ... x there [points to 3x in (2)] for exactly four y minus two within brackets [points at 4y–2 in (1) and then to (4y–2) in the first row of the student's solution] ... and then one of the variables disappears, that's the point with the whole, and we can calculate, first a value of y [points at y=4,5] and then a value of x. [points at x=10]

The teacher directs the attention to two features of the method of substitution. By pointing to the change in the number of unknowns in the equations, to generate an equation in *one* unknown is not taken for granted. The other feature is that the values of the unknowns are calculated in order, first the value of one unknown and then the value of the second unknown. By pointing at the two unknowns in turn and commenting on the order in which they are found the teacher opens the DoV regarding the main idea of the method of substitution.

One of the students points to an error in the solution which the teacher missed. The teacher corrects ”$x = 4(4{,}5 - 2)$” to ”$x = 4 \cdot 4{,}5 - 2$” and ”$x = 18 - 2$; $x = 16$”.

22:08 T: One bracket too many ... it's not often ... you can get a bracket too many.

Opened dimensions of variation SW2-L14 [11:15-22:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	1a. Main idea is not taken for granted 2a. Getting an equation in one unknown is not taken for granted

SW2-L14 [22:30-47:30]

The students continue to solve systems of equations in the textbooks. The teacher walks around and talks to some students at their desks. The limited variation between the exercises in the textbook (see above) reduces the number of DoVs that are opened. The opened DoVs are the same as when students worked on the same tasks before. One of the students shows the solution to one task from the homework on the board (linear equation in one unknown).

Opened dimensions of variation SW2-L14 [22:30-47:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5a. Rational numbers can be used as coefficients 7c. Both unknowns are not present in both equations	3b. The unknowns can be rational numbers 4a. Solution with two equal numbers possible	–

SW2-L14 [47:30-55:00]
The class is mainly dealing with other content than systems of equations. The lesson ends with a discussion of some questions concerning algebra, but not systems of equations.

Opened dimensions of variation SW2-L14 [47:30-55:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

All systems of equations shown on the board in the three SW2 lessons as well as those in the textbook have precisely one solution. This does not open the dimension of variation regarding the possible number of solutions to a system of linear equations in two unknowns.

Hong Kong 1

This eight-grade mathematics class has 39 students. It is a mixed ability class (TQ1). The teaching in general can be characterised as whole-class teaching, meaning that all students basically work with the same topic and problems throughout the lessons. As a student your main activities are either paying attention and taking part in the teacher-led instruction and discussion, or working on common problems and questions by yourself or together with the student sitting next to you. During this latter activity the teacher walks around and talks to students at their desks. As the lessons unfold there are constant shifts between these types of activities. Typically a problem is posed on the board, and the students are given some time (from 1–2 minutes up to 10–12) to think about or work on it. Students are often asked to show their solutions on the board. This is usually followed by a whole-class discussion of results and conclusions.

Overview of the lessons in Hong Kong 1

The three lessons analysed are no. 6, 7, 8 and 9.

HK1-L06
The teacher starts the lesson with a revision of the concept of a linear equation in one unknown. The concept of *system of linear equations in two unknowns* is introduced by a problem – the chicken-and-rabbit problem. It is solved by guess-and-check, by the forming of an

equation in one unknown and, finally, by the forming of two equations with two unknowns. The common solution to the two equations is found by listing solutions in tables.

HK1-L07

Three systems of equations in the same format, $x + y = a$ and $x - y = b$, with different values for a, and b are solved mentally by guess-and-check. The need for a general method becomes apparent.

The method of substitution is demonstrated and the students practice on some examples.

HK1-L08

The same system of equations is solved in four different ways. The unknown, y, from the first equation is substituted into the second equation, x from 2 to 1, y from 2 to 1 and finally x from 1 to 2.

How to start the solving – which unknown in which equation to choose for the initial substitution – is discussed for several examples, without actually doing any solving.

HK1-L09

The teacher explains the method of substitution one more time both by the use of a flowchart displayed on the board, and by describing the solution with a list of six steps.

The students practice the method of substitution.

Hong Kong 1 – Lesson 6

HK1-L06 [03:30-08:20]

The lesson starts with a short whole-class revision of the concept of linear equation in one unknown. A student gives an example ($r + 97 = 107$, $r = 10$), which is put on the board. The notions of *unknown*, *degree one* and *solution* are brought up.

07:05 T: The r here [points] means unknown.

[...]

07:40 T: What is the meaning of one degree?

[...]

07:50 T: Yes, yes. The index is one. I mean this one, this one [points above r, then writes a small 1 and directly erases it again] however, we are not used to writing the index of one explicitly. The linear equation in one 'yuan' means there is one unknown with the index of one.

We call it linear equation in one unknown. We call its answer as solution.

Opened dimensions of variation HK1-L06 [03:30-08:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

HK1-L06 [08:20-26:00]

The teacher reads aloud a question for the students to think of. He then writes it on the board.

> A farmer has some rabbits and some chickens. He does not know the exact number of rabbits and chickens, but in total there are ten heads, and there are twenty-six legs.

10:10 T: Okay, do you know how many chickens and rabbits the farmer has [...]?

In the following there are a couple of shifts between teacher-led whole-class instruction and student-work, individually and in pairs. The problem is handled in three ways. The first method used is guess-and-check. The teacher leads the way to the solution by posing questions; ”Can all of them be chickens?”, ”Can all be rabbits?”, ”Can there be five each?” etc.

Secondly, the problem is represented by the formulation of *one* equation in *one* unknown, $2x + 4(10 – x) = 26$. This is also done with firm guidance from the teacher, who assigns x to the number of chickens and $(10 – x)$ to the number of rabbits. When asked by the teacher, only five students admit they could have come up with this equation by themselves. The equation is posed but not solved, maybe because the answer is already known (as it was previously solved by guess-and-check).

The teacher then introduces simultaneous equations as the third method for solving the problem.

22:10 T: I am now going to teach you an easier method. It is also about using equations, but it is simultaneous equations in two unknowns.

The teacher assigns x to the number of chickens and y to the number of rabbits. During some minutes of teacher-led discussion the teacher writes two equations on the blackboard.

23:20 T: We are now using two 'yuan'. Ok, two 'yuan', two 'yuan' mean there are two unknowns.

[...]

24:10 T: That means, how many after adding x and y? Equals ten. [writes first equation]

[...]

24:40 T: Do the chickens have two x legs? How many legs do the rabbits have? [...] Four y. [completes the second equation]

$$\begin{cases} x + y = 10 \\ 2x + 4y = 26 \end{cases}$$

This example is used to 'define' the concept. The notions of simultaneous, linear, and two unknowns are mentioned at this time.

24:45 T: Well, okay, well, we call it simultaneous equations. Simultaneous means the equations will be listed out together. A moment ago ... Nancy has asked me why it is called linear. This is because we can draw a straight line from this kind of equation. We learnt to draw it in Form one. Okay, now we get this ... this is called two 'yuan', that means how many unknowns are there?

25:20 S: Two.

T: Two. It is simultaneous, that means the equations are put together.

The system of equations is not solved and no method for solving a system of equations is demonstrated or discussed.

25:30 T: We have already known the answer, ha, so the process, we temporary ... [the teacher writes on the board]

$$\begin{cases} x + y = 10 \\ 2x + 4y = 26 \end{cases}$$

$$\cdots\cdots$$

$$x = 7,\ y = 3$$

During the introduction the rabbit-and-chicken problem is kept invariant. The same problem is dealt with in three different ways. The method of representing and handling the problem is varied. The use of one equation is contrasted to the use of a system of equations. This contrast offers a possibility to discern some features of the new concept. There are *two* *unknowns* instead of only one. There are *two simultaneous equations* instead of just one. The characterisation of the equations as linear is mentioned only as a comment to a student's question but is not further discussed. This introduction stresses the system of equations as a method for solving problems as contrasted with the guess-and-check method and the use of one single equation in one unknown.

The example of a system of linear equations in two unknowns derived from the rabbit-and-chicken problem is used to 'define' the concept. There are no counterexamples given that could point to important features of what is not included in the concept. It is taken for granted that the system of linear equations in two unknowns is *the* system of equations. However, some DoVs related to the concept of system of linear equations in two unknowns are opened.

Opened dimensions of variation HK1-L06 [08:20-26:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one 2a. Two unknowns, not one 8a. Alternative to use of equation in one unknowns 8b. Alternative to use of guess-and-check method	–	–

HK1-L06 [26:00-34:45]

The teacher distributes worksheets and makes some remarks. The first task involves a system of equations and two tables. There are empty spaces for the corresponding y-values to $x = 0, 1, 2, 3$ for each equation.

$$\begin{cases} 4x - y = 0 \\ x + y = 10 \end{cases}$$

27:20 T: Okay, you see, there are two equations in the first question. Both of them have two unknowns so they are called two 'yuans'. The ... degree is one, therefore it is called simultaneous linear equations in two unknowns. If there are two or more equations put together, we will call it system of equations. Referring to the Hong Kong textbooks, we call them as simultaneous linear equations in two unknowns. In fact, we can also call it as system of equations of degree one in two unknowns.

The teacher guides the students through the first two y-values (corresponding to $x = 0$, $x = 1$). The students continue to work on the rest by themselves. After a minute and a half some students complete the tables that the teacher now has drawn on the board.

$4x - y = 0$

x	0	1	2	3
y	0	4	8	12

$x + y = 10$

x	0	1	2	3
y	10	9	8	7

The teacher discusses the solutions and points to the difference between solutions to the equations separately and the solution to the simultaneous equations (or systems of equations). The teacher circles the number pair 2 and 8 in both tables.

34:15 T: What is the common answer in this case? Are they two and eight? Only two and eight can satisfy these two equations. Therefore, we call x equals two and y equals eight be the solution of simultaneous equations.

The object of learning is in this episode shifted from conceptions of system of equations to the meaning of a *solution* to such a system. The pattern of variation includes keeping each equation constant while the number pairs that solve it are varied. When the focus is shifted from considering one equation at a time to considering them simultaneously the number of solutions change from many to one. The same number pair found in both tables is contrasted to the many different number pairs.

The teacher talks about the type of equations as linear and of degree one, but as no alternative is offered any appropriate meanings of these labels must be hard to discern in this episode. It is certainly not enough to use a term such as linear. Students could have been provided with opportunities to discern some of the features, for instance by examples of equations that are *not* linear. In this episode the DoV regarding the meaning of linear equation is not opened, even though the term is used.

Opened dimensions of variation HK1-L06 [26:00-34:45]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one	1a. Same solution to both equations is not taken for granted 2a. One equation has many solutions/no unique solution 2b. Two equations reduce the number of solutions to one	–

HK1-L06 [34:45-38:45]

The students start to work on the next task on the worksheet.

> Which of the given number pairs
>
> $\begin{cases} x = 2 \\ y = 7 \end{cases}$ $\begin{cases} x = 0 \\ y = 7 \end{cases}$ $\begin{cases} x = \frac{1}{3} \\ y = 6 \end{cases}$ $\begin{cases} x = 1 \\ y = 5 \end{cases}$ $\begin{cases} x = 5 \\ y = \frac{1}{2} \end{cases}$
>
> are solutons to
>
> (1) $y = 2x + 3$
>
> (2) $3x + 2y = 13$
>
> (3) $\begin{cases} y = 2x + 3 \\ 3x + 2y = 13 \end{cases}$

The lesson ends before the task is finished and it is not discussed in whole class. The five proposed solutions, however, cover all possible cases as being the solutions to none, one and both the equations.

Table 5.2 *Solutions to the task*

Equation	Number pairs				
	1	2	3	4	5
(1)	Yes	–	–	Yes	–
(2)	–	–	Yes	Yes	–
(3)	–	–	–	Yes	–

Working with the task involves keeping each number pair invariant while they are tried out in equations (1) and (2). The results form a pattern of variation shown in the table above. The differences between considering one of the equations at a time and considering them both opens some DoVs regarding the concept of solution. There is also variation in the format of the equations, when compared to the examples in the previous episodes. One of the equations is not in the 'standard format', $ax \pm by = c$.

Opened dimensions of variation HK1-L06 [34:45-38:45]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one 7a. Format of individual equations not invariant	1a. Same solution to both equations is not taken for granted 1b. Substitution of solution will get equal sides in both equations 2a. One equation has many solutions/no unique solution 2b. Two equations can reduce the number of solutions to one 3b. The unknowns can be rational numbers	–

Hong Kong 1 – Lesson 7

This lesson is in the afternoon the same day as the previous one (HK1-L06). They never return to the final task that was not finished in the morning lesson.

HK1-L07 [0:45-04:55]

The lesson starts with a short whole-class revision of the previous lesson.

00:45 T: What have you learned from the lesson this morning?

[...]

01:20 S: Systems of linear equations.

[...]

01:35 S: Simultaneous equations in two unknowns.

[...]

01:55 T: Good ... I will leave the precise definition till later ... Now I will explain it by one, two three [holds up one, two and finally three fingers] [...] ... One, can you find something with a one?

The teacher points to the board where he has written ”System of equations in two unknowns of degree one”. The Chinese characters for ’two’ and ’one’ are used. A more ’literal’ translation is ”two unknowns one degree equations system”.

02:25 T: Which place has the number one?

S: One degree.

T: Yes this is first degree [fills in character for ”one” with a chalk on the board] this is one, that is ... regardless to whether the unknown is x or y, or r it is of the first degree.

[...]

03:00 T: Yes, two unknowns. It’s quite easy to solve. It doesn’t matter if it is x, y, or m, n there should be two unknowns.

[...]

04:30 T: Yes, there are three sets.

The teacher writes ”$x + y = 3$” on the board and summarises his description by *one*, *two* and *three*. The teacher uses ’sets’ for the parts of an equation.

04:40 T: If x plus y equals three, um, two unknowns, one is first degree [points slightly above x and y on the blackboard]. Two is two unknowns [points at x and y]. Three, having three sets [points at x, y and 3]. There should be three sets after simplification.

When the teacher says ”I will explain *it* ...” one might assume that it would be the system of linear equations in two unknowns. But the example he puts on the board is a *single* linear equation in two unknowns. Also, all features that are discussed (degree *one*, *two* unknowns, *three* sets) are just as applicable to one equation as to a *system* of equations. One DoV however that is opened concerns the use of letters for unknowns. Twice the teacher points to alternatives to x and y.

Opened dimensions of variation HK1-L07 [0:45-04:55]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
6a. The letters x, y are not taken for granted	–	–

The notions of linear and degree one are mentioned several times but since no explanation is given nor any examples of something not linear or not of degree one this DoV is not opened.

HK1-L07 [04:55-07:50]
The teacher adds one more equation on the board and asks the students to ”calculate it in your head”.

$$\begin{cases} x+y=3 \\ x-y=1 \end{cases}$$

The first proposed answer (x = 2, y = 1) is checked by substitution and found to be correct. The teacher then erases 3 and 1, and writes 9 and 5 instead.

$$\begin{cases} x+y=9 \\ x-y=5 \end{cases}$$

Two incorrect answers are checked (x =4, y = 5 and x = 11, y = -2). The third suggestion (x = 7, y = 2) turns out to be correct. An exchange of 9 and 5 to 20 and 16 gives the third equation in the same format. The first answer from one of the students turns out to be correct.

$$\begin{cases} x+y=20 \\ x-y=16 \end{cases}$$

07:30 T: But what if it is very complicated? Then we have to learn a new method, method of substitution.

The three examples solved by guess-and-check mentally, are kept invariant regarding the format, while the constant terms are varied. The teacher is thinking for a while before writing down 16 in the third example, which might indicate that he is mentally calculating a suitable value and that this line of examples was not planned in detail in advance. In the second example the three proposed solutions all solve the first equation in the system, but only the third number pair also do solve the second equation. The difference between number pairs that satisfy one equation, but not the other, and a number pair that satisfies both equations would make it possible to discern this necessary feature of a solution.

Opened dimensions of variation HK1-L07 [04:55-07:50]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	1a. Same solution to both equations is not taken for granted 1b. Substitution of solution will get equal sides in both equations	–

HK1-L07 [07:50-15:40]

A worksheet is distributed to the students. The teacher solves the first task on the board.

1) $\begin{cases} x = y & \text{---(1)} \\ 8x + 2y = 10 & \text{---(2)} \end{cases}$

Substitute (1) into (2), get

$$8y + 2y = 10$$
$$10y = 10$$
$$y=1$$

Substitute $y = 1$ into (1), get

$$x=y$$
$$x=1$$
$$\begin{cases} x = 1 \\ y = 1 \end{cases}$$

The teacher points to that x and y are equal and therefore x can be replaced by y, and further that the substitution will lead to an equation in one unknown. The teacher does not indicate that $x = y$ is a special circumstance and not generally the case.

Before doing this first substitution he uses chalk of a different colour to mark the x:es in (1) and (2), maybe to indicate that they are the same. He uses another colour to mark the y in (2) and to write the first y in $8y + 2y = 10$. It is actually hard to see the difference in colour on the video, but the teacher definitely uses different chalks.

09:25 T: What is the relationship between x and y? ... They are equal. We can either write x or y. When you see x [marks x in (1) and (2)] ... you can write y [marks y in (1)], right?

The teacher writes ”Substitute (1) into (2), get 8*y* + 2*y* = 10”. The first *y* is written with different chalk, as is *y* in (1).

10:15 T: After substitution, what system of equations is it?
S: In two unknowns.
T: Huh ... in two unknowns? [points at 8y+2y=10]
S: One unknown.
T: Linear equations in one unknown because it can ... um [...] [writes 10y=10, y=1]

[...]

10:50 T: Yes we have to find x [completes the solution on the board] because x equals y. Finding x and you will get y.

[...]

12:35 T: x equals one, y equals one. This makes the value of the two sides, left and right, of the two equations of the system of linear equations in two unknowns the same.

This is repeated several times before they move on. The teacher mentions the basic idea of the method: substitution should make one of the unknowns ’disappear’.

15:00 T: Actually to be more precise, this is called the method of substitution and elimination. When you do substitution, one of the unknowns will disappear [points in the area of ”8y+2y=10, 10y=10, y=1” on the board]. If it doesn’t disappear, it renders your substitution meaningless [...] after substitution, one unknown will disappear, okay?

The method of substitution is not the only method used for solving systems of equations in this lesson. In the first part of the lesson guess-and-check is used to solve the same type of systems of equations.

Opened dimensions of variation HK1-L07 [07:50-15:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	4a. Solution with two equal numbers possible	2a. Getting an equation in one unknown is not taken for granted

HK1-L07 [15:40-29:30]

15:40 T: Please finish number two, three and four

2) $\begin{cases} x = 2y \\ 5x – 2y = 24 \end{cases}$	3) $\begin{cases} x = 3y \\ 3x – 2y = -7 \end{cases}$	4) $\begin{cases} y = 15 – 3x \\ y + 2x = 12 \end{cases}$

The students work on the tasks 2–4 in the worksheet. So far only one example of how the method of substitution is done has been shown. Seeing just one example would make it very hard for the students to separate features of the procedure that are specific to the example shown and features that are of a more general kind. There are two cases that indicate that students make mistakes trying to follow the only example shown so far.

In the only example so far x and y are equal, and this is also pointed out by the teacher. Thus, x in equation (2) is substituted for y: "$8x + 2y = 10 \Rightarrow 8y + 2y = 10$". The same procedure, substituting x for y, if employed in question 2 will look like this: "$5x - 2y = 24 \Rightarrow 5y - 2y = 24 \Rightarrow y = 8$". The value of y will thus be eight, which might explain the following conversation between the teacher and two of the students. The first student asks for help and the teacher comes and takes a look in the students' notebook.

18:00 T: Eight ... eight. y is equal to eight. Two times eight is sixteen, it's sixteen, sixteen equals two times eight, five times sixteen ... No ... five times sixteen ... five times two y ... five times two y equals ten y.

The student obviously has got $y = 8$ and the teacher starts to check the solution, and then he realises that it is not correct. A little later another student is asking for assistance. The value for y seems to be eight also in this case.

18:55 S: Sir, here, this word, x is here, this is eight, what do we write after that?

T: No ... um ... this is two y, x also equal to two y. You substituted wrongly.

S: y equals two x?

T: x equals two y, okay.

The teacher recognises the error and points to the mistake. The same error – students substitute $x = y$ incorrectly – also occurs in HK2 (see HK2-L01, 30:40-39:10).

One student is asked to do task 2 on the board.

2) $\begin{cases} x = 2y & \text{- - - (1)} \\ 5x - 2y = 24 & \text{- - - (2)} \end{cases}$

Substitute (1) into (2)

$$5(2y) - 2y = 24$$
$$10y - 2y = 24$$
$$8y = 24$$
$$y = 3$$

Substitute $y = 3$ into (1)

$$x = 2 \times 3$$
$$x = 6$$

The teacher comments the beginning of the student's solution. He draws a square around *x* in both equation (1) and (2) and a triangle around 2*y* in equation (1). He points to these and to "5(2*y*)" and "10*y*" when explaining. The choice of unknown and 'source' equation for the initial substitution is taken for granted.

20:40 T: this x and this x are the same, right? ['squares' x in (1) and (2)] ... Right? This y ['triangles' 2y in 5(2y) - 2y = 24] is originally written as 5x. We don't write 5x, we write five times two y as two y has been exchanged with x. The meaning of 5x is five times x, five times x, so you use two y to exchange for x. This is five times two y, so it is ten y. This step, I'm only asking about this step. Who got it right? Five times two y. [Counting hands] One, two, three. Any more? Four, five.

The teacher opens a dimension of variation by indicating two different *x*:es [in (1) and (2)] at the same time as he is stating that they are representing the same number. The alternative to the 'different' *x*:es being different is that they are 'the same'.

The teacher continues to walk around and another student solves no. 3 on the board. This solution is never discussed. The bell rings and the teacher ends the lesson by quickly checking the students' work. He asks them to raise their hands if their answers are correct.

28:40 T: Okay. I will ask you, for question two [...] if you get x equals six, y equals three, put up your hands.

[...]

28:57 T: On question three, put up your hands if you get x equals negative three and y equals negative one.

Only a few students have finished all six tasks in the worksheet.

1) $\begin{cases} x = y \\ 8x + 2y = 10 \end{cases}$	2) $\begin{cases} x = 2y \\ 5x – 2y = 24 \end{cases}$	3) $\begin{cases} x = 3y \\ 3x – 2y = -7 \end{cases}$
4) $\begin{cases} y = 15 – 3x \\ y + 2x = 12 \end{cases}$	5) $\begin{cases} y + 3x = 4 \\ 7x – 2y = 5 \end{cases}$	6) $x + 3y = 3x + 2y = 7$

[Solutions: (*x*, *y*) = 1: (1, 1) 2: (6, 3) 3: (-3, -1) 4: (3, 6) 5: (1, 1) 6: (1, 2)]

The first three tasks form a sequence where the format is invariant, and just the actual values of the coefficients and constants are varied. In all these tasks the same unknown from the same equation will be the obvious choice for the first substitution – *x* from (1) will be substituted into (2). The first equation in task no. 4 is different. Here *y*, instead of *x*, from (1) will be substituted into (2). Task no. 5 involves a rearrangement of the 'source' equation before substitution. The last task also involves rearranging the system of equations into the 'standard format'. Only students who reach item 5 and 6 will experience the need to rearrange the 'source' equation prior to the initial substitution.

In all tasks the most natural first step in the solving will be the same: an unknown from equation (1) will be substituted into equation (2). In task 3 there are a negative constant. Two of the tasks have solutions where $x = y$.

Opened dimensions of variation HK1-L07 [15:40-29:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
4a. 'Unknowns are the same' is not taken for granted 5b. Constants can be negative numbers not just natural 7b. A system of equations can be written in one expression	3a. The unknowns can be negative numbers 4a. The unknowns can be equal is not taken for granted 5a. Different systems with the same solution possible	5a. 'Direct' substitution is not taken for granted

Hong Kong 1 – Lesson 8

HK1-L08 [0:30-03:25]

The lesson starts with a revision of the concept of system linear equations in two unknowns. The teacher asks a number of questions. The students seem reluctant to answer. A question about the meaning of linear is not properly answered or elaborated on. The teacher repeatedly asks about the meaning of linear but does not get any answer and is finally diverted into another line of questions.

00:35	T:	Firstly, I want to ask you ... in the last lesson, we ... what did we learn?
[...]		
	S:	Simultaneous linear equations in two unknowns.
	T:	Yes thank you. What are ... ’two yuans’ ... in simultaneous linear equations in two unknowns?
[...]		
01:30	T:	Usually, we use x and y, but we can use others. Like a and b, or u and v ... Simultaneous linear equations in two unknowns ... what’s meant by linear? ... ’two yuans’ are two unknowns, what about linear ... Rebecca, what do you think?
	S:	Linear ... is it linear equations? ... linear equations.
02:10	T:	What are linear equations?
	S:	That number ... number.
	T:	Two unknowns?
	S:	Yes.
	T:	What about linear equations? Simultaneous equations ... what about system of equations ... system of equations? ... system of equations ... Nancy?
	S:	(...)
	T:	It’s made up of two equations, what about three?
[...]		
03:15	T:	When there are two equations or above, we call them systems. If you look them up in the books, you can find simultaneous linear equations in three unknowns, three unknowns or simultaneous linear equations in four unknowns.

The equations have been labelled ”linear” several times in the two previous lessons, but on no occasion has the meaning been explained. It has not been made possible for students to experience the meaning of ”linear” for instance by comparing it to the possibility for an equation of being ”not linear”. No alternatives have been introduced.

The teacher introduces alternatives regarding what letters to use at 01.30 by providing alternatives. Instead of x and y, he says, you can use a and b, or u and v. This would make it possible to discern that different letters can be used.

In similar way the teacher at 3.15 gives an alternative to the number of equations by saying ”two equations or above”. It is not taken for granted that there are two equations, there can be more. The teacher also opens up for variation regarding the number of unknowns by giving the alternatives of having three or four unknowns. It is no longer taken for granted that the number of unknowns is two. The students are offered opportunities to discern that the number of unknowns is a feature of a system of equations.

Opened dimensions of variation HK1-L08 [0:30-03:25]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1b. More than one equation 2b. Two unknowns not three (or more) 6a. The letters x, y are not taken for granted	–	–

HK1-L08 [03:25-26:00]

The students work on a problem from the textbook (no. 1, p. 114). The teacher writes it on the board. To some students (those who did not reach task 5 last lesson) this is probably the first example where one equation has to be rearranged before substitution. The first step, to choose y from (1) is indicated in the book, and the teacher also puts this start of the solution on the board. The choice of y from (1) is, at this point, taken for granted.

$$\begin{cases} 2x + y = 9 & \ldots\ldots (1) \\ x - 3y = 8 & \ldots\ldots (2) \end{cases}$$

From (1) $\quad y = 9 - 2x$

After a couple of minutes, the teacher 'squares' y in all three equations and also "9 – 2x" on the board. He shows with an arrow that y in equation (1) is substituted into (2).

$$\begin{cases} 2x + \boxed{y} = 9 & \ldots.\ (1) \\ x - 3\boxed{y} = 8 & \ldots.\ (2) \end{cases}$$

From (1) $\qquad \boxed{y} = \boxed{9 - 2x}$

(1) into (2) $\qquad x - 3(\boxed{9 - 2x}) = 8$

07:50	T:	The first step is very important. Umm ...We make y in the first equation the choice of variable. That means putting this [points to y in 'y=9-2x'] on the left and the others to the right ... and then substituting this into the other equation.
[...]		
08:40	T:	y in equation one becomes [points to y in (1)] ... nine minus two x [points to '9-2x']. The y in equation one [points to y in (1)] and the y in equation two [points to y in (2)] are the same.
[...]		
09:00	T:	Yes, add a bracket. Many students forget ...

The solving of the system of equations is then halted and not completed. Instead the teacher demonstrates three other ways in which to start the solving. The teacher 'squares' x in equation (2) and writes in a new column on the board.

