RÅDET FÖR HÖGRE UTBILDNING

COUNCIL FOR THE RENEWAL OF HIGHER EDUCATION

Project number: 044/98
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Improved understanding of mathematics through computer based problem solving

Abstract

The main purpose of the project is to develop methods and instruction material where computer based problem solving is used as a means to improve the students´ understanding of mathematics.

The target group is students in teacher training but many of the results will be applicable on other student groups as well.

Why problem solving?
Problem solving is a central focus in all mathematics education. It may be regarded as a goal in mathematics teaching. But problem solving may also be used as a means to reach understanding of mathematical concepts. In the project this aspect that can be referred to as "mathematics via problem solving" will be developed.

Why use computers in problem solving?
Problem solving involves a lot of creative activities such as devising a solving strategy and analysing the result. Carrying out the plan often implies the performance of extensive calculations and other routine operations. For many of the students in the target group this routine work may be so laborious that they are not able at the same time to think of and understand the mathematical ideas behind it. An important reason for the use of computers in this context is that the students can concentrate their mental forces on the mathematical concepts involved in the problem, thus not being confused by calculation and other routine operations necessary for carrying out the solution.

The computer usage must be combined with conventional mathematical calculations and manipulations. Students that depend too much on technical aids may fail to acquire fundamental understanding of mathematics. A certain ability to carry out manipulations by hand is necessary to reach an understanding of mathematical concepts and methods. An important didactic aspect to take into account is how to find a proper balance between the ability to work with and without technical aids in order to acquire the best possible understanding.

Update

Progress of the project

The project started one year ago and will last for another two years. The work during the first year has mainly been concentrated on developing and testing teaching material for beginner's courses in our teacher training programs.

For the Primary School Teacher program a package of six computer labs has been produced. The mathematical content in these labs is closely connected with the content in the course where the labs are a part. Mathematical concepts that are treated are for instance number sequences and patterns, divisibility and prime numbers. The computer program that is used in these labs is mainly Excel but also the computer algebra software Derive is used.

For the Secondary School Teacher program we have given priority to calculus courses. We have found that students have great difficulties with these courses when they are taught in the traditional way. We are developing problem-solving tasks where the solving requires a combination of mathematical reasoning and computer calculations. The aim is to improve the students' understanding of concepts and problem solving skills.

During the second year we have revised our teaching material according to our experiences from the first testing of it.

A quite new feature in the new teacher-training program is school located studies in mathematics courses. We have developed teaching material suitable to courses of that kind.

Another of our project ideas was to teach the students in teacher training programs to make computer-based exercises for their future pupils in school. Earlier this kind of work required knowledge of programming in a computer language. Now with the use of macros in Excel it is easy to produce programs that look very attractive. The first testing of this was very appreciated by the students. Many of them were very stimulated by the task and worked a lot more than was required both on the mathematical content and the graphic performance.

Since this part of the course was so successful we have continued and developed it further during the second project year.

During this year we hope to get our material tested in teacher training at other universities. Mathematics teachers in teacher training from all universities in Sweden meet at a conference once a year (LUMA) and there we have a good opportunity to present our project and get teachers willing to test it. Teachers who are interested in testing our material are welcome to contact us (mailto:kerstin@math.uu.se).

We have established a cooperation with another project that the Council for the Renewal of Higher Education is supporting, "To develop the ability of teacher students to reason mathematically" with Barbro Grevholm as project leader. We have found that our projects have a great deal in common and we will have a common presentation of our projects at the next LUMA conference.

During the last year we have presented ideas from our project at some conferences for mathematics teachers, for instance The Mathematics Biennale in Norrköping (Jan 2002) and a conference in Honour of Hans Wallin in Umeå (June 2002). We have also had an article "Förstå matematik med Excel" published in the journal "Datorn i utbildningen" (nr 7-8/2001).

There is a lot of developing work and research in mathematics education going on in many countries that has connections with our project. We got a good picture of that when we participated in the 2nd International Conference on the Teaching of Mathematics (Crete July 2002)

In our project work during next year we will focus on courses in the new teacher-training program that has just started.

UPPSALA UNIVERSITY 2004-05-26
The Department of Mathematics
Kerstin Ekstig

Final report on project nr 044/98

Improved understanding in mathematics through computer based problem solving

At the Department of Mathematics at Uppsala University we have worked with a project that aimed to study if problem solving with use of computers can improve the students' understanding in mathematics. The project has been supported by the Council for the Renewal of Higher Education and the target group has been students in teacher training programs. The project has been a development project with a focus on producing teaching material for different ground level courses.

