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A Note on the Cost-Benefit Ratio in Self-Enforcing Agreements*

Magnus Hennlock†

Abstract

Since the analysis of a self-enforcing agreement by Barrett (1994) it has been clear that the ratio between the slopes of the marginal cost and marginal benefit functions is conclusive for stability of self-enforcing agreements. For example Finus and Rundshagen (1998) stated: '*it turns out that all qualitative results depend only on this ratio*' as it determines the non-orthogonal free-riding response along Nash reaction functions. This note shows that this 'pure' connection between the cost-benefit ratio and non-orthogonal free-riding response occurs due to the 'anonymous contributions' property of public goods, and in such cases the cost-benefit ratio effect holds regardless the functional form of objectives, the formulation of congestion or the degree of impureness of the public good. Therefore we expect to see the cost-benefit ratio still be the conclusive component also in self-enforcing agreements based on more general functional forms than seen hitherto in the literature.

Keywords: public goods, self-enforcing agreements, reaction function, coalition theory

JEL classification: C70, H40

1 Introduction

The growing literature on self-enforcing agreements now includes Carraro and Siniscalco (1993), Hoel (1992), Barrett (1994), Botteon and Carraro (1997), Finus and Rundshagen (1998), Na and Shin (1998), Finus (2001), Barrett (2003), Ulph (2004), Rubio and Ulph (2006), Kolstad (2007), Finus and Rübbelke (2008) amongst many others. It is well-known in this literature that non-orthogonal free-riding along a player's reaction function is determined by the ratio of the slope of the marginal cost function to the slope of

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the marginal benefit function, and therefore this ratio becomes a conclusive component for stability in self-enforcing agreements. The ratio has usually been named 'benefit-cost ratio' or 'cost-benefit ratio' in the literature.[1] This has lead research to search for aspects in modeling which can expand the range of the ratio for which a coalition can be stable. Moreover, in public good models it is also well-known that this ratio is the sole determinant of the ratio of second-order derivatives (with respect to all other players' contribution and own contribution) of the objective function expressed as an unconstrained problem.[2] Usually these derivatives become complex unless functions are simple with additively separable utility from the public and the private good. In this note we show that the nature of the so-called 'anonymous contributions' to public goods (bads) will always make several terms in these second-order derivatives identical with the result that the effect of 'the cost-benefit ratio' ('the benefit-cost ratio' in the public bad case) on stability of self-enforcing agreements must hold also for more general functional forms than seen hitherto in the literature e.g. Barrett (1994), Na and Shin (1998), Finus and Rundshagen (1998), Barrett (2003) etc. In fact, the result is valid even in cases with highly nonlinear functions and even when utility of public and private goods is not additively separable, provided that the public good satisfies the 'anonymous contributions' property - a property which covers most formulations in the public goods literature since Samuelson (1954) e.g. Young (1982), Cornes and Sandler (1985), Bergstrom, Blume, and Varian (1986), Barrett (1994) and Ray and Vohra (2001) etc. Section 2 of this note provides a generalization of the effect that the cost-benefit ratio has on non-orthogonal free-riding along the reaction function, which is followed by some concluding comments in section 3.

2 Generalizing the Effect of the Cost-Benefit Ratio

Consider a general optimization problem of player i in a public good game with anonymous contributions defined as follows: There are two goods, the public good Q and the private good y_i, and N players denoted $i \in \{1, 2, 3, ..., n\} = N$. Hereinafter, we focus only on player i and its non-cooperative behavior against the $N-1$ other players. The objective function $U_i(Q, y_i)$ is maximized subject to the constraint in the following general

[1] For a consistent symmetry throughout this note we use 'the cost-benefit ratio' to denote the ratio of the slope of marginal cost function to the slope of marginal benefit function for the public goods case and the 'benefit-cost ratio' to denote the ratio of the slope of marginal benefit function to the slope of marginal cost function, which is then the symmetrical ratio in the public bad case as explained in section 3.