From (2) get $\quad x = 8 + 3y \quad \ldots.\ (3)$

10:10	T:	Now I'm going to teach you three other methods. Look. I make x in equation two the choice of variable and substitute into the first ... equation.
[...]		
11:30	T:	Okay, the students ... who have finished ... can do it again in another method ... see if the answer is the same.

The teacher continues walking around and (13:50) two students write their solutions on the board. One finishes the first approach (y from (1) into (2)) and the other does the second (x from (2) into (1)). When they are finished the teacher comments the two solutions on the board and notices that the answers are the same.

16:20 T: Look at the blackboard [...] The most important step is how substitution can eliminate the unknowns, which is to get rid of the unknowns.

[...]

16:50 T: The answer is also y equals to negative one, x equals to five, the same as that over there [...] Actually, we can do it again.

The teacher starts to show a third start to solving the same system of equations. He writes in a third column on the board.

From (2) get $x - 3y = 8$

$$-3y = 8 - x$$

$$y = \frac{8 - x}{-3} = \frac{x - 8}{3} \quad \ldots. \ (4)$$

Substitute (4) into (1) $2x + \frac{x - 8}{3} = 9$

18:25 T: From equation two [...] If we want to make y as the choice of variable.

[...]

19:55 T: Now, I've done the most important step ... huh ... making y in equation two as the choice of variable.

Then the teacher indicates with arrows how to substitute *x* from (1) into (2), the fourth possible way to start the process of solving.

20:10 T: When you finish this ... do it one more time and substitute in this way.

[...]

20:20 T: The answers ... from all four methods ... are exactly the same.

[...]

20:40 T: I want you to get the answers using all the methods. But having done that ... you'll know how to choose [...] If not, you may never know how.

One of the students finishes the third solution on the board.

25:20 T: Doing the fourth method ... multiple solutions ... in one question, that means there are more than one solution to one question [...] as long as you don't substitute the answer into the equation itself, you can solve it by using any of the methods [...] After ... you've tried these four methods ... you'll be able to choose the best one.

At 8.40 the teacher points to the y in equations (1) and (2). The two y: s present in two *different* equations, the teacher points out, are anyhow the *same*. The students are given the opportunity to discern that the unknowns in a system of equations represent the same number by considering the alternative.

During this episode the class solves one system of equations with four different approaches to the method of substitution. The first three are displayed on the board. The system is kept invariant while the process of solving is varied.

1. y is substituted from (1) into (2)
2. x is substituted from (2) into (1)
3. y is substituted from (2) into (1)
4. x is substituted from (1) into (2)

Opened dimensions of variation HK1-L08 [03:25-26:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
4a. 'Unknowns are the same' is not taken for granted	3a. The unknowns can be negative numbers 6a. 'Solution is independent of solving method' is not taken for granted	3a. The selection of unknown is not taken for granted 3b. Calculation complexity depends on the unknown selected 4a. The selection of 'source' equation is not taken for granted 4b. Calculation complexity depends on the equation selected 5a. 'Direct' substitution is not taken for granted

HK1-L08 [26:00-30:00]

They discuss the systems of equations in the textbook on page 115. These systems are not solved, the discussion is about "how to start the process of solving". They skip item 8.

1) $\begin{cases} y = x \\ y + 4x = 15 \end{cases}$	2) $\begin{cases} x = 7y \\ 9y - x = 8 \end{cases}$	3) $\begin{cases} r = 3s \\ r - 2s + 5 = 0 \end{cases}$
4) $\begin{cases} y = 3x + 1 \\ y = x + 7 \end{cases}$	5) $\begin{cases} a = 4b + 7 \\ a + 8b – 7 = 0 \end{cases}$	6) $\begin{cases} p - q = 0 \\ p + 4q + 1 = 0 \end{cases}$
7) $\begin{cases} m - m = 2 \\ m + 3n = 14 \end{cases}$	8) $3m - n - 4 = m - n = 0$	
9) $\begin{cases} 2c + 3 = d \\ c = d - 2 \end{cases}$	10) $\begin{cases} 9x - 1 = y \\ y - 8x = 3 \end{cases}$	11) $\begin{cases} a - 2b = 5 \\ 3a + b + 6 = 0 \end{cases}$

26:00 T: Question one, which one is better? Substitute equation one into equation two, or substitute equation two into equation one?

[...]

26:15 T: Can we get the answer when we do it in a reversed way?

S: Yes.

T: Yes ... but troublesome.

[...]

26:55 T: The answer of question four is, substitute two into one ... one into two is also correct, because they both use y as the choice of variable.

[...]

27:55 T: How about question eight? I'll talk about it later, I'll let you think about it first.

The teacher checks the 'success' of the lesson before it is ended.

29:40 T: Umm ... this question [points to the solutions on the blackboard] ... who can do it in all four methods, raise your hands. [About ten students raise their hands]

They go through the items from the textbook rather quickly. The question used for most items is, "which equation to select for the initial substitution?" The answer is just "substitute two into one " except for item 6 and 11 where the teacher follows up this question by asking about which unknown to choose. For item 7 the question is changed to "which is the choice of variable?" The system of equations written as one expression in item 8 is skipped for later.

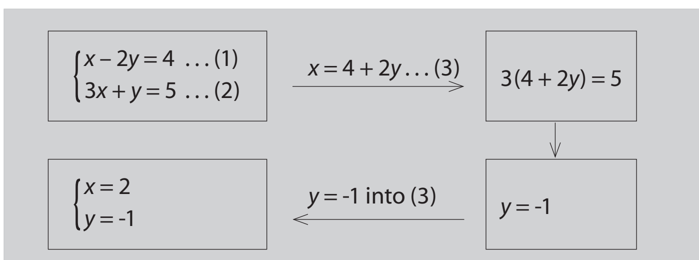
Opened dimensions of variation HK1-L08 [26:00-30:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
6a. The letters *x*, *y* are not taken for granted 7a. Format of individual equations not invariant	–	3a. The selection of unknown is not taken for granted 3b. Calculation complexity depends on the unknown selected 4a. The selection of 'source' equation is not taken for granted 4b. Calculation complexity depends on the equation selected

Hong Kong 1 – Lesson 9

HK1-L09 [0:30-12:00]

The teacher has prepared an illustration on the board showing the process of solving of a system of equations from the textbook (p. 112).



[The original text is omitted in the figure above.
Note. + *y* is missing in 3(4+2*y*) + *y*= 5. A student mentions it at 11:10]

04:05 T: Nicholas is asking, "Could you explain a bit slower sir?" Hence, I have rearranged the details again and try to explain them in another way.

[...]

04:40 T: Why do we change it like that in the first step? [points at (3)]

The teacher creates a contrast between (1) and (3). The two equations differ regarding the format. The terms are the same, but they are arranged differently.

05:30 T: Why do we choose the first one instead of the second one? [points at (1) and (2)]

The teacher creates a contrast between (1) and (2) and makes the choice explicit. The two equations are in the same format, but the coefficients and constants differ. Also the operation in the left side is different. The students seem to hesitate to answer, maybe because it is more or less just as convenient to choose (2) and express y in terms of x. The teacher seems to take the choice of (1) for granted and suddenly compare the terms x and $2y$ in (1) instead of comparing the equations (1) and (2).

06:35 T: Ah ha, we choose the first one because it is more convenient to express x in terms of y, ... which is not the case if we express y in terms of x. [points to x and 2y in (1)]

[...]

07:10 T: Which equation should be substituted to the other one?

S: Three substitutes in two.

T: Three substitutes in two. Why can't we substitute three in one?

The teacher creates a contrast between the substitutions (3) ⇒ (1) and (3) ⇒ (2), and highlights the importance of from which equation (3) originated.

07:40 T: After substitution, it has changed to an equation that you had learnt in secondary school and you can solve it.

The teacher points to the original system of equations and the new equation, $3(4 + 2y) = 5$. He creates a contrast between before and after the substitution. This highlights the change in the number of unknowns.

07:55 T: Which equation should be substituted by y equals to minus one? Should it be the first one, the second one or the third one? [points (1), (2) and (3)]

[...]

08:10 S: Substituting into the first equation.

T: Substituting into the first equation. Can't we substitute into the second equation? Can't we substitute into the third equation?

[...]

10:00 T: It can be substituted into all three equations. In fact, usually people would substitute the solution into the third equation. Why? This is because we can find the solution of x immediately.

Here the teacher creates a contrast between three different ways to substitute $y = -1$. He introduces alternatives, which makes it possible to discern how the value of the second unknown can be calculated.

Opened dimensions of variation HK1-L09 [0:30-12:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	3a. The unknowns can be negative numbers	2a. Getting an equation in one unknown is not taken for granted 3a. The selection of unknown is not taken for granted 3b. Calculation complexity depends on the unknown selected 4a. The selection of 'source' equation is not taken for granted 4b. Calculation complexity depends on the equation selected 6b. Equation (1) $\Rightarrow$ (3) should not be substituted back into (1) 7a. The equation for second substitution is not taken for granted 7b. Substitution of first unknown into equation (3) often most convenient

HK1-L09 [12:00-19:40]

The teacher hands out a paper describing the method of substitution in six steps and ask the students to look at a solved example in the textbook.

Step 1.	Choose the equation and represent one unknown in terms of the other
Step 2.	Do elimination
Step 3.	Examine whether *x* is one unknown and is the solution of the equation and explain
Step 4.	Substitute the equation *x* equals to into the third equation
Step 5.	Substitute the value of the unknowns into the two equations of the simultaneous linear equations in two unknowns at the same time to check
Step 6.	Write down the solution

[Translation from lesson-table. There is more text in the original handout]

12:10 T: You must look through page one one three, example two carefully and I will ask which step is equal to the step mentioned in the worksheet.

Example 2 (p. 113)

$$\begin{cases} 4x - 3y = 20 \ldots (1) \\ 6x + y = 8 \ldots (2) \end{cases}$$

From (2), $y = 8 - 6x$ ... (3)
Substitute (3) into (1), get

$$4x–3(8–6x) = 20$$
$$4x–24 + 18x = 20$$
$$22x = 44$$
$$x = 2$$

Substitute $x = 2$ into (3) get

$$y = 8 - 6(2)$$
$$= -4$$

The solution to the system of equations is $x = 2, \ \ y = -4$

The teacher and the class discuss the solution and relate it to the six steps in the description. The students seem to have some difficulties in identifying the different steps in the example. They find that the checking of the answer (step 5) is not done in the example.

HK1-L09 [19:40-34:30]
The students work on item 9–23 on page 115 and the teacher walks around talking to students at their desks. Some students write their solutions to

Opened dimensions of variation HK1-L09 [12:00-19:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	3a. The unknowns can be negative numbers	–

item 1–4 on the board. These tasks were discussed the previous lesson. The solutions on the board are not further discussed in whole-class.

19:40 T: Now, all of you please spend the remaining time doing questions nine to twenty-three on level 2. If you cannot finish it here, you can finish ... it at home.

9) $\begin{cases} 2c+3=d \\ c=d-2 \end{cases}$ 10) $\begin{cases} 9x-1=y \\ y-8x=3 \end{cases}$ 11) $\begin{cases} a-2b=5 \\ 3a+b+6=0 \end{cases}$

12) $\begin{cases} 2x+5y-3=0 \\ 3x-y=13 \end{cases}$ 13) $\begin{cases} x+y-2=0 \\ x+3y-8=0 \end{cases}$ 14) $\begin{cases} y=0.5x-1 \\ x+y+7=0 \end{cases}$

15) $\begin{cases} x=3-(y+2) \\ 2x+3y-5=0 \end{cases}$ 16) $\begin{cases} r-(s-1)=3 \\ r+9=7-s \end{cases}$ 17) $\begin{cases} 3c+6d=15 \\ 2d=7c-3 \end{cases}$

18) $\begin{cases} 5h-2k=17 \\ 3h+4k=5 \end{cases}$ 19) $\begin{cases} 2x+3y=9 \\ 3x+2y=11 \end{cases}$ 20) $\begin{cases} 8s-7t=13 \\ 3s-2t=28 \end{cases}$

21) $x+y=3x-y+4=0$ 22) $3p-(q-10)=p-q=0$

23) $5-x-2y=7x+5y+3=-2$

After the bell rings the teacher brings up a question from one of the students.

33:35 T: One of you asked about question nine. Uh, number nine. Have you noticed step one on my worksheet? Choose an equation with simpler coefficient. The best coefficient is?

S: One.

T: One. Okay, since the coefficient of the unknown in the second equation of question nine is one. So, it is the best. It is the best. Okay.

The teacher 'recommends' an initial substitution of equation (2) into (1), based on *c* having the coefficient of one. However, both equations are equally suitable in this regard. In (1) you have *d* to the right and in (2) you have *c* to the left. Should one also look at the coefficients of the unknown in the 'target' equation the opposite substitution might be preferred. Substitute the unknown *d* from (1) to (2) into instead of *c* from (2) to (1), and you avoid a multiplication of an expression in brackets. Anyhow, alternative choices of 'source' equation are discussed, even though the 'absolutely most convenient' alternative was not chosen.

This last set of items the class work on offers variation regarding the coefficients, constants and unknowns, as well as regarding the overall format of a system of equation.

Opened dimensions of variation HK1-L09 [19:40-34:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5a. Coefficients can be rational numbers not just natural 5b. Constants can be negative numbers not just natural 6a. The letters x, y are not taken for granted 7a. Individual equations can be in different formats 7b. The equations can be written in one expression	3a. The unknowns can be negative numbers	4a. The selection of 'source' equation is not taken for granted

Hong Kong 2

This grade eight class consists of 39 students. It is an average ability class (TQ1). The instruction in this class is mainly teacher-led in whole class. The teacher solves and discusses tasks from the textbook or worksheets on the board. This main activity is occasionally interrupted when the students are given a couple of minutes to think about a question or a problem. This will happen typically 1–3 times during a lesson. The teacher then walks around, checks the students' work and talks to individual students. Often some students are asked to show their solutions on the board. These solutions are then discussed in whole class.

All students work with the same tasks. They are just doing a few exercises by themselves during a lesson, but will, nevertheless, not get much homework (students' statements in interviews). There are some students with tutoring outside lesson time.

Overview of the lessons in Hong Kong 2

The three lessons analysed are no. 1, 2 and 3.

HK2-L01

In this lesson the concept of *system of linear equations in two unknowns* is introduced. First two equations are examined separately. They are after that merged into a system of equations, which is labelled ”simultaneous equations”. A common solution is found to the system from the lists of solutions to the two equations respectively.

The teacher demonstrates the *method of substitution* and solves with that method, all together, three systems of equations on the board.

HK2-L02

In the first episode of the lesson the teacher solves four systems of equations by the method of substitution on the board.

The students work on similar systems of equations individually. Some students show their solutions on the board. In all, eight systems of equations are solved on the board.

HK2-L03

In the first episode of the lesson the teacher solves three systems of equations by the method of substitution on the board. The solutions involve calculation with fractions.

The students work on similar system of equations individually. Three students show their solutions on the board. In all, six systems of equations are solved on the board.

Lesson L03 and L02 are very similar in structure. First the teacher demonstrates and then the students exercise/practice.

Hong Kong 2 – Lesson 1

HK2-L01 [0:00-7:30]

The teacher discusses tasks from the homework in whole class. Three items are solved on the board. The content is factorization of algebraic expressions.

Opened dimensions of variation HK2-L01 [0:00-7:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

HK2-L01 [7:40-16:30]

The teacher asks the students for the solution to the equation $3x + 2 = 8$. Two proposed answers $x = 2$ and $x = -2$ are checked by substitution into the equation. They find $x = 2$ to be correct.

Another equation is put on to the board, $2x + y = 1$. Some of its many solutions are listed. The contrast between the first two equations, $3x + 2 = 8$ and $2x + y = 1$, opens the DoV of the number of unknowns. Then a third equation, $x + 3y = -2$, and some of its solutions are listed on the board. The solutions are reached in dialogue between teacher and students.

$2x+y=1$					$x+3y=-2$				
$x = 1$	0	-2	-1	...	$x = 1$	-2	-8	4	...
$y = -1$	1	5	3	...	$y = -1$	0	2	-2	...

14:40 T: They are called linear equation in two 'yuan' ... two elements and degree is one. We know that there are indefinitely many solutions ... if I want to set the limit ... this time, I want to ... group these two equations together, I want limit the conditions ... I want to set the limit [teacher groups the equations together] ... This is called a simultaneous equation ... simultaneous means to group two unrelated equations together, to set the limit. [writes on the board]

$$\begin{cases} 2x+y=1 & (1) \\ x+3y=-2 & (2) \end{cases}$$

15:25 T: I want to find a number ... a pair of values ... that satisfy the first and the second equation.

A common solution (1, -1) is found in the two lists of number pairs. The teacher states that this is the only solution common to both equations simultaneously.

The concept system of linear equations in two unknowns (or simultaneous equations) is introduced by means of an example. The two equations are put together in order to reduce the number of solutions, and the teacher just labels this new construction ”simultaneous equations”.

However, when the two (separate) equations are put together into a system of simultaneous equations the number of solutions change from many to one. There is also a contrast between number pairs that satisfy one of the equations and the one pair that is common to both equations.

The properties of the system of equations are not elaborated on any further at this instance. Instead the teacher moves on and introduces a method for solving.

Opened dimensions of variation HK2-L01 [7:40-16:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one 2a. Two unknowns, not one 5b. Constants can be negative numbers not just natural	1a. Same solution to both equations is not taken for granted 2a. One equation has many solutions/no unique solution 2b. Two equations reduce number of solutions to one	–

HK2-L01 [16:30-30:35]

The teacher writes another system of equations on the board (it seems to be an ad hoc example from the teacher).

$$\begin{cases} x+y=4 & (1) \\ x-y=5 & (2) \end{cases}$$

The teacher states that it often is hard to guess the solution and that there are three methods for solving a system of equations called substitution, elimination and the graphical method.

19:35 T: I’ll illustrate the method of substitution by this question […] We want to find … these three methods help us find one of the unknowns first. It we’ve found the first unknown we could find the second one.

[…]

21:25 T: What's the relationship between the value of this y and the other y? [points to y in (1) and (2)] ... The same. The x here is the same as the x there [points to x in (1) and (2)], because ... these two equations are simultaneous equations.

The teacher points to the two 'different' *y*:s (and *x*:s) and states they have the same value, indicating the possibility of the opposite.

22:10 T: I'll make one unknown as the choice of variable. I'll make x or y be the choice. This time I choose x. [writes on the board]

$$x + y = 4$$
$$x = 4 - y \qquad \text{[called equation 3]}$$
Substitute $x = 4 - y$ into (2)
$$x - y = 5$$
$$(4 - y) - y = 5$$
$$4 - y - y = 5$$
$$-1 = 2y$$
$$y = \frac{-1}{2}$$
Substitute $y = \frac{-1}{2}$ into $x = 4 - y$
$$x = 4 - (-1/2)$$
$$x = 4\,1/2 \quad / \quad 4.5$$

The teacher makes some remarks while writing the solution on the board, about not substituting "$x = 4 - y$" (equation 3) back into the first equation, that the unknowns are "completely the same" in both equations, and about getting an equation in just one unknown that can be solved by previously learned methods.

22:50 T: I won't substitute it into the first equation, because this is the first equation. It's silly to substitute it back into the first equation. If we substitute it into the first equation, we're stepping backwards. We'll substitute the answer into the second equation.

[...]

23:10 T: The value of x [teacher circling 4–y] ... what will replace x [teacher circling x in x–y=5]? ... Four minus y. [teacher writes '(4–y) – y = 5']

[...]

23:30 T: But we're not substituting the original equation back to itself, but into another equation, because both values of x ... The values of x and y in these two equations are completely the same, okay? Good. Here equation one [points at '(4–y) – y = 5'] only has one unknown, not.

The teacher introduces variation in how the expression of *x* can be used and also regarding the number of unknowns in the equation they get after the substitution.

The next example is taken from the textbook (page 170).

Q. 1 $\begin{cases} y = x & (1) \\ 2x - y = 3 & (2) \end{cases}$

27:20 T: Different from the former equation, do we need to rearrange the equations? No. Which equation already helped us to choose the choice of variable? Which equation has already done it for you? The first equation, that is, y is already equals to x. This equation is more flexible. You can substitute x or y into the second equation. [writes on the board]

Here the teacher points to alternative ways to start.

28:50 T: We have found the first unknown [...] Which one should we substitute y into? ... The second one or the first one? Is there any restrictions? [...] No

Substitute (1) into (2)

$$2x - y = 3$$
$$2y - y = 3$$
$$y = 3$$

Substitute $y = 3$ into (1)

$$y = x$$
$$x = 3$$

$x = 3\,, y = 3$

The solution shows the values of the two unknowns are the same. A question from a student triggers the teacher to show an alternative solution.

30:05 T: As x equals y [points to y = x in (1)] and the two x are the same, [points to x in (1) and (2)] x is substituted into y. You can use another way.

The teacher writes the alternative to the right of the previous solution.

Substitute (1) into (2)

$2x - y = 3$	$2x - y = 3$
$2y - y = 3$	$2x - x = 3$
$y = 3$	

30:25 T: You can substitute y into x in two x minus y equals three, okay? You can do it in this way. There's no definite method. It depends on your choice.

Opened dimensions of variation HK2-L01 [16:30-30:35]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
4a. 'Unknowns are the same' is not taken for granted 7a. Format of individual equations not invariant	4a. Solution with two equal numbers possible	1a. Main idea is not taken for granted 2a. Getting an equation in one unknown is not taken for granted 3a. The selection of unknown is not taken for granted 5a. 'Direct' substitution is not taken for granted 6b. Equation (1) ⇒ (3) should not be substituted back into (1) 7a. The equation for second substitution is not taken for granted

HK2-L01 [30:40-39:10]

The students are asked to try to solve a third example themselves (in the textbook, p. 170).

Q.2 $\begin{cases} y = 5x & (1) \\ 9x - y = 4 & (2) \end{cases}$

The teacher walks around in the classroom and talks to students. He notice an error made by one student. The student obviously substitute $y = x$ (from Q. 1) instead of $y = 5x$. (This happens in HK1 too, see HK1-L07, 15:40-29:30).

31:40 T: Why does nine x change to nine y? y equals to five x. y equals to five x means both sides are the same. What's y? [...] Make y ... change y to five x.

[...]

32:20 T: Don't mix this up with the first question. In the first question, y equals to x.

After some minutes one of the students is told to show the solution on the board.

Substitute $y = 5x$ into (2)

$$9x - y = 4$$
$$9x - 5x = 4$$
$$4x = 4$$
$$x = 1$$

Substitute $x = 1$ into (1)

$$y = 5x$$
$$y = 5$$
$$y = 5\,,\, x = 1$$

The teacher discusses the solution on the board in whole class and makes some remarks.

36:55 T: [points to the first line of the student's solution] y equals five x is different ... from the first question, [points towards Q1 to the left on the board] because in the first question, y equals x can be viewed as x equals y. But in this question, y equals five x and y ... if you substitute y equals five x [points to (1)] into the second equation [points to (2)] ... y equals five x [points to (1)] ... are both y [moves his finger between y in (1) and (2) several times] ... the same? ... Yes.

[...]

37:30 T: ... x equals one. Then you can substitute it into the first equation. You can also substitute x into the second equation.

[...]

37:50 T: Some of you change ... x to y. Every question is different. In that question, [point towards Q1 to the left on the board] y equals to x, which is different from y equals five x. [underlines y=5x in (1)]

The differences between Q. 1 and Q. 2 are limited. For instance, in both tasks the equation (1) is in the format $y = ax$, which directly can be substituted into equation (2). Also, the equations (2) are in both tasks in the same format, $bx - y = c$. The possibilities for students to discerns aspects such as, both x and y can be used for the first substitution, and substituting equation (1) into equation (2) and the reversed are both possible etc, are limited. The opportunity to use differences between two tasks to open these DoVs is not taken.

Opened dimensions of variation HK2-L01 [30:40-39:10]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
4a. 'Unknowns are the same' is not taken for granted	–	7a. The equation for second substitution is not taken for granted

Hong Kong 2 – Lesson 2

HK2-L02 [01:10-2:20]

The lesson starts with a short revision by the teacher of the last lesson.

01:15 T: Yesterday we started talking about simultaneous equations [...] if there are two unknowns in one equation, we will find many pairs of solutions [...] in simultaneous equations, there are two unknowns in two equations, we will gather all these values and eventually come up with only one pair of solutions [...] This so-called method of substitution can help us solve the first unknown.

The teacher introduces variation regarding the number of equations and the number of solutions, "one equation means many solutions" and "two equations means one solution".

Opened dimensions of variation HK2-L02 [01:10-2:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one	2b. Two equations reduce the number of solutions to one	–

HK2-L02 [2:25-17:20]

The teacher solves four systems of equations from the textbook side by side on the board (No. 4, 6, 8, 9, page 170). While he is writing the solutions on the board he makes some comments.

4. $\begin{cases} y = x + 1 & (1) \\ y + 2x = 16 & (2) \end{cases}$

6. $\begin{cases} y - x = 3 & (1) \\ 2x + y = 24 & (2) \end{cases}$

8. $\begin{cases} x - 2y = 0 & (1) \\ 3x - y = 10 & (2) \end{cases}$

9. $x - 2y = x + y - 6 = 0$

(Q. 4)

02:55 T: This is a pair of simultaneous equations [points to Q. 4 on the board] [...] These are two equations, a pair of y [points to y in (1) and (2)] and a pair of x [moves hand to x in (1) and (2)], how should the pair of y values be? [moves hand back to y in (1) and (2)] ... are they equal or not?

S: Equal.

T: How about the values of this pair of x? [moves hand to x in (1) and (2)]

S: Equal.

T: They are equal. When they are solved, their values should be the same.

After the first substitution is done the teacher points to a change.

04:35 T: Originally there were two unknowns in the equation [...] There is only one unknown left.

(Q. 6)

The teacher points to a difference between the second and the first examples (Q. 6 and Q. 4).

06:40 T: This time the equation is different from the one yesterday and the one we have just finished [...] For example the y here [points at y=x+1 in Q. 4] is already expressed in terms of x. y equals x plus one. We can substitute the equation y equals x plus one directly into the other, right? But is there any here? [points at Q. 6] In the two equations, none of them are expressed in the general form. We don't have anything like x equals something or y equals something. If we face such situation, we have to express one equation of our choice in the general form.

The equation (1), $y - x = 3$, is rearranged to $y = x + 3$ and labelled (3).

07:25 T: The first equation [...] The best choice is the one with coefficient one.

The teacher argues that it is more convenient to substitute the value of the first unknown, $x = 7$, into (3), $y = x + 3$, than into (1) or (2).

10:50 T: No matter which equation you substitute x into, you can still solve the same value of y. Okay? But sometimes the equations already have their terms arranged properly.

(Q. 8)

12:10 T: ... which equation do we choose?

S: Number one.

T: Number one. If we choose one, which of the two unknowns should we express the equation in terms of?

S: x

T: x, because we can simply move the negative two y to the other side [...] Which equation should we substitute into?

The equation (1) is rearranged to $x = 2y$ and substituted into (2). None of the 'source' equation, unknown or 'target' equation are taken for granted.

(Q. 9)
The teacher comments on the fourth example (Q. 9).

14:20 T: Let's move on to number nine [...] they're written in a different way. This type of arrangement still represents simultaneous equations.

It is rearranged before it is solved by the method of substitution.

15:10 T: Do we need to rearrange the terms in one of the equations? [...] Which equation would you pick?

It turns out that the equation (1) is the same as (1) in the previous example (Q. 8) and also the solutions to item (Q. 8) and (Q. 9) are the same number pair.

16:50 T: Hum? Why? Are the equations the same?

S: They're different but the answers are the same.

The fact that two *different* systems of equations can have the *same* solution is not commented further. However this last task provides an alternative to what might be taken for granted, i.e. different systems have different solutions.

The four examples and solutions are displayed side by side on the blackboard. Some DoVs are opened by the differences between these tasks and solutions.