Problem solving is a key concept in mathematics education at all levels. In our project we have developed methods to use computers in problem solving as a means to obtain a better understanding of concepts and relations in mathematics. There are several reasons why we have chosen to use computers in problem solving. Some of them are the following:

- Problem solving involves activities of many different kinds. Some activities are very creative, such as devise a plan for the solution, analyse the result and draw conclusions. But often there is also a lot of routine work to be done when carrying out these things. This often implies performance of extensive calculations and other routine operations. For many of the students in the target group this routine work may be so laborious that they are not able at the same time to think of and understand the mathematical ideas behind it. An important reason for the use of computers in this context is that the students can focus on the mathematical concepts involved in the problem and leave the routine work to the computer.

- In traditional teaching we are often anxious to teach students the most effective methods to handle mathematical problems very soon, even when these methods are hard for the students to understand. With computers at hand we can let students develop their creativity and find out solving strategies of their own. Sometimes their method may involve laborious calculations but that doesn't matter as long as the computer takes care of that work. The main thing is that the students use a method that they understand.

- Students can develop their problem solving ability by solving more extensive and complex problems than in traditional mathematics teaching.

- The procedural knowledge the students acquire when working with the computer can help them to understand abstract mathematical concepts.

- Students can work in an exploratory way. When they have solved one problem they can do it again with slightly different assumptions. By a systematic test of how a change of a start value influences the result they can discover and formulate general conclusions. Such investigations can inspire the students to formulate and solve new problems of their own.

- While working interactively with the computer the students can easily discover when they have made a mistake and find out how to correct it. They can also control their work and get confirmation that a solution is correct.

- The students' logical ability is trained. Communication with the computer forces the students to think logically and formulate themselves correctly.

We have been aware that there are certain risks with use of computers in mathematics teaching. It is easy to give examples where unreflective use of computers or calculators has led to less understanding in mathematics. We have therefore paid special attention to that aspect.

Computer based problem solving requires new types of problems than those that are usual in textbooks in mathematics. Many of these traditional problems become trivial when powerful computer software is used. In our project we have therefore worked a great deal with developing new teaching material.

For the Primary School Teacher program a package of six computer labs has been produced. The mathematical content in these labs is closely connected with the content in the course where the labs are a part. Mathematical concepts that are treated are for instance number sequences and patterns, divisibility and prime numbers. The computer program that is used in these labs is mainly Excel but also the computer algebra software Derive is used.

For the Secondary School Teacher program we have given priority to calculus courses. We have found that students often have great difficulties with these courses when they are taught in the traditional way. We have developed problem solving tasks where the solving requires a combination of mathematical reasoning and computer calculations. The aim is to improve the students' understanding of concepts and problem solving skills.

Another of our project ideas has been to teach the students in teacher training programs to make computer based exercises for their future pupils in school. Earlier this kind of work required knowledge of programming in a computer language. Now with the use of macros in Excel it is easy to produce programs that look very attractive. Our first testing of this way of working was very appreciated by the students. Many of them were very stimulated by the task and worked a lot more than was required both on the mathematical content and the graphic performance. Since this part was so appreciated by the students we have developed it further than we planned at the beginning of the project.

Students´ reactions

The students' attitude towards use of technology was an important issue for the result of the project. A common experience from other attempts to use computers in mathematics teaching is that some of the students, often quite a small part, are very satisfied but that a majority of them find the computer applications mostly an additional burden. However, according to our experience from this project it is possible to construct computer applications that by almost all students are considered to be a valuable support in their learning. The students´ views on the project have been evaluated both by written evaluations and by interviews. The students stress that with use of computers they can work with mathematics in a more concrete and experimenting way and thereby gain a better understanding of abstract mathematical concepts. They also appreciate that skills in use of computers in mathematics is very relevant for their future work as teachers in school.

From the students´ reactions we can conclude that it is very important that the computer applications are an integrated part of the course and that they are formed so that they support their mathematical understanding. Another reason to the positive reactions from the students was the choice of computer software. The software we used, Excel and Derive, were very easy for the students to learn to handle. They felt that they could concentrate on the mathematics without trouble with the handling of the computer.

