UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

WORKING PAPERS IN ECONOMICS

No 383

Testing for a Unit Root in a Random Coefficient Panel Data Model

Joakim Westerlund and Rolf Larsson

September 2009

ISSN 1403-2473 (print)
ISSN 1403-2465 (online)

Department of Economics
School of Business, Economics and Law at University of Gothenburg
Vasagatan 1, PO Box 640, SE 405 30 Göteborg, Sweden
+46 31 786 0000, +46 31 786 1326 (fax)
www.handels.gu.se   info@handels.gu.se

Testing for a Unit Root in a Random Coefficient Panel Data Model*

Joakim Westerlund†
University of Gothenburg
Sweden

Rolf Larsson
Uppsala University
Sweden

September 25, 2009

Abstract

This paper proposes a new unit root test in the context of a random autoregressive coefficient panel data model, in which the null of a unit root corresponds to the joint restriction that the autoregressive coefficient has unit mean and zero variance. The asymptotic distribution of the test statistic is derived and simulation results are provided to suggest that it performs very well in small samples.

JEL Classification: C13; C33.

Keywords: Panel unit root test; Random coefficient autoregressive model.

1 Introduction

Consider the panel data variable y_{it}, observable for $t = 1, ..., T$ time series and $i = 1, ..., N$ cross-sectional units. The analysis of such variables has been a growing field of econometric research in recent years, with a majority of the work focusing on the issue of unit root testing, see Breitung and Pesaran (2008) for a recent review. The main reason for this being the well-known power problem of univariate tests in cases when T is small, and the potential

*Previous versions of this paper were presented at the 15^{th} International Conference on Panel Data in Bonn and at a seminar at the University of Gothenburg. The author would like to thank conference and seminar participants, and in particular Jushan Bai, Steven Bond, Jörg Breitung, Dick Durevall, Lennart Flood and Hans Christian Kongsted for many valuable comments and suggestions. Financial support from the Jan Wallander and Tom Hedelius Foundation under research grant numbers P2005–0117:1 and W2006–0068:1 is gratefully acknowledged.

†Corresponding author: Department of Economics, University of Gothenburg, P. O. Box 640, SE-405 30 Gothenburg, Sweden. Telephone: +46 31 786 5251, Fax: +46 31 786 1043, E-mail address: joakim.westerlund@economics.gu.se.

gain that can be made by pooling across a cross-section of similar units. The most common approach, pioneered by Levin *et al.* (2002), is to assume that y_{it} admits to a first-order autoregressive representation with a common slope coefficient,

$$y_{it} = \rho y_{it-1} + u_{it},$$

where u_{it} is a stationary disturbance term with zero mean. A pooled least squares t-statistic is then computed, and the null hypothesis that $\rho = 1$ is tested against the alternative that $|\rho| < 1$.

The major limitation of this approach is that ρ is restricted to be the same for all units. The null makes sense, but the alternative is too strong to be held in any interesting empirical cases. For example, when testing for price convergence, one can formulate the null as implying that none of the regions under study converges. But it does not make any sense to assume that all the regions will converge at the same rate if they do converge.

Im *et al.* (2003) relax the assumption of a common autoregressive coefficient under the alternative. The idea is very simple. Take the above model and substitute ρ_i for ρ, which in the usual formulation where ρ_i is fixed results in N separate autoregressive models, one for each unit. Thus, instead of looking at a single pooled t-statistic, we now look at N individual t-statistics, which can be combined for example by taking the average. The resulting average statistic tests the null that $\rho_i = \rho = 1$ for all i against the alternative that $|\rho_i| < 1$ for a positive fraction of N.

But this is basically the same as saying that the null should be rejected if at least one of the individual tests end up in a rejection at the appropriate significance level, which brings us back to the original problem, namely that T has to be large. But if T is large enough for valid inference at the individual level, then there is hardly no point in pooling. This leaves us with an intricate dilemma. On the one hand, we would like to exploit the additional power that becomes available when we pool, and when we do this we would like to allow for some heterogeneity in ρ_i. On the other hand, this allowance requires T to be large, in which case we can just as well go back to doing unit-by-unit inference.

The appropriate response here depends on the relative size of N and T. But if only N is large enough, then it should be possible to devise powerful tests that are informative in an average sense, even if T is small. This leads naturally to the consideration of a random specification for ρ_i. In particular, suppose that

$$\rho_i = 1 + c_i,$$

where c_i is an independently distributed random variable with mean μ_c and variance ω_c^2. Then the null of a unit root corresponds to the joint restriction that $\mu_c = \omega_c^2 = 0$, while the alternative is that $\mu_c \neq 0$ or $\omega_c^2 > 0$, or both.

This random specification of c_i has many advantages is comparison to the traditional fixed specification. Firstly, working with incompletely specified models inevitably leads to a loss of efficiency. The random specification reduces the number of parameters that need to be estimated, and is therefore expected to lead to more powerful tests. Secondly, the random specification is more general, because fixed coefficients are special random variables. Whether something is random or not should be decided by considering what would happen if we were to replicate the experiment. Is it realistic to assume that c_i stays the same under replication? If not, then the random specification is more appropriate. Thirdly, by considering not only the mean of c_i but also the variance, random coefficient tests account for more information, and are therefore expected to be more powerful. Fourthly, the alternative hypothesis does not rule out the possibility that some of the units may be explosive.

Taking this random coefficient model as our starting point, the goal of this paper is to design a procedure to test the null hypothesis that $\mu_c = \omega_c^2 = 0$, which has not received much attention in the previous literature. In fact, the only attempt that we are aware of is that of Ng (2008), who uses a random coefficient model as a basis for proposing an estimator of the fraction of units with a unit root. However, this procedure does not exploit the fact that under the null hypothesis the variance of c_i is zero, which makes it suboptimal from a power point of view. It is also rather restrictive in nature, and cannot be easily generalized to accommodate for example high-order serial correlation.

Our testing methodology is rooted in the Lagrange multiplier principle, and can be seen as a generalization of the recent time series work of Distaso (2008) and Ling (2004), who consider the problem of testing for a unit root when the autoregressive coefficient is time-varying. It is also very similar to the seminal approach of Schmidt and Phillips (1992), from which it inherits many of its distinctive features. The test is for example based on a very convenient detrending procedure that imposes the null hypothesis, and if a linear trend is included the test statistic is asymptotically invariant with respect to the presence of a level break. It is also very straightforward and easy to implement.

The asymptotic analysis reveals that the Lagrange multiplier test statistic has a limiting chi-squared distribution that is free of nuisance parameters under the null hypothesis. We also study the limiting behavior of the statistic under local alternative hypotheses. We show that in the case of either a constant that may by heterogeneous across units, or a constant and trend that are homogenous the test has power against alternatives that shrink towards the unit root at rate $\frac{1}{\sqrt{NT}}$. However, we also show that in the presence of a heterogeneous trend the test does not have any power in such neighborhoods, which is a reflection of the so-called incidental trends problem.

A small simulation study is also undertaken to evaluate the small-sample properties of the test, and the results show that the asymptotic properties are borne out well, even in very small samples.

The rest of the paper is organized as follows. Section 2 introduces the model, while Section 3 derives the Lagrange multiplier test statistic and its asymptotic properties, which are evaluated using both simulated and real data in Sections 4 and 5, respectively. Section 6 concludes. Proofs and derivations of important results are provided in the appendix.

A word on notation. The symbols $\to_w$ and $\to_p$ will be used to signify weak convergence and convergence in probability, respectively. As usual, $y_T = O_p(T^r)$ will be used to signify that y_T is at most order T^r in probability, while $y_T = o_p(T^r)$ will be used in case y_T is of smaller order in probability than T^r.[1] In the case of a double indexed sequence y_{NT}, $T, N \to \infty$ will be used to signify that the limit has been taken while passing both indices to infinity jointly. Restrictions, if any, on the relative expansion rate of T and N will be specified separately.

2 Model and assumptions

The data generating process of y_{it} is given by

$$y_{it} = d_{it} + z_{it}, \tag{1}$$

where d_{it} is the deterministic part of y_{it}, while z_{it} is the stochastic part. The typical elements of d_{it} include a constant and a linear time trend, and this is also the specification considered here. Specifically, using p to denote the lag length, then $d_{it} = \alpha_i + \beta_i(t-p)$, which nests two

[1] If y_T is deterministic, then $O_p(T^r)$ and $o_p(T^r)$ are replaced by $O(T^r)$ and $o(T^r)$, respectively.

models. In model 1, there is no trend, while in model 2, there is both an intercept and trend. The parameters α_i and β_i can be either known or unknown to be estimated along with the other parameters of the model.

The stochastic part is assumed to evolve according to a first-order autoregressive process,

$$z_{it} = \rho_i z_{it-1} + u_{it}, \tag{2}$$

or equivalently,

$$\Delta z_{it} = c_i z_{it-1} + u_{it}$$

with the error u_{it} following a stationary and invertible autoregressive process of order p,

$$\phi_i(L) u_{it} = \epsilon_{it}, \tag{3}$$

where $\phi_i(L) = 1 - \sum_{j=1}^{p} \phi_{ji} L^j$ is a polynomial in the lag operator L and ϵ_{it} is an error term that satisfies the following assumptions.

Assumption 1.

(a) *ϵ_{it} is independent across both i and t with mean zero, variance $\sigma_i^2 < \infty$ and $E(\epsilon_{it}^3) = 0$,*

(b) *$\frac{1}{N}\sum_{i=1}^{N} \kappa_i \to \kappa < \infty$, where $\kappa_i = E(\epsilon_{it}^4)/\sigma_i^4$,*

(c) *α_i, β_i and $\phi_i(L)$ are non-random with the roots of $\phi_i(L)$ falling outside the unit circle,*

(d) *$z_{i0}, ..., z_{ip}$ are $O_p(1)$.*

Assumption 2. *ϵ_{it} is normally distributed.*

The assumed independence across i is restrictive but is made here in order to make the analysis of ρ_i more manageable. Some possibilities for how to relax this condition are discussed in Section 3. Normality is also not necessary. More precisely, while needed for deriving the true Lagrange multiplier test statistic, normality is not needed when deriving its asymptotic distribution. The following assumptions are more important in that regard.

Assumption 3.

(a) *c_i is independent across i with mean μ_c and variance ω_c^2,* (*b*) c_i *and* ϵ_{it} *are mutually independent.*

Assumption 4. $\frac{N}{T} \to 0$ *as* N, $T \to \infty$.

The requirement that the mean and variance of c_i are equal across i is made for convenience, and can be relaxed as long as the cross-sectional averages of these moments have limits such as μ_c and ω_c^2, respectively. However, the assumption that c_i and ϵ_{it} are independent is crucial. Assumption 4 is standard when testing for unit roots in panels. The reason is the assumed heterogeneity in α_i, β_i, $\phi_i(L)$ and σ_i^2, whose elimination induces an estimation error in T, which is then aggravated when pooling across N. The condition that $\frac{N}{T} \to 0$ prevents the estimation from having a dominating effect, see Section 3 for a more detailed discussion and for some results when it fails.

Having laid out the assumptions we now continue to discuss the hypothesis of interest. In the conventional setup when c_i is fixed the null hypothesis of a unit root is formulated as that $\rho_i = 0$ for all i, while the alternative hypothesis is usually formulated as in Im *et al.* (2003). That is, it is assumed that $c_i < 0$ for a significant fraction of N, implying that although some of the units may be non-stationary most of them are stationary.

When c_i is random, this formulation changes. The null of a unit root now becomes

$$H_0 : \rho_i = 0 \text{ almost surely},$$

which can be written in an equivalent fashion as

$$H_0 : \mu_c = \omega_c^2 = 0.$$

A violation of this null occurs if $\mu_c \neq 0$ or $\omega_c^2 > 0$, or both, implying that while some units may be non-stationary, the probability of this happening is very small. It also implies that there are not just stationary and non-stationary units, but also explosive units, which seems like a relevant scenario in most applications, especially in financial economics, where data tend to exhibit explosive behavior.[2] Explosive behavior is also more likely if N is large, which obviously increases the probability of extreme events regardless of the application considered. There is also the question to what extent researchers can work with regular unit root tests without prior knowledge of the location of the roots.

[2]In Section 5 we consider as an example the housing market of the United States, which has recently experienced a spectacular rise in prices. Periods of hyperinflation and stock markets with rational bubbles are other examples of applications with possibly explosive data, see for example Nielsen (2008) and Phillips *et al.* (2009).

In any case, with such a formulation of the alternative hypothesis, we only learn whether the test is consistent and if so at what rate. Therefore, to be able to evaluate the power analytically, in this paper we consider an alternative in which ρ_i is local-to-unity as $N, T \to \infty$. In particular, the following formulation is adopted:

$$H_1 : \rho_i = 1 + \frac{c_i}{\sqrt{N}T},$$

where c_i again satisfies Assumption 3. This corresponds to an autoregressive coefficient that approaches one with increasing values of N and T. If $c_i < 0$, then ρ_i approaches one from below and so y_{it} is locally stationary, whereas if $c_i > 0$, then ρ_i approaches one from above and so y_{it} is locally explosive. In the limit as $N, T \to \infty$ we see that $\rho_i \to 0$, and hence the distribution of ρ_i collapses with the mean going to one and the variance going to zero.

The rate of shrinking is given by $\frac{1}{\sqrt{N}T}$. Coincidentally, this is also the rate of consistency of the pooled least squares estimator of ρ_i under the null, which is going to turn out to form the basis of our test statistic. Being an estimate of the slope of the mean function, it is logical to expect that the main effect of the local-to-unity specification of ρ_i is to induce via μ_c a non-centrality of the asymptotic distribution of the test statistic.

3 The test procedure

In this section, we first consider the true Lagrange multiplier test statistic, which is based on the assumption that the parameters of the model are all known. We then show how this analysis extends to the more realistic case when the parameters are unknown. Finally, we discuss some generalizations.

