UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

WORKING PAPERS IN ECONOMICS

No 354

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April 2009

ISSN 1403-2473 (print)
ISSN 1403-2465 (online)

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Robust Control in Global Warming Management: An Analytical Dynamic Integrated Assessment*

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Abstract

Knightian uncertainty in climate sensitivity is analyzed in a two sectoral integrated assessment model (IAM), based on an extension of DICE. A representative household that expresses ambiguity aversion uses robust control to identify robust climate policy feedback rules that work well over IPCC climate-sensitivity uncertainty range [1]. Ambiguity aversion, together with linear damage, increases carbon cost in a similar way as a low pure rate of time preference. Secondly, in combination with non-linear damage it makes policy responsive to changes in climate data observations as it makes the household concerned about misreading sudden increases in carbon concentration rate and temperature as sources to global warming. Perfect ambiguity aversion results in an infinite expected shadow carbon cost and a zero carbon-intensive consumption path. Dynamic programming identifies an analytically tractable solution to the model.

Keywords: robust control, climate change policy, carbon cost, Knightian uncertainty, ambiguity aversion, integrated assessment models

JEL classification: C73, C61, Q54

1 Introduction

An essential component subject to scientific uncertainty in climate modeling is equilibrium climate sensitivity, defined as the ratio between a steady-state change in mean atmospheric temperature ΔT_t and a steady-state change in radiative forcing ΔR_t. The IPCC Executive Summary [1] stated 'The equilibrium climate sensitivity is a measure of the climate system response to sustained radiative forcing. It is not a projection but is defined as the global

*Financial support to this project from the CLIPORE program financed by MISTRA as well as the Swedish Energy Agency is gratefully acknowledged. I am grateful to Daniel Johansson at Physical Resource Theory, Chalmers University of Technology, Sweden for valuable comments on the climate modeling.

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average surface warming following a doubling of carbon dioxide concentrations. It is *likely* to be in the range 2.0^oC to 4.5^oC with a best estimate of about 3.0^oC, and is *very unlikely* to be less than 1.5^oC. Values substantially higher than 4.5^oC cannot be excluded, but agreement of models with observations is not as good for those values.' Equilibrium climate sensitivity depends on several underlying physical feedback processes that affect equilibrium mean temperature which are hard to predict. Some of the most uncertain are the cloud effect, water vapor, albedo and vegetation effect, see e.g. [2] and [3]. An analysis by Roe and Baker [4] shows that the climate sensitivity probability distribution is highly sensitive to uncertainties in already uncertain underlying physical feedback factors. The probability distribution becomes unpredictable as well as skewed with a thicker high-temperature tail that is not likely to be reduced despite scientific progress in understanding underlying feedback factors [4]. If climate uncertainty concerns not only outcomes but also probability distributions the question is raised whether the concepts of expected utility theory and risk aversion can explain and capture all reasonings behind e.g. precautionary principles in climate change policy. In decision theory the discussion on uncertain probability distributions is not new. Already Knight [5] suggested that for many choices the assumption of known probability distributions is too strong and therefore distinguished between 'measurable uncertainty' (risk) and 'unmeasurable uncertainty', reserving the latter denotation to include also unknown probabilities. Keynes [6], in his treatise on probability, put forward the question whether we should be indifferent between two scenarios that have equal probabilities, but one of them is based on greater knowledge. Savage's Sure-Thing principle [7] argued that we could, while Ellsberg's experiment [8] showed that individuals facing two lotteries, the first one with known probabilities and the second one with unknown probabilities, tended to prefer to bet on outcomes in the first lottery to bet on outcomes in the second lottery where they had to rely on subjective probabilities, thus contradicting the Sure-Thing principle. This behavior was referred to as ambiguity (or uncertainty) aversion as a broader aversion than risk aversion. When it comes to causes of ambiguity aversion, Ellsberg's setup has been repeated several times, supporting ambiguity aversion. In e.g. Fox and Tversky [9] subjects were asked for their willingness to pay, resulting in much higher willingness to pay for the urn with known probabilities than for the ambiguous urn. However, ambiguity aversion disappeared in an experiment in which the two urns were evaluated in isolation, suggesting that the comparison of known vs unknown probabilities matters. Other experiments by e.g. Curley et al. [10] showed that fear of negative evaluation when others observe the choice and may judge the decision-maker for it, increases his ambiguity aversion, which gives it an interesting connection to social norms in policymaking, [10] and [11]. Ambiguity aversion is closely related to the 'precautionary principle' which has been raised in e.g. the Rio Declaration article 15.[1] From a normative standpoint, delayed action would then be an irresponsible choice as it does not contain the precaution necessary to avoid the irreversible increase in uncertainty. One of the most influential ways to model ambiguity aversion is by Gilboa and Schmeidler [13] who formulated a maximin expected criterion, by weakening Savage's Sure-Thing Principle.[2] The decision-maker faces a set of probability distributions, and maximizes expected utility under the belief that the worst-case probability distribution is true, which in effect implies to put more probability weight on bad outcomes. Still we could find a subjective prior *ex post* that correspond to an individual's choice, but this does not imply that it always can be fully explained by expected utility theory and pure risk aversion, as the reasoning guiding him to the choice may have other reasons, such as ambiguity aversion and precautionary principles. Kahneman and Tversky [15] found empirical evidence that individuals tend to put more weight on low-probability extreme outcomes than would be implied by expected utility. The maximin decision criterion, as modeling fears of Knightian uncertainty, has been applied before in static models with the general result that it leads to an increase in abatement effort, e.g. Chichilnisky [16] and Bretteville Froyn [17], as well as dynamic models using a robust control approach with applications to e.g. water management Roseta-Palma and Xepapadeas [18], climate change Hennlock [19] and biodiversity management Vardas and Xepapadeas [20]. In the presence of Knightian uncertainty about probabilities, the maximin criterion is an elegant way of formalizing uncertainty aversion and precaution concern that is not captured by pure risk aversion. However, in climate change policy analysis climate policy actions span over a range from zero to arbitrarily large carbon prices. Furthermore, a policy based on the most pessimistic beliefs would likely be irresponsible when feasible policy actions are connected to large expected costs of delayed action as well as costs of action. In an IAM with ambiguity aversion, modeled as a maximin criterion, we should therefore allow for the 'hypothetical minimizer' to choose over a range of worst-case beliefs and, in this range, try to find the most 'responsible level of responsibility'.

[1] *'Where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation.'* Ulph and Ulph [12] have put forward that the benefits of insuring against irreversibility effects by actions now should balanced to the benefit of awaiting better scientific information by delaying action. If there are no climate irreversibilities, the benefit of awaiting better scientific information dominates and actions should be delayed. However, they do not deny that there are climate irreversibilities, they rather question which effect is largest.

[2] The Choquet expected utility (CEU) model of Schmeidler [14] is another example.

1.1 Measurable vs Unmeasurable Uncertainty in IAMs

The simplest way to introduce 'risk' in the literature on IAMs has been the so-called 'sensitivity-analysis approach'. Uncertain parameters are varied and values of carbon cost, optimal policy and outcomes are computed from several runs. This 'deterministic approach' becomes more sophisticated by replacing uncertain input parameter values by samplings from probability distributions and then obtain policy variables, expected benefits and costs as probability distributions from which mean and variance can be calculated. Also in deterministic models, the probability distributions for policies, costs and benefits can differ significantly from assumed probability distributions for input parameters, but this merely reflects that these variables are nonlinear functions of input parameters. Two early examples based on extensions of DICE [21] resulted in 2 to 4 times higher carbon cost than the certainty case, reflecting the benefit of reducing risk of high future climate change costs, see [22] and [23]. Another example modeled catastrophic events by altering the probability distributions of damages as temperature increases [24]. The Stern Review [25] performed a similar study using PAGE2002 [26] where several parameters are represented as probability distributions, to explore consequences of e.g. high climate sensitivities of 2.4^{o}C - 5.4^{o}C for the $5-95\%$ interval. Nordhaus and Popp [23] also imposed expected utility maximization and found carbon costs slightly higher than without maximization of expected utility, reflecting risk aversion. Clearly, optimal reductions in CO_2 emissions would differ largely whether the decision is based on a climate model based on an equilibrium sensitivity of 2.0^{o}C or 4.5^{o}C or even higher. My main point (as model builder) is to be silent about this magnitude and instead leave the question unanswered by letting our decision-maker face Knightian uncertainty rather than one or another model builder's sometimes ad hoc guesses about probability distributions.