Some examples on computer based problem solving

In order to describe our main project ideas more concretely I present some examples from our teaching material and discuss in what respects they improve the students´ understanding. The complete teaching material (in Swedish) is enclosed to this report.

First some examples where computer algebra software (Derive) is used.

Example 1

Find a polynomial function *f* of the least possible degree with the following properties:

- *f* has local maxima at $x = -2$ and $x = 3$
- *f* has a local minimum at $x = 1$
- $f(-2) = 20$ and $f(2) = -20$

This task requires that the student has knowledge of fundamental relations between a function and its derivative. It has a formulation that differs from most of the problems the students have met in their textbook. In those problems the function is usually given and the task is to find the extreme values.

Many students have difficulties to formulate a solution strategy that takes all the given information into account. On the other hand they can make a start by choosing some of the properties and find a function with these properties. While working with this they get a better understanding of the situation. In this work the computer (CAS) is an excellent tool. The computer does the calculations and it is also used to test results successively during the solution work. In that way the students may discover mistakes and then go back and adjust their method.

According to my experience this kind of concrete work is a very good support for the students to reach an understanding of abstract mathematical concepts. They also find it stimulating to work in an exploratory way and they often get inspired to formulate and solve problems of their own.

Example 2

Find the value of *a* so that the function

$$f(x) = a^3(2x-1)^3 + a^2(x-2)^2 - 6(3a-2)x$$

gets a local maximum on the *y*-axis.

Also in this example the student must have knowledge of relations between a function and its derivative. The solution of this problem requires first that the student can interpret the given information and express it in mathematical terms. If the student is unable to do that, the computer will be of no use.

When performing the solution the student will have plenty of use of the computer. First, to differentiate the function with respect to x is hard to do by hand for many students – it is safer to use the computer. Then they get an equation of degree 3 with respect to a and prefer to let the computer solve it. Finally the computer is used to control the solutions and exclude values of a that do not fulfil the maximum condition.

Here is an example of a more open character that we have used in teacher training.

Example 3

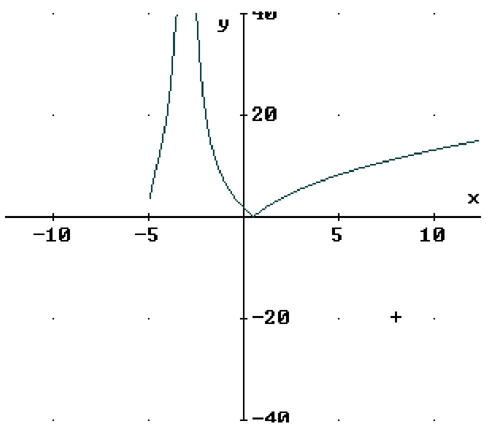
Find a function f with the following properties:

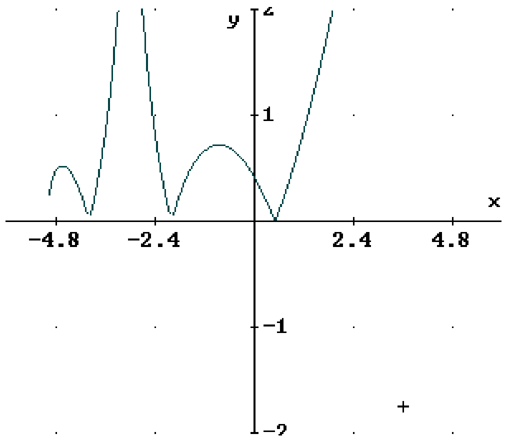
- f is continuous and non-negative
- f has a zero at $x = \frac{1}{2}$
- f is not defined at $x = -3$
- the domain of f is $[-5, \infty[$ except $x = -3$
- the graph of f passes through the point (4, 7)

Discuss different ways to form such a function.

Discuss different ways to work with the task by combining mathematical reasoning and tests on a computer. Also, discuss what mathematical understanding the task can support.

Finally, try to formulate an open problem of similar character.