3.1 The true Lagrange multiplier test statistic

Define $w_{it} = \phi_i(L)(y_{it} - d_{it})$, which in the model with a trend can be written as

$$w_{it} = \phi_i(L)\big(y_{it} - \alpha_i - \beta_i(t-p)\big) = y_{it} - \Phi_i'\mathbf{y}_{it} - \mu_i - \beta_i\phi_i(L)(t-p), \tag{4}$$

whose first difference is given by

$$\Delta w_{it} = \phi_i(L)(\Delta y_{it} - \beta_i) = \Delta y_{it} - \Phi_i'\Delta\mathbf{y}_{it} - \lambda_i, \tag{5}$$

where $\mu_i = \phi_i(1)\alpha_i + \phi_i(L)z_{ip}$, $\lambda_i = \phi_i(1)\beta_i$ and $\mathbf{y}_{it} = (y_{it-1}, ..., y_{it-p})'$ is the vector of lags with $\Phi_i = (\phi_{1i}, ..., \phi_{pi})'$ being the associated vector of slope coefficients. If there is no trend,

$\beta_i = 0$ and so $w_{it} = y_{it} - \Phi_i' \mathbf{y}_{it} - \mu_i$. In any case, by using (1) to (3),

$$\Delta w_{it} \;=\; c_i w_{it-1} + \epsilon_{it} \tag{6}$$

or, in terms of the observed variable,

$$y_{it} \;=\; y_{it} - \Delta w_{it} + c_i w_{it-1} + \epsilon_{it} \;=\; y_{it-1} + \Phi_i' \Delta \mathbf{y}_{it} + \lambda_i + c_i w_{it-1} + \epsilon_{it}.$$

Thus, letting $\mathcal{F}_{t-1}$ denote the information set available at time $t-1$,

$$E(y_{it}|\mathcal{F}_{t-1}) \;=\; y_{it-1} + \Phi_i' \Delta \mathbf{y}_{it} + \lambda_i + \mu_c w_{it-1}$$

and

$$\text{var}(y_{it}|\mathcal{F}_{t-1}) \;=\; \omega_c^2 w_{it-1}^2 + \sigma_i^2,$$

which can be used to obtain the log-likelihood function L of $y_{ip+1}, ..., y_{iT}$. In particular, suppose that ϵ_{it} is normal, then, apart from constants,

$$\begin{aligned} L &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{t=p+1}^{T} \ln\left(\text{var}(y_{it}|\mathcal{F}_{t-1})\right) - \frac{1}{2}\sum_{i=1}^{N}\sum_{t=p+1}^{T} \frac{\left(y_{it} - E(y_{it}|\mathcal{F}_{t-1})\right)^2}{\text{var}(y_{it}|\mathcal{F}_{t-1})} \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{t=p+1}^{T} \ln\left(\omega_c^2 w_{it-1}^2 + \sigma_i^2\right) - \frac{1}{2}\sum_{i=1}^{N}\sum_{t=p+1}^{T} \frac{\left((c_i - \mu_c) w_{it-1} + \epsilon_{it}\right)^2}{\omega_c^2 w_{it-1}^2 + \sigma_i^2}. \end{aligned} \tag{7}$$

In Appendix A we show that under H_0 the log-likelihood is maximized by

$$\check{\sigma}_i^2 \;=\; \frac{1}{T-p}\sum_{t=p+1}^{T} (\Delta w_{it})^2,$$

and that the Gradient and Hessian with respect to μ_c and ω_c^2 are given by

$$\mathbf{g} \;=\; \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \;=\; \sum_{i=1}^{N}\sum_{t=p+1}^{T} \begin{bmatrix} \Delta \check{e}_{it} \check{e}_{it-1} \\ \frac{1}{2}((\Delta \check{e}_{it})^2 - 1)\check{e}_{it-1}^2 \end{bmatrix}$$

and

$$\mathbf{H} \;=\; \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \;=\; -\sum_{i=1}^{N}\sum_{t=p+1}^{T} \begin{bmatrix} -\check{e}_{it-1}^2 & \Delta \check{e}_{it} \check{e}_{it-1}^3 \\ \Delta \check{e}_{it} \check{e}_{it-1}^3 & \frac{1}{2}(2(\Delta \check{e}_{it})^2 - 1)\check{e}_{it-1}^4 \end{bmatrix},$$

respectively, where $\check{e}_{it} = w_{it}/\check{\sigma}_i$. We also show that when properly normalized by N and T the Hessian is asymptotically diagonal. Thus, if all the parameters but σ_i^2 are known, then the Lagrange multiplier test statistic can be written as

$$LM \;=\; \mathbf{g}'(-\mathbf{H})^{-1}\mathbf{g} \;=\; ALM + o_p(1),$$

where

$$ALM \;=\; -\frac{g_1}{H_{11}} - \frac{g_2}{H_{22}} \;=\; \frac{\left(\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta\check{e}_{it}\check{e}_{it-1}\right)^2}{\sum_{i=1}^{N}\sum_{t=p+2}^{T}\check{e}_{it-1}^2} + \frac{\left(\sum_{i=1}^{N}\sum_{t=p+2}^{T}((\Delta\check{e}_{it})^2-1)\check{e}_{it-1}^2\right)^2}{2\sum_{i=1}^{N}\sum_{t=p+2}^{T}\left(2(\Delta\check{e}_{it})^2-1\right)(\check{e}_{it-1})^4},$$

which can be interpreted as an asymptotic Lagrange multiplier test statistic.

The formula for ALM is very simple and intuitive. In fact, a careful inspection reveals that the first part is nothing but the Lagrange multiplier test statistic for testing the null that $\mu_c = 0$ given $\omega_c^2 = 0$. That is, the first part is the Lagrange multiplier unit root statistic based on the assumption of an homogenous ρ_i. The second part is the Lagrange multiplier statistic for testing the null that $\omega_c^2 = 0$ given $\mu_c = 0$.

The formula also reveals some interesting similarities with results obtained previously in the literature. In particular, note how the first part is the squared equivalent of the panel unit root test considered by Levin *et al.* (2002).[3] The second has no direct resemblance of anything that has been proposed earlier in the panel unit root literature. However, it can be seen as a panel version of the test statistic of Leybourne *et al.* (1996), who consider the problem of testing the null of a fixed unit root against the randomized alternative in the context of a single time series. The test statistic as a whole can be regarded as a panel extension of the time series statistics discussed in Distaso (2008) and Ling (2004).

Even when ϵ_{it} is normal the exact distribution of ALM is untractable. In this paper we therefore use asymptotic theory to obtain the limiting distribution of ALM as $N, T \to \infty$. Although this means that N and T must be large for the test to be accurate, it also means that there is no need for any distributional assumptions like normality.

The asymptotic null distribution of ALM is given in the following theorem.

Theorem 1. *Under H_0 and Assumptions 1, 3 and 4,*

$$ALM \;\to_d\; X^2 + \frac{5}{24}(\kappa - 1)\,Y^2,$$

where X^2 and Y^2 are independent chi-squared random variables with one degree of freedom each.

Remarks.

(a) The theorem shows that ALM has the same limiting distribution in both models considered, and that this distribution is free of nuisance parameters, except for the dependence on κ, the average fourth normalized moment of ϵ_{it}. If ϵ_{it} is normal or if $\kappa = 3$,

[3] The first part of ALM can also be regarded as a panel version of the Lagrange multiplier unit root tests proposed in the time series literature by for example Ahn (1993) and Schmidt and Phillips (1992).

then $(\kappa-1)=2$ and hence the asymptotic distribution of ALM reduces to $X^2+\frac{5}{12}Y^2$. Thus, normality, or more generally, $\kappa=3$ implies a test distribution that is completely free of nuisance parameters.

(b) It is interesting to compare the asymptotic distribution of ALM with that obtained by Ling (2004) when testing for a unit root in a first-order autoregressive model with conditional heteroskedasticity, which can be reformulated as a random coefficient autoregressive model. The distribution of this test for cross-sectional unit i without any deterministic components is in our notation given by

$$\frac{\left(\int_0^1 W_i(r)dW_i(r)\right)^2}{\int_0^1 W_i(r)^2dr}+(\kappa_i-1)\frac{\left(\int_0^1 W_i(r)^2dV_i(r)\right)^2}{2\int_0^1 W_i(r)^4dr},$$

where $W_i(r)$ and $V_i(r)$ are two independent standard Brownian motions on $r\in[0,1]$. The asymptotic distribution of our statistic can be regarded as

$$\lim_{N\to\infty}\frac{\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N\int_0^1 W_i(r)dW_i(r)\right)^2}{\frac{1}{N}\sum_{i=1}^N\int_0^1 W_i(r)^2dr}+(\kappa-1)\lim_{N\to\infty}\frac{\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N\int_0^1 W_i(r)^2dV_i(r)\right)^2}{\frac{2}{N}\sum_{i=1}^N\int_0^1 W_i(r)^4dr}.$$

Thus, by just comparing these two distributions, we see that the main effect of summing over the cross-sectional dimension is to smooth out the Brownian motion dependency for each unit.

(c) The requirement that $\frac{N}{T}\to 0$ as $N,T\to\infty$ is needed because while Φ_i, μ_i and λ_i are assumed to be known, σ_i^2 is not and therefore has to be estimated.

Next we summarize the results obtained under H_1.

Theorem 2. *Under H_1 and Assumptions 1, 3 and 4,*

$$ALM\ \to_d\ \frac{\mu_c^2}{2}+\mu_c\sqrt{2}\,X+X^2+\frac{5}{24}(\kappa-1)\,Y^2,$$

where X and Y are as in Theorem 1.

Remarks.

(a) The first thing to note is that ω_c^2 does not enter the asymptotic distribution of the test. The reason for this originates with the rate of shrinking of the local alternative, which is determined by the normalization of the test statistic. With a composite test statistic like ours, unless the normalization of the different parts is the same, the rate of shrinking of the local alternative is given by the lowest of the normalizing orders. In our case, the appropriate normalization for the first part of the test statistic is given by $\frac{1}{\sqrt{NT}}$, while the normalization of the second part is $\frac{1}{\sqrt{N}T^{3/2}}$. The rate of shrinking is therefore just enough to manifest μ_c as a nuisance parameter in the asymptotic distribution of the first part of the statistic. The normalizing order of the second part, which represents the test of $\omega_c^2 = 0$, is higher and ω_c^2 is therefore kicked out.

(b) The specification of H_1 has two effects. The first is to shift the mean of the limiting distribution of the test. In particular, since $\mu_c^2 > 0$, this means that the mean shifts to the left as we move away from H_0, suggesting that the test is unbiased and that its asymptotic local power therefore is greater than the size. The second effect, which is captured by $\mu_c\sqrt{2}\,X \sim N(0, 2\mu_c^2)$, is to increase the variance of the limiting distribution. This effect is especially noteworthy as usually there is only the mean effect.

3.2 The feasible Lagrange multiplier test statistic

All results reported so far are based on the assumption that Φ_i, μ_i and λ_i are all known, which is of course not very realistic. Let us therefore consider using

$$\hat{w}_{it} = y_{it} - \hat{\Phi}_i' \mathbf{y}_{it} - \hat{\mu}_i - \hat{\lambda}_i(t-p) \tag{8}$$

as an estimator of w_{it}, where $\hat{\mu}_i = y_{ip+1} - \hat{\Phi}_i' \mathbf{y}_{ip} - \hat{\lambda}_i$ with $\hat{\lambda}_i$ and $\hat{\Phi}_i$ being the least squares estimators of λ_i and Φ_i, respectively, in the first-differenced regression

$$\Delta y_{it} = \lambda_i + \Phi_i' \Delta \mathbf{y}_{it} + \epsilon_{it}, \tag{9}$$

which is (5) with H_0 imposed.[4] If there is no trend, then we remove the intercept, and compute $\hat{w}_{it} = y_{it} - \hat{\Phi}_i' \mathbf{y}_{it} - \hat{\mu}_i$, where $\hat{\mu}_i = y_{ip+1} - \hat{\Phi}_i' \mathbf{y}_{ip}$.[5] The feasible Lagrange multiplier statistic in this model is given by

$$FLM_1 = \frac{\left(\sum_{i=1}^N \sum_{t=p+2}^T \Delta\hat{e}_{it}\hat{e}_{it-1}\right)^2}{\sum_{i=1}^N \sum_{t=p+2}^T \hat{e}_{it-1}^2} + \frac{12\left(\sum_{i=1}^N \sum_{t=p+2}^T ((\Delta\hat{e}_{it})^2 - 1)\hat{e}_{it-1}^2\right)^2}{5(\hat{\kappa}-1)\sum_{i=1}^N \sum_{t=p+2}^T (\Delta\hat{e}_{it})^2 \hat{e}_{it-1}^4},$$

[4] As shown in Lemma A.1 of Appendix A, under the null hypothesis $\hat{\mu}_i$, $\hat{\lambda}_i$ and $\hat{\Phi}_i$ are the feasible maximum likelihood estimators of μ_i, λ_i and Φ_i, respectively.

[5] If in addition there is no serial correlation, then $\hat{w}_{it} = y_{it} - \hat{\mu}_i$ with $\hat{\mu}_i = y_{i1}$.

where $\hat{e}_{it} = \hat{w}_{it}/\hat{\sigma}_i$, $\hat{\sigma}_i^2 = \frac{1}{T-p-1}\sum_{t=p+2}^{T}(\Delta\hat{w}_{it})^2$ and $\hat{\kappa} = \frac{1}{N(T-p-1)}\sum_{i=1}^{N}\sum_{t=p+2}^{T}(\Delta\hat{w}_{it})^4/\hat{\sigma}_i^4$. The reason for the subscript 1 is to indicate that the statistic has been computed for a particular choice of model, and that the limiting distribution depends on it. The asymptotic distribution of FLM_1 under H_0 is given in the following corollary.

Corollary 1. *Under the conditions of Theorem 1,*

$$FLM_1 \ \to_d \ X^2 + Y^2.$$

Corollary 2 provides the corresponding result under H_1.

Corollary 2. *Under the conditions of Theorem 2,*

$$FLM_1 \ \to_d \ \frac{\mu_c^2}{2} + \mu_c\sqrt{2}\,X + X^2 + Y^2.$$

Remarks.