Hennlock [19] introduced Knightian uncertainty in a IAM approach using robust control in climate-economy modeling. Weitzman independently introduced Knightian uncertainty, first in a draft to his Review of the Stern Report, and then in an early working paper of Weitzman [27]. Based on Knightian uncertainty, the main results of Hennlock [19] and Weitzman [27] seemed to tell the same story - uncertain probability distributions can justify large measures taken. In Hennlock [19] results emerged as a 'robust carbon cost markup', inducing a policymaker to take stringent measures (robust carbon pricing). Proposition 6 in Hennlock [19] showed that when a policymaker expresses *perfect* ambiguity aversion his expected shadow carbon cost becomes infinite, and hence, he 'backstop acts' by cutting carbon-generating production to zero. Weitzman's analysis, based on a static linear relationship between a utility function and a parameter with unknown probability distribution, showed that with Bayesian learning in a two-period analysis the result is an infinite expected marginal utility at zero consumption levels (the Dismal Theorem) [27].[3]

In this theoretical paper we apply the IAM approach in Hennlock [19] - Analytical Model Uncertainty in an Integrated Climate-Economy in a model with profit-maximizing firms (AMUICE-P) - but instead with a utility-maximizing household (AMUICE-C) in a model based on a two-sectoral extension of the DICE model in continuous time. The purpose of this first theoretical paper is not to perform a simulation or a full sensitivity analysis, but to present an IAM and its analytically tractable solution using dynamic programming and to introduce Knightian uncertainty and ambiguity aversion, modeled as maximin criterion within robust control, and finally comment on some major consequences in connection to the discussion following the Stern Review on discounting and the Dismal Theorem. The paper has the following organization: Section 2 presents main features of the household's Ramsey alike problem in the deterministic problem. Section 3 presents how Knightian uncertainty and ambiguity aversion are introduced. Section 4 discusses the major outcomes of the analytical solution which is followed by a summary in section 5. The appendix contains the analytical solution.

2 The Climate-Economy Model

The AMUICE-C model has its next of kin in DICE when it comes to the way it captures economic and climatic phenomena. The major choice for the representative household is whether to consume a final good, to invest in productive capitals, or to slow global climate change by abatement and investing in carbon-neutral and (more efficient) carbon-intensive technology. Besides the two-sectoral approach, the major difference with AMUICE-C is the introduction of Knightian uncertainty in climate sensitivity as one of the essential uncertain components in climate modeling. We distort the mean of the climate sensitivity probability distribution and end up with a continuum of climate sensitivity probability distributions over an arbitrarily large (but finite) range so that they can cover e.g. the IPCC uncertainty range 2.0^oC - 4.5^oC (or an even greater range) that our (possibly ambiguity averse) household is willing to imagine [1]. Given these multiple mean distorted probability distributions, which are understood as multiple priors that our household can form about climate sensitivity, the household uses robust control to identify a robust policy design that works well over a range of climate sensitivity outcomes. However, we start by presenting the deterministic version of the household problem.

[3] Nordhaus [28] comments on [27] and how the the result can depend on fat tails in the (posterior) probability distribution.

2.1 The Household Problem without Uncertainty

The representative household problem is described as a Ramsey alike problem as in DICE but the representative household owns a carbon-intensive production sector and a carbon-neutral production sector (using a natural capital stock as input). There is endogenous technology growth inspired by Romer [29] in both sectors and a climate model of the type used in DICE in continuous time. The final good is composed by a carbon-intensive input good C_t and a carbon-neutral input good G_t produced in the carbon-intensive and the carbon-neutral production sector, respectively. A CES function with constant elasticity of substitution σ and share parameter $\omega \in [0,1]$, describes how the inputs C_t and G_t compose the final good. The objective function is taken from Sterner and Persson [30]:[4]

$$\max_{C,G,q,s,r} \int_0^\infty \frac{1}{1-\eta} \left[(1-\omega)C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{(1-\eta)\sigma}{\sigma-1}} e^{-\rho t} dt \tag{1}$$

with elasticity of marginal utility of consumption (constant relative risk aversion) η from consuming the final good where a high value of η is usually interpreted as high risk aversion or inequality aversion. The household maximizes objective (1) subject to the dynamic system:

$$dK = \left[(\upsilon_K + (1-r_t)A_{Kt}^{\tau})K_t^{\alpha} L_t^{1-\alpha} - cq_t^2 - C_t - \delta K_t\right] dt \tag{2}$$

$$dA_K = \left[\nu(r_{jt}Y_{Kjt})^{\tau} A_{Kt}^{1-\tau} - \delta_K A_{Kt}\right] dt \tag{3}$$

$$dE_i = \left[(\upsilon_E + (1-s_t)A_t^{\psi})E_t^{\phi} - \frac{1}{\kappa}E_t - \Phi(T_t - T_0)E_t^{\phi} - \pi G_t\right] dt \tag{4}$$

$$dA_E = \left[\beta(r_t Y_{Ejt})^{\psi} A_{Et}^{1-\psi} - \delta_E A_{Et}\right] dt \tag{5}$$

$$dM = \left[\epsilon\varphi K_t^{\alpha} L_t^{1-\alpha} - \mu q_t - \Omega M_t\right] dt \tag{6}$$

$$R_t = \frac{\lambda_0 ln(M_t/M_0)}{ln(2)} \tag{7}$$

$$dT = \frac{1}{\tau_1}\left(R_t + O_t dt - \lambda_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)dt\right) \tag{8}$$

[4] See Hoel and Sterner [31] for a discussion on how the CES function affects the so-called Ramsey-rule.

$$d\tilde{T} = \frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right) dt \tag{9}$$

The major choice for the representative household is whether to consume the final good, to invest in productive capitals, or to slow global climate change by using the following policy variables: reduce the share of carbon-intensive composition in the final good by altering carbon-intensive share in consumption $C_t/(C_t + G_t)$, abatement effort q_t, research effort r_t in carbon-intensive research sector (more output for given emissions levels) and research effort s_t in carbon-neutral research sector. Sections 2.2 - 2.4 further describe the details of the dynamic programming problem (1) - (9). A complete list of all 32 model parameters is found in appendix A.2.

2.2 Carbon-Intensive Production Sector in (2) - (3)

The carbon-intensive production sector is described by the carbon-intensive capital growth equation (2) and the endogenous carbon-intensive technology growth equation (3). The carbon-intensive input factor is produced by using carbon-intensive capital K_t, whose accumulation (2) is determined by production $A_{Kt}^{\tau} K_t^{\alpha} L_t^{1-\alpha}$ minus research expenditure $r_t A_{Kt}^{\tau} K_t^{\alpha} L_t^{1-\alpha}$ with $r_t \in [0, 1]$, consumption of carbon-intensive good C_t and abatement cost. Applying the polluter-pays-principle, the carbon-intensive sector pays for abatement effort q_t in (2) with a quadratic cost function due to capacity constraints as more effort is employed.

Carbon-intensive technology A_K develops endogenously in (3) with research effort $r_t \in [0, 1]$ and the stock of abatement knowledge A_K as inputs in the research process. Thus a representative household whose research sector has generated many ideas in its history also has an advantage in generating new ideas relative to research sectors in less developed regions, see [29]. The 'Malthusian constraints' $0 < \tau < 1$ and $\upsilon_K > 0$ in (2) 'stabilize' the dynamics as restrictions on future technology and capital sets such that carbon-intensive growth cannot 'go on for ever'. The same restriction in (3) also suggests that it requires more than a doubling researchers in order to double the number of ideas (as researchers may come up with the same ideas). The implementation of new discoveries in the production process, implies that some of the old knowledge cannot be used in the current production process. For example, some of artisans' knowledge before the industrial revolution was lost. Imperfect substitution of knowledge over time is reflected by $\delta_K \geq 0$.

2.3 Carbon-Neutral Production Sector in (4) - (5)

Growth equations (4) and (5) describe the dynamics of the carbon-neutral sector. The carbon-neutral input good G_t is produced by using carbon-neutral (environmental or natural) capital E_t whose accumulation follows

(4). The first two terms in (4) describe a technology-enhanced natural growth function with carrying-capacity $\bar{E} = (\upsilon_K + \kappa(1 - s_t)A_{Et}^{\psi})^{1/(1-\phi)}$. Carbon-neutral technology A_E develops endogenously as in (3) and improves carbon-neutral capital growth (and raises carrying-capacity), thus counteracting the damage from temperature increases in (4). The 'Malthusian constraints' $0 < \psi < 1$ and $\upsilon_E > 0$ in (4) put restrictions on future technology and capital sets such that carbon-neutral growth cannot go on forever.

An overview of climate change impacts is found in IPCC [32] and IPCC [33]. Considered impacts are often on natural capitals; agriculture, forestry, water resources, loss of dry- and wetland (due to sea-level rise) etc. We let natural capital be damaged by an 'increasing-damage-to-scale' Cobb-Douglas function $\Phi(T_t - T_0)E_t^{\phi}$ in (4) adopted from Hennlock [34] and Hennlock [35] with Φ as a climate impact parameter.[5] The 'increasing-damage-to-scale' implies that a given temperature increase leads to a greater total damage (or gain for $\Phi < 0$) the greater is the natural capital stock. The net carrying-capacity with climate impact is then:

$$\bar{E} = \Big[\upsilon_E + \kappa\left(1 - s_t\right) A_{Et}^{\psi} - \Phi(T_t - T_0)\Big]^{\frac{1}{1-\phi}} \tag{10}$$

Carbon-neutral technology A_{Et} can then also be seen as adaption technology in (10).

