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FIRST-ORDER LOGIC

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PREFACE

Predicate logic was created by Gottlob Frege (Frege (1879)) and first-order (predicate) logic was first singled out by Charles Sanders Peirce and Ernst Schröder in the late 1800s (cf. van Heijenoort (1967)), and, following their lead, by Leopold Löwenheim (Löwenheim (1915)) and Thoralf Skolem (Skolem (1920), (1922)). Both these contributions were of decisive importance in the development of modern logic. In the case of Frege's achievement this is obvious. However, Frege's (second-order) logic is far too complicated to lend itself to the type of (mathematical) investigation that completely dominates modern logic. It turned out, however, that the first-order fragment of predicate logic, in which you can quantify over "individuals" but not, as in Frege's logic, over sets of "individuals", relations between "individuals", etc., is a logic that is strong enough for (most) mathematical purposes and still simple enough to admit of general, nontrivial theorems, the Löwenheim theorem, later sharpened and extended by Skolem, being the first example.

In stating his theorem, Löwenheim made use of the idea, introduced by Peirce and Schröder, of *satisfiability in a domain* D, i.e., an arbitrary set of "individuals" whose nature need not be specified; the cardinality of D is all that matters. This concept, a forerunner of the present-day notion of *truth in a model*, was quite foreign to the Frege-Peano-Russell tradition dominating logic at the time and its introduction and the first really significant theorem, the Löwenheim (-Skolem) theorem, may be said to mark the beginning of modern logic.

First-order logic turned out to be a very rich and fruitful subject. The most important results, which are at the same time among the most important results of logic as a whole, were obtained in the 1920's and 30's: the Löwenheim-Skolem-Tarski theorem, the first completeness theorems (Skolem (1922), (1929), Gödel (1930)), the compactness theorem (Gödel (1930) (denumerable case), Maltsev (1936)), and the undecidability of first-order logic (Church (1936b), Turing (1936)). This period also saw the beginnings of proof theory (Gentzen (1934-35), Herbrand (1930)). In fact, the main areas of research in modern logic, model theory, computability (recursion) theory, and proof theory were all inspired by and grew out of the study of first-order logic. During most of the 1940's the subject lay fallow; logic in the 1940's was dominated by computability theory and decision problems. This lasted until the rediscovery by Henkin of the compactness theorem (Henkin (1949)) – Maltsev's work was not known in the West at the time – and the subsequent numerous contributions of Alfred Tarski, Abraham Robinson, and others in the 1950's. And since then (the theory of) first-order logic has developed into a vast and technically advanced field.

But in spite of its central role in logic there still seems to be no exposition centering on first-order logic; in fact, none that covers even the material presented here. The present little book is intended to, at least partially, fill this gap in the literature.

Most of the results presented in this book were obtained before 1960 and all of them before 1970. I have confined myself (in Chapter 3) to results that will (hopefully) appear meaningful and interesting even to nonlogicians. However, sometimes the proof of a result, even the fact that it can be proved, may be more interesting than the result itself.

The reader I have in mind is thoroughly at home with the elementary aspects of first-order logic and, perhaps somewhat vaguely, aware of the basic concepts and results and would like to see exact definitions and full proofs of these. The reader is also assumed to be familiar with elementary set theory including simple cardinal arithmetic. Zorn's Lemma is used twice (and formulated explicitly) and definition by transfinite induction is used three times (and once in Appendix 5); that's all.

In Chapter 1, §7 there are several examples of first-order theories, some of them taken from "modern algebra". These are used in Chapter 3 to illustrate some of the model-theoretic results proved in that chapter. However, no knowledge of algebra is presupposed; the algebraic results, elementary and not so elementary, needed in these applications are stated without proof.

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CONTENTS

0. INTRODUCTION

Suppose you are interested in a certain mathematical object (structure, model) $\mathcal{M}$, say, the sequence of natural numbers 0, 1, 2, ... or the Euclidean plane or the family of all sets. You want to know, or be able to find out, for its own sake or for the sake of application, what is true and what is not about $\mathcal{M}$.

The first thing you have to do is then to decide on certain basic (primitive) concepts in terms of which you are going to formulate statements about $\mathcal{M}$. In the case of the natural numbers the natural choice is *addition* and *multiplication*. In the case of the Euclidean plane the concepts *point*, (*straight*) *line*, and *lies on* (a relation between points and lines) are natural choices and there are others. Finally, in set theory the obvious choice, at least since Cantor, is the *element relation*.

The primitive concepts are not defined (in terms of other concepts) – you can't define everything – but should be sufficiently clear, possibly on the basis of informal explanation. Additional concepts such as *exponentiation* and *prime number* or *triangle* and *parallel with* or *function* and *ordinal number* can then be introduced by definition.

Your goal is to be able to prove nontrivial theorems about $\mathcal{M}$. Since you cannot prove everything, you have to start by accepting certain statements about $\mathcal{M}$ as true without proof. These are your axioms. The idea, which goes back to Euclid, is then to prove theorems by showing that they follow from, or are implied by, the axioms. But “follow from” in what sense? It is in an attempt to answer this question, and related questions, that logic enters the stage.

In (mathematical) logic we want to be able to investigate, by mathematical means, mathematical statements, theories, and families of theories in much the same way as numbers are studied in number theory, points, straight lines, circles, etc. in geometry, sets in set theory, topological spaces in topology, etc.

But mathematical statements and theories in their usual form and the relation *follows from* are not sufficiently well-defined (or explicit) to be amenable to investigation of this nature. Thus, the first thing we shall have to do is replace such statements and theories by other entities sufficiently well-defined to form the subject matter of mathematical theorizing. This is achieved by formalization.

To *formalize* a theory T you first introduce a formal (artificial) language, or skeleton of a language, ℓ with a perfectly precise (and perspicuous) syntax; in other words, the “alphabet” (set of primitive symbols) including the mathematical symbols, for example, + and $\cdot$ in the case of number theory and $\in$ in the case of set theory, should be explicitly given and the rules of formation, i.e., definitions of “term” (noun phrase), “formula”, “sentence” (formula without free variables), etc. of ℓ should be explicitly stated (cf. Chapter 1, §1). In formulas we need, in addition to the mathematical symbols, among other things, certain *logical* symbols such as ¬ (not), ∧ (and), ∃ (there exists), = (equal to).

The next step is then to define a suitable semantical interpretation of ℓ. Thus, for any sentence φ of ℓ and any model $\mathcal{M}$ (appropriate) for ℓ, it should be explicitly defined what it means to say that φ is *true in* $\mathcal{M}$, or $\mathcal{M}$ is a *model of* φ (cf. Chapter 1, §2). In this definition the meaning of the logical symbols is constant (independent of $\mathcal{M}$) whereas the meaning of the mathematical symbols is determined by $\mathcal{M}$. We are going to need *true in* for all models and not just for the model we are particularly interested in. A sentence φ is *valid* if φ is true in all models. $\mathcal{M}$ is a *model of* T if the axioms of T are true in $\mathcal{M}$.

In terms of the concept *true in* we can now define one concept *follows from*: a sentence φ of ℓ *follows from* (the axioms of) T if φ is true in all models of T. (If T has only finitely many axioms, then φ follows from T iff $\chi \rightarrow \varphi$ is valid, where χ is the conjuction of the axioms of T.) Without formalization this relation could not even be precisely defined, let alone investigated by mathematical means.

The next task of logic is then to formulate suitable logical *rules of inference* by means of which theorems of T can be derived from the axioms of T. Again, without formalization, such rules could not be investigated or even precisely defined. What the logical rules are, their properties, and their relation to the concept *follows from* will in general depend on the basic concepts of your logic. In other words, there are different (classical) logics – though not different in the sense of competing – one logic may be different from another in being more powerful, having greater expressive power. The weakest mathematically interesting (in both senses) logic, and the one we shall almost exclusively be concerned with in this book, is first-order logic, L_1. It is characterized by the fact that its basic logical concepts (symbols) are the propositional connectives and the usual quantifiers (existential and universal) and that its variables are *individual variables*. And this, as it turns out, is all we need in (classical) mathematics, i.e., in mathematical definitions and proofs.

The relation between the various sets of rules of inference of L_1 – we present four such sets – and the concept *follows from* is investigated in Chapter 2, where it is shown that this relation is as satisfactory as can be: L_1 is *complete*, i.e., L_1 admits of a complete set of rules of inference; everything that follows from a first-order theory T can, at least in principle – the proof may be very long – be shown to follow from T by applying the rules of inference. In particular, if φ is valid, this can be shown by applying these rules. Extensions of L_1, on the other hand, are often not complete in this sense. For example, the logic $L_1(Q_0)$, obtained from L_1

by adding the new quantifier Q_0, “there are infinitely many”, is not complete in this sense (see Chapter 5).

Having defined a logic, one is naturally interested in its expressive power, i.e., what can and what cannot be “said” or “defined”, and how, in that logic. A class K of models is an *elementary* class – L_1 is sometimes called “elementary logic”– if K is the class of models of some first-order sentence or, more generally, some (possibly infinite) first-order theory. One question is then: What are the general properties of elementary classes and how can we tell if a class is elementary or not? Consider, for example, the class of finite models (for a given language). Is this an elementary class? This and related questions form the subject of the model theory of L_1 (cf. Chapter 3). (The class of finite models is not elementary).

Given a (first-order) sentence φ, it is often not at all clear whether or not φ is valid. We know that if φ is valid, this can be shown to be the case. But, of course, if φ is not valid, our attempts to show that it is will be inconclusive. Thus, it would be very useful, at least in principle, to have a general method by means of which, if φ isn’t valid, this can effectively be shown to be the case. Does there exist such a method? Here the question is not if such a method has been found but, rather, if such a method is at all possible. But then, if there is no such method, the question may seem unanswerable. It isn't, however: in computability (recursion) theory there is a characterization of those (sets of) problems that are computable, i.e., can be solved by a general method – in fact, there are many (equivalent) such characterizations – and examples are given of problems that are unsolvable in this sense. In Chapter 4 we borrow one such unsolvable problem from computability theory and use it to show that the answer to our question is, indeed, negative. We also give a short proof of Gödel's first incompleteness theorem.

By the results of Chapters 2, 3, there are numerous natural mathematical concepts that cannot be expressed in L_1. In L_1 we cannot say of a set (represented by a one-place predicate) that it is finite or that it is uncountable, we cannot say of a linear ordering that it is a well-ordering, etc. Thus, it is natural to extend L_1 in order to remove (some of) these “deficiencies”. This can be done in many different ways. One way is to introduce second-order variables and allow (universal and existential) quantification over these; another is to allow conjunctions and disjunctions of certain infinite sets of formulas and, possibly, quantification over certain infinite sets of (individual) variables; yet another is to add one or more so-called generalized quantifiers to L_1; for example, the quantifier Q_0 mentioned above. Etc. In Chapter 5 we define a general concept *abstract logic* such that (almost) all “standard” extensions of L_1 are abstract logics in this sense. We then prove that L_1 is unique among abstract logics in having certain fundamental properties: in other words, these properties jointly characterize L_1.

Proofs presupposing ideas not yet explained, proofs of results not belonging to the theory of L_1, and some (lengthy) examples and applications have been relegated to a number of appendices.

Notation: k, m, n, p, q, r, s are natural numbers or positive integers, unless it is clear that they are not. N is the set of natural numbers. κ, λ are *infinite* cardinals. ξ, η are ordinals. $|X|$ is the cardinality of the set X. "Denumerable" will be used to mean denumerably infinite and "countable" to mean finite or denumerable. Ø is the empty set. $X \times Y = \{\langle a,b\rangle : a \in X \ \& \ b \in Y\}$. $X^n = \{\langle a_1,\ldots,a_n\rangle : a_1,\ldots,a_n \in X\}$.

1. THE ELEMENTS OF FIRST-ORDER LOGIC

This chapter consists chiefly of a list of defintions of the basic concepts that will be studied and used throughout this book and some elementary propositions formulated in terms of these. Actually, we presuppose that the reader is already familiar, more or less, with these concepts, although perhaps not with their exact definitions, and so we can permit ourselves to be rather brief (and not overly formal).

To illustrate the scope of first-order logic, L_1, and for future use (in Chapter 3), in §7 there is a list of first-order concepts and theories.

§1. Syntax of L_1. The primitive symbols of a first-order language are the *logical symbols*:

the propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$,
the quantifiers $\exists$, $\forall$,
the equality symbol $=$,
(individual) variables x, y, z, x', x_1, y_2,...,
parentheses,

and a set of *nonlogical symbols*:

predicates, function symbols, and (individual) constants.

Each predicate and function symbol has a positive (finite) number of "places". Sometimes, when it is convenient, we also include the propositional constant $\perp$ (false) among the primitive symbols. Some definitions and results will then have to be extended or reformulated in an obvious way.

All these symbols except the nonlogical symbols are the same for all first-order languages. Thus, we may think of the *language* as the set ℓ of its nonlogical symbols. There are no restrictions on the cardinality of ℓ, except, of course, when the contrary is explicitly assumed. (What the symbols or the formulas of the language really are, symbols written on paper, natural numbers, sets etc., is of no concern to us; what is is their structure and how they are related to one another.) To be sure, for most of our results the fact that they hold for (theories in) uncountable languages is not very important. But in Chapter 3 we shall make essential use of the fact that for any (infinite) cardinal κ, ℓ may contain κ many individual constants.

The concept *term of* ℓ is defined inductively as follows: (i) variables and constants in ℓ are terms of ℓ, (ii) if f is an n-place function symbol in ℓ and $t_1,...,t_n$ are terms of ℓ, then $f(t_1,...,t_n)$ is a term of ℓ. A term t is *closed* if no variable occurs in t.

An *atomic formula of* ℓ is a formula of the form $Pt_1...t_n$ or $t_1 = t_2$, where P is an n-place predicate in ℓ and $t_1,\ldots,t_n$ are terms of ℓ. The concept *formula of* ℓ is now defined inductively as follows (we leave it to the reader to add parentheses when they are needed): (i) an atomic fomula of ℓ is a formula of ℓ, ((i') $\perp$ is a formula of ℓ) (ii) if φ and ψ are formulas of ℓ, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ are formulas of ℓ ($\varphi \leftrightarrow \psi$ is an abbreviation of $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$), (iii) if φ is a formula of ℓ, then $\exists x\varphi$ and $\forall x\varphi$ are formulas of ℓ. Note that formulas such as $\forall x(\varphi \rightarrow \exists x\psi)$ are allowed (and unambiguous). We sometimes write $\exists xy\varphi$ for $\exists x\exists y\varphi$, $\forall xyz\varphi$ for $\forall x\forall y\forall z\varphi$, etc. ℓ_φ is the set of nonlogical symbols occurring in φ.

An (occurrence of a) variable x is *free* in a formula if it is not in the scope of a quantifier expression $\exists x$ or $\forall x$; x is *bound* it it is not free. A *closed formula* or *sentence* is a formula without free variables. If a formula φ has been written as $\varphi(x_1,\ldots,x_n)$, we assume that the free variables of φ are among $x_1,\ldots,x_n$, but all these need not occur in φ; and similarly for terms $t(x_1,\ldots,x_n)$. However, φ, ψ, etc. are any formulas and t, t', etc. are any terms. The *universal closure* of a formula φ with the free variables $x_1,\ldots,x_n$ is $\forall x_1\ldots x_n\varphi$.

A formula is *existential* if it is of the form $\exists x_1,\ldots,x_n\psi$ and *universal* if it is of the form $\forall x_1,\ldots,x_n\psi$, where, in both cases, ψ is quantifier-free.

If t_i, $i = 1,\ldots,n$, are terms, then $\varphi(x_1/t_1,\ldots,x_n/t_n)$ is the formula obtained from φ by replacing all free occurrences of $x_1,\ldots,x_n$ simultaneously by $t_1,\ldots,t_n$. It is then assumed that no free occurrence of x_i lies in the scope of a quantifier containing a variable occurring in t_i. Thus, for example, in $\forall y\exists zPxyz$ we may not replace x by y or by f(z,u). If $\varphi := \varphi(x_1,\ldots,x_n)$, then $\varphi(t_1,\ldots,t_n)$ is short for $\varphi(x_1/t_1,\ldots,x_n/t_n)$.

In what follows we often use ordinary mathematical notation in (atomic) formulas. For example, if $\leq$ is a two-place predicate, we may write $x \leq y$ instead of $\leq xy$. And if + is a two-place function symbol, we may write $x + y$ instead of $+(x,y)$.

Sets of sentences, sometimes called *theories*, will be denoted by Φ, Ψ, T etc. The members of T are then the axioms of T. We always assume that the members of a set Φ are sentences of the same language ℓ_Φ.

§2. Semantics of L_1. A *model* (or *structure*) *for* ℓ is a pair $\mathcal{A} = (A,\mathcal{I})$, where A, the *domain* of $\mathcal{A}$, is a nonempty set and $\mathcal{I}$ is an *interpretation* of ℓ in A, i.e., a function on ℓ such that (i) $\mathcal{I}(P) \subseteq A^n$ if P is an n-place predicate, (ii) $\mathcal{I}(f)$ is a function on A^n into A if f is an n-place function symbol ($\mathcal{I}(f) \in A^{A^n}$), and (iii) $\mathcal{I}(c) \in A$ if c is an individual constant. $\mathcal{I}(P)$, $\mathcal{I}(f)$, $\mathcal{I}(c)$ will almost always be written as $P^{\mathcal{A}}$, $f^{\mathcal{A}}$, $c^{\mathcal{A}}$, respectively. $\ell_{\mathcal{A}} = \ell$ and $\mathcal{I}_{\mathcal{A}} = \mathcal{I}$. In what follows A, B, A', C_n etc. are the domains of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$, $\mathcal{C}_n$ etc.

A *valuation* in $\mathcal{A}$ is a function v on the set Var of variables into A, v: Var $\rightarrow$ A. The *value* $t^{\mathcal{A}}[v]$ of the term t in $\mathcal{A}$ under the valuation v is defined as follows: (i) if t is a variable x, then $t^{\mathcal{A}}[\boldsymbol{u}] = \boldsymbol{u}(x)$, (ii) if t is a constant c, then $t^{\mathcal{A}}[\boldsymbol{u}] = c^{\mathcal{A}}$ (the value of c under $\boldsymbol{u}$ is independent of $\boldsymbol{u}$), (iii) if f is an n-place function symbol of ℓ and $t_1,\dots,t_n$ are terms of ℓ, then

$$f(t_1,\dots,t_n)^{\mathcal{A}}[\boldsymbol{u}] = f^{\mathcal{A}}(t_1{}^{\mathcal{A}}[\boldsymbol{u}],\dots,t_n{}^{\mathcal{A}}[\boldsymbol{u}]).$$

Note that if $\boldsymbol{u}$ and $\boldsymbol{u}'$ coincide on the variables occurring in t, then $t^{\mathcal{A}}[\boldsymbol{u}] = t^{\mathcal{A}}[\boldsymbol{u}']$.

Example. Let + be a two-place function symbol and 1 an individual constant. Let $\mathcal{A} = (N, \mathcal{I})$ be the model for $\{+,1\}$ such that N is the set of natural numbers and $\mathcal{I}(+)$ is addition and $\mathcal{I}(1)$ is the number one. Let $\boldsymbol{u}$ be such that $\boldsymbol{u}(x) = 2$. Then

$$(x+1)^{\mathcal{A}}[\boldsymbol{u}] = x[\boldsymbol{u}] +^{\mathcal{A}} 1^{\mathcal{A}} = \boldsymbol{u}(x) + 1 = 2 + 1 = 3.$$

Here we are using + and 1 in two different senses: in the first two terms + is a formal two-place function symbol and 1 an individual constant, in the next two terms thay are used in their ordinary sense to denote addition and the number one. Similar (harmless) ambiguities will be common in what follows. ■

Our next task is to define "$\boldsymbol{u}$ *satisfies* φ in $\mathcal{A}$", in symbols, $\mathcal{A} \models \varphi[\boldsymbol{u}]$. Suppose $\boldsymbol{u}$: Var $\to$ A and $a \in A$. Then $\boldsymbol{u}(x/a)$ is the valuation $\boldsymbol{u}'$ such that $\boldsymbol{u}'(y) = \boldsymbol{u}(y)$ for $y \neq x$ and $\boldsymbol{u}'(x) = a$.

$\mathcal{A} \models \varphi[\boldsymbol{u}]$ is defined inductively as follows; P is an n-place predicate:

$\mathcal{A} \models Pt_1 \dots t_n\, [\boldsymbol{u}]$ iff $\langle t_1{}^{\mathcal{A}}[\boldsymbol{u}],\dots,t_n{}^{\mathcal{A}}[\boldsymbol{u}]\rangle \in P^{\mathcal{A}}$,

$\mathcal{A} \models t_1 = t_2[\boldsymbol{u}]$ iff $t_1{}^{\mathcal{A}}[\boldsymbol{u}] = t_2{}^{\mathcal{A}}[\boldsymbol{u}]$,

(not $\mathcal{A} \models \bot[\boldsymbol{u}]$),

$\mathcal{A} \models \neg\psi[\boldsymbol{u}]$ iff $\mathcal{A} \not\models \psi[\boldsymbol{u}]$,

$\mathcal{A} \models (\psi \wedge \theta)[\boldsymbol{u}]$, iff $\mathcal{A} \models \psi[\boldsymbol{u}]$ and $\mathcal{A} \models \theta[\boldsymbol{u}]$,

similarly for $\psi \vee \theta$ and $\psi \to \theta$,

$\mathcal{A} \models \exists x\psi[\boldsymbol{u}]$ iff $\mathcal{A} \models \psi[\boldsymbol{u}(x/a)]$ for some $a \in A$,

$\mathcal{A} \models \forall x\psi[\boldsymbol{u}]$ iff $\mathcal{A} \models \psi[\boldsymbol{u}(x/a)]$ for all $a \in A$.

If $\boldsymbol{u}$ and $\boldsymbol{u}'$ coincide on the variables free in φ, then $\mathcal{A} \models \varphi[\boldsymbol{u}]$ iff $\mathcal{A} \models \varphi[\boldsymbol{u}']$.

Example. Let E be a one-place predicate and $\cdot$ a two-place function symbol. Let $\mathcal{A} = (N, \mathcal{I})$ be the model for $\{E,\cdot\}$ such that $\mathcal{I}(E)$ is the set of even numbers and $\mathcal{I}(\cdot)$ is multiplication. Then

$\mathcal{A} \models \forall y(\exists z(x = y\cdot z) \to \neg Ey)[\boldsymbol{u}]$ iff

for every $k \in N$, $\mathcal{A} \models (\exists z(x = y\cdot z) \to \neg Ey)[\boldsymbol{u}(y/k)]$ iff

---------"---------, if $\mathcal{A} \models (\exists z(x = y\cdot z)[\boldsymbol{u}(y/k)]$, then $\mathcal{A} \models \neg Ey[\boldsymbol{u}(y/k)]$ iff

---------"---------, if there is an m such that $\mathcal{A} \models (x = y\cdot z)[\boldsymbol{u}(y/k)(z/m)]$, then $\mathcal{A} \not\models Ey[\boldsymbol{u}(y/k)]$ iff

---------"---------, if there is an m such that $x^{\mathcal{A}}[\boldsymbol{u}] = (y\cdot z)^{\mathcal{A}}[\boldsymbol{u}(y/k)(z/m)]$, then $y^{\mathcal{A}}[\boldsymbol{u}(y/k)] \notin E^{\mathcal{A}}$ iff

---------"---------, if there is an m such that $\boldsymbol{u}(x) = y^{\mathcal{A}}[\boldsymbol{u}(y/k)(z/m)] \cdot^{\mathcal{A}} z^{\mathcal{A}}[\boldsymbol{u}(y/k)(z/m)]$, then $k \notin E^{\mathcal{A}}$ iff

---------"---------, if there is an m such that $\boldsymbol{u}(x) = k\cdot m$, then k is not even iff $\boldsymbol{u}(x)$ is odd. ■

If φ is a sentence of $\ell_{\mathcal{A}}$, then $\mathcal{A} \models \varphi$, φ is *true in* $\mathcal{A}$, or $\mathcal{A}$ is a *model of* φ, if $\mathcal{A} \models \varphi[\boldsymbol{u}]$ for some $\boldsymbol{u}$: Var → A or, equivalently, $\mathcal{A} \models \varphi[\boldsymbol{u}]$ for every $\boldsymbol{u}$: Var → A. A formula φ (which may contain free variables) of ℓ is (*logically*) *valid*, $\models \varphi$, if $\mathcal{A} \models \varphi[\boldsymbol{u}]$ for every model $\mathcal{A}$ for ℓ and every valuation $\boldsymbol{u}$ in $\mathcal{A}$. Thus, φ is valid iff its universal closure is. Formulas φ and ψ are (*logically*) *equivalent* if $\models \varphi \leftrightarrow \psi$. Thus, two sentences are equivalent if they have the same models.

$\mathcal{A}$ is a *model* of Φ, $\mathcal{A} \models \Phi$, if $\mathcal{A} \models \varphi$ for every $\varphi \in \Phi$. φ is a *logical consequence of* Φ, $\Phi \models \varphi$, if for every model $\mathcal{A}$, if $\mathcal{A} \models \Phi$, then $\mathcal{A} \models \varphi$. (Thus, as is customary in model theory, and elsewhere, ⊨ is used in two, or three, different senses.) Φ and Ψ are (*logically*) *equivalent* if they have the same models.

Models $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent*, $\mathcal{A} \equiv \mathcal{B}$, if $\ell_{\mathcal{A}} = \ell_{\mathcal{B}}$ and for every sentence φ of $\ell_{\mathcal{A}}$, $\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$. ($L_1$ is also known as "elementary logic".)

Let $\mathcal{A}$ and $\mathcal{B}$ be models for ℓ. A function g is an *isomorphism of* $\mathcal{A}$ *onto* $\mathcal{B}$, g: $\mathcal{A} \cong \mathcal{B}$, if g is a function on A onto B such that for all $a_1,\ldots,a_n \in A$,

$$g(a_1) = g(a_2) \text{ iff } a_1 = a_2,$$
$$\langle g(a_1),\ldots,g(a_n)\rangle \in P^{\mathcal{B}} \text{ iff } \langle a_1,\ldots,a_n\rangle \in P^{\mathcal{A}},$$
$$c^{\mathcal{B}} = g(c^{\mathcal{A}}),$$
$$f^{\mathcal{B}}(g(a_1),\ldots,g(a_n)) = g(f^{\mathcal{A}}(a_1,\ldots,a_n)),$$

for all predicates P, constants c, and function symbols f of ℓ. $\mathcal{A}$ is *isomorphic to* $\mathcal{B}$, $\mathcal{A} \cong \mathcal{B}$, if there is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

Suppose g: A → B. If $\boldsymbol{u}$: Var → A, let $g\boldsymbol{u}$: Var → B be defined by: $g\boldsymbol{u}(x) = g(\boldsymbol{u}(x))$. $g\boldsymbol{u}(x/a) = (g\boldsymbol{u})(x/g(a))$.

The following result is really quite obvious, particularly the fact that if $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$, but we nevertheless give a complete proof.

Proposition 1. Suppose g: $\mathcal{A} \cong \mathcal{B}$ and $\boldsymbol{u}$: Var → A. Then for every term t of ℓ,

(1) $g(t^{\mathcal{A}}[\boldsymbol{u}]) = t^{\mathcal{B}}[g\boldsymbol{u}]$.

Also, for every formula φ of ℓ,

(2) $\mathcal{A} \models \varphi[\boldsymbol{u}]$ iff $\mathcal{B} \models \varphi[g\boldsymbol{u}]$.

In particular, if $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$.

Proof. (1) Induction. For a variable x we have

$$g(x^{\mathcal{A}}[\boldsymbol{a}]) = g(\boldsymbol{a}(x)) = g\boldsymbol{a}(x) = x^{\mathcal{B}}[g\boldsymbol{a}]$$

and so $g(x^{\mathcal{A}}[\boldsymbol{a}]) = x^{\mathcal{B}}[g\boldsymbol{a}]$. Thus, (1) holds if t is a variable. Next, for an individual constant c we have

$$g(c^{\mathcal{A}}[\boldsymbol{a}]) = g(c^{\mathcal{A}}) = c^{\mathcal{B}} = c^{\mathcal{B}}[g\boldsymbol{a}]$$

and so (1) holds if t is a constant. Finally, suppose $t := f(t_1,\dots,t_n)$ and (1) holds for the terms t_i, $0 < i \leq n$. Then

$$g(t^{\mathcal{A}}[\boldsymbol{a}]) = g(f^{\mathcal{A}}(t_1{}^{\mathcal{A}}[\boldsymbol{a}],\dots,t_n{}^{\mathcal{A}}[\boldsymbol{a}])) = f^{\mathcal{B}}(g(t_1{}^{\mathcal{A}}[\boldsymbol{a}]),\dots,g(t_n{}^{\mathcal{A}}[\boldsymbol{a}])) =$$
$$f^{\mathcal{B}}(t_1{}^{\mathcal{B}}[g\boldsymbol{a}],\dots,t_n{}^{\mathcal{B}}[g\boldsymbol{a}])) = t^{\mathcal{B}}[g\boldsymbol{a}])$$

and so (1) holds for t. This proves (1).

(2) Suppose φ is atomic. Then φ is of the form $t_1 = t_2$ or $Pt_1 \dots t_n$. In the first case we have, by (1),

$$\mathcal{A} \vDash t_1 = t_2[\boldsymbol{a}] \text{ iff } t_1{}^{\mathcal{A}}[\boldsymbol{a}] = t_2{}^{\mathcal{A}}[\boldsymbol{a}] \text{ iff } g(t_1{}^{\mathcal{A}}[\boldsymbol{a}]) = g(t_2{}^{\mathcal{A}}[\boldsymbol{a}]) \text{ iff } t_1{}^{\mathcal{B}}[g\boldsymbol{a}] = t_2{}^{\mathcal{B}}[g\boldsymbol{a}] \text{ iff}$$
$$\mathcal{B} \vDash t_1 = t_2[g\boldsymbol{a}]$$

and so (2) holds for $t_1 = t_2$. We also have

$$\mathcal{A} \vDash Pt_1 \dots t_n[\boldsymbol{a}] \text{ iff } \langle t_1{}^{\mathcal{A}}[\boldsymbol{a}],\dots,t_n{}^{\mathcal{A}}[\boldsymbol{a}]\rangle \in P^{\mathcal{A}} \text{ iff } \langle g(t_1{}^{\mathcal{B}}[\boldsymbol{a}]),\dots,g(t_n{}^{\mathcal{B}}[\boldsymbol{a}])\rangle \in P^{\mathcal{B}} \text{ iff}$$
$$\langle t_1{}^{\mathcal{B}}[g\boldsymbol{a}],\dots,t^{\mathcal{B}}[g\boldsymbol{a}]\rangle \in P^{\mathcal{B}} \text{ iff } \mathcal{B} \vDash Pt_1 \dots t_n[g\boldsymbol{a}]$$

and so (2) holds for $Pt_1 \dots t_n$.

The inductive cases corresponding to the propositional connectives are obvious. Suppose $\varphi := \exists x\psi$ and (2) holds for ψ. Then

$$\mathcal{A} \vDash \psi[\boldsymbol{a}(x/a)] \text{ iff } \mathcal{B} \vDash \psi[(g\boldsymbol{a})(x/g(a))].$$

Also, since g is onto B,

$$\mathcal{B} \vDash \exists x\psi[g\boldsymbol{a}] \text{ iff there is an } a \in A \text{ such } \mathcal{B} \vDash \psi[(g\boldsymbol{a})(x/g(a))].$$

It follows that $\mathcal{A} \vDash \exists x\psi[\boldsymbol{a}]$ iff $\mathcal{B} \vDash \exists x\psi[g\boldsymbol{a}]$, as desired.

The case $\varphi := \forall x\psi$ is similar. ■

It is often convenient to simplify the official notation and we shall often do so when confusion is unlikely. If $\mathcal{A} = (A, \mathcal{I})$, we may use

$$(A, P^{\mathcal{A}},\dots, f^{\mathcal{A}},\dots, c^{\mathcal{A}},\dots)$$

to refer to $\mathcal{A}$. In some cases we shall even denote models by expressions such as

$$(A, R,\dots,f,\dots,a,\dots),$$

where R is a relation on A, f is a function on A, and a is a member of A, leaving it to the reader to figure out, in case it matters, which language we have in mind and which predicate, function symbol, constant goes on which relation, function, and member of A. This is sometimes indicated by using the same symbol as in the formal language. For example, a model for $\{\leq,S,c\}$ may be written as $(A, \leq,S,c)$.

Suppose $\ell_{\mathcal{A}} \subseteq \ell_{\mathcal{B}}$. $\mathcal{B}$ is then an *expansion* of $\mathcal{A}$, and $\mathcal{A}$ a *restriction* of $\mathcal{B}$ (to $\ell_{\mathcal{A}}$), if $A = B$ and $\mathcal{A}$ and $\mathcal{B}$ ($\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{B}}$, really) coincide on $\ell_{\mathcal{A}}$. The restriction of $\mathcal{A}$ to ℓ is written as $\mathcal{A} | \ell$. If, for example, $\mathcal{I}_{\mathcal{B}} = \mathcal{I}_{\mathcal{A}} \cup \{\langle P,R\rangle\}$, we may write $(\mathcal{A}, R)$ for $\mathcal{B}$ and similarly for more than one nonlogical constant. In particular, $(\mathcal{A}, a_1,\dots,a_n)$ should be understood in this way.

§3. Prenex and negation normal form. A formula φ is in *negation normal form* (n.n.f.) if all occurrences of $\neg$ in φ apply to atomic formulas.

Proposition 2. For every formula φ, there an equivalent formula φ^n in n.n.f.

Proof. φ^n is obtained from φ by repeatedly applying the following operations:

replace $\neg\neg\psi$ by ψ,
----"---- $\neg(\psi \wedge \theta)$ by $\neg\psi \vee \neg\theta$,
----"---- $\neg(\psi \vee \theta)$ by $\neg\psi \wedge \neg\theta$,
----"---- $\neg(\psi \rightarrow \theta)$ by $\psi \wedge \neg\theta$,
----"---- $\neg\forall x\chi$ by $\exists x\neg\chi$,
----"---- $\neg\exists x\chi$ by $\forall x\neg\chi$. ■

A formula φ is in *prenex normal form* if it is of the form $Q_1x_1...Q_nx_n\psi$, where each Q_i is either $\exists$ or $\forall$ and ψ is quantifier-free.

Proposition 3. For every formula φ, there is an equivalent formula φ^p in prenex normal form.

Proof. We may assume that no two quantifier expressions $\forall x$, $\exists y$, etc. in φ contain the same variable. Next, let φ^n be a formula in n.n.f. equivalent to φ. Let φ^p be a prenex formula obtained from φ^n by repeatedly performing the following operations, where $*$ is either $\wedge$ or $\vee$ and Q is either $\forall$ or $\exists$ and Q^d is $\exists$ if Q is $\forall$ and $\forall$ if Q is $\exists$, and x is not free in θ:

replace $Qx\psi * \theta$ by $Qx(\psi * \theta)$,
----"---- $\theta * Qx\psi$ by $Qx(\theta * \psi)$,
----"---- $\theta \rightarrow Qx\psi$ by $Qx(\theta \rightarrow \psi)$,
----"---- $Qx\psi \rightarrow \theta$ by $Q^dx(\psi \rightarrow \theta)$.

φ^p is equivalent to φ. Note that φ^p is not uniquely determined by φ. ■

§4. Elimination of function symbols. Function symbols are sometimes a nuisance (and sometimes almost indispensable; see §5). They can always be eliminated in the following sense.

Let us say that an atomic formula is *primitive* if it contains at most one non-logical symbol. An arbitrary formula is *primitive* if all its atomic subformulas are primitive. Suppose, for example Pxf(c) is a subformula of φ. Let φ' be obtained from φ by replacing Pxf(c) by

$\exists yz(y = c \wedge z = f(y) \wedge Pxz)$ or
$\forall yz(y = c \wedge z = f(y) \rightarrow Pxz)$.

φ' is then equivalent to φ. In this way we can eliminate all atomic subformulas containing more than one nonlogical constant. The resulting formula is primitive and equivalent to φ. Thus, every (universal, existential) formula is equivalent to a primitive (universal, existential) formula.

Suppose φ is primitive. For every n-place function symbol f occurring in φ (and so in φ), let P_f be a new n+1-place predicate. Let φ^R be obtained from φ by replacing subformulas of the form $f(x_1,...,x_n) = y$ or $y = f(x_1,...,x_n)$ by $P_f x_1,...,x_n y$.

Let G_f be the graph of f. For any $\mathfrak{A} = (A, P^{\mathfrak{A}},...,f^{\mathfrak{A}},...,c^{\mathfrak{A}},...)$, let

$$\mathfrak{A}^R = (A, P^{\mathfrak{A}},...,G_{f^{\mathfrak{A}}},...,c^{\mathfrak{A}},...).$$

For every n+1-place predicate P let Fn(P) be the sentence saying that P is an n-place function, i.e., Fn(P) :=

$$\forall x_1...x_n \exists y \forall z (P x_1...x_n z \leftrightarrow z = y).$$

Let φ^F be the conjunction of the sentences $\mathrm{Fn}(P_f)$ for all function symbols f in φ.

Proposition 4. (a) For every $\mathfrak{A}$ and every sentence φ of $\ell_{\mathfrak{A}}$, $\mathfrak{A} \vDash \varphi$ iff $\mathfrak{A}^R \vDash \varphi^R$. Thus, the models of φ and the models of $\varphi^F \wedge \varphi^R$ are essentially the same.

(b) $\varphi^F \rightarrow \varphi^R$ is logically valid iff φ is logically valid.

This should be rather obvious.

Similarly, an individual constant c can be replaced by one-place predicates P_c plus the additional condition $\exists x \forall y (P_c y \leftrightarrow y = x)$ or, if we are only interested in validity, by a universally quantified individual variable.

Thus, function symbols (and individual constants) are dispensable in principle but in many examples and applications it would be awkward to work with predicates (and constants) only.

§5. Skolem functions. The ideas explained in this § will be important in Chapter 2, §8, and Chapter 3, §10.

Suppose $\varphi := \forall x_1...x_n \exists y \psi(x_1,...,x_n,y)$. Let f be a new n-place function symbol. Then

(*) for every model $\mathfrak{A}$ for ℓ_φ, $\mathfrak{A} \vDash \varphi$ iff there is an expansion $\mathfrak{A}' = (\mathfrak{A}, f^{\mathfrak{A}'})$ of $\mathfrak{A}$ such that $\mathfrak{A}' \vDash \forall x_1...x_n \psi(x_1,...,x_n,f(x_1,...,x_n))$.

A function $f^{\mathfrak{A}'}$ introduced in this way (and sometimes the function symbol f) is called a *Skolem function*.

Suppose φ is in prenex normal form, for example, φ :=

$$\forall x \exists y \forall z u \exists v \forall w \psi(x,y,z,u,v,w),$$

where ψ is quantifier-free. Let g_0 be new one-place function symbols and let g_1 be a new two-place function symbol. Then, by two applications of (*), for every model $\mathfrak{A}$ for ℓ_φ, $\mathfrak{A} \models \varphi$ iff there is an expansion $\mathfrak{A}' = (\mathfrak{A}, g_0^{\mathfrak{A}'}, g_1^{\mathfrak{A}'})$ of $\mathfrak{A}$ such that

$$\mathfrak{A}' \models \forall xzuw\psi(x,g_0(x),z,u,g_1(x,u),w).$$

This construction is completely general. Thus, by Proposition 3, we have the following result.

Proposition 5. For every sentence φ, we can find a universal sentence φ^S (S for Skolem) such that for every model $\mathfrak{A}$ for ℓ_φ, $\mathfrak{A} \models \varphi$ iff there is an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}' \models \varphi^S$. Thus, φ^S is satisfiable iff φ is satisfiable.

Note that φ^S is not uniquely determined by φ.

A theory T is a *Skolem theory* if for every formula $\varphi(x_1,...,x_n,y)$ of ℓ_T, there is an n-place function symbol f_φ such that the universal closure of

$$\varphi(x_1,...,x_n,y) \rightarrow \varphi(x_1,...,x_n,f_\varphi(x_1,...,x_n))$$

is a member of T. By a *Skolem model* we understand a model of a Skolem theory.

Given any theory T, we can extend T to a Skolem theory T* in the following way. We define a sequence $T_0, T_1, T_2, ...$ such that $T_0 \subseteq T_1 \subseteq T_2 \subseteq ...$ as follows. Let $T_0 = T$. Suppose T_n has been defined. Let $\{\varphi_i(x_1,...,x_{n_i},y): i \in I\}$ be the set of formulas of ℓ_{T_n} of the form indicated. For each formula $\varphi_i(x_1,...,x_{n_i},y)$, let f_{φ_i} be a new n_i-place function symbol. Finally, let $T_{n+1} =$

$$T_n \cup \{\forall x_1...x_{n_i}y(\varphi_i(x_1,...,x_{n_i},y) \rightarrow \varphi_i(x_1,...,x_{n_i},f_{\varphi_i}(x_1,...,x_{n_i}))): i \in I\}.$$

Now let $T^* = \bigcup\{T_n: n \in N\}$. Then $|\ell_{T^*}| = |\ell_T| + \aleph_0$.

It is now easily seen that:

Proposition 6. For any theory T,

(i) T* is a Skolem theory,

(ii) for every model $\mathfrak{A}$ for ℓ_T, $\mathfrak{A} \models T$ iff there is an expansion $\mathfrak{A}^*$ of $\mathfrak{A}$ to ℓ_{T^*} (a Skolem model) such that $\mathfrak{A}^* \models T^*$.

§6. Logic and set theory. The relation between set theory and (first-order) logic is a somewhat delicate matter. The question if set theory presupposes logic or if logic presupposes set theory has no easy answer. The axioms of set theory are formulated in first-order logic (§7, Example 7). On the other hand, the concepts *model*, *truth* (*in a model*), *logical validity*, etc., as these are defined above, seem to be just ordinary set-theoretic concepts (and may not even be well-defined if the concept *set* isn't).

It may also be observed that, with the present definition of validity (or "logical consequence"), it isn't obvious, although it certainly should be, that logic is applicable in set theory, where the domain (range of the variables) is a proper class and not a set. In fact, we haven't even defined “truth” (in a model) in this case. But although our definition of validity may not be intensionally correct, i.e., yield the right concept, it is extensionally correct (for L_1), i.e., the right sentences are characterized as valid (see Chapter 2, §9).

§7. **Some first-order theories.** This section consists of a list of examples that will later (in Chapter 3) be used to illustrate model-theoretic concepts and theorems. In these examples we often leave out the initial universal quantifiers of axioms.

Example 1. *Linear orderings*. Let ≤ be a two-place predicate. The theory LO of (reflexive) *linear* (or *simple*) *orderings* is the set of the following sentences (axioms).

$\forall xyz(x \le y \wedge y \le z \rightarrow x \le z)$,
$\forall xyz(x \le y \wedge y \le x \rightarrow x = y)$,
$\forall x(x \le x)$,
$\forall xy(x \le y \vee y \le x)$.

We write $x < y$ for $x \le y \wedge x \neq y$. The theory DiLO of *discrete linear orderings with no endpoints* is LO plus:

$\forall x\exists y(x < y \wedge \forall z(x < z \rightarrow y \le z))$,
$\forall x\exists y(y < x \wedge \forall z(z < x \rightarrow z \le y))$.

Let Z be the set of integers and ≤ the usual ordering of Z. (Z, ≤) is a model of DiLO.

The theory DeLO of *dense linear orderings without endpoints* is obtained from LO by adding:

$\forall xy(x < y \rightarrow \exists z(x < z \wedge z < y))$,
$\forall x\exists y(x < y)$,
$\forall x\exists y(y < x)$.

Let Ra and Re be the sets of rational and real numbers, respectively. Let ≤ be the usual ordering of Ra (Re). Then (Ra, ≤) and (Re, ≤) are models of DeLO. ■

Example 2. *The successor function*. Let S be a one-place function symbol and 0 a constant. Let $S^n(x)$ be defined by: $S^0(x) := x$, $S^{n+1}(x) := S(S^n(x))$. SF, the theory of the *successor function*, is then the set of the following sentences.

$\forall xy(S(x) = S(y) \rightarrow x = y)$,
$\forall x(S(x) \neq 0)$,
$\forall x(S^{n+1}(x) \neq x)$, $n \in N$,
$\forall x(x \neq 0 \rightarrow \exists y(x = S(y)))$.

(N, S, 0), where S is the successor function, $S(i) = i+1$, is a model of SF. ■

Example 3. *Boolean algebras*. Let $\cap$, $\cup$ be two-place function symbols, * a one-place function symbol, and 0, 1 individual constants. We write x^* for $*(x)$. The theory BA of *Boolean algebras* has the following members (axioms).

$$x \cap y = y \cap x, \quad x \cup y = y \cup x,$$
$$(x \cap y) \cap z = x \cap (y \cap z), \quad (x \cup y) \cup z = x \cup (y \cup z),$$
$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z), \quad x \cup (y \cap z) = (x \cup y) \cap (x \cup z),$$
$$x \cap x = x, \quad x \cup x = x,$$
$$x \cap (x \cup y) = x, \quad x \cup (x \cap y) = x,$$
$$(x \cap y)^* = x^* \cup y^*, \quad (x \cup y)^* = x^* \cap y^*,$$
$$x^{**} = x, \qquad 0 \neq 1,$$
$$x \cup 0 = x, \quad x \cap 0 = 0,$$
$$x \cap 1 = x, \quad x \cup 1 = 1,$$
$$x \cap x^* = 0, \quad x \cup x^* = 1.$$

In BA we can define a partial ordering $\leq$ by letting $x \leq y$ be $x \cap y = x$ or, equivalently, $x \cup y = y$.

Let At(x) (x is an *atom*) be the formula

$$x \neq 0 \wedge \forall z(z \leq x \rightarrow z = 0 \vee z = x).$$

Adding

$$\forall x(x \neq 0 \rightarrow \exists y(At(y) \wedge y \leq x))$$

to BA we get the theory AtBA of *atomic* Boolean algebras. Finite Boolean algebras are atomic.

Let X be any set, let S(X) be the set of subsets of X. Then $(S(X), \cap, \cup, {}^*, \varnothing, X)$, where $\cap$, $\cup$ are understood as usual and $Y^* = X - Y$, is an atomic Boolean algebra.

The theory NoAtBA of *atomless* Boolean algebras is obtained from BA by adding

$$\forall x \neg At(x).$$

Let PF be the set of formulas of propositional logic (in the variables p_0, p_1, p_2, ...). For every $F \in PF$, let $[F] = \{G \in PF: G \leftrightarrow F$ is a tautology$\}$. Let $[F] \cap [G] = [F \wedge G]$, $[F] \cup [G] = [F \vee G]$, $[F]^* = [\neg F]$, $0_{PF} = [\perp]$, and $1_{PF} = [H]$, where H is any tautology. Finally, let $[PF] = \{[F]: F \in PF\}$. Then $([PF], \cap, \cup, {}^*, 0_{PF}, 1_{PF})$ is an atomless Boolean algebra. ■

Example 4. *Groups*. Let + be a two-place function symbol, – a one-place function symbol, and 0 an individual constant. The theory G of *groups* has the axioms:

$$(x + y) + z = x + (y + z),$$
$$x + -x = 0, \quad -x + x = 0,$$
$$x + 0 = x, \qquad 0 + x = x.$$

In view of the first axiom, the associative law, parentheses in terms may be omitted.

The theory AG of *Abelian groups* has the additional axiom

$$x + y = y + x.$$

For every $n > 0$, let nx be and $x + x + ... + x$, with n occurrences of x. A group $\mathcal{G}$ is *torsion-free* if the sentences

$$\forall x(nx = 0 \rightarrow x = 0)$$

are true in $\mathcal{G}$. $\mathcal{G}$ is *divisible* if the sentences

$$\forall x \exists y(x = ny)$$

are true in $\mathcal{G}$:

Let TAG and DTAG be the theories of torsion-free and divisible torsion-free Abelian groups, respectively.

(Re, +, –, 0), where Re is the set of real numbers and +, – (a one-place function), and 0 are understood as usual, are models of DTAG. Another example is $(\mathrm{Re}^2, +', -', 0')$, where $\langle r,s\rangle +' \langle r',s'\rangle = \langle r+s, r'+s'\rangle$, $-'\langle r,s\rangle = \langle -r,-s\rangle$, and $0' = \langle 0,0\rangle$. ■

Example 5. *Fields*. Let +, • be two-place function symbols and let 0, 1 be individual constants. The axioms of the theory of *fields* are as follows.

$$x + y = y + x, \quad (x + y) + z = x + (y + z),$$

$$x \cdot y = y \cdot x, \quad (x \cdot y) \cdot z = x \cdot (y \cdot z),$$

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z),$$

$$x + 0 = x, \quad \exists y(x + y = 0),$$

$$x \cdot 1 = x, \quad x \cdot y = 0 \rightarrow x = 0 \vee y = 0,$$

$$0 \neq 1, \quad x \neq 0 \rightarrow \exists y(x \cdot y = 1).$$

Since + and • are associative, we may omit parentheses in terms in the usual way. For every natural number $n > 0$, let nx be the term $x + x + x + ... + x$ and let x^n be the term $x \cdot x \cdot ... \cdot x$, in both cases with n occurrences of x. A field $\mathcal{F}$ is of *characteristic* p if $p1 = 0$ is true in $\mathcal{F}$. $\mathcal{F}$ is of *characteristic* 0 if $n1 \neq 0$ is true in $\mathcal{F}$ for all $n > 0$. Every field is of characteristic p, for some prime p, or of characteristic 0. (Ra, +, •, 0, 1) and (Re, +, •, 0, 1) are fields of characteristic 0.

$\mathcal{F}$ is an *algebraically closed field* if every polynomial with coefficients in $\mathcal{F}$ has a zero in $\mathcal{F}$, i.e., the following sentences (one for each $n > 0$) are true in $\mathcal{F}$.

(1_n) $\quad x_n \neq 0 \rightarrow \exists y(x_n \cdot y^n + x_{n-1} \cdot y^{n-1} + ... + x_1 \cdot y + x_0 = 0).$

The complex numbers form an algebraically closed field of characteristic 0 (Fundamental Theorem of Algebra).

Let ACF (ACF(p), where p is 0 or a prime) be the theory of algebraically closed fields (of characteristic p).

$\mathcal{F} = (\mathcal{F}', \leq)$ is an *ordered field* if $\mathcal{F}'$ is a field and $\leq$ is a linear ordering of F and the following axioms are true in $\mathcal{F}$:

$$x \leq y \rightarrow x + z \leq y + z,$$

$$x \leq y \wedge 0 \leq z \rightarrow x \cdot z \leq y \cdot z.$$

An ordered field $\mathfrak{F}$ is *real closed* if (1_n), for n odd, and the following axiom are true in $\mathfrak{F}$:

$$0 \leq x \rightarrow \exists y(x = y^2).$$

The real numbers form a real closed ordered field. Let RCOF be the theory of real closed ordered fields. ■

Example 6. *Arithmetic*. The axioms of (first-order) Peano Arithmetic, PA, are as follows:

$$S(x) = S(y) \rightarrow x = y, \quad S(x) \neq 0,$$
$$x + 0 = x, \quad x + S(y) = S(x + y),$$
$$x \cdot 0 = 0, \quad x \cdot S(y) = (x \cdot y) + x,$$
$$\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \forall x\varphi(x),$$

where $\varphi(x)$ is any formula of the language $\{+, \cdot, S, 0\}$ of arithmetic and may contain free variables other than x. This axiom scheme is the first-order approximation of the full (second-order) axiom of induction.

Q ((Raphael) Robinson's Arithmetic) is the theory obtained from PA by dropping the induction scheme and adding the axiom

$$x \neq 0 \rightarrow \exists y(x = S(y)).$$

Exponentiation and other common number-theoretic functions and concepts can be defined in terms of + and $\cdot$. ■

Example 7. *Set theory*. We shall not give the details of the axiomatization of ZF(C), Zermelo-Fraenkel set theory (with the axiom of choice), since these details are lengthy and irrelevant for our present purposes. What is relevant, however, is the fact that ZFC is formalized in L_1 ($\ell_{ZF} = \{\in\}$). This is particularly interesting, since (practically all of) classical (non-constructive) mathematics can be developed in ZFC. In this sense, all the logic you need in mathematics is first-order logic. ■

Notes for Chapter 1. The definitions of satisfaction and truth in a model is due to Tarski (1935), (1952), but these concepts were quite well understood independently of Tarski's definitions (see, for example, Hilbert, Ackermann (1928)). That set theory can, and should, be formalized in first-order logic was pointed out by Skolem (1922).

2. COMPLETENESS

Having defined the concept *logical consequence*, $\models$, our next task is to develop systematic methods by means of which statements of the form $\Phi \models \varphi$ can be established. To this end we introduce four different formal methods (calculi) each with its advantages and disadvantages. As the reader will notice, the first three of these, FH, GS, and ND (§§ 1, 3, 6) are based on the same logical intuitions. But the formal representations of these intuitions are different leading to formal calculi with quite different properties.

Given a formal logical calculus LC it is natural to ask if it can be improved, if there are cases of logical validity or consequence that cannot be established by means of LC. For example, there may be some (simple) rule of derivation that has been overlooked or some very complicated rule(s) may be required or, worse, it may turn out that no finite set of rules will be sufficient. Given the vast variety of deductive arguments and methods of proof in the mathematical literature, there is *prima facie* no reason at all to rule out these possibilities. But, remarkably, for L_1 (though not for certain (natural) extensions of L_1; see Chapter 5) this is not the case.

A logical calculus LC for L_1 is *complete* if a sentence φ can be shown to follow from a set Φ of sentences, using only the means available in LC, whenever $\Phi \models \varphi$. The calculi presented here are complete in this sense (Corollary 1 and Theorems 7, 10, 11).

It should be observed that these calculi are defined in purely syntactical terms, with no reference to the semantical interpretation of the formulas involved (though, of course, with this semantical interpretation in mind). This is essential: our ambition is to lay bare *all* the logical intuitions that go into the construction of a derivation by means of the method in question.

In this chapter $\boldsymbol{\ell}$ is an arbitary but fixed language. Thus, in what follows, "formula", "term", etc. mean formula of $\boldsymbol{\ell}$, term of $\boldsymbol{\ell}$, etc. We assume that $\boldsymbol{\ell}$ contains an "inexhaustible" set of individual constants (parameters).

§1. Frege-Hilbert-type systems. The Frege-Hilbert system FH (for $\boldsymbol{\ell}$) has the following *logical axioms*. In these axioms $\varphi(x)$ is any formula with the one free variable x, ψ is any sentence, and t is any closed term.

A1. All closed propositional tautologies,
A2. $\forall x\varphi(x) \to \varphi(t)$,
A3. $\forall x(\psi \to \varphi(x)) \to (\psi \to \forall x\varphi(x))$,
A4. $\varphi(t) \to \exists x\varphi(x)$,
A5. $\forall x(\varphi(x) \to \psi) \to (\exists x\varphi(x) \to \psi)$.

The *identity axioms* of FH are the universal closures of the following formulas.

I1. $x = x$,

I2. $x = y \rightarrow y = x$,

I3. $x = y \wedge y = z \rightarrow x = z$,

I4. $x_1 = y_1 \wedge ... \wedge x_n = y_n \rightarrow (\varphi(x_1,...,x_n) \rightarrow \varphi(y_1,...,y_n))$,

I5. $x_1 = y_1 \wedge ... \wedge x_n = y_n \rightarrow t(x_1,...,x_n) = t(y_1,...,y_n)$.

An *axiom* of FH is either a logical axiom or an identity axiom.

There are two *rules of inference* (*derivation*).

R1. Modus Ponens: $\varphi, \varphi \rightarrow \psi / \psi$.

R2. Universal Generalization: $\varphi(c) / \forall x\varphi(x)$.

In these axioms and rules we can restrict ourselves to sentences, since free variables can always be replaced by parameters.

Let Φ be a set of sentences. A (*logical*) *derivation in* FH *of* φ *from* Φ is a sequence $\varphi_0, \varphi_1, ..., \varphi_n$ of sentences such that $\varphi_n := \varphi$ and for every $k \leq n$ either (i) φ_k is an axiom of FH or (ii) $\varphi_k \in \Phi$ or (iii) there are $i, j < k$ such that $\varphi_j := \varphi_i \rightarrow \varphi_k$ (R1) or (iv) $\varphi_k := \forall x\psi(x)$ and for some $i < k$, $\varphi_i := \psi(c)$, where c does not occur in Φ, $\psi(x)$ (R2). φ is *derivable* (*in* FH) *from* Φ, in symbols $\Phi \vdash_{FH} \varphi$, if there is a derivation of φ from Φ. If Φ is the empty set, we (may) drop all references to Φ. If we think of Φ as a theory, we shall sometimes say "proof in Φ" and "provable in Φ" instead of "derivation from Φ" and "derivable from Φ".

Among the identity axioms only I1 and I4 for $n = 1$ and φ an atomic formula are essential; given these, I2, I3, I4, I5 can be derived.

In this section and the next we write $\vdash$ for $\vdash_{FH}$.

In what follows we shall frequently (implicitly) use the following:

Lemma 1. (a) If φ is an axiom or $\varphi \in \Phi$, then $\Phi \vdash \varphi$.

(b) If $\Phi \vdash \varphi$, there is a finite subset Φ' of Φ such that $\Phi' \vdash \varphi$.

(c) If $\Phi \vdash \varphi$ and $\Phi \subseteq \Psi$, then $\Psi \vdash \varphi$.

Proof. (a) is trivial. (b) This is clear, since every derivation (from Φ) is finite.

(c) This, too, is clear except for the slight complication that a derivation of φ from Φ may not be a derivation of φ from Ψ: the applications of R2 may no longer be legal, since the constant c involved, although it doesn't occur in Φ, may occur in Ψ. But such constants can always be replaced by constants not occurring in Ψ. ■

Example 1. The sequence of the following formulas (sentences) is a derivation of

(*) $\forall x\neg\varphi(x) \rightarrow \neg\exists x\varphi(x)$

in FH. Let c be a constant not in (*).

(1) $\forall x\neg\varphi(x) \to \neg\varphi(c)$, A2

(2) $(\forall x\neg\varphi(x) \to \neg\varphi(c)) \to (\varphi(c) \to \neg\forall x\neg\varphi(x))$, A1

(3) $\varphi(c) \to \neg\forall x\neg\varphi(x)$, from (1), (2) by R1

(4) $\forall x(\varphi(x) \to \neg\forall x\neg\varphi(x))$, from (3) by R2

(5) $\forall x(\varphi(x) \to \neg\forall x\neg\varphi(x)) \to (\exists x\varphi(x) \to \neg\forall x\neg\varphi(x))$, A5

(6) $\exists x\varphi(x) \to \neg\forall x\neg\varphi(x)$, from (4), (5) by R1

(7) $(\exists x\varphi(x) \to \neg\forall x\neg\varphi(x)) \to (\forall x\neg\varphi(x) \to \neg\exists x\varphi(x))$, A1

(8) (*), from (6), (7) by R1 ■

Derivations in FH tend to be rather long and awkward. Proofs of statements of the form $\Phi \vdash \varphi$ can often be simplified by applying *derived* rules. Some examples of such rules are given in the following lemma and theorem.

Lemma 2. (a) $\Phi \vdash \varphi_i$ for $i \leq n$, and $\varphi_0 \wedge ... \wedge \varphi_n \to \psi$ is a tautology or, more generally, $\Phi \vdash \varphi_0 \wedge ... \wedge \varphi_n \to \psi$, then $\Phi \vdash \psi$.

(b) Let t be any closed term. If $\Phi \vdash \forall x\varphi(x)$, then $\Phi \vdash \varphi(t)$.

(c) Let t be any closed term. If $\Phi \vdash \varphi(t)$, then $\Phi \vdash \exists x\varphi(x)$.

(d) Let c be any constant not occurring in Φ, $\varphi(x)$, ψ. If $\Phi, \varphi(c) \vdash \psi$, then $\Phi, \exists x\varphi(x) \vdash \psi$.

Proof. (a) $\vdash (\varphi_0 \wedge ... \wedge \varphi_n \to \psi) \to (\varphi_0 \to (\varphi_1 \to ... \to (\varphi_n \to \psi)...)$, since the formula is a tautology. Now use R1 n+2 times.