(a) The first term in the formula for FLM_1 is just the feasible version of the corresponding term in the formula for ALM and does not require any explanation. The second term, however, is not as obvious. In Appendix B we show that as N, $T \to \infty$ with $\frac{N}{T} \to 0$

$$\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\big(2(\Delta\hat{e}_{it})^2 - 1\big)(\hat{e}_{it-1})^4 \ = \ \frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+2}^{T}(\Delta\hat{e}_{it})^2\hat{e}_{it-1}^4 + o_p(1) \ \to_p \ 1,$$

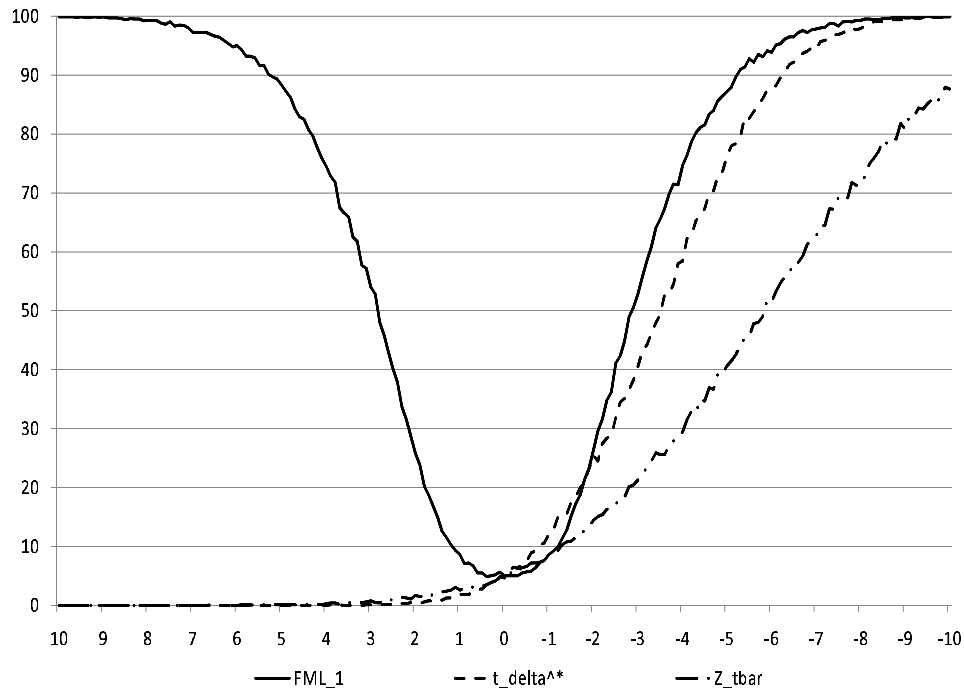
while $\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}((\Delta\hat{e}_{it})^2 - 1)\hat{e}_{it-1}^2 \ \to_d \ \sqrt{\frac{5}{12}(\kappa - 1)}\,Y$, which is the same limit as for the numerator of the second term in the formula for ALM. The second term in the formula for FLM_1 is therefore asymptotically equivalent to $\frac{24}{5}(\kappa - 1)$ times the corresponding term in ALM.

(b) As we point out in remark (a) above, FLM_1 is scale equivalent to ALM. This is very interesting because typically demeaning leads to an asymptotic bias that has to be removed in order to prevent the statistic from diverging, see for example Levin *et al.* (2002) and Im *et al.* (2003). We also see that the demeaning has no effect on the local power. This result is in agreement with the work of Moon *et al.* (2007), who develop a point optimal test statistic for the null that $\mu_c = 0$. According to their results estimation of intercepts does not affect maximal achievable power.[6]

[6]Unfortunately, the optimality property of the single parameter case does not translate directly to the present multiparameter case. The problem lies in that optimality for the single parameter case follows from maximizing power in the only direction available under the alternative hypothesis. In our case we have a power surface defined over all possible values of μ_c and ω_c^2, and hence there is no obvious direction that should be used to maximize power.

(c) It is interesting to compare the local power of the new test with the local power of the Z_{tbar} test of Im *et al.* (2003) and the t_δ^* test of Levin *et al.* (2002), two of its most natural competitors. As Moon and Perron (2008) show, under H_1 the latter statistic converges in distribution to $\frac{3}{2}\sqrt{\frac{5}{51}}\,\mu_c + N(0,1)$. The corresponding result for the former statistic is given in Harris *et al.* (2008) and is shown to be $0.282\,\mu_c + N(0,1)$, where $\frac{3}{2}\sqrt{\frac{5}{51}} > 0.282$, suggesting that t_δ^* is most powerful. This can be seen in Figure 1, which plots the power of all three tests as a function of μ_c.[7] Intuitively, when one-directional alternatives are considered one-sided tests designed for that purpose should have the highest power. But when the alternative hypothesis moves in the direction of both $\mu_c \neq 0$ and $\omega_c^2 > 0$, tests for the joint null hypothesis should have higher power. However, as the figure shows, except for the case when $-1.8 < \mu_c < 0$, FLM_1 is most powerful. The fact that the new test is most powerful even when the power is taken in the direction of only $\mu_c \neq 0$ is due to the rate of shrinking of the local alternative, which dominates the dependence upon ω_c^2, thereby effectively making the test one-directional.

Figure 1: Asymptotic local power as a function of μ_c.



[7] The figure is based on 5,000 replications.

Although unbiased in the case with a heterogeneous constant, the presence of a trend that needs to be estimated makes FLM_1 divergent. The source of this divergence is the numerator of the first term in the formula for FLM_1, which is no longer mean zero. In fact, as shown in Appendix C, $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta\hat{e}_{it}\hat{e}_{it-1} \to_p -\frac{1}{2}$ as N, $T \to \infty$ with $\frac{N}{T} \to 0$, suggesting that $\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta\hat{e}_{it}\hat{e}_{it-1}$ diverges to negative infinity at rate $\sqrt{N}$. But there is not only the mean effect, there is also a variance effect that works through the second term in the formula. Specifically, the estimation of the trend slope leads to an increase in variance, from $\frac{5}{12}(\kappa - 1)$ in model 1 to $\frac{1}{2}(\kappa - 1)$ in model 2.

In view of these concerns, a natural candidate for a feasible statistic in model 2 is to use

$$FLM_2 = \frac{\left(\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta\hat{e}_{it}\hat{e}_{it-1} + \frac{NT}{2}\right)^2}{\sum_{i=1}^{N}\sum_{t=p+2}^{T}\hat{e}_{it-1}^2} + \frac{2\left(\sum_{i=1}^{N}\sum_{t=p+2}^{T}\left((\Delta\hat{e}_{it})^2 - 1\right)\hat{e}_{it-1}^2\right)^2}{(\hat{\kappa} - 1)\sum_{i=1}^{N}\sum_{t=p+2}^{T}(\Delta\hat{e}_{it})^2\hat{e}_{it-1}^4}.$$

However, this statistic has at least two drawbacks. Firstly, quite unexpectedly the usual practice of removing the nonzero mean of the statistic does not work in the sense that the asymptotic distribution of the mean-adjusted numerator of the first term of FLM_2 is degenerate. That is,

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta\hat{e}_{it}\hat{e}_{it-1} + \frac{\sqrt{N}}{2} = o_p(1).$$

In other words, the asymptotic null distribution of FLM_2 comes only from the second term in the formula. Secondly, and even more importantly, the test has no asymptotic power against H_1. Summarizing this, we have the following theorem.

Theorem 3. *Under H_0 or H_1 and Assumptions 1, 3 and 4,*

$$FLM_2 \to_d X^2.$$

Remarks.

(a) Since the asymptotic distribution under H_1 is the same as the one that applies under H_0, the local asymptotic power of FLM_2 is equal to the size. This stands in sharp contrast to the results obtained for ALM and FLM_1, which have nontrivial asymptotic power against H_1. This difference is a manifestation of the difficulty in detecting unit roots in the presence of heterogeneous trends, commonly referred to as the incidental trend problem, see Moon and Phillips (1999). The absence of local power is therefore not due to the degeneracy of the first term in the formula for FLM_2, which might otherwise seem like a very reasonable explanation.

(b) The fact that ALM has nontrivial local power even in the presence of heterogeneous trends suggests that the problem here is not the presence of trends *per se* but rather the estimation thereof. Moon and Perron (2004, 2008), and Harris *et al.* (2008) consider the effects of incidental trends when using least squares detrending. Theorem 3 extends their results to the case of maximum likelihood demeaning.[8]

(c) Despite the absence of local power, FLM_2 is consistent against a non-local alternative in the sense that the probability of a rejection goes to one as $N, T \to \infty$ for a set of autoregressive parameters that does not depend on N or T. The rate of the divergence is $\sqrt{NT}$, which is the same as for the Levin *et al.* (2002) and Im *et al.* (2003) tests.[9]

(d) Although $\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=p+2}^{T} \Delta \hat{e}_{it} \hat{e}_{it-1} + \frac{\sqrt{N}}{2}$ is degenerate, $\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=p+2}^{T} \Delta \hat{e}_{it} \hat{e}_{it-1} + \frac{\sqrt{TN}}{2}$ is not. However, multiplication by $\sqrt{T}$ introduces nuisance parameters that are otherwise eliminated as $T \to \infty$. It also makes the test dependent upon the distribution of ϵ_{it}.

3.3 Generalizations

3.3.1 Cross-section dependence

One drawback with the above analysis is that it supposes that the cross-sectional units are independent, an assumption that is perhaps too strong to be held in many applications. Accordingly, more recent panel unit root tests such as those of Bai and Ng (2004), Moon and Perron (2004), Phillips and Sul (2003), and Pesaran (2007) relax this assumption by assuming that the dependence can be represented by a common factor model. This approach fits very well with the parametric flavor of our Lagrange multiplier framework, and it will therefore be used also in this paper.

Suppose that ϵ_{it} in (3) has the factor structure

$$\epsilon_{it} = \Theta_i' \mathbf{f}_t + v_{it}, \tag{10}$$

where we assume for simplicity that $\mathbf{f}_t = (f_{1t}, ..., f_{rt})'$ is an known r-dimensional vector of common factors with $\Theta_i = (\theta_{1i}, ..., \theta_{ri})'$ being the associated vector of factor loadings, which

[8]Consistent with the results of Moon and Perron (2008), and Moon *et al.* (2006) our preliminary calculations suggest that, although absent under H_1, the new test has nontrivial power under alternatives that shrinks towards the null hypothesis at the slower rate of $\frac{1}{N^{1/4}T}$.

[9]A formal proof of this result can be obtained from the corresponding author.

are assumed to be non-random.[10] The error v_{it} is completely idiosyncratic. Both variables are assumed to satisfy Assumption 1 with $\mathbf{f}_t$ being independent of $\Delta \mathbf{y}_{it}$. Under these conditions, (6) becomes

$$\Delta w_{it} = c_i w_{it-1} + \Theta_i' \mathbf{f}_t + v_{it},$$

which indicates that the feasible maximum likelihood estimator of Θ_i in model 2 can be obtained by running the following least squares regression:

$$\Delta y_{it} = \lambda_i + \Phi_i' \Delta \mathbf{y}_{it} + \Theta_i' \mathbf{f}_t + v_{it}. \qquad (11)$$

The factor-adjusted Lagrange multiplier test statistic is defined in exactly the same way as before but with $\hat{w}_{it}$ given by

$$\hat{w}_{it} = y_{it} - \hat{\Phi}_i' \mathbf{y}_{it} - \hat{\mu}_i - \hat{\lambda}_i (t - p) - \hat{\Theta}_i' \sum_{s=p+2}^{t} \mathbf{f}_s, \qquad (12)$$

where $\hat{\mu}_i = y_{ip+1} - \hat{\Phi}_i' \mathbf{y}_{ip} - \hat{\lambda}_i - \hat{\Theta}_i' \mathbf{f}_{p+1}$ with $\hat{\Phi}_i$, $\hat{\lambda}_i$ and $\hat{\Theta}_i$ coming from the least squares fit of (11). The asymptotic distribution of this statistic is the same as the one given in Section 3.2 for the case with cross-section independence.

If $\mathbf{f}_t$ is also unknown, then we proceed as in Bai and Ng (2004), using the method of principal components to obtain consistent estimates. The trick is to note that under H_0, $\Delta w_{it} = \Theta_i' \mathbf{f}_t + v_{it}$, which is a nothing but a static common factor model in Δw_{it}. In other words, had only Δw_{it} been known, we could have estimated $\mathbf{f}_t$ directly by the method of principal components. However, Δw_{it} is not known, and we must therefore apply the principal components method to $\Delta \hat{w}_{it}$ instead, where $\hat{w}_{it}$ is now as in (8). The testing can then be carried out as before but with $\mathbf{f}_t$ replaced by its principal components estimate.[11]

Once this estimation process has been completed, there is of course no claim of validity of the resulting test, and we do not prove here that this approach is asymptotically valid. However, intuition suggests that it should perform well in practice, and unreported simulation evidence confirms this.

[10] Of course, assuming that the common component enters via the serially uncorrelated error is by no means the only way in which the factor model can be specified. But it is convenient, see Pesaran (2007) for a detailed discussion of some alternative specifications.

[11] There are two ways to eliminate the effects of the common component, depending on the estimation Θ_i. The first is the one described in the text, which amounts to replacing $\mathbf{f}_t$ by its principal components estimate, and then to estimate Θ_i by applying least squares to the resulting first-difference regression. The second approach is to replace both $\mathbf{f}_t$ and Θ_i by their principal components estimates. Unreported simulation evidence suggests that the first approach performs best.

3.3.2 Structural breaks

Analogous with the time series statistic studied by Amsler and Lee (1995), the asymptotic null distribution of FLM_2 computed under the assumption of a linear trend but no structural break is unaffected by the presence of a break in the level of y_{it}.

Let $D_t = 1(t > \tau)$, where $1(x)$ is the indicator function and τ indicates the timing of the break, which may be unit specific. The intuition behind the above result follows from the fact that $\frac{1}{\sqrt{T}} D_t = o_p(1)$, suggesting that the break has no effect on $\frac{1}{\sqrt{T}} \hat{w}_{it}$. Moreover, since $\Delta D_t = 0$ for all t except when $t = \tau$, the effect on $\Delta \hat{w}_{it}$ is eliminated when subtracting the mean. The asymptotic null distribution of the FLM_2 is therefore unaffected.

The problem is that exclusion of the break makes the test biased towards accepting H_0. Thus, although the break does not affect the asymptotic null distribution of the test statistic, it does reduce its power. To avoid this D_t can be included in the analysis as an additional deterministic regressor, forming $d_{it} = \alpha_i + \beta_i(t-p) + \delta_i D_t$, where δ_i measures the magnitude of the break. The analysis can now be conducted exactly as before, augmenting (9) with D_t as an additional regressor.

The development of procedures that accommodate breaks that are unknown is of interest but beyond the scope of the present contribution.

3.3.3 No restrictions on the relative expansion rate of N and T

In applications when $N > T$ Assumption 4 no longer provides a reasonable approximation. In such cases we need to restrict the degree of heterogeneity that can be allowed. One way would be to assume that the heterogeneity can be regarded as random noise around an otherwise fixed mean value. But this induces a dependence on the distribution of the noise, which then has to be correctly specified. In this section we therefore go all the way and assume that α_i, β_i, $\phi_i(L)$ and σ_i^2 are completely homogeneous across i.