2.4 Climate Modeling in (6) - (9)

Equations (6) - (9) describe a continuous-time modified version of the climate model used in DICE.[6] Total emissions in the first term in (6) is proportional to carbon-intensive production fraction and thus A_K increase output for given emissions flow where $\varphi > 0$. The second term is abatement level μq_t where $\mu > 0$. Net emissions flow accumulates to the global atmospheric CO_2 stock, M_t where $\epsilon > 0$ is the marginal atmospheric retention ratio and $\Omega > 0$ the rate of assimilation. The atmospheric CO_2 stock, M_t influences global mean atmospheric temperature T_t via the change in radiative forcing R_t (Wm^{-2}) in (7) which affects the energy balance of the climate system, and hence, the global mean atmospheric temperature T_t in (8) via the deep ocean temperature $\tilde{T}_t$ in (9).[7] The parameter λ_0 is essential for equilibrium

[5]Solutions are possible also when letting physical capital carry impact of climate change. However, separating stocks to damaging (physical) capital and damaged (environmental) capital, makes a unique solution possible corresponding to the verified value function.

[6]The climate model was originally based on [36].

[7]For analytical tractability of the Isaacs-Bellman-Flemming equation, we approximate (7) in (8) by a square-root approximation

$$R_t \simeq \frac{\Lambda_0 \sqrt{M_t/M_0}}{\sqrt{2}\gamma} + \frac{\hat{\Lambda}_0 M_t/M_0}{2\gamma} \tag{11}$$

climate sensitivity, τ_1 is the thermal capacity of atmosphere and upper ocean and τ_3 is the thermal capacity of deep ocean. $1/\tau_2$ is the transfer rate from the atmosphere and upper ocean layer to the deep ocean layer.[8]

3 Robust Control in Climate Policy Design

Robust control with Knightian uncertainty is a condition of analysis when the specifications of the climate model and climate impacts are open to doubt by the decision-maker. For illustrative purposes we only introduce uncertainty in climate sensitivity, though it could also be introduced in climate impacts.[9] In temperature equation (8) there are two possible places to introduce uncertainty in probabilities over climate sensitivity outcomes - via the radiative forcing parameter λ_0 in (7) and via the climate feedback parameter λ_1 reflecting uncertainty in the underlying physical processes. Both are conclusive for equilibrium climate sensitivity in (8) and introducing uncertain probability distributions in both λ_0 and λ_1 resulted in a solution with multiple solutions.[10] For illustrative purposes, we here want a straightforward unique solution and look at a household that only forms multiple priors about *equilibrium* (steady state) climate sensitivity. Thus in what follows we let λ_0 capture all uncertainty in *equilibrium* climate sensitivity, although its uncertainty also has transitional feedback sources in λ_1 as analyzed by [4] but also their analysis is performed in equilibrium terms. We now follow Hennlock [19] and define the following unknown process:

$$B_{0t} = \hat{B}_{0t} + \int_0^t \Lambda_{0s} ds \quad \Lambda_{0s} \in [\Lambda_{0,min}, \Lambda_{0,max}] \tag{12}$$

where $d\hat{B}_0$ is the increment of the Wiener process $\hat{B}_{0t}$ on the probability space (Ξ_G, Φ_G, G) with variance $\sigma_v^2 \geq 0$ where $\{\hat{B}_{0t} : t \geq 0\}$. Moreover, $\{\Lambda_{0t} : t \geq 0\}$ is a progressively measurable drift distortion, implying that the probability distribution of B_{0t} itself is distorted and the probability measure G_0 is replaced by another unknown probability measure Q_0 on the

where γ is calibrated to fit (7). The corresponding change in equilibrium mean temperature Λ_0/Λ_1 in (8) from $M/M_0 = 2$ can still be calibrated to follow (7).

[8] The geophysical parameter values used in the discrete DICE climate model are $\Lambda_0 = 4.1$, $\Lambda_1 = 1.41$, $1/\tau_1 = 0.226$, $\tau_3/\tau_2 = 0.44$ and $1/\tau_2 = 0.02$ and $\Omega = 0.0083$. For a calibration of these parameters to continuous form see e.g. [37].

[9] Knightian uncertainty in both climate sensitivity and local climate impacts was introduced in M. Hennlock, A Robust Abatement Policy in a Climate Change Policy Model, Department of Economic and Statistics, University of Gothenburg, 2006 unpublished draft, and resulted in significantly higher expected carbon cost for a given degree of ambiguity aversion as the expected local damage becomes a function of worst-case mean distortions in both local climate impact and global climate sensitivity.

[10] M. Hennlock, A Robust Abatement Policy in a Climate Change Policy Model, Department of Economic and Statistics, University of Gothenburg, 2006, unpublished draft.

space (Ξ_G, Φ_G, Q). The sensitivity parameter process Λ_{0t} is then introduced in temperature equation (8) in the following way

$$dT = \frac{1}{\tau_1}\Big((\Lambda_{0t}dt + d\hat{B}_0)\frac{\sigma_v\sqrt{M_t/M_0}}{\gamma\sqrt{2}} + \frac{1}{2}\frac{\hat{\Lambda}_0}{\gamma}\frac{M_t}{M_0}dt + O_t dt \\ -\hat{\lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)dt\Big) \tag{13}$$

and hence, temperature equation (13) follows an analytically tractable Ito process. Since both mean and variance of drift term Λ_{0t} are uncertain, (12) yields different statistics (priors) of equilibrium climate sensitivity in (13) where the interval $[\Lambda_{0,min}, \Lambda_{0,max}]$ indicates the maximum model specification error, e.g. corresponding to the range of climate sensitivity outcomes that the household is willing to accept based on its (ambiguity averse influenced) beliefs. Setting $\sigma_0 = 0$ yields the the 'benchmark model' that the household regards as an approximation to an unknown and unspecified global climate system that generates the true data.

3.1 Ambiguity Aversion as a Dynamic Maximin Criterion

Ambiguity aversion violates the Sure-Thing Principle [7], which is essential for ensuring that conditional preferences are well-defined and consistent over time and also being a basis for Bayesian updating. We assume that a rational decision-maker instead updates her beliefs to new information by a time consistent rule derived from backward induction using a dynamic maximin criterion adopted from robust control [38].[11] We border to the problem a hypothetical minimizer that resides in the head of our household making her to think 'what if the worst about climate sensitivity turns out to be true'. We then introduce an aversion to uncertainty with $1/\theta_0 \in [0, +\infty]$ assigning how much our household listens to her 'minimizer voice'. The maximin criterion, with expectation operator ε, then takes the following form

$$\sup_{C,G,q,r,s}\inf_{\Lambda_0}\varepsilon\int_0^\infty \frac{1}{1-\eta}\left[C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{(1-\eta)\sigma}{\sigma-1}} e^{-\rho t}dt + \theta_0 R(Q_0) \tag{14}$$

which can be formulated as a zero-sum differential game between the household (the maximizer) and the hypothetical minimizer choosing the worst-case climate sensitivity prior path for the household, where the last term contains a Lagrangian multiplier θ_0 and the finite entropy (Kullback-Leibler distance) $R(Q_0)$ as a statistical measure of the distance between the benchmark climate sensitivity and the worst-case climate sensitivity priors, generated by $\{\Lambda_{0s}\}$, in what follows: Recall that the unknown process in (12) will

[11] While Gilboa and Schmeidler [13] view ambiguity aversion as a minimization of the set of probability measures, Hansen et al. [38] set a robust control problem and let its perturbations be interpreted as multiple priors in max-min expected utility theory. Epstein and M. [39] provides another updating process.

unexpectedly change the probability distribution of B_{0t}, having probability measure Q_0, relative to the distribution of $\hat{B}_{0t}$ having measure G_0. The Kullback-Leibler distance between probability measure Q_0 and G_0 is then:

$$R(Q_0) = \int_0^\infty \varepsilon_{Q_0} \left(\frac{|\Lambda_{0s}|^2}{2} \right) e^{-\rho t} ds \tag{15}$$