Opened dimensions of variation HK2-L02 [2:25-17:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
4a. 'Unknowns are the same' is not taken for granted 7a. Individual equations can be in different formats 7b. Two equations can be in one expression	5a. Different systems with the same solution possible	2a. Getting an equation in one unknown is not taken for granted 3a. The selection of unknown is not taken for granted 3b. Calculation complexity depends on the unknown selected 4a. The selection of 'source' equation is not taken for granted 5a. 'Direct' substitution is not taken for granted 6a. The 'target' equation is not taken for granted 7a. The equation for second substitution is not taken for granted

HK2-L02 [17:30-28:50]

The teacher gives the class three tasks from the textbook to solve (3, 5, 7, page 170) and the students work while the teacher walks around and talks to individual students.

3. $\begin{cases} x = 2y & (1) \\ x + 7y = 27 & (2) \end{cases}$ 5. $\begin{cases} y = 2x - 3 & (1) \\ 4x + y = 9 & (2) \end{cases}$ 7. $\begin{cases} x + y = 13 & (1) \\ x - 2y = 1 & (2) \end{cases}$

Three students show their solutions on the board. The teacher comments on some things about the solutions.

27:10 T: ... maybe some of you still haven't figured out when you can directly substitute one equation into the other.

[...]

27:45 T: ... one of the equations, at least one of them, is already in its general form. Either x or y is already the choice of the variable. Then we can substitute one equation into another directly. But for some questions like question seven [points at Q. 7 on the board], none of the two equations is in general form. Neither x nor y is directly expressed in terms of the other. In that case, you will have to pick it yourselves.

[...]

28:25 T: Some of you asked, if I express x in terms of y and vice versa, will I obtain different solutions? The answer is no. The only difference is that you will solve x first or solve y first.

Opened dimensions of variation HK2-L02 [17:30-28:50]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
7a. Individual equations can be in different formats	6a. 'Solution independent of solving method' is not taken for granted	3a. The selection of unknown is not taken for granted 4a. The selection of 'source' equation is not taken for granted

HK2-L02 [28:50-38:00]

The teacher solves question 11 from the textbook on the board together with one student.

11. $\begin{cases} b–3a = 2 & (1) \\ 5a + 2b = 15 & (2) \end{cases}$

29:10 T: ... it isn't necessary to use x and y in simultaneous equations.

The student suggests that the variable *a* in the equation (1) be used for the initial substitution. The teacher follows this suggestion even though it would be simpler to use *b* instead, which is later indicated by the teacher.

31:00 T: Sammuel has picked a to be the choice of variable. We'll follow him. If you want to pick b, you can do it there.

$$b - 3a = 2$$
$$b - 2 = 3a$$
$$\frac{b-2}{3} = a \qquad (3)$$

Substitute (3) into (2)

$$5a + 2b = 15$$
$$5\left(\frac{b-2}{3}\right) + 2b = 15$$
$$\frac{5(b-2)}{3} + 2b = 15$$
$$\frac{5b-10}{3} = 15 - 2b$$
$$5b - 10 = 45 - 6b$$
$$11b = 55$$
$$b = 5$$

Substitute $b = 5$ into (1)

$$b - 3a = 2$$
$$5 - 3a = 2$$
$$a = 1$$

$a = 1, b = 5$

The choice to use *a* instead of *b* in the beginning is commented on.

36:05 T: ... it's only that, I had used b in the first place, how would the calculation be?
S: Quicker.
T: Quicker, and?
S: Simpler.
T: Quicker and simpler. Easier. OK?

The eight tasks that are shown on the board during the lesson differ in some respects. This creates a pattern of variation that opens some DoVs. However, many features are the same in all cases, for example, all coefficients and constants are natural numbers and equation (1) is used for the first substitution in all solutions. In some cases the equation is first rearranged to an equation (3) before it is substituted into equation (2). There is no example where equation (2) is substituted into equation (1). This alternative is nevertheless discussed on several occasions. All systems of equations have one solution. There is no example with none or infinitely many solutions. There is no use of system of equations to represent or solve problems and mostly, the right hand sides of the equations are made up of a (numerical) constant – there are two exceptions only.

Opened dimensions of variation HK2-L02 [28:50-38:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
6a. The letters x, y are not taken for granted	–	3a. The selection of unknown is not taken for granted 3b. Calculation complexity depends on the unknown selected

Hong Kong 2 – Lesson 3

HK2-L03 [0:00-22:20]

The teacher solves, together with the class, three systems of equations from the textbook on the board.

15. $\begin{cases} 4r–3s = 2 & (1) \\ 3r - s = 9 & (2) \end{cases}$ 18. $\begin{cases} 2a + 3b = 2 & (1) \\ 3a + 2b = 18 & (2) \end{cases}$

20. $2x - 3y - 7 = 3x - 7y - 23 = 0$

When solving the first example (15) the teacher discusses which 'source' equation and which unknown to choose, and the advantage in choosing the variable s in equation (2) for the initial substitution. A mistake, made by the teacher, makes him comment on the importance of using brackets when substituting expressions. A question from a student causes the teacher to point out that when the first unknowns is found the value can be substituted into any of the equations (1), (2) or (3).

01:40 T: ... can we substitute the first or the second equation directly into the other?

S: No.

[...]

02:10 T: Which equation should we arrange ...? [...] Which unknown is the choice of variable? [...] We pick s to be the choice of variable [...] pick r as the variable, what will appear? [...] Fractions will turn up.

[...]

05:25 T: To which equation should we substitute 'r equals five'? [...] You can substitute it into the first, the second or even the third equation.

In Q. 18 the teacher points out that this example is different in the respect that there is no obvious choice of unknown. Calculation with fractions cannot be avoided.

08:40 T: Why you choose the second equation? Why not the first?

S: (...)

T: Oh, because you like it [...] Which is the choice of variable? a or b? [...] You may have noticed that in these two equations [...] coefficients are all attached to the unknowns, right? Therefore you can to some extent think that it doesn't matter, it doesn't matter which equation we choose. Because fractions will appear anyway.

20. $\begin{cases} 2x - 3y - 7 = 0 & (1) \\ 3x - 7y - 23 = 0 & (2) \end{cases}$

Question 20 is first rearranged into the "standard form". The solution to Q. 20 introduces variation regarding the kind of numbers that the unknowns can be. The values $x = -4$ and $y = -5$ provides an alternative to the previous solutions of natural numbers.

HK2-L03 [22:25-40:30]

The students work on three tasks from the textbook and the teacher walks around.

14. $\begin{cases} 2m - n = -7 & (1) \\ 3m - 2n = -12 & (2) \end{cases}$ 17. $\begin{cases} 3u + 4v = 4 & (1) \\ 5u - 2v = 24 & (2) \end{cases}$

19. $3x - y - 15 = 7x + 2y - 22 = 0$

The teacher remarks that he has noticed that many students end up in calculations with fractions in Q. 14 since they have not chosen to use *n* from equation (1) for the first substitution. Two students show their solutions, one to Q. 14 and one to Q. 17, on the board. The teacher

Opened dimensions of variation HK2-L03 [0:00-22:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
6a. The letters *x*, *y* are not taken for granted 7b. Two equations can be in one expression	3a. The unknowns can be negative numbers	3a. The selection of unknown is not taken for granted 3b. Calculation complexity depends on the unknown selected 4a. The selection of 'source' equation is not taken for granted 7a. The equation for second substitution is not taken for granted

comments the choice of unknown for the first substitution in Q. 14. before he ends the lesson.

38:35 T: I have noticed that many of you have used the first equation, but with m as the main term. Now fraction turns up. This isn't a serious problem but the calculations will be more complicated.

[...]

40:10 T: Tomorrow I'll teach ... I'll introduce a simpler method, that's elimination by addition/subtraction.

Question 14 has constant terms that are negative numbers. It's the second example of this. Also the solutions involve negative numbers.

Opened dimensions of variation HK2-L03 [22:25-40:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5b. Constants can be negative numbers not just natural 6a. The letters x, y are not taken for granted 7b. Two equations can be in one expression	3a. The unknowns can be negative numbers	3b. Calculation complexity depends on the unknown selected

Shanghai 1

This grade eight class consists of 50 students. It is a high ability class (TQ1). The general picture of the teaching in this class resembles the teaching in the other Shanghai classrooms. All students work simultaneously with the same tasks and problems in a cyclic shift between teacher led whole-class discussion and student-work, individual or in pairs. At the episodes of student-work the teacher walks around in the classroom, answers questions and talks to students at their desks. There will typically be 1–4 episodes with student-work in a lesson. The teacher-led instruction makes up the larger part of a lesson.

Overview of the lessons in Shanghai 1

The two lessons analysed are no. 5 and 6.

SH1-L05

The lesson begins with a short revision of single linear equation in two unknowns and its' solutions. The concept of *system of linear equations in two unknowns* is then introduced. The class works with three questions regarding the properties of the concept.

They continue with questions that problematize the meaning of a solution to a system of equations which include questions like ”is (a, b) the solution to this system of equations or not?” and ”make up a system of equations with the solution (a, b)”. Finally they solve a system of equations by listing corresponding x- and y-values in tables.

SH1-L06

This lesson starts with a short revision of concepts related to system of linear equations in two unknowns and a question about how to rearrange an expression $ax + by = c$ into $x = f(y)$.

The teacher solves two systems of equations by the method of substitution on the board. The procedure of solving is then expressed in five steps. The class works on five more tasks of increasing complexity.

Shanghai 1 – Lesson 5

SH1-L05 [0:00-04:15]

The lesson starts with a revision of (single) linear equations in two unknowns and their solutions.

00:05 T: Before we start a new topic, I would like to do some revision.

1. Linear equations in two unknowns, or not?

   a) $y = 3x^2 - 5$

   b) $\frac{2}{y} + 2x = 3$
2. How many solutions have, $2x + y = 10$?
3. $x + 3y = -4$, when $x = 2$, $y = ?$
4. If $x = -4$, $y = -5$ is a solution for $2x + ay = 7$, then $a = ?$
5. $2x + y = 10$, $x + 3y = -4$, what are the solutions for each equation?

   a) $x = 1, y = 4$ b) $x = 0, y = 10$

   c) $x = 1, y = -5/3$ d) $x = -1, y = 12$

The revision includes several things, such as, one single linear equations in two unknowns, properties of a solution, number of solutions, and the checking of a solution by substitution. Item 5 does not have all possible variation. None of the proposed solutions satisfy both equations, a) is solution to neither of the equations, b) is solution to (1), c) is solution to (2) and d) is also solution to (1). The first question provides alternatives to linear equations.

Opened dimensions of variation SH1-L05 [0:00-04:15]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
3b. x^2 is not a first degree term (non linear) 3c. $2/y$ is not a first degree term	–	–

SH1-L05 [04:15-17:30]

The teacher announces the content of today's lesson and shows a slide with three questions introducing the concept of a system of linear equations in two unknowns. The students are told to read a section in the textbook and try to answer the three questions.

04:15 T: We are going to discuss a new topic ... it is system of linear equations in two unknowns [...] After you have read it, you can think about the three questions.

1 What is a ”system of equations”?

2 How can you tell whether a system of equations is a system of linear equations in two unknowns?

3 Identify whether the given is a system of linear equations in two unknowns.

1) $\begin{cases} x+y=3 \\ x-y=1 \end{cases}$ 2) $\begin{cases} (x+y)^2=1 \\ x-y=0 \end{cases}$ 3) $\begin{cases} x=1 \\ y=1 \end{cases}$

4) $\begin{cases} x/2+y/2=0 \\ x=y \end{cases}$ 5) $\begin{cases} xy=2 \\ x=1 \end{cases}$ 6) $\begin{cases} x+1/y=1 \\ y=2 \end{cases}$

7) $u=v=0$ 8) $\begin{cases} x+y=4 \\ x-m=1 \end{cases}$

The students' answers to the first two questions are obviously the same as in the textbook [according to comments in the lesson table]. The teacher writes the answers on the board.

(Q. 1)

08:10 S: A system of equations is formed by a number of equations.

(Q. 2)

09:10 S: There are two unknowns in the equations and the indexes of the unknowns are one. This is called system of linear equations in two unknowns.

T: Oh, sit down please. He has just mentioned the definition of system of linear equations in two unknowns.

These points – two unknowns in the equations and the indexes of the unknowns are one – are then repeated a number of times in the following conversation and written on the board, before they move on to the third question.

10:45 T: ... let me summarise [...] the first point [...] these two equations should have two unknowns. The second point, the second, a classmate has just mentioned, the indexes of the unknowns ... should be one.

(Q. 3)

12:05 T: Okay, these two points, oh then, let us take a look at the following questions with these two points.

[...]

12:20 S: The first one is linear equation in two unknowns.

[...]

T: The first problem, x and y ... there is two unknowns and the indices are one.

[...]

T: The second equation, x minus y equals one, the indexes of the two unknowns x and y are one. Yes, it is a system of linear equations in two unknowns [...] let us look at the second question.

[...]

12:50 S: They are not linear equation in two unknowns.

T: Oh, he said not. Why not?

S: Because in this system of equation, the index of the term is two.

All eight items in the third question are discussed in a similar fashion, one at a time. Reasons for or against them being a system of equations in two unknowns are given in each case. Four of the examples meet the requirement and four do not.

From the revision in the beginning of the lesson it is clear that some features of the new concept are already familiar to this class. That includes notions such as the concept of (single) linear equation in two unknowns, the number of solutions to a linear equation in two unknowns and how to determine whether a proposed solution is correct or not.

The contrast created by Q1 and Q2, against the background of the revision, opens the DoVs that a system of linear equations in two unknowns includes more than one equation and that is one kind of system of equations, not *the* system of equations. It is not taken for granted that a system of equations is a system of linear equations.

During the seven and a half minutes when Q3 is discussed in a teacher-led mood of activity, a powerful pattern of variation emerges. Eight proposed systems of linear equations in two unknowns are considered. The important points that came from the first two questions – more than one equation, two unknowns with indexes of one – are attended to while question 3 is handled. Of these three points only one is really 'new'. Already in the revision part at the beginning of the lesson, when the concept of *one* linear equation in *two* unknowns was discussed, the features – *two unknowns* with *indexes of one* – were highlighted. Strangely it is these two 'previous points' that the teacher emphasises, when the new concept is introduced, rather than the 'new feature' that there is more than one equation. With the items in question 3 the teacher generates a specific pattern of variation. The choice of items could be an indication of the teacher's knowledge and experience of common student errors. The contrasts that are formed between the eight examples open a number of DoVs corresponding to aspects that very well might be critical when it comes to understanding the concept of system of linear equations in two unknowns. Some of these aspects are the following.

The equations have to be *linear* equations (the variables must be of degree one). Even if students 'know' this in principle, there are a number of cases where the mathematical symbols can be difficult to interpret correctly. Some of these instances are highlighted by the examples used.

- The expression $x + y$ is present in both (1) and (2), but $(x + y)^2$ is not of the first degree even though $x + y$ is. The contrast between the items points out a feature that distinguishes between what is and what is not an experssion of the first degree.
- xy is not of the first degree even though x and y are, when looked upon separately.
- $y/2$ is of degree one but $1/y$ is not. The difference between (4) and (6) highlights this feature.

There should be *two* unknowns, not three. In the last example (8) three letters, x, y and m, are used. Both the teacher and the students seem to interpret these letters as three unknowns, and thus, the example is judged as not meeting the requirements of a system of equations in two unknowns. The letter m is often used to denote a constant rather than a variable or unknown. This seems not to make the interpretation less straightforward.

Both unknowns must not be present in both equations. In (3) the equations $x = 1$ and $y = 1$ form a system of linear equations in two unknowns.

Other letters beside x and y can be used. The letters u and v are used in one example (7), which opens up for variation in this dimension.

Two equations can be merged together, into what might look like one equation. It is probably not evident to all students that an algebraic expression like $u = v = 0$ can be interpreted as two separate equations merged together.

The coefficients must not be natural numbers. In item (4) fractions (halves) are used.

There is, however, little variation regarding the format of the equations. All equations, but one ($x = y$ in item 4), are in a format where the right hand side consists of just a constant term.

Opened dimensions of variation SH1-L05 [04:15-17:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1b. More than one equation 2b. Two unknowns not three (or more) 3a. xy is not of first degree 3b. $(x + y)^2$ is not of the first degree 3c. $1/y$ is not of first degree 5a. Coefficients can be rational numbers not just natural 6a. The letters x, y are not taken for granted 7a. Individual equations can be in different formats 7b. Two equations can be in written in one expression 7c. Both unknowns are not present in both equations	–	–

SH1-L05 [17:30-24:40]

The lesson continues and the focus shifts to the meaning of a *solution* to a system of linear equations in two unknowns – what is required of a pair of numbers (x, y) to be a solution. The teacher shows a new slide with questions concerning the solution to a system of equations. The students are again asked to read in the textbooks and try to answer the questions.

1 What does a ”solution” mean in a system of equations?

2 Why is $x = 23$, $y = 12$ the solution of $\begin{cases} x + y = 35 \\ 2x + 4y = 94 \end{cases}$

3a Is the solution to the equation $5x - 3y = 3$ equal to the solution of $\begin{cases} 5x - 3y = 3 \\ x + 2y = 11 \end{cases}$?

3b What about vice versa?

18:25 T: Please find their answers from reading our textbooks.

(Q. 1)

20:40 T: Come on, the first question.

[...]

20:50 S: That solution is suitable for each of the equations in the pair of system of linear equations in two unknowns. [the answer is obviously the same as can be found in the textbook according to comments in lesson table].

(Q. 2)

22:15 T: Then can you explain why x equals twenty-three, y equals twelve is the solution of the system of linear equations in two unknowns? Come on.

S: It is because when we substitute x equals twenty-three and y equals twelve into the equations of x plus y equals thirty-five and two x plus four y equals ninety-four separately, the left hand side is equals to the right hand side. So that we can say this solution is suitable for this system of equations.

(Q. 3a)

23:10 S: The solution of five x minus three y equals three may not be equal to the solution of five x minus three y equals three and x plus two y equals eleven.

[...]

23:20 T: Why?

S: Because the solution of a system of linear equations in two unknowns should be suitable for both the two equations. But the solutions of five x minus three y equals three may not suitable for x plus two y equals eleven.

T: Oh, good. He has given us a very good answer.

(Q. 3b)

24:10 S: Because the solution of a system of linear equations in two unknowns must satisfy both the two equations ... satisfy two equations.

T: Mm ... satisfy two equations.

S: That means it must satisfy five x minus three y equals three.

This episode shows a similar pattern as the previous. First the concept, in this case the *solution* of a system of linear equations in two unknowns, is described and some important properties are mentioned. In this case it is ”the solution should satisfy both equations”. This is exemplified in question 2. Question 3a and 3b, further highlights the meaning of "satisfying both equations" by the difference between the solution to the system compared to the solution to one of the equations. The equations are kept invariant as the 'direction' of the question is varied. There is no rush into actual solving of systems of equations.

Opened dimensions of variation SH1-L05 [17:30-24:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one	1a. Same solution to both equations is not taken for granted 1b. Substitution of solution will get equal sides in both equations	–

SH1-L05 [24:45-28:25]

The teacher puts on a new overhead transparency onto the screen.

1 (1) Which is the solution of $\begin{cases} 3x - 2y = -2 \\ x - 6y = 2 \end{cases}$

a) $x = 1, y = -1/2$ b) $x = -1, y = 1/2$
c) $x = -1, y = -1/2$ d) $x = 1/2, y = 1/4$

(2) Which equation has the solution $x = -2, y = 3$?

a) $\begin{cases} 2x + y = -1 \\ 3x - y = 3 \end{cases}$ b) $\begin{cases} x/2 + y/3 = 2 \\ x - y = 5 \end{cases}$

c) $\begin{cases} x/3 - y/2 = -13/6 \\ x/2 - y/3 = -2 \end{cases}$ d) $\begin{cases} 3x + 2y = 0 \\ 2x + 3y = 0 \end{cases}$

(3) If $x = -2, y = 1/2$ is the solution of $\begin{cases} 3ax - 2y = 5 \\ 5x + 4by = 6 \end{cases}$

what is the value of a and b?

a) $a = 1, b = -8$ b) $a = -1, b = -8$
c) $a = 1, b = 8$ d) $a = -1, b = 8$

The students answer the questions correctly after just a short time of thinking. The answers are checked by substitution into the equations. Each of the three questions generates an interesting pattern of variation. In (1) the system of equations is invariant while the proposed solution (number pairs) varies (table 5.3).

Table 5.3 *Pattern of variation in (1)*

	Equation (1)	Equation (2)
Number pair a)	–	solve
Number pair b)	–	–
Number pair c)	solve	solve
Number pair d)	–	–

In the second question the pattern is reversed. Here the proposed solution (the number pair) is kept invariant while the systems of equations vary (table 5.4). There is also variation regarding the coefficients in the equations, some are integers, and some are fractions.

Table 5.4 *Pattern of variation in (2)*

	Equation (1)	Equation (2)
System a)	solve	–
System b)	–	–
System c)	solve	solve
System d)	solve	–

In the third question the complexity is increased. While the solution (number pair) and the overall format of the equation system are kept invariant, the coefficients are expressed by parameters. This task introduces variation regarding the coefficients beyond being specified and fixed numbers. Between the four answers to the question the absolute values of the two parameters a and b are kept invariant. It is just the sign that vary in a systematic way (table 5.5). All possible combinations are used.

Table 5.5 *Pattern of variation in (3)*

	Sign of a	Sign of b
Parameters a)	+	–
Parameters b)	–	–
Parameters c)	+	+
Parameters d)	–	+

There is no variation in the over-all format used. In all equations the right hand side is made up of a (numerical) constant. The unknowns are always in the equations left side.

Opened dimensions of variation SH1-L05 [24:45-28:25]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5a. Coefficients and constants can be rational numbers, not just natural 5b. Constants can be negative numbers, not just natural 5c. Coefficients can be parameters	1a. Same solution to both equations is not taken for granted 1b. Substitution of solution will get equal sides in both equations 3a. The unknowns can be negative numbers 3b. The unknowns can be rational numbers	–

SH1-L05 [28:25-32:45]
The teacher moves on to the next slide, two more questions are shown on the screen.

(4) Which is the solution of $\begin{cases} x - y = 5 \\ 3x - 4y = 1 \end{cases}$?

a) no solution b) infinitely many solutions
c) $x = -2, y = 3$ d) $x = 19, y = 14$

2 Given the solution $\begin{cases} x = 1 \\ y = 2 \end{cases}$, make up a system of linear equations in two unknowns

The students find the answer (d) to the first question quite immediately. Then the students are given a moment to work on the second question. Two student answers are put on the board and the class discuss how they came up with them.

$$\begin{cases} \frac{x}{2} + \frac{y}{2} = 1\frac{1}{2} \\ 2y - 2x = 2 \end{cases} \qquad \begin{cases} x + y = 3 \\ y - x = 1 \end{cases}$$

The alternatives given in question (4) open up the dimension of the *number* of solutions to a system of equations. The variation offered makes it possible to discern that a system of equations may not necessarily need to have one unique solution. The work with the second question involves keeping the solution (number pair) invariant, while there are two different systems of equations. The systems of equations provided by the students and displayed on the board are in the same general 'standard' format. They differ in respect of coefficients and constants.

Opened dimensions of variation SH1-L05 [28:25-32:45]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5a. Coefficients and constants can be rational numbers, not just natural	2c. 'One unique solution to a system' not taken for granted 3a. The unknowns can be negative numbers 5a. Different systems with the same solution is possible	–

SH1-L05 [32:45-41:15]

The teacher shows a task prepared on a small board that is lifted up in front of the main blackboard [Note: it is a bit unusual to put the negative numbers to the right].

$$\begin{cases} x - 3y = -3 \\ 2x + y = -\frac{4}{3} \end{cases}$$

$x - 3y = -3$

x	...	2	1	0	-1	-2	...
y	...						...

$2x + y = -4/3$

x	...	2	1	0	-1	-2	...
y	...						...

33:40 T: ... five values for x, and then asks you to find out those values of y. Ok, please do it now.

[...]

37:50 T: Okay, then let's check it together.

After a few minutes of student work the tables are completed. The teacher writes the answers given by students.

$x - 3y = -3$

x	...	2	1	0	-1	-2	...
y	...	$\frac{5}{3}$	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	...

$2x + y = -4/3$

x	...	2	1	0	-1	-2	...
y	...	$-5\frac{1}{3}$	$-\frac{10}{3}$	$-\frac{4}{3}$	$\frac{2}{3}$	$\frac{8}{3}$	...

39:20 T: ... can you find out the solution of this pair linear equations in two unknowns?

S: We can.

The teacher writes the solution on the board.

$$\begin{cases} x = -1 \\ y = \frac{2}{3} \end{cases}$$

39:55 T: You have to put a large bracket here. Then, you can see ... x equals [negative] one, y equals two over three, it is the solution to both equation one and equation two. So it is also this system of equations in two unknowns ...?

S: Solution.

T: Solution.

The content of the lesson is summarized before it ends.

40:25 S: Today we have learnt what a system of linear equations in two unknowns is.

[...]

S: And to justify whether it is a system of linear equations in two unknowns.

[...]

S: The meaning of the 'solution' of a system of linear equations in two unknowns.

In this episode the two equations are kept invariant and the values of x and y are varied. It is made possible to discern that one linear equation in two unknowns can have many solutions. When the values in the two tables are considered at the same time the number pair that are present in both tables can be found. The number of solutions changes from many to one when the two equations are considered simultaneously.

Opened dimensions of variation SH1-L05 [32:45-40:10]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one 5a. Coefficients and constants can be rational numbers, not just natural 5b. Constants can be negative numbers, not just natural	1a. Same solution to both equations is not taken for granted 2a. One equation has many solutions/no unique solution 2b. Two equations reduce the number of solutions to one 3a. The unknowns can be negative numbers 3b. The unknowns can be rational numbers	–

Shanghai 1 – Lesson 6

SH1-L06 [0:00-05:10]

The teacher shows a slide with some questions for revision. The students' answers are all accepted as correct by the teacher. Equation 2 in question 4 provides variation regarding the format of an equation in a system.

3 $x + y = 12$. If $y = 2x$, find the two unknowns

$x =$ $y =$

4 A student solved the following pair of equations:

$$\begin{cases} x + 3y = -5 \\ y = 2x - 4 \end{cases}$$

The solution he obtained was $x = 1$, $y = 2$.

Do you think it's correct?

Opened dimensions of variation SH1-L06 [0:00-05:10]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
7a. Format of individual equations not invariant	–	–

SH1-L06 [05:10-11:40]

The teacher reminds the students of how they, in the last lesson, had listed solutions to the two equations in a system in two tables and how they were able to find a common solution. He then introduces the method of substitution by returning to question 3.

06:15 T: However, sometimes you have found ten solutions for equation one and ten more for ... equation two, can the common solution necessarily be found? Ah, this is not absolute. So the way of listing on the table can find the solution of the linear equation of two unknowns but it's quite complicated. So now let's consider a simple way to find the solution of the linear equation in two unknowns.

[...]

08:05 T: So now let's look at the question we've just discussed [...] how can you get it? Can anyone tell me?

S: That is by substituting y equals to two x into the equation x plus y equals to twelve [...] Three x equals twelve, x equals four.

One of the students already seems to know the method of substitution.

09:20 T: Why can we substitute y equals to two x into x plus y equals to twelve ... ?

[...]

10:10 T: So look at this, are the x values in equation one and two the same? [points to x in both equations] ... It's the same unknown. So how about the y value in equation one and two? [points to y in both equations] They are also ... the same unknown [...] Therefore, we can substitute that.

[...]

10:40 T: We have substituted the first equation into the second equation, see what happens to the first equation? It has changed into ... oh, it has changed into a linear equation in one unknown, right? It has changed into a linear equation in one unknown. So this method is called ... is called method of substitution.

The actual substitution is never written on the board, just described and done mentally. However, the teacher points out some of the important features of the method of substitution. A letter – *x* and *y* – can represent the same value even though it is in two different equations. The 'status' of the letters *x* and *y* is not taken for granted.