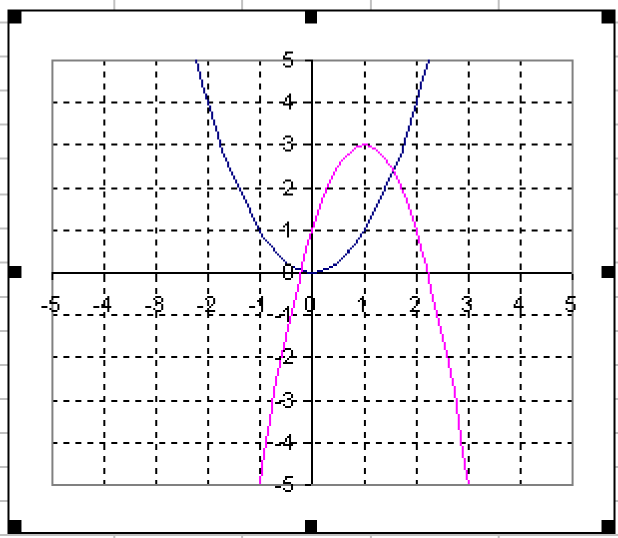


In our teacher training we work a great deal with Excel. Here I give some examples from that.

Example 4

Construct an Excel sheet for study of graphs of second degree functions. It should be formed so that the graph of an arbitrary function $f(x) = a(x-b)^2 + d$ is plotted. The values of the constants are to be placed so that they can easily be changed.

	A	B	C
1	a	-2	
2	b	1	
3	d	3	
4			
5			
6	x	$g(x) = x^2$	$f(x) = a(x-b)^2 + d$
7	-5	25	-69
8	-4,8	23,04	-64,28
9	-4,6	21,16	-59,72
10	-4,4	19,36	-55,32
11	-4,2	17,64	-51,08
12	-4	16	-47
13	-3,8	14,44	-43,08
14	-3,6	12,96	-39,32
15	-3,4	11,56	-35,72
16	-3,2	10,24	-32,28
17	-3	9	-29



Use the Excel sheet to investigate the relation between the values of the constants *a*, *b* and *d* and the look of the graph.

Formulate investigating tasks that you would like your students to work with when you teach this in school. The goal with the tasks should be that the students find out and formulate general relations between the constants and the graph.

Discuss if the following tasks are suitable when teaching this subject. Do they support the students' learning?

What can be said about the constants if the graph has a maximum at $x = 2$?

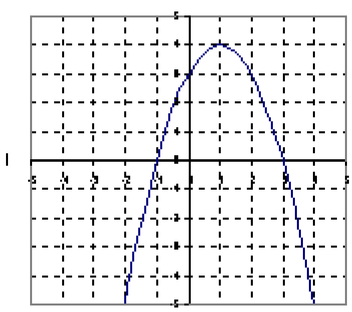
What can be said about the constants if the graph has a maximum at (2, 5)?

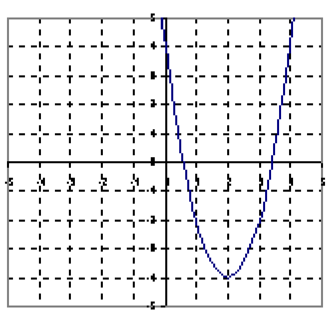
What can be said about the constants if the graph has a minimum on the y-axis?

What can be said about the constants if the graph passes through (0,3) and (4,3)?

What can be said about the graph if all the constants are positive? What if all are negative?

Find out the values of the constants in the following cases:





Construct a similar Excel model for linear functions that could be used in school. Several approaches are conceivable. Here are some suggestions:

Form an Excel model where the students can investigate the relation between the values of k and m and the graph of the function $y = kx + m$.

Form an Excel model where the students can test their knowledge about linear functions. The randomizing function in Excel can be used to draw the graph of a randomly chosen linear function $y = kx + m$ and the students are asked to tell the values of k and m.

Work with this kind of example in teacher training has several valuable effects. The students learn to work with Excel in a way that will be useful for them in their future work in school. They are inspired to think didactically about teaching mathematics. Also, when working with this kind of examples the students often discover that they have lacks in their own knowledge of fundamental school mathematics and this work gives them opportunity to improve their knowledge.

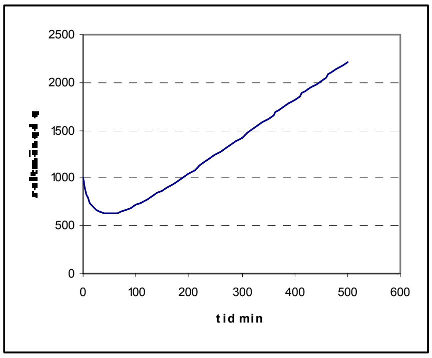
Example 5
A tank contains 100 litres of a salt solution with the concentration 5 grams per litre. You start to add salt solution with the concentration 12 grams per litre at the rate of 3 litres per minute. At the same time you open a plughole so that the mixture in the tank runs out with the rate 5 litres per minute.