As long as $R(Q_0) < \Theta_0$ in (14) is finite

$$Q_0 \left\{ \int_0^t |\Lambda_{0s}|^2 ds < \infty \right\} = 1 \tag{16}$$

which has the property that Q_0 is locally continuous with respect to G_0, implying that G_0 and Q_0 cannot be distinguished with finite data, and hence, future probability distributions cannot be inferred by using current finite climate data. Statistically this mimics the situation that current climate data from underlying physical processes is not sufficient to predict climate sensitivity probability distributions with certainty in accordance with [4]. Following [40], a maximin constraint problem as in (14) can be rewritten

$$\max_{C,G,q,r,s} \min_{\Lambda_{0i}} \varepsilon \int_0^\infty \left\{ \frac{1}{1-\eta} \left[(1-\omega) C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{(1-\eta)\sigma}{\sigma-1}} + \frac{\theta_0 \Lambda_{0t}^2}{2} \right\} e^{-\rho t} dt \tag{17}$$

where the quadratic term contains mean distortions Λ_{0t} and the minimization with respect to Λ_{0t} creates a lower (worst-case) boundary of the value function. The corresponding policy rule vector $(C_t^*, G_t^*, q_t^*, r_t^*, s_t^*)$ from the household's expected maximization would then be robust to priors that the household could imagine within the range $[0, \Lambda_{0t}^*]$. Maximizing-minimizing objective (17) subject to[12]

$$dK = \left[(\upsilon_K + (1-r_t) A_{Kt}^\tau) K_t^\alpha - c q_t^2 - C - \delta K \right] dt \tag{18}$$

$$dA_K = \left[\nu (r_t Y_{Kjt})^\tau A_{Kt}^{1-\tau} - \delta_K A_{Kt} \right] dt \tag{19}$$

$$dE = \left[(\upsilon_E + (1-s_t) A_t^\psi) E_t^\phi - \frac{1}{\kappa} E_t - \Phi(T_t - T_0) E_t^\psi - \pi G_t \right] dt \tag{20}$$

$$dA_E = \left[\beta (r_t Y_{Et})^\psi A_{Et}^{1-\psi} - \delta_E A_{Et} \right] dt \tag{21}$$

$$dM = \left[\epsilon \varphi K_t^\alpha - \mu q_t - \Omega M_t \right] dt \tag{22}$$

[12] In order to simplify, labor stock L_t is omitted in the solution hereinafter defining K_t as the amount of capital per unit labor.

$$dT = \frac{1}{\tau_1}\Bigg((\Lambda_{0t}dt + d\hat{B}_0)\frac{\sigma_v\sqrt{M_t/M_0}}{\sqrt{2}\gamma} + \hat{\Lambda}_0\frac{M_t/M_0}{2\gamma}dt + O_t dt \qquad (23)$$

$$-\hat{\lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)dt\Bigg)$$

$$d\tilde{T} = \frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right)dt \qquad (24)$$

defines the household's stochastic optimization problem (17) - (24), with the bordered hypothetical minimizer, choosing the household's upper boundary beliefs about climate-sensitivity mean distortions.

4 An Analytically Tractable Solution

The maximin dynamic programming problem (17) - (24) is solved by forming the Isaacs-Bellman-Flemming (IBF) equation in (38). Finding an analytically tractable solution to (38) by 'guessing-and-verifying' is tedious and left for appendix A.1. In short, the procedure goes as follows: Taking the first-order conditions of (38) and rearranging yield robust feedback policy rules. In order to identify shadow prices and costs, a value function that solves the IBF equation (38) needs to be identified by a guessing-and-verifying procedure. Once a value function is verified that solves (38) it can be differentiated with respect to state variables and so identify the shadow price and cost partial derivatives. An analytically tractable solution to (17) - (24) is possible by carefully specifying 6 of the 32 parameters in appendix A.2. and its corresponding value function is identified in appendix A.1. Since, the objective function in (38) is time autonomous, any robust policy feedback rule will be time consistent [41]. Moreover, certainty equivalence makes the variance distortions in (12) irrelevant and we only need to focus on its mean distortions. Taking the first-order condition of the Isaacs-Bellman-Flemming equation (38) with respect to the policy vector $(C_t^*, G_t^*, q_t^*, r_t^*, s_t^*)$ and rearranging, yield robust feedback policy rules (25) - (30) where the partial derivatives are *expected* shadow prices and costs, which in general are functions of state variables.

$$C^*(K(t)) = \left(\frac{1-\omega}{\frac{\partial W}{\partial K_t}}\right)^2 e^{-2\rho t} \qquad (25)$$

$$G^*(E(t)) = \left(\frac{\omega}{\pi\frac{\partial W}{\partial E_t}}\right)^2 e^{-2\rho t} \qquad (26)$$

$$q^*(K(t), M(t)) = -\frac{\epsilon\mu}{2c}\frac{\frac{\partial W}{\partial M_t}}{\frac{\partial W}{\partial K_t}} \geq 0 \qquad (27)$$

$$r^*(A_K(t), K(t)) = \frac{A_{Kt}^{1-\tau}(\nu\tau)^{\frac{1}{1-\tau}}}{K_t^\alpha} \left(\frac{\frac{\partial W}{\partial A_{Kt}}}{\frac{\partial W}{\partial K_t}}\right)^{\frac{1}{1-\tau}} \in [0,1] \tag{28}$$

$$s^*(A_E(t), E(t)) = \frac{A_{Et}^{1-\psi}(\beta\psi)^{\frac{1}{1-\psi}}}{E_t^\phi} \left(\frac{\frac{\partial W}{\partial A_{Et}}}{\frac{\partial W}{\partial E_t}}\right)^{\frac{1}{1-\psi}} \in [0,1] \tag{29}$$

The carbon-intensive consumption rule in (25) is determined by the shadow price of carbon-intensive capital $\partial W/\partial K_t$. The lower the shadow price, the greater is consumption. The carbon-neutral consumption rule (26) has the same structure but with the instantaneous price $\pi > 0$. The abatement rule in (27) is determined by the relative price of shadow carbon price $-\partial W/\partial M_t$ with respect to carbon-intensive capital $\partial W \partial K_t$ (due to polluter-pays-principle). Since $\partial W/\partial M_t \leq 0$ abatement will be positive. The research effort feedback rule in carbon-intensive technology in (28) reduces carbon-intensity in carbon-intensive sector by greater fuel efficiency etc. and is determined by the relative shadow price of carbon-intensive technology with respect to carbon-intensive capital. By symmetry, the relative shadow price of carbon-neutral technology with respect to carbon-neutral capital is conclusive for carbon-neutral research effort feedback rule s_t in (29).

Minimizing the IBF equation (38) with respect to Λ_0 gives the optimal feedback rule identifying the household's worst-case mean distortions, $\Lambda_0^*(M_t, T_t)$ in terms of its ambiguity aversion $1/\theta_0$ and expected shadow cost of climate change $\partial W/\partial T$:

$$\Lambda_{0t}^*(M(t), T(t)) = -\frac{\partial W}{\partial T} \frac{\sigma_v \sqrt{M_t/M_0} e^{\rho t}}{\theta_0 \tau_1 \gamma \sqrt{2}} \geq 0 \quad \Lambda_{0t}^* \in [\Lambda_{0,min}, \Lambda_{0,max}] \tag{30}$$

The optimal feedback rule (30) shows how the household updates its upper boundary of the range of climate sensitivities in (12) and $[0, \Lambda_{0t}^*]$ therefore 'stakes out the corners' of the basis used for policymaking. Why does the worst-case mean distortion $\Lambda_{0t}^*(M_t, T_t)$ depend on atmospheric CO_2 concentration rate and mean temperature? The explanation is that ambiguity aversion makes the household concerned about misreading an observed increase in CO_2 concentration rate or temperature as sources to global warming and impact and therefore the household's worst-case beliefs alter to increases in CO_2 concentration rate and mean temperature *as though* observed increases eventually will cause a greater increase in equilibrium temperature and impact than observed so far. As precaution, robust policy design becomes more responsive to changes in observed CO_2 concentration rate and mean temperature, and works as an insurance to avoid (if possible) increasing uncertainty for high-temperature outcomes up to a degree that corresponds to the household's degree of ambiguity aversion.

4.1 Ambiguity Aversion and Discounting

Ambiguity aversion is a preference of knowing (objective) probability distributions to form them subjectively in presence of Knightian uncertainty. From the analytical expression of shadow carbon cost in proposition 1 we see that ambiguity aversion has a similar effect on carbon cost path as has a low pure rate of time preference, which makes ambiguity aversion another gadget in the discussion on discounting that took place in the reviews following the Stern Review e.g. [42], [43], [44] and [30].[13]

Proposition 1 *The expected shadow carbon cost corresponding to the household's robust control problem in (38) is:*

$$\frac{-\hat{\Lambda}_0 \frac{\omega\Phi\sqrt{\frac{2/\pi}{\rho+\frac{1}{2\kappa}+\frac{\beta^2}{8(2\rho+\delta_E)^2}}}}{\rho+\frac{\hat{\lambda}_1}{\tau_1}+\frac{\tau_3}{\tau_1\tau_2}\left(1-\frac{1}{1+\rho\tau_2}\right)} - \frac{\hat{\sigma}_v^2}{2\theta_0\tau_1\gamma}\left(\frac{\omega\Phi\sqrt{\frac{2/\pi}{\rho+\frac{1}{2\kappa}+\frac{\beta^2}{8(2\rho+\delta_E)^2}}}}{\rho+\frac{\hat{\lambda}_1}{\tau_1}+\frac{\tau_3}{\tau_1\tau_2}\left(1-\frac{1}{1+\rho\tau_2}\right)}\right)^2}{2\tau_1\gamma(\rho+\Omega)M_0} \cdot e^{-\rho t} \tag{31}$$

Proof: Solving (38) by guessing-and-verifying and identifying $\partial W/\partial M_t$ by determining the undetermined coefficient in (57).