Also the core idea behind the method – to reduce the number of unknowns from two to one, and thus transforming the equation to one that can be solved – is highlighted by comparing the situation 'before' and 'after' the substitution.

Opened dimensions of variation SH1-L06 [05:10-11:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
4a. 'Unknowns are the same' is not taken for granted	–	2a. Getting an equation in one unknown is not taken for granted

SH1-L06 [11:40-17:10]

The teacher writes on the board.

$$\begin{cases} x + 2y = 1 & (1) \\ 2x - 3y = 9 & (2) \end{cases}$$

12:20	T:	Think about it ... how can we solve this equation ... system of equations? [...] What should we do first? Tell me.
	S:	Using the expression of y to indicate x. [the teacher writes on the board, "From equation (1): x = 1 – 2y (3)"]
13:10	T:	Let me ask you, why do we transform the first equation but not the second equation? [points to (1) and (2)]
	S:	It's because the coefficient of equation one is one. [teacher points to x in (1)]
[...]		
	T:	Then the numbers are much simpler.
[...]		
13:45	T:	...what should our next step be?
	S:	Substituting the third equation into the second equation.
[...]		
	T:	Let me ask, can we substitute the third equation into the first one?

The teacher offers alternatives both regarding which equation to pick and rearrange for the initial substitution as well as regarding into which equation the equation (3) should be substituted. None of these are taken for granted. The teacher and the class solve the system of equations in dialogue and the teacher writes the solution on the board.

$$\begin{cases} x + 2y = 1 & (1) \\ 2x - 3y = 9 & (2) \end{cases}$$

From (1) $\quad x = 1 - 2y \qquad (3)$

Substitute (3) into (2)

$$2(1 - 2y) - 3y = 9$$
$$2 - 4y - 3y = 9$$
$$-7y = 7$$
$$y = -1$$

Substitute $y = -1$ into (3)

$$x = 1 - 2(-1)$$
$$x = 3$$

The solution to the system of equations is $\begin{cases} x = 3 \\ y = -1 \end{cases}$

Some features are pointed out by the teacher during the process of solving. A difference from the previous example is that one of the equations has to be rearranged before the first substitution. The teacher does not point to this difference, but it still could be possible to discern that sometimes one of the equations have to be rearranged before it is suitable for substitution. While the choice of 'source' equation is not taken for granted, the choice of unknown (x) for the initial substitution is. The solution is not checked.

Opened dimensions of variation SH1-L06 [11:40-17:10]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	3a. The unknowns can be negative numbers	4a. The selection of 'source' equation is not taken for granted 6a. The selection of 'target' equation is not taken for granted 6b. Equation (1) ⇒ (3) should not be substituted back into (1)

SH1-L06 [17:10-22:00]
One more example is solved on the board. The teacher writes it to the right of the previous example.

18:15 T: ... look at this, this is the first equation and this is the second, which equation should we transform?

[...]

19:00 T: Then why do we choose equation one still? It's because the values of equation one are much simple than those in the equation two.

Also this time the teacher opens for variation in the choice of the equation for the initial substitution. The reason for choosing (1) is however merely given by a short statement. The choice of the unknown x is taken for granted.

$$\begin{cases} 2x + 3y = 4 & (1) \\ 5x - 2y = 29 & (2) \end{cases}$$

From (1) get $x = 2 - 1{,}5y$ (3)

Substitute (3) into (2)

$$5(2 - 1{,}5y) - 2y = 29$$
$$10 - 7{,}5y - 2y = 29$$
$$-9{,}5y = 19$$
$$y = -2$$

Substitute $y = -2$ into (3)

$$x = 2 - 1{,}5(-2)$$
$$x = 5$$

The solution to the system of equations is $\begin{cases} x = 5 \\ y = -2 \end{cases}$

One difference between the last two examples is that the second does not have any unknown with the coefficient of one. Similarities are:

- x in equation (1) are in both cases selected to generate equation (3),
- the main format of the systems – $\begin{cases} ax + by = c \\ dx - ey = f \end{cases}$ – is the same.

This leads to only numerical differences between the solving procedures, and in both cases the solutions are formed by a positive value of x and negative of y.

SH1-L06 [22:00-26:00]
The teacher lifts the focus from the actual solving to describing the method of substitution in more general terms. The two examples and the solutions are still on the board, side by side. The teacher and one of

Opened dimensions of variation SH1-L06 [17:10-22:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	3a. The unknowns can be negative numbers	4a. The selection of 'source' equation is not taken for granted

the students go through the solution of the first example on the board and describe the different steps in the solution.

They then look in the textbook (p. 31) where the procedure of solving is described [translation from lesson table].

> The procedures of solving linear equations in two unknowns are:
>
> 1 Rearrange an equation such that one unknown is expressed in terms of another.
> 2 Substitute the rearranged equation into another equation such that it is transformed into a linear equation in one unknown.
> 3 Solve for one of the unknowns.
> 4 Substitute the unknown solved for into the equation rearranged in step one.
> 5 Write the solution in a large bracket.

The students read the five steps from the textbook in chorus [this translation is from the transcript]

25:05 S: First, transform one equation of a system of equations into an equation, which contains only one unknown to indicate the unknown of the other. Second, use this equation to substitute the corresponding unknown value of another equation so that the linear equation in two unknowns is transformed into a linear equation in one unknown. Third, solve the linear equation in one unknown and find the solution of the unknown. Forth, substitute this solution into the equation of step one to find out the other unknown value. Fifth, write the two solutions into a big bracket.

Opened dimensions of variation SH1-L06 [22:00-26:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

SH1-L06 [26:00-37:30]
The teacher writes on the board.

$$1)\ \begin{cases} 2x+3y=1 \\ 7x-8y=6 \end{cases} \qquad 2)\ \begin{cases} \dfrac{x+y}{2}+\dfrac{x-y}{3}=6 \\ 4(x+y)-5(x-y)=2 \end{cases}$$

The teacher discusses the two examples and states that they both are systems of linear equations in two unknowns, but that the second is not written in the 'standard format', unless the system is solved for $(x + y)$ and $(x – y)$. There could be a step, added to the beginning of the list of five steps above, stating something like "put the equations into the standard format", even though that is not necessary.

27:50 T: We must make it into the standard format. [writes on the board]

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x-b_2y=c_2 \end{cases}$$

28:10 T: Then, what do a1, a2, b1, b2, c1, c2 indicates? They are constants [...] ... if a1 equals to zero, can a2 equals to zero? [...] No, it cannot. If it equals to zero here, this is not a linear equation in two unknowns.

The teacher opens up for the possibility of constants being zero, but points to a restriction. The teacher then rearranges the two equations in system 2.

$$\begin{aligned} &(1):\ 3(x+y)+2(x-y)=36 \Rightarrow 3x+3y+2x-2y=36 \Rightarrow \\ &\qquad \Rightarrow 5x+y=36 \\ &(2):\ 4x+4y-5x+5y=2 \Rightarrow -x+9y=2 \end{aligned}$$

31:15 T: Boys and girls, pay attention here, whenever it is not a standard format, we have to reorder and rewrite it.

The teacher erases the calculations and writes the reformulated second system beneath the original on the board. The students start to work on the tasks and two of them are working at the board, while the teacher walks around. Both solutions on the board are correct and follow the procedure of the previous examples. The teacher reviews and comments on the solutions in whole-class.

The difference between the two examples and the discussion of standard format – which also involves lifting the concept of system of equations from the particular to the general – highlights some features regarding the kinds of equations that can form system of equations. The dimension of variation regarding the coefficients and constants are opened by the use of parameters in the general example.

Opened dimensions of variation SH1-L06 [26:00-37:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5c. Coefficients and constants can be parameters 7a. Individual equations can be in different formats 7c. Both unknowns not present in both equations	–	–

SH1-L06 [37:30-43:50]

The teacher shows three new systems of equations. The discussion is about finding better strategies of solving. The teacher's statement at 31:15 is now totally neglected when they move on to question 3.

1) $\begin{cases} 2x + 3(5x + 7y) = 4 \\ 5x + 7y = 2 \end{cases}$ 2) $\begin{cases} 4x - 3y = 0 \\ 10x + 9y = 11 \end{cases}$

3) $\begin{cases} 3x - 2y + 5 = 0 \\ \dfrac{3x - y + 5}{4} - \dfrac{3x + 2y - 25}{11} = 0 \end{cases}$

37:35 T: For the following few questions, let's talk about the steps to solve [...] For example, in question one [...] should we solve it by the way that our classmates just used? Can you think of any other better way? Okay tell me.

S: We can substitute the second equation directly into the first one.

The method of substitution is expanded from just substitution of *x* or *y* into substitution of larger expressions from one equation into the other. They discuss some more sophisticated ways of substitution.

- The expression $5x + 7y$ in equation (1) can be exchanged by 2 from equation (2).
- From equation (1) you get $3y = 4x$ and $9y = 12x$. $9y$ in (2) can then be exchanged by $12x$.
- $3x = 2y - 5$ from equation (1) can be substituted into equation (2) at two places.

41:30 T: What will the second equation be after the substitution? What will it be? It's a linear equation of one unknown y.

The teacher comments on some strategic considerations when solving systems of equations and assigns homework before ending the lesson.

42:40 T: We look for reasonable way [...] ... should we use the expression of x to indicate the value of y or the equation of y to indicate the value of x? [...] Or should it be determined by the number before the unknown, that is the coefficient [...] ... think of the questions we have just finished, what have we done? We did substitution by three x and two y, didn't we?

In this episode the method of substitution is varied. It is used differently depending on what the equations look like.

Opened dimensions of variation SH1-L06 [37:30-43:50]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
7a. Individual equations can be in different formats	–	2a. Getting an equation in one unknown is not taken for granted 3a. The selection of unknown is not taken for granted 3c. Substitution of larger expressions is possible 4a. The selection of 'source' equation is not taken for granted

There are many dimensions of variations opened in these two lessons. There are however some aspects that are kept invariant, more or less. Firstly, there is very little variation concerning the aspect of the overall structure of the systems of equations used. They are all, with one exception (item 4 in question 3 showed on the screen at 4:15 in the previous lesson, SH1-L05) in the format $f(x, y) = a$. On the right hand side of the equations there are no unknowns, just a numeric constant.

Secondly, another invariance throughout these two lessons is the use of x and y to denote the unknowns. The only exception can be found on the same overhead transparency as mentioned above, used in the beginning of the first lesson (L05).

Shanghai 2

This grade eight class have 46 students. It is a mixed ability class (TQ1). The mode of instruction in this class is mainly teacher-led in whole class. The whole class works with the same tasks at all times. The teacher solves and discusses tasks on the board. This activity is quite regularly interrupted and the students are given a minute or two to think about a question or a problem. This will happen typically 5–8 times during a lesson. The teacher then walks around, checks the students' work and talks to individual students. Some of the students are asked to show their solutions on the board and these solutions are discussed in whole class. The repeated shifts between whole-class teacher led instruction and seatwork creates a rhythmic flow in these lessons.

Overview of the lessons in Shanghai 2

The two lessons analysed are no. 3 and 4.

SH2-L03

In this lesson the concept of *system of linear equations in two unknowns* is introduced as a way to represent a problem concerning "chickens and rabbits". A common solution is found by guess-and-check. Two more examples of systems of linear equations are solved by the *method of substitution*. The properties of a system of linear equations in two unknowns and its solutions are discussed and a third system of equations is solved by the method of substitution on the board.

SH2-L04

The first exercise is to decide whether four given examples are systems of linear equations or not. This is followed by an exercise in rearranging expressions, "represent one unknown by the other". Four systems of equations are solved by the method of substitution and shown on the board. Differences in how the same system can be solved are highlighted.

The teacher also describes ”the method of elimination by substitution” in five steps.

Shanghai 2 – Lesson 3

SH2-L03 [0:00-8:20]
Revision of (single) linear equations in two unknowns, and their solutions.

1 Please give one solution of the equation $3x - y = 6$.
2 Given the linear equation in two unknowns $7x - 2y = -5$
1) Use the equation of x to represent y, $y =$ ______
2) Use the equation of y to represent x, $x =$ ______
3) When $x = 1$, $y =$ ____
When $x = -1$, $y =$ ____
4) When $y = -2$, $x =$ ____
When $y = 0$, $x =$ ____

Opened dimensions of variation SH2-L03 [0:00-8:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

SH2-L03 [8:20-12:20]
The teacher shows a problem on the screen by an overhead projector. The exercises on the slides are apparently prepared in advance and not something the teacher comes up with 'on the spur of the moment'. Many of the examples are taken from the textbook.

There are x rabbits and y chickens in a cage. There are altogether twelve heads, and forty legs.
How many rabbits and chickens are there in the cage?

A student proposes to put up two equations. The students form equations with seemingly ease and the teacher writes on the board.

$$x + y = 12 \quad \text{[teacher changes it to]} \quad x + y = 12$$
$$4x + 2y = 40 \qquad\qquad\qquad\qquad\qquad 2x + y = 20$$

09:50 T: Good, so how many chickens and rabbits are there? ... Let me give you some pairs of numbers. [writes on the board]

$$\begin{cases} x = 2 \\ y = 4 \end{cases} \quad \begin{cases} x = 4 \\ y = 12 \end{cases} \quad \begin{cases} x = 8 \\ y = 4 \end{cases} \quad \ldots$$

The number pairs on the board are solutions to none, one and both equations. The students are given a minute to discuss. One property that a solution should have comes up.

12:00 T: ... the values of the pair of numbers have to satisfy the first equation and also the second equation.

The three proposed solutions provide variation, as they satisfy none, one and both equations. The alternatives open the DoV that a solution to a system of equations – the same number pair – should satisfy both equations. The number of equations and the number of unknowns are given. In these respects no alternatives are given.

Opened dimensions of variation SH2-L03 [8:20-12:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	1a. 'Same solution to both equations' is not taken for granted	–

SH2-L03 [12:20-16:30]

The teacher shows a new slide with the same system of linear equations that was generated from the rabbits-and-chicken problem. This example is used to 'define' the concept.

$$\begin{cases} x + y = 12 \\ 2x + y = 20 \end{cases}$$

12:25 T: In mathematics, we call this a system of linear equations ... a set of equations formed by the combination of linear equations is called a system of linear equations.

The concept of system of linear equations is introduced by this example. After the teacher has shown the example and labelled it, they discuss some of the characteristics of such a system.

13:25 T: ... so what is a system of linear equations in two unknowns? Can you tell me?

[...]

13:40 S: There are two unknowns, and the power of the unknown (...) is one.

A response from another student triggers the teacher to create a contrasting example.

14:00 T: He said that equations that have two unknowns, and the power of the unknowns is one is called a system of linear equations in two unknowns. You seem to have very different opinions.

S: Two or above (...).

T: Two and above, um. [the teacher writes on the board]

$$\begin{cases} x - y = 10 \\ y + z = 6 \\ 3x + 5y = 7 \end{cases}$$

14:30 T: So, two or above, is this system of equations still called a system of linear equations in two unknowns?

S: No.

T: Why?

S: Because there are three unknowns.

The teacher sums up the lesson and writes on the board.

15:50 T: There are ... two unknowns, and the power of the unknowns is one ... is called a system of linear equations in two unknowns.

The teacher's counter-example is a system with three unknowns. It also has three equations. This probably makes it less clear. The two examples differ in two respects/features simultaneously. Both the number of unknowns and the number of equations increase from two to three, which would make it difficult to separate the two features from each other.

Opened dimensions of variation SH2-L03 [12:20-16:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
2b. Two unknowns, not three (or more)	–	–

SH2-L03 [16:30-19:00]
The teacher changes the content from the system of equations to the solution to such a system.

16:30 T: Next, let's talk about the solutions of a system of linear equations.

A new slide showing two tables with number pairs that satisfies the two equations ($x + y = 12$ and $2x + y = 20$) separately is shown. One number pair (8, 4) is present in both tables. The requirement of a solution to be a common solution to both equations is brought forward.

18:45 T: So, the solution that satisfies all equations in the system is called the solution of the system. Then we will talk about how to solve ... the system of equations.

The difference between looking at the equations separately and together would make it possible to discern some features of a solution to a system of equations.

Opened dimensions of variation SH2-L03 [16:30-19:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one	1a. Same solution to both equations is not taken for granted 2a. One equation has many solutions/no unique solution 2b. Two equations reduce the number of solutions to one	–

SH2-L03 [19:00-27:00]

The teacher moves on to how to solve a system of equations and shows a slide with a new example. One of the equations is in a different format compared to the previous systems of equations.

$$\begin{cases} y = 2x \\ x + 2y = 15 \end{cases}$$

19:35 T: Try to solve the system, 'y equals two x', 'x plus two y equals fifteen'. Let's start now.

The students start to work with the task. The teacher gives a hint to change it to one unknown, and writes "2 $\Rightarrow$ 1" on the board. The teacher walks around and after a while he gets the notebooks from two students. The two different solutions are then projected onto the screen. The first student has listed solutions to the two equations and looked for a common pair of numbers.

22:50 T: ... then he tried to find if there is any common solution, and he gets that, but it is just like looking for something in the ocean.

The second solution is shown and the teacher discusses this approach and point to some things.

$$\begin{aligned} x + 4x &= 15 \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

23:45 T: He suggested substituting y equals two x in the first equation into the second equation. You tell us, why can you do that?

S: Because (...)

T: He said that it's because the y in the first equation is the same as the one in the second equation. Do you all agree?

S: Yes.

The teacher writes on the board and shows with arrows how to substitute $y = 2x$ into the second equation and then how $x = 3$ is substituted back into the first equation.

24:20 T: ... he changes the two unknowns [points to $x + 2y = 15$] into one unknown. [points to $x + 2 \times 2x = 15$]

[...]

$$y = 2x$$

$$x + 2y = 15$$

$$\downarrow$$

$$x + 2 \times 2x = 15$$

$$5x = 15$$

$$x = 3$$

$$y = 2 \times 3 = 6$$

The solution to the system of equations is $\begin{cases} x = 3 \\ y = 6 \end{cases}$

25:10 T: ... first of all substitute equation one into equation two, change two unknowns [points] into one unknown [points], find x, and then substitute x into the first equation, so we get the value of y, which is six.

The same system of equations is solved with two different methods. This would bring forward the method of substitution (not yet named) as just *a* method, not *the* method for solving system of equations.

That *y* in both equations are the same is not taken for granted.

Opened dimensions of variation SH2-L03 [19:00-27:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
4a. 'Unknowns are the same' is not taken for granted 7a. Format of individual equations not invariant	–	2a. Getting an equation in one unknown is not taken for granted

SH2-L03 [27:00-33:10]

The teacher gives the students a new system of equations to solve. The teacher wants the students to use the method of substitution. So far, the students have just seen one example, which would make it difficult to separate specific features of the example from the more general features of the method.

27:05 T: Have you seen the process? So you all please work on the question below.

The teacher walks around and talks to individual students. After three minutes the teacher continues the whole class discussion.

$$\begin{cases} 3x + 2y = 5 \\ x = 1 - y \end{cases}$$

30:15	T:	... which one should be substituted into which one? [...]
	S:	Substitute the second equation into the first.
	T:	Substitute the second equation into the first, can you explain why?
	S:	Because ... x, the x in equation one is the same as x in equation two.
	T:	Good, the x in equation one is the same as the x in equation two, I want to know, why did you substitute equation two into one, but not equation one into two? ... Can anyone help him?
	S:	It's because equation one equals five, and the second equation equals one minus y.

The students do not answer the question properly and the reason for why it is more convenient to use the second equation, ($x = 1 - y$) for the first substitution instead of the first one ($3x + 2y = 5$), is not really articulated. The first student points to the feature that makes the method of substitution possible to perform, namely that the unknowns are the same in both equations, but that does not answer the question. However, the teacher opens up for variation in which equation could be substituted into which by introducing the two alternatives, one into two or two into one.

A solution from one of the student's notebooks is then projected onto the screen and the teacher goes through the steps. The solution ($x = 3$, $y = -2$) introduces variation regarding the types of numbers that make up a solution. Natural numbers make up the solution in the first example. The teacher also writes the solution onto the board beside the solution to the previous task.

32:15	T:	Substitute it into the first equation, y is replaced, or x is replaced?
	S:	x.
	T:	x.

$$3(1 - y) + 2y = 5$$
$$y = -2$$

Substitute $y = -2$ into equation (2)

$$x = 1 - (-2) = 3$$

The solution to the system of equations is $\begin{cases} x = 3 \\ y = -2 \end{cases}$

The two solutions on the board differ in some respects. In the first case y from the first equation is substituted into the second equation. In the second case x from the second equation is substituted into the first equation. Also in this case the teacher provides alternatives – ”y or x is replaced?”. So even though the same method is used it is performed slightly differently. These differences would make it possible to discern that both equations as well as both unknowns can be used for the initial substitution.

Opened dimensions of variation SH2-L03 [27:00-33:10]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
4a. 'Unknowns are the same' is not taken for granted	3a. The unknowns can be negative numbers	3a. The selection of unknown is not taken for granted 4a. The selection of 'source' equation is not taken for granted

SH2-L03 [33:10-35:25]

The teacher shows a slide with four systems of equations. They return to the meaning of a system of linear equations in two unknowns.

1) $\begin{cases} x+y=5 \\ 2y-z=-1 \end{cases}$ 2) $\begin{cases} \frac{1}{2}x-\frac{2}{3}y=1 \\ x=3y \end{cases}$

3) $\begin{cases} x=-4 \\ 5x-y=-7 \end{cases}$ 4) $\begin{cases} xy=-1 \\ 2x-y=3 \end{cases}$

33:25 T: ... in the following equations, which is a system of linear equations in two unknowns.

[...]

33:50 S: The second one.

The teacher accepts this without any discussion or comment.

34:00 S: The third one.

T: The third one, tell us why? Why is it the third one? [...] Can you tell us why? [...] Are there two unknowns? [...] And the power of the unknowns is one? [...] What do you think?

[...]

S: Yes.

T: Really?

Now the teacher writes a new example on the board

$$\begin{cases} x = 3 \\ y = -2 \end{cases}$$

35:05 T: ...is this a system of linear equations in two unknowns?

S: Yes.

T: Yes, [...] such a system of equations is a system of linear equations in two unknowns.

In SH1 a similar task with eight systems of equations is used. The differences between those eight examples generates a more powerful pattern of variation than the four examples in this task. The teacher in SH1 also exploits the exercise to a higher degree by systematically discussing both the reasons why a certain example meets the requirements of a system of linear equation in two unknowns or why it does not.

In SH2 only item no. 3 is discussed. The reason why item no. 1 and 4 *are not* examples of systems of equations of linear equations in two unknowns is just taken for granted. However, the differences between the four items open some DoVs concerning the concept, although not as clearly as it was done in SH1.

Opened dimensions of variation SH2-L03 [33:10-35:25]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
2b. Two unknowns not three (or more) 3a. *xy* is not of first degree 5a. Coefficients can be rational numbers not just natural 5b. Constants can be negative numbers not just natural 7c. Both unknowns not present in both equations	–	–

SH2-L03 [35:25-37:50]
The next exercise shown on a slide is about determining whether a given number pair is a solution or not. The number pair in item 1 satisfies equation (1), but not (2). In item 2 it satisfies both equations.

1) $\begin{cases} 5x - y = 32 \\ x - 2y = 19 \end{cases}$ $\begin{cases} x = 6 \\ y = -2 \end{cases}$

2) $\begin{cases} x - \frac{1}{2}y = 4 \\ \frac{1}{3}x + \frac{1}{2}y = \frac{4}{3} \end{cases}$ $\begin{cases} x = 4 \\ y = 0 \end{cases}$

35:30 T: ... see if they are solutions of the system of equations. Think about that on your own.

After a while they look into the questions in whole class.
(1)

37:00 S: It's because x equals six, y equals negative two can only satisfy the first equation.

(2)
37:45 T: It is their common solution, so it is the solution of the system.

Opened dimensions of variation SH2-L03 [35:25-37:50]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5a. Coefficients and constants can be rational numbers not just natural	1a. Same solution to both equations is not taken for granted 3a. The unknowns can be negative numbers	–

SH2-L03 [37:50-42:10]
In the last exercise of the lesson they return to solving a system of equations. The teacher asks the students to work on solving the system shown on the screen. After a short while two students write their solutions on the board.

$$\begin{cases} x = y + 3 \\ x + 4y = 18 \end{cases}$$

41:20 T: Okay, look at the blackboard. Substitute equation one into two and we have [...] which part of the second equation should be replaced by 'y plus three'?

S: x.

T: Right, x should be replaced by y plus three. ['squares' y+3 and x]

$$\begin{cases} x = \boxed{y+3} \\ \boxed{x} + 4y = 18 \end{cases}$$

Substitute (1) into (2)

$$y + 3 + 4y = 18$$
$$5y = 15$$
$$y = 3$$

Substitute $y = 3$ into (1) get $x = 6$

The solution to the system of equations is $\begin{cases} x = 6 \\ y = 3 \end{cases}$

Teacher points to and makes some marks in one of the students' solutions on the board, while he discusses it step by step. The other student's solution is not commented on at all.

This is the third system of equations that is solved on the board this lesson. It differs from the other two, in that x from the first equation is substituted into (2). The others involved y from (1) into (2) and x from (2) into (1). The fourth example on the slide was never used in the class today, but it is of the last remaining type, where y from (2) would be substituted into (1). The four examples on the slide are varied to cover all possible cases.

However, the teacher does take the substitution of (1) into (2) for granted when he comments on the solution on the board, and no alternative is offered.

Opened dimensions of variation SH2-L03 [37:50-42:10]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

SH2-L03 [42:10-45:30]

The teacher summarises the lesson and assigns homework.

42:10 T: Okay, today we've talked about some concepts of system of linear equations in two unknowns. System of linear equations in two unknowns, its solutions and how to solve the system, we've talked about one of the way to solve the system, basically it is to change two unknowns into one unknown. Today we used the method of substitution, and we will talk about the other methods later on.

[...]

44:00 T: ... any questions about lesson of today?

One student asks if is not possible to substitute y instead of x in the last example. The teacher answers that of course – then $y = x - 3$ should be substituted into (2) – it is possible but not "reasonable" as it would cause more calculations.

Opened dimensions of variation SH2-L03 [42:10-45:30]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	3a. The selection of unknown is not taken for granted

Compared to SH1, the introduction of the concept of system of linear equations in SH2 gives a little less well-structured impression. Especially the solving of systems of equations with the method of substitution seems to come early (episode starting at 19:00).

In the lesson material for SH2-L03 there are six handouts or copies of the slides shown. The last slide (SH2-L03hout_6) contains the systems of equations solved already during the episode starting at 19:00. It looks like this slide was used too early. Maybe the teacher mixed up the slides and jumped ahead in the teaching material?

Shanghai 2 – Lesson 4

SH2-L04 [0:00-0:40]

The teacher shows a slide with four systems of equations and asks which of them is not a system of linear equations in two unknowns.

A) $\begin{cases} 3x - y = 2 \\ y = -\frac{2}{3}x \end{cases}$ B) $\begin{cases} x + 3y = 8 \\ 2x - 5y = -21 \end{cases}$

C) $\begin{cases} xy = 7 \\ x + y = 8 \end{cases}$ D) $\begin{cases} x = 4 \\ 3x - y = 6 \end{cases}$

The sound is off in the beginning of the teacher-video (0:00- 01:40). The questions are answered very shortly by a student. The teacher seems to accept the answer and it is not discussed any further. This task is similar to one used in the previous lesson (SH2-L03, 33:10). The question then was, ”which is a system of linear equations in two unknowns?”. In the present episode the question is reversed.

00:10 T: Which of the following is not a linear equation in two unknowns?

The differences between the items open some dimensions of variations that were also opened in the last lesson.