(b) A derivation of $\forall x\varphi(x)$ from Φ followed by $\forall x\varphi(x) \to \varphi(t)$, $\varphi(t)$ is a derivation of $\varphi(t)$ from Φ.

(c) A derivation of $\varphi(t)$ from Φ followed by $\varphi(t) \to \exists x\varphi(x)$, $\exists x\varphi(x)$ is a derivation of $\exists x\varphi(x)$ from Φ. ♦

To prove Lemma 2(d) we need the following:

Theorem 1 (Deduction Theorem)**.** If $\Phi, \theta \vdash \varphi$, then $\Phi \vdash \theta \to \varphi$.

Proof. Let $\varphi_0, \varphi_1, ..., \varphi_n$ be a derivation of φ from Φ,θ. We show that for all $k \leq n$,

(+) $\Phi \vdash \theta \to \varphi_k$.

If $k = 0$, this is clear. Suppose (+) holds for all $k < m \leq n$. If φ_m is an axiom of FH or a member of Φ, (+) is true for $k = m$. Suppose there are $i, j < m$ such that $\varphi_j := \varphi_i \to \varphi_m$. Then, by hypothesis, $\Phi \vdash \theta \to \varphi_i$ and $\Phi \vdash \theta \to \varphi_j$. Also, $((\theta \to \varphi_i) \wedge (\theta \to \varphi_j)) \to (\theta \to \varphi_m))$ is a tautology. But then, by Lemma 2(a), $\Phi \vdash \theta \to \varphi_m$, as desired.

Finally, suppose $\varphi_m := \forall x\psi(x)$ and for some $i < m$, $\varphi_i := \psi(c)$, where c does not occur in Φ, θ, $\psi(x)$. By hypothesis, $\Phi \vdash \theta \to \psi(c)$. Hence, by R2, $\Phi \vdash \forall x(\theta \to \psi(x))$. But also $\vdash \forall x(\theta \to \varphi(x)) \to (\theta \to \forall x\varphi(x))$ (A3) and so, by R1, $\Phi \vdash \theta \to \varphi_m$, as desired.

Thus, we have shown that (+) holds for all $k \leq n$ and so, in particular, for $k = n$; in other words, $\Phi \vdash \theta \rightarrow \varphi$, as desired. ■

Proof of Lemma 2(d). Suppose $\Phi, \varphi(c) \vdash \psi$. Then, by Theorem 1, $\Phi \vdash \varphi(c) \rightarrow \psi$. Hence, by R2, $\Phi \vdash \forall x(\varphi(x) \rightarrow \psi)$. But $\vdash \forall x(\varphi(x) \rightarrow \psi) \rightarrow (\exists x\varphi(x) \rightarrow \psi)$ (A5). And so, by R1 (twice), $\Phi, \exists x\varphi(x) \vdash \psi$. ■

Example 2. That (*) in Example 1 is derivable can now be shown as follows.

(1) $\vdash \forall x\neg\varphi(x) \rightarrow \neg\varphi(c)$, A2

(2) $\vdash \varphi(c) \rightarrow \neg\forall x\neg\varphi(x)$, (1), Lemma 2(a)

(3) $\varphi(c) \vdash \neg\forall x\neg\varphi(x)$, (2), R1

(4) $\exists x\varphi(x) \vdash \neg\forall x\neg\varphi(x)$, (3), Lemma 2(d)

(5) $\vdash$ (*), (4), Theorem 1 ■

Example 3. As a second example we show that

(**) $\vdash \forall x(\varphi(x) \rightarrow \psi(x)) \rightarrow (\exists x\varphi(x) \rightarrow \exists x\psi(x))$

is derivable. Let c be a new constant.

(1) $\forall x(\varphi(x) \rightarrow \psi(x)) \vdash (\varphi(c) \rightarrow \psi(c))$, Lemma 2(b)

(2) $\forall x(\varphi(x) \rightarrow \psi(x)), \varphi(c) \vdash \psi(c)$, (1), R1

(3) $\forall x(\varphi(x) \rightarrow \psi(x)), \varphi(c) \vdash \exists x\psi(x)$, (2), Lemma 2(c)

(4) $\forall x(\varphi(x) \rightarrow \psi(x)), \exists x\varphi(x) \vdash \exists x\psi(x)$, (3), Lemma 2(d)

(5) $\vdash$ (**), (4), Theorem 1 (twice).

§2. Soundness and completeness of FH. Of course, we want our formal system to be *sound* in the sense that $\Phi \vdash \varphi$ implies that $\Phi \models \varphi$. And this is easily established.

Theorem 2 (Soundness of FH)**.** If $\Phi \vdash \varphi$, then $\Phi \models \varphi$.

Proof. Let $\varphi_0, \varphi_1, \ldots, \varphi_n$ be a derivation of φ from Φ. We show that for all $k \leq n$,

(*) $\Phi \models \varphi_k$.

This holds for $k = 0$, since φ_0 is either an axiom of FH or a member of Φ. Suppose (*) holds for all $k < m \leq n$. We want to show that it holds for $k = m$. If φ_m is an axiom of FH or a member of Φ, this is true. Suppose there are $i, j < m$ such that $\varphi_j := \varphi_i \rightarrow \varphi_m$. Then, since, by hypothesis, $\Phi \models \varphi_i$ and $\Phi \models \varphi_j$, it follows that $\Phi \models \varphi_m$. Finally, suppose $\varphi_m := \forall x\psi(x)$ and for some $i < m$, $\varphi_i := \psi(c)$, where c does not occur in $\Phi, \psi(x)$. By hypothesis, $\Phi \models \psi(c)$. It follows that $\Phi \models \varphi_m$.

Thus, we have shown that (*) holds for all $k \leq n$ and so, in particular, for $k = n$; in other words, $\Phi \models \varphi$, as desired. ■

It may seem that this proof is circular, that we have "shown" that certain logical principles are valid by appealing to those very principles (plus mathematical induction). But that is not correct. What we have shown is not that the logical principles are valid, that is obvious, or almost, but that our formal rendering of these principles is correct.

A set Φ of sentences is *consistent* (in FH) if there is no sentence φ such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. By Lemma 1(b), if every finite subset of Φ is consistent, so is Φ.

We are now going to prove the following:

Theorem 3. If Φ is consistent, then Φ has a model.

Corollary 1 (Gödel-Henkin Completeness Theorem)**.** If $\Phi \vDash \varphi$, then $\Phi \vdash \varphi$.

The problem in proving Theorem 3 is that, given only that Φ is consistent, we have no idea what a model of Φ may look like. The main idea of the proof is to overcome this difficulty by defining a set Φ^* of sentences such that (i) $\Phi \subseteq \Phi^*$, (ii) Φ^* is consistent, $\ell_{\Phi^*} = \ell_\Phi \cup C$, where C is a set of constants (Lemma 10), (iii) Φ^* can be used in a natural way to define a model $\mathcal{A}$ (the canonical model for Φ^*, see below), and, finally, (iv) it can be shown for every sentence φ of ℓ_{Φ^*} (by induction on the length of φ), that $\mathcal{A} \vDash \varphi$ iff $\varphi \in \Phi^*$ (proof of Lemma 13). It follows that $\mathcal{A} \vDash \Phi^*$ and so $\mathcal{A} \vDash \Phi$, as desired.

Lemma 3. The following conditions are equivalent.
(i) Φ is inconsistent.
(ii) $\Phi \vdash \bot$.
(iii) For every sentence φ, $\Phi \vdash \varphi$.

Proof. (i) $\Rightarrow$ (ii). Let φ be such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. $\Phi \vdash \varphi \wedge \neg\varphi \to \bot$. But then, by Lemma 2(a), $\Phi \vdash \bot$.

(ii) $\Rightarrow$ (iii). Suppose $\Phi \vdash \bot$. Let φ be any sentence. $\Phi \vdash \bot \to \varphi$ and so $\Phi \vdash \varphi$.

(iii) $\Rightarrow$ (i). Asume (iii). Let θ be any sentence. Then $\Phi \vdash \theta \wedge \neg\theta$. It follows that $\Phi \vdash \theta$ and $\Phi \vdash \neg\theta$ and so Φ is inconsistent. ■

Lemma 4. (a) The following conditions are equivalent.
(i) $\Phi \vdash \varphi$.
(ii) $\Phi \cup \{\neg\varphi\}$ is inconsistent.

(b) The following conditions are equivalent.
(iii) $\Phi \vdash \neg\varphi$.
(iv) $\Phi \cup \{\varphi\}$ is inconsistent.

Proof. (a) (i) $\Rightarrow$ (ii). Suppose $\Phi \vdash \varphi$. Then $\Phi \cup \{\neg\varphi\} \vdash \varphi$. Also, clearly, $\Phi \cup \{\neg\varphi\} \vdash \neg\varphi$.

Thus, $\Phi \cup \{\neg\varphi\}$ is inconsistent.

(ii) $\Rightarrow$ (i). Suppose $\Phi \cup \{\neg\varphi\}$ is inconsistent. Then, by Lemma 3, $\Phi \cup \{\neg\varphi\} \vdash \bot$. By the Deduction Theorem, it follows that $\Phi \vdash \neg\varphi \rightarrow \bot$ and so $\Phi \vdash \varphi$.

The proof of (b) is similar. ■

Proof of Corollary 1. Suppose $\Phi \nvdash \varphi$. Then, by Lemma 4, $\Phi \cup \{\neg\varphi\}$ is consistent. It follows, by Theorem 3, that $\Phi \cup \{\neg\varphi\}$ has a model $\mathcal{A}$. But then $\mathcal{A} \models \Phi$ and $\mathcal{A} \not\models \varphi$ and so $\Phi \not\models \varphi$. ■

Theorem 3 can also easily be derived from Corollary 1.

A set Φ of sentences is *explicitly complete* if for every sentence, either $\varphi \in \Phi$ or $\neg\varphi \in \Phi$. (This sense of "complete" is, of course, different from that in which FH is complete.)

The following lemma is clear.

Lemma 5. If Φ is explicitly complete and consistent, then $\Phi \vdash \varphi$ iff $\varphi \in \Phi$.

Lemma 6. (a) If Φ is consistent, then for every sentence φ, either $\Phi \cup \{\varphi\}$ or $\Phi \cup \{\neg\varphi\}$ is consistent.

(b) Suppose X is a set of consistent sets of sentences and for all $\Phi_0, \Phi_1 \in X$, either $\Phi_0 \subseteq \Phi_1$ or $\Phi_1 \subseteq \Phi_0$. Then $\bigcup X$ is consistent.

Proof. (a) Suppose $\Phi \cup \{\varphi\}$ and $\Phi \cup \{\neg\varphi\}$ are inconsistent. Then $\Phi \cup \{\varphi\} \vdash \bot$ and $\Phi \cup \{\neg\varphi\} \vdash \bot$. By the Deduction Theorem, $\Phi \vdash \varphi \rightarrow \bot$ and $\Phi \vdash \neg\varphi \rightarrow \bot$. It follows that $\Phi \vdash \bot$ and so Φ is inconsistent.

(b) Every finite subset of $\bigcup X$ is included in some $\Phi \in X$. ■

Lemma 7 (Lindenbaum's Theorem)**.** If Φ is consistent, there is a set Ψ of sentences of ℓ_Φ such that $\Phi \subseteq \Psi$ and Ψ is explicitly complete and consistent.

Proof. Countable case. We first give a proof under the additional assumption that ℓ_Φ is countable and so the set of sentences is denumerable. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an enumeration of the sentences of ℓ_Φ. Let Φ_n be defined as follows: $\Phi_0 = \Phi$,

$$\begin{aligned} \Phi_{n+1} &= \Phi_n \cup \{\varphi_n\} \text{ if } \Phi_n \cup \{\varphi_n\} \text{ is consistent,} \\ &= \Phi_n \cup \{\neg\varphi_n\} \text{ otherwise.} \end{aligned}$$

Let $\Psi = \bigcup\{\Phi_n : n \in N\}$. Ψ is explicitly complete. If Φ_n is consistent, by Lemma 6(a), so is Φ_{n+1}. Thus all Φ_n are consistent. But then, by Lemma 6(b), Ψ is consistent. ♦

If ℓ_Φ is uncountable, there is no enumeration of the sentences of ℓ_Φ as above and it becomes necessary to use set theory in one form or another: ordinals and definition and proof by transfinite induction or some set-theoretical principle such as the following result.

Let X be a set of subsets of a given set. A *chain* in X is then a subset of X which is linearly ordered by $\subseteq$. A *maximal* element of X is a member of X which is not a proper subset of a member of X.

Zorn's Lemma (special case). Let X be a set of subsets of a given set such that for every chain $Y \subseteq X$, $\bigcup Y \in X$. Then X has a maximal element.

Proof of Lemma 7 (concluded). Uncountable case. Let X be the set of sets Θ of sentences of ℓ_Φ such that $\Phi \subseteq \Theta$ and Θ is consistent. By Lemma 6(b), the union of a chain in X is consistent and so is a member of X. Hence, by Zorn's Lemma, X has a maximal element Ψ. Ψ is consistent. Suppose Ψ is not explicitly complete. Let ψ be such that $\psi, \neg\psi \notin \Psi$. Then, Ψ being maximal, $\Psi \cup \{\psi\}$ and $\Psi \cup \{\neg\psi\}$ are inconsistent, contrary to Lemma 6(a). Thus, Ψ is explicitly complete. ■

Let C be a set of constants. A set Φ of sentences is *witness-complete* (with respect to C) if for every member of Φ of the form $\exists x\varphi(x)$, there is a constant c (in C), a *witness* to $\exists x\varphi(x)$, such that $\varphi(c) \in \Phi$. We shall now show that every consistent set Φ can be extended to an explicitly complete witness-complete consistent set.

Lemma 8. Suppose Φ is consistent and $\exists x\varphi(x) \in \Phi$. Let c be a constant not in ℓ_Φ. Then $\Phi \cup \{\varphi(c)\}$ is consistent.

Proof. Suppose $\Phi \cup \{\varphi(c)\}$ is inconsistent. Then $\Phi \vdash \neg\varphi(c)$ and so, by R2, $\Phi \vdash \forall x \neg\varphi(x)$. But, trivially, $\Phi \vdash \exists x\varphi(x)$. Also we have already shown that $\vdash \forall x\neg\varphi(x) \rightarrow \neg\exists x\varphi(x)$ (Examples 1, 2). It follows that $\Phi \vdash \neg\exists x\varphi(x)$. And so Φ is inconsistent, contrary to assumption. ■

Lemma 9. Suppose Φ is consistent. Let $\{\varphi_i(x_i)\colon i \in I\}$ be the set of formulas $\varphi(x)$ such that $\exists x\varphi(x) \in \Phi$. Let $\{c_i\colon i \in I\}$ be a set of constants not in ℓ_Φ. Then $\Phi \cup \{\varphi_i(c_i)\colon i \in I\}$ is consistent.

Proof. It is sufficient to show that for every finite subset J of I, $\Phi \cup \{\varphi_i(c_i)\colon i \in J\}$ is consistent. But this follows by repeated applications of Lemma 8. ■

Lemma 10. For every consistent set Φ, there is an explicitly complete witness-complete consistent set Φ^* such that $\Phi \subseteq \Phi^*$.

Proof. We define sets of sentences Φ_n, Ψ_n and sets C_n of constants as follows. Let $\Phi_0 = \Phi$ and $C_0 = \varnothing$. Suppose C_n and Φ_n have been defined and Φ_n is a consistent set of sentences of $\ell_\Phi \cup C_n$. Let $\{\varphi_i(x_i)\colon i \in I_n\}$ be the set of formulas $\varphi(x)$ of $\ell_\Phi \cup C_n$ such that $\exists x\varphi(x) \in \Phi_n$. Let $\{c_{i,n}\colon i \in I_n\}$ be a set of constants not in C_n. Let $C_{n+1} = C_n \cup \{c_{i,n}\colon i \in I\}$ and $\Psi_n = \Phi_n \cup \{\varphi_{i,n}(c_{i,n})\colon i \in I_n\}$. Then, by Lemma 9, Ψ_n is consistent. Finally, by Lemma 7, there is an explicitly complete extension Φ_{n+1} of Ψ_n in $\ell_\Phi \cup C_{n+1}$.

Now let

$$\Phi^* = \bigcup\{\Phi_n\colon n \in N\}$$

and $C = \bigcup\{C_n\colon n \in N\}$. Then Φ^* is explicitly complete and witness-complete (with respect to C). Finally, by Lemma 6(b), Φ^* is consistent, as desired. ■

Suppose Ψ, a set of sentences of $\ell_\Phi \cup C$, is consistent, explicitly complete, and witness-complete with respect to C. Let t be a closed term of $\ell_\Phi \cup C$. Since $\vdash \exists x(t = x)$, and so $\exists x(t = x) \in \Psi$, there is a $c \in C$ such that $t = c \in \Psi$.

We define the relation $\sim$ on C by:

$$c \sim d \text{ iff } c = d \in \Psi.$$

By I1, I2, I3, and since Ψ is explicitly complete and consistent, $\sim$ is an equivalence relation. Let [c] be the equivalence class of c. Let [C] be the set of such equivalence classes.

We now define the *canonical model* $\mathcal{A}$ *for* Ψ as follows. $A = [C]$. $c^{\mathcal{A}} = [c]$ for $c \in C$. If $d \in \ell_\Phi$, there is a $c \in C$ such that $c = d \in \Psi$. Let $d^{\mathcal{A}} = [c]$. If $P \in \ell_\Phi$ is an n-place predicate, let

$$P^{\mathcal{A}} = \{\langle [c_1],\ldots,[c_n]\rangle\colon c_1,\ldots,c_n \in C \;\&\; Pc_1\ldots c_n \in \Psi\}.$$

By I4, if $[c_1] = [c_1'],\ldots, [c_n] = [c_n']$, then $Pc_1\ldots c_n \in \Psi$ iff $Pc_1'\ldots c_n' \in \Psi$. And so

$$\langle [c_1],\ldots,[c_n]\rangle \in P^{\mathcal{A}} \text{ iff } Pc_1\ldots c_n \in \Psi.$$

Finally, let $f \in \ell_\Phi$ be an n-place function symbol. Suppose $c_1,\ldots,c_n \in C$. There is a $c \in C$ such that $f(c_1,\ldots,c_n) = c \in \Psi$. Let

$$f^{\mathcal{A}}([c_1],\ldots,[c_n]) = [c].$$

By I5, this is a proper definition of a function $f^{\mathcal{A}}$; in other words,

$$\text{if } [c_1] = [c_1'],\ldots, [c_n] = [c_n'], \text{ then } f^{\mathcal{A}}([c_1],\ldots,[c_n]) = f^{\mathcal{A}}([c_1'],\ldots,[c_n']).$$

Lemma 11. Suppose t is a closed term of $\ell_\Phi \cup C$ and let $c \in C$ be such that $t = c \in \Psi$. Then $t^{\mathcal{A}} = [c]$.

Proof. This is clear if t is a constant. Suppose $t = f(t_1,\ldots,t_n)$ and the statement holds for t_i, $i = 1,\ldots,n$. Let $c_i \in C$ be such that $t_i = c_i \in \Psi$ and consequently $t_i^{\mathcal{A}} = [c_i]$, $i = 1,\ldots,n$. Then $f(c_1,\ldots,c_n) = t \in \Psi$ and so $f(c_1,\ldots,c_n) = c \in \Psi$. But then $f^{\mathcal{A}}([c_1],\ldots,[c_n]) = [c]$. Finally, $t^{\mathcal{A}} = f^{\mathcal{A}}(t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}) = f^{\mathcal{A}}([c_1],\ldots[c_n])$ and so $t^{\mathcal{A}} = [c]$, as desired. ■

Lemma 12. If φ is an atomic sentence of $\ell_\Phi \cup C$, then $\mathcal{A} \models \varphi$ iff $\varphi \in \Psi$.

Proof. First, suppose φ is $t_0 = t_1$. Let $c_i \in C$ be such that $t_i = c_i \in \Psi$ and so, by Lemma 11, $t_i^{\mathcal{A}} = [c_i]$, $i = 0,1$. Then $\mathcal{A} \models \varphi$ iff $t_0^{\mathcal{A}} = t_1^{\mathcal{A}}$ iff $[c_0] = [c_1]$ iff $c_0 = c_1 \in \Psi$ iff $\varphi \in \Psi$.

Next, suppose φ is $Pt_1\ldots,t_n$. Let $c_i \in C$ be such that $t_i = c_i \in \Psi$ and so $t_i^{\mathcal{A}} = [c_i]$, $i = 0,\ldots,n$. Then $\mathcal{A} \models \varphi$ iff $\langle t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\rangle \in P^{\mathcal{A}}$ iff $\langle [c_1],\ldots,[c_n]\rangle \in P^{\mathcal{A}}$ iff $Pc_1\ldots c_n \in \Psi$ iff $\varphi \in \Psi$. ■

Lemma 13. Suppose Ψ is consistent, explicitly complete, and witness-complete and let $\mathcal{A}$ be the canonical model for Ψ. Then $\mathcal{A} \models \Psi$.

Proof. We prove, by induction, that for every sentence φ of ℓ_Ψ,
(*) $\quad \mathcal{A} \models \varphi$ iff $\varphi \in \Psi$.
For φ an atomic formula this is Lemma 12.

Suppose now φ is not atomic. We verify (*) for (i) $\varphi := \neg\psi$, (ii) $\varphi := \psi \vee \theta$, and (iii) $\varphi := \forall x\psi(x)$. The remaining cases are similar.

(i) If $\mathcal{A} \models \varphi$, then $\mathcal{A} \not\models \psi$, whence, by the inductive assumption, $\psi \notin \Psi$. Since Ψ is explicitly complete, this implies that $\varphi \in \Psi$.

Suppose $\varphi \in \Psi$. Then, Ψ being consistent, $\psi \notin \Psi$, whence $\mathcal{A} \not\models \psi$ and so $\mathcal{A} \models \varphi$.

(ii) Suppose $\mathcal{A} \models \varphi$. Then $\mathcal{A} \models \psi$ or $\mathcal{A} \models \theta$. By the inductive assumption, $\psi \in \Psi$ or $\theta \in \Psi$. It follows that $\Psi \vdash \varphi$ and so, by Lemma 5, $\varphi \in \Psi$.

Next, suppose $\varphi \in \Psi$. If $\mathcal{A} \not\models \psi$ and $\mathcal{A} \not\models \theta$, then $\psi, \theta \notin \Psi$, whence $\neg\psi, \neg\theta \in \Psi$, whence $\Psi \vdash \neg(\psi \vee \theta)$ and so Ψ is inconsistent. Thus, either $\mathcal{A} \models \psi$ or $\mathcal{A} \models \theta$.

(iii) Suppose $\mathcal{A} \models \varphi$. Suppose $\varphi \notin \Psi$. Then $\neg\varphi \in \Psi$. But $\vdash \neg\forall x\psi(x) \to \exists x\neg\psi(x)$ (we leave the proof of this to the reader). It follows that $\Psi \vdash \exists x\neg\psi(x)$ and so that $\exists x\neg\psi(x) \in \Psi$. Since Ψ is witness-complete, this implies that there is a constant c such that $\neg\psi(c) \in \Psi$ and so $\psi(c) \notin \Psi$. But then, by the inductive hypothesis, $\mathcal{A} \not\models \psi(c)$ and so $\mathcal{A} \not\models \varphi$, a contradiction. Thus, $\varphi \in \Psi$.

Next, suppose $\varphi \in \Psi$. Then, by Lemma 2(b) and Lemma 5, $\psi(c) \in \Psi$ for every constant c. But then $\mathcal{A} \models \psi(c)$ for every c. Finally, since $\mathcal{A}$ is canonical, this implies that $\mathcal{A} \models \varphi$, as desired.

This concludes the inductive proof of (*) and thereby proof of the lemma. ■

Proof of Theorem 3. Suppose Φ is consistent. By Lemma 10, there is an explicitly complete witness-complete consistent set Φ^* such that $\Phi \subseteq \Phi^*$. Let $\mathcal{A}$ be the canonical model for Φ^*. By Lemma 13, $\mathcal{A} \models \Phi^*$ and so $\mathcal{A} \models \Phi$. ■

Let φ be any valid sentence. By Corollary 1, there is then a derivation d of φ (from the empty set) in FH. It is then natural to ask if we can impose an upper bound on the length $|d|$ of d, i.e., the number of occurrences of symbols in d, in terms of the length $|\varphi|$ of φ in some (any) reasonably interesting way. Similar questions can be asked about the calculi GS and ND presented below. These questions will be answered in Chapter 4.

§3. A Gentzen-type sequent calculus. The main disadvantage of FH, in addition to the fact that it is quite unnatural, is that given that a formula φ is derivable in FH, we know next to nothing about its derivation: we know nothing about which formulas occur in the derivation nor how complicated they are. If φ has been derived by R1 from ψ and $\psi \to \varphi$, there is no way of working "backwards" to reconstruct ψ from φ or even estimate the complexity of ψ. In this section we introduce a logical calculus, GS, for which we know a great deal about the formulas occurring in any derivation (see, for example, the Subformula Property, below). On the other hand many obvious logical principles such as Modus Ponens (rule R1 of FH) and (the more general) Cut Rule (p. 30) are now difficult to derive.

In this section and the next two sections we assume, for simplicity, that there are *no function symbols*. Also, the presence of = causes certain technical problems, of limited interest in themselves, and so we restrict ourselves to *formulas not containing* =.

We now add a new symbol $\Rightarrow$ (implies) to the formal language. Γ, Δ are *finite* sets (not sequences) of sentences (not containing $\Rightarrow$). Expressions such as $\Gamma \Rightarrow \Delta$ are called *sequents*. (Γ and/or Δ may be empty. If Γ is empty, we may write $\Rightarrow \Delta$ for $\Gamma \Rightarrow \Delta$ and similarly if Δ or both Γ and Δ are empty.) The intended intuitive interpretation of $\Gamma \Rightarrow \Delta$ is that the conjunction of Γ implies the disjunction of Δ. ($\bigwedge\varnothing$ is true and $\bigvee\varnothing$ is false.) We write $\mathcal{A} \models \Gamma \Rightarrow \Delta$ to mean that if $\mathcal{A} \models \Gamma$, then $\mathcal{A} \models \varphi$ for some $\varphi \in \Delta$. $\Gamma \Rightarrow \Delta$ is (*logically*) *valid*, $\models \Gamma \Rightarrow \Delta$, if $\mathcal{A} \models \Gamma \Rightarrow \Delta$ for every $\mathcal{A}$. The union of Γ and Δ will be written as Γ, Δ. Γ, φ is $\Gamma, \{\varphi\}$.

Axioms of GS: All sequents of the form $\Gamma, \varphi \Rightarrow \Delta, \varphi$.

Rules of inference of GS:

$(\Rightarrow\neg)$ $\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$ $\quad(\neg\Rightarrow)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg\varphi \Rightarrow \Delta}$

$(\Rightarrow\wedge)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi_0 \quad \Gamma \Rightarrow \Delta, \varphi_1}{\Gamma \Rightarrow \Delta, \varphi_0 \wedge \varphi_1}$ $\quad(\wedge\Rightarrow)$ $\dfrac{\Gamma, \varphi_0, \varphi_1 \Rightarrow \Delta}{\Gamma, \varphi_0 \wedge \varphi_1 \Rightarrow \Delta}$

$(\Rightarrow\vee)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi_0, \varphi_1}{\Gamma \Rightarrow \Delta, \varphi_0 \vee \varphi_1}$ $\quad(\vee\Rightarrow)$ $\dfrac{\Gamma, \varphi_0 \Rightarrow \Delta \quad \Gamma, \varphi_1 \Rightarrow \Delta}{\Gamma, \varphi_0 \vee \varphi_1 \Rightarrow \Delta}$

$$(\Rightarrow\to)\quad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi} \qquad\qquad (\to\Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}$$

$$(\Rightarrow\exists)\quad \frac{\Gamma \Rightarrow \Delta, \psi(c)}{\Gamma \Rightarrow \Delta, \exists x\psi(x)} \qquad\qquad (\exists\Rightarrow)\quad \frac{\Gamma, \psi(c) \Rightarrow \Delta}{\Gamma, \exists x\psi(x) \Rightarrow \Delta}$$

$$(\Rightarrow\forall)\quad \frac{\Gamma \Rightarrow \Delta, \psi(c)}{\Gamma \Rightarrow \Delta, \forall x\psi(x)} \qquad\qquad (\forall\Rightarrow)\quad \frac{\Gamma, \psi(c) \Rightarrow \Delta}{\Gamma, \forall x\psi(x) \Rightarrow \Delta}$$

In $(\exists\Rightarrow)$ and $(\Rightarrow\forall)$ the individual constant c must not occur below the line.

In the conclusion of instances of each of these rules a logical constant is introduced. The formula containing this constant is called the *principal formula* of the inference; and the formula or formulas shown explicitly in the premise(s) its *active formula(s)*.

It may be observed that, unlike the rules of FH and those of ND (below), the rules of GS are inversely valid in the sense that if the conclusion is valid, then the pemise(s) is (are) valid. Another important difference is that in GS, but not in FH or ND, there are explicit rules for each of the propositional connectives.

Derivations in GS take the form of trees in a quite obvious way. We use $\vdash_{GS}$ to denote derivability in GS. In this and the following two §§ we write $\vdash$ for $\vdash_{GS}$.

Example 4. $\vdash \forall x(Fx \vee Gx) \Rightarrow \forall xFx, \exists x(\neg Fx \wedge Gx)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{Fa \Rightarrow Fa, \neg Fa \wedge Ga \qquad \dfrac{\dfrac{Ga, Fa \Rightarrow Fa}{Ga \Rightarrow Fa, \neg Fa}\,(\Rightarrow\neg) \qquad Ga \Rightarrow Fa, Ga}{Ga \Rightarrow Fa, \neg Fa \wedge Ga}\,(\Rightarrow\wedge)}{Fa \vee Ga \Rightarrow Fa, \neg Fa \wedge Ga}\,(\vee\Rightarrow)}{Fa \vee Ga \Rightarrow Fa, \exists x(\neg Fx \wedge Gx)}\,(\Rightarrow\exists)}{\forall x(Fx \vee Gx) \Rightarrow Fa, \exists x(\neg Fx \wedge Gx)}\,(\forall\Rightarrow)$$

$$\dfrac{\forall x(Fx \vee Gx) \Rightarrow Fa, \exists x(\neg Fx \wedge Gx)}{\forall x(Fx \vee Gx) \Rightarrow \forall xFx, \exists x(\neg Fx \wedge Gx)}\,(\Rightarrow\forall) \;\blacksquare$$

Example 5. Suppose ψ is a sentence.

$\vdash \psi \to \exists x\varphi(x) \Rightarrow \forall x\neg\varphi(x) \to \neg\psi$.

Let a be a new constant.

$$\dfrac{\dfrac{\dfrac{\psi, \forall x\neg\varphi(x), \Rightarrow \psi}{\forall x\neg\varphi(x), \Rightarrow \psi, \neg\psi}\,(\Rightarrow\neg)}{\Rightarrow \psi, \forall x\neg\varphi(x) \to \neg\psi}\,(\Rightarrow\to) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\varphi(a) \Rightarrow \varphi(a), \neg\psi}{\varphi(a), \neg\varphi(a) \Rightarrow \neg\psi}\,(\neg\Rightarrow)}{\varphi(a), \forall x\neg\varphi(x) \Rightarrow \neg\psi}\,(\forall\Rightarrow)}{\exists x\varphi(x), \forall x\neg\varphi(x) \Rightarrow \neg\psi}\,(\exists\Rightarrow)}{\exists x\varphi(x) \Rightarrow \forall x\neg\varphi(x) \to \neg\psi}\,(\Rightarrow\to)}{\psi \to \exists x\varphi(x) \Rightarrow \forall x\neg\varphi(x) \to \neg\psi}\,(\to\Rightarrow) \;\blacksquare$$

A derivation of a sequent $\Gamma \Rightarrow \Delta$ in GS may be though of as (the inverse of) an abortive attempt to construct a counterexample to $\Gamma \Rightarrow \Delta$, i.e., a model $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$ and $\mathfrak{A} \not\models \psi$ for every $\psi \in \Delta$. Proceeding from the bottom up we try to make all the formulas occurring to the left of $\Rightarrow$ true (in $\mathfrak{A}$) and all the formulas occurring to the right of $\Rightarrow$ false along at least one branch. And we give up only if some formula occurs both to the left and to the right of $\Rightarrow$ (as in the axioms of GS). This can always be done in such a way that the result is either a counterexample to $\Gamma \Rightarrow \Delta$ or a derivation of $\Gamma \Rightarrow \Delta$ in GS (see Examples 1, 2 in Appendix 1 and the proof of Theorem 7, below).

As is easily checked, GS has the:

Subformula Property. Every formula occurring in the derivation of a sequent S is a subformula or a formula occurring in S.

Here "subformula" is used in the somewhat technical sense: φ is a *subformula* of ψ if φ is a subformula of ψ in the usual sense or is obtained from such a formula by replacing free variables by individual constants.

From the Subformula Property it follows at once that any logical constant occurring in a derivation of S occurs in S.

It may be observed that

$$\text{if } \vdash \Gamma \Rightarrow \Delta,\ \Gamma \subseteq \Gamma',\ \text{and } \Delta \subseteq \Delta',\ \text{then } \vdash \Gamma' \Rightarrow \Delta'.$$

This is true, since the constants occurring in instances of ($\exists\Rightarrow$) and ($\Rightarrow\forall$) can always be assumed not to occur in Γ', Δ'.

Certain obviously sound principles (derived rules of inference) are rather difficult to establish for GS. The prime example is the so-called Cut Rule:

$$\text{(Cut)} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta}$$

In fact, the result that (Cut) is a derived rule of GS, the so-called Cut Elimination Theorem, is one of the major results of the proof theory of L_1. With (Cut) added the system no longer has the Subformula Property (see Appendix 1, Example 1).

The system $GS^=$ is obtained from GS by adding the sequents $\Gamma \Rightarrow \Delta, c = c$, where c is any individual constant, as new axioms and the rules of inference:

$$\frac{\Gamma, \varphi(c) \Rightarrow \Delta, \psi(c)}{\Gamma, \varphi(c'), c = c' \ [c' = c] \Rightarrow \Delta, \psi(c')}$$

Theorems 4(a) and 7, below, can be extended to $GS^=$.

For completeness we have stated the inference rules for $\rightarrow$. But in the next two sections*we do not regard $\rightarrow$ as a primitive symbol* partly because the rules of derivation for $\rightarrow$ cause problems similar to those caused by the $\neg$-rules (see below). Instead we think of $\varphi \to \psi$ as an abbreviation of $\neg\varphi \vee \psi$. The $\to$-rules are then derived rules. But we do retain all of $\wedge$, $\vee$, $\forall$, $\exists$, since we want to be able to write formulas in n.n.f.

§4. Two applications. In this § we give two applications, Theorems 4 and 5, below, of GS or, more accurately, three closely related systems GS^a, $GS^\perp$, and GS^*.

Let GS^a be obtained from GS by taking as axioms only those axioms of GS in which all formulas are atomic. Let $GS^\perp$ be obtained from GS by adding the constant $\perp$ to the language and all sequents $\Gamma, \perp \Rightarrow \Delta$ to the set of axioms. Finally, let GS^* be obtained from GS^a by replacing the rules $(\Rightarrow\wedge)$ and $(\vee\Rightarrow)$ by:

$$(\Rightarrow\wedge)^* \qquad \frac{\Gamma_0 \Rightarrow \Delta_0, \varphi_0 \qquad \Gamma_1 \Rightarrow \Delta_1, \varphi_1}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1, \varphi_0 \wedge \varphi_1} \quad \text{and} \quad (\vee\Rightarrow)^* \quad \frac{\Gamma_0, \varphi_0 \Rightarrow \Delta_0 \qquad \Gamma_1, \varphi_1 \Rightarrow \Delta_1}{\Gamma_0, \Gamma_1, \varphi_0 \vee \varphi_1 \Rightarrow \Delta_0, \Delta_1}$$

We denote derivability in GS^a, $GS^\perp$, GS^* by $\vdash^a$, $\vdash^\perp$, $\vdash^*$, respectively.

Clearly, GS^a, $GS^\perp$, GS^* have the Subformula Property.

Lemma 14. The systems GS^a, $GS^\perp$, GS^* are equivalent to GS for sequents of GS.

Proof. Obviously, $\vdash^a \Gamma \Rightarrow \Delta$ implies $\vdash \Gamma \Rightarrow \Delta$. To prove the inverse implication it is sufficient to show that all axioms of GS are derivable in GS^a. If

(1) $\quad \Gamma, \varphi \Rightarrow \Delta, \varphi$

is an axiom of GS which is not an axiom of GS^a, either some member ψ of Γ or Δ is not atomic or φ is not atomic. In both cases the non-atomic formula can be replaced by one or two simpler formulas, i.e., formulas containing fewer logical constants. If, for example, $\neg\theta$ is a member of Γ, we replace (1) by $\Gamma - \{\neg\theta\}, \varphi \Rightarrow \Delta, \theta, \varphi$ from which (1) can be derived by $(\neg\Rightarrow)$. If $\forall x\psi(x)$ is a member of Δ, let c be a new constant and replace (1) by $\Gamma, \varphi \Rightarrow \Delta - \{\forall x\psi(x)\}, \psi(c), \varphi$ from which (1) can be derived by $(\Rightarrow\forall)$. The remaining cases are similar.

If φ is not atomic, for example, $\varphi := \varphi_0 \wedge \varphi_1$ or $\varphi := \exists x\psi(x)$, then φ can be replaced by simpler formulas as follows:

$$\dfrac{\dfrac{\Gamma, \varphi_0, \varphi_1 \Rightarrow \Delta, \varphi_0}{\Gamma, \varphi_0 \wedge \varphi_1 \Rightarrow \Delta, \varphi_0}(\wedge\Rightarrow) \quad \dfrac{\Gamma, \varphi_0, \varphi_1 \Rightarrow \Delta, \varphi_1}{\Gamma, \varphi_0 \wedge \varphi_1 \Rightarrow \Delta, \varphi}(\wedge\Rightarrow)}{\Gamma, \varphi_0 \wedge \varphi_1 \Rightarrow \Delta, \varphi_0 \wedge \varphi_1}(\Rightarrow\wedge) \qquad \dfrac{\dfrac{\Gamma, \psi(c) \Rightarrow \Delta, \psi(c)}{\Gamma, \psi(c) \Rightarrow \Delta, \exists x\psi(x)}(\Rightarrow\exists)}{\Gamma, \exists x\psi(x) \Rightarrow \Delta, \exists x\psi(x)}(\exists\Rightarrow)$$

The remaining cases are similar. In this way axioms of GS containing non-atomic formulas can be replaced by axioms of GS^a.

Obviously, $\vdash S$ implies $\vdash^\perp S$. If S is a sequent of GS and $\vdash^\perp S$, then, by the Subformula Property for $GS^\perp$, $\perp$ does not occur in the derivation D of S in $GS^\perp$. But then D is a derivation in GS and so $\vdash S$.

$\vdash S$ implies $\vdash^a S$ and, obviously, $\vdash^a S$ implies $\vdash^* S$. Thus, $\vdash S$ implies $\vdash^* S$.

We have observed that if $\vdash \Gamma' \Rightarrow \Delta'$, $\Gamma' \subseteq \Gamma''$, and $\Delta' \subseteq \Delta''$, then $\vdash \Gamma'' \Rightarrow \Delta''$. It follows that $(\Rightarrow\wedge)^*$ and $(\vee\Rightarrow)^*$ are derived rules of GS. Thus, $\vdash^* S$ implies $\vdash S$. ■

As an application of $GS^\perp$ we shall now prove a (sufficiently general) special case of the following interpolation theorem.

χ is an *interpolant* for $\Gamma \Rightarrow \Delta$ in GS ($GS^\perp$) if χ is a sentence of $\ell_\Gamma \cap \ell_\Delta$ such that $\vdash \Gamma \Rightarrow \chi$ and $\vdash \chi \Rightarrow \Delta$ ($\vdash^\perp \Gamma \Rightarrow \chi$ and $\vdash^\perp \chi \Rightarrow \Delta$).

Note that if ℓ contains no predicates, the set of formulas of ℓ in GS is empty. Thus, if $\ell_\Gamma \cap \ell_\Delta$ contains no predicates, there is no interpolant for $\Gamma \Rightarrow \Delta$ in GS and the only possible interpolants for $\Gamma \Rightarrow \Delta$ in are propositional combinations of $\perp$'s.

Theorem 4. (a) (Interpolation Theorem for $GS^\perp$)**.** If $\vdash^\perp \Gamma \Rightarrow \Delta$, there is an interpolant for $\Gamma \Rightarrow \Delta$ in $GS^\perp$.

(b) If $\vdash^\perp \Gamma \Rightarrow \Delta$ and Γ, Δ have no predicates in common, then $\vdash^\perp \Gamma \Rightarrow$ or $\vdash^\perp \Rightarrow \Delta$.

The proof if this result is quite long and will only be sketched. But we shall prove Theorem 4 assuming that the members of Γ, Δ are in n.n.f.

A rule of inference *preserves the existence of interpolants* if whenever there is an interpolant for an instance of the premise of the rule or there are interpolants for the premises of an instance of the rule, there is one for its conclusion.

If $\Gamma \Rightarrow \Delta$ is an axiom, then, trivially, there is an interpolant for $\Gamma \Rightarrow \Delta$. Moreover, we have the following:

Lemma 15. The existence of interpolants is preserved under the $\wedge$-, $\vee$-, $\exists$-, $\forall$-rules.

Proof. We verify this in three cases; the remaining cases are similar. First, consider an instance of $(\vee\Rightarrow)$:

$$\frac{\Gamma, \varphi_0 \Rightarrow \Delta \qquad \Gamma, \varphi_1 \Rightarrow \Delta}{\Gamma, \varphi_0 \vee \varphi_1 \Rightarrow \Delta}$$

By assumption there are interpolants χ_i for $\Gamma, \varphi_i \Rightarrow \Delta$, i = 0, 1. Then, by $(\vee\Rightarrow)$ and $(\Rightarrow\vee)$, $\vdash^\perp \Gamma, \varphi_0 \vee \varphi_1 \Rightarrow \chi_0 \vee \chi_1$ and, by $(\vee\Rightarrow)$, $\vdash^\perp \chi_0 \vee \chi_1 \Rightarrow \Delta$. Every predicate in $\chi_0 \vee \chi_1$ occurs in $\Gamma, \varphi_0 \vee \varphi_1$ and in Δ. $\chi_0 \vee \chi_1$ is an interpolant for $\Gamma, \varphi_0 \vee \varphi_1 \Rightarrow \Delta$.

Next, consider an instance of $(\Rightarrow\exists)$:

$$\frac{\Gamma \Rightarrow \Delta, \varphi(c)}{\Gamma \Rightarrow \Delta, \exists x\varphi(x)}$$

By hypothesis there is a formula $\chi(x)$ not containing c such that $\chi(c)$ is an interpolant for $\Gamma \Rightarrow \Delta, \varphi(c)$.

There are then three cases.

Case 1. c occurs in both Γ and $\Delta, \exists x\varphi(x)$. $\chi(c)$ is an interpolant for $\Gamma \Rightarrow \Delta, \exists x\varphi(x)$.

Case 2. c does not occur in Γ. Then c does not occur in $\chi(c)$ and so $\chi(x)$ is a sentence χ. χ is an interpolant for $\Gamma \Rightarrow \Delta, \exists x\varphi(x)$.

Case 3. c does not occur in $\Delta, \exists x\varphi(x)$. Then, by $(\Rightarrow\exists)$, $\vdash^{\perp} \Gamma \Rightarrow \exists x\chi(x)$ and, by $(\exists\Rightarrow)$, $\vdash^{\perp} \exists x\chi(x) \Rightarrow \Delta, \exists x\varphi(x)$. Thus, $\exists x\chi(x)$ is an interpolant for $\Gamma \Rightarrow \Delta, \exists x\varphi(x)$.

Finally, consider an instance of $(\exists\Rightarrow)$:

$$\frac{\Gamma, \varphi(c) \Rightarrow \Delta}{\Gamma, \exists x\varphi(x) \Rightarrow \Delta}$$

By hypothesis there is an interpolant χ for $\Gamma, \varphi(c) \Rightarrow \Delta$. By $(\exists\Rightarrow)$, $\vdash^{\perp} \Gamma, \exists x\varphi(x) \Rightarrow \chi$. Thus, χ is an interpolant for $\Gamma, \exists x\varphi(x) \Rightarrow \Delta$. ■

Note that we may now infer the interpolation theorem for positive formulas, i.e., formulas with no logical constants other than $\wedge$, $\vee$, $\exists$, $\forall$.

The $\neg$-rules and $\rightarrow$-rules, too, preserve the existence of interpolants; the problem is that this is not so obvious. To deal with this difficulty one has to prove the following more complicated lemma. Let $\neg\Gamma = \{\neg\varphi\colon \varphi \in \Gamma\}$.

Lemma 16. If $\vdash^{\perp} \Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1$, there is an interpolant for $\Gamma_0, \neg\Delta_0 \Rightarrow \neg\Gamma_1, \Delta_1$.

We shall not give the (rather long) proof of this lemma.

To deal with the $\neg$-rules for formulas in n.n.f. we use the following:

Lemma 17. In any derivation (in $GS^{\perp}$), any instance of a non-$\neg$-rule R immediately followed by a $\neg$-inference, whose active formula is not the principal formula of the instance of R, can be replaced by one instance or two side-by-side instances of the $\neg$-rule followed by an instance of R.

Proof. We consider just two special cases; the other cases are similar. If R is $(\Rightarrow\wedge)$ and the $\neg$-rule is $(\neg\Rightarrow)$, the relevant part of the derivation looks as follows:

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi_0 \qquad \Gamma \Rightarrow \Delta, \varphi, \psi_1}{\Gamma \Rightarrow \Delta, \varphi, \psi_0 \wedge \psi_1}}{\Gamma, \neg\varphi \Rightarrow \Delta, \psi_0 \wedge \psi_1}$$

This can be replaced by:

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi_0}{\Gamma, \neg\varphi \Rightarrow \Delta, \psi_0} \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi_1}{\Gamma, \neg\varphi \Rightarrow \Delta, \psi_1}}{\Gamma, \neg\varphi \Rightarrow \Delta, \psi_0 \wedge \psi_1}$$

Next, suppose R is $(\Rightarrow\forall)$ and the $\neg$-rule is $(\Rightarrow\neg)$. The relevant part of the derivation then looks as follows:

$$\frac{\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi(c)}{\Gamma, \varphi \Rightarrow \Delta, \forall x\psi(x)}}{\Gamma \Rightarrow \Delta, \forall x\psi(x), \neg\varphi}$$

This can be replaced by:

$$\frac{\dfrac{\Gamma, \varphi \Rightarrow \Delta, \psi(c)}{\Gamma \Rightarrow \Delta, \psi(c), \neg\varphi}}{\Gamma \Rightarrow \Delta, \forall x\psi(x), \neg\varphi}$$ ■

Proof of Theorem 4 for n.n.f. formulas. (a) Suppose $\vdash^{\perp} \Gamma \Rightarrow \Delta$, where the members of Γ, Δ are in n.n.f. Let D be a derivation of $\Gamma \Rightarrow \Delta$ in $GS^{\perp}$. By the

Subformula Property for $GS^{\perp}$, the active formula of every instance of a ¬-rule in D is atomic and so cannot be the principal formula of a non-¬-rule. Hence, by Lemma 17, there is a derivation D' of $\Gamma \Rightarrow \Delta$ in which no non-¬-inference is followed by a ¬-inference.

In D' every branch begins with a number, possibly zero, of ¬-inferences and below these there is no ¬-inference. If a sequent is an axiom or obtained from an axiom by ¬-inferences, it is of the form $\Gamma', \varphi \Rightarrow \Delta', \varphi$ or $\Gamma', \varphi, \neg\varphi \Rightarrow \Delta'$ or $\Gamma' \Rightarrow \Delta', \varphi, \neg\varphi$ or $\Gamma', \bot \Rightarrow \Delta'$ or $\Gamma' \Rightarrow \Delta', \neg\bot$. (All formulas are in n.n.f.) In these cases φ, $\bot$, $\neg\bot$, $\bot$, $\neg\bot$, respectively, are interpolants for $\Gamma' \Rightarrow \Delta'$. Now use Lemma 15.

(b) Let us say that $\Gamma \Rightarrow \Delta$ is *hyperderivable* if either $\vdash^{\perp} \Gamma \Rightarrow$ or $\vdash^{\perp} \Rightarrow \Delta$. Let D be a derivation of $\Gamma \Rightarrow \Delta$ in which no non-¬-inference is followed by a ¬-inference. Let $\Gamma' \Rightarrow \Delta'$ be an upermost sequent in D, possibly $\Gamma \Rightarrow \Delta$, below which there is no ¬-inference. Then Γ', Δ' have no predicate in common. For if they do, the antecedent and consequent of every sequent below $\Gamma' \Rightarrow \Delta'$ will have a predicate in common. It follows that $\Gamma' \Rightarrow \Delta'$ is either an axiom $\Gamma'', \bot \Rightarrow \Delta''$ for $\bot$ or not an axiom and so is the conclusion of a ¬-inference. In the latter case $\Gamma' \Rightarrow \Delta'$ is of the form $\Gamma'', \varphi, \neg\varphi \Rightarrow \Delta''$ or $\Gamma'' \Rightarrow \Delta'', \varphi, \neg\varphi$ or $\Gamma'' \Rightarrow \Delta'', \neg\bot$. In all these cases $\Gamma' \Rightarrow \Delta'$ is hyperderivable. Furthermore, if the premise (premises) of a non-¬-inference is (are) hyperderivable, so is its conclusion. Thus, $\Gamma \Rightarrow \Delta$ is hyperderivable. ■

Even if $\bot$ does not occur in $\Gamma \Rightarrow \Delta$ and Γ, Δ have a predicate in common, the interpolant χ for $\Gamma \Rightarrow \Delta$ defined in the above proof of Theorem 4 for n.n.f. formulas may contain $\bot$. But such occurrences of $\bot$ can easily be eliminated (from the respective derivations). Thus, we have:

Corollary 2. (a) (Interpolation Theorem for GS)**.** If $\vdash \Gamma \Rightarrow \Delta$ and $\boldsymbol{\ell}_\Gamma$ and $\boldsymbol{\ell}_\Delta$ have a predicate in common, there is an interpolant for $\Gamma \Rightarrow \Delta$ in GS.

(b) If $\vdash \Gamma \Rightarrow \Delta$ and $\boldsymbol{\ell}_\Gamma \cap \boldsymbol{\ell}_\Delta = \varnothing$, then $\vdash \Gamma \Rightarrow$ or $\vdash \Rightarrow \Delta$.

Our second application, of GS*, concerns derivations of those valid sequents in which all formulas are in prenex normal form.

A derivation D in GS* is in *PQ normal form* if all propositional inferences precede all quantificational inferences and the sequents appearing in propositional inferences are quantifier-free. This implies that there is a sequent S in D such that all inferences above S are propositional and all inferences at or below S are quantificational. Since the quantificational rules are one-premise rules, it follows that the sequents below S are linearly ordered. Thus, a derivation in PQ normal form in GS is of a particularly simple and perspicuous form. The derivation in Example 4, above, is in PQ normal form.

Theorem 5. Suppose S is derivable in GS and all formulas occurring in S are prenex formulas. There is then a derivation of S in GS in PQ normal form.

A derivation D (in GS or GS*) is said to be *strict* if for every instance of $(\exists\Rightarrow)$ in D, the constant c does not occur anywhere in D except above the conclusion of that instance; and similarly for instances of $(\Rightarrow\forall)$.

Lemma 18. For every derivation in GS (GS*), there is a strict derivation in GS (GS*) of the same sequent.

Proof. Let D be any derivation in GS or GS*. For every instance

$$\dfrac{\Gamma, \varphi(c) \Rightarrow \Delta}{\Gamma, \exists x\varphi(x) \Rightarrow \Delta}$$

of $(\exists\Rightarrow)$ in D, let d be a constant not occurring in D and replace c in the formulas above $\Gamma, \exists x\varphi(x) \Rightarrow \Delta$ by d. Similarly for instances of $(\Rightarrow\forall)$. Repeating this operation we eventually obtain a strict derivation. ■

Lemma 19. In a strict derivation in GS*, an instance of a quantifier rule R, with principal formula φ, immediately followed by an instance of a propositional rule R′ in which φ is not an active formula, can be replaced by an instance of R′ immediately followed by an instance of R. The resulting derivation is strict.

Proof. If R′ is a ¬-rule, this is Lemma 17. We consider two more special cases; the remaining cases are similar.

(i) R is $(\exists\Rightarrow)$ and R′ is $(\wedge\Rightarrow)$. The relevant part of the derivation then looks as follows:

$$\dfrac{\dfrac{\Gamma, \psi_0, \psi_1, \varphi(c) \Rightarrow \Delta}{\Gamma, \psi_0, \psi_1, \exists x\varphi(x) \Rightarrow \Delta}}{\Gamma, \psi_0 \wedge \psi_1, \exists x\varphi(x) \Rightarrow \Delta}$$

This can be replaced by:

$$\dfrac{\dfrac{\Gamma, \psi_0, \psi_1, \varphi(c) \Rightarrow \Delta}{\Gamma, \psi_0 \wedge \psi_1, \varphi(c) \Rightarrow \Delta}}{\Gamma, \psi_0 \wedge \psi_1, \exists x\varphi(x) \Rightarrow \Delta}$$

(ii) R is $(\Rightarrow\forall)$ and R′ is $(\vee\Rightarrow)^*$. The relevant part of the derivation is then:

$$\dfrac{\Gamma_0, \psi_0 \Rightarrow \Delta_0 \qquad \dfrac{\Gamma_1, \psi_1 \Rightarrow \Delta_1, \varphi(c)}{\Gamma_1, \psi_1 \Rightarrow \Delta_1, \forall x\varphi(x)}}{\Gamma_0, \Gamma_1, \psi_0 \vee \psi_1 \Rightarrow \Delta_0, \Delta_1, \forall x\varphi(x)}\ (\vee\Rightarrow)^*$$

This can be replaced by:

$$\dfrac{\dfrac{\Gamma_0, \psi_0 \Rightarrow \Delta_0 \qquad \Gamma_1, \psi_1 \Rightarrow \Delta_1, \varphi(c)}{\Gamma_0, \Gamma_1, \psi_0 \vee \psi_1 \Rightarrow \Delta_0, \Delta_1, \varphi(c)}\ (\vee\Rightarrow)^*}{\Gamma_0, \Gamma_1, \psi_0 \vee \psi_1 \Rightarrow \Delta_0, \Delta_1, \forall x\varphi(x)}$$

It is then clear that the istances of ($\exists\Rightarrow$) and ($\Rightarrow\forall$) in the new derivations are legal and that these derivations are strict. ■

Lemma 20. If D is a derivation (in GS or GS*) of a sequent S such that all formulas in S are prenex (including quantifier-free formulas), then no formula containing a quantifier is an active formula of a propositional inference in D.

Proof. By the Subformula Property, ever formula occurring in D is prenex. Moreover, if an active formula of a propositional inference contains a quantifier, then the principal formula of that inference is not prenex. ■

Proof of Theorem 5. Suppose all formulas occurring in S are prenex formulas. By Lemmas 14 and 18, there is a strict derivation D′ of S in GS*. By Lemma 20, no principal formula of a quantifier inference in D′ is an active formula of a propositional inference. But then, by repeated application of Lemma 19, there is a (strict) derivation D′′ of S in GS* in PQ normal form.

Finally, it the propositional part of D′′ can be replaced by a (quantifier-free) derivation in GS. The result is a derivation of S in GS in PQ normal form. ■

Corollary 3. Suppose $\vdash \Rightarrow \varphi$ and φ is in prenex normal form. There is then a finite set Δ of quantifier-free sentences such that $\bigvee\Delta$ is a propositional tautology and $\Rightarrow \varphi$ can be obtained from $\Rightarrow \Delta$ by applying the rules ($\Rightarrow\exists$) and ($\Rightarrow\forall$).

§5. Soundness and completeness of GS. If $\Gamma \Rightarrow \Delta$ is an axiom of GS, then obviously $\vDash \Gamma \Rightarrow \Delta$. It is also easy to verify that for any instance of a rule of derivation (of GS), if the premise (premises) is (are) valid, so is the conclusion. Thus, we get:

Theorem 6 (Soundness of GS)**.** If $\vdash \Gamma \Rightarrow \Delta$, then $\vDash \Gamma \Rightarrow \Delta$.

Next, we prove the following completeness theorem. The proof is similar to that of Theorem 3, but because of the special nature of the rules of derivation of GS, we have to proceed somewhat more carefully.

Theorem 7 (Completeness Theorem for GS)**.** If $\vDash \Gamma \Rightarrow \Delta$, then $\vdash \Gamma \Rightarrow \Delta$.

It is easy to derive the completeness theorem for GS with the Cut Rule from that for FH. But then to obtain completeness of GS we need the Cut Elimination Theorem. For this reason we shall instead give a direct proof of the completeness of GS. From this it follows, of course, that the Cut Rule is redundant.

We now begin the proof of Theorem 7. Let Const(Φ) be the set of constants occurring in Φ if this set is $\neq \emptyset$; if not, let Const(Φ) = {d}, where d is an arbitary fixed constant. (This is to make sure that Const(Φ) $\neq \emptyset$.) Let (Σ,Π) be any ordered pair of (possibly infinite) sets of formulas. We shall say that a formula φ *requires attention in* (Σ,Π) if one of the following conditions is satisfied:

(i) $\varphi := \neg\psi$, $\varphi \in \Sigma$, and $\psi \notin \Pi$,
(ii) $\varphi := \neg\psi$, $\varphi \in \Pi$, and $\psi \notin \Sigma$,
(iii) $\varphi := \psi_0 \wedge \psi_1$, $\varphi \in \Sigma$, and $\psi_0 \notin \Sigma$ or $\psi_1 \notin \Sigma$,
(iv) $\varphi := \psi_0 \wedge \psi_1$, $\varphi \in \Pi$, $\psi_0 \notin \Pi$ and $\psi_1 \notin \Pi$,
(v) $\varphi := \psi_0 \vee \psi_1$, $\varphi \in \Sigma$, $\psi_0 \notin \Sigma$ and $\psi_1 \notin \Sigma$,
(vi) $\varphi := \psi_0 \vee \psi_1$, $\varphi \in \Pi$, and $\psi_0 \notin \Pi$ or $\psi_1 \notin \Pi$,
(vii) $\varphi := \forall x\psi(x)$, $\varphi \in \Sigma$, and there is a constant $c \in \mathrm{Const}(\Sigma \cup \Pi)$ such that $\psi(c) \notin \Sigma$,
(viii) $\varphi := \forall x\psi(x)$, $\varphi \in \Pi$, and there is no constant c such that $\psi(c) \in \Pi$,
(ix) $\varphi := \exists x\psi(x)$, $\varphi \in \Sigma$, and there is no constant c such that $\psi(c) \in \Sigma$,
(x) $\varphi := \exists x\psi(x)$, $\varphi \in \Pi$, and there is a constant $c \in \mathrm{Const}(\Sigma \cup \Pi)$ such that $\psi(c) \notin \Pi$.

(Φ,Ψ) is *closed* if no formula requires attention in (Φ,Ψ). $\mathcal{A}$ is a *model of* (Φ,Ψ), $\mathcal{A} \models (\Phi,\Psi)$, if $\mathcal{A} \models \Phi$ and $\mathcal{A} \not\models \psi$ for every $\psi \notin \Psi$. Thus, $\models \Gamma \Rightarrow \Delta$ iff there is no model of (Γ,Δ).

Suppose (Φ,Ψ) is closed and Φ and Ψ are disjoint. $\mathcal{A}$ is a *canonical model for* (Φ,Ψ) if the following conditions are satisfied. Let C = Const($\Phi \cup \Psi$). Then C $\neq \emptyset$.