Social carbon cost largely depends on geophysical parameters in the climate model (22) - (24) as well as economic parameters.[14] Moreover, ambiguity aversion $1/\theta_0 \in [0, +\infty]$ increases carbon cost, resulting in more stringent policy feedback rules. With no ambiguity aversion $\theta_0 \to +\infty$, and the effect of the quadratic term in (31) cancels and carbon cost and robust controls collapse to a certainty equivalent optimal control problem, using the benchmark climate sensitivity as basis in policymaking. In the other extreme, under perfect ambiguity aversion $\theta_0 \to 0$, and the household takes into account 'uncut' worst-case mean distortions and expected carbon cost becomes infinite. Its consequences are further discussed in proposition 2 and 3 in section 4.2.

Even though patience ρ and ambiguity aversion $1/\theta_0$ affect carbon cost in a similar manner in (31), ambiguity aversion has an additional effect on policy compared to low utility discounting; it makes worst-case beliefs about equilibrium climate sensitivity responsive to changes in climate data observations over time as seen in (30) and how this in turn will affect policy depends *inter alia* on the damage function. In (31) the carbon cost path is falling over time. The explanation is the way temperature deviation $T_t - T_0$ enters the Cobb-Douglas damage function in (20), which in the IBF equation (38) becomes

$$-\frac{\partial W}{\partial E_t}\Phi(T_t - T_0)E_t^{\phi} \tag{32}$$

[13] A discussion on discounting and uncertainty is also found in [45].

[14] See appendix A.2. for the list of parameters.

Normally the increase in scarcity price $\partial W/\partial E_t$, as E_t falls from a temperature increase, would increase total damage in (32), however, the reduction in capital E_t also reduces total damage in (32). In the guessed-and-verified solution to (38) this reduction in total damage gets the same rate as the increase in scarcity price and the two effects cancel each other and (32) becomes:

$$-\frac{\omega}{2}\sqrt{\frac{2/\pi}{\rho+\frac{1}{2\kappa}-\frac{\beta^2}{32(\rho+\delta_E/2)^2}}}\Phi(T_t-T_0) \tag{33}$$

which is linear in $T_t - T_0$. A non-linear formulation of temperature would call for a value function with a non-linear term in temperature which would result in a shadow carbon cost being a function of at least M and T, which would also make policy directly responsive to observed changes in carbon concentration rate and temperature via (31). Imposing a quadratic temperature term jointly with the nonlinear differential equation system for K_t, A_{Kt}, E_t and A_{Et} results in demanding calculations and is left out from this article to keep the illustration of analytical tractability straightforward.

4.2 Ambiguity Aversion and the Dismal Theorem

Proposition 6 in Hennlock [19] showed that a policymaker, who expresses *perfect* ambiguity aversion, exhibits infinite expected carbon cost and resorts to zero carbon-intensive production as precaution. In this two-sector consumer model, perfect ambiguity aversion results in a complete shift from carbon-intensive consumption to carbon-neutral consumption.

Proposition 2 *Let the household express perfect ambiguity aversion* $\theta_0 \to 0$*. Then its expected shadow cost of carbon-intensive capital* $\partial W/\partial K_t \to +\infty$*, resulting in the robust carbon-intensive consumption feedback rule*

$$\epsilon\left[\lim_{\theta_0\to 0} C^*(K^*(t)) = 0\right] \tag{34}$$

Proof: Setting $\theta_0 = 0$ in (57) gives $\lim_{\theta_0\to 0} f \to -\infty$ and $\lim_{\theta_0\to 0} a \to +\infty$ in (53). Differentiating (39) with respect to K_t gives $\lim_{\theta_0\to 0} \partial W/\partial K_t \to +\infty$ which in (25) yields (34).

Proposition 2 reminds us about the Dismal Theorem [27], though abstract and based on a static relationship between a utility function and a 'climate-sensitivity parameter' with unknown probability distribution, the dismal theorem demonstrated an infinite expected marginal utility at zero consumption level, if we believe (possibly as a result of ambiguity aversion) in fat-tail-priors.[15] In my view the dismal theorem, rather than anything else,

[15] Nordhaus [28] comments on Weitzman [27] and how the results can depend on fat tails in the (posterior) probability distribution.

highlights the importance of taking ambiguity aversion seriously in climate change policy. Ambiguity aversion pushes the incentive for taking policy measures today to avoid (if possible) high-impact outcomes with low but uncertain probability magnitudes. If the household's $\theta_0 \to 0$, its beliefs embrace 'Weitzmanian fat-tails' about worst-case mean distortions in climate sensitivity and its expected shadow carbon cost in (31) becomes infinite. An important difference here is that there is (Bayesian) learning in Weitzman [27] while there is no learning either in our model or in Hennlock [19].[16] However, neither in Weitzman's two-period model there is learning realized until we are far away (200 years?) into the future by the arrival of the second period.[17] It seems unwise for a household to give up all its wealth today in order to transfer it into an uncertain future. However, the consequence of 'Weitzmanian fat-tails' correspond to zero initial θ_0 beliefs in our model and raises the question what initial level of ambiguity aversion (or precaution) the household should have today in presence of the current scientific uncertainty. A household can express a high ambiguity aversion but it does not need to be perfect. Furthermore, even an infinite shadow carbon cost *do not necessarily* imply, as one might think at a first glance, that robust abatement and investment in technology take upper corners.

Proposition 3 *Let the household express perfect ambiguity aversion* $\theta_0 \to 0$*. Then its expected shadow cost of carbon* $\lim_{\theta_0 \to 0} \partial W/\partial M_t \to -\infty$ *resulting in the robust abatement feedback rule*

$$\epsilon\left[\lim_{\theta_0 \to 0} q^*(t, K^*(t))\right] = \frac{\frac{\mu}{c}(K_t^*)^{1/2}}{-\frac{\varphi}{(\rho+\delta/2)} + \sqrt{\left(\frac{\varphi}{(\rho+\delta/2)}\right)^2 + \frac{2\mu^2}{c(\rho+\delta/2-\frac{\nu^2}{32(\rho+\delta_K/2)})}}} \geq 0 \quad (35)$$

and the robust investment expenditure $r^* Y_t^*(K_t)$ *in carbon-intensive technology feedback rule*

$$\epsilon\left[\lim_{\theta_0 \to 0} r^* Y_t^*(K_t^*)\right] = \left(\frac{1}{2\rho + \delta_K}\right)^2 \frac{\nu^2}{4} K_t^* \geq 0 \quad (36)$$

Proof: Setting $\theta_0 = 0$ in (57) gives $\lim_{\theta_0 \to 0} f \to -\infty$ and $\lim_{\theta_0 \to 0} a \to +\infty$ in (53) and $\lim_{\theta_0 \to 0} b \to +\infty$ in (54). Differentiating (39) with respect to M_t, K_t and A_{Kt} yield $\partial W/\partial K_t \to +\infty$, $\partial W/\partial M_t \to -\infty$ and $\partial W/\partial A_{Kt} \to +\infty$ which in (27) and (28) reproduce (35) and (36).

[16] Gollier et al. [46] focus e.g. on learning and uses prudence to give an interpretation of the precautionary principle.

[17] Tol et. al. [47] admit the importance of the dismal theorem but also put a label on it: *'Warning: Not to be taken to its logical extreme in application to real world problems.'*

Thus despite that shadow price of carbon and of carbon-intensive capital explode - its ratio - the relative shadow price

$$\epsilon\left[\lim_{\theta_0\to 0} -\frac{\frac{\partial W}{\partial M_t}}{\frac{\partial W}{\partial K_t}}\right] \tag{37}$$

converges to a finite positive value, and consequently, the robust abatement feedback rule converges to a maximum (but still finite) leverage for given capital paths in (35) also for 'uncut' worst-case mean distortions and an unbounded value function. When it comes to research effort in (36), both $\partial W/\partial K_t \to +\infty$ and $\partial W/\partial A_{Kt} \to +\infty$ but again the relative shadow price, conclusive for research effort, takes a finite value as seen in (36). The results in (35) and (36) are special due to endogenous 'relative-shadow-price-driven' technology growths and that carbon-intensive capital pays for current and future abatement (polluter-pays-principle).