Opened dimensions of variation SH2-L04 [0:00-0:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
3a. xy is not of first degree 5a. Coefficients can be rational numbers not just natural 5b. Constants can be negative numbers not just natural 7c. Both unknowns not present in both equations	–	–

SH2-L04 [0:40-10:45]

In this episode the class works with rearranging equations in two unknowns. This can be viewed as a ’technical’ preparation for the method of substitution, when one of the equations has to be rearranged before it can be used for substitution. In none of the three systems that were solved the previous lesson was this necessary. The tasks are shown on the screen:

1. $x + 4y = -15$	2. $3x - 5y = 0$
3. $3x - 5y = 6$	4. $2x - y = 5$
5. $9x - 13y + 12 = 0$	6. $\frac{x}{5} - \frac{y}{2} = 2$

00:50 T: Use the substitution equation of an unknown to represent the other unknown.

Some students' answers are put on the board. In some cases (item 1, 2, 6) two or three answers are compared. For instance for item 1 two answers are put on the board.

$$y = \frac{-15 - x}{4} \qquad x = -15 - 4y$$

03:00 T: Which one do you think is simpler?
S: The second one.
T: The second one [...] Choose the one which is simpler.

No reason for why the second is simpler is given.

Opened dimensions of variation SH2-L04 [0:40-10:45]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	–

SH2-L04 [10:45-23:35]
The teacher shows a slide with a system of equations for the students to solve.

$$\begin{cases} 2x + 3y = 8 \\ x = 2y - 3 \end{cases}$$

In this example x in the second equation can be substituted directly. The rearranging that was practiced just before is not needed in this example. The teacher walks around. One student is asked to show the solution on the board.

Substitute (2) into (1)

$$\begin{aligned} 2(2y - 3) + 3y &= 8 \\ 4y - 6 + 3y &= 8 \\ 7y &= 14 \\ y &= 2 \end{aligned} \qquad \begin{aligned} x &= 2y - 3 \\ x &= 4 - 3 \\ x &= 1 \end{aligned} \qquad \begin{cases} x = 1 \\ y = 2 \end{cases}$$

The teacher makes just some small remarks.

14:25 T: To solve the equation, substitute equation two into one, we get two brackets, some of the students missed the brackets.

The teacher shows one more system of equations for the students to solve.

$$\begin{cases} x + 2y = 1 \\ 3y - 2x = -9 \end{cases}$$

The teacher walks around. Two students are sent to the board to show the beginning of their solutions. The teacher makes a comparison to the previous example where *x* from the second equation was substituted into the first. This introduces an alternative regarding the choice of 'source' equation. He then notices an error made by one student.

Substitute (1) into (2)

$$3y - 2 \times 2y = -9$$

19:15 T: So, this student substitutes equation one into equation two, three y minus two, so how about x? ... He replaces it with two y, think, from the first equation. What does x equal to?

After some questions from the teacher he gets the answer that the first equation has to be rearranged. This is highlighted by the difference between the two examples. However, the need to rearrange one of the equations is not at this point related to which unknown to choose for the initial substitution.

20:10 S: Substitute x equals one minus two y into it.

T: Oh, x can be replaced by one minus two y, where did you get that?

S: For the first equation, move the items in the first equation, um (...) move the items and you will get what x equals.

The teacher turns to the second student solution on the board and finds the correct expression for *x*.

20:55 T: ... let's look at him, he said that x equals one minus two y.

The teacher then writes a new complete solution on the board while explaining the different steps in the solution.

From (1) get $x = 1 - 2y$ (3)

Substitute (3) into (2)

$$3y - 2(1 - 2y) = -9$$
$$3y - 2 + 4y = -9$$
$$y = -1$$

There is a discussion of which is more convenient, to substitute $y = -1$ into (1), $x + 2y = 1$, or (3), $x = 1 - 2y$.

22:15 S: Substitute 'y equals negative one' into equation one.
T: Into equations one?
S: Equations three.

[...]

T: Students think about it, for y equals negative one, to find x, is it more convenient to substitute into equation one or three?
S: Three. [The teacher writes on the board]

> Substitute $y = -1$ into (3) get $x = 3$
> Solution to the systems of equations is $\begin{cases} x = 3 \\ y = -1 \end{cases}$

The two systems of equations and the solutions vary in some respects. They are the same in other respects, the same letters x, y are used for unknowns and the same unknown (x) is used for the substitution.

Opened dimensions of variation SH2-L04 [10:45-23:35]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5b. Constants can be negative numbers not just natural	3a. Solutions can be negative numbers	4a. The selection of 'source' equation selected is not taken for granted 5a. 'Direct' substitution is not taken for granted 7a. The equation for second substitution is not taken for granted 7b. Substitution of first unknown into equation (3) often most convenient

SH2-L04 [23:35-29:00]

The teacher asks the students about the steps in solving systems of equations. Based on the solution that is still on the board they discuss what is the first step.

24:15 S: The first step is to change the format.

[...]

T: ... to choose one of the unknowns [...] Among these two equations, which one is more suitable to be changed in format? [points at (1) and (2) on the board]

S: The first one. [(1) is the equation x + 2y = 1]

T: That's because the first equation is simpler.

After a short while (24:55) the teacher shows a slide with five steps listed. He reveals one step at the time after getting an acceptable answer from the students. The steps are read aloud together as they are shown.

1. Choose which equation is to be reformatted and use the substitution equation of an unknown to represent another unknown, so which equation is chosen depends on which is simpler.
2. Substitution. Change the equations into an equation in one unknown.
3. Solve the linear equation in one unknown.
4. Find the value of the other unknown.
5. Write down the solutions.

Each step is discussed and the teacher points to the main idea.

27:40 T: The basic principle of elimination by substitution is to convert the two unknowns into what?

S: One unknown. [the teacher writes "2 unknowns ⇒ 1 unknown"]

Opened dimensions of variation SH2-L04 [23:35-29:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	2a. Getting an equation in one unknown is not taken for granted 4a. The selection of 'source' equation is not taken for granted

SH2-L04 [29:00-40:40]
The teacher tells the students to solve the system of equations shown on the screen.

$$\begin{cases} 3x - 4y = 18 \\ 5x + 2y = 4 \end{cases}$$

29:00 T: Use substitution to solve the system of linear equations.

The teacher walks around, looks in the students' notebooks and talks to individual students. After a couple of minutes four students are sent to the board to write down their solutions. They show four different ways to start the solution. These are discussed in whole class.

$$x = \frac{18 + 4y}{3} \qquad y = \frac{3x - 18}{4} \qquad x = \frac{4 - 2y}{5}$$

The teacher comments the start of the first solution on the board.

37:00 T: From equation one, he got x equals four y plus eighteen over three.

Start of the second solution.

37:50 T: So, for this student, he also changed the first equation, from that he got y equals three x minus eighteen over four.

Start of the third solutionon.

38:30 T: For this student, he chose the second equation. From that he gets x equals four minus two y over five, right? Right. y equals [negative] three, x equals two, that's right too.

The teacher points to the fourth solution on the board and re-writes parts of it with larger symbols.

$$2y = 4 - 5x$$
$$3x - 2 \times 2y = 18$$
$$3x - 2(4 - 5x) = 18$$

39:10 T: Look at what she did, from the second equation she got two y equals four minus five x.

[...]

39:30 T: She changed four y into two and two y, is that okay? She changed it into two and two y, so what does two y equal? Four minus five x, see that? She got three x minus two times four minus five x [...] She

> did so well, she avoided doing many other steps. She avoided doing fraction calculations [...] well done, please clap for her. [translation: he/him is changed to she/her]

The teacher and the whole class then applaud the good solution.

The teacher looks carefully in the students notebooks before they are sent to the board. This could indicate that it is no coincidence that four *different* solutions are displayed on the board at the same time. All four different ways of solving the system of equations also obtain the same values of the unknowns. The teacher however, does not take the opportunity to highlight that all answers are identical regardless of the way the method of substitution is executed. The teacher just, mentions that the third solution is ”right too”.

Opened dimensions of variation SH2-L04 [29:00-40:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	3a. The unknowns can be negative numbers	3a. The selection of unknown is not taken for granted 3c. Substitution of larger expressions (*ax*, *by*+*c*) is possible 4a. The selection of ’source’ equation is not taken for granted

SH2-L04 [40:40-45:40]

The students work on a new task that is shown on the screen.

$$\begin{cases} 2x - y = 5 \\ 3x + 4y = 2 \end{cases}$$

After a couple of minutes two students are sent to the board. The teacher interrupts the work because the lesson is about to end. They just compare the beginnings of the two incomplete solutions on the board.

$$y = 2x - 5 \qquad\qquad x = \frac{5 + y}{2}$$

43:50 T: Both of them adopted the first equation [...] which one is more convenient to be substituted into equation two?

S: The first one.

The teacher quickly summarizes the lesson and assigns some task for homework.

44:40 T: Okay, we've talked about the first way to solve a system of linear equations in two unknowns, elimination by substitution.

This last episode is another example of how the teacher deliberately lets students, with solutions that differ in some aspects, display them on the board. In this case it is the aspect 'both unknowns can be selected for the first substitution'. Any unknown can be used even though they might not be equally 'convenient' in terms of the complexity of the following calculations.

Opened dimensions of variation SH2-L04 [40:40-45:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	3a. The selection of unknown is not taken for granted 3b. Calculation complexity depends on the unknown selected

There is little variation in the format of the equations used in this classroom. In most cases the equations have a (numerical) constant term in the right hand side. The unknowns are mostly in the left hand side. No letters other than x and y are used to denote unknowns.

Shanghai 3

This grade eight class consists of 55 students. It is a mixed ability class (TQ1). The teaching in this class looks very similar to the teaching in the other Shanghai classrooms. The students work with the same tasks and problems in a cyclic shift between teacher led whole-class discussion and shorter periods of student-work, individual or in pairs. Typically there will be 2–5 episodes with student work, 2–7 minutes in length, in a lesson, during which the teacher walks around and talks to students at their desks.

Overview of the lessons in Shanghai 3

The two lessons analysed are no. 5 and 6. The lessons take place the same day, one in the morning and one in the afternoon.

SH3-L05
The concept of *system of linear equations in two unknowns* is introduced by a problem about buying 1- and 2-dollar stamps. The class elaborate on the properties of the concept.

They continue and explore the meaning of a solution to a system of equations. Questions such as ”which of (a, b) and (c, d) are the solution to this system of equations and why?” and ”design a system of equations with the solution (a, b)” are discussed.

SH3-L06
This lesson starts with a short revision where the teacher reminds the students of how they solved the stamp problem by listing numbers in tables. Then the *method of substitution* is introduced.

Three systems are solved on the board and the method is described in 5 steps. The class works on two more tasks. Two different student solutions to the first task are shown and compared.

Shanghai 3 – Lesson 5

SH3-L05 [4:30-10:50] [*note*: the lesson starts at 04:30]
The teacher shows a problem on the screen. It is the same problem that was used earlier in this class (in lesson SH3-L01) to introduce the concept of a *single* linear equation in two unknowns. The teacher revises how they put up an equation and found four solutions.

1. Wang goes to the post office to buy a number of one-dollar and two-dollar stamps. At least one of each type of stamps will be bought. The total amount he spends on buying them is ten dollars. How many stamps of each type does Wang get from the postman?

   Let the number of 2-dollar stamps be x, and the number of 1-dollar stamps be y.

   $2x + y = 10$

x	1	2	3	4
y	8	6	4	2

   $\begin{cases} x = 1 \\ y = 8 \end{cases}$ $\begin{cases} x = 2 \\ y = 6 \end{cases}$ $\begin{cases} x = 3 \\ y = 4 \end{cases}$ $\begin{cases} x = 4 \\ y = 2 \end{cases}$

05:30 T: When we were solving this equation, we used the table method.

[...]

06:15 T: Then, what if one more condition is added into this question, that is, if the number of these two types of stamp are seven [...] what equation can we get?

This second equation is also solved by listing number pairs in a table.

$x + y = 7$

$\begin{cases} x = 1 \\ y = 6 \end{cases}$ $\begin{cases} x = 2 \\ y = 5 \end{cases}$ $\begin{cases} x = 3 \\ y = 4 \end{cases}$

$\begin{cases} x = 4 \\ y = 3 \end{cases}$ $\begin{cases} x = 5 \\ y = 2 \end{cases}$ $\begin{cases} x = 6 \\ y = 1 \end{cases}$

x	1	2	3	4	5	6
y	6	5	4	3	2	1

09:05 T: The values of unknowns x and y not only need to satisfy the first equation, but also need to satisfy the second equation.

The next picture shows the two equations together and this example is used to introduce the concept of system of equations and some properties are discussed. The actual number pair that solves both equations is not specifically identified at this point.

$$\begin{cases} 2x + y = 10 \\ x + y = 7 \end{cases}$$

09:10 T: So, we gather these two equations into one place and use a big bracket to group them together. It is called a system of equations ... that means, if there are some equations formed into one group, we will call it a system of equations ... Please observe the system of equations here, how many unknowns are there?

S: Two.

[...]

10:15 T: And the index of the unknown is one [...] That means, if there are two unknowns within a system of equations, and the indices of the unknowns are one, then we will call this system of equations a system of linear equations in two unknowns.

The original problem has many solutions. By introducing a second condition the number of solutions changes from many to just one. This common solution is however not identified, but will be when they return to the problem in a few minutes. When the teacher writes the two equations together he introduces an alternative to regarding them one at a time. This opens the DoV 'number of equations'.

Opened dimensions of variation SH3-L05 [4:30-10:50]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
1a. Two equations, not one	–	–

SH3-L05 [10:50-16:20]

The teacher shows five possible systems of linear equations in two unknowns and they discuss them in whole-class.

1) $\begin{cases} x - 2y = 1 \\ 3x + y = 5 \end{cases}$ 2) $\begin{cases} x = 2 + y \\ \frac{1}{2}x - 3y = 8 \end{cases}$

3) $\begin{cases} x + 8 = 4 \\ 5x - 7y = -2 \end{cases}$ 4) $\begin{cases} xy = 1 \\ 3x + 2y = 1 \end{cases}$ 5) $\begin{cases} x + 2y = 4 \\ x - 2z = 3 \end{cases}$

10:50 T: Then, please help me to find out whether the given system of equations is a system of linear equations in two unknowns and give me a brief explanation for your answer.

There are different opinions about item 3 in the class. The teacher asks for reasons.

11:35 T: So, what about the third pair?

S: No.

T: No?

S: Yes.

T: Yes or no? [...] please say your reason.

S: It is because the first equation of this pair of equations is not a linear equation with two unknowns.

[...]

12:30 T: Then, now please tell me the reason for yes?

[...]

S: It is because there are still two unknowns in the second equation.

[...]

13:10 T: What is the definition ... that a pair of equations should contain two unknowns ... this pair, we are not talking about each equation. Is it? So, you see, within this system of equations, the first equation has one unknown, but the second equation has two unknowns. There are still two unknowns, x and y, in this pair of equations.

The differences between five examples displayed on the screen show that not all systems of equations are system of linear equations in two unknowns.

Opened dimensions of variation SH3-L05 [10:50-16:20]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
2b. Two unknowns not three 3a. *xy* is not of first degree 5a. Coefficients can be rational numbers not just natural 5b. Constants can be negative numbers not just natural 7c. Both unknowns not present in both equations	–	–

SH3-L05 [16:20-18:25]

They return to the stamp-problem. The problem text, the system of equations and the four plus six solutions are again shown on the screen. This time the pair of numbers that satisfy both equations are identified.

16:50 T: Therefore, according to this question, the values of these unknowns x, y, not only need to satisfy the first equation 'x plus two y equals ten' ... but also need to satisfy the second equation 'x plus y equals seven' [...] then please find out whether there is this kind of solution?

S: Yes, there is.

[...]

17:45 T: We will say x is three, y is four, is the solution of this system of equations [...] The solution which can satisfy each of the equations is called the solution of this system of linear equations with two unknowns. [shows the next slide, se next page]

The unknowns have many different values when the equations are considered separately, but when the two equations are considered at the same time there is just one number pair that works. This variation makes it possible to discern that a solution must satisfy both equations in a system.

$$\begin{cases} 2x + y = 10 \\ x + y = 7 \end{cases}$$

Solution: $\begin{cases} x = 3 \\ y = 4 \end{cases}$

Opened dimensions of variation SH3-L05 [16:20-18:25]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	1a. Same solution to both equations is not taken for granted 2a. One equation has many solutions/no unique solution 2b. Two equations reduce the number of solutions to one	–

SH3-L05 [18:25-39:10]

The meaning of a solution is elaborated on by some tasks shown on next slides.

Examine the values of the following unknowns.

(1) $\begin{cases} x = -2 \\ y = 8 \end{cases}$ (2) $\begin{cases} x = 2 \\ y = 4 \end{cases}$

Which of (1) and (2) are a solution to the equation system $\begin{cases} x + y = 6 \\ 2x + y = 8 \end{cases}$?

[translation changed from lesson-table]

19:00 T: How to check whether this ordered pair is the solution of these equations?

Both number pairs satisfy the first equation but only the second pair of numbers satisfies the second equation.

The first pair of numbers is substituted into the equations. The teacher writes on the board. They find that it is not a solution since the values of the left- and right-hand sides of the second equation are not equal.

24:20 T: The first one x is negative two, y is eight satisfies the first equation, but it cannot satisfy the second equation [...] It is not the solution of this system of equation, isn't it?

Then the students check the second number pair while the teacher walks around for a couple of minutes. A correct answer from a student's notebook is shown on the screen. The next task is 'reversed'.

Among the equation systems

$$1)\ \begin{cases} 3x + y = 0 & (1) \\ x - 2y = -\frac{14}{3} & (2) \end{cases}$$

$$2)\ \begin{cases} 3x - y = -4 & (1) \\ 3x + 10y = 14 & (2) \end{cases}$$

Which one has the solution $\begin{cases} x = -\frac{2}{3} \\ y = 2 \end{cases}$?

[translation changed from lesson-table]

The students work on the task and the teacher walks around. After a couple of minutes, again, a correct solution by one student is projected onto the screen and discussed in whole-class. The number pair satisfies equation (1) in both systems but equation (2) only in the first case.

All equations in this episode are in the same format, with a numeric constant in the right side.

Opened dimensions of variation SH3-L05 [18:25-39:10]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5a. Rational numbers not just natural 5b. Negative numbers not just natural	1a. Same solution to both equations is not taken for granted 1b. Substitution of solution will get equal sides in both equations 3a. The unknowns can be negative numbers 3b. The unknowns can be rational numbers	–

SH3-L05 [39:10-45:00]
The teacher shows the next task on the screen.

> Design a system of linear equations in two unknowns with the solution: $\begin{cases} x = 1 \\ y = 2 \end{cases}$

39:35 T: To let you design a linear equation in two unknowns, but the requirement is, x is one and y is two should be the solution of this pair of equations.

The students work for a couple of minutes and the teacher walks around. Four solutions are written on the board and checked.

> 1) $\begin{cases} x + 5 = 6 \\ x + 2y = 5 \end{cases}$ 2) $\begin{cases} 3x + \frac{y}{2} = 4 \\ 2x - 5y = -8 \end{cases}$
>
> 3) $\begin{cases} 2x + y = 4 \\ x + y = 3 \end{cases}$ 4) $\begin{cases} x + 3y = 7 \\ 3x + 5y = 13 \end{cases}$

43:15 S: x plus three y equals seven; three x plus five y equals thirteen.
T: Freda, is he correct?
S: Correct.

The equations that the students come up with are all in the same general format where the right hand-side consists of just a constant numerical term. In all examples so far, with just one exception, the equations have been in this format.

In this episode the solution, the values of *x* and *y*, are kept invariant, while there are four different systems of equations shown on the board at the same time. It would make it possible to discern that different systems of equations can have the same solution.

The teacher then summarises and ends the lesson.

44:10 T: Today we have learnt the concepts of a system of linear equations in two unknowns. That means, when a pair of equations contains two unknowns and the indexes of the unknowns are one, then we will call it a system of linear equations in two unknowns. The second thing is, the concept of solution for a system of equations. That means, if a solution is suitable for both the two equations of a pair of equations, we will call it ... the solution of this system of linear equations in two unknowns.

Opened dimensions of variation SH3-L05 [39:10-45:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
5a. Rational numbers not just natural 5b. Negative numbers not just natural	5a. Different systems with the same solution is possible	–

Shanghai 3 – Lesson 6

SH3-L06 [0:00-02:00]

The lesson starts with a short revision of the lesson in the morning. The teacher once again shows the system of equations from the stamp-problem and how the solution was found by the use of tables.

$$\begin{cases} 2x + y = 10 \\ x + y = 7 \end{cases}$$

$2x + y = 10$

$$\begin{array}{c|c|c|c|c} x & 1 & 2 & 3 & 4 \\ \hline y & 8 & 6 & 4 & 2 \end{array}$$

$x + y = 7$

$$\begin{array}{c|c|c|c|c|c|c} x & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline y & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

00:05 T: Last lesson we've learnt about the concepts of linear equations in two unknowns and the solutions of the linear equations in two unknowns.

[...]

01:20 T: We say that the solution of the set of equations is x equals three, y equals four, through using the table to list the solutions of the set of equations.

Opened dimensions of variation SH3-L06 [0:00-02:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	1a. Same solution to both equations is not taken for granted 2a. One equation has many solutions/no unique solution 2b. Two equations reduce the number of solutions to one	–

The tables provide alternative values of *x* and *y*, while the number pair that's the same in both tables is singled out as the solution.

SH3-L06 [02:00-07:35]

The teacher shows a system of linear equations in two unknowns on the screen and asks the students to discuss in groups of four how to find the solution. The first equation is not in the 'standard format'.

02:00 T: So using the table is the only way to find out the solutions of the set of equations? Students please think about how to solve the set of equations 'y equals two x' and 'x plus y equals twelve'.

$$\begin{cases} y = 2x & (1) \\ x + y = 12 & (2) \end{cases}$$

04:50 S: Firstly we use y equals two x to substitute into equation two, finding out that x equals four. Then use y to represent x, x equals y over two and substitute that into equation two, finding out y equals eight.

The way to solve the system of equations proposed by the student is not quite the same as the one the teacher then shows on the screen. The power point slide has been prepared in advance.

Insert (1) into (2), get $x + 2x = 12$

$$3x = 12$$

$$x = 4$$

Insert $x = 4$ into (1), get $y = 8$

The solution to the system of equations is $\begin{cases} x = 4 \\ y = 8 \end{cases}$

07:05 T: When the first equation is substituted into the second one, the second equation becomes a linear equation in one unknown. [points at screen with a laser pen]

There is a contrast, between the solving by use of tables and solving by substitution, which shows that there is more than one method to find the solution to a system of equations. The teacher also points to the change in the number of unknowns that is caused by the substitution.

SH3-L06 [07:50-17:15]

07:35 T: We are not saying that all of the system of linear equations in two unknowns would have the first equation that is already a substitution equation of one of the unknowns to represent another unknown. So for this equation [shows the next slide] ... what should we do?

Opened dimensions of variation SH3-L06 [0:00-02:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
7a. Format of individual equations not invariant	–	2a. Getting an equation in one unknown is not taken for granted

$$\begin{cases} x + 2y = 1 & (1) \\ 2x - 3y = 9 & (2) \end{cases}$$

The students start to discuss the task. The teacher points to the difference between this example and the previous one. This time it is not possible to do a 'direct substitution'. One of the equations has to be rearranged first. One student is asked to write her solution on the board. The solution is discussed in whole-class and the teacher makes some changes along the way [he adds (3) in the first line and changes (1) to (3) in the second].

From (1) get $x = 1 - 2y$ (3)

Insert (3) into (2), get $2(1 - 2y) - 3y = 9$

$$2 - 4y - 3y = 9$$

$$-7y = 7$$

$$y = -1$$

Insert (2) into (1), get $x + 2(-1) = 1$

$$x = 3$$

Solution to the system of equations is $\begin{cases} x = 3 \\ y = -1 \end{cases}$

15:10 T: We should mark this equation as equation three. [adds (3) on the board]

[...]

15:55 T: We can substitute directly into equation three. [changes the latter part of the solution]

Insert $y = -1$ into (3), get $x = 1 - 2(-1)$

$$x = 3$$

Solution to the system of equations is $\begin{cases} x = 3 \\ y = -1 \end{cases}$

16:40 T: Why do you choose substitution equation of y to represent x?

[...]

16:55 S: If we use substitution equation of x to represent y, it will be more complicated.

T: More complicated, because the coefficient of x is one.

Opened dimensions of variation SH3-L06 [07:50-17:15]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	3a. The unknowns can be negative numbers	3a. The selection of unknown is not taken for granted 3b. Calculation complexity depends on the unknown selected 5a. 'Direct' substitution is not taken for granted

SH3-L06 [17:15-26:00]

The teacher points to another difference concerning the format of a system of equations.

$$\begin{cases} 2x + 3y = 4 & (1) \\ 5x - 2y = 29 & (2) \end{cases}$$

17:15 T: Some equations may not have similar coefficients, for example, see the following equation.

Again the students are given a couple of minutes to solve the system of equations. This time there are no unknown with a coefficient of one, which increases the complexity. After a while the teacher gives some guidance by asking about the choice of 'source' equation. After another minute or so the task is solved on the board. The teacher opens up the dimension of variation regarding the 'source' equation by asking about which equation to use, thus indicating the possible use of both.

22:00 T: How do you consider the first step? [...] Which equation do you use?

S: (...) two x plus three y equals four.

T: Use ... the first equation, right?

[...]

23:25 T: Okay let's do it together, okay?

The teacher writes on the board as they solve the system of equations.

From (1) get $x = \dfrac{4-3y}{2}$ (3)

Insert (3) into (2), get $5 \cdot \dfrac{4-3y}{2} - 2y = 29$

$$5(4-3y) - 4y = 58$$
$$20 - 15y - 4y = 58$$
$$-19y = 38$$
$$y = -2$$

Insert $y = -2$ into (3), get $x = 5$

The solution to the system of equations is $\begin{cases} x = 5 \\ y = -2 \end{cases}$

The teacher mentions that there are alternatives regarding the choice of unknown.

23:55 T: Of course, we can also use substitution equation of x to represent y, right?

In all three systems of equations solved so far, the first equation is used for the initial substitution. The difference between them lies in the need for rearrangement of the equation used for substitution. In the first example, one equation can be used without changes. In the second case one of the unknowns has the coefficient of one and is rearranged in one step. The third example could be regarded as more complicated as the solution involves expressions with fractions.

Opened dimensions of variation SH3-L06 [17:15-26:00]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	3a. The unknowns can be negative numbers	3a. The selection of unknown is not taken for granted 4a. The selection of 'source' equation is not taken for granted

SH3-L06 [26:00-32:40]

The last two solutions are kept side by side on the board. The teacher asks the students to discuss how to summarise the solving procedure in a number of steps.

26:00 T: For these two questions, can you summarize, when solving the linear equations in two unknowns for the two questions, what are the steps? [...] okay, discuss about it.

[...]

27:50 T: Okay, let's exchange our ideas [...] What do you think the steps are? Oh, this Felix.

S: (...)

T: Firstly use x to represent y, or use y to represent x.

Two students describe the steps [the transcripts are almost void here because the sound is not good enough]. Then the teacher turns to the two solutions still on the board. In both these examples, the *x* in equation (1) is used for the initial substitution. There is no variation in either the use of unknown or equation. However, the teacher provides alternatives regarding the choice of unknown for the initial substitution.

29:45 T: So, let's see, compare these two questions, the first step is to change one of the equations into substitution equation of x, or use substitution equation of y to represent the other unknown. Then, after changing the format, we substitute the equation into another equation. After substituting, students please check what kind of equation is that?

S: Linear equation in one unknown.

T: Linear equation in one unknown, everyone thinks it is a linear equation in one unknown? Then, by solving the equation, we can get the value of the one of the unknowns, then substitute this value into the equation we formatted as the first step, we can get the value of the other unknowns. Finally, as Eliza says, write down the conclusion.

The teacher shows and comments a slide that summarises the method in five steps.