A = C, $c^{\mathcal{A}} = c$. If P is an n-place predicate and $c_1,\ldots,c_n \in C$, let

$$P^{\mathcal{A}} = \{\langle c_1,\ldots,c_n\rangle : Pc_1\ldots c_n \in \Phi\}.$$

Note that it may happen that $Pc_1\ldots c_n$, $\neg Pc_1\ldots c_n \notin \Phi \cup \Psi$. In fact, you need to appeal to the Cut Rule to show that Φ, Ψ can always be defined in such a way that this does not happen. If it does, we may, but need not, put $\langle c_1,\ldots,c_n\rangle$ in $P^{\mathcal{A}}$.

Lemma 21. Suppose (Φ,Ψ) is closed and Φ and Ψ are disjoint. Let $\mathcal{A}$ be a canonical model for (Φ,Ψ). Then $\mathcal{A} \models (\Phi,\Psi)$.

Proof. We show, by induction, that for every φ,

(1) if $\varphi \in \Phi$, then $\mathcal{A} \models \varphi$,
(2) if $\varphi \in \Psi$, then $\mathcal{A} \not\models \varphi$.

This is clear for atomic φ. The inductive steps are similar to those of the proof of Lemma 13. We consider only the cases, (i) $\varphi := \neg\psi$, (ii) $\varphi := \forall x\psi(x)$; the remaining cases are similar.

(i) Suppose $\varphi \in \Phi$. Since φ does not require attention in (Φ,Ψ), it follows that $\psi \in \Psi$. But then, by the inductive assumption, $\mathcal{A} \not\models \psi$ and so $\mathcal{A} \models \varphi$.

Next, suppose $\varphi \in \Psi$. Since φ does not require attention in (Φ,Ψ), it follows that $\psi \in \Phi$. But then, by the inductive assumption, $\mathcal{A} \models \psi$ and so $\mathcal{A} \not\models \varphi$.

(ii) Suppose $\varphi \in \Phi$. Since φ does not require attention in (Φ,Ψ), $\psi(c) \in \Phi$ for every $c \in C$. It follows, by the inductive assumption, that $\mathcal{A} \models \psi(c)$ for every $c \in C$. Since $A = C$, this implies that $\mathcal{A} \models \varphi$.

Suppose $\varphi \in \Psi$. Since φ does not require attention in (Φ,Ψ), there is a constant c such that $\psi(c) \in \Psi$. But then, by the inductive assumption, $\mathcal{A} \not\models \psi(c)$ and so $\mathcal{A} \not\models \varphi$. ■

Lemma 22. If $\nvdash \Gamma \Rightarrow \Delta$, there is a closed pair (Φ,Ψ) such that $\Gamma \subseteq \Phi$, $\Delta \subseteq \Psi$, and Φ and Ψ are disjoint.

Proof. Let C be a denumerable set of individual constants not in $\ell_{\Gamma\cup\Delta}$. Let φ_0, φ_1, φ_2, ... be an enumeration of the sentences of $\ell_{\Gamma\cup\Delta} \cup C$. "First" in "first sentence", below, refers to this enumeration.

We define sequents $\Gamma_n \Rightarrow \Delta_n$, $n = 0, 1, 2, \ldots$, as follows. Let $\Gamma_0 \Rightarrow \Delta_0$ be $\Gamma \Rightarrow \Delta$. Now, suppose $\Gamma_n \Rightarrow \Delta_n$ has been defined and $\nvdash \Gamma_n \Rightarrow \Delta_n$. (This, of course, implies that Γ_n and Δ_n are disjoint.) There are then two cases. (a) (Γ_n,Δ_n) is closed. In this case the construction terminates. (b) Otherwise. In this case one of the sentences requiring attention in (Γ_n,Δ_n) *receives attention at* n but exactly which one is not important as long as

(*) for all n and φ, if φ requires attention in (Γ_n,Δ_n), then φ receives attention at some $n' \geq n$.

The reason there is a slight problem is that, in view of (vii), (x), some sentences may require attention many, even infinitely many, times. (If at n a sentence containing a new constant is added to Γ_n or Δ_n, every sentence $\forall x\psi(x)$ in Γ_{n+1} and every sentence $\exists x\psi(x)$ in Δ_{n+1} requires attention in $(\Gamma_{n+1},\Delta_{n+1})$.) But (*) can be ensured in many different ways. For example, if there is a sentence requiring attention by (vii) or (x), let φ be the first such sentence; and if there is no such sentence, let φ be the first sentence, if there is one, requiring attention (for some other reason).

Let φ be the sentence which receives attention at n. Γ_{n+1} and Δ_{n+1} are then defined as follows. Suppose $\Sigma = \Gamma_n$ and $\Pi = \Delta_n$. Then

if (i) applies, let $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n, \psi$,

if (ii) applies, let $\Gamma_{n+1} = \Gamma_n, \psi$ and $\Delta_{n+1} = \Delta_n$,

if (iii) applies, let $\Gamma_{n+1} = \Gamma_n, \psi_0, \psi_1$ and $\Delta_{n+1} = \Delta_n$,

if (iv) applies, then (since $\nvdash \Gamma_n \Rightarrow \Delta_n$) $\nvdash \Gamma_n \Rightarrow \Delta_n, \psi_i$ for $i = 0$ or $i = 1$; let j be an i for which this holds and let $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n, \psi_j$,

if (v) applies, then (since $\nvdash \Gamma_n \Rightarrow \Delta_n$) $\nvdash \Gamma_n, \psi_i \Rightarrow \Delta_n$ for $i = 0$ or $i = 1$; let j be an i for which this holds and let $\Gamma_{n+1} = \Gamma_n, \psi_j$ and $\Delta_{n+1} = \Delta_n$,

if (vi) applies, let $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n, \psi_0, \psi_1$,

if (vii) applies, let c be any constant in $\mathrm{Const}(\Gamma_n \cup \Delta_n)$ such that $\psi(c) \notin \Gamma_n$ and let $\Gamma_{n+1} = \Gamma_n, \psi(c)$ and $\Delta_{n+1} = \Delta_n$,

if (viii) applies, let c be any constant not in (Γ_n,Δ_n) and let $\Gamma_{n+1} = \Gamma_n$, $\Delta_{n+1} = \Delta_n, \psi(c)$,

if (ix) applies, let c be a constant not in (Γ_n,Δ_n), let $\Gamma_{n+1} = \Gamma_n, \psi(c)$, and $\Delta_{n+1} = \Delta_n$,

if (x) applies, let c be any constant in $\mathrm{Const}(\Gamma_n \cup \Delta_n)$ such that $\psi(c) \notin \Delta_n$ and let $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n, \psi(c)$.

It now follows that $\nvdash \Gamma_{n+1} \Rightarrow \Delta_{n+1}$. This is clear in cases (iv) and (v) and holds in the other cases, too, since then

$$\frac{\Gamma_{n+1} \Rightarrow \Delta_{n+1}}{\Gamma_n \Rightarrow \Delta_n}$$

is an instance of a rule of derivation.

Now, let Φ and Ψ be defined as follows. If case (a) applies at n, let $\Phi = \Gamma_n$ and $\Psi = \Delta_n$, if (b) applies at every n, let $\Phi = \bigcup\{\Gamma_n : n \in N\}$ and $\Psi = \bigcup\{\Delta_n : n \in N\}$. Then $\Gamma \subseteq \Phi$, $\Delta \subseteq \Psi$, Φ and Ψ are disjoint, and, by (*), (Φ,Ψ) is closed. ■

Proof of Theorem 7. Let $\Gamma \Rightarrow \Delta$ be any sequent. Suppose $\nvdash \Gamma \Rightarrow \Delta$. By Lemma 22, there is a closed pair (Φ,Ψ) such that $\Gamma \subseteq \Phi$, $\Delta \subseteq \Psi$, and Φ and Ψ are disjoint. By Lemma 21, there is a model $\mathfrak{A}$ such that $\mathfrak{A} \vDash (\Phi,\Psi)$. It follows that $\mathfrak{A} \vDash (\Gamma,\Delta)$ and so $\nvDash \Gamma \Rightarrow \Delta$, as desired. ■

Theorem 7 implies the Cut Elimination Theorem.

From Theorem 7 and Corollary 3 we get:

Corollary 4. Suppose φ is valid and in prenex normal form. There is then a finite set Δ of quantifier-free sentences such that $\bigvee\Delta$ is a propositional tautology and $\Rightarrow \varphi$ can be obtained from $\Rightarrow \Delta$ by applying the rules $(\Rightarrow\exists)$ and $(\Rightarrow\forall)$.

Combining Theorem 7 with the form of the interpolation theorem already proved and Proposition 1.2 we get the following interpolation theorem (restricted to sentences not containing = nor any function symbols).

Theorem 8 (Craig's Interpolation Theorem). Let φ, ψ be any two sentences. If $\vDash \varphi \rightarrow \psi$, there is a sentence χ such that every nonlogical constant occurring in χ occurs in both φ and ψ, $\vDash \varphi \rightarrow \chi$ and $\vDash \chi \rightarrow \psi$.

§6. Natural Deduction. The formal systems FH and GS are quite artificial and do not correspond at all closely to the way we tend to reason intuitively. In this section we present a formal system, a version ND of natural deduction – there are other versions – which does not suffer from this disadvantage. On the other hand it is more complicated than FH and GS.

There are no axioms but nine *rules of derivation*: P (premise rule), PL (rule of propositional logic), US (universal specification), UG (universal generalization), ES (existential specification), EG (existential generalization), Cond (conditionalization), and two rules I and I* of identity.

The rules UG and ES embody the same ways of reasoning as the axioms A3 and A5 and rule R2 of FH and the rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ of GS. The rule Cond is what in the present context corresponds to the Deduction Theorem for FH and the rule $(\Rightarrow\rightarrow)$ of GS. (The reader may want to look at the derivations below and in Appendix 1 to see how the various rules are applied in practice.)

A *derivation* is a (finite) column of lines:

(D) (1) φ_1 $X_1R_1Y_1$
(2) φ_2 $X_2R_2Y_2$
....
(n) φ_n $X_nR_nY_n$

Each R_k is one of the above rules of derivation, φ_k is obtained from $\{\varphi_m: m \in X_k\}$ by applying R_k, and $\{\varphi_m: m \in Y_k\}$ is the set of *premises* in (D), introduced by applying P, on which φ_k "depends", i.e., from which φ_k has been derived, except when R_k is P, in which case $X_k = \emptyset$ and $Y_k = \{k\}$. Thus, (D) is a derivation of φ_n from $\{\varphi_m: m \in Y_n\}$.

(D) is a *derivation in* ND if one of the following conditions (i) – (ix) are satisfied:

(i) R_k is P and $X_k = \emptyset$ and $Y_k = \{k\}$ (on any line you may introduce a new premise; it "depends" only on itself),

(ii) R_k is PL and φ_k is a tautological consequence of $\varphi_{k_1},\ldots,\varphi_{k_m}$, where $k_i < k$ for $i \leq m$, $X_k = \{k_1,\ldots,k_m\}$, and $Y_k = Y_{k_1} \cup\ldots\cup Y_{k_m}$,

(iii) R_k is US and there are $m < k$, $\psi(x)$, and a closed term t such that φ_m is $\forall x\psi(x)$, φ_k is $\psi(t)$, $X_k = \{m\}$, and $Y_k = Y_m$,

(iv) R_k is UG and there are $m < k$, $\psi(x)$, and c such that φ_k is $\forall x\psi(x)$, φ_m is $\psi(c)$, c does not occur in φ_k nor in any premise of φ_m, $X_k = \{m\}$, and $Y_k = Y_m$,

(v) R_k is EG, there are $m < k$, $\psi(x)$, and a closed term t such that φ_k is $\exists x\psi(x)$, φ_m is $\psi(t)$, $X_k = \{m\}$, and $Y_k = Y_m$,

(vi) R_k is ES and there are m, p, $q < k$, $\psi(x)$, and c such that φ_m is φ_k, φ_p is $\exists x\psi(x)$, φ_q is $\psi(c)$, c does not occur in $\psi(x)$ nor in φ_m or any φ_r for $r \in Y_m - \{q\}$, R_q is P (and so $X_q = \emptyset$ and $Y_q = \{q\}$), $X_k = \{m\}$, and $Y_k = (Y_m - \{q\}) \cup Y_p$,

(vii) R_k is Cond and there are m, p < k such that φ_k is $\varphi_p \rightarrow \varphi_m$, R_p is P, $X_k = \{m\}$, and $Y_k = Y_m - \{p\}$,

(viii) R_k is I and there are p, q < k such that $X_k = \{p, q\}$, φ_p is $t = t'$ for some closed terms t, t′, there is a formula $\psi(x)$ such that $\varphi_q := \psi(t)$ and $\varphi_k := \psi(t')$, and $Y_k = Y_p \cup Y_q$,

(ix) R_k is I*, $\varphi_k := t = t$. where t is any closed term, and $X_k = Y_k = \emptyset$ (the formulas $t = t$ may be entered on any line; they do not "depend" on anything).

(vi) can be explained as follows. If $\psi(c)$ is a premise and φ_m has been derived from $\psi(c)$ plus a certain set Π (possibly empty) of other premises and c does not occur in φ_m or in $\psi(x)$ or in Π, then φ_k, i.e. φ_m, can be derived from Π plus $\exists x\psi(x)$. Thus, the conclusion is not new, but now it is derived from a different set of premises, namely, Π plus the premises of $\exists x\psi(x)$.

Some of the premises occurring in a derivation may be temporary premises. Such premises are eliminated by applying ES or Cond.

We write $\Phi \vdash_{ND}\varphi$ to mean that there is a derivation as above such that $\varphi := \varphi_n$ and $\{\varphi_m\text{: } m \in Y_n\} \subseteq \Phi$. $\vdash_{ND}\varphi$ if φ is derivable (from the empty set of premises) In this section and the next $\vdash$ is short for $\vdash_{ND}$.

To illustrate the use of the rules of ND, we now given some examples.

Example 6. Derivation of A3 (see §1):

$$\forall x(\psi \rightarrow \varphi(x)) \rightarrow (\psi \rightarrow \forall x\varphi(x)).$$

Let c be a constant not occurring in this formula.

(1) $\forall x(\psi \rightarrow \varphi(x))$	$\emptyset$ P $\{1\}$
(2) ψ	$\emptyset$ P $\{2\}$
(3) $\psi \rightarrow \varphi(c)$	$\{1\}$ US $\{1\}$
(4) $\varphi(c)$	$\{2,3\}$ PL $\{1,2\}$
(5) $\forall x\varphi(x)$	$\{4\}$ UG $\{1,2\}$
(6) $\psi \rightarrow \forall x\varphi(x)$	$\{5\}$ Cond $\{1\}$
(7) $\forall x(\psi \rightarrow \varphi(x)) \rightarrow (\psi \rightarrow \forall x\varphi(x))$	$\{6\}$ Cond $\emptyset$ ■

Example 7. Derivation of A5 (see §1):

$$\forall x(\varphi(x) \rightarrow \psi) \rightarrow (\exists x\varphi(x) \rightarrow \psi).$$

Let c be a constant not occurring in this formula.

(1) $\forall x(\varphi(x) \rightarrow \psi)$	$\emptyset$ P $\{1\}$
(2) $\exists x\varphi(x)$	$\emptyset$ P $\{2\}$
(3) $\varphi(c)$	$\emptyset$ P $\{3\}$
(4) $\varphi(c) \rightarrow \psi$	$\{1\}$ US $\{1\}$

(5) ψ	{3,4} PL {1,3}
(6) ψ	{5} ES {1,2}
(7) $\exists x\varphi(x) \to \psi$	{6} Cond {1}
(8) $\forall x(\varphi(x) \to \psi) \to (\exists x\varphi(x) \to \psi)$	{7} Cond Ø ■

These derivations of A3, A5 correspond very closely to the way we reason in convincing ourselves of the validity of these principles.

Example 8. $\neg\exists x\varphi(x) \vdash \forall x\neg\varphi(x)$. Let c be a constant not in $\varphi(x)$.

(1) $\neg\exists x\varphi(x)$	Ø P {1}
(2) $\varphi(c)$	Ø P {2}
(3) $\exists x\varphi(x)$	{2} EG {2}
(4) $\neg\varphi(c)$	{1,3} PL {1,2}
(5) $\varphi(c) \to \neg\varphi(c)$	{4} Cond {1}
(6) $\neg\varphi(c)$	{5} PL {1}
(7) $\forall x\neg\varphi(x)$	{6} UG {1} ■

Example 9. $\forall x\exists yPxy$, $\forall xyz(Pxy \wedge Pxz \to y = z) \vdash \forall x\exists y\forall z(Pxz \leftrightarrow z = y)$.

(1) $\forall x\exists yPxy$	Ø P {1}
(2) $\exists yPay$	{1} US {1}
(3) Pab	Ø P {3}
(4) $\forall xyz(Pxy \wedge Pxz \to y = z)$	Ø P {4}
(5) $Pac \wedge Pab \to c = b$	{4} US (three times) {4}
(6) $Pac \to c = b$	{3,5} PL {3,4}
(7) $c = b$	Ø P {7}
(8) Pac	{3,7} I {3,7}
(9) $c = b \to Pac$	{8} Cond {3}
(10) $Pac \leftrightarrow c = b$	{6,9} PL {3,4}
(11) $\forall z(Paz \leftrightarrow z = b)$	{10} UG {3,4}
(12) $\exists y\forall z(Paz \leftrightarrow z = y)$	{11} EG {3,4}
(13) -----------"-----------	{12} ES {1,4}
(14) $\forall x\exists y\forall z(Pxz \leftrightarrow z = y)$	{13} UG {1,4} ■

That $\vdash \varphi$ for every closed tautology φ can be shown in the following (awkward) way. Let ψ be any sentence.

(1) ψ	Ø P {1}
(2) φ	{1} PL {1}
(3) $\psi \to \varphi$	{2} Cond Ø
(4) φ	{3} PL Ø

Thus, we may add the following *derived* (short-cut) rule Taut to the above definition of *derivation in* ND:

(x) R_k is Taut, φ_k is a propositional tautology, and $X_k = Y_k = \varnothing$.

For = we may add the following *derived* rules I′, I′′, I′′, I#.

(xi) R_k is I′ and there are $m < k$ and closed terms t_0, t_1 such that $\varphi_m := t_0 = t_1$ and $\varphi_k := t_1 = t_0$.

(xii) R_k is I′′ and there are p, $q < k$ and closed terms t_0, t_1, t_2 such that $\varphi_p := t_0 = t_1$, $\varphi_q := t_1 = t_2$, and $\varphi_k := t_0 = t_2$.

(xiii) R_k is I′′′ and there are $m < k$, closed terms t_0, t_1, and a term $t(x)$ such that $\varphi_m := t_0 = t_1$ and $\varphi_k := t(t_0) = t(t_1)$.

(xiv) R_k is $I^{\#}$ and there are p, $q < k$, a formula $\psi(x)$, and closed terms t, t′ such that $\varphi_p := \psi(t)$, $\varphi_q := \neg\psi(t')$, and $\varphi_k := t \neq t'$ or $\varphi_k := t' \neq t$.

That the rules I′, I′′, I′′′, $I^{\#}$ are derived rules of ND is seen as follows:

(I′)	(1)	$t_0 = t_1$	$\varnothing$ P {1}
	(2)	$t_0 = t_0$	$\varnothing$ I* $\varnothing$
	(3)	$t_1 = t_0$	{1,2} I ($\psi(x) := x = t_0$) {1}
(I′′)	(1)	$t_0 = t_1$	$\varnothing$ P {1}
	(2)	$t_1 = t_2$	$\varnothing$ P {2}
	(3)	$t_0 = t_2$	{1,2} I ($\psi(x) := t_0 = x$) {1,2}
(I′′′)	(1)	$t_0 = t_1$	$\varnothing$ P {1}
	(2)	$t(t_0) = t(t_0)$	$\varnothing$ I* $\varnothing$
	(3)	$t(t_0) = t(t_1)$	{1,2} I ($\psi(x) := t(t_0) = t(x)$) {1}
($I^{\#}$)	(1)	$\psi(t)$	$\varnothing$ P {1}
	(2)	$\neg\psi(t')$	$\varnothing$ P {2}
	(3)	$t = t'$	$\varnothing$ P {3}
	(4)	$\psi(t')$	{1,3} I {1,3}
	(5)	$t = t' \rightarrow \psi(t')$	{4} Cond {1}
	(6)	$t \neq t'$	{2,5} PL {1,2}

This takes care of $I^{\#}$ for $\varphi_k := t \neq t'$. The derivation for $\varphi_k := t' \neq t$ is similar.

Example 10. $\forall x\exists y(f(y) = x), \forall x\exists y(g(y) = x) \vdash \forall x\exists y(f(g(y)) = x)$.

(1)	$\forall x\exists y(f(y) = x)$	$\varnothing$ P {1}
(2)	$\forall x\exists y(g(y) = x)$	$\varnothing$ P {2}
(3)	$\exists y(f(y) = a)$	{1} US {1}
(4)	$f(b) = a$	$\varnothing$ P {4}
(5)	$\exists y(g(y) = b)$	{2} US {2}
(6)	$g(c) = b$	$\varnothing$ P {6}
(7)	$f(g(c)) = f(b)$	{6} I′′′ {6}
(8)	$f(g(c)) = a$	{4,7} I′′ {4,6}
(9)	$\exists y(f(g(y)) = a)$	{8} EG {4,6}

(10) -------"-------- {9} ES {2,4}
(11) -------"-------- {10} ES {1,2}
(12) $\forall x \exists y(f(g(y)) = x)$ {11} UG {1,2} ■

Further examples of derivations in ND are given in Appendix 1.

§7. Soundness and completeness of ND. If (D) (above) is a derivation in ND, then for every $k \le n$, $\{\varphi_r : r \in Y_k\} \vDash \varphi_k$. This is clear except, possibly, when R_k is ES. In that case let m, p, q, $\psi(x)$, φ_k, φ_m, φ_p, φ_q, c, and X_k be as in (vi). Suppose $\{\varphi_r : r \in Y_s\} \vDash \varphi_s$ for $s < k$. Let $\theta := \bigwedge\{\varphi_r : r \in Y_m - \{q\}\} \to \varphi_m$. Then, by assumption, $\vDash \psi(c) \to \theta$. occurs neither in $\psi(x)$ nor in θ. It follows that $\vDash \exists x\psi(x) \to \theta$, i.e., $\vDash \varphi_p \to \theta$. By assumption, $\{\varphi_r : r \in Y_p\} \vDash \varphi_p$. And so, since φ_k is φ_m, $\{\varphi_r : r \in Y_k\} \vDash \varphi_k$, as desired. Thus, we have:

Theorem 9 (Soundness of ND)**.** If $\Phi \vdash \varphi$, then $\Phi \vDash \varphi$.

Theorem 10 (Completeness Theorem for ND)**.** If $\Phi \vDash \varphi$, then $\Phi \vdash \varphi$.

The simplest, but not the most natural, way to prove this is now to prove:

Lemma 23. If $\vdash_{FH}\varphi$, then $\vdash \varphi$.

Proof. It is sufficient to show that:

(i) if φ is a logical axiom of FH, then $\vdash \varphi$,
(ii) if φ is an identity axiom of FH, then $\vdash \varphi$,
(iii) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$,
(iv) if $\vdash \psi(c)$, then $\vdash \forall x\psi(x)$, where c does not occur in $\psi(x)$.

(i) We have already shown that $\vdash$A3 and $\vdash$A5 (Examples 6, 7). The proofs that $\vdash$A1, $\vdash$A2, $\vdash$A4 are simple and are left to the reader.

(ii) The derivations of I1, I2, I3 are straightforward (cf. (I′), (I″)).

Next we show that $\vdash$I4 for n = 2.

(1) $a_1 = b_1$ Ø P {1}
(2) $\varphi(a_1,a_2) \to \varphi(b_1,a_2)$ {1} I {1}
(3) $a_1 = b_1 \to (\varphi(a_1,a_2) \to \varphi(b_1,a_2))$ {2} Cond Ø
(4) $a_2 = b_2 \to (\varphi(b_1,a_2) \to \varphi(b_1,b_2))$ similarly, Ø
(5) $a_1 = b_1 \wedge a_2 = b_2 \to (\varphi(a_1,a_2) \to \varphi(b_1,b_2))$ {3,4} PL Ø
(6) I4 with n = 2 {5} UG (twice) Ø

The derivations of I4 for $n \neq 2$ and I5 are similar.

(iii) follows by PL and (iv) by UG. ■

Proof of Theorem 10. Suppose $\Phi \vDash \varphi$. Then, by the completeness of FH, there are

$\varphi_0,\ldots,\varphi_n \in \Phi$ such that $\varphi_0 \wedge \varphi_1 \wedge\ldots\wedge \varphi_n \to \varphi$ is derivable in FH. But then, by Lemma 23, this formula is derivable in ND and so, by the rule PL, $\Phi \vdash \varphi$. ■

§8. The Skolem-Herbrand Theorem. We are interested in proving statements of the form $\Phi \vDash \varphi$. As we have seen, this can be done by deriving φ from Φ in FH or ND or, if Φ is finite, by deriving $\Phi \Rightarrow \varphi$ in GS. In this § we prove a result, the Skolem-Herbrand Theorem, which yields a related method and also forms the starting point of methods meant to be useful in practice. However, instead of proving directly that $\Phi \vDash \varphi$, it now turns out to be more convenient to prove the equivalent statement of that $\Phi \cup \{\neg\varphi\}$ is non-satisfiable.

Let term(ℓ) be the set of closed terms of ℓ plus a constant c if there is no constant in ℓ. Let Id(ℓ) be the set of quantifier-free instances of the identity axioms of FH, i.e., quantifier-free sentences obtained from identity axioms of FH by omitting the initial universally quantified variables and replacing those variables in the remaining formula by terms of ℓ.

Let φ^S be as in Proposition 1.5. Let $\Phi = \{\varphi_i\colon i \in I\}$ and let $\Phi^S = \{\varphi_i{}^S\colon i \in I\}$, where we assume that different function symbols have been used to construct the formulas φ^S for different members of Φ. Suppose $\varphi_i{}^S := \forall x_1\ldots x_{n_i}\psi_i(x_1,\ldots,x_{n_i})$, where $\psi_i(x_1,\ldots,x_{n_i})$ is quantifier-free. Let $H(\Phi) =$

$$\{\psi_i(t_1,\ldots,t_{n_i})\colon i \in I \;\&\; t_1,\ldots,t_{n_i} \in \mathrm{term}(\ell_{\Phi S})\}.$$

Theorem 11 (Skolem, Herbrand)**.** Let Φ be any set of sentences.

(a) Φ is non-satisfiable iff there is a finite subset of $H(\Phi) \cup \mathrm{Id}(\ell_{\Phi S})$ which is not consistent in propositional logic.

(b) If = does not occur in Φ, then Φ is non-satisfiable iff there is a finite subset of $H(\Phi)$ which is not consistent in propositional logic.

Theorem 11(b) is essentially equivalent to Corollary 4.

Example 11. The formula $\varphi := \forall x\exists y(Pxy \vee \forall z\neg Pyz)$ is valid. Applying the Skolem-Herbrand method this can be shown as follows (compare Appendix 1, Example 3).

$$(\neg\varphi)^S := \forall y\neg(Pcy \vee \neg Pyf(y)).$$

Thus, the inconsistent set

$$\{\neg(Pcc \vee \neg Pcf(c)), \neg(Pcf(c) \vee \neg Pf(c)f(f(c)))\}$$

is a subset of $H(\{\neg\varphi\})$. And so $\neg\varphi$ is not satisfiable. ■

Example 12. Let

$$\Phi = \{\forall x\exists yPxy, \forall xyz(Pxy \wedge Pxz \to y = z)\}.$$

We have shown that

$$\Phi \vdash_{ND} \forall x \exists y \forall z(Pxz \leftrightarrow z = y)$$

(Example 9). The Skolem-Herbrand method can be applied to this example as follows. Let

$$\Psi = \Phi \cup \{\neg\forall x \exists y \forall z(Pxz \leftrightarrow z = y)\}.$$

Then

$$\Psi^S = \{\forall x Pxf(y), \forall xyz(Pxy \wedge Pxz \rightarrow y = z), \forall y \neg(Pag(y) \leftrightarrow g(y) = y)\}.$$

The sentences

$$Paf(a),$$
$$Paf(a) \wedge Pag(f(a)) \rightarrow f(a) = g(f(a)),$$
$$\neg(Pag(f(a)) \leftrightarrow g(f(a)) = f(a))$$

are members of H(Ψ). The sentences

$$f(a) = g(f(a)) \rightarrow g(f(a)) = f(a),$$
$$g(f(a)) = f(a) \rightarrow f(a) = g(f(a)),$$
$$f(a) = g(f(a)) \rightarrow (Paf(a) \rightarrow Pag(f(a)))$$

are members of $Id(\ell_{\Psi S})$. Finally, the set of these sentences is inconsistent. And so Ψ is not satisfiable. ■

For more applications of Theorem 11, see Appendix 1, Examples 8, 9.

Proof of Theorem 11 (sketch). (a) Suppose Φ is satisfiable. Then Φ^S is satisfiable. For every $\theta \in H(\Phi)$, $\varphi^S \vDash \theta$. Also, all members of $Id(\ell_{\Phi S})$ are valid. It follows that $H(\Phi) \cup Id(\ell_{\Phi S})$ is satisfiable and so is consistent (in propositional logic).

Next, suppose every finite subset of $H(\Phi) \cup Id(\ell_{\Phi S})$ is consistent in propositional logic. We can then extend this set to a consistent set Θ of quantifier-free sentences of $\ell_{\Phi S}$ such that for every such sentence θ, either $\theta \in \Theta$ or $\neg\theta \in \Theta$. Let $\mathcal{A}$ be the canonical model for Θ. Then $\mathcal{A} \vDash \Theta$ and so $\mathcal{A} \vDash H(\Phi)$. Since $\mathcal{A}$ is canonical, it follows that $\mathcal{A} \vDash \Phi^S$. Thus, Φ^S is satisfiable and so Φ is satisfiable, as desired.

This proves (a). (b) follows from (a). ■

§9. Validity and provability. Practically all proofs in mathematics can be seen as valid arguments based of the axioms of ZFC. We can now see that all such arguments can be carried out using only the rules of, say, ND.

One important consequence of any one of the above completeness theorems is that the definition of (logical) validity in Chapter 1 is extensionally correct (see Chapter 1, §6). Let Val be the set of intuitively valid sentences, let V be the set of sentences valid in the sense of Chapter 1, and let Pr be the set of sentences provable in, say, FH. It is then clear that Pr ⊆ Val ⊆ V. Also, by (the proof of) the Completeness Theorem for, say, FH, V ⊆ Pr. Thus, although the definitions of Val and V are not completely satisfactory, it follows that Val = V = Pr.

Notes for Chapter 2. The Completeness Theorem for FH (Theorem 3 and Corollary 1) with Φ countable is due to Gödel (1930); the problem was formulated in Hilbert, Ackermann (1928). The generalization to uncountable Φ and the present proof are due to Henkin (1949).

The sequent calculus is GS due to Gentzen (1934-35). The active formulas of an inference are usually called “side formulas” of that inference. Lemmas 17 and 19 are special cases of a general result due to Kleene (1952a). The Interpolation Theorem (Theorem 4), for a different (complete) logical calculus, is due to Craig (1957a).Theorem 5 is due to Gentzen (1934-35). Gentzen proved that GS is complete (Theorem 7) by showing that if φ is provable in FH, then ⇒ φ is derivable in GS with the Cut Rule and then that all applications of (Cut) can be eliminated from the derivation, Gentzen's Hauptsatz or Cut Elimination Theorem. Direct proofs of essentially Theorem 7, similar to the present proof, were given by Beth (1955), Hintikka (1955), and Kanger (1957). The idea of thinking of a derivation of a sequent S in GS as the result of an abortive attempt to define a counterexample to S is due to these authors.

Systems of natural deduction were first defined by Jaskowski (1934) and Gentzen (1934-35). There are now several such systems (cf. Prawitz (1965)). ND is essentially taken from Mates (1965). ND is closely related to the system of Gentzen (1934-35); more “natural”, perhaps, if only marginally. On the other hand Gentzen’s system (like GS) lends itself more easily to proof-theoretical investigations (cf. Prawitz (1965)).

Theorem 11, for finite Φ, was first proved by Skolem (1922), (1929) and, with “non-satisfiable” replaced by “refutable in FH”, by Herbrand (1930) (see the introductions to Skolem (1928) and Herbrand (1930) in van Heijenoort (1967)).

3. MODEL THEORY

So far the relation $\models$, as in $\mathcal{A} \models \varphi$, has only been used as a tool in defining logical validity and logical consequence and proving the various completeness theorems. In this chapter we shall study $\models$ for its own sake.

It is one of the basic tasks of model theory (of L_1) to investigate the "expressive power" of L_1; in particular, to show that, whereas many important mathematical conditions (on models) can be expressed in L_1 (see Chapter 1, §7), there are natural conditions that cannot. The most fundamental results of this type are those presented in §§1, 2. Also, for example, if a theory T is complete (see §6) and $\mathcal{A}$, $\mathcal{B}$ are models of T, we may conclude that there is no first-order sentence true in $\mathcal{A}$ and false in $\mathcal{B}$.

In this chapter we use $\vdash$ to mean $\vdash_{FH}$. We also sometimes replace $\models$ by $\vdash$, and *vice versa*, and use "is consistent" and "has a model" interchangably. This is justified by the Completeness Theorem for FH (Theorem 2.3). We always assume that the theory referred to by "T" is consistent. For illustrations and applications of some of the results proved in this Chapter, see §13.

§1. Basic concepts. In this section we define some of the basic concepts of model theory and prove some elementary results.

$\mathrm{Th}(\mathcal{A})$, the *theory of* $\mathcal{A}$, is the set $\{\varphi: \mathcal{A} \models \varphi\}$. Thus, $\mathcal{A} \equiv \mathcal{B}$ if $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$. If $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$ (Proposition 1.1). Clearly, $\equiv$ is an equivalence relation.

Let K be a class of models. We always assume that all members of K are models for the same language ℓ_K. Th(K), the theory of K, is the set of sentences φ such that $\mathcal{A} \models \varphi$ for every $\mathcal{A} \in K$. $\mathrm{Mod}(\varphi)$ is the class of models of φ. $\mathrm{Mod}(\Phi)$ is the class of models of Φ.

$\mathcal{A}$ is a *submodel* of $\mathcal{B}$, and $\mathcal{B}$ an *extension* of $\mathcal{A}$, $\mathcal{A} \subseteq \mathcal{B}$, if $\ell_{\mathcal{A}} = \ell_{\mathcal{B}}$, $A \subseteq B$, $P^{\mathcal{A}} = P^{\mathcal{B}} \cap A^n$ for every n-place predicate $P \in \ell_{\mathcal{A}}$, $f^{\mathcal{A}} = f^{\mathcal{B}} \restriction A$ for every n-place function symbol $f \in \ell_{\mathcal{A}}$, and $c^{\mathcal{A}} = c^{\mathcal{B}}$ for every individual constant $c \in \ell_{\mathcal{A}}$. (If X is a subset of the domain of the function g, then $g \restriction X = \{\langle a_1,\ldots,a_n, g(a_1,\ldots,a_n)\rangle: a_1,\ldots,a_n \in X\}$.)

A set X is *closed under* a function g if $g(a_1,\ldots,a_n) \in X$ for all $a_1,\ldots,a_n \in X$. If $X \subseteq A$, $\mathcal{A}$ has a submodel with domain X, written as $\mathcal{A} \restriction X$, iff $c^{\mathcal{A}} \in X$ for every individual constant $c \in \ell_{\mathcal{A}}$ and X is closed under $f^{\mathcal{A}}$ for every function symbol $f \in \ell_{\mathcal{A}}$.

Suppose $\mathcal{A} = (A, \mathcal{J})$ and $X \subseteq A$. For every $a \in X$, let c_a be a new constant. Let $\ell_{\mathcal{A}}(X) = \ell_{\mathcal{A}} \cup \{c_a: a \in X\}$. Let $\mathcal{A}_X$ be the expansion of $\mathcal{A}$ to $\ell_{\mathcal{A}}(X)$ such that $c_a^{\mathcal{A}_X} = a$ for $a \in X$.

$\mathcal{A}$ is an *elementary submodel* of $\mathcal{B}$, and $\mathcal{B}$ an *elementary extension* of $\mathcal{A}$, $\mathcal{A} \preccurlyeq \mathcal{B}$, if $A \subseteq B$ and for every formula φ and every valuation $\mathfrak{v}$: Var $\to$ A, $\mathcal{A} \models \varphi[\mathfrak{v}]$ iff $\mathcal{B} \models \varphi[\mathfrak{v}]$. Clearly, $\preccurlyeq$ is reflexive, and transitive and if $\mathcal{A} \preccurlyeq \mathcal{B}$, then $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \equiv \mathcal{B}$. Also, $\mathcal{A} \preccurlyeq \mathcal{B}$ iff $A \subseteq B$ and $\mathcal{A}_A \equiv \mathcal{B}_A$.

We now prove two basic lemmas on the relation $\preccurlyeq$.

Lemma 1. Suppose $\mathcal{A} \subseteq \mathcal{B}$. Then $\mathcal{A} \preccurlyeq \mathcal{B}$ iff for every formula $\exists x\varphi$ of $\ell_{\mathcal{A}}$ and every $\boldsymbol{u}$: Var $\rightarrow$ A, if $\mathcal{B} \vDash \exists x\varphi[\boldsymbol{u}]$, there is an $a \in A$ such that $\mathcal{B} \vDash \varphi[\boldsymbol{u}(x/a)]$.

Proof. Suppose first $\mathcal{A} \preccurlyeq \mathcal{B}$. Suppose $\exists x\varphi$ is a formula of $\ell_{\mathcal{A}}$ such that $\mathcal{B} \vDash \exists x\varphi[\boldsymbol{u}]$. Then $\mathcal{A} \vDash \exists x\varphi[\boldsymbol{u}]$. It follows that there is an $a \in A$ such that $\mathcal{A} \vDash \varphi[\boldsymbol{u}(x/a)]$. But then $\mathcal{B} \vDash \varphi[\boldsymbol{u}(x/a)]$, as desired.

We prove the converse implication by induction. Let ψ be a formula of $\ell_{\mathcal{A}}$ and suppose $\boldsymbol{u}$: Var $\rightarrow$ A. We have to show that $\mathcal{A} \vDash \psi[\boldsymbol{u}]$ iff $\mathcal{B} \vDash \psi[\boldsymbol{u}]$. Since $\mathcal{A} \subseteq \mathcal{B}$, this holds if ψ is atomic. The cases corresponding to the propositional connectives are straightforward. Suppose $\psi := \exists x\varphi$. If $\mathcal{A} \vDash \exists x\varphi[\boldsymbol{u}]$, there is an $a \in A$ such that $\mathcal{A} \vDash \varphi[\boldsymbol{u}(x/a)]$. But then, by the inductive assumption, $\mathcal{B} \vDash \varphi[\boldsymbol{u}(x/a)]$ and so $\mathcal{B} \vDash \exists x\varphi[\boldsymbol{u}]$. Next, suppose $\mathcal{B} \vDash \exists x\varphi[\boldsymbol{u}]$. Then, by assumption, there is an $a \in A$ such that $\mathcal{B} \vDash \varphi[\boldsymbol{u}(x/a)]$. By the inductive assumption, $\mathcal{A} \vDash \varphi[\boldsymbol{u}(x/a)]$ and so $\mathcal{A} \vDash \exists x\varphi[\boldsymbol{u}]$. The case $\psi := \forall x\varphi$ is similar. ■

The point of Lemma 1 is that the conclusion $\mathcal{A} \preccurlyeq \mathcal{B}$ follows from a condition that doesn't mention satisfaction in $\mathcal{A}$.

An *automorphism* of $\mathcal{A}$ is an isomorphism of $\mathcal{A}$ onto $\mathcal{A}$.

Corollary 1. Suppose $\mathcal{A} \subseteq \mathcal{B}$ and for any n, any $a_1,\ldots,a_n \in A$, and any $b \in B$, there is an automorphism f of $\mathcal{B}$ such that $f(a_i) = a_i$ for $0 < i \leq n$ and $f(b) \in A$. Then $\mathcal{A} \preccurlyeq \mathcal{B}$.

To simplify the notation in what follows we shall often make no distinction between the members of a model and the corresponding individual constants (or regard the members of $\mathcal{A}$ as names of themselves). Thus, if $\varphi(x_1,\ldots,x_n)$ is a formula and $a_1,\ldots,a_n \in A$, we may write $\varphi(a_1,\ldots,a_n)$ for $\varphi(c_{a_1},\ldots,c_{a_n})$ and $\mathcal{A} \vDash \varphi(a_1,\ldots,a_n)$ for $\mathcal{A} \vDash \varphi[\boldsymbol{u}]$, where $\boldsymbol{u}(x_i) = a_i$ for $i \leq n$. But, of course, if $\mathcal{B} \neq \mathcal{A}$, then $\mathcal{B} \vDash \varphi(a_1,\ldots,a_n)$ is not (automatically) defined.

Proof of Corollary 1. Suppose $\exists y\psi(x_1,\ldots,x_n,y)$ is a formula of $\ell_{\mathcal{A}}$ and suppose $a_1,\ldots,a_n \in A$ are such that $\mathcal{B} \vDash \exists y\psi(a_1,\ldots,a_n,y)$. There is then a $b \in B$ such that $\mathcal{B} \vDash \psi(a_1,\ldots,a_n,b)$. Let f be an automorphism f of $\mathcal{B}$ such that $f(a_i) = a_i$ for $0 < i \leq n$ and $f(b) \in A$. Then $\mathcal{B} \vDash \psi(a_1,\ldots,a_n,f(b))$. Thus, the condition of Lemma 1 is satisfied and so $\mathcal{A} \preccurlyeq \mathcal{B}$. ■

The following is a simple illustration of Corollary 1. We want to show that (Ra, $\leq$) $\preccurlyeq$ (Re, $\leq$) (cf. Chapter 1, §7, Example 1). Suppose $r_1,\ldots,r_n \in \mathrm{Ra}$, $s \in \mathrm{Re}$, and $r_1 \leq \ldots \leq r_i < s < r_{i+1} \leq \ldots \leq r_n$. Let $r \in \mathrm{Ra}$ be such that $r_i < r < r_{i+1}$. Now, let f be any increasing function on Re onto Re such that $f(r_j) = r_j$ for $j = 1,\ldots,n$ and $f(s) = r$. Then f is an automorphism of (Re, $\leq$) and $f(s) \in \mathrm{Ra}$. And so, by Corollary 1, (Ra, $\leq$) $\preccurlyeq$ (Re, $\leq$), as desired.

Let $\{\mathcal{A}_i\colon i \in I\}$ be a set of models. Suppose $\{\mathcal{A}_i\colon i \in I\}$ is a *chain* in the sense that for all i, $j \in I$, either $\mathcal{A}_i \subseteq \mathcal{A}_j$ or $\mathcal{A}_j \subseteq \mathcal{A}_i$. The union of $\{\mathcal{A}_i\colon i \in I\}$, $\bigcup\{\mathcal{A}_i\colon i \in I\}$, is then the model $\mathcal{A}$ such that $A = \bigcup\{A_i\colon i \in I\}$, $P^{\mathcal{A}} = \bigcup\{P^{\mathcal{A}_i}\colon i \in I\}$, $f^{\mathcal{A}} = \bigcup\{f^{\mathcal{A}_i}\colon i \in I\}$, and $c^{\mathcal{A}} = c^{\mathcal{A}_i}$ for every predicate P, function symbol f, and individual constant c. Note that $f^{\mathcal{A}}$ is a function. $\{\mathcal{A}_i\colon i \in I\}$ is an *elementary chain* if $\mathcal{A}_i \preccurlyeq \mathcal{A}_j$ or $\mathcal{A}_j \preccurlyeq \mathcal{A}_i$ for any i, $j \in I$.

Lemma 2 (Tarski's Lemma)**.** If $\{\mathcal{A}_i\colon i \in I\}$ is an elementary chain, then for every $j \in I$, $\mathcal{A}_j \preccurlyeq \bigcup\{\mathcal{A}_i\colon i \in I\}$.

Proof. Let $\mathcal{A} = \bigcup\{\mathcal{A}_i\colon i \in I\}$. We show, by induction, that for every sentence φ,

(*) for every $j \in I$, formula φ, and $\boldsymbol{v}\colon \mathrm{Var} \to A_j$, $\mathcal{A}_j \models \varphi[\boldsymbol{v}]$ iff $\mathcal{A} \models \varphi[\boldsymbol{v}]$.

This is clear if φ is atomic. The inductive steps corresponding to the propositional connectives are evident. Let $\varphi := \exists x\psi$. First suppose $\mathcal{A}_j \models \varphi[\boldsymbol{v}]$. There is then an $a \in A$ such that $\mathcal{A}_j \models \psi[\boldsymbol{v}(x/a)]$. But then, by the inductive assumption, $\mathcal{A} \models \psi[\boldsymbol{v}(x/a)]$ and so $\mathcal{A} \models \varphi[\boldsymbol{v}]$. Finally, suppose $\mathcal{A} \models \varphi[\boldsymbol{v}]$. Let $b \in A$ be such that $\mathcal{A} \models \psi[\boldsymbol{v}(x/b)]$. Since $\{\mathcal{A}_i\colon i \in I\}$ is a chain, there is $k \in I$ such that $\mathcal{A}_j \preccurlyeq \mathcal{A}_k$ and $b \in A_k$. By the inductive assumption, $\mathcal{A}_k \models \psi[\boldsymbol{v}(x/b)]$, whence $\mathcal{A}_k \models \varphi[\boldsymbol{v}]$ and so $\mathcal{A}_j \models \varphi[\boldsymbol{v}]$, as desired. The case $\varphi := \forall x\psi$ is similar. ■

A formula is *basic* if it is atomic or the negation of an atomic formual. The (*basic*) *diagram* of $\mathcal{A}$, D($\mathcal{A}$), is then set of basic sentences of $\ell_{\mathcal{A}}(A)$ obtained from primitive formulas of $\ell_{\mathcal{A}}$ by replacing free variables by constants c_a for $a \in A$ and true in $\mathcal{A}_A$. The *elementary diagram* of $\mathcal{A}$, ED($\mathcal{A}$), is the set of sentences of $\ell_{\mathcal{A}}(A)$ true in $\mathcal{A}_A$. The *universal diagram* of $\mathcal{A}$, UD($\mathcal{A}$), is the set of universal sentences of $\ell_{\mathcal{A}}(A)$ true in $\mathcal{A}_A$. If, as in Lemma 3(d), below, we consider diagrams of two models $\mathcal{A}$ and $\mathcal{B}$ at the same time, we assume that the same constants are used in the two diagrams to correspond to the same objects (common elements of $\mathcal{A}$ and $\mathcal{B}$).

f is an *embedding of* $\mathcal{A}$ *in* $\mathcal{B}$ if $f(\mathcal{A}) \subseteq \mathcal{B}$, $\mathcal{A}$ is *embeddable* in $\mathcal{B}$ if there is an embedding of $\mathcal{A}$ in $\mathcal{B}$. Thus, $\mathcal{A}$ is embeddable in $\mathcal{B}$ iff there is a model $\mathcal{C} \cong \mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{C}$. f is an *elementary embedding of* $\mathcal{A}$ *in* $\mathcal{B}$ if $f\colon A \to B$ and for every formula φ and every valuation $\boldsymbol{v}\colon \mathrm{Var} \to A$, $\mathcal{A} \models \varphi[\boldsymbol{v}]$ iff $\mathcal{B} \models \varphi[f(\boldsymbol{v})]$; in other words, there is an elementary submodel $\mathcal{C}$ of $\mathcal{B}$ such that $f\colon \mathcal{A} \cong \mathcal{C}$. $\mathcal{A}$ is *elementarily embeddable in* $\mathcal{B}$ if there is an elementary embedding of $\mathcal{A}$ in $\mathcal{B}$.

An existential formula $\exists x_1 \ldots x_n\psi$ is *simple* if ψ is a conjunction of primitive basic formulas. Every existential formula is equivalent to a disjunction of simple existential formulas. We write $\mathcal{A} \preccurlyeq_1 \mathcal{B}$ to mean that $\mathcal{A} \subseteq \mathcal{B}$ and for every simple existential formula $\varphi(x_1,...,x_n)$ of $\ell_{\mathcal{A}}$ and all $a_1,...,a_n \in A$, if $\mathcal{B} \models \varphi(a_1,...,a_n)$, then $\mathcal{A} \models \varphi(a_1,...,a_n)$. (Thus, in fact, $\mathcal{A} \preccurlyeq_1 \mathcal{B}$ iff every existential sentence of $\ell_{\mathcal{A}}(A)$ true in $\mathcal{B}_A$ is true in $\mathcal{A}_A$.) A function g is an *existential embedding of* $\mathcal{A}$ *in* $\mathcal{B}$ if g is an embedding of $\mathcal{A}$ in $\mathcal{B}$ and $g(\mathcal{A}) \preccurlyeq_1 \mathcal{B}$, where $g(\mathcal{A})$ is the image of $\mathcal{A}$ under g.

Lemma 3. (a) If $\mathcal{B} \models D(\mathcal{A})$, then $\mathcal{A}$ is embeddable in $\mathcal{B}$.

(b) If $\mathcal{B} \models ED(\mathcal{A})$, then $\mathcal{A}$ is elementarily embeddable in $\mathcal{B}$.

(c) If $\mathcal{B} \models UD(\mathcal{A})$, then $\mathcal{A}$ is existentially embeddable in $\mathcal{B}$.

(d) If $A \subseteq B$ and $\mathcal{C} \models ED(\mathcal{A}) \cup D(\mathcal{B})$, there is an embedding g of $\mathcal{B}$ in $\mathcal{C}$ such that $g|A$ is an elementary embedding of $\mathcal{A}$ in $\mathcal{C}$.

Here we have taken the liberty of saying, for example, that $\mathcal{A}$ is embeddable in $\mathcal{B}$ when, strictly speaking, we should have said that $\mathcal{A}$ is embeddable in $\mathcal{B}|\ell_{\mathcal{A}}$.

Proof. This is really evident but we nevertheless give a detailed proof of (a); the remaining cases are similar. Suppose $\mathcal{B} \models D(\mathcal{A})$. Let $g\colon A \to B$ be such that $g(a) = c_a{}^{\mathcal{B}}$. Now, suppose, for example $P \in \ell_{\mathcal{A}}$ is a two-place predicate and $a, b \in A$. If $\langle a,b\rangle \in P^{\mathcal{A}}$, then $Pc_ac_b \in D(\mathcal{A})$. It follows that $\mathcal{B} \models Pc_ac_b$ and so $\langle c_a{}^{\mathcal{B}}, c_b{}^{\mathcal{B}}\rangle \in P^{\mathcal{B}}$, whence $\langle g(a),g(b)\rangle \in P^{\mathcal{B}}$. Similarly, if $\langle a,b\rangle \notin P^{\mathcal{A}}$, then $\langle g(a),g(b)\rangle \notin P^{\mathcal{B}}$. Next, suppose $f \in \ell_{\mathcal{A}}$ is, say, a one-place function symbol. Suppose $f^{\mathcal{A}}(a) = b$, where $a, b \in A$. Then $f(c_a) = c_b \in D(\mathcal{A})$ and so $\mathcal{B} \models f(c_a) = c_b$, whence $f^{\mathcal{B}}(c_a{}^{\mathcal{B}}) = c_b{}^{\mathcal{B}}$ and so $f^{\mathcal{B}}(g(a)) = g(f^{\mathcal{A}}(a))$. Finally, let $c \in \ell_{\mathcal{A}}$ be a constant. Let a be such that $c^{\mathcal{A}} = a$. Then $c = c_a \in D(\mathcal{A})$ and so $\mathcal{B} \models c = c_a$. But then $g(c^{\mathcal{A}}) = c_a{}^{\mathcal{B}} = c^{\mathcal{B}}$. Thus, g is an embedding of $\mathcal{A}$ in $\mathcal{B}$. ■

The converse of Lemma 3(a) is also true: If $\mathcal{A}$ is embeddable in $\mathcal{B}$, then (some expansion of) $\mathcal{B}$ is a model of $D(\mathcal{A})$; and similarly for Lemma 3(b), (c), (d). The verification of this is left to the reader.

§2. Compactness and cardinality theorems. In model-theoretic proofs we frequently want to show that a certain set of sentences has a model. For example, suppose we want to show that a certain set Φ has a model which is an extension of a given model $\mathcal{A}$. By Lemma 3(a), it is then sufficient (and necessary) to show that $\Phi \cup D(\mathcal{A})$ has a model. But this may be far from obvious and at the same time it may be quite clear, or at least (much) easier to prove, that every finite subset of $\Phi \cup D(\mathcal{A})$ has a model. It is in situations like this, and they occur very often, that the following theorem is indispensable.

Theorem 1 (Compactness Theorem)**.** For any set of sentences Φ, if every finite subset of Φ has a model, then Φ has a model.

Proof. This follows at once from the Completeness Theorem for FH (Theorem 2.3): Suppose every finite subset of Φ has a model. Then every finite subset of Φ is consistent. But then Φ is consistent and so Φ has a model. ■

Theorem 1 can also be proved directly, without going via a completeness theorem, by just replacing "$\Psi \vdash \psi$" by "there is a finite subset Ψ' of Ψ such that "$\Psi' \vDash \psi$" in the proof of Theorem 2.3. "Ψ is consistent" then becomes "every finite subset of Ψ has a model". In §9 we give a quite different proof of Theorem 1.

From the proof of Theorem 1 we obtain the following:

Theorem 2 (Löwenheim-Skolem Theorem)**.** If Φ is a countable set of sentences and Φ has an infinite model, then Φ has a denumerable model.

For any formula $\varphi(x)$, let $\exists^{>n}x\varphi(x) :=$

$$\exists x_0 \ldots x_n(\bigwedge\{x_i \neq x_j \colon i < j \leq n\} \wedge \bigwedge\{\varphi(x_i) \colon i \leq n\}).$$

Let

$$\mathrm{INF} = \{\exists^{>n}x(x{=}x) \colon n \in \mathrm{N}\}.$$

Proof. The model of $\Phi \cup \mathrm{INF}$ constructed in the proof of the Completeness (or Compactness) Theorem is denumerable. ■

As a first application of Theorem 1 we have the following:

Proposition 1. If Φ is any set of sentences and Φ has arbitrarily large finite models, then Φ has an infinite model.

Proof. By assumption every finite subset of $\Phi \cup \mathrm{INF}$ has a model. It follows that the whole set has a model $\mathcal{A}$. $\mathcal{A}$ is an infinite model of Φ. ■

Let $\mathcal{N} = (\mathrm{N}, +, \cdot, \mathrm{S}, 0)$ be the standard model of arithmetic; S is the successor function.

Corollary 2. (Skolem) $\mathrm{Th}(\mathcal{N})$ has a denumerable model not isomorphic to $\mathcal{N}$.

Proof. Let c be a new constant and let $\Phi = \mathrm{Th}(\mathcal{N}) \cup \{c \neq 0, c \neq \mathrm{S}(0), c \neq \mathrm{S}(\mathrm{S}(0)), \ldots\}$. Clearly, every finite subset of Φ has a model. But then, by Theorems 1, 2, Φ has a denumerable model $\mathcal{A}$. Obviously, $\mathcal{A} \ncong \mathcal{N}$. ■

Our next result is a generalization and strengthening of Theorem 2.

Theorem 3 (Downward Löwenheim-Skolem-Tarski (LST) Theorem)**.** Suppose $|\ell_{\mathcal{A}}| \leq \kappa$, $X \subseteq A$, and $|X| \leq \kappa \leq |A|$. There is then a model $\mathcal{B}$ such that $X \subseteq B$, $|B| = \kappa$, and $\mathcal{B} \preccurlyeq \mathcal{A}$.

Proof. We define an increasing sequence Y_0, Y_1, Y_2, ... of subsets of A as follows. Let $Y_0 \subseteq A$ be such that $X \subseteq Y_0$ and $|Y_0| = \kappa$. Now suppose Y_n has been defined and $|Y_n| = \kappa$. Let $\{\exists x_i\varphi_i\colon i \in I\}$ be the set of sentences of $\mathcal{L}_{\mathfrak{A}}(Y_n)$ of the form indicated and true in $\mathfrak{A}_A$. Then $|I| = \kappa$. For every $i \in I$, there is an $a_i \in A$ such that $\mathfrak{A}_A \vDash \varphi(x_i/a_i)$. Let $Y_{n+1} = Y_n \cup \{a_i\colon i \in I\}$. Then $|Y_{n+1}| = \kappa$.

Let $Y = \bigcup\{Y_n\colon n \in N\}$. Then $|Y| = \kappa$ and Y is closed under the functions $f^{\mathfrak{A}}$, where $f \in \mathcal{L}_{\mathfrak{A}}$, and $c^{\mathfrak{A}} \in Y$ for every constant $c \in \mathcal{L}_{\mathfrak{A}}$. Let $\mathfrak{B} = \mathfrak{A}|Y$. Then $X \subseteq B$ and $|B| = \kappa$. Suppose $\exists x\varphi$ is a sentence of $\mathcal{L}(B)$ such that $\mathfrak{A}_A \vDash \exists x\varphi$. $\exists x\varphi$ is a sentence of $\mathcal{L}_{\mathfrak{A}}(Y_n)$ for some n. It follows that there is an $a \in Y_{n+1}$ such that $\mathfrak{A}_A \vDash \varphi(x/a)$. Hence, by Lemma 1, $\mathfrak{B} \preccurlyeq \mathfrak{A}$. ■

A somewhat different proof of Theorem 3 is as follows. Let $\mathfrak{A}$ be any Skolem model and suppose $\varnothing \neq X \subseteq A$. The *Skolem hull* of X in $\mathfrak{A}$, $H_{\mathfrak{A}}(X)$, is then the least set Y containing X such that $c^{\mathfrak{A}} \in Y$ for all individual constants in $\mathcal{L}_{\mathfrak{A}}$ and Y is closed under a functions $f^{\mathfrak{A}}$ such that f is a function symbol in $\mathcal{L}_{\mathfrak{A}}$. It follows that $|X| \leq |H_{\mathfrak{A}}(X)| \leq |X| + |\mathcal{L}_{\mathfrak{A}}| + \aleph_0$. Let $\mathcal{H}_{\mathfrak{A}}(X) = \mathfrak{A}|H_{\mathfrak{A}}(X)$. In what follows we write H(X), $\mathcal{H}(X)$ for $H_{\mathfrak{A}}(X)$, $\mathcal{H}_{\mathfrak{A}}(X)$, respectively.

Proposition 2. If $\mathfrak{A}$ is a Skolem model and $\varnothing \neq X \subseteq A$, then $\mathcal{H}(X) \preccurlyeq \mathfrak{A}$.

Proof. Suppose $a_1,...,a_n \in H(X)$ and $\mathfrak{A} \vDash \exists y\varphi(a_1,...,a_n,y)$. It follows that $\mathfrak{A} \vDash \varphi(a_1,...,a_n,f_\varphi(a_1,...,a_n))$. But $f_\varphi(a_1,...,a_n) \in H(X)$. Thus, by Lemma 1, $\mathcal{H}(X) \preccurlyeq \mathfrak{A}$. ■

Theorem 3 is an immediate consequence of this and Proposition 1.6.

Theorem 4 (Upward LST Theorem)**.** Suppose $|\mathcal{L}_{\mathfrak{A}}| \leq \kappa$ and $\aleph_0 \leq |A| \leq \kappa$. There is then an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $|B| = \kappa$.

Proof. Let $\{c_i\colon i \in I\}$, where $|I| = \kappa$, be a set of new individual constants. Let $\Phi = ED(\mathfrak{A}) \cup \{\neg c_i = c_j\colon i, j \in I \ \&\ i \neq j\}$. Since $\mathfrak{A}$ is infinite, every finite subset of Φ has a model. By the Compactness Theorem, it follows that Φ has a model $\mathfrak{C}$. By Lemma 3(b), we may assume that $\mathfrak{A} \preccurlyeq \mathfrak{C}|\mathcal{L}_{\mathfrak{A}}$. Clearly, $|C| \geq \kappa$. By Theorem 3, there is a model $\mathfrak{B}$ such that $A \subseteq B$, $\mathfrak{B} \preccurlyeq \mathfrak{C}|\mathcal{L}_{\mathfrak{A}}$, and $|B| = \kappa$. But then $\mathfrak{A} \preccurlyeq \mathfrak{B}$ and so $\mathfrak{B}$ is as desired. (This application of Theorem 3 is not really necessary, since the model of Φ defined in the proof of the Compactness Theorem is of cardinality κ.) ■

Corollary 3 (LST or Cardinality Theorem)**.** Let Φ be any set of sentences and suppose $\kappa \geq |\Phi|$. If Φ has an infinite model, then Φ has a model of cardinality κ.