4.3 How much Precaution is too Cautious?

Still, proposition 2 and 3, the Dismal Theorem and proposition 6 in Hennlock [19] bring up the question how much precaution is justified. Weitzman applies the value of statistical life (VSL) parameter to generate a lower boundary on consumption [27]. In our case, a finite upper boundary on ambiguity aversion $1/\theta_0$, seen as precaution, would cut the considered worst-case mean distortion and leave a minimized finite value of the value function. A possible upper boundary would be to set θ_0 sufficiently high (as an upper boundary for precaution) to make it difficult to statistically distinguish alternative worst-case climate sensitivity outcomes from a benchmark sensitivity properly set within the IPCC climate sensitivity range. But that brings us back to the difficult question of a normative statement how much aversion to uncertainty should be involved in climate change policy for it to be consistent with a 'precautionary principle' as formulated by e.g. article 15 of the Rio Declaration? To get a feeling for what θ_0 levels we are talking about, we take some upper boundaries 4.5^oC and 6.0^oC as mentioned in IPCC [1] and derive corresponding lower boundaries for θ_0 by rearranging the optimal feedback rule (30). This suggests boundaries for θ_0 in the interval 1 - 2 percent which in (31) suggests a carbon cost path that is 3 to 5 times higher than the certainty case, compared to Nordhaus and Popp [23] who got up to 4 times the certainty case due to pure risk aversion. Recall that our result (for simple-tractability reason) uses an elasticity of marginal utility of only $\eta = 0.5$ and *linear* one-sector damage while Nordhaus and Popp [23] use $\eta = 1$ and non-linear damage, suggesting that ambiguity aversion corresponding to the IPCC range up to 4.5^oC together with a higher η and non-linear damage could justify more stringency than what pure risk aversion has shown in IAMs so far. The effect of ambiguity aversion on carbon cost comes in addition and independently of a low utility discounting (here 0.03) as the latter increases both the certainty and the uncertainty cost by the same percentage. As a precursor, table 2 shows derived values for $\hat{\theta}_0$ and robust shadow carbon cost (as a shadow ambiguity premium) expressed as markups (times the certainty carbon cost path) for various boundaries of Λ_0, here against a benchmark of $\hat{\Lambda}_0$ corresponding to 1.5^oC.[18] The point with a statistical approach would be to calibrate θ_0 by calculating

Range (^oC)	$\hat{\theta}_0$	Shadow Ambiguity Premium
1.5 - 1.5	$+\infty$	1
1.5 - 3.0	0.039	2.31
1.5 - 4.5	0.019	3.62
1.5 - 6.0	0.013	4.93
1.5 - 7.0	0.011	5.81
1.5 - 8.0	0.009	6.68
1.5 - 9.0	0.0078	7.56
1.5 - 10.0	0.0068	8.43
1.5 - 20.0	0.0031	17.17

Table 1: Derivation of initial $\hat{\theta}_0$ and Shadow Ambiguity Premium

overall detection error probabilities $p(1/\theta_0)$ for distinguishing a benchmark climate model from worst-case climate models by varying $1/\theta_0$.[19] Ambiguity aversion then translates to what detection error probabilities that we are willing to accept for distinguishing worst-case models from an approximative benchmark model. In the case science should narrow predictions of climate sensitivity in the future, the corresponding adjustment $\hat{\theta}_{0t}$ over time replaces learning by updating the range of climate-sensitivity outcomes used as basis for robust policy design. Precaution then becomes a function of waiting time for enough data to discriminate worst-case climate sensitivity from approximative benchmark sensitivity.

4.4 Analytical IAMs and Tractability vs Complexity

A technical feature of our IAM is analytically tractable dynamic programming solutions in continuous time instead of computer-based numerical sim-

[18] The calculations are based on a pure rate of time preference 0.03, elasticity of marginal utility $\eta = 0.5$, $\Phi = 0.10$ and $\omega = 0.5$. Geophysical parameter values are based on calibrations by the author as well as [37] for a continuous-time version of the climate model in DICE. The parameter values used are $\sigma_v = 1$, $\Lambda_0 = 3.38$, $\gamma = 0.5719$, $\lambda_1 = 1.41$, $1/\tau_1 = 0.226$, $\tau_3/\tau_2 = 0.44$ and $1/\tau_2 = 0.02$ and $\Omega = 0.0083$.

[19] Hansen and Sargent [40] suggest that a robustness parameter θ should be set sufficiently high for it to take long time series to distinguish the benchmark model from worst-case models. By calculating likelihood ratio under benchmark and worst-case models Hansen and Sargent [48] suggest calculating overall detection error probability using detection error probabilities conditional on each model, respectively. For $1/\theta_0 = 0$ models are identical and $p = 0.5$. In general the greater is $1/\theta_0$, the lower is then p.

ulations in discrete time as usual seen in IAM. Analytically tractable solutions to non-linear differential games, using guessing-and-verifying methods in dynamic programming, are usually extremely difficult to identify. Still an analytical solution usually has better reliability, allows for deeper understandings as trajectories can be traced down to their explicit functional forms, and the model can also serve as a basis for which more complex extensions gradually can be added as seen in this paper: e.g. endogenous technology growth in both carbon-intensive and carbon-neutral sectors, Knightian uncertainty and ambiguity aversion in a differential game. To obtain a unique analytically tractable solution some major simplifications have been made e.g. (i) specifications have been carefully chosen for 6 of the 32 parameters, that amongst other things make the objective function additively separable (see appendix A.2.) while remaining 26 parameters are free to be varied for sensitivity analysis and simulations, (ii) linear damage only in the non-carbon-generating sector (two-sector damage resulted in multiple and not straightforward analytically tractable solutions),[20] (iii) Knightian uncertainty only in the radiative forcing parameter, and (iv) a slightly modified temperature equation to make it follow an Ito process, but still it can be calibrated to follow the original temperature equation used in DICE closely.

5 Summary

The article initiated a discussion on Knightian uncertainty and ambiguity aversion, modeled as a maximin criterion, in IAMs. We applied the robust IAM approach in Hennlock [19] which differ from existing 'risk analysis' in IAMs in that it does not fix one or another model builder's sometimes ad hoc (subjective) probability distributions to uncertain parameters. Instead it introduced multiple priors, over equilibrium climate sensitivity as illustration, using a distorted continuous-time version of the climate model used in DICE. Statistically this mimics a decision-making process in which finite climate data from underlying physical processes is statistically insufficient to predict climate sensitivity probability distributions in accordance with Roe and Baker [4]. We applied the approach to a two-sectoral extension of DICE - with a carbon-intensive and a carbon-neutral sector and a representative household that expresses ambiguity aversion and therefore uses robust control to identify robust policy feedback rules. We find that ambiguity aversion puts forward a responsibility of taking action via a 'shadow ambiguity premium' that lifts the expected shadow carbon cost path. The result is a robust policy feedback design, more stringent than optimal policy under certainty equivalence as a special case, avoiding (if possible) putting into effect increased uncertainty following high-temperature outcomes. As

[20] M. Hennlock, A Robust Abatement Policy in a Climate Change Policy Model, Department of Economic and Statistics, University of Gothenburg, 2006, unpublished draft.

an upper boundary for precautionary priors, we proposed the statistical approach, that θ_0 should be set sufficiently high to make it statistically difficult to distinguish worst-case models from benchmark models using detection error probabilities. Ambiguity aversion has two effects; firstly with linear damage, it increases shadow carbon cost by a 'shadow ambiguity premium' in a similar way as a low pure rate of time preference, and secondly, in combination with non-linear damage it will make policy feedback rules responsive to changes in climate data observations as it makes the household concerned about misreading a sudden increase in observed CO_2 concentration rate or temperature as sources to global warming. This behavior cannot be calibrated by a low pure rate of time preference.

There are several interesting ways to extend robust control in climate change policy analysis. Thinking about Roe and Baker [4], who stated that it is not likely that scientific progress or learning will narrow the thicker high-temperature tail, strengthens the argument for a robust policy design as a complement to Bayesian learning or 'risk analysis' in IAMs.[21] An interesting extension would be to combine ambiguity aversion and learning under Knightian uncertainty and balance benefits of insuring against irreversible *uncertainty* by action to benefits of awaiting better scientific information by delaying action, as has already been done within risk analysis, see e.g. [46]. A scientific-oriented extension is to develop and adapt statistical methods based on detection error probabilities for discriminating worst-case climate models from approximative benchmark climate models within the IPPC best-estimation ranges, to simulate and calibrate for reasonable empirical parameter values. Other extensions are non-linear damage in both natural and physical capital, which together with ambiguity aversion would make policy responsive to climate data observations as discussed in section 4.1, to extend the game dimension with multiple regional representative households as well as to repeat the maximin criterion to include several uncertain parameters.[22] Model uncertainty in both climate sensitivity and local impact resulted in significantly higher expected carbon cost for a given degree of ambiguity aversion as expected local impacts become functions of worst-case mean distortions in both local climate impact as well as global climate sensitivity outcomes.[23]

[21] Scientific research that reduces uncertainty in the underlying physical processes has little effect in reducing uncertainty at high-temperature outcomes [4]. For values above the interval 2.0^{o}C - 4.5^{o}C, the upper tail of the probability distribution would remain thick despite progress in understanding the underlying physical processes and Roe and Baker [4] therefore conclude *'We do not therefore expect the range presented in the next IPCC report to be different from that in the 2007 report'* and *'we are constrained by the inevitable: the more likely a large warming is for a given forcing (i.e. the greater the positive feedbacks) the greater the uncertainty will be in the magnitude of that warming.'*

[22] A differential game (AMUICE-P) with multiple regional producers and regulators with independent feedback strategies is analyzed in [19].