31:00 T: The first step, change one of the equations into the substitution equation of one of the unknowns to represent another unknown. Secondly, use this substitution equation to replace the related unknown in another equation, making the linear equation in two unknowns into a linear equation in one unknown. Thirdly, find the solution of this linear equation in one unknown, finding the value of one of the unknowns, substituting this value into the equation we have got in the first step, to get the value of another unknown. Finally, Eliza says, use a big bracket to indicate the values of the two unknowns, getting the solutions of the equations.

32:20 T: Using such a method to solve the linear equation in two unknowns, we call it the method of elimination, we can also call it the method of substitution. Using this, we can eliminate the unknowns, changing the linear equation in two unknowns into linear equation in one unknown. Two unknowns become one unknown

The second step in the list and the teacher both point to the change in the number of unknowns when the method of substitution is performed – the initial substitution will change the number of unknowns from two to one.

Opened dimensions of variation SH3-L06 [26:00-32:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
–	–	2a. Getting an equation in one unknown is not taken for granted 3a. The selection of unknown is not taken for granted

SH3-L06 [32:40-43:40]

Two more systems of linear equations are shown on the screen.

$$1)\ \begin{cases} 3x+2y=13 & (1) \\ x-y+3=0 & (2) \end{cases} \qquad 2)\ \begin{cases} 3a+4b=0 & (1) \\ 2a-2b=7 & (2) \end{cases}$$

32:45 T: ... students please look, using the substitution method to solve the two equations.

The students work and the teacher walks around for a couple of minutes. The teacher talks to students and looks at their solutions. He chooses, seemingly with care, which students' notebooks to show.

39:15 T: Let's look at the first one.

The first student has chosen to start with x from equation (1). Both students have come to the correct answer, but the solving procedures differ in the beginning.

[Student 1]	From (1) get	$x = \frac{13 - 2y}{3}$	(3)
[Student 2]	From (2) get	$x = y - 3$	(3)

40:45 T: Let's compare together, which student's working is more reasonable, the first or the second?

S: The second.

[...]

T: Also we can use substitution equation of x to represent y, right? [...] When, working on the questions we have to choose which equation is to be formatted.

One student solution for item 2 is looked at very briefly before the teacher summarises and ends the lesson.

43:05 T: The basic concepts for the method of elimination is through substitution, changing linear equations in two unknowns into one unknown, achieving the aim of solving the equations.

The letters used for unknowns are varied in the second example, but the same general format is used. The two student solutions shown vary in the choice of equation to begin with. They appear to be chosen by the teacher deliberately to make a point about the difference in computational complexity.

Opened dimensions of variation SH3-L06 [32:40-43:40]

System of linear equations in two unknowns	Solution to a system of linear equations in two unknowns	Method of substitution
6a. The letters x, y are not taken for granted	–	2a. Getting an equation in one unknown is not taken for granted 3a. The selection of unknown is not taken for granted 4a. The selection of 'source' equation is not taken for granted 4b. Calculation complexity depends on the equation selected

Three enacted objects of learning

In this section I summarise the results from the analysis of the sixteen lessons by discussing differences in the enacted objects of learning, one at a time. How the mathematical content was handled in the six classrooms is compared by means of dimensions of variation. The opened DoVs formed a space of learning for each set of lessons.

The system of linear equations in two unknowns

The eight DoVs related to the concept of system of linear equations in two unknowns, which I found were opened in at least one of the analysed lessons, are of three different types. The first category consists of five DoVs related to the 'properties' of the system of linear equations in two unknowns. The second category, with two DoVs, relates to the 'appearance' and the third, with one DoV, relates to the 'use'. Table 5.6 shows the differences in how the concept of the system of linear equations in two unknowns was handled in the six classrooms.

In all six classrooms the DoVs regarding the 'number of equations' and the 'number of unknowns' were opened. The students were given the possibility to experience that a system of equations in two unknowns is different from a single equation in one unknown. The alternatives provided were just slightly different.

The third DoV concerns the 'type of equations'. In SW2 the type of equations was completely taken for granted. The term linear was never even used. In SW2 the vocabulary 'system of equations' was used. It was never specified as for instance 'simultaneous first degree equations in two unknowns' or 'system of linear equations in two unknowns'. The SW2-vocabulary did not in any way indicate that there are different kinds of systems of equations. The system of linear equations in two unknowns was simply treated as *the* (only) system of equations.

In the two HK-classrooms the type of equations were called linear or of the first degree at some occasions. The students were however not provided with any variation in this respect. It was not made possible for the students to discern any specific properties of a linear equation and there were no examples given of what a linear equation is *not*.

In HK1 the notion of first degree was discussed when revising linear equations in one unknown in the beginning of HK1-L06, before the introduction of systems of equations. The teacher wrote the digit 1 as exponent to the unknown r in the equation $r + 9 = 107$ on the board. It was removed immediately and the teacher stated that the index of one would not be written explicitly. The revision was concluded and the teacher used index of one to explain the notion of linear.

(HK1-L06)

08:00 T: The linear equation in one 'yuan' means there is one unknown with the index of one. We call it linear equation in one unknown.

In the beginning of the next lesson (HK1-L07) the teacher described the equations by using "one, two, three". He said (01:55): "Now I will explain it by *one*, *two*, *three*". The *one* was meant to indicate that unknowns have to be of degree one or first degree. In the four lessons from HK1 included in this study there were at no point any alternatives given to the index of one, no variation was provided. The teacher might be referring to index of one as something that already was, or ought to be, familiar to the students. However, in the present analysis, only what occured in the studied lessons can be regarded. All equations dealt with in the actual lessons were linear and the lack of variation would make it really hard to discern any significant features of linear from what took place in these four lessons. The difficulty to pick up the meaning of linear or first degree for students was manifested in the beginning of HK1-L08. The lesson started with a revision and the teacher repeatedly posed a question about the meaning of linear. The questions were never answered by any of the students. Nor did the teacher, who eventually got distracted and moved on to another question, answer it.

In HK2 the term linear was used three times in the first lesson (HK2-L01) when the system of equations was introduced. After having examined the two equations, $2x + y = 1$ and $x + 3y = -2$, separately the teacher made a brief comment before putting them together into a system of equations.

(HK2-L01)

14:40 T: They are called linear equation in two 'yuan' ... two elements and degree one. We know that there are indefinitely many solutions.

The focus here was on the number of solutions rather than on any other feature of the equations. There was no effort to elaborate on the meaning of linear or degree one from the teacher. In the following two lessons neither linear nor degree one were used or even mentioned. The systems dealt with were referred to as just simultaneous equations without any further reference to linear or degree one.

There were noticeable differences between the Shanghai classrooms and the other classrooms in the DoV regarding the type of equations. In all three Shanghai classrooms the students discussed, more or less thoroughly, the properties of the system of linear equations in two unknowns. In all three classrooms the students were given a number of potential systems of equations where the task was to determine whether the examples met the requirements or not. The variation, created by the comparison of the examples and counterexamples, would highlight a number of properties of the concept. The contrasts would not only bring forward that there can be different types of systems of equations, but also many of the other aspects regarding the concept. The most powerful pattern of variation was created in SH1 where eight proposed systems of linear equations in two unknowns were compared. Three of the items provided alternatives to the linear equations and exemplified what a linear equation might not be.

Table 5.6 *Spaces of learning for the concept of system of equations*

	SW2	HK1	HK2	SH1	SH2	SH3
DoVs concerning 'properties'						
1 Number of equations (two or more equations)						
1a. Two equations, not one	x	x	x	x	x	x
1b. More than one equation		x		x		
2 Number of unknowns (two unknowns)						
2a. Two unknowns not one	x	x	x			
2b. Two unknowns not three (or more)		x		x	x	x
3 Type of equations (linear equations,first degree unknowns)						
3a. xy is not first degree				x	x	x
3b. x^2 , $(x+y)^2$, is not first degree				x		
3c. $1/y$ is not first degree				x		
4 An unknown represents the same number in both equations						
4a. 'Unknowns are the same' is not taken for granted		x	x	x	x	
5 Constants and coefficients can be different types of numbers						
5a. Rational numbers not just natural	x	x		x	x	x
5b. Negative numbers not just natural	x	x	x	x	x	x
5c. Parameters not just specified numbers				x		
DoVs concerning 'appearance'						
6 Different letters may be used						
6a. The letters x, y are not taken for granted		x	x	x		x
7 A system of equations can be in different formats						
7a. Format of individual equations not invariant	x	x	x	x	x	x
7b. One expression		x	x	x		
7c. Both unknowns not present in both equations	x			x	x	x
DoV concerning 'use'						
8 A way of representing and solving problems						
8a. Alternative to use of equation in one unknown		x				
8b. Alternative to use of guess-and-check method		x				

Note. A grey space indicates that the dimension was opened for variation, and x indicates which alternatives were provided

Also in SH2 and SH3 similar tasks were used, but these only contained four or five examples. These tasks were also discussed, but not as thoroughly as in SH1. There were anyhow counterexamples showing non-linear equations that would provide alternatives to the linear equations, and thus open the DoV regarding the type of equations.

The fourth DoV, 'an unknown represents the same number in both equations', must be quite obvious to everyone familiar with the concept of systems of equations. It is manifested in many ways, for instance by the simultaneousness of the equations, the values of the unknowns that form the solution, and in the method of substitution where an unknown in one equation is exchanged for an expression of the same unknown taken from the other equation. The idea that unknowns represent the same number in both equations is certainly central to the concept of system of equations. Perhaps this circumstance also makes it easy for an expert, like a mathematics teacher, to take this aspect for granted?

If 'an unknown represents the same number' is obvious to someone familiar with systems of equations it may be the opposite for a novice in this area. If you have never experienced anything but single equations you may just as well take the opposite for granted, i.e. that x in one equation is different from x in another equation. Why would the two x:es in the two equations in a system be the same, when this was hardly ever the case before? The findings from the study by Vaiyavutjamai, Ellerton and Clements (2005) that students were not sure if the x:es in different positions in a quadratic equation really represented the same numbers supports this line of reasoning (see chapter 2).

'An unknown represents the same number in both equations' was taken for granted in two of the classrooms (SW2 and SH3). The teachers of course, treated the unknowns as being the same but this was never commented or elaborated on in a way that the dimension of variation was opened in these classrooms.

The other teachers did not take this for granted. In HK1 the teacher at one occasion (HK1-L07, 20:40) 'squared' the x:es in both equation (1) and equation (2) and made an explicit comparison and stated, "this x and this x are the same". The following lesson (HK1-L08, 08:40) the teacher once again created a pattern of variation by pointing first to y in one of the equations and then to y in the other, and stating that they nevertheless are the same – the y:s are the same even though they are situated in different equations.

The teachers in HK2, SH1 and SH2 acted similarly on one or more occasions. They indicated one unknown, first in one equation, and then by moving the hand they indicated the same unknown in the other equation. At the same time the teachers stated that the unknowns are the same, or asked the students whether the two unknowns are the same or not. In this way the DoV was opened.

The range of numbers that can be used in formulating the equations in a system was in all classrooms expanded beyond just natural numbers. Both negative whole numbers (integers), as well as positive and negative rational numbers were used as coefficients and/or constant terms in the equations. There was just one exception. In HK2 there were no rational numbers as coefficients or constants in any of the examples. SH1 was the only classroom where the DoV regarding the coefficients and constants went beyond specified numbers. The use of parameters as coefficients and constants introduced a pattern of variation that expanded the range in this dimension.

The next two DoVs relate to the 'appearance' of the system of equations. The first concerns the letters used to denote the unknowns. The most frequent letters were of course x and y. They were used in all six classrooms. In two of the classrooms no other letters were used in any examples or tasks, nor were the teachers in any way indicating that there might be alternatives to x and y. There was no variation regarding the letters used in SW2 and SH2. Thus, the DoV 'different letters can be used' was not opened in these two classrooms.

The second DoV related to 'appearance' concerns the format of the equations. Students in all classrooms were given opportunities to discern that 'a system of equations can be in different formats'. In all classrooms the format of the individual equations were varied. There was an interesting difference, however, between the most common format in SW2 and the other classes. In almost all cases in SW2 one of the equations were expressed as $y = f(x)$ or $x = f(y)$. This meant that this equation could be used for the initial substitution directly. In the other classes most equations were expressed as $ax \pm by = c$, which in some cases were referred to as the 'standard format'.

In three of the classrooms (HK1, HK2 and SH1) there was at least one example where the two equations were written together in one expression, $ax \pm by = cx \pm dy = e$.

In four classrooms (SW1, SH1, SH2 and SH3) there was at least one example where one of the equations contained just one of the unknowns. In SW2 there was one such example among the tasks in the textbook. In this classroom this was never discussed in whole class. It could of course be possible for students to discern the aspect just by working in the textbook. In the three Shanghai classes, on the other hand, this aspect was both highlighted by examples and by discussions in whole class. In the Hong Kong lessons it was taken for granted that both equations contain both unknowns.

There was only one classroom where the use of systems of equations in problem solving was explicitly brought forward in a pattern of variation. The introduction of the concept in HK1 was done by solving the same problem in three different ways. The use of a system of equations in two unknowns was shown as an alternative to two other methods. This would open the DoV 'systems of equations as a way to represent and solve problems'. In the other classrooms this aspect was taken for granted. Also in SW2 and SH1 problems were used in the introduction of systems of equations in two unknowns, but the use of systems of equations as a way of solving problem was just taken for granted. In the other classrooms problem solving by systems of equations was never even touched upon. In all six classes problem solving by systems of equations was regarded as a content of its own and it would be handled after the systems of equations and the methods for solving had been covered. This is shown in table 4.1, where 'problem solving with systems of equations' appears in many of the last recorded lessons. Also in SW2 'problem solving with systems of equations' was treated as a particular content, even though it is not visible in the table. There is a section dealing with problem solving in the SW2 textbook at the end of the chapter on systems of equations.

Solution to a system of linear equations in two unknowns

I have found six different DoVs related to the concept of 'solution to a system of linear equations in two unknowns' in the analysed lessons (see table 5.7).

The first DoV – the same values of the two unknowns satisfy both equations simultaneously – concerns a fundamental property of the solution. It is closely connected to the fourth DoV related to the concept of system of equations – the unknowns represent the same number in both equations – and it may be hard to distinguish between them. However, the distinction made here is that the first one relates to the *solution* and the second one to the *system of equations*. The DoV 'the same values of the two unknowns satisfy both equations simultaneously' was opened in all classrooms.

In SW2 there was some confusion initially when the teacher aborted the first demonstration of how to solve a system of equations when the value of *one* unknown had been found. The teacher clearly took for granted that a solution contains the values of both unknowns and that these values should satisfy both equations simultaneously. The reason for the unfinished solution was probably the unfortunate circumstance that the two unknown numbers used to formulate the systems of equations were revealed. The teacher formed the equations from two 'secret' numbers. A student 'thought of a number' and the teacher provided the other. Unfortunately another student happened to notice which the numbers were and, sort of, spoiled the teacher's plan. The values of the unknowns supposed to be gained by the process of solving were thus already known and this might cause the teacher to think that there were no reasons to continue and finish the solving. As a probable consequence of the unfinished solving there was at least one student who showed uncertainty about the meaning of solving a system of equations and asked the teacher if she was supposed to calculate *both* values. The obvious thing to someone familiar with this mathematical content – a solution to a system of equations in *two* unknowns must include the values to *both* unknowns – was apparently not that obvious to a novice. In the next system of equations that was solved on the board in SW2 both values were included in the solution. *How* the second value was found, however, was not shown until the teacher discussed the third example on the board.

Table 5.7 *Spaces of learning for the solution to a system of equations*

	SW2	HK1	HK2	SH1	SH2	SH3
1 Same values of the unknowns satisfy both equations simultaneously						
1a Same solution to both equations is not taken for granted		x	x	x	x	x
1b Substitution of solution will get equal sides in both equations	x	x		x		x
2 Number of solutions						
2a One equation has many solutions / no unique solution	x	x	x	x	x	x
2b Two equations reduce number of solutions to one		x	x	x	x	x
2c One unique solution to a system not taken for granted				x		
3 Types of numbers						
3a The unknowns can be negative numbers	x	x	x	x	x	x
3b The unknowns can be rational numbers	x	x		x		x
4 Two different unknowns (x, y) can represent the same number ($x=y$)						
4a Solution with two equal numbers possible	x	x	x			
5 The same values of the unknowns can satisfy different systems						
5a Different systems with the same solution possible		x	x	x		x
6 The solution is independent of the process of solving						
6a Solution independent is not taken for granted		x	x			

Note. A grey space indicates that the dimension was opened for variation, and x indicates which alternatives were provided

The next DoV is 'the number of solutions to a system of linear equations in two unknowns'. This aspect was elaborated on in all classrooms, but in quite different ways. However, in all classrooms the number of solutions to one single equation was related to the number of solutions to a system with two equations. Equations have many solutions when considered separately in contrast to the single solution when two equations are considered together. This is however not really mathematically correct (see chapter 2). There are at least three possible situations. A system of equations of this type can have none, one or infinitely many solutions. In all but one classroom it was taken for granted that a system of linear equations in two unknowns always has one unique solution.

In SH1-L05 there was a multiple-choice question about the solutions to a system of equations. Two of the alternatives were A) no solution and B) infinitely many solutions. A student gave the correct answer which in this case was a specific number pair given by alternative D) $x = 19, y = 14$. The first two alternatives were not further commented on and the class moved on to the next question. However, these alternatives opened for possible variation regarding the number of solutions to a system of linear equations in two unknowns.

The third DoV concerns what types of numbers that can form a solution. In all classrooms there was variation in this respect, with just minor differences. Beside natural numbers there were examples where the solution was made up of negative numbers and rational numbers (the latter not in HK2 and SH2).

The last three DoVs are related to what perhaps can be considered as special cases. Nevertheless, these aspects were taken for granted in some classrooms but not in others. The fourth DoV – the two *different* unknowns can be the *same* number – was opened by the use of examples or tasks with a solution where both unknowns are the same number. This was done in SW2, HK1 and HK2.

The fifth DoV – the *same* solution can satisfy *different* systems of equations – was opened in four classrooms. It could be done just briefly, as in HK1-L07 where two systems of equations on a worksheet had the same solution, without any special attention drawn to it. In HK2-L02 two examples discussed in whole class and put on the board at the same time happened to have the same solution. The teacher seemed a bit surprised when he realised this (HK2-L02, 16:45): ”Hum? Why? Are the equations the same?” A student answered: ”They're different but the answers are the same ...”. It was not commented on further. In two of the classrooms in Shanghai, on the other hand, this aspect was elaborated on in a more explicit way. In SH1 and SH3 the students were given the task to make up systems of equations from a given pair of values of the unknowns. The pattern of variation generated by the *same* solution satisfying many *different* systems of equations would bring forward this aspect in a powerful way. In the remaining two classrooms (SW2 and SH2) this aspect was taken for granted.

The last DoV related to the solution of a system of equations is 'the solution is independent of the process of solving'. This aspect was taken for granted in four classrooms. However, in HK1 the teacher actually solved the same system of equations in four different ways on the board and thus created a pattern of variation where the system of equations and the solution were invariant while the method of solving varied. The teacher also pointed out explicitly that the answers would be the same regardless of method used. In HK2 this DoV was opened due to a question from a student. In all other classrooms this aspect was taken for granted.

The method of substitution

Regarding the third enacted object of learning in the analysed lessons, which is related to the understanding of and ability to use the method of substitution, I have found seven DoVs that were opened in at least one of the classrooms (see table 5.8). This object of learning is different from the previous two. The first two objects of learning are related to the understanding of mathematical concepts. This third object of learning relates to a procedure or a method, rather than a concept. The DoVs in this last category thus, are related to what to do and why. The DoVs are of three types. The first category contains one DoV related to the main idea of the method of substitution. The second category concerns the DoVs related to finding the value of the first unknown and the DoV in the last category is related to finding the value of the second unknown.

The main idea of the method of substitution – first find the value of one unknown, then use this to find the value of the second unknown – was of course visible every time the method of substitution was executed. That was, if you were familiar with the method. A novice who never had experienced the use of the method of substitution before, would be less likely to see the main idea behind the different steps, unless attention was directed towards it. Strangely the main idea behind the method of substitution was not getting any particular attention in any of the classrooms. However, there were some instances when teachers appeared to 'incidentally happened to mention' that the main idea is to firstly find the value of one of the unknowns and then secondly the value of the other unknown. The teacher in SW2 at one occasion directed the attention to this aspect when he in SW2-L14 discussed a student's solution on the board. He pointed to the solution on the board, first to $y = 4{,}5$ and then to $x = 10$, and said "... we can calculate, first a value of y and then

Table 5.8 *Spaces of learning for the method of substitution*

	SW2	HK1	HK2	SH1	SH2	SH3
DoV concerning 'main idea'						
1 Find value of one unknown, use this to find value of second unknown						
1a Main idea is not taken for granted	x		x			
DoVs concerning 'finding value of first unknown'						
2 The initial substitution shall generate an equation in one unknown						
2a An equation in one unknown is not taken for granted	x	x	x	x	x	x
3 Any unknown can be used in the initial substitution						
3a The selection of unknown is not taken for granted		x	x	x	x	x
3b Calculation complexity depends on the unknown selected		x	x		x	x
3c Substitution of larger expressions (ax, $by+c$) is possible				x	x	
4 Any equation can be used in the initial substitution						
4a The selection of 'source' equation is not taken for granted		x	x	x	x	x
4b Calculation complexity depends on the equation selected		x				x
5 Rearrangement of 'source' equation sometimes needed						
5a 'Direct' substitution is not taken for granted		x	x		x	x
6 Substitution has to be done in 'the other' equation						
6a The 'target' equation is not taken for granted			x	x		
6b Equation (1) $\Rightarrow$ (3) should not be substituted back into (1)	x	x	x	x		
DoVs concerning 'finding value of second unknown'						
7 Value of first unknown can be substituted into any equation						
7a The equation for second substitution is not taken for granted		x	x		x	
7b Substitution into equation (3) often most convenient		x			x	

Note. A grey space indicates that the dimension was opened for variation, and x indicates which alternatives were provided

a value of x". It was not taken for granted that the two unknowns were calculated in succession.

In HK2-L01 the teacher described the main idea behind different methods for solving systems of equations in two unknowns before showing the method of substitution for the first time.

(HK2-L01)

19:40 T: We want to find ... these three methods help us find one of the unknowns first. If we've found the first unknown we could find the second one.

In all three Shanghai classrooms the method of substitution was summed up and described in a number of steps. When they discussed these steps the main idea would be expressed, but the attention was not really directed to this particular aspect.

The next five DoVs are related to the finding of the first unknown. This step in the procedure of solving involves the exchange of the unknown in one equation for an expression for that unknown generated by the other equation. The point of this is to eliminate one of the unknowns and to get an equation in *one* unknown that can be solved. This is obvious to everyone familiar with this method for solving systems of equations, but not necessarily to a novice. As previously discussed, the fact that something is perfectly clear to the teacher will not guarantee that it will be given any attention in the classroom. However, in this case this aspect was not taken for granted in any of the classrooms. All teachers did highlight at some occasion the change from equations in *two* unknowns to an equation in *one* unknown.

The next two DoVs concern the choice of unknown for the first substitution. This involves two different aspects, 'any unknown can be used' and 'any equation can be used'. These aspects were taken for granted in SW2 but highlighted in different ways in the other classrooms.

In SW2 five different systems of equations were solved on the board. Four of them were discussed in whole class and in every case the choice of unknown and 'source' equation was taken for granted. There were no indications that there could be any alternative ways to begin. One explanation for this may be found in the format of the systems of equations. In four of the five examples on the board and in all examples in the students' textbook, one of the equations was in the format $y = f(x)$ or $x = f(y)$. This could have made the choice of unknown and 'source' equation so completely obvious that the opportunity to discuss alternatives would be overlooked.

In none of the other classrooms was the choice of unknown taken for granted. In HK1, HK2, SH2, and SH3 also the issue of different calculation complexity was raised in connection to which unknown to select and use for the first substitution.

In two classrooms SH1 and SH2 the method was expanded beyond the substitution of a single unknown to the substitution of larger expressions. In SH1 this was done by examples where the same expression could be found in both equations, for instance in a system with the expression $5x + 7y$ in both equations.

SH1-L06 [37:30]

$$\begin{cases} 2x + 3(5x + 7y) = 4 \\ 5x + 7y = 2 \end{cases}$$

In SH2 the teacher noticed that a student had substituted an expression for 2*y* and took the opportunity to highlight this by letting the student display her solution on the board (SH2-L04, 39:30). The discussion of differences of calculation complexity and the possibility to substitute larger expressions would open the DoV 'any unknown can be used in the initial substitution'. The choice of 'source' equation was not taken for granted in any of the Hong Kong and Shanghai classrooms.

The next DoV – rearrangement is sometimes necessary before substitution – is related both to the choice of unknown and the choice of equation as well as to calculation complexity. This aspect was taken for granted in two of the classrooms, but in quite different ways. In SW2 there was only one example where one of the equations could not be used for substitution right away. The necessary rearrangement of the 'source' in this example was not given any attention and was thus taken for granted. In SH1 on the other hand, there was only one example where 'direct' substitution was possible. It was the first example, which was discussed but never noted on the board. All other examples involved rearrangement before the initial substitution. This difference between the first and the other examples was not given any attention and rearrangement of the 'source' equation was taken for granted. There was no variation in this respect regarding the examples that went onto the board. One could perhaps argue that this dimension was opened for variation also in SW2 and SH1, but I mean that the students' attention was not directed to this aspect. At least not as clearly as in the other classrooms. In HK1 this DoV was opened through a sequence of tasks. There was a difference between the first four and the fifth task. However, all students might not reach item 5. In the other three classrooms this DoV was opened more clearly. In HK2 two items were compared. (HK2-L02, 27:20) Teacher: "Different from the former equation, do we need to rearrange the equation?" Also in SH2 two items were compared. There were two different student solutions to the same system of equations on the board side by side. In one of them the student had made an error which could be interpreted as if the aspect concerning the necessity to rearrange the equation was not discerned. Another student explained and the difference between the two solutions created a contrast that opened this dimension of variation. In SH3 a direct comparison of two items was made by the teacher. (SH3-L06, 07:35) Teacher: "[not] all [...] would have [...] a substitution equation of one of the unknowns to represent another unknown. So for this equation ... what should we do?"

The last aspect in this category is 'substitution has to be done into the *other equation*'. Also this must be considered an obvious feature to anyone familiar to the method of substitution. It was taken for granted by teachers in two of the Shanghai classrooms.

In SW2 this DoV was opened due to a poor suggestion from a student while solving a system of equations in whole class. In SW2-L13 the equation (1) was rearranged into an expression suitable for substitution, called (3). A student proposed that (3) be substituted into equation (1). The teacher followed the suggestion from the student and finished the operation. Teacher commented the result: ”we will get fifteen equals fifteen ... it won’t get us any more [...] we sort of used (1) to get that ... then we insert it into (2)”. This counter-example opened the DoV related to the selection of the ’target’ equation in the solving procedure.

In other classrooms (HK1, HK2 and SH1) the teacher directed the attention to this aspect by indicating possible alternatives. The teacher did this by asking why or why not the equation (3) can be substituted into (1) – equations (1) and (3) being equivalent – or by *not* just selecting the ’target’ equation without comment.

The very last DoV relates to the finding of the value of the second unknown. When the value of the first unknown is found it will be used to find the second unknown. This is done by substituting this value into any of the equations in order to generate a solvable equation in one unknown. Even though the substitution of the value of the first unknown, in principle, can be done into any of the equations the most convenient is to use equation (3) (if there is one, of course) as it already expresses the wanted unknown in terms of the other. This aspect – the value of first unknown can be substituted into any of the equations – was taken for granted in three of the classrooms.

In HK1-L09 the teacher, obviously on request from one student, had prepared a diagram on the board showing the different steps in the method of substitution. When the class went through the steps in the procedure the teacher opened this dimension of variation by asking which of the three equations the value of y (the first calculated unknown) should be substituted into, ”Should it be the first one, the second one or the third one?”, thus providing alternatives. This discussion was a little later followed up by the conclusion that all three equations were possible, but that the third equation was most suitable. The teachers in HK2 and SH2 also offered alternatives when discussing this step in the solving procedure. In the other classroom this aspect was taken for granted.