§3. Elementary and projective classes. A class K is *elementary*, $K \in EC$, if there is a sentence φ such that $K = Mod(\varphi)$. K is Δ-*elementary*, $K \in EC_\Delta$, if there is a set Φ of sentences such that $K = Mod(\Phi)$. $K^c = \{\mathfrak{A}: \ell_{\mathfrak{A}} = \ell_K \;\&\; \mathfrak{A} \notin K\}$.

Proposition 3. If $K = Mod(\Phi)$ and $K \in EC$, there is a finite subset Ψ of Φ such that $K = Mod(\Psi)$.

Proof. Let φ be such that $K = Mod(\varphi)$. Then $\Phi \cup \{\neg\varphi\}$ has no model. But then there is a finite subset Ψ of Φ such that $\Psi \cup \{\neg\varphi\}$ has no model. It follows that $Mod(\Phi) \subseteq Mod(\Psi)$ and $Mod(\Psi) \cap Mod(\neg\varphi) = \emptyset$ and so $K = Mod(\Psi)$. ■

From Propositions 1, 3 we obtain the following:

Corollary 4. Suppose $K \in EC_\Delta$ and K has arbitrarily large finite members. Then
(i) K has an infinite member,
(ii) the class of infinite members of K is EC_Δ but not EC.

Proof. (i) This is Proposition 1.

(ii) If $Mod(\Phi \cup INF)$ is EC, by Proposition 3, there is a member φ of INF such that $K = Mod(\Phi \cup \varphi)$, which is not true. ■

By Corollary 4, the classes of finite linear orderings, finite Boolean algebras, finite groups etc. are not EC_Δ and the classes of infinite linear orderings, Boolean algebras, groups etc., although EC_Δ, are not EC.

That there are classes $K \in EC_\Delta - EC$ can also be shown without using the Compactness Theorem. If we allow ℓ_K to be infinite, this is clear.

A slightly less trivial example is this. Let P be a one-place predicate and let K be any EC_Δ class of infinite models for {P}. Every sentence of {P} which has a model has a finite model (see, §13, Example 1). It follows that $K \notin EC$.

Yet another way of showing that $EC_\Delta \neq EC$, for a quite different but still rather trivial reason, is this. Suppose ℓ is finite and contains a two-place predicate (or a one-place function symbol). Then there are $2^{\aleph_0}$ EC_Δ classes of models for ℓ but only $\aleph_0$ EC classes of such models.

K is *closed under isomorphisms* if for all $\mathfrak{A}$, $\mathfrak{B}$, if $\mathfrak{A} \in K$ and $\mathfrak{B} \cong \mathfrak{A}$, then $\mathfrak{B} \in K$. K is *closed under* $\equiv$ if for all $\mathfrak{A}$, $\mathfrak{B}$, if $\mathfrak{A} \in K$ and $\mathfrak{B} \equiv \mathfrak{A}$, then $\mathfrak{B} \in K$. By Proposition 1.1, if K is closed under $\equiv$, then K is closed under isomorphisms. Clearly, if $K \in EC_\Delta$, then K is closed under $\equiv$.

There are classes $K \notin EC_\Delta$ that are closed under $\equiv$. By Corollary 4, one example is the class of finite models for a given language. But this, too, is true for cardinality reasons. If ℓ is countable and contains a two-place predicate, there are

$2^{2^{\aleph_0}}$ classes of models for ℓ closed under $\equiv$ but only $2^{\aleph_0}$ such EC_Δ classes.

Proposition 4. (a) If K_0, $K_1 \in EC_\Delta$ and $K_0 \cap K_1 = \varnothing$, there is a class $K \in EC$ such that $K_0 \subseteq K$ and $K \cap K_1 = \varnothing$.

(b) $K \in EC$ iff $K \in EC_\Delta$ and $K^c \in EC_\Delta$.

Proof. (a) Let Φ_i be such that $K_i = Mod(\Phi_i)$, $i = 0, 1$. Then $\Phi_0 \cup \Phi_1$ has no model. It follows that there is a finite subset Ψ of Φ_0 such that $\Psi \cup \Phi_1$ has no model. Let $K = Mod(\bigwedge\Psi)$.

(b) "$\Rightarrow$" is clear. "$\Leftarrow$" follows from (a). ■

K is a *projective* (or *pseudo-elementary*) class, $K \in PC$, if there is a $K' \in EC$ such that $K = \{\mathcal{A} \mid \ell_K\colon \mathcal{A} \in K'\}$. In other words, there is a sentence φ of $\ell_{K'}$ such that K is the class of models of the second-order sentence obtained from φ by existentially quantifying the members of $\ell_{K'} - \ell$. $K \in PC_\Delta$ if there is a $K' \in EC_\Delta$ such that $K = \{\mathcal{A} \mid \ell_K\colon \mathcal{A} \in K'\}$. Clearly, $EC \subseteq PC$ and $EC_\Delta \subseteq PC_\Delta$.

From Corollary 4 it follows that:

Corollary 5. If $K \in PC_\Delta$ and K has arbitrarily large finite members, then K has an infinite member. In particular, the class of finite models for a given language in not PC_Δ.

On the other hand, the class of infinite members of an EC_Δ (PC, PC_Δ) class is still EC_Δ (PC, PC_Δ).

PC classes need not be closed under $\equiv$. One example is as follows. Let $<$ and P be a two-place and a one-place predicate, respectively. Let φ be a sentence saying that "$<$ is a linear ordering and P is nonempty and has no $<$-smallest element". Let $K = Mod(\varphi) \mid \{<\}$. Then $K \in PC$ and $(A, <) \in K$ iff $(A, <)$ is a linear ordering which is not a well-ordering. Now let $(A', <')$ be any infinite well-ordering. $(A', <') \notin K$. Let c_n, $n \in N$, be individual constants. Let $\Phi = Th((A', <')) \cup \{c_k < c_m\colon m < k\}$. Every finite subset of Φ has a model and so Φ has a model $(A'', <'', a_n)_{n \in N}$. $(A'', <'') \equiv (A', <')$ and $(A'', <'') \in K$. Thus, K is not closed under $\equiv$. It follows that $PC \nsubseteq EC_\Delta$.

Proposition 4 can be strengthened as follows.

Proposition 5. If K_0, $K_1 \in PC_\Delta$ and $K_0 \cap K_1 = \varnothing$, there is a class $K \in EC$ such that $K_0 \subseteq K$ and $K \cap K_1 = \varnothing$. Thus, in particular, if K and K^c are PC_Δ, then K is EC.

Proof. Let Φ_i be such that $K_i = \{\mathcal{A} \mid \ell\colon \mathcal{A} \models \Phi_i\}$, $i = 0, 1$. We may assume that $\ell_{\Phi_0} \cap \ell_{\Phi_1} = \ell$. Since $K_0 \cap K_1 = \varnothing$, there are finite subsets Ψ_0 and Ψ_1 of Φ_0 and Φ_1, respectively, such that $\Psi_0 \cup \Psi_1$ has no model. Let $\psi_i := \bigwedge\Psi_i$. Then $\vDash \psi_0 \to \neg\psi_1$. By the Interpolation Theorem (Theorem 2.8; cf. also Theorem 8, below), there is a sentence θ of ℓ such that $\vDash \psi_0 \to \theta$ and $\vDash \theta \to \neg\psi_1$. Let $K = \mathrm{Mod}(\theta)$. ■

The relation between PC_Δ and EC_Δ is given in the following:

Proposition 6. $K \in EC_\Delta$ iff $K \in PC_\Delta$ and K is closed under $\equiv$.

Proof. "⇒" is clear. "⇐". Let $\Phi = \mathrm{Th}(K)$. It is sufficient to show that $K = \mathrm{Mod}(\Phi)$. Clearly, $K \subseteq \mathrm{Mod}(\Phi)$. To show that $\mathrm{Mod}(\Phi) \subseteq K$, suppose $\mathcal{A} \vDash \Phi$. Every finite subset of $\mathrm{Th}(\mathcal{A})$ has a model in K. Indeed, suppose there is a sentence φ such that $\mathcal{A} \vDash \varphi$ and φ has no model in K. Then $\neg\varphi \in \Phi$ and so $\mathcal{A} \vDash \neg\varphi$, a contradiction. Let Ψ be such that $K = \mathrm{Mod}(\Psi) \mid \ell_K$. Then every finite subset of $\mathrm{Th}(\mathcal{A}) \cup \Psi$ has a model. Let $\mathcal{B}$ be a model of $\mathrm{Th}(\mathcal{A}) \cup \Psi$ and let $\mathcal{C} = \mathcal{B} \mid \ell_K$. Then $\mathcal{C} \in K$ and $\mathcal{A} \equiv \mathcal{C}$. It follows that $\mathcal{A} \in K$, as desired. ■

§4. Preservation theorems. One important aspect of model theory is the study of the relation between syntactic properties of (sets of) sentences and algebraic properties of the corresponding classes of models. So-called preservation theorems are particularly clear examples of results of this type.

A class K of models is *closed under submodels* if for all $\mathcal{A}$, $\mathcal{B}$, if $\mathcal{A} \in K$ and $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B} \in K$. Let φ be a universal sentence, $\varphi := \forall x_1 \ldots x_n \psi(x_1,\ldots,x_n)$, where $\psi(x_1,\ldots,x_n)$ is quantifier-free. Suppose $\mathcal{A} \vDash \varphi$ and $\mathcal{B} \subseteq \mathcal{A}$. Let $b_1,\ldots,b_n$ be any members of B. Then $\mathcal{A} \vDash \psi(b_1,\ldots,b_n)$. Since $\psi(x_1,\ldots,x_n)$ is quantifier-free, we have $\mathcal{B} \vDash \psi(b_1,\ldots,b_n)$. It follows that $\mathcal{B} \vDash \varphi$. Thus, φ is *preserved under submodels* in the sense that $\mathrm{Mod}(\varphi)$ is closed under submodels. A theory is *universal* if all its members (axioms) are universal. It follows that a universal theory T is *preserved under submodels* in the sense that Mod(T) is closed under submodels.

We now show that the converse of this is also true and so we have the following:

Theorem 5 (Łoś -Tarski Theorem)**.** T is preserved under submodels iff T is equivalent to a universal theory.

Proof. "⇒". Let T′ be the set of universal sentences φ such that $T \vdash \varphi$. Then $\mathrm{Mod}(T) \subseteq \mathrm{Mod}(T')$. To show that $\mathrm{Mod}(T') \subseteq \mathrm{Mod}(T)$, suppose $\mathcal{A} \vDash T'$. We want to show that $T \cup D(\mathcal{A})$ has a model. Suppose not. There is then a conjunction $\psi(c_1,\ldots,c_n)$ of members of $D(\mathcal{A})$, where $\psi(x_1,\ldots,x_n)$ is a formula of ℓ_T, such that $T \vdash \neg\psi(c_1,\ldots,c_n)$. Let $\theta := \forall x_1 \ldots x_n \neg\psi(x_1,\ldots,x_n)$. Then $T \vdash \theta$, whence $\theta \in T'$ and so $\mathcal{A} \vDash \theta$. On the other hand, since $\psi(c_1,...,c_n)$ is a conjunction of members of $D(\mathfrak{A})$, it is clear that $\mathfrak{A} \vDash \neg\theta$, a contradiction. It now follows that $T \cup D(\mathfrak{A})$ has a model $\mathfrak{B}$. By Lemma 3(a), $\mathfrak{A}$ is embeddable in $\mathfrak{B}$. But then, since T is preserved under submodels, $\mathfrak{A} \vDash T$, as desired. ■

The conjunction of two universal sentences is equivalent to a universal sentence. Thus, combining Theorem 5 and Proposition 3 we get:

Corollary 6. For any sentence φ, φ is preserved under submodels iff φ is equivalent to a universal sentence.

A simple illustration of Corollary 6 is as follows. Let $\varphi :=$

$$\exists xy\forall z(f(z) = x \vee f(z) = y).$$

Clearly, φ is preserved under submodels. Thus, there is a universal sentence equivalent to φ. And, of course, φ is equivalent to

$$\forall xyz(f(x) = f(y) \vee f(x) = f(z) \vee f(y) = f(z)).$$

A formula is *universal-existential* or $\forall\exists$ if it is of the form

(*) $\quad \forall x_1...x_n\psi(x_1,...,x_n)$,

where $\psi(x_1,...,x_n)$ is existential. (Universal and existential formulas are $\forall\exists$.) A set of sentences is $\forall\exists$ if all its members are.

A class K is *closed under unions of chains* if for any chain $\{\mathfrak{A}_i: i \in I\} \subseteq K$, $\bigcup\{\mathfrak{A}_i: i \in I\} \in K$. T is *preserved under unions of chains* if Mod(T) is closed under unions of chains. Note that if T is preserved under submodels, then T is preserved under unions of chains.

Suppose $\{\mathfrak{A}_i: i \in I\}$ is a chain of models of (*). Let $\mathfrak{A} = \bigcup\{\mathfrak{A}_i: i \in I\}$. To see that $\mathfrak{A}$ is then a model of (*), let $a_1,...,a_n$ be any members of A. Since $\{\mathfrak{A}_i: i \in I\}$ is a chain, there is an $\mathfrak{A}_i$ such that $a_1,...,a_n \in A_i$. Since $\mathfrak{A}_i$ is a model of (*), we have $\mathfrak{A}_i \vDash \psi(a_1,...,a_n)$ and so, since $\psi(x_1,...,x_n)$ is existential and $\mathfrak{A}_i \subseteq \mathfrak{A}$, $\mathfrak{A} \vDash \psi(a_1,...,a_n)$. Thus, $\mathfrak{A}$ is a model of (*). It follows that any $\forall\exists$ theory is preserved under unions of chains.

We now show that the converse of this is also true and so we have the following:

Theorem 6 (Chang-Łoś -Suszko Theorem)**.** T is preserved under unions of chains iff T is equivalent to an $\forall\exists$ theory.

Lemma 4. Let T′ be the set of $\forall\exists$ sentences φ such that $T \vdash \varphi$. Suppose $\mathfrak{A} \vDash T'$. There is then a model $\mathfrak{B}$ such that $\mathfrak{B} \vDash T$ and $\mathfrak{A} \preccurlyeq_1 \mathfrak{B}$.

Proof. By Lemma 3(c), it is sufficient to show that $T \cup UD(\mathcal{A})$ has a model. Suppose not. There are then a universal formula $\psi(x_1,...,x_n)$ and constants $c_1,...,c_n$ not occurring in T such that $\mathcal{A}_A \models \psi(c_1,...,c_n)$ and $T \vdash \neg\psi(c_1,...,c_n)$. Let $\varphi :=$ $\forall x_1...x_n\neg\psi(x_1,...,x_n)$. Then φ is (equivalent to) an $\forall\exists$ sentence, $T \vdash \varphi$, and $\mathcal{A} \not\models \varphi$, contrary to assumption. ■

Lemma 5. Suppose $\mathcal{A} \preccurlyeq_1 \mathcal{B}$. Then there is a model $\mathcal{C}$ such that $\mathcal{A} \preccurlyeq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{C}$.

Proof. By Lemma 3(d), it is sufficient to show that $ED(\mathcal{A}) \cup D(\mathcal{B})$ has a model. Suppose not. There is then a conjunction $\varphi(a_1,...,a_k,b_1,...,b_n)$ of members of $D(\mathcal{B})$, where $\varphi(x_1,...,x_k,y_1,...,y_n)$ is a formula of $\ell_{\mathcal{A}}$, $a_1,...,a_k \in A$, $b_1,...,b_n \in B - A$ such that $ED(\mathcal{A}) \vdash \neg\exists y_1...y_n\varphi(a_1,...,a_k,y_1,...,y_n)$. Let $\psi(x_1,...,x_k) := \exists y_1...y_n\varphi(x_1,...,x_k,y_1,...,y_n)$. It follows that $\mathcal{A} \models \neg\psi(a_1,...,a_k)$. On the other hand $\psi(x_1,...,x_k)$ is simple and $\mathcal{B} \models \psi(a_1,...,a_k)$ and so, since $\mathcal{A} \preccurlyeq_1 \mathcal{B}$, $\mathcal{A} \models \psi(a_1,...,a_k)$, a contradiction. ■

Proof of Theorem 6. "$\Rightarrow$". Let T′ be as in Lemma 4. Then $Mod(T) \subseteq Mod(T')$. To show that $Mod(T') \subseteq Mod(T)$, suppose $\mathcal{A} \models T'$. We are going to show that there is a model $\mathcal{B}$ of T such that $\mathcal{A} \preccurlyeq \mathcal{B}$. From this it follows that $\mathcal{A} \models T$. We construct a sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$, ... of models such that $\mathcal{A}_0 = \mathcal{A}$ and for all n,

(1) $\mathcal{A}_{2n} \models T'$,

(2) $\mathcal{A}_{2n+1} \models T$,

(3) $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$,

(4) $\mathcal{A}_{2n} \preccurlyeq \mathcal{A}_{2n+2}$.

Suppose $\mathcal{A}_{2n}$ has been defined and (1) holds; this is true for n = 0. Then, by Lemma 4, there is a model $\mathcal{A}_{2n+1}$ such that (2) holds and $\mathcal{A}_{2n} \preccurlyeq_1 \mathcal{A}_{2n+1}$. Finally, by Lemma 5, this implies that there is a model $\mathcal{A}_{2n+2}$ such that (4) holds, whence $\mathcal{A}_{2n+2} \models T'$, and $\mathcal{A}_{2n+1} \subseteq \mathcal{A}_{2n+2}$.

Now, let

$$\mathcal{B} = \bigcup\{\mathcal{A}_n : n \in N\}.$$

Then, by (3),

$$\mathcal{B} = \bigcup\{\mathcal{A}_{2n} : n \in N\} = \bigcup\{\mathcal{A}_{2n+1} : n \in N\}.$$

By (4) and Lemma 2, $\mathcal{A} \preccurlyeq \mathcal{B}$. By (2) and the fact that T is preserved under unions of chains, $\mathcal{B} \models T$. It follows that $\mathcal{A} \models T$, as desired. ■

Corollary 7. A sentence is preserved under unions of chains iff it is equivalent to an $\forall\exists$ sentence.

A simple illustration of Corollary 7 is as follows. Let $\varphi :=$

$$\forall x\exists y\forall z(Pxz \leftrightarrow z = y).$$

φ says that "the relation P is a function". The union of a chain of functions is a function. Thus, φ is preserved under unions of chains. And, of course, φ is equivalent to the conjunction of

$\forall x \exists y Pxy$ and

$\forall xyz(Pxy \wedge Pxz \to y = z)$

and this conjunction is equivalent to an $\forall\exists$ sentence.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are models for ℓ. A function h is a *homomorphism of $\mathcal{A}$ onto $\mathcal{B}$* if h is a function on A onto B such that for all $a_1,\ldots,a_n \in A$,

$$\text{if } \langle a_1,\ldots,a_n\rangle \in P^{\mathcal{A}}, \text{ then } \langle h(a_1),\ldots,h(a_n)\rangle \in P^{\mathcal{B}},$$

$$h(c^{\mathcal{A}}) = c^{\mathcal{B}},$$

$$h(f^{\mathcal{A}}(a_1,\ldots,a_n)) = f^{\mathcal{B}}(h(a_1),\ldots,h(a_n)),$$

for predicates P, constants c, and function symbols f of ℓ.

$\mathcal{B}$ is a *homomorphic image* of $\mathcal{A}$ if there is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Note that homomorphic images of the same model need not be isomorphic, not even if the homomorphisms in question are the same.

A formula φ is *positive* if the only logical symbols occurring in φ are $\wedge$, $\vee$, $\forall$, $\exists$, $=$. T is *positive* if all its axioms are positive. We write $\mathcal{A}\Sigma^+\mathcal{B}$ to mean that every positive sentence true in $\mathcal{A}$ is true in $\mathcal{B}$.

The proof of the following lemma is straightforward (compare the proof of Proposition 1.1).

Lemma 6. Suppose h is a homomorphism on $\mathcal{A}$ onto $\mathcal{B}$ and all $\boldsymbol{u} : \mathrm{Var} \to A$. Then for every term t of ℓ,

$$h(t^{\mathcal{A}}[\boldsymbol{u}]) = t^{\mathcal{B}}[h\boldsymbol{u}].$$

Also, for every positive formula φ of ℓ,

$$\text{if } \mathcal{A} \vDash \varphi[\boldsymbol{u}], \text{ then } \mathcal{B} \vDash \varphi[h\boldsymbol{u}].$$

In particular, if $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$, then $\mathcal{A}\Sigma^+\mathcal{B}$.

K is *closed under homomorphisms* if for all $\mathcal{A}$, $\mathcal{B}$, if $\mathcal{A} \in K$ and $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$, then $\mathcal{B} \in K$. A sentence φ (theory T) is *preserved under homomorphisms* if $\mathrm{Mod}(\varphi)$ ($\mathrm{Mod}(T)$) is closed under homomorphisms. By Lemma 6, positive sentences and, therefore, positive theories are preserved under homomorphisms.

The converse of this is also true and so we have the following:

Theorem 7 (Lyndon)**.** T is preserved under homomorphisms iff T is equivalent to a positive theory.

Lemma 7. If $\mathfrak{A}_0\Sigma^+\mathfrak{A}_1$, there are models $\mathcal{B}_0$, $\mathcal{B}_1$ such that $\mathcal{B}_i \equiv \mathfrak{A}_i$, i = 0, 1, and $\mathcal{B}_1$ is a homomorphic image of $\mathcal{B}_0$.

This can be proved in a number of different ways. Our proof, in Appendix 2, uses a variant of some ideas that will be presented in §7 and in Chapter 5.

Proof of Theorem 7. "⇒". Suppose T is preserved under homomorphisms. Let T′ be the set of positive sentences φ such that $T \models \varphi$. Then $\mathrm{Mod}(T) \subseteq \mathrm{Mod}(T')$. To prove that $\mathrm{Mod}(T') \subseteq \mathrm{Mod}(T)$, suppose $\mathfrak{A} \models T'$. Let

$$\Phi = T \cup \{\neg\varphi\colon \varphi \text{ positive and } \mathfrak{A} \models \neg\varphi\}.$$

Φ has a model. For, suppose not. There are then positive sentences $\varphi_0,\ldots,\varphi_n$ such that $\mathfrak{A} \models \neg\varphi_k$, for $k \leq n$, and $T \models \varphi_0 \vee \ldots \vee \varphi_n$ and so $\mathfrak{A} \models \varphi_0 \vee \ldots \vee \varphi_n$, which is a contradiction.

Let $\mathcal{B}$ be a model of Φ. Then $\mathcal{B} \models T$ and $\mathcal{B}\Sigma^+\mathfrak{A}$. By Lemma 7, there are $\mathfrak{A}'$, $\mathcal{B}'$ such that $\mathfrak{A}' \equiv \mathfrak{A}$, $\mathcal{B}' \equiv \mathcal{B}$, and $\mathfrak{A}'$ is a homomorphic image of $\mathcal{B}'$. Thus, $\mathcal{B}' \models T$ and so $\mathfrak{A}' \models T$ and so $\mathfrak{A} \models T$, as desired. ■

Corollary 8. A sentence is preserved under homomorphisms iff it is equivalent to a positive sentence.

The preservation theorems prove in this section are exemplified by some of the theories defined in Chapter 1, §7. Thus, for example, (i) the classes of linear orderings, Boolean algebras, and (Abelian) groups are closed under submodels and the corresponding theories are universal; (ii) the classes of fields, algebraically closed fields, ordered fields, and real closed ordered fields are closed under unions of chains and the corresponding theories are ∀∃; (iii) the classes of groups and Abelian groups and, if we omit the axiom $0 \neq 1$, Boolean algebras are closed under homomorphisms and the corresponding theories are positive.

§5. Interpolation and definability. In Chapter 2, §4 we outlined a proof of the following result (Theorem 2.8).

Theorem 8 (Craig's Interpolation Theorem). Let φ, ψ be any two sentences. If $\models \varphi \to \psi$, there is sentence θ of $\ell_\varphi \cap \ell_\psi$ such that $\models \varphi \to \theta$ and $\models \theta \to \psi$.

We shall now give a purely model-theoretic proof of this. We derive Theorem 8 from the following result.

A set Φ is *complete* if for every sentence φ of ℓ_Φ, either $\Phi \models \varphi$ or $\Phi \models \neg\varphi$.

Theorem 9 (Robinson's Consistency Theorem)**.** Let $\ell = \ell_0 \cap \ell_1$ and let Φ be a complete consistent set of sentences of ℓ. Let Φ_i be a consistent set of sentences of ℓ_i such that $\Phi \subseteq \Phi_i$, i = 0, 1. Then $\Phi_0 \cup \Phi_1$ is consistent.

Lemma 8. Suppose $\mathcal{A}_i$ is a model for ℓ_i, i = 0, 1, and let $\ell = \ell_0 \cap \ell_1$.

(a) If $\mathcal{B}$ is a model for ℓ_1 and $\mathcal{A}_0 | \ell \equiv \mathcal{B} | \ell$, there is a model $\mathcal{C}$ for ℓ_1 such that $\mathcal{A}_0 | \ell \preccurlyeq \mathcal{C} | \ell$ and $\mathcal{C} \equiv \mathcal{B}$.

(b) If $\mathcal{A}_0 | \ell \preccurlyeq \mathcal{A}_1 | \ell$, there is a model $\mathcal{B}$ for ℓ_0 such that $\mathcal{A}_0 \preccurlyeq \mathcal{B}$ and $\mathcal{A}_1 | \ell \preccurlyeq \mathcal{B} | \ell$.

Proof. (a) By Lemma 3, it suffices to show that the set $\mathrm{ED}(\mathcal{A}_0 | \ell) \cup \mathrm{Th}(\mathcal{B})$ has a model. Suppose not. There are then a formula $\varphi(x_1,\ldots,x_n)$ of ℓ and constants $c_1,\ldots,c_n$ not in ℓ_1 such that $\varphi(c_1,\ldots,c_n) \in \mathrm{ED}(\mathcal{A}_0 | \ell)$ and $\mathrm{Th}(\mathcal{B}) \vDash \neg\varphi(c_1,\ldots,c_n)$. It follows that $\mathrm{Th}(\mathcal{B}) \cup \{\exists x_1 \ldots x_k \varphi\}$ has no model. But this is not true, since the fact that $\mathcal{A}_0 | \ell \equiv \mathcal{B} | \ell$ implies that $\mathcal{B}$ is a model of this set.

(b) By Lemma 3, it suffices to show that the set $\mathrm{ED}(\mathcal{A}_0) \cup \mathrm{ED}(\mathcal{A}_1 | \ell)$ has a model. Suppose not. There are then a formula $\varphi(x_1,\ldots,x_n)$ of $\ell(A_1)$ and constants $c_1,\ldots,c_n$ not in $\ell_0(A_0)$ such that $\varphi(c_1,\ldots,c_n) \in \mathrm{ED}(\mathcal{A}_1 | \ell)$ and $\mathrm{ED}(\mathcal{A}_0) \vDash \neg\varphi(c_1,\ldots,c_n)$. It follows that $\mathrm{ED}(\mathcal{A}_0) \cup \{\exists x_1 \ldots x_k \varphi\}$ has no model. But this is not true, since the fact that $\mathcal{A}_0 | \ell \preccurlyeq \mathcal{A}_1 | \ell$ implies that $\mathcal{A}_{0A_0}$ is a model of this set. ■

Proof of Theorem 9. Let $\mathcal{A}_0$ be any model of Φ_0. There is a model $\mathcal{B}$ of Φ_1. $\mathcal{A}_0 | \ell$ and $\mathcal{B} | \ell$ are models of Φ and so $\mathcal{A}_0 | \ell \equiv \mathcal{B} | \ell$. By Lemma 8(a), it follows that there is a model $\mathcal{A}_0 | \ell \preccurlyeq \mathcal{A}_1 | \ell$ and $\mathcal{A}_1 \equiv \mathcal{B}$. It follows that $\mathcal{A}_1 \vDash \Phi_1$. Starting from $\mathcal{A}_0$, $\mathcal{A}_1$ and applying Lemma 8(b) we can now define models $\mathcal{A}_n$ for $n \geq 2$ such that for all n,

(1) $\mathcal{A}_{2n} \vDash \Phi_0$,

(2) $\mathcal{A}_{2n+1} \vDash \Phi_1$,

(3) $\mathcal{A}_n \preccurlyeq \mathcal{A}_{n+2}$,

(4) $\mathcal{A}_n | \ell \preccurlyeq \mathcal{A}_{n+1} | \ell$.

Now let

$$\mathcal{C}_0 = \bigcup\{\mathcal{A}_{2n} \colon n \in \mathrm{N}\} \text{ and } \mathcal{C}_1 = \bigcup\{\mathcal{A}_{2n+1} \colon n \in \mathrm{N}\}.$$

Then, by (1), (2), (3), and Lemma 2, $\mathcal{C}_i \vDash \Phi_i$, i = 0, 1. Also, clearly, $\mathcal{C}_0 | \ell = \mathcal{C}_1 | \ell$. It follows that $\mathcal{C}_0$ and $\mathcal{C}_1$ have a common expansion $\mathcal{C}$ to $\ell_0 \cup \ell_1$. $\mathcal{C} \vDash \Phi_0 \cup \Phi_1$. ■

There is an alternative proof of Theorem 9 in Appendix 2.

Proof of Theorem 8. Let

$$\Theta = \{\theta \colon \theta \text{ sentence of } \ell_\varphi \cap \ell_\psi \text{ and } \vDash \varphi \to \theta\}.$$

We are going to show that $\Theta \cup \{\neg\psi\}$ has no model. Suppose it does and let $\mathcal{A}$ be such a model. Then $\mathrm{Th}(\mathcal{A}) \cup \{\neg\psi\}$ is consistent. Also $\mathrm{Th}(\mathcal{A}) \cup \{\varphi\}$ has a model. For suppose not. Then there is a sentence $\chi \in \mathrm{Th}(\mathcal{A})$ such that $\vDash \varphi \to \neg\chi$. But then $\neg\chi \in \Theta$ and so $\mathcal{A} \vDash \neg\chi$, a contradiction. Thus, $\mathrm{Th}(\mathcal{A}) \cup \{\varphi\}$ has a model. Applying

Theorem 9 we may now conclude that $\{\varphi, \neg\psi\}$ (in fact, $\mathrm{Th}(\mathcal{A}) \cup \{\varphi, \neg\psi\}$) is consistent. But then $\nvDash \varphi \rightarrow \psi$, contrary to hypothesis.

This shows that $\Theta \cup \{\neg\psi\}$ has no model. Since Θ is closed under conjunction, it follows that there is a $\theta \in \Theta$ such that $\vDash \theta \rightarrow \psi$. But also $\vDash \varphi \rightarrow \theta$ and θ is a sentence of ℓ. Thus, θ is as desired. ■

We have derived Theorem 8 from Theorem 9. It is worth noting that we can also derive Theorem 9 from Theorem 8: Assume Theorem 8 and let Φ, ℓ, Φ_i, ℓ_i, $i = 0, 1$, be as assumed in Theorem 9. Suppose $\Phi_0 \cup \Phi_1$ has no model. There are then conjunctions φ_0, φ_1 of members of Φ_0 and Φ_1, respectively, such that $\vDash \varphi_0 \rightarrow \neg\varphi_1$. But then, by Theorem 8, there is a sentence ψ of ℓ such that $\vDash \varphi_0 \rightarrow \varphi$ and $\vDash \varphi \rightarrow \neg\varphi_1$. It follows that $\Phi_0 \vDash \varphi$ and $\Phi_1 \vDash \neg\varphi$, whence $\Phi \vDash \varphi$ and $\Phi \vDash \neg\varphi$, contrary to assumption.

We shall now use Theorem 8 to prove a basic result on definability. We shall only discuss definability of predicates but our discussion can easily be extended to function symbols and individual constants.

A predicate F is *implicitly defined* in T if for any two models $\mathcal{A}$ and $\mathcal{B}$ of T, if $\mathcal{A} \mid (\ell_T - \{F\}) = \mathcal{B} \mid (\ell_T - \{F\})$, then $F^{\mathcal{A}} = F^{\mathcal{B}}$. F is *explicitly definable* in T if there is a formula $\varphi(x_1,\ldots,x_n)$ of $\ell_T - \{F\}$ such that

(*) $\quad T \vdash \forall x_1 \ldots x_n (Fx_1 \ldots x_n \leftrightarrow \varphi(x_1,\ldots,x_n))$.

Note that, by compactness, if F is implicitly defined in T, then there is a finite subtheory T′ of T such that F is implicitly defined in T′.

Suppose (*) is true. Let T′′ be obtained from T by replacing $Fx_1 \ldots x_n$ everywhere by $\varphi(x_1,\ldots,x_n)$. (We assume that no variable occurring in T is bound in $\varphi(x_1,\ldots,x_n)$.) Then T′′ plus (*) is equivalent to T.

Clearly, explicit definability implies implicit definability. Thus, to show that F is not explicitly definable in T it is sufficient to show that it is not implicitly defined in T. This is known as Padoa's method. (Of course, nowadays this is just common sense.) The question arises if Padoa's method is complete in the sense that the converse of this holds so that if F is not explicitly definable, this can at least in principle be shown by applying Padoa's method. In view of the following result the answer is affirmative.

Theorem 10 (Beth's Definability Theorem)**.** Suppose F is an n-place predicate implicitly defined in T. Then F is explicitly definable in T.

Proof. Let F′ be an n-place predicate not in ℓ_T. For every formula ψ of ℓ_T, let ψ' be obtained from ψ by replacing F by F′. Let $T' = \{\psi' : \psi \in T\}$. Let $c_1,\ldots,c_n$ be new individual constants. Since F is implicitly defined in T, it follows that

$$T \cup T' \vdash Fc_1...c_n \rightarrow F'c_1...c_n.$$

But then there is a sentence ψ of ℓ_T such that $T \vdash \psi$ and

$$\vdash \psi \wedge Fc_1...c_n \rightarrow (\psi' \rightarrow F'c_1...c_n).$$

By Theorem 8, there is then a formula $\varphi(x_1,...,x_n)$ of ℓ_T–{F} such that

$$\vdash \psi \wedge Fc_1...c_n \rightarrow \varphi(c_1,...,c_n),$$

$$\vdash \varphi(c_1,...,c_n) \rightarrow (\psi' \rightarrow F'c_1...c_n).$$

From the latter of these and since F′ does not occur in $\varphi(x_1,...,x_n)$, it follows that

$$\vdash \psi \rightarrow (\varphi(c_1,...,c_n) \rightarrow Fc_1...c_n).$$

Thus we get

$$\vdash \psi \rightarrow (Fc_1...c_n \leftrightarrow \varphi(c_1,...,c_n))$$

and so, since $T \vdash \psi$ and $c_1,...,c_n$ do not occur in T, (*) holds, as desired. ■

A result similar to Theorem 10 but dealing with a different notion of (explicit) definability is proved in Appendix 3.

§6. Completeness and model completeness. Let $\mathfrak{A}$ be any model. Suppose that we want to know what is true and what is not about $\mathfrak{A}$ (restricted to first-order sentences). We may approach this problem by writing down a number of sentences that (we know) are true of $\mathfrak{A}$; call these the axioms of our (tentative) theory T of $\mathfrak{A}$. We will then want to know if these axioms are sufficient, in other words, if $\mathfrak{A} \vDash \varphi$ implies that $T \vdash \varphi$. It does iff the theory T is *complete* in the sense that for every sentence φ of ℓ_T, either $T \vdash \varphi$ or $T \vdash \neg\varphi$. There are several ways of proving that (first-order) theories are complete; two of these will be presented in this section, a third method in the next.

A result to the effect that a given theory T is complete can also be regarded as a (negative) result on the expressive power of L_1: if $\mathfrak{A}$, $\mathfrak{B}$ are any models of T, there is no sentence φ such that φ is true in $\mathfrak{A}$ but not in $\mathfrak{B}$.

The proof of the following lemma is straightforward.

Lemma 9. The following conditions are equivalent.

(i) T is complete.

(ii) Any two models of T are elementarily equivalent.

(iii) $T = Th(\mathfrak{A})$ for some model $\mathfrak{A}$.

Clearly, isomorphic models are of the same cardinality. Thus, by Corollary 3, no theory which has an infinite model is categorical in the sense that all its models are isomorphic. But there is an interesting weaker notion: a theory T is κ-*categorical* (or *categorical in* κ) if T has a model of cardinality κ and all models of T of cardinality κ are isomorphic.

Theorem 11 (Łoś -Vaught Test)**.** If all models of T are infinite and T is κ-categorical for some $\kappa \geq |T|$, then T is complete.

Proof. Suppose T is not complete. Let φ be a sentence such that $T \nvdash \varphi$ and $T \nvdash \neg\varphi$. Then $T \cup \{\neg\varphi\}$ and $T \cup \{\varphi\}$ are consistent. By Corollary 3, $T \cup \{\neg\varphi\}$ has a model $\mathcal{A}$ of cardinality κ. Similarly, $T \cup \{\varphi\}$ has a model $\mathcal{B}$ of cardinality κ. Clearly, $\mathcal{A} \not\equiv \mathcal{B}$ and so T is not κ-categorical. ■

By a classical theorem of Steinitz, ACF(p), the theory of algebraically closed fields of characteristic p, where p is 0 or a prime, is κ-categorical for every $\kappa > \aleph_0$. Since ACF(p) has no finite models, by Theorem 11, it follows that:

Theorem 12. ACF(p), where p is 0 or a prime, is complete.

For further applications of Theorem 11, see §13, Examples 2, 4, 6, 7.

But, of course, there are (important) complete theories that are not κ-categorical for any κ. To deal with some of these we introduce the concept of model completeness, which is also of independent interest.

A theory T is *model-complete* if for any two models $\mathcal{A}$, $\mathcal{B}$ of T, if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} \preccurlyeq \mathcal{B}$. $\mathcal{A}$ is an *existentially closed model for* T if $\mathcal{A} \vDash T$ and for every model $\mathcal{B}$ of T, if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} \preccurlyeq_1 \mathcal{B}$.

Theorem 13 (Robinson's Test)**.** T is model-complete iff every model of T is existentially closed for T.

Proof. "⇒" is clear. To prove "⇐" suppose $\mathcal{A}$, $\mathcal{A}'$ are models of T and $\mathcal{A} \subseteq \mathcal{A}'$. We show that there are models $\mathcal{A}_n$ of T, n = 0, 1, 2, ..., such that $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_1 = \mathcal{A}'$ and for all n,

(1) $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$,

(2) $\mathcal{A}_n \preccurlyeq \mathcal{A}_{n+2}$.

Suppose $\mathcal{A}_n$ and $\mathcal{A}_{n+1}$ have been defined. Since they are models of T, (1) holds, and $\mathcal{A}_n$ is existentially closed for T, it follows that $\mathcal{A}_n \preccurlyeq_1 \mathcal{A}_{n+1}$. But then, by Lemma 5, there is a model $\mathcal{A}_{n+2}$ such that $\mathcal{A}_n \preccurlyeq \mathcal{A}_{n+2}$ and $\mathcal{A}_{n+1} \subseteq \mathcal{A}_{n+2}$. Thus, $\mathcal{A}_{n+2}$ is as desired.

Now let

$$\mathcal{B} = \bigcup\{\mathcal{A}_{2n} : n \in N\} = \bigcup\{\mathcal{A}_{2n+1} : n \in N\}.$$

Then, by (2) and Lemma 2, $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{A}' \preccurlyeq \mathcal{B}$. It follows that $\mathcal{A} \preccurlyeq \mathcal{A}'$, as desired. ■

For some simple direct applications of Robinson's test, see §13, Examples 2, 4.

A theory T was originally defined to be model-complete if $T \cup D(\mathcal{A})$ is complete for every model $\mathcal{A}$ of T. The two definitions are easily equivalent.

One of the most important applications of Theorem 13 is the proof of the following theorem; RCOF, the theory of real closed ordered fields, is not κ-categorical for any κ.

Theorem 14. RCOF is model-complete.

This is proved, modulo an algebraic lemma, in Appendix 4.

A model-complete theory need not be complete (see Theorem 18, below, and §13, Example 2). But we do have the following result. $\mathcal{A}$ is a *prime* model of a theory T if $\mathcal{A} \models T$ and $\mathcal{A}$ is embeddable in every model of T.

Theorem 15. If T is model-complete and has a prime model, then T is complete.

Proof. Let $\mathcal{A}$, $\mathcal{B}$ be any models of T. By Lemma 9, it is sufficient to show that $\mathcal{A} \equiv \mathcal{B}$. By hypothesis, there is a model $\mathcal{C}$ of T which is embeddable in $\mathcal{A}$ and $\mathcal{B}$. Since T is model-complete, this implies that $\mathcal{C} \equiv \mathcal{A}$ and $\mathcal{C} \equiv \mathcal{B}$ and so $\mathcal{A} \equiv \mathcal{B}$. ■

RCOF has a prime model, the ordered field of real algebraic numbers. Thus, from Theorems 14 and 15 it follows that:

Theorem 16 (Tarski's Theorem)**.** RCOF is complete.

Combining Lemma 2 and Theorem 6 we get:

Corollary 9. If T is model-complete, then T is equivalent to an ∀∃ theory.

Given Corollary 9 the question arises if every complete ∀∃ theory is model-complete. This, however, is not true (see §13, Example 8) but we do have the following:

Theorem 17. Suppose T is ∀∃, all models of T are infinite, and T is κ-categorical for some $\kappa \geq |T|$. Then T is model-complete.

Lemma 10. Suppose $\kappa \geq |T|$ and all models of T are infinite. If every model of T of cardinality κ is existentially closed for T, then T is model-complete.

Proof. Suppose T is not model-complete. By Theorem 13, there are then infinite models $\mathcal{A}$, $\mathcal{B}$ of T such that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \not\preccurlyeq_1 \mathcal{B}$. By the LST theorem (Corollary 3), there is a model $(\mathcal{C},D) \equiv (\mathcal{B},A)$ such that $|D| = \kappa$. $\mathcal{C}|D$ is a model of T of cardinality

κ which is not existentially closed for T. ■

Lemma 11. Suppose T is preserved under unions of chains and $\kappa \geq |T|$.Then for every model $\mathcal{A}$ of T such that $|A| = \kappa$, there is an existentially closed model $\mathcal{A}^*$ for T such that $\mathcal{A} \subseteq \mathcal{A}^*$ and $|A^*| = \kappa$.

Proof. We first show that

(*) for every model $\mathcal{B}$ of T such that $|B| = \kappa$, there is a model $\mathcal{B}'$ of T such that $\mathcal{B} \subseteq \mathcal{B}'$, $|B'| = \kappa$, and for every existential sentence of $\ell_T(B)$ and every model $\mathcal{C}$ of T such that $\mathcal{B}' \subseteq \mathcal{C}$, if $\mathcal{C} \models \varphi$, then $\mathcal{B}' \models \varphi$.

Let $\{\varphi_\xi \colon \xi < \kappa\}$ be the set of existential sentences of $\ell_T(B)$. We define models $\mathcal{B}_\xi$ of T of cardinality κ for $\xi < \kappa$ as follows. $\mathcal{B}_0 = \mathcal{B}$. Suppose $\mathcal{B}_\xi$ has been defined. If there is a model $\mathcal{C}$ of T extending $\mathcal{B}_\xi$ such that $\mathcal{C} \models \varphi_\xi$, let $\mathcal{B}_{\xi+1}$ be such a model of cardinality κ. Otherwise, let $\mathcal{B}_{\xi+1} = \mathcal{B}_\xi$. If $\xi < \kappa$ is a limit ordinal, let $\mathcal{B}_\xi = \bigcup\{\mathcal{B}_\eta \colon \eta < \xi\}$. Then $\mathcal{B}_\xi \models T$.

Now let $\mathcal{B}' = \bigcup\{\mathcal{B}_\xi \colon \xi < \kappa\}$. Then $\mathcal{B}' \models T$. Let φ be any existential sentence of $\ell_T(B)$ and suppose there is a model $\mathcal{C}$ of T such that $\mathcal{B}' \subseteq \mathcal{C}$ and $\mathcal{C} \models \varphi$. $\varphi := \varphi_\xi$ for some $\xi < \kappa$. $\mathcal{B}_\xi \subseteq \mathcal{C}$. It follows that $\mathcal{B}_{\xi+1} \models \varphi_\xi$ and so $\mathcal{B}' \models \varphi_\xi$. Thus, $\mathcal{B}'$ is as claimed in (*).

We now define $\mathcal{A}_n$ such that $\mathcal{A}_n \models T$, for $n \in N$, as follows. Let $\mathcal{A}_0 = \mathcal{A}$. Suppose $\mathcal{A}_n$ has been defined. Let $\mathcal{A}_{n+1}$ be a model related to $\mathcal{A}_n$ the way $\mathcal{B}'$ is related to $\mathcal{B}$ in (*). Finally, let

$$\mathcal{A}^* = \bigcup\{\mathcal{A}_n \colon n \in N\}.$$

Then $\mathcal{A}^* \models T$. Now let φ be any existential sentence of $\ell_T(A^*)$ and suppose there is a model $\mathcal{B}$ of T such that $\mathcal{A}^* \subseteq \mathcal{B}$ and $\mathcal{B} \models \varphi$. φ is a sentence of $\ell_T(A_n)$ for some n. $\mathcal{A}_n \subseteq \mathcal{B}$. It follows that $\mathcal{A}_{n+1} \models \varphi$ and so $\mathcal{A}^* \models \varphi$. Thus, $\mathcal{A}^*$ is as desired. ■

Theorem 17 now follows from Lemmas 10, 11. ■

ACF(p) is $\forall\exists$ and κ-categorical for every $\kappa > \aleph_0$ and has no finite models. It follows, by Theorem 17, that ACF(p) is model-complete. Moreover, any extension of a field of characteristic p is of characteristic p. Thus, we have the following:

Theorem 18. ACF is model-complete.

The fact that a theory T is model-complete is sometimes interesting not only because it may be used to prove that T is complete. One reason is the following:

Theorem 19. If T is model-complete iff for every formula $\varphi(x_1,...,x_n)$, there is an existential formula $\chi(x_1,...,x_n)$ such that

$$T \vdash \varphi(x_1,...,x_n) \leftrightarrow \chi(x_1,...,x_n).$$

This follows from our next lemma (compare Corollary 6).

Lemma 12. Let $\varphi(x_1,...,x_n)$ be any formula of ℓ_T. Suppose for any models $\mathcal{A}$, $\mathcal{B}$ of T and any $a_1,...,a_n \in A$, if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \vDash \varphi(a_1,...,a_n)$, then $\mathcal{B} \vDash \varphi(a_1,...,a_n)$. There is then an existential formula $\psi(x_1,...,x_n)$ such that

$$T \vdash \varphi(x_1,...,x_n) \leftrightarrow \psi(x_1,...,x_n).$$

In what follows we sometimes write $\underline{x}$, $\underline{y}$, ... for finite sequences $x_1,...,x_k$, $y_1,...,y_m$, ... of variables, $\underline{c}$, $\underline{d}$, ... for finite sequences $c_1,...,c_k$, $d_1,...,d_m$, ... of constants, and $\underline{a}$, $\underline{b}$, ... for finite sequences $a_1,...,a_k$, $b_1,...,b_m$, ... of elements of models. The length of these sequences will, in case it matters, be determined by the context. Thus, for example, $\exists\underline{y}\varphi(\underline{c},\underline{y})$ is (or may be) short for $\exists y_1...y_n\varphi(c_1,...,c_k,y_1,...,y_n)$.

Proof of Lemma 12. Let

$$\Phi = \{\chi(\underline{x}) \colon \chi \text{ is existential \& } T \vdash \chi(\underline{x}) \rightarrow \varphi(\underline{x})\}.$$

Let $c_1,...,c_n$ be new individual constants and let

$$\Psi = T \cup \{\varphi(\underline{c})\} \cup \{\neg\chi(\underline{c}) \colon \chi \in \Phi\}.$$

If Ψ has no model, we are done. Thus, suppose it does and let $(\mathcal{A}, \underline{a}) \vDash \Psi$. Then

$$\Theta = T \cup \{\neg\varphi(\underline{c})\} \cup D((\mathcal{A}, \underline{a}))$$

has a model. Indeed, if not, there are $d_1,...,d_m \notin \ell_T \cup \{c_1,...,c_n\}$ and a quantifier-free formula $\psi(\underline{x},\underline{y})$ of ℓ_T such that $\chi(\underline{c},\underline{d})$ is a conjunction of members of $D((\mathcal{A}, \underline{a}))$ and $T \vdash \chi(\underline{c},\underline{d}) \rightarrow \varphi(\underline{c})$ and so

$$T \vdash \exists\underline{y}\chi(\underline{x},\underline{y}) \rightarrow \varphi(\underline{x}).$$

It follows that $\exists\underline{y}\chi(\underline{x},\underline{y}) \in \Phi$ and so

$$(\mathcal{A}, \underline{a}) \vDash \neg\exists\underline{y}\chi(\underline{c},\underline{y}).$$

But also $(\mathcal{A}, \underline{a}) \vDash \exists\underline{y}\chi(\underline{c},\underline{y})$, which is a contradiction.

Thus, given that $(\mathcal{A}, \underline{a}) \vDash \Psi$, Θ has a model, which we may assume to be of the form $(\mathcal{B}, \underline{a})$, where $\mathcal{A} \subseteq \mathcal{B}$. But then $\varphi(\underline{c})$ is true in $(\mathcal{A}, \underline{a})$ but not in $(\mathcal{B}, \underline{a})$, contrary to assumption. It follows that Ψ has no model and so the proof is complete. ■

Proof of Theorem 19. "⇐". Suppose $\mathcal{A}$, $\mathcal{B}$ are models of T and $\mathcal{A} \subseteq \mathcal{B}$. Let $\varphi(\underline{x})$ be any formula and let $a_1,...,a_n \in A$ be such that $\mathcal{A} \vDash \varphi(\underline{a})$. Let $\chi(\underline{x})$ be an existential formula such that $T \vdash \varphi \leftrightarrow \chi$. Then $\mathcal{A} \vDash \chi(\underline{a})$. Since χ is existential, it follows that $\mathcal{B} \vDash \chi(\underline{a})$ and so that $\mathcal{B} \vDash \varphi(\underline{a})$. This shows that $\mathcal{A} \preccurlyeq \mathcal{B}$, as desired.

"⇒". This follows from Lemma 12. ■

By Theorem 19, given that T is model-complete, we know a great deal about what sets and relations are definable in models of T. Thus, even if, as in the case of ACF(p), we already know that T is complete it may still be of interest to show that T is model-complete.

In view of Theorems, 14, 19, every formula of RCOF is equivalent in RCOF to an existential formula. In fact, this can be improved: RCOF admits quantifier elimination, i.e., every formula of RCOF is equivalent to a quantifier-free formula. Similarly, ACF admits quantifier elimination.

We conclude this § by showing that a model-complete theory T is uniquely determined by the set of universal sentences provable in T.

Let us say that T and T′ are *u-equivalent* if they prove the same universal sentences.

Theorem 20. If T and T′ are model-complete and u-equivalent, then T and T′ are equivalent.

The proof of the following Lemma is almost the same as that of Theorem 5.

Lemma 13. Suppose $\ell_T = \ell_{T'}$ and for every universal sentence φ, if $T \vdash \varphi$, then $T' \vdash \varphi$. Then for every $\mathcal{A} \models T'$, there is a $\mathcal{B} \models T$ such that $\mathcal{A} \subseteq \mathcal{B}$.

Proof of Theorem 20. By symmetry, it is sufficient to show that every model of T is a model of T′. Suppose $\mathcal{A} \models T$. Then, by Lemma 13, there are models $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$ such that $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_{2n} \models T$, and $\mathcal{A}_{2n+1} \models T'$. Let

$$\mathcal{B} = \bigcup\{\mathcal{A}_{2n}: n \in N\} = \bigcup\{\mathcal{A}_{2n+1}: n \in N\}.$$

Then, since T and T′ are model-complete, $\mathcal{A}_0 \preccurlyeq \mathcal{B} \models T'$ and so $\mathcal{A} \models T'$, as desired. ■

T is *u-complete* if for every universal sentence φ of ℓ_T, either $T \vdash \varphi$ or $T \vdash \neg\varphi$. T is u-complete if every existential sentence true in some model of T is true in a model (not necessarily of T) which is embeddable in every model of T.

Corollary 10. (i) If T is model-complete and u-complete, then T is complete.

(ii) If T is u-complete, then T has (up to equivalence) at most one (consistent) model-complete extension.

§7. The Fraïssé-Ehrenfeucht criterion. Of course, there are many complete theories which are neither κ-categorical nor model-complete (see e.g. Examples 3, 5 in §13). In such cases one can sometimes use the following (quite elementary) method.

In this §*we now restrict ourselves to models* $\mathcal{A}$ *and theories* T *such that* $\ell_{\mathcal{A}}$ *and*

ℓ_T *are finite.*

For any formula φ the *quantifier depth* of φ, $qd(\varphi)$, is defined as follows:

$qd(\varphi) = 0$ if φ is atomic, $qd(\neg\varphi) = qd(\varphi)$, $qd(\varphi \wedge \psi) = qd(\varphi \vee \psi) = qd(\varphi \rightarrow \psi) = \max\{qd(\varphi),qd(\psi)\}$, $qd(\exists x\psi) = qd(\forall x\psi) = qd(\psi) + 1$.

For any two models $\mathcal{A}$, $\mathcal{B}$ we write $\mathcal{A} \equiv_n \mathcal{B}$ to mean that for every primitive sentence φ, if $qd(\varphi) \leq n$, then $\mathcal{A} \vDash \varphi$ iff $\mathcal{B} \vDash \varphi$. Thus, every sentence being equivalent to a primitive sentence (Chapter 1, §4), $\mathcal{A} \equiv \mathcal{B}$ iff for every n, $\mathcal{A} \equiv_n \mathcal{B}$.

We use **s**, **t** to denote finite sequences. $|\mathbf{s}|$ is the length of **s**. Suppse $\mathbf{s} = \langle a_1,...,a_n\rangle$. Then $\mathbf{s}a = \langle a_1,...,a_n,a\rangle$ and $a\mathbf{s} = \langle a,a_1,...,a_n\rangle$. $\langle\rangle$ is the empty sequence. $(\mathcal{A}, \mathbf{s}) = (\mathcal{A}, a_1,...,a_n)$. If $\varphi(x_1,...,x_n)$ is a formula with no free variables except $x_1,...,x_n$, then $\mathcal{A} \vDash \varphi(\mathbf{s})$ iff $\mathcal{A} \vDash \varphi(a_1,...,a_n)$.

Let $\mathcal{A}$, $\mathcal{B}$ be any models (for ℓ). The relation I is an *n-isomorphism of* $\mathcal{A}$ *onto* $\mathcal{B}$, I: $\mathcal{A} \cong_n \mathcal{B}$, if $I \subseteq \bigcup\{A^k x B^k: k \leq n\}$, $\langle\rangle I \langle\rangle$, and

if $|\mathbf{s}| = |\mathbf{t}| < n$ and $\mathbf{s}I\mathbf{t}$, then for every $a \in A$ ($b \in B$), there is a $b \in B$ ($a \in A$) such that $\mathbf{s}aI\mathbf{t}b$, and

if $\mathbf{s}I\mathbf{t}$, then $(\mathcal{A}, \mathbf{s}) \equiv_0 (\mathcal{B}, \mathbf{t})$.

We write $\mathcal{A} \cong_n \mathcal{B}$ to mean that there is an n-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

By an *(n,n)-condition* we understand a primitive atomic formula of ℓ in the variables $x_1,...,x_n$. For every formula φ, let $\varphi^i := \varphi$ if $i = 0$ and $:= \neg\varphi$ if $i = 1$. If $\varphi_0,...,\varphi_m$ are all (n,k)-conditions, then for all $i_0,...,i_m$, ${\varphi_0}^{i_0} \wedge ... \wedge {\varphi_m}^{i_m}$ is a *complete (n,k)-condition*. Finally, if φ is a complete (n,k+1)-condition, then $\exists x_{k+1}\varphi$ is an *(n,k)-condition*. Thus, (complete) (n,k)-conditions are primitive. The free variables of (n,k)-conditions are $x_1,...,x_k$ and (n,k)-conditions are formulas of quantifier depth $n-k$.

Lemma 14. For all n, $\mathcal{A} \equiv_n \mathcal{B}$ iff $\mathcal{A} \cong_n \mathcal{B}$.

Proof. "$\Leftarrow$". Induction. This is true for $n = 0$. Suppose it holds for n. Let I: $\mathcal{A} \cong_{n+1} \mathcal{B}$. Suppose φ is primitive and $qd(\varphi) = n+1$. Then φ is equivalent to a truth-functional combination of sentences of the form $\exists x\psi(x)$, where $qd(\psi) = n$, and sentences of quantifier depth $\leq n$, all of which are primitive. Thus, it is sufficient to consider sentences of the former kind. Suppose $\mathcal{A} \vDash \exists x\psi(x)$. Let a be such that $\mathcal{A} \vDash \psi(a)$. There is a b such that $\langle a\rangle I \langle b\rangle$. Let I′ be defined by: $\mathbf{s}I'\mathbf{t}$ iff $a\mathbf{s}Ib\mathbf{t}$. Then I′: $(\mathcal{A}, a) \cong_n (\mathcal{B}, b)$. By the inductive assumption, it follows that $\mathcal{B} \vDash \psi(b)$ and so $\mathcal{B} \vDash \exists x\psi(x)$. Similarly, if $\mathcal{B} \vDash \exists x\psi(x)$, then $\mathcal{A} \vDash \exists x\psi(x)$.

"$\Rightarrow$". Let I be defined by:

$\mathbf{s}I\mathbf{t}$ iff there is a $k \leq n$ such that $|\mathbf{s}| = |\mathbf{t}| = k$ and for every (n,k)-condiditon φ, $\mathcal{A} \vDash \varphi(\mathbf{s})$ iff $\mathcal{B} \vDash \varphi(\mathbf{t})$.

Then $\langle\rangle I\langle\rangle$, since $\mathcal{A} \equiv_n \mathcal{B}$. Suppose **s**I**t**, where $|\mathbf{s}| = |\mathbf{t}| = k < n$, and $a \in A$. Let ψ be the complete (n,k+1)-condition such that $\mathcal{A} \vDash \psi(\mathbf{s}a)$. (This is where we need the assumption that $\ell_{\mathcal{A}}$ is finite.) Let $\theta := \exists x_{k+1}\psi$. Then θ is an (n,k)-condition and $\mathcal{A} \vDash \theta(\mathbf{s})$. By assumption, it follows that $\mathcal{B} \vDash \theta(\mathbf{t})$. Let $b \in B$ be such that $\mathcal{B} \vDash \psi(\mathbf{t}b)$. Then **s**aI**t**b. Similarly, for every $b \in B$, there is an $a \in A$ such that **s**aI**t**b.

Finally, it is clear that if **s**I**t**, then $(\mathcal{A}, \mathbf{s}) \equiv_0 (\mathcal{B}, \mathbf{t})$. Thus, I: $\mathcal{A} \cong_n \mathcal{B}$. ■

From Lemma 14 it follows at once that:

Theorem 21 (Fraïssé, Ehrenfeucht). For all models $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \equiv \mathcal{B}$ iff for all n, $\mathcal{A} \cong_n \mathcal{B}$.

A theory T is complete iff any two models of T are elementarily equivalent. Thus:

Corollary 11. T is complete iff for all models $\mathcal{A}$, $\mathcal{B}$ of T and all n, $\mathcal{A} \cong_n \mathcal{B}$.

Corollary 12. For all models $\mathcal{A}$, $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} \preccurlyeq \mathcal{B}$ iff for every finite sequence $a_1,\ldots,a_k$ of members of A and every n, $(\mathcal{A}, a_1,\ldots,a_k) \cong_n (\mathcal{B}, a_1,\ldots,a_k)$.