The resulting test statistic is computed as before but with $\hat{w}_{it}$ given by

$$\hat{w}_{it} \;=\; y_{it} - \hat{\Phi}' \mathbf{y}_{it} - \hat{\mu} - \hat{\beta} \left((t-p) - \sum_{j=1}^{p} \hat{\phi}_j (t-p-j) \right). \tag{13}$$

where $\hat{\mu} = \overline{y}_{p+1} - \hat{\Phi}' \overline{\mathbf{y}}_p - \hat{\lambda}$ with $\overline{y}_{p+1} = \frac{1}{N} \sum_{i=1}^{N} y_{ip+1}$ and an analogous definition of $\overline{\mathbf{y}}_p$. Here $\hat{\lambda}$ and $\hat{\Phi} = (\hat{\phi}_1, ..., \hat{\phi}_p)'$ are the pooled least squares intercept and slope estimators in a regression of Δy_{it} onto a constant and $\Delta \mathbf{y}_{it}$. The homogenous trend slope β cannot be estimated directly, but it can be inferred from $\hat{\lambda}$ and $\hat{\Phi}$. Specifically, by using $\lambda = \phi(1)\beta$ and a first-order Taylor expansion, it is not difficult to see that $\hat{\beta} = \frac{\hat{\lambda}}{(1-\sum_{j=1}^{p}\hat{\phi}_j)}$ should be consistent for β.

Replacing $\hat{\sigma}_i^2$ with $\hat{\sigma}^2 = \frac{1}{N(T-p-1)}\sum_{i=1}^{N}\sum_{t=p+1}^{T}(\Delta\hat{w}_{it})^2$, the feasible homogenous Lagrange multiplier test statistic in models 1 and 2 has the same form as FLM_1, and the asymptotic distributions under H_0 and H_1 are the same as the ones given in Corollaries 1 and 2, respectively. Thus, imposing homogeneity of the trend coefficient not only relaxes Assumption 4 but also removes the incidental trends problem and the absence of local power in $\frac{1}{\sqrt{NT}}$ neighborhoods.

4 Simulations

In this section, we investigate the small-sample properties of the new test through a small simulation study using (1)–(3) to generate the data. For simplicity, we assume that $\phi_i(L) = 1 - \phi L$, $\alpha_i = \beta_i = 1$ and $\varepsilon_{it} \sim N(0,1)$.

A total of seven configurations of the autoregressive parameter ρ_i and the drift parameter c_i are considered, where the latter is assumed to be generated as $c_i \sim U(a,b)$. The first configuration is for analyzing the size of the test, while the remaining six are for analyzing the power. Three of these are local to the null hypothesis and three are non-local. Specifically, the following cases are considered:

1. $\rho_i = 1$ for all i;
2. $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$ with $c_i = -10$ for all i;
3. $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$ with $c_i \sim U(-20,0)$;
4. $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$ with $c_i \sim U(-40,20)$;
5. $\rho_i = 1 + c_i$ with $c_i = -0.05$ for all i;
6. $\rho_i = 1 + c_i$ with $c_i \sim U(-0.1,0)$;
7. $\rho_i = 1 + c_i$ with $c_i \sim U(-0.15,0.05)$.

Note that with this specification of c_i, $\mu_c = \frac{1}{2}(a+b)$ and $\omega_c^2 = \frac{1}{12}(a-b)^2$. Hence, μ_c is the same in cases 2–4, and also in cases 5–7. The only thing that separates for example case

1 from cases 2 and 3 is therefore the variance, which goes from zero in case 1 to 33.33 in case 2 to 300 in case 3. This direction away from the null is interesting to consider since in our random coefficient setting there is not just the mean of c_i that matters but also the variance. The data in all six cases are generated for 5,000 panels with $T + 100$ time series observations, where the first 100 are disregarded to reduce the effect of the initial value, which is set to zero.

For the sake of comparison, the Levin *et al.* (2002) t^*_δ statistic and Im *et al.* (2003) Z_{tbar} statistic are also simulated. As explained earlier, both are constructed as t-ratios of the null that $\rho_i = 1$ for all i. The difference is that while t^*_δ is based on the t-ratio of the pooled least squares estimator of ρ, Z_{tbar} is based on the average of the individual t-ratios of the least squares estimator of ρ_i. As with the new test, Z_{tbar} is fully parametric with respect to the serial correlation properties of the data, and hence only requires lag augmentation. By contrast, t^*_δ not only requires lag augmentation but also semiparametric estimation of the so-called long-run variance of u_{it}.

In the simulations the lag length is selected using the Schwarz Bayesian information criterion, which facilitates a data dependent choice. Consistent with the results of Ng and Perron (1995), the maximum number of lags is allowed to increase with T at the rate $4(T/100)^{2/9}$. To also allow for the possibility of heterogeneous lag lengths, the criterion is evaluated once for each unit. As for the semiparametric estimation needed for computing t^*_δ, we follow the recommendation of Levin *et al.* (2002) and use the Bartlett kernel with the bandwidth parameter set equal to $3.21T^{1/3}$.

The t^*_δ and Z_{tbar} statistics can be constructed in two ways depending on the choice of mean and variance adjustment, which can be either asymptotic or sample-specific. Our test is asymptotic, suggesting that the most appropriate comparison here is obtained by using the former adjustments. However, since Im *et al.* (2003) do not provide any asymptotic results for their test, t^*_δ and Z_t are simulated based on the small-sample moments.[12] For brevity, we only report the size and power at the 5% significance level. Also, since size accuracy is not perfect, all powers are adjusted so that each test has the same level of 5% when the null hypothesis is true. All computational work has been performed in GAUSS.[13]

[12]The use of sample-specific adjustment terms is expected to lead to better performance in the simulations, which is also what we find. The comparison with the Lagrange multiplier test is therefore biased in favor of its competitors.

[13]In addition to the results reported here, we have experimented with a large number of different parame-

Consider first the size results for model 1, which are reported in Table 1. It is seen that among the three tests considered the best size accuracy is generally obtained by using Z_{tbar}, with FLM_1 performing only marginally worse. In fact, our results suggest that these tests are remarkably robust even to quite high degrees of serial correlation, a valuable property that is not very common. Of course, the accuracy is not perfect, and some distortions remain. In particular, we see that there is a slight tendency for the test to become oversized as N increases, although the distortions vanish quickly as T increases. The t^*_δ test performs worst with massive size distortions, even when $\phi = 0$ and there is no serial correlation.

The results from the local power of the tests in cases 2–4 are even more encouraging. Indeed, as Table 1 shows, the new test is almost uniformly more powerful than the other tests, and this holds even when the power is taken in the direction of only $\mu_c < 0$, which is consistent with our asymptotic results on this point, as summarized by Figure 1. We also see that increasing the variance does not lead to any increase in power, as should be expected from remark (c) following Corollary 2. Moreover, although there is a small increase among the smaller values of N and T, the power is quite flat in the sample size, which is in accordance with our expectations, since asymptotically there is no dependence on N and T.

The poor performance of the t^*_δ test is due to overfitting, which apparently can cause drastic reductions in power. This is in agreement with the results of Westerlund (2009), who shows that the power of t^*_δ depends heavily on the choice of lag length, and to an even greater extent on the choice of bandwidth. The fact that the new test seems much more robust in this regard is of some importance from an applied standpoint because these are difficult choices.

Focusing now on the non-local power, we see that while FLM_1 keeps its relative advantage in cases 5 and 7, in case 6 Z_{tbar} is ranked first, although not by much. We also see that while the level of the power is higher than in cases 2–4, in relative terms the t^*_δ test is still dominated by the others. As indicated in remark (c) of Theorem 3, under the non-local alternatives considered here, all three tests diverge at rate $\sqrt{NT}$. In agreement with this we see that increasing values of N and T lead to roughly the same increase in power, and that the magnitude of the increase is roughly of the expected rate.

Consider next the results reported in Table 2 for model 2. The first thing to notice is the

terizations of the data generating process, including negative autoregressive errors, moving average errors, and heterogeneous deterministic intercept and trend terms. Except possibly for the usual distortions in the case with negative moving average errors, the conclusions were not altered. These results are available from the corresponding author upon request. Some results based on alternative lag selection rules are also available.

power in cases 2–4, which is almost absent. The theoretical result that the distribution of the new statistic is the same under the null and local alternative hypotheses implies that the power should be roughly equal to size, or 5%. Our results are quite suggestive of this. The results for the non-local alternatives of cases 5–7 are more promising, but the power is still very low, especially among the smaller values of N and T.

5 An empirical illustration

In a well-functioning market, an increase in demand brought about by for example higher income should be accompanied by a one-for-one increase in supply, with prices being left unchanged. By contrast, in a poorly functioning market, demand and supply do not move one-for-one and therefore prices rise. This is presently the situation in many housing markets around the world. In the United States, prices have grown so fast that it has raised fears of speculative bubbles with real prices moving away from real income.[14]

This development is illustrated in Figure 2, which plots the cross-sectional mean, range and normal 95% confidence bands for the log of the price-to-income ratio for 49 states between 1975 and 2003.[15] As can be seen, the ratio first increased but then in the early 1980's, a period largely consistent with the NBER business cycle peak of January 1980, it started to decline, levelling off in the early 1990's. The sharp increase in the end of the sample is consistent with the NBER peak of March 2001.

Figure 2 suggests that if the absence of speculative bubbles is to be interpreted as a mean-reversion of the price-to-income ratio, then there is little evidence to support it. It is observations like this that have recently led many researchers to question the health of the United States housing market. One such study is that of Holly *et al.* (2009), in which the authors deduce evidence of a stable long-run one-to-one relationship between prices and income, suggesting that the market is actually in good health. However, their unit root test is based on the assumption that the data are integrated of at most order one, which naturally raises the question of how the conclusions hold up in case of a violation.[16]

[14]These concerns culminated with the eruption of the sub-prime mortgage crisis in mid-2006, which have lead to plunging property prices and a slowdown in the United States economy.

[15]The data are taken from Holly *et al.* (2009), and include for each state the real house price and income, which are both transformed by taking logs.

[16]Preliminary evidence at the state level indicates that the fully stationary alternative is inappropriate. Take for example prices, for which the estimated first-order autoregressive coefficient can be as high as 1.17 for some states, suggesting the presence of explosive behavior.

Figure 2: The cross-sectional mean of the price-to-income ratio.

In this section, we try to shed some light on this issue by reevaluating the results of Holly *et al.* (2009) based on the new test. The appropriate number of lags to use is determined as in Section 4, using Schwarz Bayesian information criterion. Because the of the strong co-movement in the data the test is implemented while allowing for up to four common factors with the exact number determined using the IC_2 criterion of Bai and Ng (2002).[17] Most of the price and income series also seem to be trending, implying that model 1 with only an intercept might not provide an accurate description of the data. The approach taken here is very simple and is based on using the Ayat and Burridge (2000) approach to determine the significance of the individual trend slopes. Only if the zero slope hypothesis is accepted for all states do we conclude that model 1 is appropriate. The results suggest that for a majority of the states the zero slope hypothesis must be rejected. We therefore focus on model 2, but include the results for model 1 for comparison.

The results are presented in Table 3. In agreement with the findings of Holly *et al.* (2009), we see that the factor-adjusted test is unable to reject the unit root null at the 5% level for

[17] In addition to using the principal components approach of Bai and Ng (2004) to estimate the factors we tried the cross-sectional average approach of Pesaran (2007) with very little differences in the results.

prices and income in their levels, but not in their first differences. The fact that the unadjusted test always rejects is not totally unexpected given the well-known size distortions of so-called first-generation panel unit root tests in the presence of unattended cross-section dependence. The results for the price-to-income ratio, which also agree with Holly *et al.* (2009), show that the variable is trend-stationary, suggesting the presence of a stable long-run one-to-one relationship between prices and income.

6 Conclusion

This paper has developed a new procedure for testing the null hypothesis of a unit root in panels where the heterogeneity of the autoregressive coefficient can be assumed to be random across the cross-section. This is quite important since in most, if not all, related work, whenever heterogeneity is allowed, it is assumed to be non-random. This means that each individual coefficient has to be estimated separately, leading to excess variation in the test. The purpose of the current paper was to device a test that exploits the information that under the null hypothesis of a unit root, when a random approach is used, the autoregressive coefficients have unit mean and zero variance. This led us naturally to the consideration of the Lagrange multiplier, or score, principle.

We have shown that with individual constants, the proposed Lagrange multiplier test has power in a local neighborhood that shrinks towards the null hypothesis at rate $\frac{1}{\sqrt{NT}}$. The limiting distribution of the new test statistic is chi-squared and therefore no special table is required to compute p-values. We have also shown that in the presence of heterogeneous trends that have to be estimated, although still consistent against non-local alternatives, the local power of the test is equal to the size.

Finally, we have provided simulation evidence that supports our theoretical results. In particular, we have shown that when no estimation of deterministic trends is necessary the new test has good size accuracy and excellent power in comparison to other tests. When such estimation is necessary, the test typically has no power beyond size.

References

Abadir, K. M. and J. R. Magnus (2005). Matrix Algebra. In K. M. Abadir, J. R. Magnus and P. C. B. Phillips (Eds.), *Econometric Exercises*, Volume 1. Cambridge University Press, New York.

Ahn, S. K. (1993). Some Tests for Unit Roots in Autoregressive-Integrated-Moving Average Models with Deterministic Trends. *Biometrica* **80**, 855–68.

Amsler, C., and J. Lee (1995). An LM Test for a Unit Root in the Presence of a Structural Break. *Econometric Theory* **11**, 359–68.

Ayat, L., and P. Burridge (2000). Unit Root Tests in the Presence of Uncertainty about the Nonstochastic Trend. *Journal of Econometrics* **95**, 71–96.

Bai, J., and S. Ng (2002). Determining the Number of Factors in Approximate Factor Models. *Econometrica* **70**, 191–221.

Bai, J., and S. Ng (2004). A Panic Attack on Unit Roots and Cointegration. *Econometrica* **72**, 1127–1177.

Breitung, J., and M. H. Pesaran (2008). Unit Roots and Cointegration in Panels. In Mátyás, L., and P. Sevestre (Eds.), *The Econometrics of Panel Data*, Chapter 9, 279–322. Klüwer Academic Publishers.