Chapter 6
Discussion and conclusion

Discussion of the result

Critical dimensions of variation?

The complexity of a mathematics classroom is considerable. It must be remembered that this complexity is reduced to just a few features in a study like this. The discussion of the outcomes is more or less restricted to these features and can only account for one of many possible ways of seeing and describing the studied activities. In the previous chapter it was shown that there are differences in which dimensions of variations were opened in the six classrooms. There are differences in which aspects of the objects of learning have been made possible to discern. This implies that what was made possible for students to learn, the space of learning, was different.

Are the different *dimensions of variations* critical? The dimensions of variations – DoVs – correspond to aspects of the objects of learning, but these aspects are not necessarily 'critical' in the phenomenographic sense that they will distinguish between different conceptions. Can they be critical in any other sense? Can they for instance distinguish between whether a certain task can be successfully solved or not? Yes, in my opinion, that is possible. I give two examples.

1 A student had not experienced variation in the kind of letters used. All examples the student had met used x and y. Would it be possible for this student to understand (and solve) a system of equations where the unknowns are u and v?

2 The aspect 'the different unknowns can be the same number' was taken for granted. The corresponding DoV was never opened. A student solved a system of equations and obtained $x = 2$ and $y = 2$. Would the student accept this as the solution?

Questions like these can of course not be answered based on the results from the present study. Nevertheless, I suggest that students that participated in mathematics classrooms where the DoVs in the examples were opened more likely would have discerned the corresponding aspects and thus would cope with the situations in better ways than students that had not. In this sense the differences in DoVs possibly could be critical. Remember that I will only refer to differences in relation to the studied lessons. The students in this study certainly would have learnt a lot of mathematics before (and after) the studied lessons, as well as outside of school. I only discuss what tasks were possible to solve in relation to the DoVs from the lessons in the study.

Different spaces of learning

The analysis of how the mathematics was handled has been conducted mainly on the 'public' parts of the lessons and paid little attention to differences between individual students. A student can take part in many types of activities during a mathematics lesson, for instance discussing with classmates or maybe having a 'private' discussion with the teacher at the desk. Would it not be possible to learn something from this? Yes, of course. However, dimensions of variation that were opened at these events were just opened to the limited number of students involved at these particular instances. The focus of this study was differences in how the content was handled, which basically implies how the teachers handled the content in relation to the whole group of students. The focus was not primarily on what individual students might experience. What DoVs were opened in conversations between students was mostly beyond teacher control. Teaching cannot rely on students opening certain dimensions in their 'private conversation'. The teacher cannot just neglect some aspects of the object of learning and hope that they will 'come up anyway'. The private conversations between the teacher and individual students at their desks that were available in the data have not been analysed in detail. There were instances where the content of a private conversation was made public by the teacher later and thus was included in the analysis. When the teacher was not making such private content public those aspects were regarded as taken for granted in relation to the other students. Thus, the analysis did not cover the case for every individual student. The aspects taken for granted by the teacher (and what was kept invariant in exercises in textbooks and worksheets), as reported in the previous chapter, should be regarded as taken for granted in general. Thus, the result from the analysis concerned the space of learning on the level of the classroom, not on the level of each individual student.

Are the differences in the *spaces of learning* critical? On the one hand, when a DoV is opened, it is still possible that students miss what is going on. A few minutes of 'daydreaming' and lost concentration, and the opportunity to discern a certain aspect of the object of learning may be missed. This line of reasoning will lead to the conclusion that regardless of what extensive pattern of variation is offered during teaching, there can be no guarantees concerning what students will actually learn.

On the other hand, when a DoV is not opened, we still cannot completely rule out that some students discern the corresponding aspect anyhow. DoVs that are not opened will not be possible to discern from the action in the classrooms *alone*. These aspects are not made possible for all students to experience. However, it is possible for individual students to experience variation regarding an aspect, even though the DoV is not opened in the lesson. Students may generate the necessary variation themselves by the use of earlier experience. The possible learning is then not only related to how the mathematical content is handled in the classroom but also to how individual students may relate their immediate experience to previous ones in a fruitful way. So, is the conclusion that differences in the spaces of learning do not matter? No, high-quality teaching must address the learning of all students and not depend on their previous experience. The possible variation individual students may generate cannot be a reason for not opening important DoVs regarding the object of learning. The DoVs opened in the classroom provide all students with opportunities to learn. I also mean that teaching have a special responsibility to students, with lesser abilities to open DoVs by means of their previous experience, and to students that cannot do so in interaction outside of school. It is the least resourceful students that will suffer most from teaching that takes a lot for granted concerning the objects of learning. So, even if an opened DoV in no way can guarantee that a student will learn, the probability that the student actually learn will be far greater than if the DoV is not opened.

Space of learning in Sweden 2

The data in this study did not allow for comparisons on a national or cultural level. However, I will comment on some differences without making any general claims.

Was there anything that distinguished the Swedish classroom from the others? Without going into the specific handling of the content, all six classrooms appeared quite similar. The organisation of the teaching had many similar features. There were episodes of whole class teaching, when the teacher wrote on the board and talked to the whole class. There were episodes when students worked with tasks individually or in pairs. One could also find episodes in all classrooms when students were at the front writing their solutions on the board. Discernable differences on this superficial level were, for instance, the number of students. There were significantly fewer students in SW2, where there were 24 students. There were 38 students in both Hong Kong classes and between 46 and 55 in the three Shanghai classes. Another difference that could be noted concerned the age of the students. Until the early 1990's almost all children in Sweden started school the year they turned 7, which is one year later than in Hong Kong and Shanghai. In SW2 there were some students that begun at the age of six. This meant that in this grade nine class the age of the students was 14–16 years compared to 13–14 years in the classes from Hong Kong and Shanghai. Neither the class size nor the age differences seemed to have any influence of the general organisation of the teaching.

A closer look revealed that the students in SW2 worked more freely with the tasks in their textbooks. In the Chinese classes the teachers made use of a quite strict whole class approach and all students worked with the same exercises at all times. Regardless of whether they worked in whole-class, in pairs or individually they worked with the same tasks and questions. In Sweden the 'fast' students would eventually complete all tasks in the actual topic and move on to do something else. In this sense the instruction was more individual in SW2 than in the other classrooms and was to a higher degree dependent on the textbook. Unfortunately the tasks in the textbook in SW2 were very similar to each other. The variation created by the teaching material was a noticeable difference between the classrooms. The tasks provided by the textbook in SW2 did not open many dimensions of variation. In SW2 the tasks were so similar that an almost identical solving procedure could be applied every time. This was in sharp contrast to the exercises used in the Chinese classrooms where the tasks and examples often showed a completely exhaustive pattern of variation in many respects. The teachers in HK and even more so in SH used variation in what seemed to be a deliberate way to direct the attention to certain features of the content. Even though the textbook tasks in SW2 provided little variation they still would open some DoVs. The teacher, however, did not make any comments regarding these aspects and seemed not to vary his own examples and tasks in the same deliberate way as it was done in HK and SH. In SW it seemed to be more important that the students were engaged in the creation of a problem to be solved together on the board than to use a problem that the teacher had prepared in advance. My impression was that almost 'any task would do'. The exercises and tasks in the Chinese classrooms were chosen with much more care and showed insight in the particular difficulties students usually would encounter with mastering the content.

The teacher in SW2 appeared to be aiming at involving the students in the handling of the content. This meant that the students' participation was taken seriously and even when a student provided an incorrect or less suitable answer the teacher would not dismiss it right away and move on (see Emanuelsson and Sahlström (2008) for a discussion of this phenomenon). There was at least one instance when a contribution from a student opened up a dimension of variation, a dimension that probably would not have been opened otherwise. It was in SW2-L13 a student suggested that the equation (3), which came from equation (1) in a system of equations, be substituted back into (1). The teacher followed this suggestion even though it would not lead to the solving of the system. By being sensitive to input from students in this way the handling of the content became more elaborated. In this particular case the input from the student opened a dimension of variation regarding the choice of target equation. Examples of input from students that opened DoVs were found in other classes too. In general, however, the teachers in the Chinese classes tended to display more control of the content and the input from students came more often via the teacher. The teachers walked around and carefully chose the student input, apparently in order to introduce variation in a certain respect. This was frequent in the Shanghai classrooms where the teachers often walked around and looked for particular student solutions to display on the board. One such example was in SH2-L03 when the teacher showed two different solutions by students and thus introduced alternative ways to solve the same task.

One obvious characteristic of the Swedish classroom was the procedural approach. Little attention was paid to the mathematical concepts. Immediately after the first example of a system of equations was written on the board the teacher began to solve it. All examples on the board and in the textbook seemed to have the same purpose, namely to practice the procedure of solving a system of equations with the method of substitution. This *procedural* focus stood in contrast to the time and attention that were devoted to discussing the properties of the concept of simultaneous equations, as well as those of a solution, in the other classrooms. This *conceptual* focus was particularly clear in the Shanghai classrooms. As a consequence one might expect that many DoVs concerning the method of substitution would have been opened in SW2. However it was quite the opposite. In spite of the focus on the method it was still not handled in a way that would open more than a few DoVs related to the method of substitution. Part of the explanation could be found in the limited pattern of variation in the exercises used.

Space of learning in the Chinese classrooms

The analysis has shown that variation and contrasts are used frequently in the Shanghai classrooms. One explanation for the presence of these complex patterns of variation is that they were used deliberately in order to direct students' attention to important features of the mathematics. A lot of the variation was created by the use of well-prepared worksheets and sets of tasks. These seemed to be designed to generate patterns of variation that would open important DoVs and make the corresponding aspects of the object of learning possible to discern. The worksheets and tasks often used variation both to show all possible cases and to 'reverse' the tasks. There were several examples of both types. In SH there were also examples of the use of contrasts, for instance when the concept of 'system of equations' was introduced. By comparing examples of *what is* and *what is not* 'systems of linear equations in two unknowns' many dimensions of variations were opened. The data in this study do not allow for any conclusions regarding what is typical in SH classrooms. However, the characteristics discussed here are similar to what have been reported elsewhere (cf. Gu, Huang & Marton, 2004; Huang, 2002; Huang & Leung, 2002). "In China, 'teaching with variation' has been applied consciously or intuitively for a long time" (Gu, Huang & Marton, 2004, p. 314). For instance, the concept of systems of linear equations was not introduced as *the* system of equations in any of the SH classrooms. It was not taken for granted that there are different types of systems. A parallel finding was reported by Huang and Leung (2002) in their study of how the Pythagoras' theorem was taught. "The Shanghai teacher started the lesson by asking the questions 'what is the relationship between the three sides of a triangle?' and 'what is the relationship between the three sides of a right-angled triangle?'" (Huang & Leung, 2002, p. 270). In this case the Shanghai teacher did not take for granted that the triangles were of a certain kind but rather created a pattern of variation that would open a DoV regarding what type of triangles the Pythagoras' theorem can be applied to.

Differences in the space of learning indicate that what was made possible to learn was different. It would however be too simple to rate the quality of teaching based solely on these differences. For instance, considering the hierarchy of mathematics it is always necessary to take many things for granted in teaching. It is not possible to 'start from scratch' every time. Things that students already are aware of and familiar with, can be taken as starting points when new content is introduced. The more advanced the group of students you are teaching is, the more can reasonably be taken for granted. This means that a simple comparison of the spaces of learning found in different classrooms probably would be unfavourable to classrooms with the more advanced students. The students in the SH classes were apparently already familiar with many aspects of the mathematics taught. It would then have been possible to take many aspects for granted and to start the teaching from a more 'advanced' level. However, in the studied lessons it turned out that little was taken for granted in SH. Many DoVs were opened deliberately and explicitly.

Carefully designed tasks and exercises in textbooks, worksheets and other material can introduce a pattern of variation that can open many DoVs. Teachers are not necessarily always aware of these aspects and the potential of the tasks may not be exploited fully. The three teachers in Shanghai, for instance, used a similar kind of task where students were to examine a set of potential systems of linear equations in two unknowns and decide whether or not they met the requirements. However the three teachers exploited the pattern of variation provided by the items differently. The several aspects that might be discerned from the sets of tasks were discussed with quite different emphasis in whole class discussion. It is clear that if this kind of material – well designed exercises that have the potential to open certain DoVs – is not accompanied by teacher awareness, its full potential may not be exploited.

Results related to the lived object of learning

Even though the lived object of learning has not been in the focus of this study there are some instances that could be interpreted in the light of the relation between teaching and learning provided by the Variation theory. One example is from SW2. The teacher took for granted that the values of both unknowns make up the solution of a system of equations when the first system was solved and displayed on the board. The students then started to work on similar problems in the textbook. A few minutes later a student asked if you were supposed to get *both* values. Apparently there was at least some uncertainty regarding what constitutes a solution to a system of equations. An interpretation is that as the corresponding DoV was not opened, the student was not given the opportunity to experience variation in this respect, which brought about the uncertainty. To the teacher, and to anyone else familiar with systems of equations, it must have been obvious that the solution should contain the values of both unknowns. Maybe this could explain why the teacher in this case took this for granted.

Another example is from SH2. At the point in time, when no examples requiring the rearrangement of the source equation had been shown, a student substituted x in the second equation with $2y$ from the first equation. The first equation is $x + 2y = 1$. A correct procedure would be to exchange x with $1 - 2y$, which would be obtained after rearranging the original equation. So far in this classroom it had been taken for granted that an equation could be used for direct substitution and no variation in this respect had been at hand. This DoV had not been opened. A way to interpret the error is that the student could not discern this aspect of the method of substitution from what had taken place so far.

A third example is from HK1 where the teacher used the term *linear* when he introduced the system of equations. However, the meaning of the term was taken for granted. It was just renamed as *first degree*. All equations used were also invariant regarding this aspect. There were no examples of what a linear equation is *not*. As this DoV was not opened it would not, according to Variation theory, be possible to discern any of the properties that characterise a linear equation unless you could make use of some relevant previous experience of your own. Later it transpired that the teacher could not persuade any of the students to answer the question of what is meant by linear. Maybe this is an indication of the difficulties these students had in discerning any appropriate aspects regarding the meaning of linear as this dimension of variation was not opened.

Critical examination of study and results

One aim of this study was to examine the handling of the mathematical content in different classrooms and to describe in detail the differences. Another aim was to trial a model for analysing the handling of the mathematics based on Variation theory. The method of analysis has shown to be able to detect and describe differences in how the specific mathematical content was handled. By providing extensive descriptions of how the mathematics was handled in the different classrooms as well as supplying direct citations from the transcripts I have made the question of evidence (as well as validity and reliability) for the findings as explicit as possible for the reader to make judgement.

The analysis was made of the *enacted object of learning*. The enacted object of learning is the researcher's description of what the students met in the classroom. How the content was handled in the classroom was described by means of which dimensions of variation regarding the object of learning were opened. This means that, even though the descriptions of the enacted objects of learning are the researcher's, the analysis was made from the perspective of a 'fictitious student' in the classroom.

As discussed previously it will not be possible to tell with certainty what students actually learn from a mathematics lesson. I have shown, however, that the *differences* in what was *made possible to learn* can be detected and described. Some of the aspects made possible for students to discern were not really possible for *all* students to discern. For example, there were DoVs that were opened through the sequences of students' exercises. These DoVs would be opened only for students that did the actual tasks and not to students that omitted some of the tasks. In SW2 this was the case regarding the aspect 'both unknowns must not be present in both equations of a system'. This dimension of variation was opened by the thirteenth task in the sequence of tasks in the textbook. The difference between this task and the previous ones opened this DoV. Students that, for any reason, did not do this task would miss the opportunity to experience this variation. A similar situation was found in HK1-L07 (15:40–29:30) where the DoV 'a system of equations can be written in the form of one expression' only would be opened if students actually did task number 6 in the worksheet.

The classes in this study were average or above average in mathematical ability, as stated by the respective teachers. All students were 'well behaved' in general and seemed to be motivated to learn mathematics. There were no students causing disturbances during the lessons. The teachers did not have to bother with disciplinary matters, but could fully concentrate on teaching the *mathematics*. There were no obvious reasons for them to adjust the subject matter or the exercises in order to 'keep students busy so they won't disturb the lesson'. In more 'troublesome' classes where teachers experience difficulties regarding student interest, motivation and abilities in mathematics a 'survival strategy' can be to 'streamline' exercises in such a way that students only have to take 'small steps' and never get stuck. A teacher may also provide detailed prescriptions and/or foolproof rules of thumb that will take students through the exercises as painlessly as possible. However, in the classrooms in this study I did not find any reasons that would divert the teachers from concentrating on teaching the mathematics. All teachers in the study were properly educated, experienced and also recognised as good teachers. They taught students that were very well behaved and still much was taken for granted.

Range of the results

In the study Variation theory was employed on many levels. Firstly, on the level of the particular mathematics and what aspects of the object of learning had been made possible to discern based on the particular pattern of variation. It was necessary to compare teaching of the *same* mathematics. By keeping the content invariant it was possible to discern differences that could be related to the mathematics. It would not have been meaningful to compare the teaching of different mathematics. This implies that the results tightly related to the teaching of systems of equations can be hard to generalise to the teaching of other contents.

Secondly, Variation theory was employed on the level of comparing classrooms and the researcher's ability to discern differences regarding the spaces of learning. In the same way, as aspects that were kept invariant during teaching were not made possible for students to discern, the researcher would be less likely to discern aspects of teaching (in relation to how the content was handled) that were invariant when comparing different classrooms. However, also in the case of the researcher, previous experience would make it possible to open up dimensions of variations that were not possible to experience from just the immediate comparison of classrooms. Aspects of the objects of learning that were not previously experienced by the researcher and at the same time invariant in the comparison would be 'out of reach' within this approach and not possible to discern. This will then imply that there could be features of how the specific mathematics was handled that were not discerned. By comparing teaching in classrooms, that one could presume are quite different, this limitation of the approach can be reduced.

Missing dimensions of variation

A good reason for a teacher to take some aspects of the object of learning for granted could be that they are already well known by the students. For instance, the fact that letters represent numbers (in different ways) may be one such thing in the actual classrooms. Students' different conceptions of the letters used in algebra have in many studies been showed to be 'critical' when it comes to what tasks students can handle successfully (see e.g. Kieran, 1992; Kücheman, 1981). The present analysis of teaching did not reveal any particular attention to this matter in any of the classrooms. The students in this study had earlier used letters both in equations and algebraic expressions. It could therefore be judged as reasonable for a teacher not to elaborate especially around this aspect. As already discussed, the hierarchical structure of mathematics makes it not just possible but also necessary to consider concepts a 'bit lower down' in the structure as 'known' and to use them as starting points for further teaching. Teaching cannot start from the beginning every time. The point is that the teacher should not take things for granted that are not properly well known by students. In order to judge what can be taken for granted and what cannot, teacher knowledge of student thinking regarding the actual topic, both in general and concerning the particular group of students, is vital. It is probably a wise teaching strategy to give students opportunities to express themselves about the mathematics in order to avoid taking 'obvious aspects' for granted. One way of doing this is discussed by Watson and Mason (2005). They discuss how the use of 'learners' generated examples' can aid teachers in revealing how students understand and think about mathematical concepts.

Connections to previous research

Results from studies of students' conceptions of algebraic concepts and of students' difficulties have shown a number of aspects that could be relevant to consider when teaching systems of equations. One of these was discussed by Drijvers and Herwaarden (2000). They found that many students could not regard and handle an algebraic expression, for instance in the format $ax + b$ as an 'object', which was necessary in order to perform the method of substitution successfully. The obstacle could be described in the terms of the procedural-conceptual duality of mathematical concepts and symbols. Students who only could interpret the expression as something that should be 'calculated', and not as if it represented 'the answer' of the calculation, had difficulty in carrying out the method of substitution. In the classrooms in this study such expressions ($ax + b$) were taken for granted as objects that can be treated as entities. This was also the case in SH1 and SH3 where the method of substitution on some occasions was expanded to involve quite large expressions.

Another example is students' difficulties regarding the transitivity property of equality ($a = b$ and $b = c \Rightarrow a = c$) found by Filloy, Rojano and Solares (2003, 2004). This property of equality was taken for granted in all classrooms in this study. It was used when the method of substitution was employed but no variation was introduced regarding this aspect.

A third finding concerns students' difficulties in recognising that the same letters appearing at different 'places' still represent the same number. This was found by Vaiyavutjamai, Ellerton and Clements (2005) in their study of students solving quadratic equations. In quadratic equations a letter x generally appears twice, for instance as in $2x^2 + 4x - 7 = 0$ or $(x - 4)(x + 3) = 0$. Many students believed the x:es represented different numbers. This result can easily be transformed to the context of systems of equations where the 'same letters' appear at two different places, in this case in two different equations. The DoV 'the same letter represent the same number in both equations' was taken for granted in two of the studied classrooms, but opened in the other four.

Didactical implications

Can the results from this study be useful for practice? I suggest that this is the case. Parts of the results can be directly related to the particular mathematics taught. Other parts are related to more general features. The study provides in-depth descriptions of how DoVs in relation to the systems of equations, the solution and the method of substitution are opened in many different ways. This can inform teachers and teacher education of possible ways to handle the actual content. By taking part of such descriptions, teachers and teacher educators can become aware of their own experiences and conceptions of how to teach the actual content and relate it to other possible ways. Some of the DoVs that are described in chapter 5 may also correspond to aspects of the content that teachers have not considered before. Such aspects are likely to be taken for granted in teaching and becoming aware of them can thus be helpful in developing the teaching of systems of equations. Personally, I realise that I, for a long time, must have taken the aspect 'the same letter represents the same number in both equations' for granted. I cannot remember that I ever have thought of this as something that must be addressed in teaching.

I further suggest that the idea of 'deliberate and systematic variation' to bring out critical aspects of the content can be used in different teaching traditions and in different teaching conditions. In this study there are many examples of how this can be done. The idea can be powerful regardless of how many students there are in a class, how the teaching is organised etc. How the idea is implemented will of course be dependent on, for instance, the teacher's knowledge of common students conceptions in relation to the actual mathematical content. Aspects that the teacher takes for granted are unlikely to be subject to any deliberate variation.

Teachers' knowledge of students' conceptions

This study shows that even experienced teachers handled the same mathematical content differently. How can these differences in how the same mathematical content was handled be understood and explained? Well, the answer to this question is really beyond the scope of the study but there is room for some speculations. One way to start is to consider the relation between the teachers' intended object of learning and the enacted objected object of learning. Teachers' beliefs regarding 'what is mathematical proficiency', 'what is important to teach', and 'procedural or conceptual approach' will no doubt influence the intended object of learning which in turn surely will influence the enacted object of learning.

Another aspect is the teachers' awareness of common student conceptions and difficulties with the content at hand. A particular way to handle the content can be deliberate and grounded in the teachers' knowledge of the students' previous knowledge and in a particular idea of how the content ought to be presented. On the other hand some aspects might be taken for granted unintentionally. One explanation of the latter can be that teachers suffer from the 'blind spot of mathematics teaching'. The notion "expertise blind spot" is used by Nathan and Petrosino (2003) for their hypotheses that an advanced mathematics expertise can make teachers blind to students' learning processes and make them tend to view student development as corresponding to the mathematical structure. Nathan and Petrosino argue that this can lead to an idea that it is important to present the content in a mathematically correct way, overlooking other aspects. All teachers in this study have expertise in the area of systems of equations. The 'expertise blind spot' here could be that some of the 'elementary' features of the object of learning tend to be so obvious that they no longer are kept in focal awareness, but rather have 'disappeared' into the background. This could explain why some aspects were taken for granted. Unintentionally as it may be, it will nevertheless affect the quality of the teaching in a negative way. Could experiencing this deficiency in mathematics teaching be the origin of quotes like the one by John von Neuman [as quoted in (Zukav, 1981), footnote in page 249], "Young man, in mathematics you don't understand things. You just get used to them"? The quote indicates that in mathematics teaching many things are hidden from students, but eventually you might get used to them. It is not good enough for teachers to take their own way of seeing the object of learning as a starting point. As a teacher it is necessary to make an effort in trying to see the object of learning from the perspective of the students.

I suggest that an important part of the teacher competence lies in the ability to problematise the aspects that are obvious to you as a teacher. Avoiding taking critical features of the object of learning for granted requires insight into common content specific student errors and conceptions. Many times there is a need to re-experience aspects of the object of learning that may have lost their actuality and are no longer in focal awareness. It is obviously quite natural to take things, that are no longer discerned, for granted. To help teachers avoid this is one important component of the education of mathematics teachers. Teacher education should help teacher students to discover powerful ways of understanding teaching of mathematics. I mean that teachers' awareness of what may constitute critical aspects of the object of learning, and of common student conceptions regarding the object of learning, is a necessary requirement in order to plan and carry out teaching where important aspects are made possible for students to discern.

Teachers' knowledge of mathematics

Another explanation as to why some aspects are taken for granted can be that teachers are not aware of crucial features of the objects of learning. Teachers may not have discerned all aspects themselves. Perhaps some teachers just have learned to 'do things in a certain way', to follow a certain procedure, without ever being provided with any alternatives. The results from the study of Ma (1999) point in this direction. Many of the American teachers in her study seemed to have little knowledge beyond memorised procedures. Aspects of the object of learning that are not present in the teachers' awareness will of course most likely be taken for granted in teaching. It is hardly controversial to suggest that teachers must well understand the content they are about to teach.

Competing aims

In Sweden there seems to be a contradiction between teaching for 'subject matter knowledge' and teaching for 'social development and values'. There has been a strong emphasis on *values, social development* and *equity* while the development of students' 'cognitive capacities' and 'knowledge' have been given less attention. This is manifested in the removal of subject matter groups and seminars in many schools. In these seminars teachers could discuss the teaching of certain content. Instead 'teacher teams' have been established. These teams are formed by teachers that teach the same group of students, not by teachers that teach the same subjects. It means that a team of teachers in many cases contain just one mathematics teacher (other subjects face the same problematic situation too). Such groups are not the most appropriate for cooperative development of the teaching of *mathematics*. The organisation of teachers in cross-subject teams probably will promote a shift in focus from *subject matter* to *students*. When teachers are to cooperate in the teams and look for a common object of learning it will easily be related to the students' general development, and issues of values and attitudes rather than to students' subject matter knowledge.

In Sweden one can sometimes hear teachers say that firstly one need to cater for a positive 'learning climate' in the classroom, before it is any use to bother about teaching the content. I would like to turn this around and suggest that if the handling of the content is neglected the learning climate will suffer. A thoughtful handling of the content that opens critical dimensions of variation and give students opportunities to learn will make more students enjoy themselves and find interest in school. It is boring not to understand! A thoughtful handling of the content will also provide a good foundation for the social development of students. There is in my opinion no real contradiction between teaching 'social development and values' and a wise treatment of the mathematical content.

Different teaching approaches

An approach to teaching where teachers rely too heavily on students' input and on textbooks concerning the handling of the mathematical content can be problematic. For instance, a strong dependency on students' input will be especially unfavourable for classes with less 'advanced' students. If the teacher does not make sure that the important dimensions of variation are opened the likelihood that students' contribution will open them is probably lower in classes where less students have suitable previous experience. Contrary to what often may be assumed, student-centred approaches will not necessary benefit 'weaker' students. If a large part of the constitution of the mathematical content in the classroom is handed over to the students there is a risk that less resourceful students will not be given the opportunity to discern critical aspects of the content. This means that students that have less previous experience rely more on the actual teaching and how the content is handled in the classroom by initiative from the teacher. However, a thoughtful handling of the content that does not take important features for granted, but instead opens these dimensions of variation will provide students with better opportunities to learn and understand. I also suggest that it is possible to find powerful ways to handle the content regardless of the organisation of the classrooms, the number of students etc.