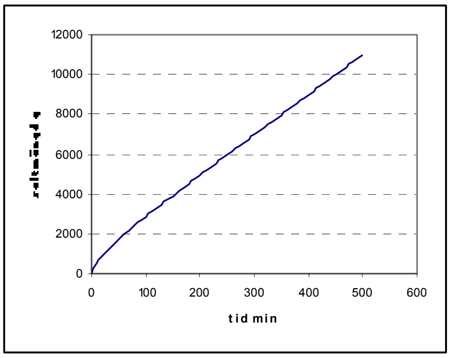
Form a differential equation for determining the amount of salt in the tank as a function of time.

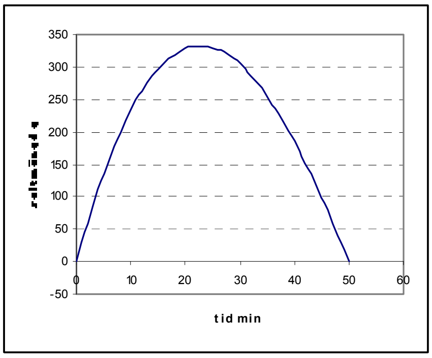
Solve the differential equation by Euler's method. Use Excel to do the calculations and to draw the graph of the solution function. Make the Excel model so that the given values can be changed easily.

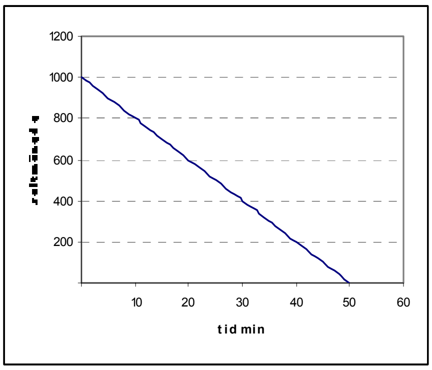
Use the Excel model to investigate how the solution curve depends on the start values. Find out what happens if the rate of the inflow is greater than, less than or equal to the rate of the outflow.

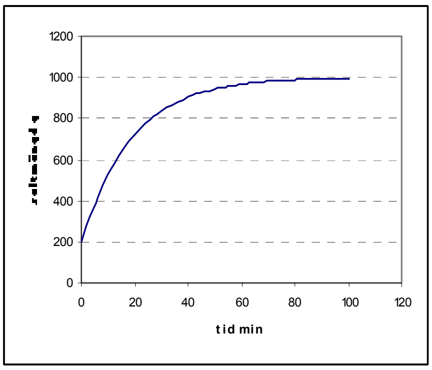
Find out what values of concentrations and rates that give the following solution graphs.