Distaso, W. (2008). Testing for Unit Root Processes in Random Coefficient Autoregressive Models. *Journal of Econometrics* **142**, 581–609.

Holly, S., M. Pesaran, and T. Yamagata (2007). A Spatio-Temporal Model of House Prices in the US. Forthcoming in *Journal of Econometrics*.

Leybourne, S. J., B. M. P. McCabe, and A. R. Tremayne (1996). Can Economic Time Series be Differenced to Stationarity? *Journal of Business and Economic Statistics* **14**, 435–446.

Levin, A., C. Lin, and C.-J. Chu (2002). Unit Root Tests in Panel Data: Asymptotic and Finite-sample Properties. *Journal of Econometrics* **108**, 1–24.

Ling, S. (2004). Estimation and Testing Stationarity for Double-Autoregressive Models. *Journal of the Royal Statistical Society: Series B* **66**, 63–78.

McCabe, B. M. P., and A. R. Tremayne (1995). Testing a Time Series for Difference Stationarity. *The Annals of Statistics* **23**, 1015–1028.

Moon, H. R., and B. Perron (2004). Testing for Unit Root in Panels with Dynamic Factors. *Journal of Econometrics* **122**, 81–126.

Moon, H. R., and B. Perron (2008). Asymptotic Local Power of Pooled t-Ratio Tests for Unit Roots in Panels with Fixed Effects. *Econometrics Journal* **11**, 80–104.

Moon, H. R., and P. C. B. Phillips (1999). Maximum Likelihood Estimation in Panels with Incidental Trends. *Oxford Bulletin of Economics and Statistics* **61**, 771–748.

Moon, H. R., B. Perron and P. C. B. Phillips (2007). Incidental Trends and the Power of Panel Unit Root Tests. *Journal of Econometrics* **141**, 416–459.

Nielsen, B. (2008). On the Explosive Nature of Hyper-Inflation Data. *Economics: The Open-Access, Open-Assessment E-Journal* **2**, 2008–21.

Park, J. Y., and P. C. B. Phillips (1999). Statistical Inference in Regressions with Integrated Processes: Part 1. *Econometric Theory* **4**, 468–497.

Pesaran, M. H. (2007). A Simple Panel Unit Root Test in the Presence of Cross Section Dependence. *Journal of Applied Econometrics* **22**, 265–312.

Phillips, P. C. B., and H. R. Moon (1999). Linear Regression Limit Theory of Nonstationary Panel Data. *Econometrica* **67**, 1057–1111.

Phillips, P. C. B. and D. Sul (2003). Dynamic Panel Estimation and Homogeneity Testing Under Cross Section Dependence. *Econometrics Journal* **6**, 217–259.

Phillips, P. C. B., Y. Wu and J. Yu (2009). Explosive Behavior in the 1990s Nasdaq: When did Exuberance Escalate Asset Values? Cowles Foundation Discussion Paper No. 1699.

Schmidt, P., and P. C. B. Phillips (1992). LM Tests for a Unit Root in the Presence of Deterministic Trends. *Oxford Bulletin of Economics and Statistics* **54**, 257–287.

Appendix A: Derivation of the true Lagrange multiplier statistic

In this appendix we derive the true Lagrange multiplier statistic. For brevity, the results are only provided for the case with a trend, which is the most general deterministic specification considered.

Lemma A.1. *Under H_0 and Assumptions 1–3 in the model with a trend the maximum likelihood estimators of σ_i^2, μ_i, λ_i and Φ_i are given by*

$$(a)\quad \check{\sigma}_i^2 \;=\; \frac{1}{T-p}\sum_{t=p+1}^{T}\epsilon_{it}^2,$$

$$(b)\quad \check{\mu}_i \;=\; y_{ip+1}-\Phi_i'\mathbf{y}_{ip}-\lambda_i,$$

$$(c)\quad \check{\lambda}_i \;=\; \Delta\overline{y}_i-\Phi_i'\Delta\overline{\mathbf{y}}_i,$$

$$(d)\quad \hat{\Phi}_i \;=\; \left(\sum_{t=p+2}^{T}(\Delta\mathbf{y}_{it}-\Delta\overline{\mathbf{y}}_i)(\Delta\mathbf{y}_{it}-\Delta\overline{\mathbf{y}}_i)'\right)^{-1}\sum_{t=p+2}^{T}(\Delta\mathbf{y}_{it}-\Delta\overline{\mathbf{y}}_i)(\Delta y_{it}-\Delta\overline{y}_i),$$

where $\Delta\overline{y}_i = \frac{1}{T-p-1}\sum_{t=p+2}^{T}\Delta y_{it}$ with an analogous definition of $\Delta\overline{\mathbf{y}}_i$.

Proof of Lemma A.1.

Consider (a). With the trend specification of d_{it},

$$\phi_i(L)y_{ip+1} \;=\; \phi_i(1)(\alpha_i+\beta_i)+\rho_i\phi_i(L)z_{ip}+\epsilon_{ip+1}$$

for $t=p+1$, and for $t=p+2,...,T$,

$$\phi_i(L)y_{it} \;=\; (1-\rho_i)\phi_i(L)\big(\alpha_i+\beta_i(t-p)\big)+\rho_i\phi_i(L)(y_{it-1}+\beta_i)+\epsilon_{it}.$$

Under H_0 these two equations reduce to

$$\phi_i(L)y_{ip+1} \;=\; \phi_i(L)(\alpha_i+\beta_i+z_{ip})+\epsilon_{ip+1} \;=\; \mu_i+\lambda_i+\epsilon_{ip+1}, \tag{A1}$$

$$\phi_i(L)y_{it} \;=\; \phi_i(L)(y_{it-1}+\beta_i)+\epsilon_{it} \;=\; \phi_i(L)y_{it-1}+\lambda_i+\epsilon_{it}. \tag{A2}$$

Moreover, under H_0 the log-likelihood function in (7) reduces to

$$L \;=\; -\frac{T-p}{2}\sum_{i=1}^{N}\ln(\sigma_i^2)-\frac{1}{2}\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\sum_{t=p+1}^{T}\epsilon_{it}^2. \tag{A3}$$

Clearly,

$$\frac{\partial L}{\partial\sigma_i^2} \;=\; -\frac{T-p}{2\sigma_i^2}+\frac{1}{2\sigma_i^4}\sum_{t=p+1}^{T}\epsilon_{it}^2,$$

which can be put equal to zero, and then solved for σ_i^2, proving (a).

Consider (b). By imposing H_0 and then concentrating with respect to σ_i^2,

$$L = -\frac{T-p}{2}\sum_{i=1}^{N}\ln(\check{\sigma}_i^2) = -\frac{T-p}{2}\sum_{i=1}^{N}\ln\left(\frac{1}{T-p}\sum_{t=p+1}^{T}\epsilon_{it}^2\right), \tag{A4}$$

where, making use of (A1) and (A2),

$$\sum_{t=p+1}^{T}\epsilon_{it}^2 = \epsilon_{ip+1}^2 + \sum_{t=p+2}^{T}\epsilon_{it}^2 = \big(\phi_i(L)y_{ip+1} - \mu_i - \lambda_i\big)^2 + \sum_{t=p+2}^{T}\big(\phi_i(L)\Delta y_{it} - \lambda_i\big)^2.$$

It follows that

$$\frac{\partial L}{\partial \mu_i} = \frac{1}{\check{\sigma}_i^2}\big(\phi_i(L)y_{ip+1} - \mu_i - \lambda_i\big),$$

implying $\check{\mu}_i = \phi_i(L)y_{ip+1} - \lambda_i = y_{ip+1} - \Phi_i'\mathbf{y}_{ip} - \lambda_i$.

Moreover, since ϵ_{ip+1}, and therefore also ϵ_{ip+1}^2, is zero when evaluated at $\mu_i = \hat{\mu}_i$ and $\sigma_i^2 = \check{\sigma}_i^2$,

$$\frac{\partial L}{\partial \lambda_i} = \frac{1}{\check{\sigma}_i^2}\sum_{t=p+2}^{T}\big(\phi_i(L)\Delta y_{it} - \lambda_i\big),$$

giving $\check{\lambda}_i = \frac{1}{T-p-1}\sum_{t=p+2}^{T}\phi_i(L)\Delta y_{it} = \Delta\overline{y}_i - \Phi_i'\Delta\overline{\mathbf{y}}_i$, which establishes (c).

Similarly, by using $\check{\lambda}_i$ in place of λ_i,

$$\sum_{t=p+1}^{T}\epsilon_{it}^2 = \sum_{t=p+2}^{T}\big(\phi_i(L)\Delta y_{it} - \hat{\lambda}_i\big)^2 = \sum_{t=p+2}^{T}\big((\Delta y_{it} - \Delta\overline{y}_i) - \Phi_i'(\Delta\mathbf{y}_{it} - \Delta\overline{\mathbf{y}}_i)\big)^2,$$

and so

$$\frac{\partial L}{\partial \Phi_i} = \frac{1}{\check{\sigma}_i^2}\sum_{t=p+2}^{T}\big((\Delta y_{it} - \Delta y_i) - \Phi_i'(\Delta\mathbf{y}_{it} - \Delta\mathbf{y}_i)\big)(\Delta\mathbf{y}_{it} - \Delta\mathbf{y}_i)',$$

from which we deduce that

$$\hat{\Phi}_i = \left(\sum_{t=p+2}^{T}(\Delta\mathbf{y}_{it} - \Delta\overline{\mathbf{y}}_i)(\Delta\mathbf{y}_{it} - \Delta\overline{\mathbf{y}}_i)'\right)^{-1}\sum_{t=p+2}^{T}(\Delta\mathbf{y}_{it} - \Delta\overline{\mathbf{y}}_i)(\Delta y_{it} - \Delta\overline{y}_i).$$

This establishes (d) and hence the proof of the lemma is complete. ■

Lemma A.2. *Under the conditions of Lemma A.1,*

$$(a)\quad \frac{\partial L}{\partial \mu_c} = \sum_{i=1}^{N}\sum_{t=p+1}^{T}\Delta\check{e}_{it}\check{e}_{it-1},$$

$$(b) \quad \frac{\partial^2 L}{(\partial \mu_c)^2} \;=\; -\sum_{i=1}^{N}\sum_{t=p+1}^{T} \check{e}_{it-1}^2,$$

$$(c) \quad \frac{\partial L}{\partial \omega_c^2} \;=\; \frac{1}{2}\sum_{i=1}^{N}\sum_{t=p+1}^{T} \big((\Delta \check{e}_{it})^2 - 1\big)\check{e}_{it-1}^2,$$

$$(d) \quad \frac{\partial^2 L}{(\partial \omega_c^2)^2} \;=\; -\frac{1}{2}\sum_{i=1}^{N}\sum_{t=p+1}^{T} \big(2(\Delta \check{e}_{it})^2 - 1\big)\check{e}_{it-1}^4,$$

$$(e) \quad \frac{\partial^2 L}{\partial \mu_c \partial \omega_c^2} \;=\; -\sum_{i=1}^{N}\sum_{t=p+1}^{T} \Delta \check{e}_{it}\check{e}_{it-1}^3.$$

Proof of Lemma A.2.

We prove (a). The proofs of (b) to (e) follow by similar arguments.

From (7),

$$\frac{\partial L}{\partial \mu_c} \;=\; \sum_{i=1}^{N}\sum_{t=p+1}^{T} \frac{\big((c_i - \mu_c)w_{it-1} + \epsilon_{it}\big)w_{it-1}}{\omega_c^2 w_{it-1}^2 + \sigma_i^2}.$$

By dividing both the numerator and the denominator by σ_i^2, and then imposing H_0, we obtain

$$\frac{\partial L}{\partial \mu_c} \;=\; \sum_{i=1}^{N}\sum_{t=p+1}^{T} \frac{\big((c_i - \mu_c)e_{it-1} + \varepsilon_{it}\big)e_{it-1}}{\omega_c^2 e_{it-1}^2 + 1} \;=\; \sum_{i=1}^{N}\sum_{t=p+1}^{T} \varepsilon_{it}e_{it-1} \;=\; \sum_{i=1}^{N}\sum_{t=p+1}^{T} \Delta e_{it}e_{it-1},$$

where $e_{it} = w_{it}/\sigma_i$ and $\varepsilon_{it} = \epsilon_{it}/\sigma_i$. The required result is obtained by concentrating the above expression with respect to σ_i^2. ■

The Lagrange multiplier statistic is defined as

$$LM \;=\; \mathbf{g}'(-\mathbf{H})^{-1}\mathbf{g}, \qquad \text{(A5)}$$

where by Lemma A.2,

$$\mathbf{g} \;=\; \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \;=\; \begin{bmatrix} \frac{\partial L}{\partial \mu_c} \\ \frac{\partial L}{\partial \omega_c^2} \end{bmatrix}, \quad \mathbf{H} \;=\; \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \;=\; \begin{bmatrix} \frac{\partial^2 L}{(\partial \mu_c)^2} & \frac{\partial^2 L}{\partial \mu_c \partial \omega_c^2} \\ \frac{\partial^2 L}{\partial \mu_c \partial \omega_c^2} & \frac{\partial^2 L}{(\partial \omega_c^2)^2} \end{bmatrix}.$$

We now show that when properly normalized **H** is asymptotically diagonal, which yields the desired result after substituting for **g** and **H** in (A5). Let us therefore consider

$$LM \;=\; -(\mathbf{G}^{-1}\mathbf{g})'\big(\mathbf{G}^{-1}\mathbf{H}\mathbf{G}^{-1}\big)^{-1}(\mathbf{G}^{-1}\mathbf{g}),$$

where

$$\mathbf{G} \;=\; \begin{bmatrix} \sqrt{N}T & 0 \\ 0 & \sqrt{N}T^{3/2} \end{bmatrix}$$

is the normalizing matrix. The off-diagonal element of $\mathbf{G}^{-1}\mathbf{H}\mathbf{G}^{-1}$ is given by

$$\begin{aligned} \frac{1}{NT^{5/2}} H_{12} &= -\frac{1}{NT^{5/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} \Delta \check{e}_{it} \check{e}_{it-1}^{3} \;=\; -\frac{1}{NT^{5/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} w_i^2 \varepsilon_{it} s_{it-1}^{3} \\ &= \frac{1}{NT^{5/2}} H_{12}^{\circ} + R, \end{aligned}$$

where $H_{12}^{\circ} = \sum_{i=1}^{N} \sum_{t=p+1}^{T} \varepsilon_{it} s_{it-1}^{3}$, $R = \frac{1}{NT^{5/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} (w_i^2 - 1) \varepsilon_{it} s_{it-1}^{3}$, $w_i = \sigma_i^2 / \check{\sigma}_i^2$ and $s_{it} = \sum_{k=p+1}^{t} \varepsilon_{ik}$.