Corollary 13. T is model-complete iff for all models $\mathcal{A}$, $\mathcal{B}$ of T such that $\mathcal{A} \subseteq \mathcal{B}$, all sequences $a_1,\ldots,a_k$ of members of A and all n, $(\mathcal{A}, a_1,\ldots,a_k) \cong_n (\mathcal{B}, a_1,\ldots,a_k)$.

Applications of Corollaries 11, 13 can be found in §13, Examples 2, 3, 5, 6, 7.

Theorem 21 and Corollary 11 can be applied to arbitrary models and theories in view of the obvious fact that for any models $\mathcal{C}$, $\mathcal{D}$, $\mathcal{C} \equiv \mathcal{D}$ iff for every finite language $\ell \subseteq \ell_{\mathcal{C}}$, $\mathcal{C}|\ell \equiv \mathcal{D}|\ell$; and similarly for Corollaries 12 and 13.

Theorem 21 can also be applied to problems of definability in the following way. Let $\mathcal{A}$ be a model for ℓ and let $R \subseteq A^n$. Then R is *definable in* $\mathcal{A}$ if there is a formula $\varphi(x_1,\ldots,x_n)$ of ℓ such that $R = \{\langle a_1,\ldots,a_n\rangle : \mathcal{A} \vDash \varphi(a_1,\ldots,a_n)\}$.

Corollary 14. Let $\mathcal{A}$ be a model for ℓ and let $R \subseteq A^n$. Then R is definable in $\mathcal{A}$ iff there is a k such that for all $a_1,\ldots,a_n,a_1',\ldots,a_n' \in A$, if $(\mathcal{A}, a_1,\ldots,a_n) \cong_k (\mathcal{A}, a_1',\ldots,a_n')$, then $\langle a_1,\ldots,a_n\rangle \in R$ iff $\langle a_1',\ldots,a_n'\rangle \in R$.

I is an ω-*isomorphism of* $\mathcal{A}$ *onto* $\mathcal{B}$, I: $\mathcal{A} \cong_\omega \mathcal{B}$, if for every n, the relation $\{\langle \mathbf{s},\mathbf{t}\rangle : \mathbf{s}I\mathbf{t}\ \&\ |\mathbf{s}| = |\mathbf{t}| \leq n\}$ is an n-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. $\mathcal{A} \cong_\omega \mathcal{B}$ means that there is an ω-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

The following lemma is occasionally useful.

Lemma 15. If $\mathcal{A}$ and $\mathcal{B}$ are countable and $\mathcal{A} \cong_\omega \mathcal{B}$, then $\mathcal{A} \cong \mathcal{B}$.

Proof. Let I: $\mathcal{A} \cong_{\omega}\mathcal{B}$. Let a_0, a_1, a_2,... be an enumeration of A and let b_0, b_1, b_2,... be an enumeration of B (with repetitions if the set is finite). It is completely straightforward to define c_n and d_n in such a way that for every n, $c_{2n} = a_n$, $d_{2n+1} = b_n$, and $\langle c_0,\ldots,c_n\rangle I \langle d_0,\ldots,d_n\rangle$. Let f: A → B be such that $f(c_n) = d_n$. Then f: $\mathcal{A} \cong \mathcal{B}$. ■

As an example of an application of Theorem 21 we now prove that direct products preserve elementary equivalence.

Let $\mathcal{A}$, $\mathcal{B}$ be models for the same language ℓ. The *direct product*, $\mathcal{A}\times\mathcal{B}$ is then the model $\mathcal{C}$ for ℓ defined as follows:

$C = A \times B$,

$P^{\mathcal{C}} = \{\langle\langle a_1,b_1\rangle,\ldots,\langle a_n,b_n\rangle\rangle\colon \langle a_1,\ldots,a_n\rangle \in P^{\mathcal{A}}\ \&\ \langle b_1,\ldots,b_n\rangle \in P^{\mathcal{B}}\}$,

$f^{\mathcal{C}}(\langle a_1,b_1\rangle,\ldots,\langle a_n,b_n\rangle) = \langle f^{\mathcal{A}}(a_1,\ldots,a_n), f^{\mathcal{B}}(b_1,\ldots,b_n)\rangle$,

$c^{\mathcal{C}} = \langle c^{\mathcal{A}},c^{\mathcal{B}}\rangle$.

Direct products with arbitrarily many "factors" can be defined in a similar way.

Proposition 7. If $\mathcal{A}_0 \equiv \mathcal{B}_0$ and $\mathcal{A}_1 \equiv \mathcal{B}_1$, then $\mathcal{A}_0\times\mathcal{A}_1 \equiv \mathcal{B}_0\times\mathcal{B}_1$.

Proof. Let I^i: $\mathcal{A}_i \cong_n \mathcal{B}_i$, i = 0, 1. For $\mathbf{s} = \langle a_0,\ldots,a_k\rangle$ and $\mathbf{t} = \langle b_0,\ldots,b_k\rangle$ let $\mathbf{s}\times\mathbf{t} = \langle\langle a_0,b_0\rangle,\ldots,\langle a_k,b_k\rangle\rangle$. Let I be defined by:

$\mathbf{s}_0\times\mathbf{s}_1 I \mathbf{t}_0\times\mathbf{t}_1$ iff $\mathbf{s}_0 I^0 \mathbf{t}_0$ and $\mathbf{s}_1 I^1 \mathbf{t}_1$.

Then I: $\mathcal{A}_0\times\mathcal{A}_1 \cong_n \mathcal{B}_0\times\mathcal{B}_1$. ■

This has a straightforward extension to direct products with arbitrarily many "factors". Similar results can be proved in almost the same way for many other "sums" and "products" of models.

§8. Omitting types and $\aleph_0$-categoricity. We are interested in the expressive power of L_1. We know that, because of the Cardinality Theorem (Corollary 3), no infinite model can be characterized (up to isomorphism) in L_1. But we may still ask, for any κ, which models of cardinality κ can be so characterized among models of cardinality κ. In this section we answer this question in the simplest case $\kappa = \aleph_0$ (Corollary 18). To do this, we need a new way of constructing models (Theorems 22, 22′, below).

Let ℓ be any language. A *type* in ℓ in the variables $x_1,\ldots,x_n$ is a set $\Phi(\underline{x})$ of formulas of ℓ with the free variables $x_1,\ldots,x_n$. $\Phi(\underline{x})$ is a *type over* T if $\Phi(\underline{x})$ is consistent with T in the sense that $T \cup \Phi(\underline{c})$, where c is a sequence of constants not in T or $\Phi(\underline{x})$, is consistent. A type $\Phi(\underline{x})$ is (*explicitly*) *complete* if for every formula $\varphi(\underline{x}) \in \Phi(\underline{x})$ or $\neg\varphi(\underline{x}) \in \Phi(\underline{x})$. We write $\mathcal{A} \models \Phi(a_1,\ldots,a_n)$, where $a_1,\ldots,a_n \in A$, to mean that $\mathcal{A} \models \varphi(a_1,\ldots,a_n)$ for every $\varphi(\underline{x}) \in \Phi(\underline{x})$. A model $\mathcal{A}$ *realizes* $\Phi(\underline{x})$ if there are $a_1,\ldots,a_n \in A$ such that $\mathcal{A} \models \Phi(a_1,\ldots,a_n)$. $\mathcal{A}$ *omits* $\Phi(\underline{x})$ if $\mathcal{A}$ does not realize $\Phi(\underline{x})$.

Examples. 1. Let $\mathcal{N} = (N, +, \cdot, S, 0)$ be the standard model of arithmetic. Let $S^0(0) := 0$ and $S^{n+1}(0) := S(S^n(0))$. Let Pr be the set of prime numbers. For each $p \in Pr$, let $\varphi_p(x) := \exists y(x = S^p(0) \cdot y)$. (Thus, $\varphi_p(x)$ says that x is divisible by p.) For $X \subseteq Pr$, let

$$\Phi_X(x) = \{\varphi_p(x) : p \in X\} \cup \{\neg\varphi_p(x) : p \notin X\}.$$

Then $\Phi_X(x)$ is a type over Th($\mathcal{N}$). It follows that there are $2^{\aleph_0}$ complete types over Th($\mathcal{N}$).

2. The types $\{x = S(0)\}$, $n \in N$, and $\{x \neq S^n(0) : n \in N\}$ can be extended in only one way to complete types over SF. Thus, there are denumerably many complete types $\Phi(x)$ over SF. For every n, there are denumerably many complete types over SF in the variables $x_1, \ldots, x_n$.

3. There is only one complete type $\Phi(x)$ over DeLO. For every n, there are finitely many complete types over DeLO in the variables $x_1, \ldots, x_n$. ■

Lemma 16. (a) Every type over T can be extended to a complete type over T.

(b) Every type over T is realized in a model of T.

The proof of this is straightforward.

In the rest of this section *ℓ is a countable language*, T *is a theory in* ℓ, *and* $\mathcal{A}$, $\mathcal{B}$ *are models for* ℓ. T *locally omits* $\Phi(\underline{x})$ if for every formula $\psi(\underline{x})$ (of ℓ), if $T \cup \{\exists\underline{x}\psi(\underline{x})\}$ is consistent, then there is a $\varphi(\underline{x}) \in \Phi(\underline{x})$ such that $T \cup \{\exists\underline{x}(\psi(\underline{x}) \wedge \neg\varphi(\underline{x}))\}$ is consistent.

Theorem 22 (Omitting Types Theorem)**.** If T locally omits $\Phi(\underline{x})$, then T has a countable model omitting $\Phi(\underline{x})$.

Proof. We consider only the case where $\underline{x}$ is a single variable x; the general case is similar. Let $C = \{c_n : n \in N\}$ be a set of new individual constants. By Lemma 2.13, it is then sufficient to show that there is a set T^* of sentences of $\ell \cup C$ such that $T \subseteq T^*$ and

(1) T^* is consistent, (explicitly) complete, and witness-complete (w.r.t. C),

(2) for every n, there is a formula $\varphi(x) \in \Phi(x)$ such that $\neg\varphi(c_n) \in T^*$.

Indeed, then the canonical model of T^* omits $\Phi(x)$.

Let $\psi_0(y), \psi_1(y), \psi_2(y), \ldots$ be an enumeration of all formulas of $\ell \cup C$ with the free variable y. We (may) assume that c_k occurs in $\psi_n(y)$ only if $k < n$.

We now define an increasing sequence $T_0, T_1, T_2, \ldots$ of consistent theories such that for every n, c_k occurs in T_n only if $k < n$ and

(3) there is a formula $\varphi(x) \in \Phi(x)$ such that $\neg\varphi(c_n) \in T_{n+1}$,

(4) $\exists y\psi_n(y) \to \psi_n(c_n) \in T_{n+1}$.

Let $T_0 = T$. Suppose T_n has been defined. Let $T_n = T \cup \{\theta_0,\ldots,\theta_m\}$ and let $\theta := \theta_0 \wedge\ldots\wedge \theta_m$, or $\neg\bot$ if $n = 0$.

$$T \cup \{\theta, \exists x(\exists y\psi_n(y) \to \psi_n(x))\}$$

is consistent. Let θ', $\psi_n'(y)$ be obtained from θ, $\psi_n(y)$ by replacing c_k for $k < n$ by a new variable z_k. Then

$$T \cup \{\exists x\exists z_0\ldots z_{n-1}(\theta' \wedge (\exists y\psi_n'(y) \to \psi_n'(x)))\}$$

is consistent. By hypothesis, this implies that there is a formula $\varphi(x) \in \Phi(x)$ such that

$$T_n \cup \{\exists x(\exists z_0\ldots z_{n-1}(\theta' \wedge (\exists y\psi_n'(y) \to \psi_n'(x))) \wedge \neg\varphi(x))\}$$

is consistent. Let

$$T_{n+1} = T_n \cup \{\theta, \exists y\psi_n(y) \to \psi_n(c_n), \neg\varphi(c_n)\}.$$

Then T_{n+1} is consistent and (3) and (4) are satisfied.

Now, let T^* be a complete, consistent extension of $\bigcup\{T_n: n \in N\}$. Then (1) and (2) are satisfied, as desired. ■

Theorem 22 can be extended as follows; the proof is almost the same.

Theorem 22′ (Extended Omitting Types Theorem)**.** If T locally omits $\Phi_n(\underline{x})$, for $n \in N$, then T has a countable model omitting each $\Phi_n(\underline{x})$.

A formula $\varphi(\underline{x})$ is an *atom* of T (not to be confused with an atomic formula) if $\exists\underline{x}\varphi(\underline{x})$ is consistent with T and for every formula $\psi(\underline{x})$ of ℓ, either $T \vdash \varphi(\underline{x}) \to \psi(\underline{x})$ or $T \vdash \varphi(\underline{x}) \to \neg\psi(\underline{x})$. Thus, two atoms of T are either equivalent or incompatible in T. $\Phi(\underline{x})$ is a *principal* type of T if there is a formula $\psi(\underline{x}) \in \Phi(\underline{x})$ such that $T \vdash \psi(\underline{x}) \to \varphi(\underline{x})$ for every $\varphi(\underline{x}) \in \Phi(\underline{x})$. If this holds and $\Phi(\underline{x})$ is complete, then $\psi(\underline{x})$ is an atom. Clearly, if $\varphi(\underline{x})$ is an atom and $\varphi(\underline{x}) \in \Phi(\underline{x})$, then $\Phi(\underline{x})$ is principal. Thus, a complete type is principal iff it contains an atom.

Corollary 15. If $\Phi_n(\underline{x})$, $n \in N$, are complete nonprincipal types over T, then T has a countable model omitting each $\Phi_n(\underline{x})$.

Proof. We show that T locally omits every $\Phi_n(\underline{x})$. Suppose not. There is then a formula $\psi(\underline{x})$ such that $T \cup \{\exists\underline{x}\psi(\underline{x})\}$ is consistent and $T \vdash \psi(\underline{x}) \to \varphi(\underline{x})$ for every $\varphi(\underline{x}) \in \Phi_n(\underline{x})$. But then, $\Phi_n(\underline{x})$ being complete, $\psi(\underline{x})$ is an atom of T. Since $\Phi_n(\underline{x})$ is non-principal, it follows that $\psi(\underline{x}) \notin \Phi_n(\underline{x})$ and so $\neg\,\psi(\underline{x}) \in \Phi_n(\underline{x})$. But then $T \vdash \psi(\underline{x}) \to \neg\psi(\underline{x})$ and so $T \cup \{\exists\underline{x}\psi(\underline{x})\}$ is inconsistent, a contradiction. It follows that T locally omits every $\Phi_n(\underline{x})$. Now use Theorem 22′. ■

Let us say that $\mathcal{A}$ is *atomic* if for any n and any n-tuple $\underline{a}$ of members of A, there is an atom $\varphi(\underline{x})$ of $Th(\mathcal{A})$ such that $\mathcal{A} \models \varphi(\underline{a})$. Thus, $\mathcal{A}$ is atomic iff all complete types realized in $\mathcal{A}$ are principal.

Examples. 4. Suppose T is a complete extension of PA. Let $\varphi(x)$ be such that $T \vdash \exists x\varphi(x)$ and let $\psi(x) :=$

$$\varphi(x) \wedge \forall y(y < x \rightarrow \neg\varphi(y)),$$

where $y < x := \exists z(z \neq 0 \wedge y + z = x))$. Then $\psi(x)$ is an atom of T. If $\mathcal{A} \models T$ and B is the set of members of A satisfying an atom of T in $\mathcal{A}$, then $\mathcal{A} | B \preccurlyeq \mathcal{A}$ (Lemma 1). $\mathcal{A} | B$ is an atomic model of T. In particular, $\mathcal{N}$ is an atomic model of $Th(\mathcal{N})$.

5. The formula $x = x$ is an atom of the theory DeLO. Every model of DeLO is atomic. ■

We are going to need the following:

Lemma 17. Suppose $\mathcal{A} \equiv \mathcal{B}$.

(a) If $\mathcal{A}$ is denumerable and atomic, then $\mathcal{A}$ is elementarily embeddable in $\mathcal{B}$.

(b) If $\mathcal{A}$ and $\mathcal{B}$ are denumerable and atomic, then $\mathcal{A} \cong \mathcal{B}$.

Proof. (a) Let $A = \{a_n: n \in N\}$. We show that there are $b_0, b_1, b_2, \ldots \in B$ such that for all n,

$$(\mathcal{A}, a_0,\ldots,a_{n-1}) \equiv (\mathcal{B}, b_0,\ldots,b_{n-1}).$$

For $n = 0$ this holds by assumption. Suppose it holds for a certain n. By hypothesis, there is an atom $\varphi(x_0,\ldots,x_n)$ of $Th(\mathcal{A})$ such that $\mathcal{A} \models \varphi(a_0,\ldots,a_n)$. It follows that

$$\mathcal{A} \models \exists x_n\varphi(a_0,\ldots,a_{n-1},x_n)$$

and so, by the inductive assumption,

$$\mathcal{B} \models \exists x_n\varphi(b_0,\ldots,b_{n-1},x_n).$$

Let b_n be such that

(1) $\mathcal{B} \models \varphi(b_0,\ldots,b_n)$.

Let $\psi(x_0,\ldots,x_n)$ be any formula of ℓ such that $\mathcal{A} \models \psi(a_0,\ldots,a_n)$. Since φ is an atom, it follows that $\mathcal{A} \models \varphi \rightarrow \psi$ and so, by (1) and since $\mathcal{B} \equiv \mathcal{A}$, $\mathcal{B} \models \psi(b_0,\ldots,b_n)$. Thus, $(\mathcal{A}, a_0,\ldots,a_n) \equiv (\mathcal{B}, b_0,\ldots,b_n)$, as desired.

Now let $f: A \rightarrow B$ be such that $f(a_n) = b_n$. Then f is an elementary embedding of $\mathcal{A}$ in $\mathcal{B}$. ♦

(b) Let $A = \{a_n: n \in N\}$ and $B = \{b_n: n \in N\}$. In much the same way as under (a) we can then show that there are d_n and e_n, $n = 0, 1, 2, \ldots$, such that for all n, $a_n = d_{2n}$, $b_n = e_{2n+1}$, and $(\mathcal{A}, d_0,\ldots,d_n) \equiv (\mathcal{B}, e_0,\ldots,e_n)$. Let f be such that $f(d_n) = e_n$. Then $f: \mathcal{A} \cong \mathcal{B}$. ■

By Lemma 17(a), every countable atomic model $\mathcal{A}$ is *elementarily prime* in the sense that $\mathcal{A}$ is elementarily embeddable in every model $\mathcal{B} \equiv \mathcal{A}$. In fact:

Theorem 23. For every countable model $\mathcal{A}$, $\mathcal{A}$ is atomic iff $\mathcal{A}$ is elementarily prime.

Proof. "If". Suppose $\mathcal{A}$ is not atomic. Let $\underline{a} \in A$ be such that there is no atom $\varphi(\underline{x})$ of $Th(\mathcal{A})$ such that $\mathcal{A} \vDash \varphi(\underline{a})$. Let

$$\Psi(\underline{x}) = \{\psi(\underline{x}) : \mathcal{A} \vDash \psi(\underline{a})\}.$$

Then $\Psi(\underline{x})$ is a complete nonprincipal type; it contains no atom. Hence, by Corollary 15, $Th(\mathcal{A})$ has a model $\mathcal{B}$ omitting $\Psi(\underline{x})$. Clearly $\mathcal{A}$ is not elementarily embeddable in $\mathcal{B}$. And so $\mathcal{A}$ is not elementarily prime. ■

From Lemma 17(b) and Theorem 23 we get:

Corollary 16. Any two elementarily equivalent elementarily prime models for a countable language are isomorphic.

Prime models need not be elementarily prime. In Appendix 5 we given an example of a complete theory which has a prime model but no elementarily prime model.

Lemma 18. The following conditions are equivalent:
(i) For every n, there is a finite set $\Psi_n(\underline{x})$ of formulas such that every formula $\varphi(\underline{x})$ of ℓ is equivalent in T to a member of $\Psi_n(\underline{x})$.
(ii) For every n, there is a finite set $\Theta_n(\underline{x})$ of (incompatible) atoms of T such that

$$T \vdash \bigvee \Theta_n(\underline{x}).$$

(iii) Every complete type over T is principal.
(iv) For every n, there are only finitely many complete types $\Phi(\underline{x})$ over T.

Proof. (i) ⇒ (ii). Any conjunction of some members of $\Psi_n(\underline{x})$ and negations of the remaining members of $\Psi_n(\underline{x})$ consistent with T is an atom of T and the disjunction of these atoms is provable in T.

(ii) ⇒ (i). Suppose (ii) holds. Let $\varphi(\underline{x})$ be any formula. Let $\Theta(\underline{x})$ be the set of atoms $\theta(\underline{x})$ of T such that $T \vdash \theta(\underline{x}) \to \varphi(\underline{x})$. (This set may be empty.) Then $T \vdash \bigvee \Theta(\underline{x}) \to \varphi(\underline{x})$. ($\bigvee \varnothing = \bot$.) Let $\Theta^*(\underline{x})$ be the set of atoms $\theta(\underline{x})$ of T not in $\Theta(\underline{x})$. Then $T \vdash \bigvee \Theta^*(\underline{x}) \to \neg\varphi(\underline{x})$, whence $T \vdash \varphi(\underline{x}) \to \neg\bigvee \Theta^*(\underline{x})$. Since $T \vdash \neg\bigvee \Theta^*(\underline{x}) \to \bigvee \Theta(\underline{x})$, it follows that $T \vdash \varphi(\underline{x}) \leftrightarrow \bigvee \Theta(\underline{x})$. Thus, every formula $\varphi(\underline{x})$ is equivalent to a disjunction of atoms. This implies (i).

(ii) ⇒ (iii), (iv). Since every complete type in the variables $\underline{x}$ contains exactly one member of $\Theta_n(\underline{x})$.

(iii) ⇒ (ii). Suppose (ii) is false. Let

$\Psi(\underline{x}) = \{\neg\psi(\underline{x})\colon \psi(\underline{x})$ atom of T$\}$.

Then $\Psi(\underline{x})$ is a type over T. Let $\Phi(\underline{x})$ be any complete type of T extending $\Psi(\underline{x})$ (see Lemma 16(a)). Obviously, $\Phi(\underline{x})$ is not principal and so (iii) is false.

(iv) ⇒ (i). This is true, since, by Lemma 16(a), any two formulas belonging to the same complete types are equivalent in T. ■

We can now characterize the $\aleph_0$-categorical theories as follows.

Theorem 24. Suppose T is complete and has an infinite model. The following conditions are equivalent:

(i) T is $\aleph_0$-categorical.

(ii) For every n, there is a finite set $\Psi_n(\underline{x})$ of formulas such that every formula $\varphi(\underline{x})$ of $\mathcal{L}$ is equivalent in T to a member of $\Psi_n(\underline{x})$ (Lemma 18(i)).

Proof. (i) ⇒ (ii). Suppose (ii) is false. By Lemma 18, there is then a nonprincipal complete type $\Phi(\underline{x})$ over T. By Lemma 16(b), T has a denumerable model realizing $\Phi(\underline{x})$ and, by Corollary 15, T has a denumerable model omitting $\Phi(\underline{x})$. These models are not isomorphic and so (i) is false.

(ii) ⇒ (i). Suppose (ii) holds. Then (ii) of Lemma 18 is true. But this implies that every model of T is atomic. And so, by Lemma 17(b), (i) is true. ■

Corollary 17. Suppose $\mathcal{A}$ is denumerable andTh($\mathcal{A}$) is $\aleph_0$-categorical.

(a) If $\mathcal{L}' \subseteq \mathcal{L}$, then Th($\mathcal{A} | \mathcal{L}'$) is $\aleph_0$-categorical.

(b) If $a_0,\dots,a_n \in A$, then Th(($\mathcal{A}$, $a_0,\dots,a_n$)) is $\aleph_0$-categorical.

Recall that we wanted to know for which denumerable models $\mathcal{A}$ the theory Th($\mathcal{A}$) is $\aleph_0$-categorical. The following corollary answers this question.

Corollary 18. Suppose $\mathcal{A}$ is denumerable. Then the following conditions are equivalent:

(i) Th($\mathcal{A}$) is $\aleph_0$-categorical.

(ii) For every n, the equivalence relation $\sim_{\mathcal{A},n}$ on A^n defined by

$$\langle a_1,\dots,a_n\rangle \sim_{\mathcal{A},n} \langle b_1,\dots,b_n\rangle \text{ iff } (\mathcal{A}, a_1,\dots,a_n) \cong (\mathcal{A}, b_1,\dots,b_n)$$

has only finitely many equivalence classes.

Proof. (i) ⇒ (ii). Suppose (i) is true. Let $\Theta_n = \{\theta_{i,n}(\underline{x})\colon i \le k_n\}$ be as in condition (ii) of Lemma 18 with T = Th($\mathcal{A}$). Suppose there is an $i \le k_n$ such that $\mathcal{A} \models \theta_{i,n}(a_1,\dots,a_n)$ and $\mathcal{A} \models \theta_{i,n}(b_1,\dots,b_n)$. Then $(\mathcal{A}, a_1,\dots,a_n) \equiv (\mathcal{A}, b_1,\dots,b_n)$ and so, by Corollary 17(b), $\langle a_1,\dots,a_n\rangle \sim_{\mathcal{A},n} \langle b_1,\dots,b_n\rangle$. This implies (ii).

(ii) $\Rightarrow$ (i). Formulas $\varphi(\underline{x})$ satisfied in $\mathcal{A}$ by $\sim_{\mathcal{A},n}$-equivalent n-tuples are equivalent in Th($\mathcal{A}$). Thus, by Theorem 24, (i) follows from (ii). ■

Corollary 18 can be applied, for example, to direct products as follows.

Proposition 8. Suppose $\mathcal{A}$, $\mathcal{B}$ are denumerable and Th($\mathcal{A}$), Th($\mathcal{B}$) are $\aleph_0$-categorical. Then Th($\mathcal{A}\times\mathcal{B}$) is $\aleph_0$-categorical.

Proof. Suppose $\sim_{\mathcal{A},n}$ and $\sim_{\mathcal{B},n}$ have k and m equivalence classes. If $\mathbf{s}\sim_{\mathcal{A},n}\mathbf{s}'$ and $\mathbf{t}\sim_{\mathcal{B},n}\mathbf{t}'$, then $\mathbf{s}\times\mathbf{t}\sim_{\mathcal{A}\times\mathcal{B},n}\mathbf{s}'\times\mathbf{t}'$. Hence $\sim_{\mathcal{A}\times\mathcal{B},n}$ has at most k•m equivalence classes. ■

Finally, it may be observed that Theorem 9 (Robinson's Consistency Theorem) can be proved by applying the Omitting Types Theorem. The idea is as follows. Let ℓ, ℓ_i, Φ, Φ_i be as assumed in Theorem 9. It is sufficient to consider the case that the languages ℓ_i are countable. Let $\psi_0(x_0)$, $\psi_1(x_0,x_1)$, $\psi_2(x_0,x_1,x_2)$, ... be an enumeration of the formulas of ℓ. Let ℓ^i be obtained from ℓ by replacing the predicates P, function symbols f, and individual constants c of ℓ by new predicates P^i, function symbols f^i, and constants c^i. For each formula $\psi_n(x_0,\ldots,x_n)$ let $\psi_n{}^i(x_0,\ldots,x_n)$ be obtained by replacing each nonlogical symbol by the corresponding symbol in ℓ^i. Let Φ^i be obtained in this way from Φ_i. Let $c_n{}^i$ be new individual constants and let

$$\Psi = \Phi^0 \cup \Phi^1 \cup \{\psi_n{}^0(c_0{}^0,\ldots,c_n{}^0) \leftrightarrow \psi_n{}^1(c_0{}^1,\ldots,c_n{}^1)\colon n\in N\}.$$

Then it suffices to show that Ψ has a model omitting $\{x \neq c_k{}^0 \vee x \neq c_m{}^1\colon k, m\in N\}$.

Theorem 7 can be proved in a similar way.

§9. Ultraproducts. We now return to the Compactness Theorem. In §2 this result was obtained as an immediate consequence of (the proof of) Theorem 2.3. It is, however, also natural to ask if, assuming that all finite subsets of a set Φ of sentences has a model, a model of Φ can, somehow, be put together from these models. In this § we show that this is, indeed, the case. To describe how this is done we need some new concepts and results.

Let X be any non-empty set. A set D of subsets of X has the *finite intersection property* if $D \neq \varnothing$ and the intersection of any finite number of elements of D is nonempty. D is a *filter* on X if (i) D has the finite intersection property, (ii) for any $Y, Z\in D$, $Y \cap Z\in D$, and (iii) if $Y\in D$ and $Y \subseteq Z \subseteq X$, then $Z\in D$. Thus, for example, if X is infinite, then the set of subsets Y of X such that $X - Y$ is finite is a filter on X. Also, trivially, {X} is a filter on X. If D is a filter on X, then $X\in D$ and $\varnothing\notin D$.

In what follows we sometimes omit the references to the set X.

Lemma 19. If E has the finite intersection property, then there is a filter F such that $E \subseteq F$.

Proof. Let F be the set of subsets Y of X for which there are $Y_0, Y_1, \ldots, Y_n \in E$ such that $Y_0 \cap Y_1 \cap \ldots \cap Y_n \subseteq Y$. ■

D is a *maximal filter* on X if there is no filter on X which properly includes D. D is an *ultrafilter* on X if for every $Y \subseteq X$, either $Y \in D$ or $X - Y \in D$. For example, if $a \in X$, then $\{Y \subseteq X: a \in Y\}$ is a *principal* (and trivial) ultrafilter on X.

If D is a filter, then for any $Y, Z \subseteq X$, $Y \cap Z \in D$ iff $Y \in D$ and $Z \in D$. Also, if D is an ultrafilter, then $Y \in D$ iff $X - Y \notin D$.

Lemma 20. The following conditions are equivalent.

(i) D is a maximal filter.

(ii) D is an ultrafilter.

Proof. We are only going to need the fact that (i) implies (ii). And so we leave the (very simple) proof of the inverse implication to the reader. Let D be any filter. Suppose (ii) is false, i.e., D is not an ultrafilter. Let $Y \subseteq X$ be such that $Y, X - Y \notin D$. Then either $D \cup \{Y\}$ or $D \cup \{X - Y\}$ has the finite intersection property. For if not, there are a $Z, U \in D$ such that $Z \cap Y = \varnothing$ and $U \cap (X - Y) = \varnothing$. But then $Z \cap U = \varnothing$, contrary to assumption. If $D \cup \{Y\}$ has the finite intersection property, by Lemma 19, there is a filter F such that $D \subseteq F$. Since $F \neq D$, D is not maximal. Similarly, if $D \cup \{X - Y\}$ has the finite intersection property, D is not maximal. Thus, D is not maximal, i.e., (i) is false, as desired. ■

Lemma 21. If E has the finite intersection property, there is an ultrafilter D such that $E \subseteq D$.

Proof. By Lemma 19, there is a filter F on X such that $E \subseteq F$. Consider the set S of filters extending F partially ordered by inclusion. As is easily verified, the union of a chain of filters is again a filter. Thus, by Zorn's Lemma, S has a maximal member D. $E \subseteq D$. Also, D is a maximal filter and so, by Lemma 20, D is an ultrafilter, as desired. ■

Let $\mathcal{A}_i$ for $i \in I$ be models for the common language ℓ and let $\langle \mathcal{A}_i: i \in I \rangle$ be the function f on I such that $f(i) = \mathcal{A}_i$ for $i \in I$. Let D be an ultrafilter on I. The *ultraproduct* $\Pi\langle \mathcal{A}_i: i \in I \rangle / D$ determined by D is then defined as follows. Since the difficulties caused by the presence of function symbols are uninteresting we shall assume that ℓ contains no such symbols. (And, of course, they can always first be eliminated and then reintroduced.) In fact, to further simplify the discussion we shall assume that $\ell = \{P, c\}$, where P is a two-place predicate and c an individual constant. It will be quite clear that the following considerations can be extended to the general case.

Let $\Pi\langle A_i: i \in I\rangle$ be the set of functions on I such that $f(i) \in A_i$ for $i \in I$. On this set we define a relation $\sim_D$ as follows:

$$f \sim_D g \text{ iff } \{i \in I: f(i) = g(i)\} \in D.$$

$\sim_D$ is an equivalence relation. Symmetry and reflexivity are trivial. To see that $\sim_D$ is transitive, suppose $f \sim_D g$ and $g \sim_D h$. Then $\{i \in I: f(i) = g(i)\} \in D$ and $\{i \in I: g(i) = h(i)\} \in D$. Also,

$$\{i \in I: f(i) = g(i)\} \cap \{i \in I: g(i) = h(i)\} \subseteq \{i \in I: f(i) = h(i)\}.$$

Since D is a filter, it follows that $\{i \in I: f(i) = h(i)\} \in D$ and so $f \sim_D h$, as desired.

For f in $\Pi\langle A_i: i \in I\rangle$ let f/D be the $\sim_D$-equivalence class of f. Let $A = \Pi\langle A_i: i \in I\rangle/D$ be the set of such equivalence classes. Let the relation R on A and the member a of A be defines as follows.

$$\langle f/D, g/D\rangle \in R \text{ iff } \{i \in I: \langle f(i), g(i)\rangle \in P^{\mathcal{A}_i}\} \in D,$$

$$a = f_c/D, \text{ where } f_c \text{ is the function on I such that } f_c(i) = c^{\mathcal{A}_i} \text{ for } i \in I.$$

Here R is well-determined, since whether or not $\langle f/D, g/D\rangle \in R$ is independent of the representatives f, g of f/D, g/D, In other words, if

$$\{i \in I: \langle f(i), g(i)\rangle \in P^{\mathcal{A}_i}\} \in D,$$

$f'/D = f/D$, and $g'/D = g/D$, then

$$\{i \in I: \langle f'(i), g'(i)\rangle \in P^{\mathcal{A}_i}\} \in D.$$

This is true since

$$\{i \in I: \langle f(i), g(i)\rangle \in P^{\mathcal{A}_i}\} \cap \{i \in I: f(i) = f'(i)\} \cap \{i \in I: g(i) = g'(i)\} \subseteq \{i \in I: \langle f'(i), g'(i)\rangle \in P^{\mathcal{A}_i}\}.$$

And so if the sets on the left are members of D, so is the set on the right.

Finally, let

$$\Pi\langle \mathcal{A}_i: i \in I\rangle/D = (A, R, a).$$

Note that if D is the principal ultrafilter $\{J \subseteq I: j \in J\}$, where $j \in I$, then the ultraproduct is isomorphic to $\mathcal{A}_j$. Thus, finite index sets I are not interesting, since every ultrafilter on a finite set is principal.

The principal reason why the ultraproduct is an interesting construction is the following:

Theorem 25 (Łos'). Let $\mathcal{A}_i$ for $i \in I$ be any models for ℓ and D any ultrafilter on I. Then for every sentence φ of ℓ,

$$\Pi\langle \mathcal{A}_i: i \in I\rangle/D \models \varphi \text{ iff } \{i \in I: \mathcal{A}_i \models \varphi\} \in D.$$

More generally, for all formulas $\varphi(x_1,\ldots,x_n)$ of ℓ and all $f_1,\ldots,f_n \in \Pi\langle A_i: i \in I\rangle$,

$$\Pi\langle \mathcal{A}_i: i \in I\rangle/D \models \varphi(f_1/D,\ldots,f_n/D) \text{ iff } \{i \in I: \mathcal{A}_i \models \varphi(f_1(i),\ldots,f_n(i))\} \in D.$$

Proof. By induction on the length of $\varphi(x_1,\ldots,x_n)$. For simplicity we assume that the logical constants of $\varphi(x_1,\ldots,x_n)$ are $\neg$, $\wedge$, $\exists$. Let $\mathfrak{A} = \Pi\langle\mathfrak{A}_i\colon i\in I\rangle/D$.

For atomic formulas of the form Px_1x_2 or $x_1 = x_2$ the statement holds by definition. For atomic formulas of the form $x = c$ we have

$$\mathfrak{A} \vDash f/D = c \text{ iff } f/D = f_c/D \text{ iff } \{i\in I\colon f(i) = f_c(i)\}\in D \text{ iff } \{i\in I\colon \mathfrak{A}_i \vDash f(i) = c\}\in D.$$

The cases where the atomic formula is $c = x$ or Pxc or Pcx are similar.

Inductive step. Let $\underline{f(i)}$ be $f_1(i),\ldots,f_n(i)$ and let $\underline{f/D}$ be $f_1/D,\ldots,f_n/D$. First suppose $\varphi(\underline{x}) := \neg\psi(\underline{x})$ and the statement holds for ψ. Then

$\mathfrak{A} \vDash \varphi(\underline{f/D})$ iff

$\mathfrak{A} \nvDash \psi(\underline{f/D})$ iff

$\{i\in I\colon \mathfrak{A}_i \vDash \psi(\underline{f(i)}\} \notin D$ iff

$\{i\in I\colon \mathfrak{A}_i \vDash \varphi(\underline{f(i)}\} \in D$,

where the last "only if" holds because D is ultra.

Next, suppose $\varphi(\underline{x}) := \psi(\underline{x}) \wedge \theta(\underline{x})$. Then

$\mathfrak{A} \vDash \varphi(\underline{f/D})$ iff

$\mathfrak{A} \vDash \psi(\underline{f/D})$ and $\mathfrak{A} \vDash \theta(\underline{f/D})$ iff

$\{i\in I\colon \mathfrak{A}_i \vDash \psi(\underline{f(i)})\}\in D$ and $\{i\in I\colon \mathfrak{A}_i \vDash \theta(\underline{f(i)})\}\in D$ iff

$\{i\in I\colon \mathfrak{A}_i \vDash \psi(\underline{f(i)})\} \cap \{i\in I\colon \mathfrak{A}_i \vDash \theta(\underline{f(i)})\}\in D$ iff

$\{i\in I\colon \mathfrak{A}_i \vDash \varphi(\underline{f(i)})\}\in D$.

Finally, suppose $\varphi(\underline{x}) := \exists y\chi(\underline{x},y)$. Suppose

(1) $\mathfrak{A} \vDash \varphi(\underline{f/D})$.

There is then a function $g\in\Pi\langle A_i\colon i\in I\rangle$ such that

(2) $\mathfrak{A} \vDash \chi(\underline{f/D},g/D)$.

By assumption, this implies that

(3) $\{i\in I\colon \mathfrak{A}_i \vDash \chi(\underline{f(i)},g(i))\}\in D$.

But also

$$\{i\in I\colon \mathfrak{A}_i \vDash \chi(\underline{f(i)},g(i))\} \subseteq \{i\in I\colon \mathfrak{A}_i \vDash \varphi(\underline{f(i)})\}.$$

It follows that

(4) $\{i\in I\colon \mathfrak{A}_i \vDash \varphi(\underline{f(i)})\}\in D$.

Next, suppose (4) holds. For every i such that $\mathfrak{A}_i \vDash \varphi(\underline{f(i)})$, there is an $a_i\in A_i$ such that $\mathfrak{A}_i \vDash \chi(\underline{f(i)},a_i\}$. Let $g\in\Pi\langle A_i\colon i\in I\rangle$ be such that $g(i) = a_i$ whenever $\mathfrak{A}_i \vDash \varphi(\underline{f(i)})$. Then (3) holds, whence, by hypothesis, (2) follows. And so (1) is true, as desired. ■

If all the models $\mathfrak{A}_i$ are the same, $\mathfrak{A}_i = \mathfrak{A}$. the ultraproduct is written $\mathfrak{A}^I/D$ and is called an *ultrapower* of $\mathfrak{A}$.

Theorem 25 has the following immediate:

Corollary 19. If D is an ultrafilter on I, then $\mathfrak{A}^I/D \equiv \mathfrak{A}$.

In fact, we can say at bit more. For $a \in A$, let d_a be the function on I such that $d_a(i) = a$ for $i \in I$. Let d be the function on A such that $d(a) = d_a/D$ for $a \in A$.

Corollary 20. If D is an ultrafilter on I, then d is an elementary embedding of $\mathcal{A}$ in $\mathcal{A}^I/D$.

Proof. Let $\mathcal{A}_i = \mathcal{A}$ for $i \in I$. For every formula $\varphi(x_1,...,x_n)$ and all $a_1,...,a_n \in A$,

$$\mathcal{A} \models \varphi(a_1,...,a_n) \text{ iff } \{i \in I: \mathcal{A}_i \models \varphi(d_{a_1}(i),...,d_{a_n}(i))\} \in D \text{ iff } \mathcal{A}^I/D \models \varphi(d(a_1),... d(a_n)). \blacksquare$$

In forming an ultrapower of a model $\mathcal{A}$ we can take I to be A. Then if D is a non-principal ultrafilter on A, then d maps $\mathcal{A}$ into a proper submodel of $\mathcal{A}^A/D$. For let i_A be the identity function on A. Then for every $a \in A$, $\{b \in A: i_A(b) = d_a(b)\} = \{a\} \notin D$ and so $i_A/D \neq d(a)$.

A further immediate corollary to Theorem 25 is as follows:

Corollary 21. (a) If $K \in EC_\Delta$, then K is closed under ultraproducts and K^c is closed under ultrapowers.

(b) If $K \in EC$, then K and K^c are closed under ultraproducts.

Proof. (a) Suppose $K \in EC_\Delta$, $K = Mod(\Phi)$, $\mathcal{A}_i \in K$ for $i \in I$ and D is an ultrafilter on I. If $\varphi \in \Phi$, then $\{i \in I: \mathcal{A}_i \models \varphi\} = I \in D$ and so $\Pi\langle \mathcal{A}_i: i \in I\rangle/D \models \varphi$. Thus, $\Pi\langle \mathcal{A}_i: i \in I\rangle/D \in K$. That K^c is closed under ultrapowers follows from Corollary 19.

(b) follows from (a). ■

It may be observed that if $\mathcal{A}_i$, for $i \in I$, are models for ℓ, $\mathcal{B}_i$, for $i \in I$, are models for ℓ', and each $\mathcal{B}_i$ is an expansion of $\mathcal{A}_i$, then $\Pi\langle \mathcal{B}_i: i \in I\rangle/D$ is an expansion of $\Pi\langle \mathcal{A}_i: i \in I\rangle/D$. Thus, from Corollary 21(a) it follows that if $K \in PC_\Delta$, then K is closed under ultraproducts.

We can now prove the promised version of the Compactness Theorem.

Theorem 26. Suppose every finite subset of Φ has a model. Let I be the set of finite subsets of Φ. For every $i \in I$, let $\mathcal{A}_i$ be a model of i. There is then an ultrafilter D on I such that $\Pi\langle \mathcal{A}_i: i \in I\rangle/D$ is a model of Φ.

Proof. For every sentence $\varphi \in \Phi$, let $J(\varphi) = \{i \in I: \mathcal{A}_i \models \varphi\}$. Let $E = \{J(\varphi): \varphi \in \Phi\}$. Then E has the finite intersection property. For suppose $J(\varphi_1),...,J(\varphi_n) \in E$. Then

$$\{\varphi_1,...,\varphi_n\} \in J(\varphi_1) \cap ... \cap J(\varphi_n).$$

By Lemma 21, there is an ultrafilter D on I such that $E \subseteq D$. For every $\varphi \in \Phi$, $\{i \in I: \mathcal{A}_i \models \varphi\} = J(\varphi) \in D$. Thus, by Theorem 25, $\Pi\langle \mathcal{A}_i: i \in I\rangle/D \models \Phi$, as desired. ■

Corollary 22. If $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A}$ is elementarily embeddable in an ultrapower of $\mathfrak{B}$.

Proof. Every finite subset of Th($\mathfrak{B}$) ∪ ED($\mathfrak{A}$) has a model which is an expansion of $\mathfrak{B}$. By Theorem 26, there is a model of Th($\mathfrak{B}$) ∪ ED($\mathfrak{A}$) which is an expansion of an ultrapower $\mathfrak{B}^I/D$ of $\mathfrak{B}$. By Lemma 3(b), $\mathfrak{A}$ is elementarily embeddable in $\mathfrak{B}^I/D$. ■

We can now also improve Corollary 21 as follows.

Theorem 27. Let K be any class of models.

(a) $K \in EC_\Delta$ iff K is closed under ultraproducts and ≡.

(b) $K \in EC$ iff K and K^c are closed under ultraproducts and ≡.

Proof. (a) "Only if" is now clear. To prove "if", it suffices to show that K = Mod(Th(K)). Clearly K ⊆ Mod(Th(K)). To prove the inverse inclusion, let $\mathfrak{A}$ be any model of Th(K). Let I be the set of finite subsets of Th($\mathfrak{A}$). For every $i \in I$, there is a model $\mathfrak{A}_i$ of i such that $\mathfrak{A}_i \in K$. Indeed, if not and $i = \{\varphi_1,\ldots, \varphi_n\}$, then $\neg(\varphi_1 \wedge\ldots\wedge \varphi_n)$ is true in $\mathfrak{A}$ and $\varphi_1 \wedge\ldots\wedge \varphi_n$ is a member of Th(K), which is a contradiction. By Theorem 26, there is an ultrafilter on I such that $\mathfrak{B} = \Pi\langle\mathfrak{A}_i\colon i \in I\rangle/D$ is a model of Th($\mathfrak{A}$); in other words, $\mathfrak{B} \equiv \mathfrak{A}$. But also $\mathfrak{B} \in K$ and so $\mathfrak{A} \in K$, as desired.

(b) This follows from (a) and Proposition 4(b). ■

There is an extensive theory of ultraproducts but the results generally fall outside the scope of this book. However, one result must be mentioned: For any two models $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic ultrapowers, i.e., there are I, J and ultrafilters D, E on I, J, respectively, such that $\mathfrak{A}^I/D \cong \mathfrak{B}^J/E$. (Of course, "if" follows from Corollary 19.) From this it follows that K is closed under ≡ iff K and K^c are closed under ultrapowers and isomorphisms. And this together with Theorem 27 implies that (a) $K \in EC_\Delta$ iff K is closed under ultraproducts and isomorphisms and K^c is closed under ultrapowers, and (b) $K \in EC$ iff K and K^c are closed under ultraproducts and isomorphisms.

§10. Löwenheim-Skolem theorems for two cardinals. Let ℓ be an arbitrary *countable* language and $U \notin \ell$ a one-place predicate. In this § we assume that T is a theory in $\ell \cup \{U\}$ and that all models are models for $\ell \cup \{U\}$. $\mathfrak{A}$ is of (*cardinality*) *type* (κ,λ) if $|A| = \kappa$ and $|U^{\mathfrak{A}}| = \lambda$. T *admits* (κ,λ) if T has a model of type (κ,λ). A Löwenheim-Skolem theorem for two cardinals (two-cardinal theorem) is a result to the effect that if T satisfies certain conditions, then T admits (κ,λ) for certain (κ,λ). We write $(\kappa,\lambda) \to (\kappa',\lambda')$ to mean that for every model of type (κ,λ), there is an equivalent model of type (κ′,λ′). And so if T admits (κ,λ), then T admits (κ′,λ′).

Suppose $\kappa > \mu \geq \lambda$. Then, by Theorem 3, $(\kappa,\lambda) \rightarrow (\mu,\lambda)$. It is also clear that $(\kappa,\lambda) \rightarrow (\mu,\mu)$ for all κ, λ, μ such that $\kappa \geq \lambda$.

Theorem 28. If $\kappa > \lambda$, then $(\kappa,\lambda) \rightarrow (\aleph_1,\aleph_0)$.

In the opposite direction we have the following propostion.

Let κ^+ be the smallest cardinal $> \kappa$. We define $\kappa^{(n)}$ by: $\kappa^{(0)} = \kappa$, $\kappa^{(n+1)} = \kappa^{(n)+}$. Let $2_n(\kappa)$ be defined by: $2_0(\kappa) = \kappa$, $2_{n+1}(\kappa) = 2^{2_n(\kappa)}$.

Proposition 9. (a) For every n, there is a sentence φ_n which admits (κ,λ) iff $\kappa \leq \lambda^{(n)}$.

(b) For every n, there is a sentence ψ_n which admits (κ,λ) iff $\kappa \leq 2_n(\lambda)$.

Proof. (a) For $n = 0$ this is trivial. Let $<$ be a two-place predicate and f a two-place function symbol. Let φ_1 be the sentence saying that:

> $<$ is a linear ordering and for every x, the function f_x such that $f_x(y) = f(x,y)$ maps U onto $\{y: y < x\}$.

If $\kappa \leq \lambda^{(1)}$, φ_1 has a model of type (κ,λ). If $\kappa = \lambda$, this is trivial. Suppose $\kappa = \lambda^{(1)}$. Let $\mathcal{A} = (\kappa, U^{\mathcal{A}}, <^{\mathcal{A}}, f^{\mathcal{A}})$ where $<^{\mathcal{A}}$ is the usual well-ordering of κ, $U^{\mathcal{A}}$ is any subset of κ of cardinality λ and for every ordinal $\xi < \kappa$, $\{\langle\eta,f^{\mathcal{A}}(\xi,\eta)\rangle: \eta \in U^{\mathcal{A}}\}$ maps $U^{\mathcal{A}}$ onto the set of predecessors of ξ. Then $\mathcal{A}$ is a model of φ_1.

Next, let $\mathcal{B}$ be any model of φ_1. Let $|B| = \kappa$ and $|U^{\mathcal{B}}| = \lambda$. Then every proper initial $<^{\mathcal{B}}$-segment of B is of cardinality $\leq \lambda$. But this implies that $|B| \leq \lambda^+$. (Consider the set of 1-1 functions on initial $<^{\mathcal{B}}$-segments of B (including B) onto initial segments of the set of ordinals $< \lambda^+$ partially ordered by inclusion and use Zorn's Lemma.) Thus, $\kappa \leq \lambda^{(1)}$, as desired.

This proves the statement for $n = 1$. The statement for arbitrary n is proved by repeating this construction. We leave the details to the reader. ♦

(b) Again, we only prove this for $n = 1$. Let P be a two-place predicate. Let ψ_1 be a sentence saying that

> for all x, y, if $\{z: zPx \wedge Uz\} = \{z: zPy \wedge Uz\}$, then $x = y$.

In any model $\mathcal{A}$ of ψ_1, there are at most as many elements of A as there are subsets of $U^{\mathcal{A}}$ and so $|A| \leq 2^{|U^{\mathcal{A}}|}$, as desired. ■

We now begin the proof of Theorem 28. We write $\mathcal{A} \preccurlyeq^* \mathcal{B}$ to mean that for every finite subset X of A, there is an isomorphism f of $\mathcal{A}$ onto $\mathcal{B}$ such that $f(a) = a$ for $a \in X$. Clearly, $\preccurlyeq^*$ is transitive.

Lemma 22. If for all n, $\mathcal{A}_n$ is countable and $\mathcal{A}_n \preccurlyeq^* \mathcal{A}_{n+1}$, then $\mathcal{A}_0 \preccurlyeq^* \bigcup\{\mathcal{A}_n: n \in N\}$.

Proof. Let $\mathfrak{A} = \bigcup\{\mathfrak{A}_n: n \in N\}$. Let $A_0 = \{a_n: n \in N\}$ and $A = \{b_n: n \in N\}$. Let $X = \{a_n: n \leq m\}$ be any finite subset of A_0. We define d_n, e_n, functions f_n, and numbers k_n in such a way that the following conditions are satisfied:
(i) $d_{2n} = a_n$, $e_{2n+1} = b_n$,
(ii) $f_n: (\mathfrak{A}_0, a_0,\ldots,a_m,d_0,\ldots,d_{n-1}) \cong (\mathfrak{A}_{k_n}, a_0,\ldots,a_m,e_0,\ldots,e_{n-1})$.
Let $k_0 = 0$ and let f_0 be the identity function on A_0. Now, suppose $d_0,\ldots,d_{n-1}$, $e_0,\ldots,e_{n-1}$, f_n, and k_n have been defined.

Case 1. n even, $n = 2p$. Let $d_n = a_p$, $e_n = f_n(d_n)$, $f_{n+1} = f_n$, and $k_{n+1} = k_n$.

Case 2. n odd, $n = 2p+1$. Let $e_n = b_p$. Let $k_{n+1} \geq k_n$ be such that $e_n \in A_{k_{n+1}}$. There is an isomorphism f of $\mathfrak{A}_{k_n}$ onto $\mathfrak{A}_{k_{n+1}}$ which is the identity on $\{a_0,\ldots,a_m,e_0,\ldots,e_{n-1}\}$. Let $f_{n+1} = ff_n$ and let d_n be such that $f_{n+1}(d_n) = e_n$.

Then (i) and (ii) are satisfied for n+1.

Let g be such that $g(a_n) = a_n$ for $n \leq m$ and $g(d_n) = e_n$. Then $g: (\mathfrak{A}_0, a_0,\ldots,a_m) \cong (\mathfrak{A}, a_0,\ldots,a_m)$. ■

Lemma 23. Let $\mathfrak{A}$ be a model of type (κ,λ), where $\kappa > \lambda$. There are then denumerable models $\mathfrak{A}_0$, $\mathfrak{A}_1$ such that $\mathfrak{A}_0 \equiv \mathfrak{A}$, $\mathfrak{A}_0 \neq \mathfrak{A}_1$, $\mathfrak{A}_0 \preccurlyeq^* \mathfrak{A}_1$, and $U^{\mathfrak{A}_0} = U^{\mathfrak{A}_1}$.

This can be proved in a number of different ways. Our proof, in Appendix 2, uses a variant of an idea that will be explained in Chapter 5.
Proof of Theorem 28. Suppose $\mathfrak{A}$ is of type (κ,λ). Let $\mathfrak{A}_0$, $\mathfrak{A}_1$ be as in Lemma 23. We define $\mathfrak{A}_\xi$ such that $\mathfrak{A}_\xi \cong \mathfrak{A}_0$ and $U^{\mathfrak{A}_\xi} = U^{\mathfrak{A}_0}$ for $1 < \xi < \omega_1$ as follows. (ω_1 is the first uncountable ordinal.) Let $\mathfrak{A}_{\xi+1}$ be such that $(\mathfrak{A}_{\xi+1}, A_\xi) \cong (\mathfrak{A}_1, A_0)$. If $\eta < \omega_1$ is a limit ordinal, let $\mathfrak{A}_\eta = \bigcup\{\mathfrak{A}_\xi: \xi < \eta\}$. Then $U^{\mathfrak{A}_\eta} = U^{\mathfrak{A}_0}$ and, by Lemma 22, $\mathfrak{A}_\eta \cong \mathfrak{A}_0$.

Now, let $\mathfrak{B} = \bigcup\{\mathfrak{A}_\xi: \xi < \omega_1\}$. Then $\mathfrak{B} \equiv \mathfrak{A}$. Finally, $\mathfrak{A}_\xi \subsetneq \mathfrak{A}_{\xi+1}$ and $U^{\mathfrak{A}_\xi} = U^{\mathfrak{A}_0}$ for every $\xi < \omega_1$. It follows that $|B| = \aleph_1$ and $U^{\mathfrak{B}} = U^{\mathfrak{A}_0}$ and so $\mathfrak{B}$ is of type $(\aleph_1,\aleph_0)$. ■

There are in the literature a number of results similar to Theorem 28 but, with one exception, they will not be discussed here. As it turns out, however, many two-cardinal questions cannot be settled in set theory (ZFC). This is true, for example, even of such, seemingly simple, questions as if $(\aleph_1,\aleph_0) \rightarrow (\aleph_2,\aleph_1)$ or if any model of type $(\aleph_2,\aleph_1)$ has an elementary submodel of type $(\aleph_1,\aleph_0)$.

The following result will be proved in the next section.

Theorem 29. Suppose for every n, there are κ_n, λ_n such that $\kappa_n \geq 2_n(\lambda_n)$ and T admits (κ_n,λ_n). Then for all κ, λ such that $\kappa > \lambda$, T admits (κ,λ).

In view of Proposition 9(b), this result is best possible.

Let $2_\omega(\kappa)$ be the least upper bound of $\{2_n(\kappa): n \in N\}$.

Corollary 25. If $\kappa \geq 2_\omega(\lambda)$ and $\mu \geq \nu$, then $(\kappa,\lambda) \to (\mu,\nu)$.

§11. Indiscernibles. Models realizing "many" (complete) types can be obtained by applying the Compactness Theorem and countable models realizing "few" types can often be obtained by applying the (Extended) Omitting Types Theorem. In this section we introduce a new idea which will enable us to construct uncountable models realizing "few" types (Theorem 32, below) and which can also be applied to prove Theorem 29.

Let $(X, <)$ be a simply (linearly) ordered set. Let $\mathcal{A}$ be any model such that $X \subseteq A$. $(X, <)$ is then a set of (*order*) *indiscernibles in* $\mathcal{A}$ if for any n, any members $a_1,\ldots,a_n$, $b_1,\ldots,b_n$ of X, and any formula $\varphi(x_1,\ldots,x_n)$ of $\ell_{\mathcal{A}}$, if $a_1 <\ldots< a_n$ and $b_1 <\ldots< b_n$, then

$$\mathcal{A} \models \varphi(a_1,\ldots,a_n) \leftrightarrow \varphi(b_1,\ldots,b_n).$$

In what follows "$\{a_1 <\ldots< a_n\}$" is short for "$\{a_1,\ldots,a_n\}$ where $a_1 <\ldots< a_n$".

Theorem 30 (Ehrenfeucht-Mostowski). Suppose T has infinite models and let $(X, <)$ be any simply ordered set. Then T has a model $\mathcal{A}$ such that X is a set of indiscernibles in $\mathcal{A}$.

The proof of this depends on the following well-known combinatorial theorem.

For any set X, let

$$X^{[n]} = \{Y: Y \subseteq X \;\&\; |Y| = n\}.$$

Theorem 31 (Ramsey's Theorem). If X is infinite and $X^{[n]} = Z_0 \cup\ldots\cup Z_m$, there are an infinite subset Y of X and an $i \leq m$ such that $Y^{[n]} \subseteq Z_i$.

This is proved in Appendix 6.

Proof of Theorem 30. Let $\{c_a: a \in X\}$ be a set of constants not in ℓ_T. Let

$$\Phi = \{\varphi(c_{a_1},\ldots,c_{a_n}) \leftrightarrow \varphi(c_{b_1},\ldots,c_{b_n}): \varphi(x_1,\ldots,x_n) \text{ formula of } \ell_T,\ n > 0, \text{ and}$$
$$\{a_1 <\ldots< a_n\}, \{b_1 <\ldots< b_n\} \in X^{[n]}\} \cup \{c_a \neq c_b: a \neq b,\ a, b \in X\}.$$

It is then sufficient to show that $T \cup \Phi$ has a model. If this holds if ℓ_T is countable, it holds in general. Thus, we may assume that ℓ_T is countable.

Since T has an infinite model, T has a model $\mathcal{B}$ such that $X \subseteq B$. Let $\varphi_n(x_1,\ldots,x_n)$, $n = 1, 2, \ldots$, be all formulas of ℓ_T. (The variables $x_1,\ldots,x_n$ need not all occur in $\varphi_n(x_1,\ldots,x_n)$.) We show that

(*) there are infinite subsets Y_n of X, $n \in N$, such that (i) $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \ldots$ and

(ii) for every $n > 0$ and all $\{a_1 <...< a_n\}, \{b_1 <...< b_n\} \in Y_n^{[n]}$,

$\mathcal{B} \vDash \varphi_n(a_1,...,a_n)$ iff $\mathcal{B} \vDash \varphi_n(b_1,...,b_n)$.

Let $Y_0 = X$. (ii) is trivial for $n = 0$. Suppose Y_n has been defined. Let

$Z_i = \{\{a_1 <...< a_{n+1}\} \in Y_n^{[n+1]}: \mathcal{B} \vDash \varphi_n{}^i(a_1,...,a_{n+1})\}$, $i = 0, 1$.

By Ramsey's Theorem, there are an infinite subset Y_{n+1} of Y_n and a j such that $Y_{n+1}^{[n+1]} \subseteq Z_j$. Y_{n+1} is as desired.

Finally, from (*) it follows that every finite subset of $T \cup \Phi$ has a model and so $T \cup \Phi$ has a model. ■

In Theorem 30 the order < is completely arbitrary. But we cannot in general omit the references to < and claim that $\mathcal{A} \vDash \varphi(a_1,...,a_n) \leftrightarrow \varphi(b_1,...,b_n)$ whenever $\{a_1,...,a_n\}, \{b_1,...,b_n\} \in X^{[n]}$: for $n = 2$ the theory of simple orderings is a counter-example.