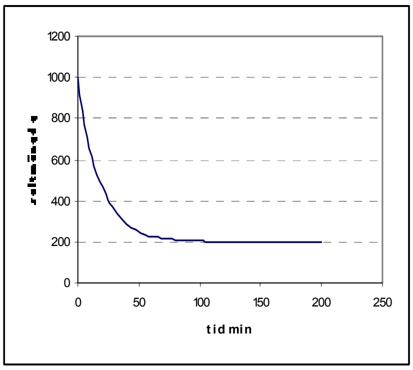






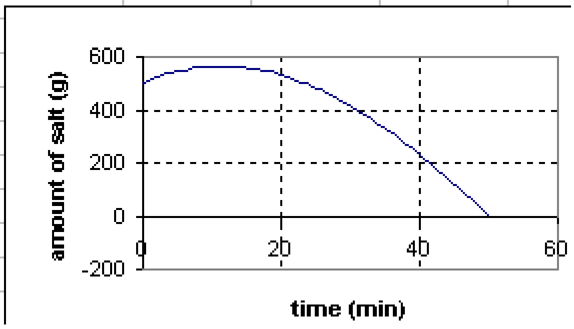






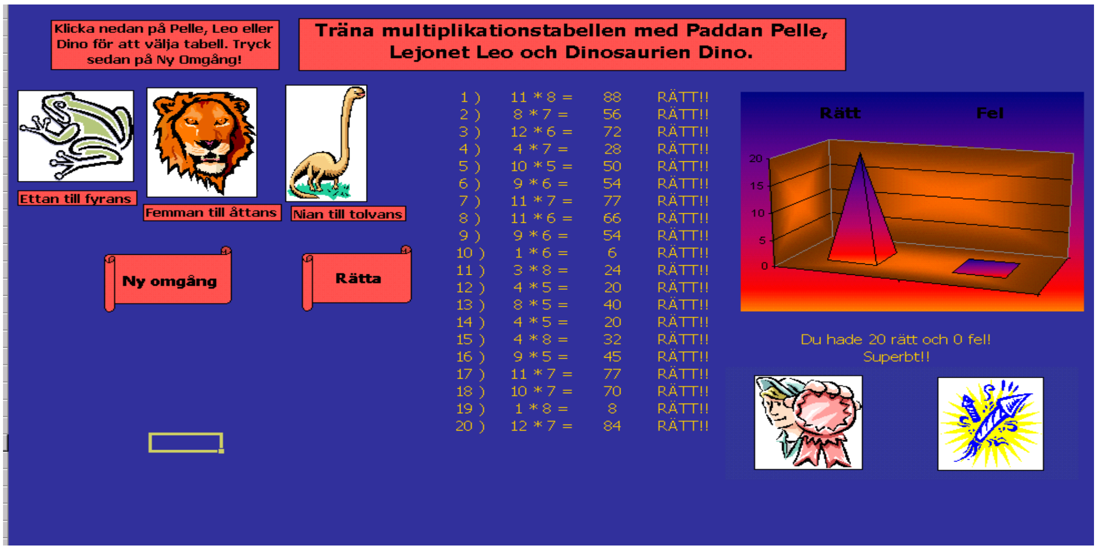
An Excel model may be formed like this:

	A	B	C
1	start concentration	5	
2	rate of inflow	3	
3	conc of inflow	12	
4	rate of outflow	5	
5			
6	t	y	y´
7	0	500	11
8	1	511	9,9286
9	2	521	8,8683
10	3	530	7,8193



Solving this task involves a lot of mathematical understanding. First the student must be able to form a differential equation that describes the situation. Then she must understand Euler´s method and use it to construct an Excel model. After that the student will use the Excel model to work in an investigating way and find out relations between the different parameters and the form of the solution graph. The student is also given opportunity to test her understanding by answering what values of the parameters that result in the given graphs.

Another of our project ideas has been to teach the students in teacher training programs to make computer based exercises for their future pupils in school. Earlier this kind of work required knowledge of programming in a computer language. Now with the use of macros in Excel it is easy to produce programs that look very attractive. The first testing of this was very appreciated by the students. Many of them were very stimulated by the task and worked a lot more than was required both on the mathematical content and on the graphic performance. Here are some examples from student works:



The students learn a lot of mathematics of different kinds in this work. First of all they get excellent training in logical thinking. Building up programs like these requires that they make a clear structure of it and that they instruct the computer what to do in all possible situations. Secondly, this work is very creative for the students. They are free to choose mathematical content and graphic performance, which is very stimulating for them. They also experience a great motivation in this work because they feel that they will have use of this knowledge as teachers in school. Thirdly, this work arouses many didactic thoughts. When designing programs of this kind the students must decide on how a certain skill should be taught in school and choose suitable exercises.

Experiences and continuation of the project

As is evident from this report we have many positive experiences from our project. We find it very satisfying that almost all students in the teacher training program appreciate it and feel that it gives them knowledge that will be valuable in their future work as teachers.

Among negative experiences we can mention difficulties to transfer our ideas to teacher training programs at other universities. We have presented our project at several meetings with colleges from other universities and many have shown great interest but still they have not been prepared to implement it in their courses.

Another experience from the project is that it requires teachers that are engaged and have good knowledge of computer use. When working with computers the students often have a lot of questions that they expect the teacher to be able to answer.

Looking back at the project now we realise that it would have been better to involve more participants in it from the beginning.

In our department the teaching methods and the teaching material that we have developed in this project have successively been integrated into the mathematics courses for the students in the target group and are now a regular part of these courses. There is also great interest from other education programs at our university to introduce similar methods. We will next year work with a local project that aims to develop use of computers in the Science program.

Kerstin Ekstig, project director