Consider H_{12}°. Since ε_{it} is independent of s_{it-1} as well as across both i and t,

$$\frac{1}{NT^{5/2}} E(H_{12}^{\circ}) \;=\; \frac{1}{NT^{5/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} E(\varepsilon_{it}) E(s_{it-1}^{3}) \;=\; 0.$$

Also, by a functional central limit theorem, $\frac{1}{\sqrt{T}} s_{it-1} = O_p(1)$, which in turn suggests that a central limit theorem should apply to $\frac{1}{\sqrt{N}T^2} \sum_{i=1}^{N} \sum_{t=p+1}^{T} \varepsilon_{it} s_{it-1}^{3}$. Hence,

$$\frac{1}{NT^{5/2}} H_{12}^{\circ} \;=\; O_p\left(\frac{1}{\sqrt{NT}}\right)$$

and by the Cauchy–Schwarz inequality,

$$\begin{aligned} R &= \frac{1}{NT^{5/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} (w_i^2 - 1) \varepsilon_{it} s_{it-1}^{3} \\ &\leq \left[\frac{1}{N} \sum_{i=1}^{N} (w_i^2 - 1)^2 \right]^{1/2} \left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{T^{5/2}} \sum_{t=p+1}^{T} \varepsilon_{it} s_{it-1}^{3} \right)^2 \right]^{1/2} \;=\; O_p\left(\frac{1}{T}\right), \end{aligned}$$

where we have made used of the fact that $w_i - 1 = O_p(1/\sqrt{T}\,)$, as follows from a first-order Taylor expansion of the inverse of $\check{\sigma}_i^2$, which is such that

$$\check{\sigma}_i^2 \;=\; \frac{1}{T-p} \sum_{t=p+1}^{T} \epsilon_{it}^2 \;=\; \sigma_i^2 + O_p\left(\frac{1}{\sqrt{T}}\right). \tag{A6}$$

In fact, we even have $E(\check{\sigma}_i^2) = \frac{1}{T-p} \sum_{t=p+1}^{T} E(\epsilon_{it}^2) = \sigma_i^2$, meaning that $\check{\sigma}_i^2$ is not only consistent but also unbiased.

It follows that

$$\frac{1}{NT^{5/2}}\, H_{12} \;=\; O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{T}\right),$$

proving that the Hessian is indeed asymptotically diagonal. Lemma A.3 further shows that minus the Hessian for σ_i^2, μ_i, λ_i and Φ_i tends to a positive definite matrix, verifying that $\check{\sigma}_i^2$, $\check{\mu}_i$, $\check{\lambda}_i$ and $\hat{\Phi}_i$ maximizes the log-likelihood function.

Lemma A.3. *Under the conditions of Lemma A.1, as $T \to \infty$*

$$-(\mathbf{G}^*)^{-1}\mathbf{H}^*(\mathbf{G}^*)^{-1} \;\to\; \mathbf{H}^\circ > 0,$$

where

$$\mathbf{H}^* = \begin{bmatrix} \frac{\partial^2 L}{(\partial\sigma^2)^2} & \frac{\partial^2 L}{\partial\sigma_i^2\partial\mu_i} & \frac{\partial^2 L}{\partial\sigma_i^2\partial\lambda_i} & \frac{\partial^2 L}{\partial\sigma_i^2(\partial\Phi_i)'} \\ \frac{\partial^2 L}{\partial\mu_i\partial\sigma^2} & \frac{\partial^2 L}{(\partial\mu_i)^2} & \frac{\partial^2 L}{\partial\mu_i\partial\lambda_i} & \frac{\partial^2 L}{\partial\mu_i(\partial\Phi_i)'} \\ \frac{\partial^2 L}{\partial\lambda_i\partial\sigma^2} & \frac{\partial^2 L}{\partial\lambda_i\partial\mu_i} & \frac{\partial^2 L}{(\partial\lambda_i)^2} & \frac{\partial^2 L}{\partial\lambda_i(\partial\Phi_i)'} \\ \frac{\partial^2 L}{\partial\Phi_i\partial\sigma^2} & \frac{\partial^2 L}{\partial\Phi_i\partial\mu_i} & \frac{\partial^2 L}{\partial\Phi_i\partial\lambda_i} & \frac{\partial^2 L}{\partial\Phi_i(\partial\Phi_i)'} \end{bmatrix}, \quad \mathbf{G}^* = \begin{bmatrix} \sqrt{T} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{T} & 0 \\ 0 & 0 & 0 & \sqrt{T} \end{bmatrix},$$

$$\mathbf{H}^\circ = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sigma_i^2 \mathrm{cov}(\Delta\mathbf{y}_{it}). \end{bmatrix}.$$

Proof of Lemma A.3.

From the proof of Lemma A.1 we have that when evaluated at $\check{\sigma}_i^2$, $\check{\mu}_i$, $\check{\lambda}_i$ and $\hat{\Phi}_i$ minus the Hessian becomes

$$-\mathbf{H}^* = \frac{1}{\check{\sigma}_i^4}\begin{bmatrix} \frac{1}{2}(T-p) & 0 & 0 & 0 \\ 0 & 1 & 1 & \check{\sigma}_i^2\mathbf{y}_{ip}' \\ 0 & 1 & T-p-1 & \check{\sigma}_i^2(T-p-1)\Delta\overline{\mathbf{y}}_i' \\ 0 & \check{\sigma}_i^2\mathbf{y}_{ip} & \check{\sigma}_i^2(T-p-1)\Delta\overline{\mathbf{y}}_i & \check{\sigma}_i^2\sum_{t=p+2}^{T}(\Delta\mathbf{y}_{it}-\Delta\overline{\mathbf{y}}_i)(\Delta\mathbf{y}_{it}-\Delta\overline{\mathbf{y}}_i)' \end{bmatrix}$$
$$= \frac{1}{\check{\sigma}_i^4}\mathbf{G}^*\mathbf{D}\mathbf{G}^*,$$

where

$$\mathbf{D} = \begin{bmatrix} \frac{1}{2}(1-\frac{p}{T}) & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sqrt{T}} & \frac{1}{\sqrt{T}}\check{\sigma}_i^2\mathbf{y}_{ip}' \\ 0 & \frac{1}{\sqrt{T}} & 1-\frac{p+1}{T} & \check{\sigma}_i^2(1-\frac{p+1}{T})\Delta\overline{\mathbf{y}}_i' \\ 0 & \frac{1}{\sqrt{T}}\check{\sigma}_i^2\mathbf{y}_{ip} & \check{\sigma}_i^2(1-\frac{p+1}{T})\Delta\overline{\mathbf{y}}_i & \check{\sigma}_i^2\frac{1}{T}\sum_{t=p+2}^{T}(\Delta\mathbf{y}_{it}-\Delta\overline{\mathbf{y}}_i)(\Delta\mathbf{y}_{it}-\Delta\overline{\mathbf{y}}_i)' \end{bmatrix}.$$

Note that $\mathbf{G}^* > 0$. Thus, by using the results of Abadir and Magnus (2005), if we can show that $\mathbf{D} > 0$, then $-\mathbf{H}^* > 0$. Towards this end, note that $\mathbf{D} \to \mathbf{H}^\circ$ as $T \to \infty$, where $\mathrm{cov}(\Delta\mathbf{y}_{it})$ is a diagional matrix, which implies $\det(\mathrm{cov}(\Delta\mathbf{y}_{it})) > 0$. Hence, because all the leading principal minors of $\mathbf{H}^\circ$ are positive definite, $\mathbf{H}^\circ > 0$. ■

Appendix B: Asymptotic properties of the true Lagrange multiplier statistic

Proof of Theorem 1.

Write

$$\begin{aligned} ALM &= \frac{\left(\sum_{i=1}^N \sum_{t=p+1}^T \Delta\check{e}_{it}\check{e}_{it-1}\right)^2}{\sum_{i=1}^N \sum_{t=p+1}^T \check{e}_{it-1}^2} + \frac{\left(\sum_{i=1}^N \sum_{t=p+1}^T ((\Delta\check{e}_{it})^2-1)\check{e}_{it-1}^2\right)^2}{2\sum_{i=1}^N \sum_{t=p+1}^T (2(\Delta\check{e}_{it})^2-1)\check{e}_{it-1}^4} \\ &= \frac{\left(\sum_{i=1}^N \sum_{t=p+1}^T w_i\varepsilon_{it}s_{it-1}\right)^2}{\sum_{i=1}^N \sum_{t=p+1}^T w_i s_{it-1}^2} + \frac{\left(\sum_{i=1}^N \sum_{t=p+1}^T w_i^2(\varepsilon_{it}^2-w_i^{-1})s_{it-1}^2\right)^2}{2\sum_{i=1}^N \sum_{t=p+1}^T w_i^3(2\varepsilon_{it}^2-w_i^{-1})s_{it-1}^4} = I + II, \end{aligned}$$

where w_i and s_{it} are as in Appendix A.

Consider I, which we write as

$$I = \frac{\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^N \sum_{t=p+1}^T w_i\varepsilon_{it}s_{it-1}\right)^2}{\frac{1}{NT^2}\sum_{i=1}^N \sum_{t=p+1}^T w_i s_{it-1}^2} = \frac{I_1^2}{I_2},$$

where $I_1 = I_1^\circ + R_1$ with $I_1^\circ = \frac{1}{\sqrt{N}T}\sum_{i=1}^N \sum_{t=p+1}^T \varepsilon_{it}s_{it-1}$ and

$$\begin{aligned} R_1 &= \frac{1}{\sqrt{N}T}\sum_{i=1}^N \sum_{t=p+1}^T (w_i-1)\varepsilon_{it}s_{it-1} \\ &\leq \left[\frac{1}{\sqrt{N}}\sum_{i=1}^N (w_i-1)^2\right]^{1/2}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^N\left(\frac{1}{T}\sum_{t=p+1}^T \varepsilon_{it}s_{it-1}\right)^2\right]^{1/2} = O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right), \end{aligned}$$

which goes to zero if $\frac{N}{T} \to 0$ as $N,\, T \to \infty$. It follows that I_1 is asymptotically equivalent to I_1°, whose expectation is given by

$$E(I_1^\circ) = \frac{1}{\sqrt{N}T}\sum_{i=1}^N \sum_{t=p+1}^T E(\varepsilon_{it})E(s_{it-1}) = 0.$$

The computation of the variance is simplified by noting that as $T \to \infty$

$$\frac{1}{T}\sum_{t=p+1}^T \varepsilon_{it}s_{it-1} \to_w \int_0^1 W_i(r)dW_i(r),$$

where $W_i(r)$ is a standard Brownian motion on $r \in [0,1]$. Thus, by the continuous mapping theorem,

$$\begin{aligned}
\text{var}(I_1^\circ) &\to E\left[\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\int_0^1 W_i(r)dW_i(r)\right)^2\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^1\int_0^1 E(W_i(r)W_j(u))E(dW_i(r)dW_j(u)) \\
&= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 E(W_i(r)^2)E(dW_i(r)^2) = \int_0^1 rdr = \frac{1}{2},
\end{aligned}$$

where the second equality follows from the fact that $E(dW_i(r)dW_j(u)) = 0$ for all $i \neq j$ and $r \neq u$, while the third uses $E(W_i(r)^2) = r$ and $dW_i(r)^2 = dr$.

Define $X_i = \frac{\sqrt{2}}{\sqrt{N}T}\sum_{t=p+1}^{T}\varepsilon_{it}s_{it-1}$, which is independent across i with mean zero and variance $\text{var}(X_i) = O(1/N)$. Therefore, according to Theorem 2 of Phillips and Moon (1999), if we can show that for all $\delta > 0$,

$$\lim_{N,T\to\infty}\sum_{i=1}^{N} E\big(X_i^2 1(|X_i| > \delta)\big) = 0,$$

where $1(x)$ is the indicator function, then $\sum_{i=1}^{N} X_i \to_d X \sim N(0,1)$ as $N, T \to \infty$.

To verify this condition we make use of the Cauchy–Schwarz inequality, which yields

$$E\big(X_i^2 1(|X_i| > \delta)\big) \leq \sqrt{E(X_i^4)E(1(|X_i| > \delta))}$$

and by further application of the Markov inequality, $E(1(|X_i| > \delta)) \leq \frac{1}{\delta^2}E(X_i^2)$. Thus,

$$\begin{aligned}
\sum_{i=1}^{N} E\big(X_i^2 1(|X_i| > \delta)\big) &\leq \frac{1}{\delta}\sum_{i=1}^{N}\sqrt{E(X_i^4)E(X_i^2)} \leq \frac{1}{\delta}\left(\sum_{i=1}^{N}E(X_i^4)\right)^{1/2}\left(\sum_{i=1}^{N}E(X_i^2)\right)^{1/2} \\
&= O\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

Therefore, since the above condition holds, as $N, T \to \infty$

$$I_1^\circ = \frac{1}{\sqrt{2}}\sum_{i=1}^{N} X_i \to_d \frac{1}{\sqrt{2}}X, \tag{A7}$$

which, via the continuous mapping theorem, gives $(I_1^\circ)^2 \to_d \frac{1}{2}X^2$.

Consider $I_2^\circ = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=p+1}^{T} s_{it-1}^2$. As $T \to \infty$

$$\frac{1}{T^2}\sum_{t=p+1}^{T} s_{it-1}^2 \to_w \int_0^1 W_i(r)^2 dr,$$

and therefore

$$E(I_2^\circ) \;\to\; \frac{1}{N}\sum_{i=1}^{N}\int_0^1 E(W_i(r)^2)dr \;=\; \int_0^1 r dr \;=\; \frac{1}{2}.$$

Thus, by Corollary 1 of Phillips and Moon (1999), if $\frac{1}{T^2}\sum_{t=p+1}^{T} s_{it-1}^2$ is uniformly integrable in T, then $I_2^\circ \to_p \frac{1}{2}$ as $N,\, T \to \infty$. But $\frac{1}{T^2}\sum_{t=p+1}^{T} s_{it-1}^2 \to_w \int_0^1 W_i(r)^2 dr$, and therefore uniform integrability is a direct consequence of

$$E\left(\frac{1}{T^2}\sum_{t=p+1}^{T} s_{it-1}^2\right) \;=\; tr\left(\frac{1}{T^2}\sum_{t=p+1}^{T} E(s_{it-1}^2)\right) \;\to\; E\left(\int_0^1 W_i(r)^2 dr\right),$$

see Appendix C of Phillips and Moon (1999).