[23] M. Hennlock, A Robust Abatement Policy in a Climate Change Policy Model, De-

Appendix

A.1. The Dynamic Programming Problem

The section presents a solution structure to AMUICE-C with a parameter-setting that allows for an analytical tractable solution to the zero-sum differential game defined by objective (17) and dynamic system (18) - (24). Forming the Isaacs-Bellman-Fleming (IBF) dynamic programming equation [49]:

$$
\begin{aligned}
-\frac{\partial W}{\partial t} = \quad & (38)\\
\max_{C,G,q,r,s} \min_{\Lambda_0} \left\{ \frac{1}{1-\eta} \left[(1-\omega) C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{(1-\eta)\sigma}{\sigma-1}} + \frac{\theta_0 \Lambda_{0t}^2}{2} \right\} e^{-\rho t} & \\
+ \frac{\partial W}{\partial K} \left[(\upsilon_K + (1-r_t) A_{Kt}^{\tau}) K_t^{\alpha} - c q_t^2 - C_t - \delta K_t \right] & \\
+ \frac{\partial W}{\partial A_K} \left[\nu (s_t Y_{Kt})^{\tau} A_{Kt}^{1-\tau} - \delta_K A_{Kt} \right] & \\
+ \frac{\partial W}{\partial E_K} \left[(\upsilon_E + (1-s_t) A_t^{\Psi}) E_t^{\phi} - \frac{1}{\kappa} E_t - \Phi(T_t - T_0) E_t^{\phi} - p G_t \right] & \\
+ \frac{\partial W}{\partial A_E} \left[\beta (r_t Y_{Et})^{\psi} A_{Et}^{1-\psi} - \delta_E A_{Et} \right] + \frac{\partial W}{\partial M} \left[\epsilon \varphi K_t^{\alpha} L_t^{1-\alpha} - \mu q_t - \Omega M_t \right] & \\
+ \frac{\partial W}{\partial T} \frac{1}{\tau_1} \left[\Lambda_{0t} \frac{\sigma_v \sqrt{M_t/M_0}}{\sqrt{2}\gamma} + \hat{\Lambda}_0 \frac{M_t/M_0}{2\gamma} + O_t - \hat{\lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t) \right] & \\
+ \frac{1}{2} \frac{\partial^2 W}{\partial T^2} \sigma_v^2 M_t + \frac{\partial W}{\partial \tilde{T}} \left[\frac{1}{\tau_3} \left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t) \right) \right] &
\end{aligned}
$$

The robust control vector $\Gamma_t^* = (C_t, G_t, q_t, r_t, s_t)$ is given by maximizing the partial differential equation (38) with respect to policy variables and minimizing with respect to Λ_{0t} and solving for feedback rules.

Proposition 4 *The value function* $W(K, A_K, E, A_E, M, T, \tilde{T}, t)$

$$
= \left(aK^{1-\alpha} + bA_K^{\tau} + dE^{1-\phi} + eA_E^{\psi} + fM + gT + h\tilde{T} + k \right) e^{-\rho t} \qquad (39)
$$

satisfy the differential equation system formed by (38).

Proof: An analytically tractable solution is possible by setting $\sigma = 2$ and $\eta = \tau = \alpha = \phi = \psi = 1/2$ while the remaining 26 parameters, listed in appendix A.2., can be set free. Substituting (25) to (30) into (38) and

partment of Economic and Statistics, University of Gothenburg, 2006, unpublished draft.

collecting terms forms the indirect Isaacs-Bellman-Fleming equation. The carbon-intensive consumption feedback rule is

$$C^*(K_t) = \frac{4(1-\omega)^2}{a^2} K_t \geq 0 \tag{40}$$

where a is defined in (53). The carbon-neutral consumption feedback rule is

$$G^*(E_t) = \frac{4\omega^2}{(\pi d)^2} E_t \geq 0 \tag{41}$$

where d is defined in (55). The abatement feedback rule is

$$q^*(K_t) = -\frac{\epsilon\mu}{c}\frac{f}{a} K_t^{1/2} \geq 0 \tag{42}$$

where a is defined in (53) and f in (57). The research effort feedback rule in the carbon-intensive sector is

$$r^*(K_t) = \left(\frac{b\nu}{2a}\right)^2 \left(\frac{K_t}{A_{Kt}}\right)^{1/2} \in [0,1] \tag{43}$$

where b is defined in (54). The research effort feedback rule in the carbon-neutral sector is

$$s^*(E_t) = \left(\frac{e\beta}{2d}\right)^2 \left(\frac{E_t}{A_{Et}}\right)^{1/2} \in [0,1] \tag{44}$$

where e is defined in (56). The minimizer's feedback rule is

$$\Lambda^*_{0t}(M(t)) = \frac{\frac{d}{2}\Phi}{\rho + \frac{\hat{\lambda}_1}{\tau_1} + \frac{\tau_3}{\tau_1\tau_2}(1 - \frac{1}{1+\rho\tau_2})} \frac{\sigma_v\sqrt{M_t/M_0}}{\theta_0\tau_1\gamma\sqrt{2}} \geq 0 \tag{45}$$

Using value function (39) and substituting robust feedback rules (40) to (45) in (38) yield the equation system

$$\rho a = \frac{2(1-\omega)^2}{a} - a\frac{\delta}{2} + \frac{a\nu^2}{32(\rho + \delta_K/2)^2} + \frac{(f\epsilon\mu)^2}{2ac} + f\epsilon\varphi \tag{46}$$

$$\rho b = \frac{a}{2} L_K^{1/2} - b\frac{\delta_K}{2} \tag{47}$$

$$\rho d = \frac{2\omega^2}{\pi} + \frac{(e\beta)^2}{8d_i} - \frac{1}{2\kappa} \tag{48}$$

$$\rho e = \frac{d}{2} - e\frac{\delta_E}{2} \tag{49}$$

$$\rho f = -\frac{1}{2\theta_0 M_0}\left(\frac{g\sigma_v}{\tau_1\gamma\sqrt{2}}\right)^2 - f\Omega + \frac{g\hat{\Lambda}_0}{2\tau_1\gamma M_0} \tag{50}$$

$$\rho g = -\frac{d\Phi}{2} - g_i\frac{\hat{\Lambda}_1}{\tau_1} - g\frac{\tau_3}{\tau_1\tau_2} + h\frac{1}{\tau_2} \tag{51}$$

$$\rho h = g\frac{\tau_3}{\tau_1\tau_2} - h\frac{1}{\tau_2} \tag{52}$$

Solving the equation system (46) - (52) for undetermined coefficients gives the coefficients in terms of parameter values

$$a = \frac{f\epsilon\varphi}{2\left(\rho + \frac{\delta}{2} - \frac{\nu^2}{32(\rho+\delta_K/2)^2}\right)} \tag{53}$$
$$+\frac{1}{2}\sqrt{\left(\frac{f\epsilon\varphi}{\rho + \frac{\delta}{2} - \frac{\nu^2}{32(\rho+\delta_K/2)^2}}\right)^2 + \frac{8(1-\omega)^2 + 2\frac{(f\epsilon\mu)^2}{c}}{\rho + \frac{\delta}{2} - \frac{\nu^2}{32(\rho+\delta_K/2)^2}}}$$

$$b = \frac{a}{2\rho + \delta_K} \tag{54}$$

$$d = \omega\sqrt{\frac{2/\pi}{\rho + \frac{1}{2\kappa} - \frac{\beta^2}{32(\rho+\delta_E/2)^2}}} \tag{55}$$

$$e = \frac{d}{2\rho + \delta_E} \tag{56}$$

$$f = \frac{g\left(\hat{\Lambda}_0 - \frac{g\sigma_v^2}{2\theta_{0i}\tau_1\gamma}\right)}{2\tau_1\gamma(\rho + \Omega)M_0} \tag{57}$$

$$g = \frac{-\frac{d}{2}\Phi}{\rho + \frac{\hat{\lambda}_1}{\tau_1} + \frac{\tau_3}{\tau_1\tau_2}(1 - \frac{1}{1+\rho\tau_2})} \tag{58}$$

$$h = g\frac{\tau_3}{\tau_1\tau_2(\rho + \frac{1}{\tau_2})} \tag{59}$$

The coefficients in (53) - (59) are uniquely defined, and hence, feedback rules (40) to (45) corresponding to value function (39) are unique. The coefficient

k in proposition 4 is uniquely determined by (53) - (59).