[2] Let the objective function be $U(q_i, Q_{-i})$ and the reaction function $R(Q_{-i})$, then $\frac{dR_i}{dQ_{-i}} \equiv -\frac{d^2U_i}{dq_i dQ_{-i}} / \frac{d^2U_i}{dq_i^2}$

optimization problem:

$$\max_{q_i} U_i\left(Q, y_i\right) \tag{1}$$

$$\text{s.t} \quad y_i = y_i(q_i, w_i, p) \tag{2}$$

$$Q = Q(q_i, Q_{-i}) \quad j \neq i \tag{3}$$

$$q_i \geq 0 \quad \forall i \in N \tag{4}$$

The objective function (1) is continuous and at least twice differentiable and quasi-concave in Q and y_i with $\partial^2 U_i/\partial y_i \partial Q \equiv \partial^2 U_i/\partial Q \partial y_i > 0$. The constraint (2) is continuous, at least once differentiable and depicts the frontier of a convex set in the (q_i, y_i) space, reflecting the loss in consumption of the private good y_i when contributing $q_i \geq 0$ to the public good Q where (3) is a monotonous concave function, at least once differentiable. The parameters w_i and p are the endowment level and the relative price of q_i, respectively, which together determine the locus of $y_i(q_i, w_i, p)$ in the (q_i, y_i) space. Then consider the following definition:

Definition 1 *A public good Q fulfills the anonymous contributions property iff the gradient $\nabla Q(\boldsymbol{q})$ of (3) satisfies*

$$\frac{\partial Q}{\partial q_i}(Q_{-i}, q_i) \equiv \frac{\partial Q}{\partial q_j}(Q_{-i}, q_i) \equiv \frac{\partial Q}{\partial Q_{-i}}(Q_{-i}, q_i) > 0 \quad \forall i \neq j \tag{5}$$

and the Hessian $D^2 Q(\boldsymbol{q})$ satisfies

$$\frac{\partial^2 Q}{\partial q_i^2}(Q_{-i}, q_i) \equiv \frac{\partial^2 Q}{\partial q_j^2}(Q_{-i}, q_i) \equiv \frac{\partial^2 Q}{\partial q_i \partial Q_{-i}}(Q_{-i}, q_i) \leq 0 \quad \forall i \neq j \tag{6}$$

If (6) is satisfied with equality, the public good Q fulfills the weakly anonymous contributions property.

Note that definition 1 e.g. covers the standard additive public good formulation $Q = \sum_i q_i$ that is common in the literature on public goods e.g. Samuelson (1954), Bergstrom, Blume, and Varian (1986), Cornes and Sandler (1996) and Ray and Vohra (2001) amongst others.

Theorem 1 *Let a public good game be defined by (1) to (6) and let the cost-benefit ratio $\gamma_i(Q_{-i}, q_i)$ be defined as the ratio of the slope of the marginal cost function $MC_i(Q_{-i}, q_i)$ to the slope of the marginal benefit function $MB_i(Q_{-i}, q_i)$ according to:*

$$\gamma_i(Q_{-i}, q_i) \equiv \frac{\frac{\partial MC_i(Q_{-i}, q_i)}{\partial q_i}}{\frac{\partial MB_i(Q_{-i}, q_i)}{\partial q_i}} \tag{7}$$

then $\gamma_i(Q_{-i}, q_i)$ determines the slope of player i's reaction function $R_i(Q_{-i})$ via the mechanism:[3]

$$\frac{dR_i}{dQ_{-i}} = -\frac{1}{1+\gamma_i(Q_{-i}, q_i^*)} \tag{8}$$

Proof: Assuming that player i is a positive contributor, $q_i^* > 0$ and takes Q_{-i} as given under Nash conjectures, the first-order conditions of the (unconstrained) problem (1) - (6) with respect to q_i is:

$$\frac{dU_i}{dq_i} = \underbrace{\frac{\partial U_i}{\partial Q}(Q, y_i)\frac{\partial Q}{\partial q_i}}_{MB} + \underbrace{\frac{\partial U_i}{\partial y_i}(Q, y_i) \cdot \frac{\partial y_i}{\partial q_i}(q_i, w_i)}_{MC} = 0 \tag{9}$$