Hence, because $I_2 = I_2^\circ + R_2$, where

$$\begin{aligned} R_2 \;&=\; \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=p+1}^{T}(w_i-1)s_{it-1}^2 \\ &\le\; \left[\frac{1}{N}\sum_{i=1}^{N}(w_i-1)^2\right]^{1/2}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T^2}\sum_{t=p+1}^{T}s_{it-1}^2\right)^2\right]^{1/2} \;=\; O_p\left(\frac{1}{\sqrt{T}}\right), \end{aligned}$$

by Taylor expansion and then passing $N,\, T \to \infty$ with $\frac{N}{T} \to 0$, we obtain

$$I \;=\; \frac{I_1^2}{I_2} \;=\; \frac{(I_1^\circ)^2}{I_2^\circ} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) \;\to_d\; X^2. \tag{A8}$$

Next, consider II, which can be written as

$$II \;=\; \frac{\left(\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\sum_{t=p+1}^{T} w_i^2(\varepsilon_{it}^2 - w_i^{-1})s_{it-1}^2\right)^2}{\frac{2}{NT^3}\sum_{i=1}^{N}\sum_{t=p+1}^{T} w_i^3(2\varepsilon_{it}^2 - w_i^{-1})s_{it-1}^4} \;=\; \frac{II_1^2}{2\,II_2}.$$

By the same steps used for evaluating I_1,

$$II_1 \;=\; \frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\sum_{t=p+1}^{T}(\varepsilon_{it}^2 - w_i^{-1})s_{it-1}^2 + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) \;=\; II_1^\circ + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right),$$

implying that II_1 is asymptotically equivalent to II_1° as $N,\, T \to \infty$ with $\frac{N}{T} \to \infty$. In order to compute the mean of this quantity note that by the unbiasedness of $\check{\sigma}_i^2$, $E(\epsilon_{it}^2 - \check{\sigma}_i^2) = 0$, which in turn implies that

$$E(II_1^\circ) \;=\; \frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\sum_{t=p+1}^{T} E(\varepsilon_{it}^2 - w_i^{-1})E(s_{it-1}^2) \;=\; 0.$$

The computation of the variance is simplified by rewriting II_1° in the following way:

$$\begin{aligned} II_1^\circ &= \frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} (\varepsilon_{it}^2 - w_i^{-1}) \left(s_{it-1}^2 - \frac{1}{T} \sum_{k=p+1}^{T} s_{ik-1}^2 \right) \\ &= \frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} (\varepsilon_{it}^2 - 1) \left(s_{it-1}^2 - \frac{1}{T} \sum_{k=p+1}^{T} s_{ik-1}^2 \right) + O_p\left(\frac{\sqrt{N}}{\sqrt{T}} \right), \end{aligned}$$

which uses deviations from means. Note that

$$\begin{aligned} \operatorname{var}\left(\frac{1}{\sqrt{T}} \sum_{k=p+1}^{T} (\varepsilon_{ik}^2 - 1) \right) &= \frac{1}{T} \sum_{k=p+1}^{T} \sum_{t=p+1}^{T} E\big((\varepsilon_{ik}^2 - 1)(\varepsilon_{it}^2 - 1)\big) = \frac{1}{T} \sum_{k=p+1}^{T} E\big((\varepsilon_{ik}^2 - 1)^2\big) \\ &= \frac{1}{T} \sum_{k=p+1}^{T} (E(\varepsilon_{ik}^4) - 1) = \kappa_i - 1, \end{aligned}$$

suggesting that

$$\frac{1}{\sqrt{T}} \sum_{k=p+1}^{t} (\varepsilon_{ik}^2 - 1) \to_w \sqrt{\kappa_i - 1}\, V_i(r)$$

as $T \to \infty$, where $V_i(r)$ is a standard Brownian motion that is independent of $W_i(r)$, see Lemma A1 of McCabe and Tremayne (1995). It follows that

$$\frac{1}{T^{3/2}} \sum_{t=p+1}^{T} (\varepsilon_{it}^2 - 1) \left(s_{it-1}^2 - \frac{1}{T} \sum_{k=p+1}^{T} s_{ik-1}^2 \right) \to_w \sqrt{\kappa_i - 1} \int_0^1 \left(W_i(r)^2 - \int_0^1 W_i(u)^2 du \right) dV_i(r),$$

from which we deduce

$$\begin{aligned} \operatorname{var}(II_1^\circ) &\to E\left[\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sqrt{\kappa_i - 1} \int_0^1 \left(W_i(r)^2 - \int_0^1 W_i(u)^2 du \right) dV_i(r) \right)^2 \right] \\ &= \frac{1}{N} \sum_{i=1}^{N} (\kappa_i - 1) \int_0^1 E\left[\left(W_i(r)^2 - \int_0^1 W_i(u)^2 du \right)^2 \right] dr, \end{aligned}$$

where

$$\begin{aligned} \int_0^1 E\left[\left(W_i(r)^2 - \int_0^1 W_i(u)^2 du \right)^2 \right] dr &= \int_0^1 E(W_i(r)^4) dr - E\left[\left(\int_0^1 W_i(r)^2 dr \right)^2 \right] \\ &= \int_0^1 E(W_i(r)^4) dr - \int_0^1 \int_0^1 E(W_i(r)^2 W_i(u)^2) dr du. \end{aligned}$$

By using the moments of Brownian motion,

$$\begin{aligned} \int_0^1 E(W_i(r)^4) dr &= 3 \int_0^1 r^2 dr = 1, \\ \int_0^1 \int_0^1 E(W_i(r)^2 W_i(u)^2) dr du &= \int_0^1 \int_0^1 (ru + 2\min\{u^2, r^2\}) dr du = 2 \int_0^1 \int_0^r (ru + 2u^2) dr du \\ &= \frac{7}{3} \int_0^1 r^3 dr = \frac{7}{12}, \end{aligned}$$

and therefore

$$\text{var}(II_1^\circ) \;\to\; \frac{5}{12}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}(\kappa_i-1) \;=\; \frac{5}{12}(\kappa-1)$$

as $N,\,T\to\infty$. The results for the mean and variance of II_1°, together with Theorem 2 of Phillips and Moon (1999), yield $II_1^\circ \to_d \sqrt{\frac{5}{12}(\kappa-1)}\,Y$ as $N,\,T\to\infty$ with $\frac{N}{T}\to\infty$, where $Y\sim N(0,1)$, implying $(II_1^\circ)^2 \to_d \frac{5}{12}(\kappa-1)\,Y^2$.

As for II_2, note that

$$\begin{aligned}
II_2 &= \frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+1}^{T} w_i^3(2\varepsilon_{it}^2 - w_i^{-1})s_{it-1}^4 \\
&= \frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+1}^{T} w_i^3(\varepsilon_{it}^2 - w_i^{-1})s_{it-1}^4 + \frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+1}^{T} w_i^3\varepsilon_{it}^2 s_{it-1}^4 \\
&= \frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+1}^{T} w_i^3\varepsilon_{it}^2 s_{it-1}^4 + O_p\left(\frac{1}{\sqrt{NT}}\right) \;=\; \frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+1}^{T} \varepsilon_{it}^2 s_{it-1}^4 + O_p\left(\frac{1}{\sqrt{T}}\right) \\
&= II_2^\circ + O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}$$

where the last equality follows from the fact that $II_1 = O_p(1)$. But

$$\frac{1}{T^3}\sum_{t=p+1}^{T}\varepsilon_{it}^2 s_{it-1}^4 \;\to_w\; \int_0^1 W_i(r)^4 dW_i(r)^2 \;=\; \int_0^1 W_i(r)^4 dr$$

as $T\to\infty$, suggesting that by Corollary 1 of Phillips and Moon (1999), as $N,\,T\to\infty$

$$II_2^\circ \;\to_p\; \int_0^1 E\big(W_i(r)^4\big)dr \;=\; 3\int_0^1 r^2 dr \;=\; 1.$$

Consequently,

$$II \;=\; \frac{II_1^2}{2II_2} \;=\; \frac{(II_1^\circ)^2}{2II_2^\circ} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) \;\to_d\; \frac{5}{24}(\kappa-1)\,Y^2 \tag{A9}$$

as $N,\,T\to\infty$ with $\frac{N}{T}\to 0$.

It remains to show that X^2 and Y^2 are independent. Define

$$X_N \;=\; \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\int_0^1 W_i(r)dW_i(r)}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\int_0^1 W_i(r)^2 dr}}, \qquad Y_N \;=\; \frac{\sqrt{\frac{12}{5}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\int_0^1 W_i(r)^2 dV_i(r)}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\int_0^1 W_i(r)^4 dr}}$$

such that $X_N \to_d X$ and $Y_N \to_d Y$ as $N\to\infty$. Hence,

$$ALM \;\to_d\; \lim_{N\to\infty} X_N^2 + \frac{5}{24}(\kappa-1)\lim_{N\to\infty} Y_N^2$$

as $N, T \to \infty$ with $\frac{N}{T} \to 0$. By Corollary 5.3 or Park and Phillips (1988), X_1^2 and Y_1^2 are independent such that their sum is chi-squared distributed with two degrees of freedom. But $W_i(r)$ and $V_j(r)$ are independent for all i and j, and therefore X_N^2 and Y_N^2 are also independent. The proof is completed by noting that the independence is preserved as $N \to \infty$. ■

Proof of Theorem 2.

By combining (2) and (3) we get

$$w_{it} = \rho_i^t \phi_i(L) z_{ip} + \sum_{k=p+1}^{t} \rho_i^{t-k} \epsilon_{ik},$$

which, via Taylor expansion and insertion of $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$, can be rewritten as

$$\begin{aligned} \frac{1}{\sqrt{T}} w_{it} &= \frac{1}{\sqrt{T}} \sum_{k=p+1}^{t} \epsilon_{ik} + \frac{1}{\sqrt{T}} \phi_i(L) z_{ip} + \frac{c_i}{\sqrt{NT}} \left(\frac{t}{T} \phi_i(L) z_{ip} + \sum_{k=p+1}^{t} \frac{t-k}{T} \epsilon_{ik} \right) + o_p(1) \\ &= \frac{1}{\sqrt{T}} \sum_{k=p+1}^{t} \epsilon_{ik} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + o_p(1). \end{aligned} \quad \text{(A10)}$$

Hence, just as under H_0, if we assume that $N, T \to \infty$, then $\frac{1}{\sqrt{T}} e_{it} \to_w W_i(r)$. But we also have $\Delta e_{it} = (\rho_i - 1) e_{it-1} + \varepsilon_{it}$, from which it follows that

$$\begin{aligned} (\Delta e_{it})^2 &= ((\rho_i - 1) e_{it-1} + \varepsilon_{it})^2 = (\rho_i - 1)^2 e_{it-1}^2 + 2(\rho_i - 1) e_{it-1} \varepsilon_{it} + \varepsilon_{it}^2 \\ &= \frac{c_i^2}{NT^2} e_{it-1}^2 + 2 \frac{c_i}{\sqrt{NT}} e_{it-1} \varepsilon_{it} + \varepsilon_{it}^2. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{k=p+1}^{t} ((\Delta e_{ik})^2 - 1) &= \frac{1}{\sqrt{T}} \sum_{k=p+1}^{t} (\varepsilon_{ik}^2 - 1) + \frac{c_i}{NT^{5/2}} \sum_{k=p+1}^{t} e_{ik-1}^2 + 2 \frac{c_i}{\sqrt{N} T^{3/2}} \sum_{k=p+1}^{t} e_{ik-1} \varepsilon_{ik} \\ &= \frac{1}{\sqrt{T}} \sum_{k=p+1}^{t} (\varepsilon_{ik}^2 - 1) + O_p\left(\frac{1}{\sqrt{TN}}\right) \to_w \sqrt{\kappa_i - 1}\, V_i(r) \end{aligned} \quad \text{(A11)}$$

as $T \to \infty$, and by a similar calculation,

$$\begin{aligned} \check{\sigma}_i^2 &= \frac{1}{T-p} \sum_{t=p+1}^{T} (\Delta w_{it})^2 = \frac{c_i^2}{NT^3} \sum_{t=p+2}^{T} w_{it-1}^2 + 2 \frac{c_i}{\sqrt{N} T^2} \sum_{t=p+2}^{T} w_{it-1} \epsilon_{it} + \frac{1}{T} \sum_{t=p+2}^{T} \epsilon_{it}^2 \\ &= \frac{1}{T} \sum_{t=p+2}^{T} \epsilon_{it}^2 + O_p\left(\frac{1}{\sqrt{NT}}\right) = \sigma_i^2 + O_p\left(\frac{1}{\sqrt{T}}\right). \end{aligned} \quad \text{(A12)}$$

Equations (A10) to (A12) imply that the asymptotic results obtained for $(I_1^\circ)^2$, I_2°, $(II_1^\circ)^2$ and II_2° under H_0 apply also under H_1.