One application of indiscernibles (and Skolem functions) is this.

Theorem 32. Suppose T is countable and has an infinite model. Then for every κ, T has a model of cardinality κ realizing only denumerably many complete types.

Proof. We prove this for types with one free variable and leave the general case to the reader. Let T* be a Skolem extension of T and let (X, <) be any simply ordered set of cardinality κ. By Theorem 30, T* has a model $\mathcal{A}$ of cardinality κ in which X is a set of indiscernibles. By Proposition 2, X is a set of indiscernibles of $\mathcal{H}(X)$. Let $\mathcal{B} = \mathcal{H}(X) \mid \ell_T$. Then $\mathcal{B}$ is a model of T of cardinality κ. Any member of $\mathcal{B}$ is of the form $t^{\mathcal{H}(X)}(a_1,...,a_n)$, where $a_1 <...< a_n$. Let φ(x) be any formula of ℓ_T. Let $a = t^{\mathcal{H}(X)}(a_1,...,a_n)$ and $b = t^{\mathcal{H}(X)}(b_1,...,b_n)$, where $\{a_1 <...< a_n\}, \{b_1 <...< b_n\} \in X^{[n]}$. Then $\mathcal{B} \vDash \varphi(a)$ iff $\mathcal{B} \vDash \varphi(b)$. Thus a, b realize the same complete type in $\mathcal{B}$. Since there are only countably many terms of ℓ_{T^*}, it follows that only denumerably many complete types (with one free variable) are realized in $\mathcal{B}$. ■

Corollary 23. If T is countable and κ-categorical, there are only denumerably many complete types over T.

Proof. By Theorem 32, T has a model $\mathcal{A}$ of cardinality κ realizing only denumerably many complete types. Thus, if the conclusion is false, there is a complete type over T not realized in $\mathcal{A}$. This type is realized in some model $\mathcal{B}$ of T of cardinality κ. But then $\mathcal{A} \not\cong \mathcal{B}$ and so T is not κ-categorical. ■

Corollary 24. Suppose T is countable and all models of T are infinite. If T is κ-categorical, then T has an (atomic) prime model.

Proof. By Corollaries 15, 19, T has a denumerable model $\mathcal{A}$ omitting all complete nonprincipal types. But then $\mathcal{A}$ is atomic and so, by Lemma 17(a), $\mathcal{A}$ is prime (in fact, elementarily prime). ■

For $\kappa = \aleph_1$ Theorem 32 and therefore Corollaries 19, 20 can also be proved by observing that the model $\mathcal{B}$ defined in the proof of Theorem 28, except that we now omit all references to U, realizes only denumerably many complete types.

Finally, we now turn to the proof of Theorem 29. In this proof we shall need the following extension of Ramsey's Theorem:

Theorem 33 (Erdös–Rado)**.** Suppose $|X| > 2_n(\kappa)$ and $X^{[n+1]} = \bigcup\{Z_i\colon i \in I\}$, where $|I| \leq \kappa$. There are then a set $Y \subseteq X$ and a $j \in I$ such that $|Y| > \kappa$ and $Y^{[n+1]} \subseteq Z_j$.

This is proved in Appendix 6.

Proof of Theorem 29. Suppose $\kappa > \lambda$. Let T^* be a Skolem extension of T. The idea is to show that there are a model $\mathcal{B}$ of T^* of type (κ,κ), a subset Y of $A - U^{\mathcal{B}}$ of cardinality κ, and a linear ordering $<'$ of Y such that for every term $t(x_1,\dots,x_k,y_1,\dots,y_m)$ of ℓ_{T^*}, all $a_1,\dots,a_k \in U^{\mathcal{B}}$, and all $b_1,\dots,b_m,b_1',\dots,b_m' \in Y$,

if $b_1 <' \dots <' b_m$, $b_1' <' \dots <' b_m'$, and $t^{\mathcal{B}}(\underline{a},\underline{b})$, $t^{\mathcal{B}}(\underline{a},\underline{b}') \in U^{\mathcal{B}}$, then $t^{\mathcal{B}}(\underline{a},\underline{b}) = t^{\mathcal{B}}(\underline{a},\underline{b}')$.

It follows that if X is any subset of $U^{\mathcal{B}}$ of cardinality λ, then $\mathcal{H}_{\mathcal{B}}(X \cup Y)$ is of type (κ,λ).

Let $t_1, t_2, t_3,\dots$ be an enumeration of all terms of ℓ_{T^*}. We may assume that t_n can be written as $t_n(x_1,\dots,x_n,y_1,\dots,y_n)$. (The variables $x_1,\dots,x_n,y_1,\dots,y_n$ need not all occur in t_n.) For $n > 0$, let $\sigma_n(\underline{y},\underline{z})$ be the formula

$$\forall x_1\dots x_n(Ux_1 \wedge\dots\wedge Ux_n \wedge Ut_n(\underline{x},\underline{y}) \wedge Ut_n(\underline{x},\underline{z}) \rightarrow t_n(\underline{x},\underline{y}) = t_n(\underline{x},\underline{z})).$$

Let c_n, $n \in N$, be new individual constants. Let T^+ be obtained from T^* by adding the following sentences:

$\neg Uc_n$ for $n \in N$,
$c_k \neq c_m$ for $k < m$,
$\exists^{>n}xUx$ for $n \in N$,
all sentences $\sigma_n(c_{k_1},\dots,c_{k_n},c_{m_1},\dots c_{m_n})$, where $k_1 <\dots< k_n$ and $m_1 <\dots< m_n$.

Our first task is now to show that

(1) T^+ is consistent.

We prove (1) by applying Theorem 33. Fix n. Define p_k for $k \leq n$ by: $p_n = 0$, $p_k = p_{k+1}+k$ for $k < n$. Let $m = p_0+2$. Every model of T has an expansion to a model of T^*. Thus, there is a model $\mathcal{A}$ of T^* such that $|A| \geq 2_m(|U^{\mathcal{A}}|)$. Let $V = U^{\mathcal{A}}$. Let $\mu = 2^{|V|}$. Let $<'$ be a linear ordering of A.

To prove (1) we show that

(2) there is a sequence $X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n$ of subsets of $A - V$ such that for every $k \le n$, (i) $|X_k| > 2_{p_k}(\mu)$ and (ii) if $k > 0$ and $\{a_1 <' \ldots <' a_k\}$, $\{b_1 <' \ldots <' b_k\} \in X_k^{[k]}$, then $\mathcal{A} \vDash \sigma_k(\underline{a}, \underline{b})$.

Let $X_0 = A - V$. $|X_0| = |A| \ge 2_m(|V|) = 2_{p_0+1}(\mu) > 2_{p_0}(\mu)$. Thus, (i) holds for $k = 0$. (ii) is trivial for $k = 0$. Suppose (i) holds for a certain $k < n$. Let $\sim_k$ be the relation on $X_k^{[k+1]}$ defined by:

$$\{a_1 <' \ldots <' a_{k+1}\} \sim_k \{b_1 <' \ldots <' b_{k+1}\} \text{ iff } \mathcal{A} \vDash \sigma_{k+1}(a_1,\ldots,a_{k+1},b_1,\ldots,b_{k+1}).$$

$\sim_k$ is an equivalence relation. Let $\{Z_i : i \in I\}$ be the set of equivalence classes of $\sim_k$. Then $X_k^{[k+1]} = \bigcup\{Z_i : i \in I\}$. Let e be any object not in V. The $\sim_k$-equivalence class of $\{a_1 <' \ldots <' a_{k+1}\}$ is then uniquely determined by the function $f: V^{[k+1]} \to V \cup \{a\}$ defined by:

$$f(d_1,\ldots,d_{k+1}) = t_{k+1}^{\mathcal{A}}(d_1,\ldots d_{k+1},a_1,\ldots,a_{k+1}) \text{ if this is in V,}$$
$$= e \text{ otherwise.}$$

There are $\le \mu$ such functions. Thus, $|I| \le \mu$. $|X_k| > 2_{p_k}(\mu) = 2_k(2_{p_{k+1}}(\mu))$. It follows, by Theorem 33, that there are a set $X_{k+1} \subseteq X_k$ and a $j \in I$ such that $|X_{k+1}| > 2_{p_{k+1}}(\mu)$ and $X_{k+1}^{[k+1]} \subseteq Z_j$. X_{k+1} is as desired.

This proves (2). From (2), since n is any number and the set X_n is infinite, it follows that every finite subset of T^+ has a model. By compactness, this implies that (1) is true.

Now, suppose $\kappa > \lambda$. We replace T^+ by a related theory $T^\#$. Let d_ξ, $\xi < \kappa$, be new individual constants. $T^\#$ is obtained from T^* by adding the following sentences:

$\neg U d_\xi$ for $\xi < \kappa$,

$d_\xi \ne d_\eta$ for $\xi < \eta$,

$\exists^{>n} x U x$ for $n \in N$,

all sentences $\sigma_n(d_{\xi_1},\ldots,d_{\xi_n},d_{\eta_1},\ldots d_{\eta_n})$, where $\xi_1 < \ldots < \xi_n$ and $\eta_1 < \ldots < \eta_n$.

By (1) and compactness, $T^\#$ has a model $(\mathcal{B}, e_\xi)_{\xi < \kappa}$ of type (κ,κ). Let X be any subset of $U^{\mathcal{B}}$ of cardinality λ. Let $Y = \{e_\xi : \xi < \kappa\}$. Then $X \cap Y = \varnothing$. Let $\mathcal{C} = \mathcal{H}_{\mathcal{B}}(X \cup Y) | \ell_T$. Then $\mathcal{C}$ is a model of T of cardinality κ.

For every member c of $\mathcal{C}$, there are a term $t_n(x_1,\ldots x_n,y_1,\ldots,y_n)$, members $a_1,\ldots,a_n$ of X, and members $e_{\xi_1},\ldots,e_{\xi_n}$ of Y, where $\xi_1 < \ldots < \xi_n$, such that $c = t_n^{\mathcal{B}}(a_1,\ldots a_n,e_{\xi_1},\ldots,e_{\xi_n})$. Since $\mathcal{B} \vDash \sigma_n(e_{\eta_1},\ldots,e_{\eta_n},e_{\xi_1},\ldots,e_{\xi_n})$ whenever $\eta_1 < \ldots < \eta_n$, it follows that if $c \in U^{\mathcal{C}}$, then $t_n^{\mathcal{B}}(a_1,\ldots a_n,e_{\xi_1},\ldots,e_{\xi_n})$ is uniquely determined by t_n and $a_1,\ldots,a_n$, in other words, it is independent of $e_{\xi_1},\ldots,e_{\xi_n}$. Since $|X| = \lambda$ and there are only countably many terms t_n, this implies that $|U^{\mathcal{C}}| = \lambda$. Thus, $\mathcal{C}$ is a model of T of type (κ,λ), as desired. ■

§12. An illustration. In this section we prove a small result illustrating how some of the results and methods developed in the preceding sections can be combined.

Let us say that a model $\mathcal{A}$ of T is *minimal* if $\mathcal{A}$ is a prime model of T and no proper submodel of $\mathcal{A}$ is a model of T. For example, ACF(0) has a minimal model (the field of algebraic numbers). This is a special case of the following result.

Theorem 34. Suppose T is countable and all models of T are infinite. If T is $\forall\exists$, $\aleph_1$-categorical, and not $\aleph_0$-categorical, then T has a minimal model.

Suppose T is countable and $\aleph_0$-categorical. Let $\mathcal{A}$ be any denumerable model of T. There is a denumerable proper elementary extension $\mathcal{A}'$ of $\mathcal{A}$. $\mathcal{A}'$ is not a minimal model of T. Since $\mathcal{A} \cong \mathcal{A}'$, it follows that $\mathcal{A}$ is not a minimal model of T. Thus, T has no minimal model.

Proof of Theorem 34. By Corollary 24, T has an atomic prime model $\mathcal{A}$. We are going to show that $\mathcal{A}$ is minimal. Suppose not. There is then a proper submodel $\mathcal{A}'$ of $\mathcal{A}$ which is a model of T. By Theorem 17, $\mathcal{A}' \preccurlyeq \mathcal{A}$. But then $\mathcal{A}'$ is atomic, and so, by Lemma 17(b), $\mathcal{A}' \cong \mathcal{A}$.

We now define an elementary chain of denumerable isomorphic atomic models $\mathcal{B}_\xi$, where $\xi < \omega_1$, as follows. Let $\mathcal{B}_0 = \mathcal{A}'$ and $\mathcal{B}_1 = \mathcal{A}$. Suppose $\eta \geq 2$ and $\mathcal{B}_\xi$ has been defined for $\xi < \eta$. Suppose η is a successor ordinal; $\eta = \xi+1$. By hypothesis, $\mathcal{B}_\xi \cong \mathcal{B}_0$. $\mathcal{B}_0$ has a proper elementary extension $\mathcal{B}_1$ isomorphic to $\mathcal{B}_0$. Thus, $\mathcal{B}_\xi$ has a proper elementary extension $\mathcal{B}_\eta$ isomorphic to $\mathcal{B}_\xi$. If η is a limit ordinal, let $\mathcal{B}_\eta = \bigcup\{\mathcal{B}_\xi : \xi < \eta\}$. In this case every type realized in $\mathcal{B}_\eta$ is realized in some $\mathcal{B}_\xi$ with $\xi < \eta$. Hence, every complete type realized in $\mathcal{B}_\eta$ is principal. Thus, $\mathcal{B}_\eta$ is atomic and so, by Lemma 17(b), $\mathcal{B}_\eta \cong \mathcal{B}_0$.

Now let $\mathcal{B} = \bigcup\{\mathcal{B}_\xi : \xi < \omega_1\}$. Then every complete type realized in $\mathcal{B}$ is principal (and so $\mathcal{B}$ is atomic). On the other hand, since T is not $\aleph_0$-categorical, by Theorem 24 and Lemma 18, there is a nonprincipal complete type over T. This type is realized in some model $\mathcal{B}'$ of T of cardinality $\aleph_1$. But then $\mathcal{B}' \ncong \mathcal{B}$, contradicting the assumption that T is $\aleph_1$-categorical. It now follows that $\mathcal{A}$ is a minimal model of T. ■

§13. Examples. The theories discussed but not defined in this § are defined in Chapter 1, §7. In the following examples Z is the (ordered) set of integers.

Example 1. Let $\mathcal{A} = (A, X)$ and $\mathcal{B} = (B, Y)$, where $X \subseteq A$ and $Y \subseteq B$. Suppose either $|X| = |Y|$ or $|X|, |Y| \geq n$ and either $|A - X| = |B - Y|$ or $|A - X|, |B - Y| \geq n$. Let I be defined by: $\langle\rangle I \langle\rangle$ and $\langle a_1,\ldots,a_k\rangle I \langle b_1,\ldots,b_k\rangle$, where $k \leq n$, $a_1,\ldots,a_k \in A$ and $b_1,\ldots,b_k \in B$, iff $a_i \in X$ iff $b_i \in Y$ and $a_i = a_j$ iff $b_i = b_j$ for $i, j \leq k$. Then I is an n-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ and so $\mathcal{A} \cong_n \mathcal{B}$. It follows by Lemma 14, that if φ is a sentence of {P}, where P is a one-place predicate, and φ has a model, then φ has a finite model. The generalization of this to sentences containing several one-place predicates is straightforward. ■

Example 2. All models of DeLO are infinite. By a classical result of Cantor, DeLO is $\aleph_0$-categorical. Thus, by Theorem 11, DeLO is complete.

DeLO is model-complete. To show this, let $\mathcal{A} = (A, \leq)$ and $\mathcal{B} = (B, \leq)$ be any models of DeLO such that $\mathcal{A} \subseteq \mathcal{B}$. By Theorem 13, it is then sufficient to show that $\mathcal{A} \preccurlyeq_1 \mathcal{B}$. Let $\varphi := \exists y_1,\ldots,y_k \psi(x_1,\ldots,x_n,y_1,\ldots,y_k)$, where ψ is quantifier-free, be any simple existential formula (of $\{\leq\}$). Let $a_1,\ldots,a_n \in A$ be such that $\mathcal{B} \vDash \varphi(a_1,\ldots,a_n)$. Let $b_1,\ldots,b_k \in B$ be such that

$$\mathcal{B} \vDash \psi(a_1,\ldots,a_n,b_1,\ldots,b_k).$$

Then, since $\leq$ is dense and has no endpoints, there are $c_1,\ldots,c_k \in A$ such that for $i \leq n$ and $j, j' \leq k$, $a_i \leq c_j$ iff $a_i \leq b_j$ and $c_j \leq c_{j'}$ iff $b_j \leq b_{j'}$. But then $\mathcal{A} \vDash \psi(a_1,\ldots,a_n,c_1,\ldots,c_k)$ and so $\mathcal{A} \vDash \varphi(a_1,\ldots,a_n)$. Thus, $\mathcal{A} \preccurlyeq_1 \mathcal{B}$, as desired.

That DeLO is model-complete also follows, by Theorem 17, from the facts that it is $\aleph_0$-categorical and $\forall\exists$. This can also be shown in the following elementary way. Suppose $\mathcal{A} = (A, \leq)$ and $\mathcal{B} = (B, \leq)$ are models of DeLO. Let I be defined by: $\langle\rangle I \langle\rangle$ and $\langle a_1,\ldots,a_n\rangle I \langle b_1,\ldots,b_n\rangle$ iff for $i, j \leq n$, $a_i \leq a_j$ iff $b_i \leq b_j$. It is then easy to verify that $I\colon \mathcal{A} \cong_\omega \mathcal{B}$. Suppose $\mathcal{A} \subseteq \mathcal{B}$. Clearly, for any $a_1,\ldots,a_n \in A$, $\langle a_1,\ldots,a_n\rangle I \langle a_1,\ldots,a_n\rangle$. It follows, by Corollaries 11, 13, that DeLO is complete and model-complete.

If $\mathcal{A}$ is denumerable and $\mathcal{B} = \mathcal{A}$, this also shows that for every n, the relation $\sim_{\mathcal{A},n}$ has only finitely many equivalences classes (compare Corollary 18).

Every existential sentence true in one linear ordering is true in all infinite linear orderings. This implies that the theory ILO of infinite linear orderings is u-complete. Since DeLO is model-complete, it follows, by Corollary 10(ii), that ILO has exactly one model-complete extension, namely DeLO.

Let c_0, c_1 be individual constants. Let T be the theory in $\{\leq,c_0,c_1\}$ whose axioms are those of DeLO. Then T is model-complete but not complete. ■

Example 3. The models of DiLO consist of a linearly ordered set of copies of Z. DiLO is complete but not categorical in any infinite cardinality. DiLO is not model-complete. Let $\mathcal{A}$ be any model of DiLO, a any member of $\mathcal{A}$ and a′ its immediate $\leq^{\mathcal{A}}$-successor. We can then insert a new element "between" a and a′. The result is a model $\mathcal{B}$ of DiLO (in fact, isomorphic to $\mathcal{A}$), an extension of $\mathcal{A}$, but not an elementary extension of $\mathcal{A}$; the (simple existential) formula $\exists x(a < x \wedge x < a')$ is true in $(\mathcal{B}, a,a')$ but not in $(\mathcal{A}, a,a')$.

Thus, neither Theorem 11 nor Theorem 15 applies. Instead we are going to use Corollary 11. Let $\mathcal{A} = (A, \leq)$ and $\mathcal{B} = (B, \leq)$ be any models of DiLO. It is sufficient to show that for every n, $\mathcal{A} \cong_n \mathcal{B}$. For a_0, $a_1 \in A$, let $|a_0,a_1|_{\mathcal{A}}$ be the cardinality of the set $\{a \in A\colon a_0 \leq^{\mathcal{A}} a <^{\mathcal{A}} a_1\}$. Similarly, for b_0, $b_1 \in B$, let $|b_0,b_1|_{\mathcal{B}}$ be the cardinality of the set $\{b \in B\colon b_0 \leq^{\mathcal{B}} b <^{\mathcal{B}} b_1\}$. ($|a_0,a_1|_{\mathcal{A}}$ and $|b_0,b_1|_{\mathcal{B}}$ may be infinite.) Thus, $|a_0,a_1|_{\mathcal{A}} > 0$

iff $a_0 <^{\mathcal{A}} a_1$ and $|b_0,b_1|_{\mathcal{B}} > 0$ iff $b_0 <^{\mathcal{B}} b_1$.

Now, fix n. Let I^n be defined by: $\langle\rangle I^n \langle\rangle$ and

$\langle a_1,\ldots,a_k\rangle I^n \langle b_1,\ldots,b_k\rangle$, where $k \le n$, iff for $i, j \le k$, $|a_i,a_j|_{\mathcal{A}}$ and $|b_i,b_j|_{\mathcal{B}}$ are either equal or both $\ge 2^{n-k}$.

Then

(*) $I^n\colon \mathcal{A} \cong_n \mathcal{B}$.

The verification of (*) is left to the reader.

By (*), it is also clear that if $\mathcal{A}$, $\mathcal{B}$ are models of DiLO, $\mathcal{A} \subseteq \mathcal{B}$, and no member of $\mathcal{B}$ lies between two adjacent members of $\mathcal{A}$, then $\mathcal{A} \preccurlyeq \mathcal{B}$. It follows that $(Z, \le)$ is an atomic and therefore elementarily prime model of DiLO.

Let $\mathcal{C} = (Z, \le, 0)$. Suppose $X \subseteq Z$ and X is definable in $\mathcal{C}$. By (*), if $|0,a|_{\mathcal{C}}$ and $|0,a'|_{\mathcal{C}}$ or $|a,0|_{\mathcal{C}}$ and $|a',0|_{\mathcal{C}}$ are large enough, then $a \in X$ iff $a' \in X$. Thus, for example, the set E of even integers is not definable in $\mathcal{C}$. By Theorem 10, this implies that there are models $\mathcal{D} = (D, \le', b, X)$ and $\mathcal{D}' = (D, \le', b, X')$ equivalent to $(\mathcal{C}, E)$ such that $X \ne X'$. Such models can be obtained as follows. Let $D = Z \cup \{d_i\colon i \in Z\}$, where $d_i \notin Z$. Let $\le'$ be the extension of $\le$ to D such that $i \le' d_j$ for all i, j and $d_i \le' d_j$ iff $i \le j$. Let $b = 0$, Finally, let $X = E \cup \{d_i\colon i \text{ even}\}$ and $X' = E \cup \{d_i\colon i \text{ odd}\}$. Then ($\mathcal{D} \cong \mathcal{D}'$ and) $\mathcal{D}$, $\mathcal{D}'$ are as desired.

It follows from (*) that E is not definable in DiLO using parameters. By the Theorem proved in Appendix 3, this implies that there are $> \aleph_0$ isomorphic denumerable models equivalent to $(\mathcal{C}, E)$. The definition of such models is left to the reader.

Example 4. SF is not finitely axiomatizable, in other words, $\mathrm{Mod(SF)} \notin \mathrm{EC}$. For, suppose it were. There would then, by Proposition 3, be a finite subset Φ of SF such that $\mathrm{Mod}(\Phi) = \mathrm{Mod(SF)}$, which is clearly not the case.

Let X be any set. Let

$A^{(X)} = N \cup X{\times}Z$,
$f^{(X)}(n) = n+1$ for $n \in N$,
$f^{(X)}(\langle a,i\rangle) = \langle a,i+1\rangle$ for $\langle a,i\rangle \in X{\times}Z$.

Let $\mathcal{A}^{(X)} = (A^{(X)}, f^{(X)}, 0)$.Thus, $\mathcal{A}^{(\varnothing)} = (N, S, 0)$. Every model of SF is isomorphic to a model of this form.

SF (= Th((N, S, 0))) is model-complete (but Th((N, S)) is not.) This can be shown in a number of different ways. One is as follows. let $\mathcal{B}$, $\mathcal{B}'$ be any models of SF such that $\mathcal{B} \subseteq \mathcal{B}'$. We (may) assume that there are X, X′ such that $X \subseteq X'$, $\mathcal{B} = \mathcal{A}^{(X)}$, and $\mathcal{B}' = \mathcal{A}^{(X')}$. By Theorem 13, it is sufficient to show that $\mathcal{B} \preccurlyeq_1 \mathcal{B}'$. If $\varphi(x_1,\ldots,x_n)$ is a simple existential formula, $a_1,\ldots,a_n \in A^{(X)}$, and $\mathcal{B}' \vDash \varphi(a_1,\ldots,a_n)$, there is a finite subset X^* of $X' - X$ such that $a_1,\ldots,a_n \in A^{(X \cup X^*)}$ and $\mathcal{A}^{(X \cup X^*)} \vDash \varphi(a_1,\ldots,a_n)$. And so it is sufficient to show that $\mathcal{A}^{(X)} \preccurlyeq_1 \mathcal{A}^{(X \cup X^*)}$. But then, since $\preccurlyeq_1$ is transitive, it suffices to show that for any $a \notin X$, $\mathcal{A}^{(X)} \preccurlyeq_1 \mathcal{A}^{(X \cup \{a\})}$.

Let $\varphi := \exists y_1,\dots,y_k \psi(x_1,\dots,x_n,y_1,\dots,y_k)$, where ψ is quantifier-free, be any simple existential formula (of {S, 0}). Let $a_1,\dots,a_n \in A^{(X)}$ be such that $\mathcal{A}^{(X \cup \{a\})} \vDash \varphi(a_1,\dots,a_n)$. We want to show that $\mathcal{A}^{(X)} \vDash \varphi(a_1,\dots,a_n)$. Let $b_1,\dots,b_k \in A^{(X \cup \{a\})}$ be such that

$$\mathcal{A}^{(X \cup \{a\})} \vDash \psi(a_1,\dots,a_n,b_1,\dots,b_k).$$

We (may) assume that $b_1,\dots,b_k \in A^{(X \cup \{a\})} - A^{(X)}$. (If, for example, $b_1 \in A^{(X)}$, we replace φ by $\exists y_2,\dots,y_k \psi(x_1,\dots,x_n,x_{n+1},y_2,\dots,y_k)$.) It follows that every basic conjunct of ψ containing an x_i and a y_j is the negation of an atomic formula. Let $b_r = \langle a,i_r \rangle$, $0 < r \leq k$. Let p be such that, for $0 < r \leq k$, $p + i_r > 0$, whence $p+i_r \in N$, and $p+i_r > a_s$ for every a_s in N. Then

$$\mathcal{A}^{(X)} \vDash \psi(a_1,\dots,a_n,p+i_1,\dots,p+i_k)$$

and so $\mathcal{A}^{(X)} \vDash \varphi(a_1,\dots,a_n)$. Thus, we have shown that $\mathcal{A}^{(X)} \preccurlyeq_1 \mathcal{A}^{(X \cup \{a\})}$, as desired.

(N, S, 0) is a prime model of SF and so, by Theorem 15, that SF is complete.

Suppose $|X| = |Y|$. Let g be a one-one function on X onto Y. Let h be the function on $A^{(X)}$ onto $A^{(Y)}$ defined by: $h(n) = n$, for $n \in N$, and $h(\langle a,i \rangle) = \langle g(a),i \rangle$, for $\langle a,i \rangle \in X \times Z$. Then h: $\mathcal{A}^{(X)} \cong \mathcal{A}^{(Y)}$. Thus, the isomorphism type of $\mathcal{A}^{(X)}$ is determined by the cardinality of X. If $|X| > \aleph_0$, then $|A^{(X)}| = |X|$. It follows that SF is κ-categorical for every $\kappa > \aleph_0$ (but not $\aleph_0$-categorical). And so, by Theorem 11, SF is complete. Moreover, SF is $\forall\exists$ and so, by Theorem 17, SF is model-complete as well. SF can also, easily, be shown to be complete and model-complete by applying Corollaries 11, 13.

There are only countably many complete types (in x) over SF (§8, Example 2). (N, S, 0) is a prime and a minimal model of SF (compare Corollaries 23, 24 and Theorem 34). ■

Example 5. Let IBA be the theory of infinite Boolean algebras, in other words, the axioms of IBA are the axioms of BA plus the sentences $\exists^{>n}x(x = x)$, $n \in N$. IBA is not finitely axiomatizable.

Let IAtBA be the theory of infinite atomic Boolean algebras. IAtBA is not categorical in any infinite cardinality and it is not model-complete. However, IAtBA is complete. Let $\mathcal{A}$, $\mathcal{B}$ be any two models of IAtBA. Fix n. We define an n-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ and leave the verification to the reader.

We use the same symbols for the functions and elements of $\mathcal{A}$ and $\mathcal{B}$. For $a \in A$, let $a^i = a$ if $i = 0$ and $= a^*$ if $i = 1$. Define b^i similarly for $b \in B$. For $a \in A$ let $|a|$ be the cardinality of the set of atoms a' of $\mathcal{A}$ such that $\mathcal{A} \vDash a \cap a' \neq 0$ and define $|b|$ for $b \in B$ similarly. Now, define I by: $\langle \rangle I \langle \rangle$ and

$\langle a_1,\dots,a_k \rangle I \langle b_1,\dots,b_k \rangle$, where $k \leq n$, iff for every sequence $i_1,\dots,i_k$ of 0's and 1's, $|a_1^{i_1} \cap \dots \cap a_k^{i_k}|$ and $|b_1^{i_1} \cap \dots \cap b_k^{i_k}|$ are either equal or both $\geq 2^{n-k}$.

It now follows, by Corollary 11, that IAtBA is complete. ■

Example 6. Since $0 \neq 1$ in any Boolean algebra, all models of NoAtBA are infinite. Let $\mathcal{A}$, $\mathcal{B}$ be any two atomless Boolean algebras. Let I be defined by: $\langle\rangle I \langle\rangle$ and

$\langle a_1,\dots,a_n\rangle I \langle b_1,\dots,b_n\rangle$ iff for every sequence $i_1,\dots,i_n$ of 0's and 1's,

$\mathcal{A} \models a_1^{i_1} \cap \dots \cap a_n^{i_n} = 0$ iff $\mathcal{B} \models b_1^{i_1} \cap \dots \cap b_n^{i_n} = 0$.

Then I: $\mathcal{A} \cong_\omega \mathcal{B}$. It follows, by Lemma 15, that NoATBA is $\aleph_0$-categorical and so, by Theorem 11, it is complete. Also, NoAtBA is ∀∃ and so, by Theorem 17, it is model-complete. This also follows directly from the fact that I: $\mathcal{A} \cong_\omega \mathcal{B}$ and Corollary 13.

IBA is u-complete. (Every existential sentence true in a Boolean algebra is true in a finite Boolean algebra and every finite Boolean algebra is embeddable in every infinite Boolean algebra.) It follows, by Corollary 10(ii), that IBA has exactly one model-complete extension, NoAtBA. ■

Example 7. DTAG is not finitely axiomatizable. DTAG is ∀∃ and κ-categorical for $\kappa > \aleph_0$ and so, by Theorems 11 and 17, is complete and model-complete.

This can also be shown using Theorem 13; this proof is similar to the first proof above that SF is model-complete, and also by using Corollaries 11, 13 as follows. Let p be any integer. Then $|p| = p$ if $p \geq 0$ and $|p| = -p$ if $p < 0$. Let $\mathcal{A}$, $\mathcal{B}$ be models of DTAG. The term pa is as in Chapter 1, §7, Example 4 if $p > 0$, 0a is 0, and pa is $-(-pa)$ if $p < 0$. Now define I by: $\langle\rangle I \langle\rangle$ and for $k \leq n$

$\langle a_1,\dots,a_k\rangle I \langle b_1,\dots,b_k\rangle$ iff for all $p_1,\dots,p_k$, if $|p_i| \leq 2^{2^{n-k}-1}$, $i = 1,\dots,k$, then

$\mathcal{A} \models p_1a_1 + \dots + p_ka_k = 0$ iff $\mathcal{B} \models p_1b_1 + \dots + p_kb_k = 0$.

Then I: $\mathcal{A} \cong_n \mathcal{B}$.

The theory TAG is u-complete. (Every existential sentence true in some model of TAG is true in (Z,+,–,0) and this group is embeddable in every model of TAG.) Thus, by Corollary 10(ii), TAG has exactly one model-complete extension, namely DTAG. (Ra, +, –, 0) (Chapter 1, §7, Example 4) is a minimal model of DTAG (compare Theorem 34). ■

Example 8. Example of a complete ∀∃ theory which is not model-complete. Let f, g, h be one-place function symbols and c a constant. Let $f^n(x)$ be defined by: $f^0(x) := x$, $f^{n+1}(x) := f(f^n(x))$; and similarly for $g^n(x)$ and $h^n(x)$. Let T be the theory whose axioms are:

$\forall xy(f(x) = f(y) \to x = y)$,
$\forall x(f^{n+1}(x) \neq x)$, $\forall x(f(g(x)) = x)$,
$\forall xy(h(x) = h(y) \to x = y)$,
$\forall x(h(f(x)) = f(h(x)))$, $\forall x(h(x) \neq c)$,
$\forall x(h^{n+1}(x) \neq f^k(x))$, $\forall x(h^{n+1}(x) \neq g^k(x))$,

where $k, n \in N$. T is universal and so ∀∃. We leave the task of showing that T is complete but not model-complete to the reader. [Hint: To show that T is complete, let C be a set of constants of cardinality $\aleph_1$ and define a set Φ of sentences of $\ell_T \cup C$ such that (a) if $\mathcal{A}, \mathcal{B} \models T \cup \Phi$ and $|A| = |B| = \aleph_1$, then $\mathcal{A} | \ell_T \cong \mathcal{B} | \ell_T$ and (b) if Φ' is any finite subset of Φ and $\mathcal{A} \models T$, then $\mathcal{A}$ can be expanded to a model of $T \cup \Phi'$.] ■

Example 9. The complete theories discussed so far all have prime models (in fact, elementarily prime models). Here is an example of a complete theory with no prime model. Let P_k, $k \in N$, be one-place predicates. Let T be the set of sentences

$$\exists x(P_{i_0}x \wedge \ldots \wedge P_{i_m}x \wedge \ldots \wedge \neg P_{j_0}x \wedge \ldots \wedge \neg P_{j_n}x),$$

where $i_0, \ldots, i_m, j_0, \ldots, j_n$ are all distinct. If $\mathcal{A}, \mathcal{B}$ are countable models of T, for every n, $\mathcal{A} | \{P_k : k \leq n\} \cong \mathcal{B} | \{P_k : k \leq n\}$. It follows that T is complete.

Suppose $\mathcal{A} \models T$. Let $a \in A$ and let $\mathcal{B}$ be obtained from $\mathcal{A}$ by leaving out all $b \in A$ such that $\{k : b \in P_k^{\mathcal{A}}\} = \{k : a \in P_k^{\mathcal{A}}\}$. Then $\mathcal{B} \models T$. Also, clearly, $\mathcal{A}$ is not embeddable in $\mathcal{B}$ and so $\mathcal{A}$ is not a prime model of T. Thus, T has no prime model.

ℓ_T is infinite. In Appendix 5 we give an example of a complete theory in a finite language with no prime model. We also give an example of a complete theory (in a finite language) which has a prime model but no elementarily prime model. ■

Example 10. PA is not finitely axiomatizable nor is any consistent extension of PA. PA is not complete (see Theorem 4.5) and so PA $\neq$ Th($\mathcal{N}$). Th($\mathcal{N}$) is complete, of course, but not model-complete: Th($\mathcal{N}$) is not equivalent to a set of $\forall\exists$ sentences (cf. Corollary 9); in fact, if T is any consistent extension of PA, there are sentences in T with (if written in prenex normal form) arbitrarily many quantifier alternations.

Th((N, S, 0)) (= SF) is not $\aleph_0$-categorical (Example 4). It follows, by Corollary 17(a), that Th($\mathcal{N}$) is not $\aleph_0$-categorical (Corollary 2). By Example 1 in §8 and Corollary 23, Th($\mathcal{N}$) is not κ-categorical for any κ. This can also be more easily proved by showing that for every $\kappa > \aleph_0$, Th($\mathcal{N}$) has models $\mathcal{A}, \mathcal{B}$ of cardinality κ such that the ordering of $\mathcal{A}$ is of cofinality ω and the ordering of $\mathcal{B}$ is not. (A linear ordering (A, <) is of *cofinality* ω if there is a denumerable set $X \subseteq A$ such that for every $a \in A$, there is a $b \in X$ such that $a < b$. An ordering of order type ω_1, for example, is not of cofinality ω.) Let $\mathcal{A}, \mathcal{B}$ be unions of suitable elementary chains.

$\cdot$ is not explicitly definable in terms of {+, S, 0} in Th($\mathcal{N}$). This can be shown by a direct model-theoretic argument, e.g., one based on Corollary 14. We can also use the fact that Th((N, +, S, 0)) is axiomatizable (and therefore decidable (Presburger's theorem); cf. Proposition 4.1) but Th($\mathcal{N}$) is not (Theorem 4.5).) By Theorem 10, it follows that Th($\mathcal{N}$) has (denumerable) models $\mathcal{A}, \mathcal{B}$ such that A = B, $+^{\mathcal{A}} = +^{\mathcal{B}}$ (and consequently $S^{\mathcal{A}} = S^{\mathcal{B}}$, $0^{\mathcal{A}} = 0^{\mathcal{B}}$) but $\cdot^{\mathcal{A}} \neq \cdot^{\mathcal{B}}$. ■

Example 11. In ZF it can be proved that there are uncountably many sets (of natural numbers). Nevertheless, if ZF(C) is consistent – this is, perhaps, not absolutely certain – by the Löwenheim-Skolem Theorem, it has a denumerable model. In fact, if we somewhat strengthen our assumption, ZF(C) has a denumerable model $\mathcal{A} = (A, E)$ which is *standard* in the sense that E is element relation $\in$ restricted to A, and A is *transitive*, in other words, if $a \in b \in A$, then $a \in A$. (To derive this conclusion we need to assume that ZF(C) has a standard model.) This is the so-called "Skolem paradox", the first and, in combination with Skolem's discovery that set theory can be formalized in first-order logic, still one of the most striking applications of model theory to set theory and, thereby, to mathematics in general. This is not a real paradox, however; the explanation is simply that the function on N onto A, that exists, since A is denumerable, is not a member of A; if fact, although $N \in A$, not every subset of N is a member of A. Thus, the statement that there is a function on N onto A, although true in "the real world", is not true in $\mathcal{A}$. And $\mathcal{A}$ is an "unintended" model of ZF(C).

Another straightforward application of model theory to set theory is as follows. In a standard model of ZF there is no sequence $a_0, a_1, a_2, \ldots$ of sets such that for every n, $a_{n+1} \in a_n$. Indeed, the sentence to this effect is provable in ZF. Nevertheless, there are models $\mathcal{B}$ of ZF(C) which have members $b_0, b_1, b_2, \ldots$ such that for every n, $b_{n+1} \in^{\mathcal{B}} b_n$. No such model is standard.

To obtain such a model $\mathcal{B}$ we introduce individual constants $c_0, c_1, c_2, \ldots$ and consider the set $\Phi = ZF(C) \cup \{c_{n+1} \in c_n : n \in N\}$. Every finite subset of Φ has a model. Thus, Φ has a model $\mathcal{B}$ (another unintended model of ZF(C)). Let $b_n = c_n^{\mathcal{B}}$. ■

Notes for Chapter 3. The basic concepts of model theory are due to Tarski (1952) (cf. also Tarski, Vaught (1957)). Lemmas 1, 2 are due to Tarski (cf. Tarski, Vaught (1957)). Theorem 1 is due to Gödel (1930) (for denumerable Φ), Maltsev (1936), and Henkin (1949). Theorem 2 was first proved by Löwenheim (1915) for single sentences and then in general by Skolem (1920), (1922), (1928), in a number of different ways, for example, using Skolem functions. Corollary 2 is due to Skolem (1934), with a quite different proof. The fact that Corollary 2 is an immediate consequence of the (Gödel's) denumerable compactness theorem was overlooked at the time, even by Gödel, and was first pointed out by Henkin (1949). Theorems 3, 4 are from Tarski, Vaught (1957).

Theorem 5 is due to Łoś (1955) and Tarski (1954). Theorem 6 is due to Chang (1959) and Łoś , Suszko (1957); the present proof is due to Robinson (1959). Theorem 7 is due to Lyndon (1959). The proof of Lemma 7 (in Appendix 2) is a special case of a general idea explained in Lindström (1966a). For a related general idea, see Keisler (1965).

Theorem 8 is due to Craig (1957a) with a proof-theoretic proof. Theorem 9 is due to Robinson (1956a). Theorem 10 is due to Beth (1953) with a proof-theoretic proof. Robinson proved Theorem 9 to give a less complicated, model-theoretic, proof of Beth's Theorem. The result referred to at the end of this section is essentially due to Chang (1964) and Makkai (1964) using the Generalized Continuum Hypothesis; a proof not using the GCH is given in Chang, Keisler (1990).

Theorem 11 is due to Łoś (1954) and Vaught (1954). The results that the theories ACF(p) and RCOF are complete (Theorems 12, 16) are due to Tarski (see Tarski, McKinsey (1948)). The completeness of ACF(p) was proved by a model-theoretic argument in Robinson (1951). It was later derived from Theorem 11 by Łoś (1954) and Vaught (1954). Tarski's proofs, by so-called quantifier elimination, are quite long. Robinson (1955), (1956b), in a (successful) attempt to give shorter and more perspicuous (model-theoretic) proofs, introduced the concept *model-complete* (with a different definition; see p. 63). Robinson proved Theorem 13 (the present proof is from Lindström (1964)) and used this result to show that RCOF and a number of other theories, e.g. ACF (Theorem 18) and DTAG (§13, Example 7) are model-complete. This can then be combined with Theorem 15, also due to Robinson (1955), (1956b), to obtain results on completeness. (Note that our terminology in Theorem 15 and later is not the same as in Chang, Keisler (1990).) Later, in some cases, simpler proofs were found. But the proof that RCOF is model-complete (Theorem 14) in Appendix 4 is due to Robinson. Theorem 17 is due to Lindström (1964). Theorem 19 is due to Robinson (1956b). Actually, Tarski's proofs, too, show that ACF(p) and RCOF are model-complete, indeed, that these theories admit elimination of quantifiers; model-theoretic proofs of these stronger results were given by Robinson (1958). Theorem 20 is essentially due to Robinson (1958).

The main ideas and results of §7 are due to Fraïssé (1954) and Ehrenfeucht (1961). ω-isomorphisms are usually called partial isomorphisms. Proposition 7 is a special case of a result of Feferman and Vaught (1959), with a quite different and much more informative proof.

Theorem 22 is due to Grzegorczyk, Mostowski, Ryll-Nardzewski (1961); special cases had earlier been obtained by Henkin (1954), Orey (1956), Svenonius (1959), and others. Lemma 17, Theorem 23, and Corollary 16 are due to Svenonius (1960a) and Vaught (1961). Theorem 24 is due to Engeler (1959), Ryll-Nardzewski (1959), Svenonius (1959), and Vaught (1961). Corollary 18 is due to Svenonius (1960b).

Lemma 21 was first proved by Tarski (1930). Ultraproducts were introduced by Łos' (1955) and Theorem 25 is due to him. A 'restricted' ultrapower was used by

Skolem (1934) in his proof of Corollary 2. Theorem 26 was proved in Frayne, Morel, Scott (1962). The results mentioned at the end of this section were first proved by Keisler (1961) assuming the Generalized Continuum Hypothesis. This assumption was eliminated by Shelah (1972). See also Chang, Keisler (1990).

Theorem 28 is due to Vaught (see Morley, Vaught (1962)); it is the first nontrivial two-cardinal theorem. Theorem 29 is due to Vaught (1965) with a quite different proof; the present proof (in §11) is essentially due to Morley (1965b) (see also Chang, Keisler (1990)).

Theorems 30, 32 are due to Ehrenfeucht, Mostowski (1956). Theorem 31 is due to Ramsey (1930). Theorem 33 is due to Erdös, Rado (1956). The idea of applying the Erdös-Rado Theorem as, for example, in the proof of Theorem 29 is due to Morley (1965b).

The reader may have noticed that the complete theories T mentioned in this chapter are either (i) not categorical in any infinite cardinality or (ii) categorical in $\aleph_0$ but not in κ for any $\kappa > \aleph_0$ or (iii) categorical in all $\kappa > \aleph_0$ but not in $\aleph_0$. There are also (complete) theories categorical in all infinite cardinalities, for example, the trivial theory $\{\exists^{>n}x(x = x): n \in N\}$. Thus, if T is categorical in some $\kappa > \aleph_0$, then T is categorical in all $\kappa > \aleph_0$. This was observed by Łos' (1954), who conjectured that it is true for all countable theories. This conjecture played an important role in model theory in the early 1960s. It was eventually proved by Morley (1965a) (also cf. Chang, Keisler (1990)). The ideas introduced by Morley in this proof have been of decisive importance to the subsequent development of model theory (cf. Chang, Keisler (1990)).

By Morley's theorem, $\aleph_1$ in Theorem 34 can be replaced by any cardinal $\kappa > \aleph_0$.

4. UNDECIDABILITY

Let F be a formula of propositional logic. There is an algorithm (independent of F) which allows us to decide whether or not F is logically valid (a tautology). We just assign truth-values in all possible ways to the variables in F. For each such assignment we compute the corresponding truth-value of F. If this value is "true" for all assignments, F is valid, if it is "false" for some assignment, F is not valid. This is an algorithm, a purely mechanical or effective or routine method, and the computation can be carried out by anyone who understands the instructions, in fact, by a (suitably programmed) computer. In this sense propositional logic is *decidable*. The question arises if L_1 is decidable. This is what is known as the Entscheidungsproblem (decision problem) proposed by David Hilbert as the fundamental problem of mathematical logic. The answer is negative: there is no algorithm by means of which it can be decided for any given formula of L_1 whether or not it is logically valid (or, equivalently, derivable in, say, FH). This will be derived from a result on unsolvability stated in §1.

§1. An unsolvable problem. We need some notation and terminology. Let X be a finite set of symbols, e.g. X = {0,1,h,q} (as in Lemma 1, below). An X-*word* is a finite string of (occurrences of) members of X. uv is the concatenation of the words u and v. It is convenient to assume that there is an *empty* word ø. uø = øu = u. Let X^* be the set of X-words. An n-place function $f(u_1,...,u_n)$ on X^* into Y^* is *computable* (or (*effectively*) *calculable*) if there is an algorithm by means of which $f(u_1,...,u_n)$ can be found (computed) for any given $u_1,...,u_n \in X^*$.

In what follows we shall think of natural numbers as (represented by) strings of 1's: n is the string 1^{n+1} of n+1 1's. $N = \{1\}^* - \{ø\}$.

Suppose $Y \subseteq X^*$. The *characteristic function* of Y is the function f on X^* such that

$$f(u) = 1 \text{ if } u \in Y,$$
$$= 11 \text{ if } u \notin Y.$$

More generally, the *characteristic function* of the relation $R(u_1,...,u_n)$ on X^* is the function f on X^* such that

$$f(u_1,...,u_n) = 1 \text{ if } R(u_1,...,u_n),$$
$$= 11 \text{ if not } R(u_1,...,u_n).$$

Y ($R(u_1,...,u_n)$) is *computable* if its characteristic function is computable. Thus, Y is computable iff there is an algorithm by means of which we can decide for any $u \in X^*$ whether or not $u \in Y$; and similarly for relations.

Computability is usually defined for functions on N (the set of natural numbers) into N and relations on N (subsets of N). The concept is then extended to functions on syntactic expressions etc. via an effective (Gödel) numbering of the latter. In the present context, however, it is more convenient to take the syntactic expressions as the basic objects.

Suppose $Y \subseteq X^*$. Y is *semicomputable* if there are a finite set Z of symbols and a computable relation $R(u,v) \subseteq X^* x Z^*$ such that

$$Y = \{u: \exists v R(u,v)\}.$$

Thus, Y is semicomputable if there is an effective method by means of which it can be shown that $u \in Y$ exactly when this is the case, but which may yield no answer at all when $u \notin Y$. This method is simply to run through, in some systematic way, the members v of Z^* and for each of these to check whether or not $R(u,v)$.

Y is *computably enumerable* (c.e.) if Y is empty or there is a *computable enumeration* of Y, i.e., a computable function f on N such that $Y = \{f(n): n \in N\}$. The terms "semicomputable" and "computably enumerable" can also be applied to relations in the obvious way.

We have not given an exact mathematical definition of "computable function" but certain principles, e.g. (I) – (IV) below, can be proved even without such a definition.

(I) Y is semicomputable iff Y is c.e.

To see this, suppose first Y is c.e. and let f be the computable function enumerating Y. The relation $u = f(n)$ ($\subseteq X^* x N$) is computable and

$$Y = \{u: \exists n(u = f(n)\}.$$

Thus, Y is semicomputable.

Next, suppose Y is semicomputable and let $R(u,v)$ be a computable relation such that

$$Y = \{u: \exists v R(u,v)\}.$$

If $Y = \emptyset$, then Y is c.e. Thus, suppose $Y \neq \emptyset$ and let $w \in Y$. Let $\langle u_0,v_0 \rangle$, $\langle u_1,v_1 \rangle$, $\langle u_2,v_2 \rangle$, ... be a computable enumeration of the set of ordered pairs of members of X^*. Let f be defined by:

$$f(n) = u_n \text{ if } R(u_n,v_n),$$
$$= w \text{ if not } R(u_n,v_n).$$

Then f is computable and $Y = \{u: \exists n(u = f(n)\}$. Thus, Y is c.e.

Clearly, any computable set is semicomputable and if Y is computable so is $X^*–Y$. Also, if Y_0 and Y_1 are (semi)computable, so are $Y_0 \cup Y_1$ and $Y_0 \cap Y_1$.

It is a fundamental observation that:

(II) Y is computable if Y and $X^* – Y$ are semicomputable.

Indeed, suppose

$$Y = \{u: \exists v R(u,v)\},$$
$$X^* – Y = \{u: \exists v R'(u,v)\},$$

where R(u,v) and R'(u,v) are computable relations on $X^*\times Z^*$. Let v_0, v_1, v_2, ... be a computable enumeration of Z^*. Now, given u, for each n, check whether or not $R(u,v_n)$ or $R'(u,v_n)$. Sooner or later, you will find a v_n such that either $R(u,v_n)$ or $R'(u,v_n)$ (but not both). In the former case $u \in Y$, in the latter $u \notin Y$.

It is easily seen that

(III) if f is computable and Y is (semi)computable, then $\{u: f(u) \in Y\}$ is (semi)computable.

Thus, if f is computable, for every $u \in X^*$, $u \in Y$ iff $f(u) \in Z$, and Y is not (semi)computable, then Z is not (semi)computable. This fact will be used in the proofs of Theorems 3 – 6 below.

Let $|u|$ be the length of u, i.e., the number of occurrences of symbols in u. It is then clear that

(IV) if $f: X^* \rightarrow N$ and R(u,v) are computable, then $\{u: \exists v(|v| \leq f(u) \ \& \ R(u,v)\}$ is computable.

It is important to realize that a set Y may be semicomputable (c.e) but not computable, in which case, by (II), $X^* - Y$ is not semicomputable (c.e.). This follows from:

Theorem 1 (Kleene's Enumeration Theorem)**.** There are semicomputable subsets W_k, $k \in N$, of N such that the two-place relation $n \in W_k$ is semicomputable and for every semicomputable set $X \subseteq N$, there is a number e for which $X = W_e$; in other words, W_0, W_1, W_2, ... is an enumeration (with repetitions) of the semicomputable (c.e.) subsets of N.

Corollary 1. Let $K = \{n: n \in W_n\}$. Then K is a semicomputable but not computable subset of N.

Proof. Clearly, K is semicomputable. Suppose K is computable. Then N – K is (semi)computable. Hence, by Theorem 1, there is a number e such that $N - K = W_e$. But then $e \in N - K$ iff $e \in W_e$ iff $e \in K$; a contradiction. ■

A *production* (over X) is an expression of the form

(*) $\alpha u \beta \rightarrow \alpha v \beta$.

Here α, β are variables and u, v are fixed X-words and $u \neq \emptyset$. (Productions may be thought of as rules of derivation.) The expression $w_0 \rightarrow w_1$ is an *instance* of (*) if there are words w, w' (including the empty word) such that $w_0 = wuw'$ and $w_1 = wvw'$. A *combinatorial system* C (over X) is determined by a finite set of productions (over X) and an *initial word* (axiom) $w^i \in X^*$. A *derivation* in C is a sequence $w_0,...,w_m$ of X-words such that $w_0 = w^i$ and for $k < m$, $w_k \rightarrow w_{k+1}$ is an instance of a production of C. A word w is *derivable* in C, $C \vdash w$, if there is a derivation $w_0,\dots,w_m$ in C such that $w_m = w$.

For every C, the set $D = \{u: C \vdash u\}$ is semicomputable (but not necessarily computable). To see this, let # be a symbol not in X. The set $D^{\#}$ of words $w_0\#w_1\#\dots\#w_m$ such that $w_0,w_1,\dots,w_m$ is a derivation in C is clearly computable. Thus, the relation R such that

$$R(u,v) \text{ iff } u \in X^* \;\&\; v \in D^{\#} \;\&\; \exists w(v = w\#u)$$

is computable. (Given u and v it can be checked by means of algorithm whether or not $\exists w(v = w\#u)$.) But $D = \{u: \exists vR(u,v)\}$ and so D is semicomputable.

We borrow the following result from computability theory.

Lemma 1. There is a finite set of productions

(P) $\alpha u_i\beta \to \alpha v_i\beta, \quad i \leq p,$

over {0,1,h,q} such that the following holds. For every n, let C_n be the combinatorial system with these productions and whose initial word is $hq1q1^{n+1}h$. Then

(i) the set $\{n: C_n \vdash 0\}$ is not computable,

(ii) $\{w: C_n \vdash w\}$ is finite iff $C_n \vdash 0$.

Thus, the problem of finding an algorithm by means of which it can be decided for every n whether or not $C_n \vdash 0$ is unsolvable; there is no such algorithm. The set $\{n: C_n \vdash 0\}$ is semicomputable. And so (i) and (II) imply that $\{n: C_n \nvdash 0\}$ is not semicomputable.

Lemma 1 is proved by showing that for every semicomputable set $Y \subseteq N$, there is a set of productions (P) such that $Y = \{n: C_n \vdash 0\}$ (and (ii) holds) and then applying Corollary 1. The reason why (ii) is true is that for every n, there is sequence σ_n of words such that (a) the derivations in C_n are exactly the initial segments of σ_n, and so the words derivable in C_n are exactly the members of σ_n, (b) if $C_n \vdash 0$, then 0 is the last member of σ_n, and so σ_n is finite, and, finally, (c) if $C_n \nvdash 0$, then σ_n is infinite.

Some of the details of Lemma 1 are not essential in the present context. For example, the exact shape of the initial word is not important. We could also have allowed "productions" with just one or more than two variables. (ii) is not relevant in the proof of Theorem 3 but it is essential in the proof of Theorem 4.

§2. Undecidability of L_1. Let ℓ be a finite language. Since we can take the individual variables to be, say, x, x', x'', ..., formulas of ℓ are X-words for a certain finite set X of symbols. The relation "d is a derivation of φ in FH", where φ is a sentence of ℓ is clearly computable. But then, in view of Corollary 2.1, we have:

Theorem 2. The set of valid first-order sentences of any finite language is semicomputable.

L_1 is *undecidable* if there is a finite language $\boldsymbol{\ell}$ such that the set of valid sentences of $\boldsymbol{\ell}$ is not computable. We shall now use Lemma 1(i) to prove the following:

Theorem 3 (Church, Turing)**.** L_1 is undecidable.

Proof. Let $\boldsymbol{\ell} = \{F, \circ, ø, 0, 1, h, q\}$, where F is a one-place predicate, $\circ$ is a two-place function symbol, and ø, 0, 1, h, q are individual constants. (The interpretation of $\circ$ we have in mind is concatenation). For $w := s_0 s_1 \dots s_n$, where $s_0, s_1, \dots, s_n$ are single symbols, let $[w] = (s_0 \circ (s_1 \circ (s_2 \circ \dots \circ s_n) \dots)$; $[s_i] := s_i$ and $[ø] := ø$.

Let the productions (P) be as in Lemma 1. For $i \leq p$, let

$$\pi_i(x,y) := \exists zz'(x = (z \circ [u_i]) \circ z' \wedge y = (z \circ [v_i]) \circ z'),$$

$$\pi(x,y) := \bigvee\{\pi_i(x,y) \colon i \leq p\}.$$

The intuitive meaning of $\pi(x,y)$ is then: "$x \to y$ is an instance of a production (P)".

Now, let θ be the conjunction of the following sentences:

(1) $\forall xyz((x \circ y) \circ z = x \circ (y \circ z))$,

(2) $\forall x(x \circ ø = ø \circ x = x)$,

(3) $\forall xy(Fx \wedge \pi(x,y) \to Fy)$.

Intuitively, (3) says that "the set F is closed under the productions (P)".

All models appearing in this proof are models of (1). We may therefore drop parentheses in terms.

Let $t(x) := [hq1q] \circ x \circ h$. Let $\chi(x) :=$

$$\theta \wedge Ft(x) \to F0.$$

Finally, let $\underline{n} = [1^{n+1}]$.

We are going to show that for every n,

(*) $\models \chi(\underline{n})$ iff $C_n \vdash 0$.

First, suppose $C_n \vdash 0$. Let $\mathcal{A}$ be any model for $\boldsymbol{\ell}$ such that

$$\mathcal{A} \models \theta \wedge Ft(\underline{n}).$$

Let $w_0, w_1, \dots, w_m$ be a derivation of 0 in C_n. $w_0 = hq1q1^{n+1}h$ and $w_m = 0$. Thus, $\mathcal{A} \models t(\underline{n}) = [w_0]$. It follows that $\mathcal{A} \models F[w_0]$.

Suppose $k < m$ and $\mathcal{A} \models F[w_k]$. There are w, w' and $i \leq p$ such that $w_k = wu_iw'$ and $w_{k+1} := wv_iw'$. But then, by (1),

$$\mathcal{A} \models [w_k] = [w] \circ [u_i] \circ [w'],$$

$$\mathcal{A} \models [w_{k+1}] = [w] \circ [v_i] \circ [w'].$$

It follows that $\mathcal{A} \models \pi_i([w_k], [w_{k+1}])$ and so $\mathcal{A} \models F[w_{k+1}]$.

We may now conclude that $\mathcal{A} \vDash F[w_k]$ for all $k \leq m$ and so, in particular, $\mathcal{A} \vDash F[w_m]$, in other words, $\mathcal{A} \vDash F0$. Thus, we have shown that $\mathcal{A} \vDash \chi(\underline{n})$. Since $\mathcal{A}$ was any model for ℓ, it now follows that $\vDash \chi(\underline{n})$, as desired.

Next, suppose $C_n \nvdash 0$. Let

$$\mathcal{A} = (A, F^{\mathcal{A}}, o^{\mathcal{A}}, ø, 0, 1, h, q),$$

where $A = \{0,1,h,q\}^*$, $F^{\mathcal{A}} = \{w: C_n \vdash w\}$ and $o^{\mathcal{A}}$ is concatenation. Then $\mathcal{A} \vDash \theta \wedge Ft(\underline{n})$ and $\mathcal{A} \nvDash F0$, whence $\mathcal{A} \nvDash \chi(\underline{n})$. And so $\nvDash \chi(\underline{n})$. This proves (*).

Finally, by (*), (III), and Lemma 1(i), L_1 is undecidable, as desired. ■

Corollary 2. There is a finite set ℓ of predicates such that the set of =-free valid sentences of ℓ (in prenex normal form) is not computable.

Proof. By Theorem 3 and Proposition 1.4(a), there is a finite set ℓ' of predicates such that the set of valid sentences of ℓ' is not computable. Let ψ be any sentence of ℓ'. Let θ be the conjunction of the identity axioms I1, I2 I3 of FH and those axioms I4 in which $\varphi(x_1,...,x_n)$ is a subformula of ψ. Next, let χ be the result of replacing = in $\theta \rightarrow \psi$ by a new two-place predicate E. Finally, write χ in prenex normal form. The resulting sentence is then computable from ψ and valid iff ψ is valid. Thus, $\ell = \ell' \cup \{E\}$ is as desired. ■

In fact, Corollary 2 holds with $\ell = \{P\}$, where P is a two-place predicate.