The term MB is the marginal benefit of q_i, which is positive due to the properties of (1) - (6). The term MC is the marginal cost of q_i, which is negative due to the same properties. Recall that, by definition, a reaction function, $R_i(Q_{-i})$, collects the union of the set of q_i and the set of Q_{-i} in the (Q_{-i}, q_i) space that preserves the optimal condition in (9) for player i under Nash conjectures. Thus any point that is off $R_i(Q_{-i})$ results in a violated optimal condition (9) for player i. The derivative of $R_i(Q_{-i})$ evaluated in the neighborhood of a point (Q_{-i}, q_i^*) is then given by totally differentiating (9) with respect to q_i and Q_{-i} given the equality in (9) be preserved. Rearranging then yields the well-known expression for the slope of the reaction function in the (Q_{-i}, q_i) space:

$$\frac{dR_i}{dQ_{-i}} \equiv -\frac{\frac{d^2U_i}{dq_i dQ_{-i}}}{\frac{d^2U_i}{dq_i^2}} \tag{10}$$

In order to shorten coming expressions we can alternatively express (10) by using the underbrace denotations in (9):

$$\frac{dR_i}{dQ_{-i}} = -\frac{MB'(Q_{-i}) + MC'(Q_{-i})}{MB'(q_i) + MC'(q_i)} \tag{11}$$

where $MB'(Q_{-i})$ and $MB'(q_i)$ denote the first-order derivatives of the term MB with respect to Q_{-i} and q_i respectively. The total second-order derivatives in (10) or (11) can now be decomposed using the denotations in (9):[4]

$$\frac{d^2U_i}{dq_i^2} = \underbrace{\left(\frac{\partial^2 U_i}{\partial Q^2} \cdot \frac{\partial Q}{\partial q_i} + \frac{\partial^2 U_i}{\partial Q \partial y_i} \cdot \frac{\partial y_i}{\partial q_i}\right)\frac{\partial Q}{\partial q_i} + \frac{\partial U_i}{\partial Q} \cdot \frac{\partial^2 Q}{\partial q_i^2}}_{MB'(q_i)} + \tag{12}$$

[3] Hence, $R_i(Q_{-i})$ can be expressed in terms of $\gamma_i(Q_{-i}, q_i)$ as $R_i(Q_{-i}) = \arg\max_{q_i} U_i|_{Q_{-i}=0} - \int_0^{Q_{-i}} \frac{1}{1+\gamma_i(Q_{-i}, q_i)} dQ_{-i}$ by integrating (8) over Q_{-i} using $\bar{q}_i = \arg\max U_i|_{Q_{-i}=0}$ as boundary condition.

[4] The sign of the total cross derivative in (13) determines whether the actions q_i and Q_{-i} are strategic complements or strategic substitutes (Bulow, Geanakoplos, and Klemperer, 1985). From concavity properties of (1) - (6) follow that (12) and (13) are negative and q_i for all $i \in N$ are strategic substitutes.

$$\underbrace{\left(\frac{\partial^2 U_i}{\partial y_i \partial Q} \cdot \frac{\partial Q}{\partial q_i} + \frac{\partial^2 U_i}{\partial y_i^2} \cdot \frac{\partial y_i}{\partial q_i}\right) \frac{\partial y_i}{\partial q_i} + \frac{\partial U_i}{\partial y_i} \cdot \frac{\partial^2 y_i}{\partial q_i^2}}_{MC'(q_i)} < 0$$

$$\frac{d^2 U_i}{dq_i dQ_{-i}} = \underbrace{\frac{\partial^2 U_i}{\partial Q^2} \cdot \frac{\partial Q}{\partial Q_{-i}} \cdot \frac{\partial Q}{\partial q_i} + \frac{\partial U_i}{\partial Q} \cdot \frac{\partial^2 Q}{\partial q_i \partial Q_{-i}}}_{MB'(Q_{-i})} + \underbrace{\frac{\partial^2 U_i}{\partial y_i \partial Q} \cdot \frac{\partial Q}{\partial Q_{-i}} \cdot \frac{\partial y_i}{\partial q_i}}_{MC'(Q_{-i})} < 0 \tag{13}$$