A.1.1. Transitional Dynamics

To find the optimal trajectories in the dynamic system, the feedback rules (40) to (45) are substituted in dynamic system (18) - (24) which then gives

$$dK = \left[(\upsilon_K + A_{Kt}^{1/2})K_t^{1/2} - \left(\frac{b}{a}\right)^2 \frac{\nu^2}{4} K_t - c\left(-\frac{\epsilon\mu}{c}\frac{f}{a}K_t^{1/2}\right)^2 \right. \\ \left. - \frac{4(1-\omega)^2}{a^2} K_t - \delta K_t\right] dt \tag{60}$$

$$dA_K = \left[\nu \left(\left(\frac{b}{a}\right)^2 \frac{\nu^2}{4} K_t\right)^{1/2} A_{Kt}^{1/2} - \delta_K A_{Kt}\right] dt \tag{61}$$

$$dE_i = \left[(\upsilon_E + A_t^{1/2})E_t^{1/2} - \left(\frac{e}{d}\right)^2 \frac{\beta^2}{4} E_t - \frac{1}{\kappa} E_t \right. \\ \left. - \Phi(T_t - T_0)E_t^{1/2} - \pi \frac{4\omega^2}{(\pi d)^2} E_t\right] dt \tag{62}$$

$$dA_E = \left[\beta \left(\left(\frac{e}{d}\right)^2 \frac{\beta^2}{4} E_t\right)^{1/2} A_{Et}^{1/2} - \delta_E A_{Et}\right] dt \tag{63}$$

$$dM = \left[\epsilon\varphi K_t^{1/2} L_t^{1/2} + \mu \frac{\epsilon\mu}{c}\frac{f}{a} K_t^{1/2} - \Omega M_t\right] dt \tag{64}$$

$$dT = \frac{1}{\tau_1}\Bigg(\frac{\sigma_v\sqrt{M_t/M_0}}{\sqrt{2}\gamma} d\hat{B}_0 - \frac{g\sigma_v^2 M_t/M_0}{2\theta_0\tau_1\gamma^2} + \hat{\Lambda}_0 \frac{M_t/M_0}{2\gamma} dt + O_t dt \\ - \hat{\Lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t) dt\Bigg) \tag{65}$$

$$d\tilde{T} = \frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right) dt \tag{66}$$

To simplify expressions, define the following parameters

$$\Pi_K \equiv \left(\frac{b}{a}\right)^2 \frac{\nu^2}{4} - c\left(-\frac{\epsilon\mu}{c}\frac{f}{a}\right)^2 - \frac{4(1-\omega)^2}{a^2} - \delta \quad \Gamma_K \equiv \left(\frac{b}{a}\right)^2 \frac{\nu^2}{2} \tag{67}$$

$$\Pi_E \equiv \left(\frac{e}{d}\right)^2 \frac{\beta^2}{4} E_t - \frac{1}{\kappa} - \pi \frac{4\omega^2}{(\pi d)^2} \quad \Gamma_E \equiv \left(\frac{e}{d}\right)^2 \frac{\beta^2}{2} \tag{68}$$

$$\Theta \equiv \epsilon\varphi + \mu \frac{\epsilon\mu}{c} \frac{f}{a} \quad \chi \equiv \frac{\hat{\Lambda}_0}{2\tau_1\gamma} \quad \Xi \equiv \frac{\hat{\Lambda}_1}{\tau_1} + \frac{\tau_3}{\tau_1\tau_2} \tag{69}$$

And the dynamic system (60) - (66) can be rewritten as

$$dK = \left[(\upsilon_K + A_{Kt}^{1/2})K_t^{1/2} - \Pi_K K_t\right] dt \tag{70}$$

$$dA_K = \left[\Gamma_K A_{Kt}^{1/2} K_t^{1/2} - \delta_K A_{Kt}\right] dt \tag{71}$$

$$dE = \left[(\upsilon_E + A_{Et}^{1/2})E_t^{1/2} - \Pi_E E_t - \Phi(T_t - T_0)E_t^{1/2}\right] dt \tag{72}$$

$$dA_E = \left[\Gamma_E A_{Et}^{1/2} E_t^{1/2} - \delta_E A_{Et}\right] dt \tag{73}$$

$$dM = \left[\Theta K_t^{1/2} - \Omega M_t\right] dt \tag{74}$$

$$dT = \chi M_t - \Xi T_t - \frac{\tau_3}{\tau_1\tau_2}\tilde{T} \tag{75}$$

$$d\tilde{T} = \frac{1}{\tau_2}\left(T_t - \tilde{T}_t\right) dt \tag{76}$$

Besides computer-based methods an analytical solution to (70) - (76) can be found by using transformations $\hat{K} \equiv K^{1/2}$, $\hat{A}_K \equiv A_K^{1/2}$, $\hat{E} \equiv E^{1/2}$ and $\hat{A}_E \equiv A_E^{1/2}$, transforming the system to a linear system in which eigenvalues and eigenvectors can be defined before transforming back to the nonlinear system.

A.1.2. Steady States

From (60) to (66) the steady states $(\bar{K}, \bar{A}_K, \bar{E}, \bar{A}_E, \bar{M}, \bar{T}, \bar{\tilde{T}})$ as $t \to \infty$ in the $4N+3$ state space can be derived in terms of parameter values. Applying certainty equivalence in (38) yields

$$\bar{K} = \left(\frac{\upsilon_K}{\Pi_K - \frac{\Gamma_K}{\delta_K}} \right)^2 \quad \bar{A}_K = \left(\frac{\frac{\Gamma_K}{\delta_K}}{\Pi_K - \frac{\Gamma_K}{\delta_K}} \right)^2 \tag{77}$$

$$\bar{E} = \left(\frac{\upsilon_E + \Phi(\bar{T}_t - T_0)}{\Pi_E - \frac{\Gamma_E}{\delta_E}} \right)^2 \quad \bar{A}_E = \left(\frac{\frac{\Gamma_E}{\delta_E}}{\Pi_E - \frac{\Gamma_E}{\delta_E}} \right)^2 \tag{78}$$

$$\bar{M} = \frac{\Theta}{\Omega} \frac{\upsilon_K}{\Pi_K - \frac{\Gamma_K}{\delta_K}} \quad \bar{T} = \frac{\chi}{\Xi} \frac{\Theta}{\Omega} \frac{\upsilon_K}{\Pi_K - \frac{\Gamma_K}{\delta_K}} \quad \bar{\tilde{T}} = \bar{T} \tag{79}$$

The corresponding steady state policy variables as $t \to \infty$ are found by substituting (77) - (79) in (40) - (44).

A.2. List of Parameters

The analytically tractable solution required 6 of 32 parameters to be fixed as specified below. Remaining 26 parameters below are free to be varied.

Free Parameters

$\rho > 0$	pure time preference
$\omega \in [0,1]$	share parameter in objective function
$1/\theta_0 \in [0, +\infty]$	degree of ambiguity aversion
$c \geq 0$	abatement cost parameter
Θ	climate impact parameter
$\pi \geq 0$	relative price carbon-neutral input good
$\nu \geq 0$	carbon-intensive research sector efficiency parameter
$\beta \geq 0$	carbon-neutral research sector efficiency parameter
$\delta \geq 0$	depreciation rate carbon-intensive capital
$1/\kappa \geq 0$	depreciation rate carbon-neutral capital
$\delta_K \geq 0$	depreciation rate carbon-intensive technology
$\delta_A \geq 0$	depreciation rate carbon-neutral technology
$\varphi \geq 0$	carbon-intensity
$\mu \geq 0$	abatement effort efficiency
$\upsilon_K \geq 0$	Malthusian constraint carbon-intensive sector
$\upsilon_E \geq 0$	Malthusian constraint carbon-neutral sector

Free Parameters in the Climate Model

$\hat{\lambda}_0 \geq 0$	radiative forcing parameter
$\hat{\lambda}_1 \geq 0$	climate feedback parameter
$\tau_1 \geq 0$	thermal capacity of atmospheric layer
$\tau_3 \geq 0$	thermal capacity of deep ocean layer
$1/\tau_2 \geq 0$	transfer rate from the upper layer to the deeper ocean layer
$1/\Omega \geq 0$	transfer rate of CO_2 from atmosphere to other reservoirs
$\epsilon \geq 0$	marginal atmospheric retention ratio
$M_0 \geq 0$	initial CO_2 concentration rate
$T_0 \geq 0$	initial atmospheric mean temperature
$\tilde{T}_0 \geq 0$	initial deep ocean mean temperature

Specified Parameters in the Analytically Tractable Solution

$\sigma = 2$	elasticity of substitution
$\eta = 0.5$	elasticity of marginal utility of final good
$\alpha = 0.5$	capital intensity carbon-intensive production
$\phi = 0.5$	capital intensity carbon-neutral production
$\tau = 0.5$	Malthusian exponent constraint carbon-intensive sector
$\psi = 0.5$	Malthusian exponent constraint carbon-neutral sector

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