We can now answer the questions raised in Chapter 2, end of §2. For each of the calculi FH, GS, ND the relation "d is a derivation of" φ is a computable relation. Let f be any function on N into N. Suppose every sentence φ derivable in FH etc. has a derivation d such that $|d| \leq f(|\varphi|)$. Then, by the completeness theorems for FH etc., φ is valid iff there is a derivation d of φ such that $|d| \leq f(|\varphi|)$. If f is computable, by (IV), this implies that the set of valid sentences is computable, contradicting Corollary 2. Thus, f cannot be a computable function. A similar conclusion is true of the Skolem-Herbrand method. It also follows that there is no computable function f such that $f(|\varphi|)$ is an upper bound of the cardinality of the set Δ, where φ, Δ are as in Corollary 2.4.

Our next result will be needed in Chapter 5. A sentence φ is *finitely valid* if it is true in all finite models. Note that the set of not finitely valid sentences of any given finite language is semicomputable.

Theorem 4 (Trakhtenbrot)**.** There is a finite language ℓ such that the set of finitely valid sentences of ℓ is not semicomputable.

This will follow if we can show that there is a formula $\chi'(x)$ such that

(**) $\chi'(\underline{n})$ has a finite model iff $C_n \vdash 0$.

Let θ, $t(x)$ be as above. One reason we cannot simply take $\chi'(x)$ to be $\theta \wedge Ft(x)$ is that, even if $C_n \nvdash 0$, $\theta \wedge Ft(\underline{n})$ may still have a finite model. Moreover, if we modify θ, replacing it by some sentence θ' so that this cannot happen, we have to make sure that if $C_n \vdash 0$, then $\theta' \wedge Ft(\underline{n})$ has a finite model.

Proof of Theorem 4. Let $\ell = \{F, \circ, \varnothing, 0, 1, h, q, e\}$, where $F, \circ, \varnothing, 0, 1, h, q$ are as above and e is an individual constant. Let $\sigma(x) :=$

$$x = 0 \vee x = 1 \vee x = h \vee x = q.$$

Let θ' be the conjunction of the following sentences, where $\pi(y,z)$ is as in the proof of Theorem 3:

(1) $\forall xyz((x \circ y) \circ z = x \circ (y \circ z))$,

(2) $\forall x(x \circ \varnothing = \varnothing \circ x = x)$,

(3) $\forall x(x \circ y = \varnothing \rightarrow x = \varnothing)$,

(4) $\forall x(x \circ e = e)$,

(5) $\forall xyz(x \circ y = x \circ z \neq e \rightarrow y = z)$,

(6) $\forall xyzu(\sigma(x) \wedge \sigma(y) \wedge x \circ z = y \circ u \neq e \rightarrow x = y)$,

(7) $\neg Fe$ and "$\varnothing$, 0, 1, h, q, e are all different",

(8) $\forall xy(Fx \wedge \pi(x,y) \rightarrow Fy)$.

As before we may drop parentheses in terms. Again let $t(x) := [hq1q] \circ x \circ h$. Let $\chi'(x) :=$

$$\theta' \wedge Ft(x).$$

We now prove (**). First, suppose $C_n \nvdash 0$. Let $\mathcal{A}$ be a model of $\chi'(\underline{n})$. By Lemma 1(ii), $\{w: C_n \vdash w\}$ is infinite. As in the proof of Theorem 3 we have $\mathcal{A} \models F[w]$ for every w such that $C_n \vdash w$. Hence, by (7), $\mathcal{A} \models [w] \neq e$ for all such w. But then, by (3) – (7), $\mathcal{A} \models [u] \neq [v]$ whenever $C_n \vdash u$, $C_n \vdash v$, and $u \neq v$. It follows that $F^{\mathcal{A}}$ is infinite, and so $\mathcal{A}$ is infinite, as desired.

Next suppose $C_n \vdash 0$. By Lemma 1(ii), $\{w: C_n \vdash w\}$ is finite. For every word w, let $|w|$ be the length of w; $|\varnothing| = 0$.Let $r = \max\{|w| : C_n \vdash w\}$. Let $A' = \{w \in \{0,1,h,q\}^*: |w| \leq r\}$.

Let $A = A' \cup \{e\}$. Let $F^{\mathcal{A}} = \{w: C_n \vdash w\}$. Finally, for $u, v \in A$, let

$$u \circ^{\mathcal{A}} v = uv \text{ if } uv \in A',$$
$$= e \text{ otherwise.}$$

We now claim that

(9) $\mathcal{A} = (A, F^{\mathcal{A}}, \circ^{\mathcal{A}}, \varnothing, 0, 1, h, q, e)$ is a (finite) model of $\chi'(\underline{n})$.

For example, consider (1). Suppose $u, v, w \in A'$. If $u, v, w \in A'$ and $|uvw| \leq r$, then $(u \circ^{\mathcal{A}} v) \circ^{\mathcal{A}} w = u \circ^{\mathcal{A}} (v \circ^{\mathcal{A}} w) = uvw$; if, on the other hand, $u, v, w \in A'$ and $|uvw| > r$ or one of u, v, w is e, then $u \circ^{\mathcal{A}} (v \circ^{\mathcal{A}} w) = u \circ^{\mathcal{A}} (v \circ^{\mathcal{A}} w) = e$. Thus, (1) is true in $\mathcal{A}$.

The verification of (2) – (7) in $\mathcal{A}$ is straightforward.

To see that (8) holds in $\mathcal{A}$, let u, $v \in A$ be such that $\mathcal{A} \vDash Fu \wedge \pi(u,v)$. Then $C_n \vdash u$. Let i be such that $\mathcal{A} \vDash \pi_i(u,v)$. There are then uniquely determined w, w' such that $u = wu_iw'$. It follows that $C_n \vdash wv_iw'$ and $v = wv_iw'$. But then $C_n \vdash v$ and so $\mathcal{A} \vDash Fv$, as desired. Thus, (8) holds in $\mathcal{A}$.

Finally, let $w_n = hq1q1^{n+1}h$, the initial word of C_n. Then $\mathcal{A} \vDash Fw_n$ and $\mathcal{A} \vDash w_n = t(\underline{n})$, whence $\mathcal{A} \vDash Ft(\underline{n})$. It follows that $\mathcal{A} \vDash \chi'(\underline{n})$. And so (9) is proved.

This proves (**). By (**), for every n, $\neg\chi'(\underline{n})$ is finitely valid iff $C_n \nvdash 0$. It follows that the set of finitely valid sentences of ℓ is not semicomputable, as desired. ■

Corollary 3. There is a finite set ℓ of predicates such that the set of finitely valid sentences of ℓ is not semicomputable.

In fact, this holds with $\ell = \{P\}$, where P is a two-place predicate.

§3. The Incompleteness Theorem. We can now give a short proof of (a version of) Gödel's first incompleteness theorem.

A theory T is (*computably*) *axiomatizable* if there is a computable theory, an *axiomatization* of T, equivalent to T.

Theorem 5 (Gödel)**.** Th($\mathcal{N}$) is not axiomatizable.

Thus, the natural and *prima facie* entirely reasonable project of axiomatizing first-order arithmetic (together with the logic used in proofs) turns out to be impossible: every axiomatizaion of a subtheory T of Th($\mathcal{N}$) is *incomplete* in the sense that there are sentences φ of $\ell_{\mathcal{N}}$ which are *undecidable* in T, i.e., such that neither $T \vdash \varphi$ nor $T \vdash \neg\varphi$.

The reason we require the axiomatization T to be computable is that we want to be able to recognize a proof in T when we see one (without solving any additional mathematical problems). It is, however, not required in Theorem 5 that we be able to see that the axioms of T are true.

Suppose T is computable. Then the relation "p is a proof of φ in T" is computable. It follows that Th(T), the set of *theorems* of T, is semicomputable. (In fact, if T is semicomputable, so is Th(T).) Thus, to prove Theorem 5, it is sufficient to show that Th($\mathcal{N}$) is not semicomputable.

The following number-theoretic lemma is proved in Appendix 7. We write km for k•m and xy for x•y.

Lemma 2. For all m, n, and k_i, $i \leq n$, there are r, s such that for all $i \leq n$ and all $k \leq m$, $k = k_i$ iff $\exists q{\leq}r(r = q(1+(i+1)s) + k)$.

Lemma 3. Let R be any two-place relation on $\{k: k \leq m\}$. There are then numbers n, r, s, r', s' such that

$$R = \{\langle k,k'\rangle: k, k' \leq m \;\&\; \exists i{\leq}n(\exists q{\leq}r(r = q(1+(i+1)s + k) \;\&\; \exists q'{\leq}r'(r' = q'(1+(i+1)s' + k'))\}.$$

Proof. Let n, k_i, k_i' be such that $R = \{\langle k_i,k_i'\rangle: i \leq n\}$. Let r, s be as in Lemma 2 and let r', s' be as in that lemma with k_i replaced by k_i'. ■

Clearly, there is a result similar to Lemma 3 for n-place relations for every n.

Of course, the usual ordering ≤ of N can be defined in terms of +. But, for simplicity we now add the two-place predicate ≤ to the language of arithmetic. Let $\mathcal{N}' = (\mathcal{N}, \leq)$. Clearly, Th($\mathcal{N}$) is axiomatizable iff Th($\mathcal{N}'$) is.

In what follows let $\exists x{\leq}y\varphi := \exists x(x \leq y \wedge \varphi)$ and $\forall x{\leq}y\varphi := \forall x(x \leq y \rightarrow \varphi)$.

Lemma 4. For any sentence φ containing no function symbols or constants, we can construct an arithmetical sentence φ^* such that $\mathcal{N}' \vDash \varphi^*$ iff φ is finitely valid.

Proof. We explicitly deal only with the case where φ contains only one two-place predicate P; the extension to the general case is straightforward. We (may) assume that the variables y, y', z, z', u, u', v, v', w do not occur in φ. Let φ'(u) be the result of relativizing all quantifiers in φ to ". ≤ u", i.e., replacing ∃x by ∃x≤u and ∀x by ∀x≤u. Let $\rho(y,z,y',z',u',x,x') :=$

$$\exists w{\leq}u'(\exists v{\leq}y(y = v(1+(w+1)z + x) \;\&\; \exists v'{\leq}y'(y' = v'(1+(w+1)z' + x'))$$

(compare Lemma 3).

Next, replace Pxx', for any variables x, x', everywhere in φ'(u) by $\rho(y,z,y',z',u',x,x')$. Let $\varphi''(y,z,y',z',u,u')$ be the result.

Now let m be any number and let R be any two-place relation on $\{k: k \leq m\}$. Let n, r, s, r', s' be as in Lemma 3. Then

$$R = \{\langle k,k'\rangle: k, k' \leq m \;\&\; \mathcal{N}' \vDash \rho(r,s,r',s',m,k,k')\}.$$

It follows that

$$(\{k: k \leq m\}, R) \vDash \varphi \text{ iff } \mathcal{N}' \vDash \varphi''(r,s,r',s',m,n).$$

Finally, let $\varphi^* :=$

$$\forall yy'zz'uu'\varphi''(y,z,y',z',u,u').$$

Then, by Lemma 3, if $\mathcal{N}' \vDash \varphi^*$, then φ is finitely valid. The converse implication follows from the fact that all quantifiers of φ'(u) are relativized to ". ≤ u". ■

Proof of Theorem 5. Clearly, the function mapping φ on φ^* as in Lemma 4 is computable. Suppose Th($\mathcal{N}$) is semicomputable. Then so is Th($\mathcal{N}'$). But then, by Lemma 4 and (III), the set of finitely valid sentences would be semicomputable, contradicting Corollary 3. It follows that Th($\mathcal{N}$) is not axiomatizable. ■

An arithmetical formula φ is *bounded* if every quantifier expression in φ is bounded, i.e., of the form $\forall x \leq y$ or $\exists x \leq y$, where x, y are any variables. φ is *essentially universal* (*e.u.*) if it is of the form $\forall x_1 \ldots x_n \psi$, where ψ is bounded.

Every e.u. formula is equivalent (in Th($\mathcal{N}'$)) to an e.u. formula of the form $\forall x \psi$, where ψ is bounded. For suppose x is a variable not in ψ. Then $\forall x_1 \ldots x_n \psi$ is equivalent to $\forall x \forall x_1 \leq x \ldots \forall x_n \leq x \psi$.

The formula φ'' defined in the proof of Lemma 4 is bounded and so φ^* is e.u. Thus, (the proof of) Theorem 5 has the following:

Corollary 4. The set of e.u. sentences true in $\mathcal{N}'$ is not semicomputable.

On the other hand, every false e.u. sentence is (rather trivially) disprovable in Q. Thus, if T is a consistent axiomatizable extension of Q, then every e.u. sentence provable in T is true (even if T is not true). And so, by Corollary 4, it follows that:

Corollary 5. If T is a consistent axiomatizable extension of Q, there is a true e.u. sentence not provable in T.

Let HF be the set of hereditarily finite sets, i.e., finite sets whose members are finite, whose members of members are finite, etc. Let $\mathcal{HF} = (\mathrm{HF}, \in)$.

On the present approach the following result, essentially equivalent to Theorem 5, is particularly easy to prove.

Theorem 6. Th($\mathcal{HF}$) is not axiomatizable.

Proof. Let φ be any sentence as in the proof of Theorem 5. Suppose the variables u, v do not occur in φ. Let $\varphi'(u)$ be the result of relativizing all quantifiers in φ to "$. \in u$". As usual let $\langle x,y \rangle = \{\{x\},\{x,y\}\}$. Let $\varphi''(u,v)$ be obtained from $\varphi'(u)$ by replacing Pxy by $\langle x,y \rangle \in v$. Let $\varphi^* := \forall uv\varphi''(u,v)$. It is then clear that $\mathcal{HF} \models \varphi^*$ iff φ is finitely valid. And so Th($\mathcal{HF}$) is not axiomatizable. ■

§4. Completeness and decidability. One reason why it may be interesting to know that a given theory T is complete is as follows.

A theory T is *decidable* if Th(T) is computable.

Proposition 1. If T is axiomatizable and complete, then T is decidable.

Proof. Th(T) is c.e. Let $\psi_0, \psi_1, \psi_2, \ldots$ be a computable enumeration of Th(T). Let φ be any sentence of ℓ_T. Since T is complete, in the enumeration $\psi_0, \psi_1, \psi_2, \ldots$ we will, sooner or later, either come accross φ or $\neg\varphi$. In the former case $T \vdash_{FH} \varphi$, in the latter case $T \nvdash_{FH} \varphi$. ■

The theories DeLO, DiLO, SF, IAtBA, NoAtBA, DTAG, ACF(p), RCOF are complete and axiomatizable (Chapter 3, Theorems 13, 17 and §12). Thus, all these theories are decidable. It also follows that another way of proving Gödel's incompleteness theorem would be to show that no consistent extension of, say, Q is decidable. This can be done, but then we have to translate computability theory into arithmetic.

§5. The Church-Turing Thesis. "Positive" results such as (I) – (IV) can often, but not always, be established on the basis of an intuitive pre-mathematical concept of computabilty. But in the proofs of Theorem 1 and "negative" results such as Corollary 1 and Lemma 1 (and Theorems 3 – 6) you need a mathematical characterization (or analysis or definition or explication) of computability. In the 1930's a number of such characterizations were put forward – all of them provably equivalent – by Church (1936), Turing (1936) (independently), and others. The most convincing of these, perhaps the only really convincing characterization, is the one proposed by Turing in terms of (what is now known as) Turing Machines. On the basis of his analysis Turing proposed the following thesis – Church is mentioned here because he was the first to suggest (what turned out to be) an equivalent thesis:

> *The Church-Turing Thesis*: Every intuitively calculable function is *Turing computable*, i.e., computable by a Turing Machine.

The converse of this is clearly true.

The above arguments remain valid if the (informal) concept *computable* is replaced by that of *Turing-computability*. And so "computable" may be understood as an abbreviation of "Turing computable".

(Turing) computable functions and relations (sets) are, for historical reasons, usually, but inadequately, called "recursive" functions and relations (sets) and semicomputable (c.e.) relations (sets) "recursively enumerable (r.e.) relations (sets)" (whence the term "recursion theory").

Notes for Chapter 4. Theorem 1 is a fundamental result of Kleene (see Kleene (1943), (1952), Davis (1958), Soare (1978)); it is equivalent to the existence of a so-called *universal Turing Machine* proved in Turing (1936). Corollary 1, with different examples, is due to Church (1936a) and Turing (1936). Lemma 1 is essentially due to Post (1947) (cf. also Kleene (1952) and Davis (1958)).

Theorem 3 is due to Church (1936b) and Turing (1936). Another way of proving Theorem 3 is to show that Q is undecidable (cf. e.g. Tarski, Mostowski, Robinson (1953)). Theorem 4 is due to Trakhtenbrot (1950).

Theorem 5 is, of course, essentially due to Gödel (1931), but more recent formulations, including the present one, presuppose the Church-Turing thesis, which was not known to Gödel at the time. However, Gödel's original result covers all theories, including PA and ZFC, that are at all likely to be used in mathematical proofs. The present (simple) proof of Theorem 5 is different from the proofs in the literature, but also, in various respects, less informative.

Corollaries 4, 5 can be improved. The set of true universal sentences of the form $\forall x_1 \ldots x_n(P_1(x_1,\ldots,x_n) \neq P_1(x_1,\ldots,x_n))$, where $P_i(x_1,\ldots,x_n)$, $i = 1, 2$, are polynomials with positive integral coefficients, is not semicomputable (cf. Davis (1973)). It follows that if T is a consistent axiomatizable extension of Q, there is a true sentence of this form not provable in T.

5. CHARACTERIZATIONS OF FIRST-ORDER LOGIC

Let K be a class of models. If $K \notin EC$, this can, as we have seen, often be proved by showing that the assumption that $K \in EC$ is incompatible with some basic theorem, or combination of theorems, about L_1. For example, suppose there are $\mathcal{A} \in K_0$ such that $P^{\mathcal{A}}$ is any finite set but none such that $P^{\mathcal{A}}$ is infinite. Then the assumption that $K_0 \in EC$ contradicts the Compactness Theorem. Similarly, if K_1 has infinite members but no denumerable member, $K_1 \notin EC$ follows from the Löwenheim (-Skolem) Theorem. (But this theorem may not rule out that $K_0 \in EC$ and the Compactness Theorem may not rule out that $K_1 \in EC$.) The Upward LST Theorem (Theorem 3.4) and Theorem 4.2, too, can be used in this way. For example, from the latter result (together with Theorem 4.4) it follows that $K_0 \notin EC$. Thus, the question arises if the fact, assuming it is one, that $K \notin EC$ always follows in this way from what we already know about L_1 or if we need some, as yet unknown, stronger theorem. In this Chapter we prove that there is no such stronger theorem. To formulate (and prove) this result we need a sufficiently general concept (*abstract*) *logic*. Such a concept is defined in §1.

To avoid certain purely notational complications, we shall in what follows restrict our discussion to languages ℓ containing no function symbols or individual constants. These can be eliminated and then reintroduced in the same way as in L_1 (see Chapter 1, §4).

§1. Extensions of L_1. There are several ways of constructing extensions of L_1. We may, for example, introduce second-order variables and allow universal and existential quantification with respect to these variables. The result is second-order logic, L_2. Weak second-oder logic, wL_2, is obtained by interpreting the second order variables as ranging over finite sets (and relations). (Thus, we distinguish between predicates and predicate variables.) Another way is to allow disjunctions and conjunctions of certain infinite sets of formulas and, possibly, universal and existential quantification over certain infinite sets of variables. A third possibility, and the only one to be discussed here in any detail – the reasons for this will become clear – is to add one or more so-called generalized quantifiers.

Each of these "logics" can be thought of as a pair $L = (\Phi_L, \models_L)$, where Φ_L is the set of "formulas" of L and $\models_L$ is the "satisfaction relation" of L. But, in general, to count as a logic, L will have to satisfy certain additional conditions. One very basic such condition is the following: If φ is a sentence of L and $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \models_L \varphi$ iff $\mathcal{B} \models_L \varphi$. (It will be clear or assumed that all logics considered in what follows satisfy this condition.) It is also, in the present context, natural to require that L have the "finite occurrence property", i.e., informally, that no formula of L contain infinitely many nonlogical constants. We shall also require that L be closed under such opperations as negation, conjunction, and universal and existential quantification, in the obvious sense. Etc.

In this way, by listing a number of natural (but numerous and somewhat awkward) requirements, one can define a workable concept (*abstract*) *logic*. A somewhat different approach – and the one we shall follow here – is to define a family of more "concrete" logics and then argue that that family is general enough for our present purposes.

Let a *signature* σ be a finite sequence $\langle k_0,\ldots,k_n\rangle$ of positive integers. By a *relational system* of signature σ we understand a sequence $(A, R_0,\ldots,R_n)$ such that A is a nonempty set and for each $i \leq n$, R_i is a k_i-place relation on A. (Thus a relational system is a (relational) model except that no language is involved.) A (*generalized*) *quantifier* of signature σ is a class Q of relational systems of signature σ closed under isomorphisms (in the obvious sense).

For example, suppose $\boldsymbol{\ell} = \{P_0,P_1\}$, where P_0 is a one-place and P_1 a two-place predicate. Let K be a class of models for $\boldsymbol{\ell}$ closed under isomorphisms. Let Q =

$$\{(A, R_0,R_1)\colon (A, \{\langle P_0,R_0\rangle, \langle P_1,R_1\rangle\}) \in K\}.$$

Then Q is a quantifier of signature $\langle 1,2\rangle$.

The logic $L_1(Q)$ can now be defined as follows. To the inductive definition of formula of L_1 we add the following clause: If $\varphi(x)$, $\psi(y,z)$ are *formulas of* $L_1(Q)$, which may contain free variables other than those displayed, then

$$Qx;yz(\varphi(x);\ \psi(y,z))$$

is a *formula of* $L_1(Q)$.

If $\varphi(x)$, $\psi(y,z)$ are formulas of $L_1(Q)$ and $\boldsymbol{u}$ is a valuation in $\mathcal{A}$, let

$$\varphi^{\mathcal{A},x}[\boldsymbol{u}] = \{a \in A\colon \mathcal{A} \models \varphi[\boldsymbol{u}(x/a)]\},$$

$$\psi^{\mathcal{A},y,z}[\boldsymbol{u}] = \{\langle a,b\rangle \in A^2\colon \mathcal{A} \models \psi[\boldsymbol{u}(y/a,z/b)]\}.$$

The definition of "$\boldsymbol{u}$ *satisfies* φ in $\mathcal{A}$", in symbols, $\mathcal{A} \models \varphi[\boldsymbol{u}]$, for $L_1(Q)$ is then obtained from that of $\models$ for L_1 by adding the following clause:

$$\mathcal{A} \models Qx;y,z(\varphi(x);\ \psi(y,z))[\boldsymbol{u}] \text{ iff } (A, \varphi^{\mathcal{A},x}[\boldsymbol{u}], \psi^{\mathcal{A},y,z}[\boldsymbol{u}]) \in Q.$$

Thus, we use the same symbol as a formal symbol of $L_1(Q)$ and to denote the quantifier in question.

It can now be seen that if $\mathcal{A}$ is any model for $\boldsymbol{\ell}$, then

$$\mathcal{A} \models Qx;yz(P_0x;P_1yz) \text{ iff } \mathcal{A} \in K.$$

And so K can be characterized in $L_1(Q)$. In fact, $L_1(Q)$ would seem to be the "weakest" reasonably natural logic in which K can be characterized.

This generalizes in an obvious way to quantifiers of any signature. Of course, we can also add several quantifiers to L_1 and extend the concept of a formula and the relation $\models$ in the obvious way.

On this analysis, $\forall$ and $\exists$ are quantifiers of signature $\langle 1 \rangle$:

$\forall = \{(A, X): X = A\}$,

$\exists = \{(A, X): \varnothing \neq X \subseteq A\}$.

With this definition there is, of course, an inexhaustible supply of quantifiers (not definable in L_1). A number of these have been discussed in the literature, for example:

$\mathsf{Q}_\alpha = \{(A, X): X \subseteq A \;\&\; |X| \geq \aleph_\alpha\}$, α any ordinal,

$\mathsf{F} = \{(A, X, Y): X \subseteq A \;\&\; Y \subseteq A \;\&\; |X| < |Y|\}$,

$\mathsf{W} = \{(A, R): R$ is a well-ordering of $A\}$,

$\mathsf{Cof}_\omega = \{(A, R): R$ is a linear ordering of its domain of cofinality $\omega\}$.

This list is far from complete but it is sufficient in the present context.

A *quantifier logic* is a logic of the form $L_1(q)$, where q is a family of quantifiers and $L_1(q)$ has been obtained from L_1 by adding the members of q.

Let L, L' be any (quantifier) logics. K is an L-*class* if there is a sentence φ of L such that $K = \{\mathcal{A}: \ell_{\mathcal{A}} = \ell_\varphi \;\&\; \mathcal{A} \models_L \varphi\}$. L is *included in* L' (or a *sublogic of* L') and L' an *extension of* L, in symbols $L \leq L'$, if every L-class is an L'-class. $L \simeq L'$, L is *equivalent* to L', if $L \leq L' \leq L$.

It is now not difficult to see that every "natural" extension L of L_1 with the finite occurrence property, for example, those mentioned in the first paragraph of this section, is equivalent (in an often rather uninteresting way) to a quantifier logic: Let φ be any sentence of L with, say, $\ell_\varphi = \{P_0, P_1\}$, where P_0 is a one-place and P_1 a two-place predicate. Let

$\mathsf{Q}_\varphi = \{(A, R_0, R_1): (A, \{\langle P_0, R_0\rangle, \langle P_1, R_1\rangle\}) \models \varphi\}$.

Then

$\mathcal{A} \models \mathsf{Q}_\varphi x;yz(P_0x;P_1yz)$ iff $\mathcal{A} \models_L \varphi$.

Let L' be obtained from L_1 by adding in this way a new quantifier Q_φ for every sentence φ of L. Then $L \leq L'$. Also, if all formulas of L_1 are formulas of L and L is closed under "replacing atomic formulas by arbitrary formulas", then $L' \leq L$ and so $L \simeq L'$. In this sense, the family of quantifier logics is sufficiently general for our present purposes. (But, of course, in replacing L by L' we may loose sight of syntactic properties of the formulas of L that may be essential in the study of L.) Thus, by an *abstract logic* we shall simply understand a quantifier logic. (But it will be clear from the presentation in §3 exactly what we need to assume about L for the various proofs to go through.) For example, L_2 and wL_2 are (equivalent to) abstract logics (in this sense).

§2. Properties of logics. In view of what has been shown in the preceding chapters it is natural to consider *inter alia* the following properties of logics:

The *Löwenheim property*. L has the Löwenheim property if every sentence of L, which has an infinite model, has a denumerable model.

The *Löwenheim-Skolem property*. L has the Löwenheim-Skolem property if every countable set of sentences of L, which has an infinite model, has a denumerable model.

The *Tarski property*. L has the Tarski property if every sentence of L, which has a denumerable model, has an uncountable model.

Compactness. L is compact if for every set Φ of sentences of L, if every finite subset of Φ has a model, then Φ has a model.

κ-compactness. L is κ-compact if for every set Φ of sentences of L, if $|\Phi| \leq \kappa$ and every finite subset of Φ has a model, then Φ has a model.

Completeness. L is complete (in an abstract sense) if for every finite language ℓ, the set of valid sentences of ℓ of L is semicomputable.

L_1 has all the above properties. These properties are *preserved under* ≤ in the sense that if $L \leq L'$ and L' has the property, so does L. It follows from this that in the results in §3 it is sufficient to consider the case where q is a single quantifier.

There are many more quite natural model-theoretic properties of logics, some of which are not preserved under ≤, that turn out to be relevant and interesting. For example, for every model-theoretic result on L_1 that can meaningfully (even if not correctly) be transferred to any logic, such as the Robinson Consistency Theorem, the Interpolation Theorem, and the Beth Definability Theorem, there is a corresponding property of logics. And there are numerous variations of the ones we have defined, but these are sufficient for our present purposes.

$L_1(Q_0)$ has the Löwenheim-Skolem property. (The proof is similar to the proof of Theorem 3.3.) However, $L_1(Q_0)$ is not $\aleph_0$-compact: Every finite subset of

$$\{\neg Q_0 xPx\} \cup \{\exists^{>n} xPx : n \in N\}$$

has a model, but the whole set does not. $L_1(Q_0)$ does not have the Tarski property: Let φ be the conjunction of the axioms of LO, the theory of linear orderings. Then $\varphi \wedge \forall x \neg Q_0 y(y \leq x)$ has a denumerable model but no uncountable model. Let ψ be any sentence of L_1 and let P be a one-place predicate not in ψ. Then ψ is finitely valid iff $\neg Q_0 xPx \rightarrow \psi^{(P)}$ is valid. Thus, by Corollary 4.3, $L_1(Q_0)$ is not complete.

$L_1(Q_1)$, on the other hand, is complete and $\aleph_0$-compact. But, clearly, $L_1(Q_1)$ does not have the Löwenheim or Tarski properties nor is it ($\aleph_1$-) compact.

Q_0 is definable in $L_1(F)$, namely, $Q_0 xPx$ is equivalent to

$$\exists x(Px \wedge \neg Fy;z(Py \wedge y \neq x; Pz)).$$

It follows that $L_1(F)$ does not have the Tarski property nor is it complete or $\aleph_0$-compact. It is also easy to see that $L_1(F)$ does not have the Löwenheim property.

$L_1(W)$ has the Löwenheim-Skolem property, but it does not have the Tarski property nor is it complete or $\aleph_0$-compact.

Finally, $L_1(\mathsf{Cof}_\omega)$ is (fully) compact and complete. But it does not have the Löwenheim property: any linear ordering with no last element and not of cofinality ω is uncountable.

Q_0, F, W (and Q_1, Cof_ω) are definable in L_2. It follows that L_2 has none of the properties enumerated above. Q_0 is definable in wL_2. This implies that wL_2 does not have the Tarski property, is not $\aleph_0$-compact, and is not complete. But wL_2 does have the Löwenheim-Skolem property.

It may be observed that none of the above logics have all the most basic properties of L_1. In the next § we show that this is no coincidence.

§3. Characterizations. We shall now prove the (three) results mentioned at the beginning of this chapter.

In what follows L is any abstract logic.

Theorem 1. If L has the Löwenheim-Skolem property and L is $\aleph_0$-compact, then $L \simeq L_1$.

To prove this we assume that $L \not\leq L_1$ and L has the Löwenheim-Skolem property and construct a counterexample to $\aleph_0$-compactness.

Let $L \leq_{\text{inf}} L'$ mean that for every sentence φ of L, there is a sentence ψ of L' such that φ and ψ have the same infinite models. $L \simeq_{\text{inf}} L'$ if $L \leq_{\text{inf}} L' \leq_{\text{inf}} L$.

Lemma 1. Suppose $L \not\leq L_1$ and $L \leq_{\text{inf}} L_1$. There is then a sentence φ of L such that φ has arbitrarily large finite models but no infinite model.

Proof. Let ψ be any sentence of L which is not equivalent to any sentence of L_1. There is a sentence θ of L_1 which has the same infinite models as ψ. Let $\varphi :=$ $\neg(\psi \leftrightarrow \theta)$. Then φ has no infinite model. Suppose for some n, all models of φ are of cardinality $\leq n$. ℓ_ψ is finite. It follows that the number of isomorphism types t of models of ψ of cardinality $\leq n$ is finite and each of them can be characterized by a sentence μ_t of L_1. Let μ be the disjunction of the μ_t. Finally, let $\chi := (\exists^{>n}x(x=x) \wedge \theta)$ $\vee$ $(\neg\exists^{>n}x(x=x) \wedge \mu)$. Then χ is a sentence of L_1 with the same models as ψ, contrary to assumption. Thus, φ is as desired. ■

Lemma 2. Suppose $L \not\leq_{\text{inf}} L_1$ and L has the Löwenheim property. There is then a sentence φ of L such that for every n, there are denumerable models $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \models \varphi$, $\mathfrak{B} \models \neg\varphi$, and $\mathfrak{A} \cong_n \mathfrak{B}$.

Proof. Let φ be any sentence of L which does not have the same infinite models as any sentence of L_1. Let ψ be any sentence of ℓ_φ of L_1. Then $\neg(\varphi \leftrightarrow \psi)$ is a sentence of L and has an infinite model. It follows that $\neg(\varphi \leftrightarrow \psi)$ has a denumerable model and so that φ and ψ do not have the same denumerable models. Thus φ does not have the same denumerable models as any sentence of L_1.

In particular, φ does not have the same denumerable models as any disjunction of complete (n,0)-conditions of ℓ (cf. Chapter 3, §7). Different complete (n,0)-conditions are incompatible and the disjunction of these conditions is valid. It follows that there are denumerable models $\mathcal{A}$, $\mathcal{B}$ of the same complete (n,0)-condition such that $\mathcal{A} \vDash \varphi$, $\mathcal{B} \vDash \neg\varphi$. By the proof of the "$\Rightarrow$"-part of Lemma 3.13, $\mathcal{A} \cong_n \mathcal{B}$. ■

If $L \not\leq_{inf} L_1$, there is a quantifier Q of signature (1,2), say, of L such that $Qx;yz(P_0x;P_1yz)$ does not have the same infinite models as any sentence of L_1. Thus, in Lemma 2 we can take φ to be this sentence.

Proof of Theorem 1. Suppose $L \not\leq L_1$ and L has the Löwenheim-Skolem property. We are going to show that L is not $\aleph_0$-compact.

Case 1. $L \leq_{inf} L_1$. Let φ be as in Lemma 1 and let

$$\Phi = \{\varphi\} \cup \{\exists^{>n}x(x=x) \colon n \in N\}.$$

Then every finite subset of Φ has a model but Φ has no model, as desired.

Case 2. $L \not\leq_{inf} L_1$. Let φ be as in Lemma 2. Let $\ell = \{P\}$, $\ell' = \{P'\}$, where P, P' are two-place predicates. For simplicity we assume that φ is a sentence of ℓ; the general case is essentially the same. Let φ' be obtained from φ by replacing P by P'. For $k > 0$, let I_k be a 2k-place predicate. Let Ψ be the set of universal closures of the following formulas, where $0 < i, j \leq k$ (compare the conditions satisfied by an ω-isomorphism):

$\exists y(xI_1y)$, $\exists x(xI_1y)$,

$x_1...x_kI_ky_1...y_k \rightarrow \exists y(x_1...x_kxI_{k+1}y_1...y_ky)$,

---------"--------- $\rightarrow \exists x(x_1...x_kxI_{k+1}y_1...y_ky)$,

---------"--------- $\rightarrow (x_i = x_j \leftrightarrow y_i = y_j)$,

---------"--------- $\rightarrow (Px_ix_j \leftrightarrow P'y_iy_j)$.

Let

$$\Phi = \Psi \cup \{\varphi, \neg\varphi'\}.$$

We now show that Φ is a counterexample to $\aleph_0$-compactness.

Let Θ be any finite subset of Φ. Let n be such that I_k occurs in Θ only if $k \leq n$. By Lemma 2, there are models $\mathcal{A}$, $\mathcal{B}$ such that $A = B = N$, $\mathcal{A} \vDash \varphi$, $\mathcal{B} \vDash \neg\varphi$, and $\mathcal{A} \cong_n \mathcal{B}$. Let $I \colon \mathcal{A} \cong_n \mathcal{B}$. Let $\mathcal{C}$ be such that

$$C = N,\ P^{\mathcal{C}} = P^{\mathcal{A}},\ \ P'^{\mathcal{C}} = P^{\mathcal{B}},$$

$$I_k^{\mathcal{C}} = \{\langle a_1,...,a_k,b_1,...,b_k\rangle \colon \langle a_1,...,a_k\rangle I \langle b_1,...,b_k\rangle\} \text{ for } k \leq n.$$

Then, as is easily verified, $\mathfrak{C} \models \Theta$. Thus, every finite subset of Φ has a model.

Now, suppose Φ has a model. Then, since L has the Löwenheim-Skolem property, Φ has a countable model $\mathfrak{D}$. Let $\mathfrak{A}$, $\mathfrak{B}$ be the models for ℓ such that A = B = D, $P^{\mathfrak{A}} = P^{\mathfrak{D}}$, and $P^{\mathfrak{B}} = P'^{\mathfrak{D}}$. Then $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$ and so $\mathfrak{A} \not\equiv \mathfrak{B}$.

Let I be such that for every k,

$\langle a_1,\dots,a_k\rangle I\langle b_1,\dots,b_k\rangle$ iff $a_1,\dots,a_k I_k^{\mathfrak{D}} b_1,\dots,b_k$.

Then I: $\mathfrak{A} \cong_\omega \mathfrak{B}$, and so, by Lemma 3.14, $\mathfrak{A} \cong \mathfrak{B}$, a contradiction. It follows that Φ has no model. Thus, Φ is a counterexample to $\aleph_0$-compactness for L. ■

Since $L_1(\mathsf{Cof}_\omega)$ is compact, we cannot in Theorem 1 omit the assumption that L has the Löwenheim-Skolem property, not even if we replace $\aleph_0$-compactness by full compactness.

The set Φ appearing in the above proof contains infinitely many nonlogical symbols. We now want to replace Φ by a (particulary simple) set containing only finitely many such. (This will also be needed in the proof of Theorem 3, below.) One way of doing this is as follows.

Let $\ell = \{P\}$ and $\ell' = \{P'\}$ be as above. Let F be a one-place predicate, $\prec$ a two-place predicate, and R, S three-place predicates.

Consider the following conditions which can all be expressed in first-order logic:

(*)
- $\prec$ is an irreflexive linear ordering in which there is a first element and every element has an immediate successor.
- F is a nonempty initial segment of $\prec$.
- For every a, $g_a = \{\langle x,y\rangle: aRxy\}$ is a function on F.
- If x is the $\prec$-first member of F, there are a, b such that xRab.
- If xSab, Fx, and y is the immediate $\prec$-successor of x in F, then for every z, there are a', b' such that ySa'b', $g_{a'}(y) = z$ and $g_{a'}(y') = g_a(y')$ and $g_{b'}(y') = g_b(y')$ for $y' \neq y$.
- Like the preceding condition except that $g_{a'}(y) = z$ is replaced by $g_{b'}(y) = z$.
- If xSab and u,v $\prec$ x, then

  $g_a(u) = g_a(v)$ iff $g_b(u) = g_b(v)$,

  $Pg_a(u)g_a(v)$ iff $P'g_b(u)g_b(v)$.

The point of all this is explained in the following:

Lemma 3. Let θ be the conjunction of the first-order sentences expressing the above conditions (*).

(a) Suppose I: $\mathfrak{A} \cong_n \mathfrak{B}$, where A = B = N. Let $\mathfrak{C}$ be such that

$C = N$, $F^{\mathfrak{C}} = \{k: k \le n\}$,

$\prec^{\mathfrak{C}}$ the usual ordering of N.

Let g be a function on N onto the set of functions on $F^{\mathcal{C}}$ into N. We write $g_a(x)$ for $(g(a))(x)$.

$\mathcal{C} \mid \ell = \mathcal{A}$, $P'^{\mathcal{C}} = P^{\mathcal{B}}$,

$R^{\mathcal{C}} = \{\langle a,x,y\rangle: x \in F^{\mathcal{C}}\ \&\ g_a(x) = y\}$,

$S^{\mathcal{C}} = \{\langle k,a,b\rangle: k \in F^{\mathcal{C}}\ \&\ \langle g_a(0),\ldots,g_a(k-1)\rangle I \langle g_b(0),\ldots,g_b(k-1)\rangle$.

Then $\mathcal{C} \models \theta$ and $|F^{\mathcal{C}}| = n+1$.

(b) Suppose $\mathcal{C} \models \theta$ and $F^{\mathcal{C}}$ is infinite. Let c_k be the member of $F^{\mathcal{C}}$ with exactly k $<^{\mathcal{C}}$-predecessors. Next define the relation I by: for every k,

$\langle a_0,\ldots,a_{k-1}\rangle I \langle b_0,\ldots,b_{k-1}\rangle$ iff

there are a, b such that $c_k Rab$, $g_a(c_m) = a_m$, and $g_b(c_m) = b_m$ for $m < k$.

Let $\mathcal{A} = \mathcal{C} \mid \ell$ and let $\mathcal{B}$ be obtained from $\mathcal{C} \mid \ell'$ by replacing P' by P. Then I: $\mathcal{A} \cong_\omega \mathcal{B}$.

The proof of Lemma 3 is a matter of straightforward verification.

Lemma 4. Suppose $L \not\leq_{\text{inf}} L_1$ and L has the Löwenheim property. There is then a sentence ψ of L containing a one-place predicate F such that

(i) for every n, ψ has a model $\mathcal{C}_n$ such that $C_n = N$ and $|F^{\mathcal{C}_n}| = n+1$,

(ii) for every model $\mathcal{C}$ of ψ, $F^{\mathcal{C}}$ is nonempty and finite.

Proof. Let φ be as in Lemma 2. Let θ be as in Lemma 3. Finally, let

$\psi := \theta \wedge \varphi \wedge \neg\varphi'$,

where, as before, φ' is obtained from φ by replacing P by P'.

(i) Let $\mathcal{A}$, $\mathcal{B}$ be as in Lemma 2. We may assume that A = B = N. Let I: $\mathcal{A} \cong_n \mathcal{B}$. Let $\mathcal{C}_n$ be the model $\mathcal{C}$ mentioned in Lemma 3(a). Then $\mathcal{C}_n \models \psi$ and $|F^{\mathcal{C}_n}| = n+1$, as desired.

(ii) Suppose $\mathcal{C} \models \psi$. Then $F^{\mathcal{C}} \neq \emptyset$. Suppose $F^{\mathcal{C}}$ is infinite. Let G be a two-place predicate not in ψ. Let χ be a sentence of L_1 saying that "G is a function mapping a proper subset of F onto F". Then $\psi \wedge \chi$ has an infinite model. It follows that $\psi \wedge \chi$ has a denumerable model $\mathcal{D}$ such that $F^{\mathcal{D}}$ is infinite. Thus, we may assume that $\mathcal{C}$ is denumerable.

Let $\mathcal{A}$, $\mathcal{B}$, I be as in Lemma 3(b) (with the present model $\mathcal{C}$). Then I: $\mathcal{A} \cong_\omega \mathcal{B}$ and so, by Lemma 3.14, $\mathcal{A} \cong \mathcal{B}$. But, on the other hand, $\mathcal{A} \models \varphi$ and $\mathcal{B} \models \neg\varphi$ and so $\mathcal{A} \not\cong \mathcal{B}$, a contradiction. Thus, $F^{\mathcal{C}}$ is finite. ■

Lemma 5. Suppose $L \not\leq L_1$ and L has the Löwenheim property. There is then a sentence ψ of L containing a one-place predicate F such that

(i) for every n, ψ has a model $\mathcal{C}$ such that and $|F^{\mathcal{C}}| = n+1$,

(ii) for every model $\mathcal{C}$ of ψ, $F^{\mathcal{C}}$ is nonempty and finite.

Proof. If $L \leq_{\inf} L_1$ let φ be as in Lemma 1 and let $\psi := \varphi \wedge \exists x Fx$. If $L \nleq_{\inf} L_1$, let ψ be as in Lemma 4. ■

We can now somewhat improve Theorem 1:

Theorem 1'. If L has the Löwenheim property and L is $\aleph_0$-compact, then $L \simeq L_1$.

Proof. Suppose $L \nleq L_1$ and L has the Löwenheim property. Let ψ and F be as in Lemma 5. Let

$$\Psi = \{\psi\} \cup \{\exists^{>n} x Fx \colon n \in N\}.$$

Then every finite subset of Ψ has a model but Ψ has no model. Thus, L is not $\aleph_0$-compact. ■

Our next objective is to show that L_1 can (almost) be characterized in terms of the Löwenheim and Tarski properties.

Let ψ be any sentence and y be a variable not in ψ. For every n-place predicate G in ψ, replace $Gx_1...x_n$ by $G^+x_1...x_ny$. Let $\psi'(y)$ be the result. Finally, let $\psi^+ := \forall y \psi'(y)$. If $\mathcal{l}_{\mathcal{A}} = \mathcal{l}_{\psi^+}$ and $a \in A$, let $\mathcal{A}^{(a)}$ be defined by:

$$A^{(a)} = A,$$

$$G^{\mathcal{A}^{(a)}} = \{\langle a_1,...,a_n\rangle \colon \langle a_1,...,a_n,a\rangle \in G^{+\mathcal{A}}\},$$

where $G \in \mathcal{l}_\psi$. Then $\mathcal{A} \models \psi'(a)$ iff $\mathcal{A}^{(a)} \models \psi$. It follows that $\mathcal{A} \models \psi^+$ iff $\mathcal{A}^{(a)} \models \psi$ for every $a \in A$.

Lemma 6. Suppose $L \nleq_{\inf} L_1$ and L has the Löwenheim property. Then there are a finite language $\mathcal{l}^+ = \{\preccurlyeq, ...\}$, where $\preccurlyeq$ is a two-place predicate, and a sentence θ of $\mathcal{l}^+$ of L such that θ has a model and if $\mathcal{A} \models \theta$, then $\mathcal{A} | \{\preccurlyeq\} \cong (N, \leq)$.

Proof. Let ψ, F, and $\mathcal{C}_n$ be as in Lemma 4. Let $\preccurlyeq$ be F^+. Let χ be the sentence of L_1 saying that "$\preccurlyeq$ is a linear ordering with a smallest element, in which each element has an immediate successor". Let $\theta := \chi \wedge \psi^+$.

Now define $\mathcal{C}$, where $\mathcal{l}_{\mathcal{C}} = \mathcal{l}_{\psi^+}$, by:

$$C = N,$$

$$\mathcal{C}^{(n)} = \mathcal{C}_n \text{ for every n.}$$

Then $\mathcal{C} \models \psi^+$ iff $\mathcal{C}^{(n)} \models \psi$ for every n. It follows that $\mathcal{C} \models \psi^+$. Also, clearly, $\mathcal{C} \models \chi$ and so $\mathcal{C} \models \theta$. Thus, θ has a model.

Finally, if $\mathcal{A} \models \theta$, then $\mathcal{A} \models \chi$ and, by Lemma 4, each initial segment of the ordering $\preccurlyeq^{\mathcal{A}}$ is finite. It follows that $(A, \preccurlyeq^{\mathcal{A}}) \cong (N, \leq)$. ■

The sentence θ in Lemma 6 has a denumerable model but no uncountable model. Thus, we get:

Theorem 2. If L has the Löwenheim and Tarski properties, then $L \simeq_{\text{inf}} L_1$.

Theorem 2 isn't quite a characterization of L_1, since the conclusion is not $L \simeq L_1$. One way to obtain this conclusion is to add the assumption that any sentence of L which has arbitrarily large finite models has an infinite model (see Lemma 1). Another is to assume that L *relativizes* in the sense that for every sentence φ of L and one-place predicate F not in φ, there is a sentence $\varphi^{[F]}$ of $\ell_\varphi \cup \{F\}$ such that if $\mathfrak{A}$ is a model for ℓ_φ, then $(\mathfrak{A}, X) \models \varphi^{[F]}$ iff $\mathfrak{A} \mid X \models \varphi$.

Of the logics mentioned above all but $L_1(W)$ relativize. A (minimal) relativizing extension of $L_1(W)$ is obtained by replacing W by the quantifier W^{rel} such that $W^{rel}x;yz(Fx;y\preccurlyeq z)$ means that "$\preccurlyeq$ is a well-ordering of F". And similarly for any (nonrelativizing) abstract logic.

Lemma 7. If $L \leq_{\text{inf}} L_1$ and L relativizes, then $L \simeq L_1$.

Proof. By Lemma 1, it suffices to show that every sentence φ of L with arbitrarily large finite models has an infinite model. Let F a one-place predicate not in φ. Then, for every n, $\varphi^{[F]}$ has an infinite model $\mathfrak{A}$ such that $|F^{\mathfrak{A}}| \geq n$. Since $L \leq_{\text{inf}} L_1$, it follows that $\varphi^{[F]}$ has a model $\mathfrak{B}$ such that $F^{\mathfrak{B}}$ is infinite. But then $\mathfrak{B} \mid F^{\mathfrak{B}}$ is an infinite model of φ. ■

From Theorem 2 and Lemma 7 we get:

Theorem 2'. If L relativizes and has the Löwenheim and Tarski properties, then $L \simeq L_1$.

If in Theorems 1, 1' we add the assumption that L relativizes, the proofs can be somewhat simplified. We can replace φ, φ' by $\varphi^{[F]}$, $\varphi^{[F']}$, where F, F' are one-place predicates and modify the sentences in Ψ accordingly. There is then no need to treat the two cases $L \leq_{\text{inf}} L_1$ and $L \nleq_{\text{inf}} L_1$ separately.

Finally, it turns out that the combination of, what are arguably the two most interesting properties of L_1, at least from a philosophical point of view, can also be used to characterize L_1.

Theorem 3. If L has the Löwenheim property and L is complete, then $L \simeq L_1$.

Proof. Suppose $L \nleq L_1$ and L has the Löwenheim property. Let ψ and F be as in Lemma 5. Let ℓ be any finite language not containing F nor any nonlogical constant occurring in ψ. Let φ be any sentence of ℓ of L_1. Let $\varphi^{(F)}$ be obtained from

φ by replacing $\exists x\psi(x)$ by $\exists x(Fx \wedge \psi(x))$ and $\forall x\psi(x)$ by $\forall x(Fx \to \psi(x))$ for all subformulas $\exists x\psi(x)$, $\forall x\psi(x)$ of φ. It is then clear that

φ is finitely valid iff $\psi \to \varphi^{(F)}$ is valid.

It follows, by Corollary 4.3, that L is not complete, as desired. ■

Since, for example, $L_1(Q_1)$ is complete, we cannot in Theorem 3 omit the assumption that L has the Löwenheim property.

If in Theorem 3 we add the assumption that L relativizes, then $L \simeq L_1$ follows at once from Lemmas 6, 7 and Theorem 4.5 (Gödel's Incompleteness Theorem).

Notes for Chapter 5. Generalized quantifiers were first defined in Mostowski (1957) and, in greater generality, in Lindström (1966b) (see also Ebbinghaus (1985)). Proofs of the completeness and ($\aleph_0$-) compactness of $L_1(Q_1)$ and $L_1(\mathrm{Cof}_\omega)$ and numerous related results can be found in Barwise, Feferman (1985). The characterizations of L_1 proved in §3 are due to Lindström (1966b), (1969). These results are almost always formulated in terms of some concept *abstract logic* more general than the one defined here; see e.g. Ebbinghaus, Flum, Thomas (1984), Flum (1985), Chang, Keisler (1990). The first such concept was introduced in Lindström (1969); but see also Svenonius (1960).

APPENDIX 1

Example 1. The sequent

(1) $\forall x(Fx \vee Gx) \Rightarrow \neg\exists x\neg Gx, \forall xFx$

is not valid. A counterexample can be found as follows.

(2) $\forall x(Fx \vee Gx) \Rightarrow \neg\exists x\neg Gx, Fa$
(3) $Fa \vee Ga, \forall x(Fx \vee Gx) \Rightarrow \neg\exists x\neg Gx, Fa$
(4) $Fa \vee Ga, \forall x(Fx \vee Gx), \exists x\neg Gx \Rightarrow Fa$
(5) $Fa \vee Ga, \forall x(Fx \vee Gx), \neg Gb \Rightarrow Fa$
(6) $Fa \vee Ga, Fb \vee Gb, \neg Gb \Rightarrow Fa$
(7) $Fa \vee Ga, Fb \vee Gb \Rightarrow Fa, Gb$
(81) $Fa, Fb \vee Gb \Rightarrow Fa, Gb$ $Ga, Fb \vee Gb \Rightarrow Fa, Gb$ (82)
(91) $Ga, Fb \Rightarrow Fa, Gb$ $Ga, Gb \Rightarrow Fa, Gb$ (92)

$\forall xFx$ occurs to the right of $\Rightarrow$ in (1) and so is false in the (prospective) counterexample to (1). Thus, we introduce a (new) constant a and add Fa to the right of $\Rightarrow$. Then Fa will be false and so $\forall xFx$ will be false. This yields (2). $\forall x(Fx \vee Gx)$ is true and so we have to add $Fa \vee Ga$ on the left, as in (3). $\neg\exists x\neg Gx$ is false and so $\exists x\neg Gx$ is true and we add $\exists x\neg Gx$ on the left as in (4). Since $\exists x\neg Gx$ is true, we add $\neg Gb$, where b is a (new) constant, on the left as in (5). But then, since we want $\forall x(Fx \vee Gx)$ to be true, we also have to add $Fb \vee Gb$ on the left as in (6). Also, $\neg Gb$ is true, whence Gb is false and so we add Gb on the right as in (7). $Fa \vee Ga$ is true and so there are two possibilities (81) and (82). But (81) is an axiom and so cannot yield a counterexample. In (82) either Fb or Gb is true. And so we get (91) and (92). And now there is no more we can do. (92) is an axiom but (91) isn't. And as is easily checked, and holds on general grounds (see the proof of Theorem 2.7), the two-element model in which Ga, Fb are true and Fa, Gb are false is a conterexample to (1). ■

Applying this method to a valid sequent S, for example, the sequents in Examples 2.4, 2.5 and Example 3, below, we do not, of course, get a counterexample to S; what we get is (the inverse of) a derivation of S in GS.

The above counterexample is finite. But, of course, this not always the case:

Example 2. Applying the method of Example 1 to the non-valid sequent $\forall x\exists yPxy \Rightarrow$ we get:

$\forall x\exists yPxy \Rightarrow$
$\forall x\exists yPxy, \exists yPc_0y \Rightarrow$
$\forall x\exists yPxy, \exists yPc_0y, Pc_0c_1 \Rightarrow$
$\forall x\exists yPxy, \exists yPc_0y, Pc_0c_1, \exists yPc_1y \Rightarrow$ etc.

Thus, although in this case there is a (very simple) finite counterexample, the counterexample obtained by applying the above method is infinite. ■

Example 3. $\vdash_{GS+(Cut)} \Rightarrow \forall x\exists y(Pxy \vee \forall z\neg Pyz)$ (compare Example 2.11).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pab \Rightarrow Pab, \forall z\neg Pbz}{Pab \Rightarrow Pab \vee \forall z\neg Pbz}\,(\Rightarrow\vee)}{Pab \Rightarrow \exists y(Pay \vee \forall z\neg Pyz)}\,(\Rightarrow\exists)}{\exists yPay \Rightarrow \exists y(Pay \vee \forall z\neg Pyz)}\,(\exists\Rightarrow) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pab \Rightarrow Pab, Paa}{Pab \Rightarrow \exists yPay, Paa}\,(\Rightarrow\exists)}{\Rightarrow \exists yPay, Paa, \neg Pab}\,(\Rightarrow\neg)}{\Rightarrow \exists yPay, Paa, \forall z\neg Paz}\,(\Rightarrow\forall)}{\Rightarrow \exists yPay, Paa \vee \forall z\neg Paz}\,(\Rightarrow\vee)}{\Rightarrow \exists yPay, \exists y(Pay \vee \forall z\neg Pyz)}\,(\Rightarrow\exists)}{\Rightarrow \exists y(Pay \vee \forall z\neg Pyz)}\,(\text{Cut})}{\Rightarrow \forall x\exists y(Pxy \vee \forall z\neg Pyz)}\,(\Rightarrow\forall)$$

Note that $\exists yPay$ is not a subformula of the end-sequent. Of course, this sequent can also be derived in GS. ■

Example 4. $\exists y\forall x(Qxy \rightarrow Px), \forall yy'z(Qzy \rightarrow Qzy') \vdash_{ND} \forall x(\exists yQxy \rightarrow Px)$.

(1)	$\exists y\forall x(Qxy \rightarrow Px)$	Ø P {1}
(2)	$\forall yy'z(Qzy \rightarrow Qzy')$	Ø P {2}
(3)	$\exists yQay$	Ø P {3}
(4)	Qab	Ø P {4}
(5)	$\forall x(Qxb' \rightarrow Px)$	Ø P {5}
(6)	$Qab' \rightarrow Pa$	{5} US {5}
(7)	$Qab \rightarrow Qab'$	{2} US (three times) {2}
(8)	Pa	{4,6,7} PL {2,4,5}
(9)	Pa	{8} ES {1,2,4}
(10)	Pa	{9} ES {1,2,3}
(11)	$\exists yQay \rightarrow Pa$	{10} Cond {1,2}
(12)	$\forall x(\exists yQxy \rightarrow Px)$	{11} UG {1,2} ■

Example 5. $\forall xy\exists z(Pxz \wedge Pyz), \forall x\neg Pxx \vdash_{ND} \forall x\exists yz(y \neq z \wedge Pxy \wedge Pxz)$.

(1)	$\forall xy\exists z(Pxz \wedge Pyz)$	Ø P {1}
(2)	$\exists z(Paz \wedge Paz)$	{1} US (twice) {1}
(3)	$Pab \wedge Pab$	Ø P {3}
(4)	Pab	{3} PL {3}
(5)	$\exists z(Paz \wedge Pbz)$	{1} US (twice) {1}
(6)	$Pac \wedge Pbc$	Ø P {6}
(7)	Pbc	{6} PL {6}

(8) $\forall x \neg Pxx$ $\varnothing$ P {8}
(9) $\neg Pcc$ {8} US {8}
(10) $b \neq c$ {7,9} $\mathrm{I}^{\#}$ {6,8}
(11) $b \neq c \wedge Pab \wedge Pac$ {4,6,10} PL {3,6,8}
(12) $\exists yz(y \neq z \wedge Pay \wedge Paz)$ {11} EG (twice) {3,6,8}
(13) --------------"--------------- {13} ES {1,3,8}
(14) --------------"--------------- {14} ES {1,8}
(15) $\forall x \exists yz(y \neq z \wedge Pxy \wedge Pxz)$ {15} UG {1,8} ■

Derivations in ND can often be simplified by using the following two derived rules of ND, which we state informally as follows. According to the first rule, QR, we may infer (i) $\exists x \neg \varphi(x)$ from $\neg \forall x \varphi(x)$ and conversely and (ii) $\forall x \neg \varphi(x)$ from $\neg \exists x \varphi(x)$ and conversely. According to the second rule, RA (*reductio ad absurdum*), if we have inferred either $\neg\varphi$ or both ψ and $\neg\psi$, for some ψ, from $\Pi \cup \{\varphi\}$, we may infer $\neg\varphi$ from Π. Example 2.8 shows that one case of QR is a derived rule. The proofs of the remaining cases of QR and RA are left to the reader.

Example 6. The sentence

$$\exists x \forall y(y \in x \leftrightarrow \neg \exists z(y \in z \wedge z \in y))$$

is not satisfiable (an instance of the extended Russell paradox). Indeed, suppose c is such that

$$\forall y(y \in c \leftrightarrow \neg \exists z(y \in z \wedge z \in y)).$$

If $c \in c$, then $c \in c \wedge c \in c$, whence $\exists z(c \in z \wedge z \in c)$ and so $c \notin c$. Contradiction. Thus, $c \notin c$. But then $\exists z(c \in z \wedge z \in c)$. Let d be such that $c \in d \wedge d \in c$. Since $d \in c$, it follows that $\neg \exists z(d \in z \wedge z \in d)$, whence $\neg(c \in d \wedge d \in c)$, again a contradiction.

Formalized in ND this argument looks as follows (see also Example 8, below).