From the 'anonymous contributions property' in (5) and (6) and that any pair of symmetric partial cross derivatives are identical[5] it follows that $MB'(q_i)$ in (12) and $MB'(Q_{-i}) + MC'(Q_{-i})$ in (13) must always be identical terms for *any* $U_i(Q, y_i)$ and $y_i(q_i, w_i, p)$ that satisfy (1) - (6), thus, we have the identity:

$$MB'(q_i) \equiv MB'(Q_{-i}) + MC'(Q_{-i}) \tag{14}$$

That is, a change in *marginal benefit* of another unit of *own* contribution q_i must always be identical to a change in *marginal net benefit* of another unit of *other* players' contributions Q_{-i}. Using identity (14) in (11) to replace $MB'(Q_{-i}) + MC'(Q_{-i})$ and multiplying numerator and denominator by $1/MB'(q_i)$ we have:

$$\frac{dR_i}{dQ_{-i}} = -\frac{1}{1 + \frac{MC'(q_i)}{MB'(q_i)}} \tag{15}$$

Then, define the cost-benefit ratio:[6]

$$\gamma_i(Q_{-i}, q_i) \equiv \frac{MC'(q_i)}{MB'(q_i)} \in [-\infty, +\infty] \tag{16}$$

Substituting (16) in (15) yields the effect of cost-benefit ratio (8). **Q.E.D.**

3 Concluding Comments

Already in Barrett (1994) it was clear that 'the cost-benefit ratio' is conclusive for players' non-orthogonal free-riding response along their Nash reaction

[5] Applying Young's theorem (see e.g. Chiang (1984)) in (12) and (13) stating that any pair of symmetric cross partial derivatives are identical whenever they exist and are continuous.

[6] Note that γ_i can be negative, i.e. the response to a unit increase in Q_{-i} is greater than one unit decrease in q_i. This may occur e.g. with a concave total cost function without violating second-order conditions in (12) and (13) though it may violate stability of Nash equilibrium.

functions, and hence, conclusive for stability of self-enforcing agreements. Theorem 1 in this note showed that this result can be generalized due to the 'anonymous contributions' property (5) and (6) as it makes the cost-benefit ratio $\gamma_i(Q_{-i}, q_i^*)$ always carry full information about a player's free-riding best response along his Nash reaction function regardless functional forms of his objective and constraint in (1) - (2), and thus, even when his utility of public and private goods is not additively separable in (1).[7] Hennlock (2005) showed that theorem 1 is valid not only in highly nonlinear models but also in congestion models, which adds a general congestion function $C_i(Q, g)$, as well as in impure public good (bad) models, which follow the 'characteristic approach' in Sandmo (1973) and Cornes and Sandler (1996).[8] Therefore we should expect to see the cost-benefit ratio (or the benefit-cost ratio in public bad models) still be the conclusive component for stability also in self-enforcing agreements based on more general functional forms than seen hitherto in the literature.

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[7] The rationale for creating the cost-benefit ratio by 'normalizing' by the first derivative of the marginal benefit function in (7) is that the externality (3) enters in the public good argument of objective function (1). Conversely in public bad models, where externality enters in the public bad argument, the symmetrical proof of theorem 1 was shown in Hennlock (2005) that the ratio instead is defined as a 'benefit-cost ratio' by 'normalizing' by the first derivative of the marginal cost function, as used in public bad formulations in e.g. Finus and Rundshagen (1998).

[8] In the weak form of anonymous contributions when (6) holds with equality, it was shown in Hennlock (2005) that the effect of the generalized cost-benefit ratio in theorem 1 holds also for non-anonymous contribution by multiplying $\gamma_i(Q_{-i}, q_i^*)$ with a corresponding agent-specific weight parameter.

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