Implications for further research

Research on the teaching of school algebra

I was a bit surprised by the result from my review of research in the field of teaching and learning of school algebra. The amount of research on teaching seemed to be much smaller than research that focused other aspects such as students' learning and conceptions. An explanation for the relatively little research on teaching could, at least in part, be a lack of methods for analysis that will be sensitive to differences in the handling of the content. In this study I have showed that an analysis based on Variation theory can be fruitful in this respect and this may contribute to the research on teaching. The approach of analysis employed in this study has a clear focus on the mathematics taught. As mentioned before the *same* mathematical content is vital for this kind of comparison. Differences may be discerned if the content is kept invariant.

The result of the review of mathematics classrooms research showed little attention to the specific mathematics taught, which was surprising too. Perhaps the present approach to studying mathematics classrooms can contribute to this field as well, by supplying tools that make it possible to let the mathematical content attract more attention and significance than what obviously was the case.

This approach is at the same time not sensitive to distal factors or variables on the political level that are compared in many large scale comparative studies. I suggest that this approach in many respects is closer to the practise of teaching and that findings perhaps can more directly be embraced by teachers. In the 12th ICMI study on *The future of the teaching and learning of algebra* Doerr (2004) discusses the need for new methodological approaches that can contribute to a knowledge base for the teaching of algebra.

> As we continue to find and develop new methodological approaches to effectively investigate the practices of teachers of algebra, we need to do this in a way that will contribute to a growing and sustainable knowledge base for teaching. This implies that research investigations will need to become more deeply embedded in the practical knowledge of teachers. (Doerr, 2004, p. 285)

The approach employed in this study may hopefully contribute to this area as well as within the community of the Learners' Perspective Study. For instance, it is quite possible to expand the present study both to include more classrooms from the LPS database where the same topics – systems of equations – are taught, as well as using the same approach to compare and explore differences in how other topics are taught in different classrooms.

Research on textbooks

The differences in space of learning between the classrooms in the study were related to some extent to the patterns of variation present in the textbooks. Especially the tasks used in SH showed an extensive variation in many relevant aspects whereas the textbook used in SW2 contained very similar tasks. I suggest that Variation theory can provide a framework for textbook analysis – an approach of analysis with a clear focus on how the mathematical content is presented. A comparison of the patterns of variation within and between the tasks in different textbooks could make it possible to detect and describe differences in which and whether DoVs are opened or not. This could be used as a base for discussing possible advantages and disadvantages of different ways to present certain mathematical content in relation to the 'intended object of learning'.

Research on the relation between teaching and learning

What the students actually did learn in the different classrooms was beyond the scope of this study. The attention has been on what was made possible to learn rather than what was actually learnt. According to Variation theory the experience of variation is necessary for the discernment of an aspect of the object of learning. It means that from a theoretical perspective the interesting point is whether a DoV is opened or not – that is the critical difference. From the theoretical perspective it is not possible to discriminate between different ways to open a DoV. Nor is it possible to discriminate between the number of times a DoV is opened during number a lessons. If a DoV is opened it means that the corresponding aspect has been made possible for students to discern. However the number of students that actually will experience variation in the aspect in question will probably vary depending on both *in what way* and *how many times* a DoV is opened. Thus, from a practical teaching perspective it would be quite interesting to compare different ways of opening a DoV in relation to what students actually learn.

A DoV can be opened in many different ways. Here are some examples related to the aspect 'different letters can be used for unknowns'. In the first one the aspect is taken for granted.

- x and y are used in all instances (DoV not opened).
- x and y are used in all instances, but the teacher provides an alternative by stating that other letters (e.g. u and v) can also be used.
- x and y are used in all instances except one where u and v are used (e.g. in a task in the textbook).
- x and y are used in all instances except one (u and v) which is put on the board but not commented by the teacher.
- x and y are used in all instances except one (u and v) which is put on the board and the teacher points to the difference.

- three different systems are displayed side by side on the board (using x, y and u, v and r, s).
- three examples of the same system except for the letters used are displayed side by side on the board (x, y and u, v and r, s).

Does it matter how explicit the DoV is opened? Will some ways be better? Questions like these can be studied in 'Learning studies' (cf. Holmqvist, 2006; Lo, Pong & Chik, 2005). A learning study is a systematic attempt to develop the teaching in relation to a certain object of learning. It is done in a cyclic manner similar to and inspired by the Japanese 'lesson study'. A difference is that a learning study is based on Variation theory. It will never be possible to predict in detail what students will learn from a lesson or an activity. The relations between differences in learning and the ways a certain aspect is handled in the classroom can nevertheless be studied systematically.

Comparative research on teaching

The data in this study does not permit me to make any claims regarding general differences between the teaching of mathematics in Sweden and China, or between mathematics teaching in 'the west' and 'the east'. There are numerous research that quite persistently have reported of the good performance of students from countries in East Asia in relation to Western students. There have also been many attempts to find explanations to these differences in performance. As I discussed in chapter 1 and 2, I find it reasonable to look for, at least, part of the explanation inside the classrooms. The findings in this study, especially the differences between the SW2 classroom and the Shanghai classrooms, indicate that differences in how the content is handled can be important. Based on this I will formulate the following hypothesis: a deliberate and thoughtful use of variation to bring forward key aspects of the objects of learning is 'typical' of mathematics teaching in China, the patterns of variation that this teaching approach generates will be crucial in promoting students' learning and further that this can explain a great deal of the excellent performance of Chinese students. Research aiming at testing this hypothesis would be most welcome.

In the TIMSS videos studies there was an aim to find critical differences between mathematics teaching in the US and six high performing countries. As described earlier it was not until the research team analysed in detail how the content was handled (in this case it was how the 'making-connections-tasks' were used) that they found anything that could distinguish the teaching in US from the teaching in the other countries. The method for in-depth analysis used in the present study, with its definite focus on the handling of the content, could be most useful when testing my hypothesis above on a larger international scale. In this study I have shown that the analytical approach is quite sensitive to differences in how the content is handled. Further, it would be most interesting if the relation between the 'teaching with variation' and successful learning could be established. Thoughtful and systematic use of variation that opens up dimensions of variation that corresponds to critical aspects of understanding the mathematics, is an idea that is applicable everywhere mathematics is taught, regardless of class size, age of students, classroom organisation, and cultural circumstances.

References

Adolfsson, L. (1997). Är svenska elever dåliga i algebra och geometri? *Nämnaren*, 24 (1), 21–26.

Amit, M. & Fried, M. N. (2005). Authority and authority relations in mathematics education: a view from an 8th grade classroom. *Educational Studies in Mathematics*, 58 (2), 145–168.

Araï, D. (1999). *Introduktion till kognitiv psykologi*. Lund: Studentlitteratur.

Aufmann, R. N., Barker, V. C. & Lockwood, J. S. (1999). *Beginning algebra with applications* (5th ed.). Boston: Houghton Mifflin.

Aufmann, R. N., Barker, V. C. & Lockwood, J. S. (2000). *Intermediate algebra with applications* (5th ed ed.). Boston: Houghton Mifflin.

Becker, J. P. & Jacob, B. (1998). Math war developments in the United States (California). *ICME Bulletin* (44).

Bednarz, N., Kieran, C. & Lee, L. (1996). *Approaches to algebra: perspectives for research and teaching*. Dordrecht: Kluwer.

Bergsten, C. (1990). *Matematisk operativitet: en analys av relationen mellan form och innehåll i skolmatematik* [Mathematical operativity: an anlysis of the relation between form and content in school mathematics] (Ph.D thesis). Dept. of Education and Psychology, Linköping University.

Bush, G. W. (2006). *Executive order: national mathematics advisory panel*. Retrieved August 27, 2007, from http://www.whitehouse.gov/news/releases/2006/04/20060418-5.html

Cai, J. & Lester, J. F. K. (2005). Solution representations and pedagogical representations in Chinese and U.S. classrooms. *The Journal of Mathematical Behavior*, 24 (3-4), 221–237.

Chen, X. (1996). *Quality schooling with limited resources: an international comparison of mathematics and science education in China, Korea and Hungary*. Institute of International Education, Stockholm University.

Clapham, C. & Nicholson, J. (2005). *The concise Oxford dictionary of mathematics* (3rd revised ed.). New York: Oxford University Press.

Clarke, D. (2000). *The Learner's Perspective Study. Research design*. University of Melbourne.

Clarke, D. J., Shimizu, Y., Ulep, S., A., Gallos, F., L., Sethole, G., et al. (2006). Cultural diversity and the learner's perspective: attending to voice and context. In F. K. S. Leung, K.-D. Graf & F. J. Lopez-Real (Eds.), *Mathematics education in different cultural traditions – comparative study of East Asia and the West: the 13th ICMI study* (pp. 353–380). New York: Springer.

Clarke, D., Emanuelsson, J., Jablonka, E. & Mok, I. A. C. (2006). *Making connections: comparing mathematics classrooms around the world*. Rotterdam: Sense Publishers.

Doerr, H. M. (2004). Teachers' knowledge and the teaching of algebra. In K. Stacey, H. Chick & M. Kendal (Eds.), *The future of the teaching and learning of algebra: the 12th ICMI study* (pp. 267–290). Boston: Kluwer Academic Publishers.

Downing, D. (1995). *Dictionary of mathematics terms* (2nd ed ed.). Hauppauge, NY: Barron's.

Drijvers, P. & Herwaarden, O. (2000). Instrumentation of ICT-tools: the case of algebra in a computer algebra environment. *The International Journal of Computer Algebra in Mathematics Education*, 7 (4), 255–275.

Emanuelsson, J. (2001). *En fråga om frågor: hur lärares frågor i klassrummet gör det möjligt att få reda på elevernas sätt att förstå det som undervisningen behandlar i matematik och naturvetenskap* [A question about questions: How teachers' questioning makes it possible to learn about the students' ways of understanding the content taught in mathematics and science]. Göteborg: Acta Universitatis Gothoburgensis. Retrieved April 2, 2008 from https://gupea.ub.gu.se/dspace/handle/2077/8448

Emanuelsson, J. & Sahlström, F. (2006). Same from the outside, different from the inside: Swedish mathematics classrooms from students' points of view. In D. Clarke, C. Keitel & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: the insiders' perspective* (pp. 307–322). Rotterdam: Sense publishers.

Emanuelsson, J. & Sahlström, F. (2008). The price of participation: teacher control versus student participation in classroom interaction. *Scandinavian Journal of Educational Research*, 52 (2), 205–223.

Emanuelsson, J., Häggblom, J., Häggström, J., Liljestrand, J., Lindblad, S., et al. (2002). *The way we do things in Sweden: on school cultures and maths lessons from different perspectives* (Presentation to the Learner's Perspective Study symposium in Melbourne, December 2002). Retrieved August 14, 2003 from http://www.ped.uu.se/kult/Texter/texter.asp

Emanuelsson, J., Häggblom, J., Häggström, J., Liljestrand, J., Lindblad, S., et al. (2003, August). *Interaction and variation of content in a Swedish and a US mathematics classroom: Talking the talk or getting it right*? Paper presented at the symposia, "Social interaction and learning in mathematics classrooms in Australia, Germany, Hong Kong, Japan, Sweden, and the United States", at the EARLI conference.

Emanuelsson, J., Sahlström, F. & Liljestrand, J. (2003, December). *Contrastive perspectives on constructing learning – in classrooms and in research*. Paper presented at the LPS Team Conference, Melbourne.

Fan, L. (2004). *How Chinese learn mathematics: perspectives from insiders*. New Jersey; London: World Scientific.

Filloy, E., Rojano, T. & Solares, A. (2003). Two meanings of the 'equal' sign and senses of comparison and substitution methods. In N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds.), *Proceedings of the 27th conference of the international group for the Psychology of Mathematics Education* (Vol. 4, pp. 223–229). Honolulu, HI: University of Hawai'i.

Filloy, E., Rojano, T. & Solares, A. (2004). Arithmetic/algebraic problem-solving and the representation of two unknown quantities. In M. Johnsen Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the international group for the Psychology of Mathematics Education* (Vol. 2, pp. 391–398). Bergen University College.

Fried, M. N. & Amit, M. (2003). Some reflections on mathematics classroom notebooks and their relationship to the public and private nature of student practices. *Educational Studies in Mathematics*, 53 (2), 91–112.

Givvin, K. (2004). Video surveys: how the TIMSS studies drew on the marriage of two research traditions and how their findings are being used to change teaching practice. In M. Johnsen Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the international group for the Psychology of Mathematics Education* (Vol. 1, pp. 206–211). Bergen University College.

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity, and flexibility: a "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, 25 (2), 116–140.

Grouws, D. A. (Ed.). (1992). *Handbook of research on mathematics teaching and learning*. New York: Macmillan.

Gu, L., Huang, R. & Marton, F. (2004). Teaching with variation: a Chinese way of promoting effective mathematics learning. In L. Fan, N.-Y. Wong, J. Cai & S. Li (Eds.), *How Chinese learn mathematics: perspectives from insiders* (pp. 309–345). Singapore: Worls Scientific.

Gustafsson, J.-E. (2003). What do we know about effects of school resources on educational results? *Swedish Economic Policy Review*, 10, 77–110.

Harper, E. (1987). Ghosts of Diophantus. *Educational Studies in Mathematics*, 18 (1), 75–90.

Hart, K. M. (1981). *Children's understanding of mathematics: 11–16*. London: Murray.

Hiebert, J. (Ed.). (1986). *Conceptual and procedural knowledge: the case of mathematics*. Hillsdale, N.J.: Erlbaum.

Hiebert, J. & Stigler, J. W. (2004). *A world of difference: classrooms abroad provide lessons in teaching math and science*. JSD, 25 (4). Retrieved April 2, 2008 from http://www.nsdc.org/library/publications/jsd/hiebert254.cfm

Hiebert, J., Gallimore, R., Garnier, H., Bogard Givvin, K., Hollingworth, H. et al. (2003). *Teaching mathematics in seven countries: results from the TIMSS 1999 video study*. Washington, DC: NCES.

Holmqvist, M. (2006). *Lärande i skolan: Learning Study som skolutvecklings-modell*. Lund: Studentlitteratur.

Huang, R. (2002). *Mathematics teaching in Hong Kong and Shanghai* (Unpublished Ph.D. dissertation). University of Hong Kong.

Huang, R. & Leung, F. K. S. (2002). How Pythagoras' theorem is taught in Czech Republic, Hong Kong and Shanghai: a case study. *Zentralblatt für Didaktik der Mathematik*, 34 (6), 268–277.

Häggström, J. (2004). KULT-projektet. Matematikundervisning i Sverige i internationell belysning. In C. Bergsten & B. Grevholm (Eds.), *Mathematics and language. Proceedings of MADIF 4. The 4th Swedish Mathematics Education Research Seminar, Malmö, January 21–22, 2004*. Linköping: SMDF.

Häggström, J. (2006a, March). *Interaction on mathematics*. Paper presented at the 34th Congress of the Nordic Educational Research Association, Örebro, Sweden. Retrieved October 10, 2006 from http://www.ped.gu.se/personal/johan.haggstrom/konferens.htm

Häggström, J. (2006b). The introduction of new content: what is possible to learn? In D. Clarke, J. Emanuelsson, E. Jablonka & I. A. C. Mok (Eds.), *Making connections: comparing mathematics classrooms around the world* (pp. 185–200). Rotterdam: Sense Publishers.

Häggström, J. (2007). The same topic – Different opportunities to learn. In C. Bergsten & B. Grevholm (Eds.), *Developing and researching quality in mathematics teaching and learning. Proceedings of MADIF5 – The 5th Swedish Mathematics Education Research Seminar, Malmö, January 24–25, 2006* (pp. 69–79). Linköping: SMDF.

ICCR. (2007). *International Centre for Classroom Research*. Retrieved September 25, 2007 from http://www.edfac.unimelb.edu.au/ict/iccr/

IEA. (2005). *International Association for the Evaluation of Educational Achievement*. Retrieved November 9, 2005, from www.iea.nl

Jacobs, J. K. & Morita, E. (2002). Japanese and American teachers' evaluations of videotaped mathematics lessons. *Journal for Research in Mathematics Education*, 33 (3), 154–175.

Jacobson, C. & Lehrer, R. (2000). Teacher appropriation and student learning of geometry through design. *Journal for Research in Mathematics Education*, 31 (1), 71–88.

Justitiedepartementet. (1998). *Personuppgiftslag (1998:204)*. Retrieved September 26, 2007 from http://www.riksdagen.se/webbnav/index.aspx?nid=3911&bet=1998:204

Kawanaka, T. & Stigler, J. W. (1999). Teachers' use of questions in eighth-grade mathematics classrooms in Germany, Japan, and the United States. *Mathematical Thinking and Learning*, 1 (4), 255–278.

Keitel, C., Clarke, D. & Shimizu, Y. (2006). *Mathematics classrooms in twelve countries: the insiders perspective*. Rotterdam: Sense Publishers.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12 (3), 317–326.
Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.
Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Charlotte, NC: Information Age Pub.
Kilborn, W. (1979). *PUMP-projektet: bakgrund och erfarenheter*. Stockholm: LiberLäromedel/Utbildningsförlaget.
Kilpatrick, J. (1993). Beyond face value: assessing research in mathematics education. In M. Blomhøj & G. Nissen (Eds.), *Criteria for scientific quality and relevance in the didactics of mathematics: report from symposium held in Gilleleje, Denmark, April 27 to May 2, 1992* (pp. 15–34). IMFUFA, Roskilde University.
KULT. (2003). *Svensk skolkultur – klassrumspraktik i komparativ belysning*. Retrieved October 29, 2003 from http://www.ped.uu.se/kult/default.asp
Kücheman, D. (1981). Algebra. In K. M. Hart (Ed.), *Children's understanding of mathematics: 11–16* (pp. 102–119). London: Murray.
Lester, F. K. (Ed.). (2007). *Second handbook of research on mathematics teaching and learning*. Charlotte, NC: Information Age Pub.
Leung, F. K. S. (2005). Some characteristics of East Asian mathematics classrooms based on data from the TIMSS 1999 video study. *Educational Studies in Mathematics*, 60 (2), 199–215.
Leung, F. K. S. (2006). Mathematics education in East Asia and the West: does culture matter? In F. K. S. Leung, K.-D. Graf & F. J. Lopez-Real (Eds.), *Mathematics education in different cultural traditions – comparative study of East Asia and the West: the 13th ICMI study* (pp. 21–46). New York: Springer.
Leung, F. K. S., Graf, K.-D. & Lopez-Real, F. J. (Eds.). (2006). *Mathematics education in different cultural traditions – a comparative study of East Asia and the West: the 13th ICMI study* (1st ed.). New York, PA: Springer.
Liljestrand, J. & Runesson, U. (2006). Interaction, organisation, tasks and possibilities for learning about mathematical relationships: a Swedish classroom compared with a US classroom. In D. Clarke, J. Emanuelsson, E. Jablonka & I. A. C. Mok (Eds.), *Making connections: comparing mathematics classrooms around the world* (pp. 165–183). Rotterdam.: Sense Publishers.
Lo, M. L., Pong, W. Y. & Chik, P. M. P. (2005). *For each and everyone: catering for individual differences through Learning Studies*. Hong Kong University Press.
Lopez-Real, F. & Leung, A. (2001). Reflections on a TIMSS geometry lesson. *For the learning of mathematics*, 21 (3), 25–31.

LPS. (2003). *The Learner's Perspective Study*. Retrieved October 29, 2003 from http://extranet.edfac.unimelb.edu.au/DSME/lps/index.shtml

Lybeck, L. (1981). *Arkimedes i klassen: en ämnespedagogisk berättelse*. Göteborg: Acta Universitatis Gothoburgensis.

Löwing, M. (2004). *Matematikundervisningens konkreta gestaltning: en studie av kommunikationen lärare - elev och matematiklektionens didaktiska ramar*. Göteborg: Acta Universitatis Gothoburgensis.

Ma, L. (1999). *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, N.J.: Lawrence Erlbaum Associates.

MacGregor, M. & Stacey, K. (1997). Students' understanding of algebraic notation: 11–15. *Educational Studies in Mathematics*, 33 (1), 1–19.

Martin-Gay, K. E. (2003). *Intermediate algebra* (2nd ed.). Upper Saddle River, N.J.: Prentice Hall.

Marton, F. & Booth, S. (1997). *Learning and awareness*. Mahwah, N.J.: Erlbaum.

Marton, F. & Morris, P. (2001). *What matters? Discovering critical conditions of classroom learning*. Göteborg: Acta Universitatis Gothoburgensis.

Marton, F. & Pong, W. Y. (2005). On the unit of description in phenomenography. *Higher Education Research and Development*, 24 (4), 335–348.

Marton, F., Runesson, U. & Tsui, A. (2004). The space of learning. In F. Marton & A. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3–40). Mahwah, N.J.: Lawrence Erlbaum.

Marton, F. & Tsui, A. (2004). *Classroom discourse and the space of learning*. Mahwah, N.J.: Lawrence Erlbaum.

Matematikdelegationen. (2004). *Att lyfta matematiken: intresse, lärande, kompetens (betänkande)*. Stockholm: Fritzes offentliga publikationer.

Montis, K. (2000). Language development and concept flexibility in dyscalculia: a case study. *Journal for Research in Mathematics Education*, 31 (5), 541–556.

Nathan, M. J. & Petrosino, A. (2003). Expert blind spot among preservice teachers. *American Educational Research Journal*, 40 (4), 905–928.

Neubrand, J. (2006). The TIMSS 1995 and 1999 video studies. In F. K. S. Leung, K.-D. Graf & F. J. Lopez-Real (Eds.), *Mathematics education in different cultural traditions – a comparative study of East Asia and the West: the 13th ICMI study* (pp. 291–318). New York: Springer.

Nyström, P. (2005). TIMSS 2003. Trender i svenska elevers matematikkunskaper i ett internationellt perspektiv. *Nämnaren*, 32 (1), 13–16.

OECD. (2005). PISA. *The Programme for International Student Assessment*. Retrieved November 9, 2005 from www.pisa.oecd.org

Olteanu, C. (2007). *"Vad skulle x kunna vara?": andragradsekvation och andragradsfunktion som objekt för lärande* ["What could *x* be?": second degree equation and quadratic function as objects of learning]. Institutionen för beteendevetenskap, Högskolan i Kristianstad.
Pong, W. Y. & Morris, P. (2002). Accounting for differences in achievement. In F. Marton & P. Morris (Eds.), *What matters? Discovering critical conditions of classroom learning*. Göteborg: Acta Universitatis Gothoburgensis.
Powell, A. B., Francisco, J. M. & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, 22 (4), 405–435.
Quinlan, C. (2001). Importance of view of regarding algebraic symbols. In H. L. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *The future of the teaching and learning of algebra. Proceedings of the 12th ICMI Study Conference* (Vol. 2, pp. 507–514). Dept. of Science and Mathematics Education, University of Melbourne.
Runesson, U. (1999). *Variationens pedagogik: skilda sätt att behandla ett matematiskt innehåll* [The pedagogy of variation: different ways of handling mathematical topic]. Göteborg: Acta Universitatis Gothoburgensis.
Runesson, U. & Marton, F. (2002). The object of learning and the space variation. In F. Marton & P. Morris (Eds.), *What matters? Discovering critical conditions of classroom learning* (pp. 19–37). Göteborg: Acta Universitatis Gothoburgensis.
Runesson, U. & Mok, I. A. C. (2005). The teaching of fractions – a comparative study of a Swedish and a Hong Kong classroom. *Nordic Studies in Mathematics Education*, 10 (2), 1–15.
Santagata, R. & Barbieri, A. (2005). Mathematics teaching in Italy: a cross-cultural video analysis. *Mathematical Thinking and Learning*, 7 (4), 291–312.
Saxe, G. B., Taylor, E. V., McIntosh, C. & Gearhart, M. (2005). Representing fractions with standard notation: a developmental analysis. *Journal for Research in Mathematics Education*, 36 (2), 137–157.
Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22 (1), 1–36.
Sfard, A. (1992). Operational origins of mathematical objects and the quandry of reification – the case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: aspects of epistemology and pedagogy* (pp. 59–84). Washington, DC: Mathematical Association of America.
Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification – the case of algebra. *Educational Studies in Mathematics*, 26 (2-3), 191–228.
Skolverket. (2004a). *Nationella utvärderingen av grundskolan 2003 – Sammanfattande huvudrapport*. Retrieved September 5, 2007 from http://www.skolverket.se/publikationer?id=1362

Skolverket. (2004b). *Pisa 2003 – svenska femtonåringars kunskaper och attityder i ett internationellt perspektiv* (Rapport 254). Retrieved December 15, 2004 from http://www.skolverket.se/publikationer?id=1390

Skolverket. (2004c). *TIMSS 2003. Svenska elevers kunskaper i matematik och naturvetenskap i skolår 8 i ett nationellt och internationellt perspektiv* (Rapport 255). Retrieved February 15, 2005 from http://www.umu.se/edmeas/timss2003/publ/TIMSS2003.pdf

Smith, K. J. (1991). *The nature of mathematics* (6th ed.). Pacific Grove, Calif.: Brooks/Cole Publ. Co.

Stacey, K., Chick, H. & Kendal, M. (2004). *The future of the teaching and learning of algebra: the 12th ICMI study*. Boston: Kluwer Academic Publishers.

Stevenson, H. W. & Stigler, J. W. (1992). *The learning gap: why our schools are failing and what we can learn from Japanese and Chinese education*. New York: Summit Books.

Stigler, J. W. & Hiebert, J. (1999). *The teaching gap: best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Stigler, J. W. & Hiebert, J. (2004). Improving mathematics teaching. *Educational Leadership*, 61(5), 12–17.

Sutherland, R. (2002). *A comparative study of algebra curricula*. London: Qualifications and Curriculum Authority.

Sutherland, R., Rojano, T., Bell, A. & Lins, R. C. (Eds.). (2000). *Perspectives on school algebra*. Boston: Kluwer Academic Publishers.

Thompson, J. (1985). Är dagens matematikundervisning historielös? In G. Emanuelsson, A. Sörensen & L. Wettstam (Eds.), *Studium matematik NT. Lärarbok* (pp. 169–176). Solna: Esselte studium.

Thompson, J. (1996). *Matematiken i historien* [Mathematics in history]. Lund: Studentlitteratur.

Thompson, J., Martinsson, T., Thompson, J. & Martinsson, P.-G. (1994). *Wahlström & Widstrands matematiklexikon* (Ny utg. ed.). Stockholm: Wahlström & Widstrand.

TIMSS. (2005). *The Trends in International Mathematics and Science Study*. Retrieved November 9, 2005 from http://timss.bc.edu/

Tirosh, D., Even, R. & Robinson, N. (1998). Simplifying algebraic expressions: teacher awareness and teaching approaches. *Educational Studies in Mathematics*, 35(1), 51–64.

Travers, K. J. & Weinzweig, A. I. (1999). The second international mathematics study. In E. Luna, I. Huntley & G. Kaiser (Eds.), *International comparisons in mathematics education*. London: Falmer Press.

Vaiyavutjamai, P., Ellerton, N. F. & Clements, K. (2005). Students' attempts to solve two elementary quadratic equations: a study in three nations. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), *Building connections: theory research and practice. Proceedings of the 28th conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 735–742). Sydney: MERGA Inc.

Vetenskapsrådet. (2002). *Forskningsetiska principer inom humanistisk-samhällsvetenskaplig forskning*. Stockholm: Vetenskapsrådet.

Wang, J. & Lin, E. (2005). Comparative studies on U.S. and Chinese mathematics learning and the implications for Standards-based mathematics teaching reform. *Educational Researcher*, 34 (5), 3–13.

Watson, A. & Mason, J. (2005). *Mathematics as a constructive activity: learners generating examples*. Mahwah, N.J.: Lawrence Erlbaum.

Watson, A. & Mason, J. (2006). Seeing an exercise as a single mathematical object: using variation to structure sense-making. *Mathematical Thinking and Learning*, 8 (2), 91–111.

Zukav, G. (1981). *De dansande Wu Li-mästarna: en översikt över den nya fysiken*. Stockholm: Askild & Kärnekull.