(1) $\forall y(y \in c \leftrightarrow \neg \exists z(y \in z \wedge z \in y))$ $\varnothing$ P {1}
(2) $c \in c$ $\varnothing$ P {2}
(3) $c \in c \wedge c \in c$ {2} PL {2}
(4) $\exists z(c \in z \wedge z \in c)$ {3} EG {2}
(5) $c \in c \leftrightarrow \neg \exists z(c \in z \wedge z \in c)$ {1} US {1}
(6) $\neg c \in c$ {4,5} US {1,2}
(7) $\neg c \in c$ {6} RA {1}
(8) $\exists z(c \in z \wedge z \in c)$ {7,5} PL {1}
(9) $c \in d \wedge d \in c$ $\varnothing$ P {9}
(10) $d \in c$ {9} PL {9}
(11) $d \in c \leftrightarrow \neg \exists z(d \in z \wedge z \in d)$ {1} US {1}
(12) $\neg \exists z(d \in z \wedge z \in d)$ {10,11} PL {1,9}
(13) $\forall z \neg(d \in z \wedge z \in d)$ {12} QR {1,9}

(14) $\neg(d\in c \land c\in d)$	{13} US {1,9}
(15) $\neg(1)$	{14} RA {9}
(16) $\neg(1)$	{15} ES {1}
(17) $\neg(1)$	{16} RA Ø
(18) $\forall x\neg\forall y(y\in x \leftrightarrow \neg\exists z(y\in z \land z\in y))$	{17} UG Ø
(19) $\neg\exists x\forall y(y\in x \leftrightarrow \neg\exists z(y\in z \land z\in y))$	{18} QR Ø ■

Example 7. Yablo's paradox is as follows. Imagine an infinite line of people. It has a first member, a second member, etc. At a certain point of time everyone in the line says: What everyone behind me says right now is false. Let P be any person in the line. Suppose what P says is true. There is a person P′ behind P. Since what P says is true, what P′ says is false. But then there is a person P′′ behind P′ who says something true. Now P′′ stands behind P. And so what P says is false, a contradiction. Thus, what P says is false. But this is true of every person in the line. And so what everyone in the line says is true. Paradox.

Now think of $x < y$ as saying that "x stands in front of y" and Tx as saying that "what x says is true". The (necessary) premises and conclusion of the argument are:

Premise 1. $\forall xyz(x < y \land y < z \rightarrow x < z)$

Premise 2. $\forall x\exists y(x < y)$

Conclusion. $\neg\forall x(Tx \leftrightarrow \forall y(x < y \rightarrow \neg Ty))$

(1) $\forall x(Tx \leftrightarrow \forall y(x < y \rightarrow \neg Ty))$	Ø P {1}
(2) Ta	Ø P {2}
(3) $Ta \leftrightarrow \forall y(a < y \rightarrow \neg Ty)$	{1} US {1}
(4) $\forall y(a < y \rightarrow \neg Ty)$	{2,3} PL {1,2}
(5) Premise 2	Ø P {5}
(6) $\exists y(a < y)$	{5} US {5}
(7) $a < b$	Ø P {7}
(8) $a < b \rightarrow \neg Tb$	{4} US {1,2}
(9) $Tb \leftrightarrow \forall y(b < y \rightarrow \neg Ty)$	{1} US {1}
(10) $\neg\forall y(b < y \rightarrow \neg Ty)$	{7,8,9} PL {1,2,7}
(11) $\exists y\neg(b < y \rightarrow \neg Ty)$	{10} QR {1,2,7}
(12) $\neg(b < c \rightarrow \neg Tc)$	Ø P {12}
(13) $b < c$	{12} PL {12}
(14) Tc	{12} PL {12}
(15) Premise 1	Ø P {15}
(16) $a < b \land b < c \rightarrow a < c$	{15} US (three times) {15}
(17) $a < c$	{7,13,16} PL {7,12,15}

(18) $\neg(a < c \to \neg Tc)$ {14,17} PL {7,12,15}
(19) $\exists y \neg(a < y \to \neg Ty)$ {18} EG {7,12,15}
(20) $\neg\forall y(a < y \to \neg Ty)$ {19} QR {7,12,15}
(21) $\neg Ta$ {3,20} PL {3,7,12,15}
(22) $\neg Ta$ {21} ES (twice) {1,2,5,15}
(23) $\neg Ta$ {22} RA {1,5,15}
(24) $\forall x \neg Tx$ {23} UG {1,5,15}
(25) $\neg Td$ {24} US {1,5,15}
(26) $e < d \to \neg Td$ {25} PL {1,5,15}
(27) $\forall y(e < y \to \neg Ty)$ {26} UG {1,5,15}
(28) $Te \leftrightarrow \forall y(e < y \to \neg Ty)$ {1} US {1}
(29) Te {27,28} PL {1,5,15}
(30) $\neg Te$ {24} US {1,5,15}
(31) $\neg\forall x(Tx \leftrightarrow \forall y(x < y \to \neg Ty))$ {29,30} RA {5,15} ■

Example 8. The sentence $\varphi :=$

$$\exists x \forall y(y \in x \leftrightarrow \neg\exists z(y \in z \wedge z \in y))$$

is not satisfiable (Example 6). The Skolem-Herbrand method can be applied to show this as follows.

Rewriting φ in one way in prenex normal form (with the matrix in conjunctive normal form) we get $\psi :=$

$$\exists x \forall y \exists z' \forall z(\chi_0(x.y,z) \wedge \chi_1(x.y,z') \wedge \chi_2(x.y,z')),$$

where

$$\chi_0(x.y,z) := y \notin x \vee z \notin y \vee y \notin z,$$
$$\chi_1(x.y,z') := y \in x \vee z' \in y,$$
$$\chi_2(x.y,z') := y \in x \vee y \in z'.$$

Thus, $\psi^S := \forall yz\theta(y,z)$, where

$$\theta(y,z) := \chi_0(c,y,z) \wedge \chi_1(c,y,f(y)) \wedge \chi_2(c,y,f(y)).$$

$\{\theta(c,c), \theta(f(c),c)\}$ is an inconsistent subset of $H(\{\psi\})$. Thus, ψ is not satisfiable and so φ is not satisfiable. ■

Example 9. Let

$$\Phi = \{\forall xy \exists z(Pxz \wedge Pyz), \forall x \neg Pxx\}.$$

We have shown that

$$\Phi \vdash_{ND} \forall x \exists yz(y \neq z \wedge Pxy \wedge Pxz)$$

(Example 5). The Skolem-Herbrand method can be applied to this example as follows. Let

$$\Psi = \Phi \cup \{\neg\forall x \exists yz(y \neq z \wedge Pxy \wedge Pxz)\}.$$

Then

$$\Psi^S = \{\forall xy(Pxf(x,y) \wedge Pyf(x,y), \forall x\neg Pxx, \forall yz\neg(y \neq z \wedge Pay \wedge Paz)\}.$$

Let $t := f(a,a)$ and $t' := f(a,t)$. The sentences

$$Pat \wedge Pat, \quad Pat' \wedge Ptt',$$
$$\neg Pt't', \quad \neg(t \neq t' \wedge Pat \wedge Pat'),$$

are members of $H(\Psi)$. The sentence

$$t = t' \to (Ptt' \to Pt't')$$

is a member of $\mathrm{Id}(\boldsymbol{\ell}_{\Psi^S})$. The set of these sentences is inconsistent. Thus, Ψ is not satisfiable. ■

APPENDIX 2

Using the idea in Case 2 of the proof of Theorem 5.1 we now give an:
Alternative proof of Theorem 3.9. We first consider the case where ℓ_0, ℓ_1 are finite. The case that Φ has no infinite model is trivial, since then all models of Φ are isomorphic. Thus suppose Φ has an infinite model. Then Φ_0, Φ_1 have denumerable models. We explicitly consider only the special case that $\ell = \{P,f,c\}$, where P is a two-place predicate, f a one-place function symbol, and c an individual constant, but the idea of the construction is perfectly general. Let $\ell' = \{P',f',c'\}$.

Let Φ_1' be obtained from Φ_1 by replacing P, f, c by the corresponding member of ℓ'. For every k, let I_k be a new 2(k+1)-place predicate. Let Ψ be as in the proof of Theorem 5.1 except that we now add the universal closures of the following formulas for $k > 0$ and $0 < i, j \le k$:

$x_1 \ldots x_k I_k y_1 \ldots y_k \rightarrow (f(x_i) = x_j \leftrightarrow f'(y_i) = y_j)$,

---------"-------- $\rightarrow (x_i = c \leftrightarrow y_i = c')$.

Let

$$\Theta = \Psi \cup \Phi_0 \cup \Phi_1'.$$

Then, as in the proof of Theorem 5.1, every finite subset of Θ has a model and so Θ has a denumerable model $\mathcal{A}$. Finally, from $\mathcal{A}$, again as in the proof of Theorem 5.1, we obtain models $\mathcal{B}_i$ of Φ_i, i = 0, 1, such that $\mathcal{B}_0 | \ell \cong \mathcal{B}_1 | \ell$. And from this it follows that $\Phi_0 \cup \Phi_1$ has a model.

This proof extends in an obvious way to the case where ℓ, ℓ_0, ℓ_1 are countable. In the general case let g be a new one-place function symbol. Let Γ be the set of sentences saying in a model $\mathcal{A}$ for $\ell \cup \ell' \cup \{g\}$ that "g is an isomorphism of $\mathcal{A} | \ell$ onto $\mathcal{A} | \ell'$" (in the obvious sense). Let

$$\Delta = \Gamma \cup \Phi_0 \cup \Phi_1'.$$

Then, by what have already shown, every finite subset of Δ has a model. It follows that Δ has a model. But from this model we can obtain a model of $\Phi_0 \cup \Phi_1$ in the same way as before. ■

We shall now show how the idea of the above proof can be adapted to yield proofs of Lemmas 3.7, 3.22.

For any two models $\mathcal{A}$, $\mathcal{B}$ we write $\mathcal{A}\Sigma_n^+\mathcal{B}$ to mean that for every primitive positive sentence φ, if $qd(\varphi) \le n$ and $\mathcal{A} \models \varphi$, then $\mathcal{B} \models \varphi$. Thus, $\mathcal{A}\Sigma^+\mathcal{B}$ iff for every n, $\mathcal{A}\Sigma_n^+\mathcal{B}$. The relation H is an *n-homomorphism* of $\mathcal{A}$ onto $\mathcal{B}$, H: $\mathcal{A} \Rightarrow_n \mathcal{B}$, if $H \subseteq \bigcup\{A^k x B^k : k \le n\}$, $\langle\rangle H \langle\rangle$, and

if $|\mathbf{s}| = |\mathbf{t}| < n$ and $\mathbf{s}H\mathbf{t}$, then for every $a \in A$ ($b \in B$), there is a $b \in B$ ($a \in A$) such that $\mathbf{s}aH\mathbf{t}b$, if $\mathbf{s}H\mathbf{t}$, then $(\mathcal{A}, \mathbf{s})\Sigma_0^+(\mathcal{B}, \mathbf{t})$.

We write $\mathcal{A} \Rightarrow_n \mathcal{B}$ to mean that there is an n-homomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

Suppose the language ℓ of the models $\mathcal{A}$, $\mathcal{B}$ is finite. By an *$(n,n)^+$-condition* we understand a primitive atomic formula of ℓ in the variables $x_1,\ldots,x_n$. If $\varphi_0,\ldots,\varphi_m$ are $(n,k+1)^+$-conditions, $\exists x_{k+1}(\varphi_0 \wedge\ldots\wedge \varphi_m)$ and $\forall x_{k+1}(\varphi_0 \vee\ldots\vee \varphi_m)$ are *$(n,k)^+$-conditions.*

Lemma 1. If $\mathcal{A}\Sigma_n^+ \mathcal{B}$, then $\mathcal{A} \Rightarrow_n \mathcal{B}$.

Proof. Let H be defined by:

> $\mathbf{s}H\mathbf{t}$ iff there is a $k \leq n$ such that $|\mathbf{s}| = |\mathbf{t}| = k$ and for every $(n,k)^+$-condiditon φ, if $\mathcal{A} \models \varphi(\mathbf{s})$, then $\mathcal{B} \models \varphi(\mathbf{t})$.

Then $\langle\rangle H\langle\rangle$, since $\mathcal{A}\Sigma_n^+ \mathcal{B}$. Suppose $\mathbf{s}H\mathbf{t}$, where $|\mathbf{s}| = |\mathbf{t}| = k < n$. Suppose $a \in A$. Let ψ be the conjunction of the $(n,k+1)^+$-conditions φ such that $\mathcal{A} \models \varphi(\mathbf{s}a)$. Let $\theta := \exists x_k\psi$. Then θ is an $(n,k)^+$-condition and $\mathcal{A} \models \theta(\mathbf{s})$. By assumption, it follows that $\mathcal{B} \models \theta(\mathbf{t})$. Let $b \in B$ be such that $\mathcal{B} \models \psi(\mathbf{t}b)$. Then $\mathbf{s}aH\mathbf{t}b$.

Next, suppose $b \in B$. Let ψ be the disjunction of the $(n,k+1)^+$-conditions φ such that $\mathcal{B} \models \neg\varphi(\mathbf{t}b)$. Let $\theta := \forall x_{k+1}\psi$. Then θ is an $(n,k)^+$-condition and $\mathcal{B} \models \neg\theta(\mathbf{t})$. It follows that $\mathcal{A} \models \neg\theta(\mathbf{s})$. Let $a \in A$ be such that $\mathcal{A} \models \neg\psi(\mathbf{s}a)$. Then $\mathbf{s}aH\mathbf{t}b$.

Finally, it is clear that if $\mathbf{s}H\mathbf{t}$, then $(\mathcal{A}, \mathbf{s})\Sigma_0^+(\mathcal{B}, \mathbf{t})$. Thus, H: $\mathcal{A} \Rightarrow_n \mathcal{B}$. ■

The converse of Lemma 1, too, is true (compare Lemma 3.13) but will not be needed here.

H is an *ω-homomorphism* of $\mathcal{A}$ onto $\mathcal{B}$, H: $\mathcal{A} \Rightarrow_\omega \mathcal{B}$, if for every n, the relation $\{\langle \mathbf{s},\mathbf{t}\rangle: \mathbf{s}H\mathbf{t}\ \&\ |\mathbf{s}| = |\mathbf{t}| \leq n\}$ is an n-homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. $\mathcal{A} \Rightarrow_\omega \mathcal{B}$ means that there is an ω-homomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

The following lemma is the analogue in the present context of Lemma 3.14.

Lemma 2. If $\mathcal{A}$, $\mathcal{B}$ are countable and $\mathcal{A} \Rightarrow_\omega \mathcal{B}$, then $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$.

Proof. Let H: $\mathcal{A} \Rightarrow_\omega \mathcal{B}$. Let a_0, a_1, a_2,... be an enumeration of A and let b_0, b_1, b_2,... be an enumeration of B (in both cases with repetitions if the set is finite). It is completely straightforward to define c_n and d_n in such a way that for every n, $c_{2n} = a_n$, $d_{2n+1} = b_n$. and $\langle c_0,\ldots,c_n\rangle H\langle d_0,\ldots,d_n\rangle$. Let $f: A \to B$ be such that $f(c_n) = d_n$. Then f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. ■

Proof of Lemma 3.7. First we assume that the language ℓ of the models $\mathcal{A}_i$ is finite. For example, let $\ell = \{P,f,c\}$ be as above. Let $\ell_i = \{P_i,f_i,c_i\}$, i = 0, 1. Let $\mathcal{A}_i'$ be obtained from $\mathcal{A}_i$ by replacing P, f, c by P_i ,f_i ,c_i, respectively. Let U_0, U_1 be new one-place predicates. Let $\mathcal{A}$ be a model for $\ell_0 \cup \ell_1 \cup \{U_0,U_1\}$ such that $(\mathcal{A} | U_i^{\mathcal{A}}) | \ell_i = \mathcal{A}_i'$. For $n > 0$, let H_n be a new 2n-place predicate. Let Ψ be the set of universal closures of the following formulas, where $1 \leq j, k \leq n$:

$U_0x \to \exists y(U_1y \wedge xH_1y)$, $U_1y \to \exists x(U_0x \wedge xH_1y)$,

$x_1...x_nH_ny_1...y_n \to (U_0x \to \exists y(U_1y \wedge x_1...x_nxH_{n+1}y_1...y_ny))$,

---------"--------- $\to (U_1y \to \exists x(U_0x \wedge x_1...x_nxH_{n+1}y_1...y_ny))$,

---------"--------- $\to (x_j = x_k \to y_j = y_k)$,

---------"--------- $\to (P_0x_jx_k \to P_1y_jy_k)$,

---------"--------- $\to (f_0(x_j) = x_k \to f_1(y_j) = y_k)$,

---------"--------- $\to (x_j = c_0 \to y_j = c_1)$.

By Lemma 1, every finite subset of $\mathrm{Th}(\mathcal{A}) \cup \Psi$ has a model. It follows that this set has a countable model $\mathcal{B}$. Let $\mathcal{B}_i$ be the result of replacing P_i, f_i, c_i by P, f, c, respectively, in $(\mathcal{B} \mid U_i^{\mathcal{B}}) \mid \ell_i$. Then $\mathcal{B}_i \equiv \mathcal{A}_i$, $i = 0, 1$. Let H be such that for every n,

$\langle a_1,...,a_n\rangle H \langle b_1,...,b_n\rangle$ iff $a_1,...,a_nH_n^{\mathcal{B}}b_1,...,b_n$.

Then $H\colon \mathcal{B}_0 \Rightarrow_\omega \mathcal{B}_1$. But then, by Lemma 2, $\mathcal{B}_1$ is a homomorphic image of $\mathcal{B}_0$, as desired.

This proves the lemma for models for a finite language; in fact, for every countable language. The full result can now be proved by arguing in much he same way as in the final paragraph of the alternative proof of Theorem 3.9. ■

Proof of Lemma 3.23. There is a model $\mathcal{B} \preccurlyeq \mathcal{A}$ such that $U^{\mathcal{A}} \subseteq B$, and so $U^{\mathcal{B}} = U^{\mathcal{A}}$, and $|B| = \lambda$, and so $B \neq A$. Let $\ell \cup \{V\}$ be the language of $(\mathcal{A}, B)$. As before we consider a special case: $\ell = \{U,P,f,c\}$, where P, f, c are as above.

For $k > 0$, let I_k be a new 2k-place predicate. Let Ψ be the set of universal closures of the following formulas, where $0 < i, j \leq k$:

$\exists y(Vy \wedge xI_1y)$, $Vy \to \exists x(xI_1y)$,

$x_1...x_kI_ky_1...y_k \to \exists y(Vy \wedge x_1...x_kxI_{k+1}y_1...y_ky)$,

---------"-------- $\to (Vy \to \exists x(x_1...x_kxI_{k+1}y_1...y_ky))$,

---------"-------- $\to (x_i = x_j \leftrightarrow y_i = y_j)$,

---------"-------- $\to (Px_ix_j \leftrightarrow Py_iy_j)$,

---------"-------- $\to (f(x_i) = x_j \leftrightarrow f(y_i) = y_j)$,

---------"-------- $\to (x_i = c \leftrightarrow y_i = c)$,

---------"-------- $\to (Ux_i \leftrightarrow Uy_i)$,

$Vx_1 \wedge ... \wedge Vx_k \to x_1...x_kI_kx_1...x_k$.

Next let

$\Phi = \mathrm{Th}((\mathcal{A}, B)) \cup \Psi$.

Since $\mathcal{A} \mid B \preccurlyeq \mathcal{A}$, every finite subset of Φ has a model. Thus, Φ has a denumerable model $\mathcal{C}$. Let $\mathcal{A}_0 = (\mathcal{C} \mid \ell) \mid V^{\mathcal{C}}$ and $\mathcal{A}_1 = \mathcal{C} \mid \ell$. Since $\mathcal{C} \models \mathrm{Th}((\mathcal{A}, B))$, it is clear that $\mathcal{A}_0 \equiv \mathcal{A}$, $\mathcal{A}_0 \neq \mathcal{A}_1$, and $U^{\mathcal{A}_0} = U^{\mathcal{A}_1} \subseteq A_0$. Finally, since $\mathcal{C} \models \Psi$, it follows that $\mathcal{A}_0 \preccurlyeq^* \mathcal{A}_1$. ■

APPENDIX 3

In addition to (ordinary) explicit definability there is a related weaker notion: F is (*explicitly*) *definable using parameters* in T if there is a formula $\varphi(x_1,\ldots,x_n,y_1,\ldots,y_k)$ of $\ell_T - \{F\}$ such that

(**) $\quad T \vdash \exists y_1 \ldots y_k \forall x_1 \ldots x_n (Fx_1 \ldots x_n \leftrightarrow \varphi(x_1,\ldots,x_n,y_1,\ldots,y_k))$.

A natural (model-theoretic) question is then if there is a characterization, in terms of some property of the class of models of T, analogous to Theorem 10, of those theories T in which F is definable in this sense. This question is answered for complete theories in the following:

Theorem (Chang, Makkai)**.** Suppose T is countable and complete and has no finite model. Then F is definable using parameters in T iff for every countable model $(\mathcal{A}, R)$, where $\ell_{\mathcal{A}} = \ell_T - \{F\}$, of T,

$$|\{S: (\mathcal{A}, S) \cong (\mathcal{A}, R)\}| \leq \aleph_0.$$

Proof. $\Rightarrow$. Suppose (**) holds. Let $\mathcal{A}$ be any countable model for ℓ. For every sequence $a_1,\ldots,a_k$ of members of A, there is at most one relation S such that $(\mathcal{A}, S) \vDash T$ and

$$(\mathcal{A}, S) \vDash \forall x_1 \ldots x_n (Fx_1 \ldots x_n \leftrightarrow \varphi(x_1,\ldots,x_n,a_1,\ldots,a_k)).$$

Since there are denumerably many such sequences and denumerably many formulas $\varphi(x_1,\ldots,y_k)$, the desired conclusion (even with $\cong$ replaced by $\equiv$) follows. In fact, for every (infinite) model $(\mathcal{A}, R)$ of T,

$$|\{S: (\mathcal{A}, S) \cong (\mathcal{A}, R)\}| \leq |A|.$$

$\Leftarrow$. Suppose F is not definable using parameters in T. For simplicity we assume that F is a one-place predicate. The idea is to find a model $(\mathcal{A}, X)$ of T for which the following is true: For every ω-sequence σ of 0's and 1's, there is an automorphism f_σ of $\mathcal{A}$ such that if for some number k, $\sigma(3k) \neq \sigma'(3k)$, then $f_\sigma(X) \neq f_{\sigma'}(X)$.

We assume, for simplicity, that $\ell_T = \{P,g,c\}$, where P is a two-place predicate, g is a one-place function symbol, and c is a constant. Let $\mathcal{B}$ be any model of T. Then for all n and all $b_1,\ldots,b_k \in B$, $F^{\mathcal{B}}$ is not definable in $(\mathcal{B} | \ell, b_1,\ldots,b_k)$. By Corollary 3.14, it follows that

(+) $\quad$ for all n, k and all $b_1,\ldots,b_k \in B$, there are $b, b' \in B$ such that $(\mathcal{B} | \ell, b_1,\ldots,b_k,b) \cong_n (\mathcal{B} | \ell, b_1,\ldots,b_k,b')$, $b \in F^{\mathcal{B}}$, and $b' \notin F^{\mathcal{B}}$.

Let Ψ be the set of universal closures of the following formulas:

$\exists y(xI_1y)$, $\exists x(xI_1y)$,

$x_1 \ldots x_k I_k y_1 \ldots y_k \rightarrow \exists y(x_1 \ldots x_k x I_{k+1} y_1 \ldots y_k y)$,

---------"--------- $\rightarrow \exists x(x_1 \ldots x_k x I_{k+1} y_1 \ldots y_k y)$,

---------"--------- $\rightarrow (x_i = x_j \leftrightarrow y_i = y_j)$,

---------"--------- $\rightarrow (Px_ix_j \leftrightarrow Py_iy_j)$,
--------"---------- $\rightarrow (g(x_i) = x_j \leftrightarrow g(y_i) = y_j)$,
---------"--------- $\rightarrow (x_i = c \leftrightarrow y_i = c)$.

Let Φ be the union of $T \cup \Psi$ and the set of the following sentences:

"I_k is an equivalence relation", $k = 1, 2, \ldots$
$\forall x_1 \ldots x_k \exists y_1 y_2 (Fy_1 \wedge \neg Fy_2 \wedge x_1 \ldots x_k y_1 I_{k+1} x_1 \ldots x_k y_2)$, $k = 1, 2, \ldots$

By (+) and (the proof of) Lemma 3.13, every finite subset of Φ has a model. Thus, Φ has a countable model $\mathcal{C}$. Let $\mathcal{A} = \mathcal{C} | \ell$ and $X = F^{\mathcal{C}}$. Let $I = \bigcup\{I_k^{\mathcal{C}}: k = 1, 2, \ldots\}$. Then

(1) $I: \mathcal{A} \cong_{\omega} \mathcal{A}$,
(2) I is an equivalence relation,
(3) for all $c_1, \ldots, c_n \in A$, there are $c, c' \in A$ such that $\langle c_1, \ldots, c_n, c\rangle I \langle c_1, \ldots, c_n, c'\rangle$, $c \in X$, and $c' \notin X$.

Let $A = \{a_n: n \in N\}$. We use s to denote finite sequences of 0's and 1's. si is the sequence s followed by i. Let $|s|$ be the length of s, and for $n \leq |s|$, let $s|n$ be the sequence of the first n members of s. For each s we now define inductively $b_s \in A$ and a function f_s so that

(4) $f_s \subseteq f_{si}$, $i = 0, 1$,
(5) $\langle b_{s|1}, \ldots, b_{s|n}\rangle I \langle f_s(b_{s|1}), \ldots, f_s(b_{s|n})\rangle$,
(6) if $|s| = 3k$, then $b_{s0} = b_{s1} = a_k$,
(7) if $|s| = 3k+1$, then $b_{s0} = b_{s1}$ and $f_{s0}(b_{s0}) = f_{s1}(b_{s1}) = a_k$,
(8) if $|s| = 3k+2$, then $b_{s0} \in X$, $b_{s1} \notin X$, $f_{s0}(b_{s0}) = f_{s1}(b_{s1})$.

Let $f_{\emptyset}$ be the empty function. Now, suppose f_s has been defined and $|s| = n-1$. We define b_{si} and f_{si} as follows. We always assume that (4) is satisfied.

First suppose $n = 3k$. Let $b_{s0} = b_{s1} = a_k$. By (5), there is an $a \in A$ such that

$$\langle b_{s|1}, \ldots, b_{s|n}, a_k\rangle I \langle f_s(b_{s|1}), \ldots, f_s(b_{s|n}), a\rangle.$$

Let $f_{s0}(a_k) = f_{s1}(a_k) = a$. Then (5), (6) and, trivially, (7), (8) are satisfied.

Next suppose $n = 3k+1$. There is a $b \in A$ such that

$$\langle b_{s|1}, \ldots, b_{s|n}, b\rangle I \langle f_s(b_{s|1}), \ldots, f_s(b_{s|n}), a_k\rangle.$$

Let $b_{s0} = b_{s1} = b$ and $f_{s0}(b) = f_{s1}(b) = a_k$. (Thus, if $b = b_{s|k}$ for some $k \leq n$, then $f_{s0}(b) = f_{s1}(b) = f_s(b_{s|k})$.) Then (5), (7) are satisfied.

Finally suppose $n = 3k+2$. By (3), there are b, b' such that $b \in X$, $b' \notin X$,

$$\langle b_{s|1}, \ldots, b_{s|n}, b\rangle I \langle b_{s|1}, \ldots, b_{s|n}, b'\rangle.$$

By (5), there is an $a \in A$ such that

$$\langle b_{s|1}, \ldots, b_{s|n}, b\rangle I \langle f_s(b_{s|1}), \ldots, f_s(b_{s|n}), a\rangle.$$

But then, by (2),

$$\langle b_{s|1}, \ldots, b_{s|n}, b'\rangle I \langle f_s(b_{s|1}), \ldots, f_s(b_{s|n}), a\rangle.$$

Let $b_{s0} = b$, $b_{s1} = b'$, and $f_{si} = f_s \cup \{\langle b_{si}, a\rangle\}$, $i = 0, 1$. Then (5), (8) are satisfied.

For every ω-sequence σ of 0's and 1's, let

$f_\sigma = \bigcup\{f_s\text{: s initial segment of } \sigma\}$

and let $Y_\sigma = f_\sigma(X)$. Then f_σ: $(\mathcal{A}, X) \cong (\mathcal{A}, Y_\sigma)$. Suppose there is a number 3k such that $\sigma(3k) \neq \sigma'(3k)$. Let n be the least such number. We may assume that $\sigma(n) = 0$ and $\sigma'(n) = 1$. By (6), (7), (8), there are $a \in X$, $a' \notin X$ such that $f_\sigma(a) = f_{\sigma'}(a')$. $f_\sigma(a) \in Y_\sigma$. Also, since $a' \notin X$ and $f_{\sigma'}$ is 1-1, $f_{\sigma'}(a') \notin Y_{\sigma'}$ and so $Y_\sigma \neq Y_{\sigma'}$. Since there are $> \aleph_0$, in fact $2^{\aleph_0}$, sequences σ which differ on some number 3k, it follows that there are $> \aleph_0$ sets Y such that $(\mathcal{A}, Y) \cong (\mathcal{A}, X)$, as desired. ■

APPENDIX 4

Suppose $\mathscr{F}$, $\mathscr{F}'$ are (ordered) fields. $\mathscr{F}'$ is a *real closure* of $\mathscr{F}$ if $\mathscr{F}'$ is real closed, $\mathscr{F} \subseteq \mathscr{F}'$, and there is no real closed (ordered) field $\mathscr{F}''$ such that $\mathscr{F} \subseteq \mathscr{F}'' \subsetneq \mathscr{F}'$.

To prove Theorem 3.14 we need the following:

Algebraic Lemma. (a) If $\mathscr{F}$, $\mathscr{F}'$ are (ordered) fields, $\mathscr{F} \subseteq \mathscr{F}'$, and $X \subseteq F'$, there is a least (ordered) subfield $\mathscr{F}(X) = (F(X),...)$ of $\mathscr{F}'$ such that $\mathscr{F} \subseteq \mathscr{F}(X)$ and $X \subseteq F(X)$. (We write $\mathscr{F}(a_1,...,a_n)$ for $\mathscr{F}(\{a_1,...,a_n\})$.

(b) If $\mathscr{F} \subseteq \mathscr{F}^*$ and $\mathscr{F}^*$ is a real closed ordered field, there is a unique subfield $\mathscr{F}'$ of $\mathscr{F}^*$ which is a real closure of $\mathscr{F}$; $\mathscr{F}'$ is *the* real closure of $\mathscr{F}$ in $\mathscr{F}^*$.

(c) Suppose $\mathscr{F}_0$, $\mathscr{F}_1$, $\mathscr{F}_0'$, $\mathscr{F}_1'$ are real closed ordered fields, $\mathscr{F}_0 \subseteq \mathscr{F}_0'$, $\mathscr{F}_1 \subseteq \mathscr{F}_1'$, $d_0 \in F_0' - F_0$, $d_1 \in F_1' - F_1$, and $f: \mathscr{F}_0 \cong \mathscr{F}_1$ is such that for all $a \in F_0$,

$$a \leq^{\mathscr{F}_0'} d_0 \text{ iff } f(a) \leq^{\mathscr{F}_1'} d_1.$$

Then f can be extended to an isomorphism $g: \mathscr{F}_0(d_0) \cong \mathscr{F}_1(d_1)$ such that $g(d_0) = d_1$.

(d) Suppose $\mathscr{F}_0$, $\mathscr{F}_1$ are ordered fields, $\mathscr{F}_0'$, $\mathscr{F}_1'$ are real closures of $\mathscr{F}_0$, $\mathscr{F}_1$, respectively, and $g: \mathscr{F}_0 \cong \mathscr{F}_1$. Then g can be extended to an isomorphism $h: \mathscr{F}_0' \cong \mathscr{F}_1'$.

Here (a), (b) are clear but (c), (d) are substantial (classical) mathematical results.

Proof of Theorem 3.14. Let $\mathscr{F}$, $\mathscr{F}'$ be any models of RCOF and suppose $\mathscr{F} \subseteq \mathscr{F}'$. By Theorem 3.13, it is sufficient to show that $\mathscr{F} \preccurlyeq_1 \mathscr{F}'$.

Let φ be any simple existential sentence of $\ell_{\mathscr{F}}(F)$ such that $\mathscr{F}'_F \vDash \varphi$; $\varphi := \exists x_1...x_n\psi(x_1,...,x_n)$, where $\psi(x_1,...,x_n)$ is quantifier-free. There are then $a_1,...,a_n \in F'$ such that $\mathscr{F}' \vDash \psi(a_1,...,a_n)$. It follows that $\mathscr{F}(a_1,...,a_n)_F \vDash \varphi$. Let $\mathscr{F}_0 = \mathscr{F}$ and let $\mathscr{F}_{k+1}$ be the real closure of $\mathscr{F}_k(a_k)$ in $\mathscr{F}'$, $k < n$. Then $\mathscr{F}(a_1,...,a_n) = \mathscr{F}_n$ and so $\mathscr{F}_{nF} \vDash \varphi$. $\preccurlyeq_1$ is a transitive relation. It follows that if $\mathscr{F}_k \preccurlyeq_1 \mathscr{F}_{k+1}$ for $k < n$, then $\mathscr{F} \preccurlyeq_1 \mathscr{F}_n$ and so $\mathscr{F}_F \vDash \varphi$, as desired.

In view of this, it is sufficient to consider the case where $\mathscr{F}'$ is a real closure of $\mathscr{F}(d)$ for some $d \in F' - F$.

Let c be a new constant and let

$$\Phi = \{c_a < c: a \in F \ \& \ (\mathscr{F}',d) \vDash c_a < c\} \cup \{c < c_a: a \in F \ \& \ (\mathscr{F}',d) \vDash c < c_a\}.$$

Next, let

$$\Psi = \text{RCOF} \cup \Phi \cup D(\mathscr{F}).$$

We want to show that $\Psi \vDash \varphi$. Let $\mathscr{G}$ be any model of Ψ. We are going to show that $\mathscr{G} \vDash \varphi$. (By assumption $\mathscr{F}'_F \vDash \varphi$.) For $a \in F$, let $f(a) = c_a^{\mathscr{G}}$. Let $\mathscr{G}'$ be the image of $\mathscr{F}$ under f. Then $f: \mathscr{F} \cong \mathscr{G}'$. Let $e = c^{\mathscr{G}}$. $d \in F' - F$. Since $\mathscr{G} \vDash \text{RCOF} \cup \Phi$, this implies that $e \notin \mathscr{G}'$. Also, for all $a \in F$,

$$a \leq^{\mathcal{F}'} d \text{ iff } f(a) \leq^{\mathcal{G}} e.$$

By (c) of the Algebraic Lemma, it follows that f can be extended to an isomorphism g: $\mathcal{F}(d) \cong \mathcal{G}'(e)$ such that $g(d) = e$. Let $\mathcal{G}''$ be the real closure of $\mathcal{G}'(e)$ in $\mathcal{G}$. Then, by (d) of the Algebraic Lemma, g can be extended to an isomorphism h: $\mathcal{F}' \cong \mathcal{G}''$. Since $\mathcal{F}'_F \vDash \varphi$, it follows that $\mathcal{G}''_{hF} \vDash \varphi$ and so $\mathcal{G} \vDash \varphi$, as desired. This shows that $\Psi \vDash \varphi$.

By compactness, it now follows that there are a, $a' \in F$ such that

$$\mathrm{RCOF} \cup \{\psi(c)\} \cup D(\mathcal{F}) \vDash \varphi,$$

where $\psi(x) := c_a < x$ or $\psi(x) := x < c_{a'}$ or $\psi(x) := c_a < x \wedge x < c_{a'}$, and $\mathcal{F}'_F \vDash \psi(c)$. In the first case let $b = a+1$, in the second case let $b = a'-1$. In the third case it follows that $\mathcal{F}'_F \vDash c_a < c_{a'}$, whence $\mathcal{F}_F \vDash c_a < c_{a'}$. Let $b = (a + a')/2$ (in $\mathcal{F}$). Then $(\mathcal{F}_F, b) \vDash \psi(c)$. Since $\mathcal{F}_F \vDash \mathrm{RCOF} \cup D(\mathcal{F})$, it now follows that $\mathcal{F}_F \vDash \varphi$, as desired.

Thus, we have shown that $\mathcal{F} \preccurlyeq_1 \mathcal{F}'$ and the proof is complete. ■

APPENDIX 5

The theory in Chapter 3, §13, Example 9, having no prime model has infinitely many nonlogical symbols. The question arises if there is a complete theory T such that ℓ_T is finite and T has no prime model. We are going to show that the answer is affirmative.

Let Z be the set of integers, let S be the successor relation on Z, and let $\mathcal{S}$ = (Z, S). We are going to define a subset X of Z such that the theory of T = Th(($\mathcal{S}$, X)) has no prime model.

If f is an embedding of $\mathcal{S}$ in $\mathcal{S}$, there is a $j \in Z$ such that for all $i \in Z$, $f(i) = i+j$; and so f: $\mathcal{S} \cong \mathcal{S}$. It follows that for any two sets X, Y, if ($\mathcal{S}$, X) is embeddable in ($\mathcal{S}$, Y), then $(\mathcal{S}, X) \cong (\mathcal{S}, Y)$. Thus, it is sufficient to show that there are sets X and Y such that $(\mathcal{S}, X) \equiv (\mathcal{S}, Y)$ and $(\mathcal{S}, X) \not\cong (\mathcal{S}, Y)$. We do this by using a, particularly simple, form of *forcing*, as follows.

Let $\ell = \{S, F\}$, where F is a one-place predicate. In what follows i, j are arbitrary integers. Let c_i, $i \in Z$, be individual constants. A (*forcing*) *condition* is a consistent finite set (including the empty set Ø) of formulas of the form Fc_i and $\neg Fc_i$. Let p, q, r ... be forcing conditions.

We now assume that ¬, ∨, ∃ are the only propositional connectives and quantifier; ∧, →, ∀ are regarded as defined symbols. The relation *forces*, $\Vdash$, between conditions and sentences of $\ell \cup \{c_i\colon i \in Z\}$ is defined as follows.

$p \Vdash Fc_i$ iff $Fc_i \in p$,

$p \Vdash c_i = c_j$ iff $i = j$,

$p \Vdash Sc_ic_j$ iff $j = i+1$,

$p \Vdash \varphi \vee \psi$ iff $p \Vdash \varphi$ or $p \Vdash \psi$,

$p \Vdash \exists x\varphi(x)$ iff there is a c_i such that $p \Vdash \varphi(c_i)$,

$p \Vdash \neg\varphi$ iff there is no condition q such that $p \subseteq q$ and $q \Vdash \varphi$.

From this definition it follows at once, for example, that (i) $p \Vdash \neg Fc_i$ iff $\neg Fc_i \in p$, (ii) $p \Vdash c_i \neq c_j$ iff $i \neq j$, $p \Vdash \neg Sc_ic_j$ iff $j \neq i+1$, (iii) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \neg(\neg\varphi \vee \neg\psi)$ iff for every $q \supseteq p$, there are r, r' such that $q \subseteq r$, $q \subseteq r'$, $r \Vdash \varphi$ and $r' \Vdash \psi$, (iv) $p \Vdash \forall x\varphi(x)$ iff for every $q \supseteq p$ and every c_i, there is an $r_i \supseteq q$ such that $r_i \Vdash \varphi(c_i)$.

Note that (v) the fact that φ is valid does not imply that $p \Vdash \varphi$ and that (vi) the fact that $\varphi \to \psi$ is valid does not imply that if $p \Vdash \varphi$, then $p \Vdash \psi$; in particular, $p \Vdash \neg\neg\varphi$ does not imply $p \Vdash \varphi$. For example, $\varnothing \Vdash \neg\neg(Fc_i \vee \neg Fc_i)$ and $\varnothing \nVdash Fc_i \vee \neg Fc_i$.

The following three lemmas are standard.

Lemma 1. (i) For all sentences φ and forcing conditions p, if $p \Vdash \neg\varphi$, then $p \nVdash \varphi$.

(ii) For all sentences φ and forcing conditions p, q, if $p \Vdash \varphi$ and $p \subseteq q$, then $q \Vdash \varphi$.

Proof. (i) is clear. We prove (ii) by induction on the complexity of φ. The statement is clear for atomic φ. The inductive steps corresponding to $\vee$ and $\exists$ are straightforward. Finally, suppose $\varphi := \neg\psi$ and the statement is true of ψ. Let p, q be such that $p \Vdash \varphi$ and $p \subseteq q$. Then for every $r \supseteq p$, $r \nVdash \psi$. But then for every $r \supseteq q$, $r \nVdash \psi$. Hence $q \Vdash \varphi$. ■

From this lemma it follows at once that if $p \Vdash \varphi$ and $p \Vdash \psi$, then $p \Vdash \varphi \wedge \psi$ and that if $p \Vdash \varphi(c_i)$ for every i, then $p \Vdash \forall x\varphi(x)$. It also follows that $p \Vdash \varphi$ implies $p \Vdash \neg\neg\varphi$ and that $p \Vdash \neg\neg\neg\varphi$ implies $p \Vdash \neg\varphi$.

Let C be a set of forcing conditions. We write $C \Vdash \varphi$ to mean that $p \Vdash \varphi$ for some $p \in C$. C is (*forcing-*) *consistent* if there is no sentence φ such that $C \Vdash \varphi$ and $C \Vdash \neg\varphi$. C is (*forcing-*) *complete* if for every sentence φ, either $C \Vdash \varphi$ or $C \Vdash \neg\varphi$.

Lemma 2. For every condition p, there is a complete consistent set C of conditions such that $p \in C$.

Proof. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an enumeration of all sentences of $\ell \cup \{c_i : i \in Z\}$. We now define p_n, $n \in N$, as follows. Let $p_0 = p$. Suppose p_n has been defined. Either $p_n \Vdash \neg\varphi_n$ or $p_n \nVdash \neg\varphi_n$. In the first case let $p_{n+1} = p_n$. In the second case there is a $q \supseteq p$ such that $q \Vdash \varphi_n$. Let p_{n+1} be some such q. Let $C = \{p_n : n \in N\}$. Then $p \in C$ and C is complete: for every n, either $p_{n+1} \Vdash \varphi_n$ or $p_{n+1} \Vdash \neg\varphi_n$. Finally, since $p_m \subseteq p_n$ for $m \leq n$, by Lemma 1, C is consistent. ■

In $(\mathcal{S}, X)_Z$ for every $i \in Z$, i is denoted by c_i. If C is a set of conditions, let $[C] = \{i \in Z : C \Vdash Fc_i\}$.

Lemma 3. If C is complete and consistent, for every sentence φ, $(\mathcal{S}, [C])_Z \models \varphi$ iff $C \Vdash \varphi$.

Proof. By induction. The statement is clear for atomic sentences φ. The inductive steps corresponding to $\vee$ and $\exists$ are straightforward. Finally, suppose $\varphi := \neg\psi$ and the statement is true for ψ. First, suppose $(\mathcal{S}, [C])_Z \models \varphi$. Then $(\mathcal{S}, [C])_Z \nvDash \psi$, whence, by the inductive assumption, $C \nVdash \psi$. But then, C being complete, $C \Vdash \varphi$. Next, suppose $C \Vdash \varphi$. Then, C being consistent, $C \nVdash \psi$, whence $(\mathcal{S}, [C])_Z \nvDash \psi$ and so $(\mathcal{S}, [C])_Z \models \varphi$, as desired. ■

It may be observed that if we had defined $p \Vdash \forall x\varphi(x)$ to mean that for every c_i, $p \Vdash \varphi(c_i)$, Lemma 3 would not have been true.

A subset X of Z is *generic* if X = [C] for some complete consistent set C. From Lemmas 1, 2, 3 it follows that for every sentence φ, $\varnothing \Vdash \neg\neg\varphi$ iff for every generic set X, $(\mathcal{S}, X)_Z \models \varphi$.

By Lemma 2, there is a generic set. We need at bit more:

Lemma 4. There are generic sets X, Y such that $(\mathcal{S}, X) \not\equiv (\mathcal{S}, Y)$.

Proof. Let $\varphi_0, \varphi_1, \varphi_2, \dots$ be as in the proof of Lemma 2. Let $j_0, j_1, j_2, \dots$ be an enumeration of all integers. We define p_n, q_n and integers i_n as follows. Let $p_0 = q_0 = \varnothing$. Suppose p_n, q_n have been defined. Let i_n be such that $\neg Fc_{i_n} \notin p_n$ and $Fc_{i_n+j_n} \notin q_n$. Next, as in the proof of Lemma 2, let p_{n+1} be such that $p_n \cup \{Fc_{i_n}\} \subseteq p_{n+1}$ and either $p_{n+1} \Vdash \varphi_n$ or $p_{n+1} \Vdash \neg\varphi_n$. Similarly, let q_{n+1} be such that $q_n \cup \{\neg Fc_{i_n+j_n}\} \subseteq q_{n+1}$ and either $q_{n+1} \Vdash \varphi_n$ or $q_{n+1} \Vdash \neg\varphi_n$. Let $C = \{p_n : n \in N\}$, $D = \{q_n : n \in N\}$. Then C, D are complete and consistent. Let $X = [C]$ and $Y = [D]$. Then X, Y are generic.

Suppose $f: \mathcal{S} \cong \mathcal{S},$. There is then a j such that for all i, $f(i) = i+j$. For some n, $j = j_n$. Now, $Fc_{i_n} \in p_{n+1}$ and $\neg Fc_{i_n+j_n} \in q_{n+1}$, whence $C \Vdash Fc_{i_n}$ and $D \Vdash \neg Fc_{i_n+j_n}$ and so $i_n \in X$ and $f(i_n) = i_n+j \notin Y$. It follows that f is not an isomorphism of $(\mathcal{S}, X)$ onto $(\mathcal{S}, Y)$. And so $(\mathcal{S}, X) \not\equiv (\mathcal{S}, Y)$, as desired. ■

Lemma 4 can easily be strengthened: There are $2^{\aleph_0}$ many generic sets. Since for every X, there are only denumerably many Y for which $(\mathcal{S}, X) \cong (\mathcal{S}, Y)$, it follows that there is a set G of generic sets of cardinality $2^{\aleph_0}$ such that if $X, Y \in G$ and $X \neq Y$, then $(\mathcal{S}, X) \not\equiv (\mathcal{S}, Y)$.

For any formula φ and integer i, let φ^i be obtained from φ by replacing each constant c_j occurring in φ by c_{j+i}. Let $p^i = \{\varphi^i : \varphi \in p\}$.

Lemma 5. $p^i \Vdash \varphi^i$ iff $p \Vdash \varphi$.

Proof. Induction. Since $(\varphi^i)^{-i} := \varphi$ and $(p^i)^{-i} = p$, it is sufficient to prove "if". The statement is clear for atomic φ. The inductive step corresponding to $\vee$ is easy, since $(\varphi \vee \psi)^i := \varphi^i \vee \psi^i$. Suppose $p \Vdash \exists x\psi(x)$. Let c_j be such that $p \Vdash \psi(c_j)$. By the inductive assumption, $p^i \Vdash \psi(c_j)^i$ and so $p^i \Vdash \psi^i(c_{j+i})$ and so $p^i \Vdash (\exists x\psi(x))^i$. Finally, suppose $p^i \nVdash \neg\psi^i$. Let $q \supseteq p^i$ be such that $q \Vdash \psi^i$. By the inductive assumption (with i replaced by –i), $q^{-i} \Vdash \psi$. Also $p \subseteq q^{-i}$. It follows that $p \nVdash \neg\psi$, as desired. ■

Lemma 6. If X, Y are generic, then $(\mathcal{S}, X) \equiv (\mathcal{S}, Y)$.

Proof. Suppose $(\mathcal{S}, X) \not\equiv (\mathcal{S}, Y)$. Let φ be a sentence of ℓ such that $(\mathcal{S}, X) \vDash \varphi$ and $(\mathcal{S}, Y) \vDash \neg\varphi$. Let C, D be complete consistent sets of conditions such that $X = [C]$ and $Y = [D]$. By Lemma 3, $C \Vdash \varphi$ and $D \Vdash \neg\varphi$. Let $p \in C$ and $q \in D$ be such that $p \Vdash \varphi$ and $q \Vdash \neg\varphi$. Since p, q are finite, there is a j such that $r = p \cup q^j$ is a condition. Since φ contains no c_i, it follows, by Lemma 5, that $q^j \Vdash \neg\varphi$. But then, by Lemma 1(ii), $r \Vdash \varphi$ and $r \Vdash \neg\varphi$, contradicting Lemma 1(i). ■

Let X be a generic set and let $T = \mathrm{Th}((\mathcal{S}, X))$. By Lemma 4, there is a generic set Y such that $(\mathcal{S}, X) \not\equiv (\mathcal{S}, Y)$. By Lemma 6, $(\mathcal{S}, Y) \vDash T$. Now, let $\mathcal{A}$ be any model of T. If $\mathcal{A}$ is embeddable in $(\mathcal{S}, X)$, then, as mentioned above, $\mathcal{A} \cong (\mathcal{S}, X)$, whence $\mathcal{A} \not\equiv (\mathcal{S}, Y)$ and so $\mathcal{A}$ is not embeddable in $(\mathcal{S}, Y)$. Thus, $\mathcal{A}$ is not a prime model of T. And so T is a complete theory in a finite language and T has no prime model, as desired.

Modifying this example, we now define a model $\mathcal{B}$ (for a finite language) such that $\mathrm{Th}(\mathcal{B})$ has a prime model but no elementarily prime model. Let $\mathcal{S}^* = (Z, \leq)$, where $\leq$ is understood as usual. Let the *forcing* relation $\Vdash^*$ be defined in the same way as $\Vdash$ except that the third clause is replaced by:

$$p \Vdash^* c_i \leq c_j \text{ iff } i \leq j.$$

The (new) notions *complete* and *consistent* set of forcing conditions and *generic* (subset of Z) are then defined in terms of $\Vdash^*$ in the same way as before. As is easily checked, Lemmas 1 – 6 (with $\mathcal{S}$ replaced by $\mathcal{S}^*$ and $\Vdash$ by $\Vdash^*$) carry over to this setting.

Let X be a generic set and let $\mathcal{B} = (\mathcal{S}^*, X)$ and $T^* = \mathrm{Th}(\mathcal{B})$. Let φ be the conjunction of the sentences

$$\forall x(\exists y(x \leq y \wedge Fy) \wedge \exists y(x \leq y \wedge \neg Fy)),$$
$$\forall x(\exists y(y \leq x \wedge Fy) \wedge \exists y(y \leq x \wedge \neg Fy)).$$

Then $\varnothing \Vdash^* \varphi$ and so $\mathcal{B} \vDash \varphi$. It is not difficult to see that any model of the form $(\mathcal{S}^*, Y)$ is embeddable in any model of this form in which φ is true. It follows that every such model of T^* is a prime model of T^*. And so T^* has a prime model; in fact, T^* has $2^{\aleph_0}$ many non-isomorphic prime models. Finally, if f is an elementary embedding of $\mathcal{S}^*$ in $\mathcal{S}^*$, then $f\colon \mathcal{S}^* \cong \mathcal{S}^*$. But then, reasoning in the same way as above, we may conclude that T^* has no elementarily prime model.

APPENDIX 6

Proof of Theorem 3.31. If this holds for an infinite set X, it holds for all sets of cardinality $\geq |X|$. Thus, it is sufficient to consider the case where X is the set of natural numbers N.

For $n = 1$ the statement is clear. We assume it holds for n and show that it holds for n+1. Suppose $N^{[n+1]} = Z_0 \cup ... \cup Z_m$. We (may) assume that the sets Z_i are disjoint.

We define subsets X_k of N, numbers m_k, and a function f: $N \to \{0,...,m\}$ such that $X_0 \supseteq X_1 \supseteq X_2 ...$, each X_k is infinite, m_k is the least member of X_k, and $m_0 < m_1 < m_2 < ...$ in the following way. The idea is to make sure that if $k < k_1 < ... < k_n$, then the i for which $\{m_k, m_{k_1}, ..., m_{k_n}\} \in Z_i$ is determined by k.

Let $X_0 = N$ and $m_0 = 0$. Suppose X_k and m_k have been defined. Let

$$Z_{k,i} = \{\{r_1,...,r_n\} \in X_k^{[n]} : m_k < r_1 < ... < r_n \ \& \ \{m_k, r_1, ..., r_n\} \in Z_i\}.$$

Then $Z_{k,0} \cup ... \cup Z_{k,m} = (\{r: m_k < r\} \cap X_k)^{[n]}$. Thus, by assumption, there are an infinite subset X_{k+1} of $\{r: m_k < r\} \cap X_k$ and a number $i \leq m$ such that $X_{k+1}^{[n]} \subseteq Z_{k,i}$. It follows that

$$\{\{m_k, r_1, ..., r_n\}: r_1, ..., r_n \in X_{k+1}\} \subseteq Z_i.$$

Let m_{k+1} be the least member of X_{k+1} and let $f(k) = i$. This completes the definition of the sets X_k, numbers m_k, and the function f.

Let $j \leq m$ be such that $\{k: f(k) = j\}$ is infinite. Let $Y = \{m_k: f(k) = j\}$. Y is infinite. Finally, $Y^{[n+1]} \subseteq Z_j$. For suppose $m_k, m_{k_1}, ..., m_{k_n} \in Y$, where $k < k_1 < ... < k_n$. Then $f(k) = j$. $m_{k_i} \in X_{k_i}$ and $k_i > k$, $i = 1,...,n$. It follows that $\{m_{k_1}, ..., m_{k_n}\} \in X_{k+1}^{[n]}$. Now $X_{k+1}^{[n]} \subseteq Z_{k,j}$ and so $\{m_{k_1}, ..., m_{k_n}\} \in Z_{k,j}$. But then $\{m_k, m_{k_1}, ..., m_{k_n}\} \in Z_j$, as was to be shown. ■

Proof of Theorem 3.33. This is clear for $n = 0$. We assume the statement for n and show that it holds for n+1. We (may) assume that the sets Z_i, $i \in I$, are disjoint. Let $\lambda_k = 2_k(\kappa)$. For $Z \subseteq X$ we define the equivalence relation $\sim_Z$ on $X - Z$ as follows:

$$a \sim_Z b \text{ iff } a, b \in X - Z \text{ and}$$
$$\text{for all } c_0, ..., c_n \in Z \text{ and all } i \in I, \{c_0, ..., c_n, a\} \in Z_i \text{ iff } \{c_0, ..., c_n, b\} \in Z_i.$$

(If $|Z| \leq n$, then trivially $a \sim_Z b$ for all $a, b \in X - Z$.) For every $a \in X - Z$, the $\sim_Z$-equivalence class of a is uniquely determined by the function f: $Z^{n+1} \to I$ such that

$$f(c_0, ..., c_n) = i \text{ if } \{c_0, ..., c_n, a\} \in Z_i.$$

There are $|I|^{|Z^{n+1}|}$ such functions. Hence

(1) if $|Z| \leq \lambda_n$, then $\sim_Z$ has $\leq \lambda_{n+1}$ equivalence classes.

We can now define sets $X_\xi \subseteq X$, $\xi < (\lambda_n)^+$, such that $|X_\xi| = \lambda_{n+1}$, $X_\xi \subseteq X_\eta$ for $\xi < \eta$, and

(2) for every $Z \subseteq X_\xi$ such that $|Z| \leq \lambda_n$ and every $a \in X - X_\xi$, there is a $b \in X_{\xi+1}$ such that $a \sim_Z b$.

Let X_0 be any subset of X of cardinality λ_{n+1}. Suppose $0 < \xi < (\lambda_n)^+$ and X_η has been defined for $\eta < \xi$. Suppose ξ is a successor ordinal; $\xi = \eta+1$. For every $Z \subseteq X_\eta$ such that $|Z| \leq \lambda_n$, let Y_Z be a set containing exactly one member of each equivalence class of $\sim_Z$. Then, by (1), $|Y_Z| \leq \lambda_{n+1}$. Let

$$X_\xi = X_\eta \cup \bigcup\{Y_Z: Z \subseteq X_\eta \text{ and } |Z| \leq \lambda_n\}.$$

X_η has λ_{n+1} subsets of cardinality $\leq \lambda_n$. Thus, X_ξ is as desired.

If ξ is a limit ordinal, let $X_\xi = \bigcup\{X_\eta: \eta < \xi\}$. Then again X_ξ is as desired.

Now let

$$X' = \bigcup\{X_\xi: \xi < (\lambda_n)^+\}.$$

Then $|X'| = \lambda_{n+1}$. By hypothesis $|X| > \lambda_{n+1}$ and so $X - X' \neq \emptyset$. Let $c \in X - X'$.

By (2), for each $\xi < (\lambda_n)^+$, there is a $b_\xi \in X_{\xi+1}$ such that $b_\xi \sim_{\{b_\eta:\, \eta<\xi\}} c$ and $b_\xi \neq b_\eta$ for $\eta < \xi$. Let $U = \{b_\xi: \xi < (\lambda_n)^+\}$. Then $|U| = (\lambda_n)^+$. Let $V_i \subseteq U^{[n+1]}$, $i \in I$, be such that for $d_0,\dots,d_n \in U$,

$$\{d_0,\dots,d_n\} \in V_i \text{ iff } \{d_0,\dots,d_n,c\} \in Z_i.$$

Then $U^{[n+1]} = \bigcup\{V_i: i \in I\}$. Thus, by the inductive hypothesis, there are a $j \in I$ and a set $Y \subseteq U$ such that $|Y| > \kappa$ and $Y^{[n+1]} \subseteq V_j$.

Suppose now $b_{\xi_0},\dots,b_{\xi_n},b_{\xi_{n+1}} \in Y$, where $\xi_0 < \dots < \xi_n < \xi_{n+1}$. Then $\{b_{\xi_0},\dots,b_{\xi_n}\} \in V_j$ and so $\{b_{\xi_0},\dots,b_{\xi_n},c\} \in Z_j$. But $\{b_\eta: \eta < \xi_{n+1}\} \subseteq X_{\xi_{n+1}}$ and $|\{b_\eta: \eta < \xi_{n+1}\}| \leq \lambda_n$. It follows that $b_{\xi_{n+1}} \sim_{\{b_\eta:\, \eta<\xi_{n+1}\}} c$ and so $\{b_{\xi_0},\dots,b_{\xi_n},b_{\xi_{n+1}}\} \in Z_j$. Thus, $Y^{[n+2]} \subseteq Z_j$ and the proof is complete. ■

APPENDIX 7

Two (natural) numbers k, n are *relatively prime* if they have no (prime) factor in common. Let rm(k,n), where $n \neq 0$, denote the remainder of k upon division by n, i.e., the number $r < n$ for which there is a number q such that $k = qn + r$.

Number-theoretic Lemma. If k, n are relatively prime and $n > 1$, there is a number q such that $rm(kq,n) = 1$.

Proof. Suppose $q < q' < n$ and $rm(kq,n) = rm(kq',n)$. Then n divides $kq' - kq = k(q' - q)$. But this is impossible, since n has no factor in common with k and $0 < q' - q < n$. Thus, the numbers rm(kq,n) for $q < n$ are all different and $< n$. It follows that one of them is 1. ■

Chinese Remainder Theorem. Suppose the numbers n_i, $i \leq m$, are pairwise relatively prime and suppose $1 < n_i$ and $k_i < n_i$, $i \leq m$. There is then a number r such that $rm(r,n_i) = k_i$, $i \leq m$.

Proof. Let $q_i = n_0 \ldots n_{i-1} n_{i+1} \ldots n_m$. Then n_i and q_i are relatively prime. By the Number-theoretic Lemma, there are numbers p_i such that $rm(q_i p_i, n_i) = 1$, $i \leq m$. Let $r = k_0 q_0 p_0 + \ldots + k_m q_m p_m$. Then $rm(k_i q_i p_i, n_i) = k_i$ and $rm(k_j q_j p_j, n_i) = 0$ for $j \neq i$. It follows that $rm(r,n_i) = k_i$. ■

Proof of Lemma 4.2. Suppose m, n, and k_i, $i \leq n$, are given. Let $m' = \max\{m,n,k_0,\ldots,k_n\} + 1$ and let $s = 1 \cdot 2 \cdot \ldots \cdot m'$. The numbers $1+(i+1)s$, $i = 0,\ldots,n$, are then > 1 and relatively prime. For suppose $i < j \leq n$ and that p is a prime dividing $1+(i+1)s$ and $1+(j+1)s$. Then p divides $(j-i)s$. Since $j - i \leq n \leq m'$ and p either divides $j - i$ or s, it follows that $p \leq m'$. But then p divides s and so p does not divide $1+(i+1)s$, a contradiction. Since $k_i < 1+(i+1)s$, it now follows, by the Chinese Remainder Theorem, that there is a number r such that for $i \leq n$, $rm(r,1+(i+1)s) = k_i$. Finally, if $k \neq k_i$ and there is a q such that $r = q(1+(i+1)s) + k$, then $k - k_i$ is $\geq s$ and so $k > m$. ■

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SYMBOLS