Let us therefore decompose ALM into $I = \frac{I_1^2}{I_2}$ and $II = \frac{II_1^2}{II_2}$, where I_1^2, I_2, II_1^2 and II_2 are the relevant numerator and denominator terms. We begin by considering I, where

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{t=p+2}^{T} \Delta \check{e}_{it} \check{e}_{it-1} = \frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{t=p+2}^{T} w_i \Delta e_{it} e_{it-1} \\ &= \frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{t=p+2}^{T} \Delta e_{it} e_{it-1} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) \\ &= \frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{t=p+2}^{T} ((\rho_i - 1) e_{it-1}^2 + e_{it-1}\varepsilon_{it}) + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) = R_1 + I_1^\circ + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right), \end{aligned}$$

where I_1° is as in the proof of Theorem 1, while

$$E(R_1) = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=p+2}^{T} E(c_i e_{it-1}^2) = \mu_c \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=p+2}^{T} E(e_{it-1}^2) \to \frac{\mu_c}{2},$$

which, via Corollary 1 of Phillips and Moon (1999), gives $R_1 \to_p \frac{\mu_c}{2}$ as $N, T \to \infty$. Also, from the proof of Theorem 1, $I_1^\circ \to_d \frac{1}{\sqrt{2}} X$, giving

$$\begin{aligned} I_1^2 &= (R_1 + I_1^\circ)^2 + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) = R_1^2 + 2 R_1 I_1^\circ + (I_1^\circ)^2 + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) \\ &\to_d \frac{\mu_c^2}{4} + \frac{\mu_c}{\sqrt{2}} X + \frac{1}{2} X^2. \end{aligned}$$

But we also have that $I_2 \to_p \frac{1}{2}$ as $N, T \to \infty$ with $\frac{N}{T} \to 0$, and so

$$I \to_d \frac{\mu_c^2}{2} + \mu_c \sqrt{2} X + X^2. \tag{A13}$$

Next, consider II, where

$$\begin{aligned} II_1 &= \frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \sum_{t=p+2}^{T} ((\Delta \check{e}_{it})^2 - 1) \check{e}_{it-1}^2 \\ &= \frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \sum_{t=p+2}^{T} ((\rho_i - 1)^2 e_{it-1}^2 + 2(\rho_i - 1) e_{it-1} \varepsilon_{it} - (\varepsilon_{it}^2 - w_i^{-1})) w_i^2 e_{it-1}^2 \\ &= R_1 + R_2 + II_1^\circ. \end{aligned}$$

From the proof of Theorem 1 we know that $II_1^\circ \to_d \sqrt{\frac{5}{12}(\kappa - 1)}\, Y$ as $N, T \to \infty$ with $\frac{N}{T} \to 0$. Also,

$$\begin{aligned} R_1 &= \frac{1}{N^{3/2} T^{7/2}} \sum_{i=1}^{N} \sum_{t=p+2}^{T} c_i^2 w_i^2 e_{it-1}^4 = O_p\left(\frac{1}{\sqrt{NT}}\right), \\ R_2 &= \frac{2}{NT^{5/2}} \sum_{i=1}^{N} \sum_{t=p+2}^{T} c_i w_i^2 e_{it-1}^3 \varepsilon_{it} = O_p\left(\frac{1}{\sqrt{NT}}\right). \end{aligned}$$

Part II_2 can be written as

$$\begin{aligned} II_2 &= \frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+2}^{T}(\Delta\check{e}_{it})^2\check{e}_{it-1}^4 + O_p\left(\frac{1}{\sqrt{NT}}\right) \\ &= \frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\left((\rho_i-1)^2 e_{it-1}^2 + 2(\rho_i-1)e_{it-1}\varepsilon_{it} + \varepsilon_{it}^2\right)w_i^3 e_{it-1}^4 + O_p\left(\frac{1}{\sqrt{NT}}\right) \\ &= R_1 + R_2 + II_2^\circ + O_p\left(\frac{1}{\sqrt{NT}}\right), \end{aligned}$$

where $II_2^\circ \to_p 1$, while

$$\begin{aligned} R_1 &= \frac{1}{N^2T^5}\sum_{i=1}^{N}\sum_{t=p+2}^{T} c_i^2 w_i^3 e_{it-1}^6 = O_p\left(\frac{1}{NT}\right), \\ R_2 &= \frac{2}{N^{3/2}T^{7/2}}\sum_{i=1}^{N}\sum_{t=p+2}^{T} c_i w_i^3 e_{it-1}^5\varepsilon_{it} = O_p\left(\frac{1}{\sqrt{T}N}\right). \end{aligned}$$

Thus, by Taylor expansion,

$$II = \frac{II_1^2}{II_2} = \frac{(II_1^\circ)^2}{II_2^\circ} + O_p\left(\frac{1}{\sqrt{NT}}\right) \to_d \frac{5}{24}(\kappa-1)\,Y^2. \tag{A14}$$

which, together with (A13), establishes the required result. ∎

Appendix C: Asymptotic properties of the feasible Lagrange multiplier statistic

Lemma C.1. *Under the conditions of Corollary 1 and in the model with a trend,*

$$(a)\quad \hat{w}_{it} = \sum_{k=p+1}^{t}(\varepsilon_{ik}-\bar{\varepsilon}_i) + O_p(1),$$

$$(b)\quad \Delta\hat{w}_{it} = \epsilon_{it} + O_p\left(\frac{1}{\sqrt{T}}\right),$$

where $\bar{\epsilon}_i = \frac{1}{T-p-1}\sum_{t=p+2}^{T}\epsilon_{it}$.

Proof of Lemma C.1.

We begin with (a). From (8) and Lemma A.1,

$$\begin{aligned} \hat{w}_{it} &= y_{it} - \hat{\Phi}_i'\mathbf{y}_{it} - \hat{\mu}_i - \hat{\lambda}_i(t-p) \\ &= \sum_{k=p+1}^{t}\epsilon_{ik} - (\hat{\Phi}_i - \Phi_i)'\mathbf{y}_{it} - (\hat{\mu}_i - \mu_i) - (\hat{\lambda}_i - \beta_i\phi_i(L))(t-p) \\ &= \sum_{k=p+1}^{t}\epsilon_{ik} - (\hat{\Phi}_i - \Phi_i)'\mathbf{y}_{it} - (\hat{\mu}_i - \mu_i) - (\hat{\lambda}_i - \lambda_i)(t-p) - \beta_i\phi_i^*(1), \end{aligned}$$

where the last equality uses the Beveridge–Nelson decomposition of $\phi_i(L)$ as $\phi_i(L) = \phi_i(1) + \phi_i^*(L)(1-L)$.

Consider $\hat{\lambda}_i$. From (A2) we have that $\phi_i(L)\Delta y_{it} = \lambda_i + \varepsilon_{it}$, or

$$\Delta y_{it} \;=\; \Phi_i'\Delta\mathbf{y}_{it} + \lambda_i + \epsilon_{it}. \tag{A15}$$

Hence, $(\Delta y_{it} - \Delta\overline{y}_i) = \Phi_i'(\Delta\mathbf{y}_{it} - \Delta\overline{\mathbf{y}}_i) + \epsilon_{it} - \overline{\epsilon}_i$, which is a stationary regression with asymptotically exogenous regressors. It follows that as $T \to \infty$

$$\begin{aligned}\hat{\Phi}_i \;&=\; \Phi_i + \left(\sum_{t=p+2}^{T}(\Delta\mathbf{y}_{it} - \Delta\overline{\mathbf{y}}_i)(\Delta\mathbf{y}_{it} - \Delta\overline{\mathbf{y}}_i)'\right)^{-1}\sum_{t=p+2}^{T}(\Delta\mathbf{y}_{it} - \Delta\overline{\mathbf{y}}_i)(\epsilon_{it} - \overline{\epsilon}_i)\\ &=\; \Phi_i + O_p\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

Hence, since $\overline{\epsilon}_i = O_p(1/\sqrt{T})$,

$$\hat{\lambda}_i \;=\; \Delta\overline{y}_i - \hat{\Phi}_i'\Delta\overline{\mathbf{y}}_i \;=\; \lambda_i - (\hat{\Phi}_i - \Phi_i)'\Delta\overline{\mathbf{y}}_i + \overline{\epsilon}_i \;=\; \lambda_i + O_p\left(\frac{1}{\sqrt{T}}\right).$$

A similar calculation reveals that

$$\hat{\mu}_i \;=\; y_{ip+1} - \hat{\Phi}_i'\mathbf{y}_{ip} - \hat{\lambda}_i \;=\; \mu_i - (\hat{\Phi}_i - \Phi_i)'\mathbf{y}_{ip} - (\hat{\lambda}_i - \lambda_i) \;=\; \mu_i + O_p\left(\frac{1}{\sqrt{T}}\right).$$

By putting everything together,

$$\hat{w}_{it} \;=\; \sum_{k=p+1}^{t}(\epsilon_{ik} - \overline{\epsilon}_i) + O_p(1),$$

where we have made use of the fact that $\sum_{k=p+1}^{t}\epsilon_{ik} - (\hat{\lambda}_i - \lambda_i)(t-p) = \sum_{k=p+1}^{t}(\epsilon_{ik} - \overline{\epsilon}_i) + o_p(1)$. This establishes (a), and by similar arguments,

$$\Delta\hat{w}_{it} \;=\; \Delta y_{it} - \hat{\Phi}_i'\Delta\mathbf{y}_{it} - \hat{\lambda}_i \;=\; \epsilon_{it} - (\hat{\Phi}_i - \Phi_i)'\Delta\mathbf{y}_{it} - (\hat{\lambda}_i - \lambda_i) \;=\; \epsilon_{it} + O_p\left(\frac{1}{\sqrt{T}}\right),$$

which establishes (b). ■

Note that if there is no trend in the model, then $(\hat{\lambda}_i - \lambda_i)(t-p)$ drops out in the equation for $\hat{w}_{it}$, and therefore so does $\overline{\epsilon}_i$. Hence, $\sum_{k=p+1}^{t}(\epsilon_{ik} - \overline{\epsilon}_i)$ reduces to $\sum_{k=p+1}^{t}\epsilon_{ik}$.

Note also that since the feasible maximum likelihood estimators converge to their unfeasible counterparts, and since ϵ_{ip+1} is zero when evaluated at the unfeasible estimators, observation $t = p+1$ can be disregarded when forming the feasible Lagrange multiplier test statistic.

Proof of Corollary 1.

This proof follows by a simple adaptation of the proof of Theorem 2. We begin by writing the test statistic as

$$
\begin{aligned}
FLM_1 &= \frac{\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta\hat{e}_{it}\hat{e}_{it-1}\right)^2}{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\hat{e}_{it-1}^2} + \frac{24\left(\frac{1}{NT^{3/2}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\left((\Delta\hat{e}_{it})^2-1\right)\hat{e}_{it-1}^2\right)^2}{5(\hat{\kappa}-1)\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+2}^{T}(\Delta\hat{e}_{it})^2\hat{e}_{it-1}^4} \\
&= \frac{I_1^2}{I_2} + \frac{24}{5(\hat{\kappa}-1)}\frac{II_1^2}{II_2} = I + \frac{24}{5(\hat{\kappa}-1)}II
\end{aligned}
$$

with an obvious definition of I_1, I_2, II_1 and II_2.

Consider I_1. By Lemma C.1,

$$
\hat{\sigma}_i^2 = \frac{1}{T}\sum_{t=p+2}^{T}(\Delta\hat{w}_{it})^2 = \check{\sigma}_i^2 + O_p\left(\frac{1}{\sqrt{T}}\right) = \sigma_i^2 + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{A16}
$$

which we can use to obtain

$$
I_1 = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta\hat{e}_{it}\hat{e}_{it-1} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\varepsilon_{it}s_{it-1} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) = I_1^{\circ} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right)
$$

where the second equality uses the same trick as in the proof of Theorem 2, and where I_1° is the same as in that proof.

Similarly,

$$
I_2 = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\hat{e}_{it-1}^2 = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=p+2}^{T}s_{it-1}^2 + O_p\left(\frac{1}{\sqrt{T}}\right) = I_2^{\circ} + O_p\left(\frac{1}{\sqrt{T}}\right),
$$

where I_2° is the same as before. It follows that

$$
I = \frac{(I_1^{\circ})^2}{I_2^{\circ}} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) \to_d X^2
$$

as N, $T \to \infty$ with $\frac{N}{T} \to 0$. But the same steps can be applied to show that $II \to_d \frac{5}{12}(\kappa - 1)\, X^2$. The proof is completed by noting that $\hat{\kappa} = \kappa + o_p(1)$. ■

Proof of Corollary 2.

We omit this proof in the paper. The required result is obtained by adapting the proof of Theorem 2 in the same way as the proof of Theorem 1 was adapted to establish Corollary 1. ■

Proof of Theorem 3.

Consider first the case when H_0 holds. Write

$$FLM_2 = \frac{\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta\hat{e}_{it}\hat{e}_{it-1} + \frac{\sqrt{N}}{2}\right)^2}{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\hat{e}_{it-1}^2} + \frac{\left(\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\sum_{t=p+1}^{T}\left((\Delta\hat{e}_{it})^2 - 1\right)\hat{e}_{it-1}^2\right)^2}{(\hat{\kappa}-1)\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=p+1}^{T}\Delta\hat{e}_{it}^2\hat{e}_{it-1}^4}$$
$$= \frac{I_1^2}{I_2} + \frac{1}{\hat{\kappa}-1}\frac{II_1^2}{II_2} = I + \frac{1}{\hat{\kappa}-1}II.$$

Let $g_{it} = \sum_{k=p+1}^{t}(\varepsilon_{ik} - \bar{\varepsilon}_i)$. By using Lemma C.1 and the technique of Theorem 2,

$$I = \frac{\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\varepsilon_{it}g_{it-1} + \frac{\sqrt{N}}{2}\right)^2}{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=p+2}^{T}g_{it-1}^2} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) = \frac{(I_1^\circ)^2}{I_2^\circ} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right),$$

where $g_{it}^2 = (g_{it-1} + \Delta g_{it})^2 = g_{it-1}^2 + (\Delta g_{it})^2 + 2g_{it-1}\Delta g_{it}$ with $g_{iT} = g_{ip+1} = 0$, giving

$$\sum_{t=p+2}^{T} g_{it-1}\Delta g_{it} = \frac{1}{2}(g_{iT}^2 - g_{ip+1}^2) - \frac{1}{2}\sum_{t=p+2}^{T}(\Delta g_{it})^2 = -\frac{1}{2}\sum_{t=p+2}^{T}(\Delta g_{it})^2.$$

But $\Delta g_{it} = \varepsilon_{it} + O_p(1/\sqrt{T})$ and hence

$$I_1^\circ = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\varepsilon_{it}g_{it-1} + \frac{\sqrt{N}}{2} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta g_{it}g_{it-1} + \frac{\sqrt{N}}{2} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right)$$
$$= -\frac{1}{2}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}((\Delta g_{it})^2 - 1) + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right)$$
$$= -\frac{1}{2}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}(\varepsilon_{it}^2 - 1) + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) = O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right),$$

which uses the fact that $\frac{1}{T}\sum_{t=p+2}^{T}\varepsilon_{it}^2 \to_p 1$ as $T \to \infty$.

Moreover, note that $\frac{1}{\sqrt{T}}g_{it-1} \to_w W_i(r) - rW_i(1)$ with $\frac{t}{T} \to r$ as $T \to \infty$, and so

$$\frac{1}{T^2}\sum_{t=p+2}^{T}E(g_{it-1}^2) \to \int_0^1 E\big(W_i(r)^2 - 2rW_i(r)W_i(1) + r^2W_i(1)^2\big)dr = \frac{1}{6},$$

where we have used that $E(W_i(r)^2) = E(W_i(r)W_i(1)) = r$, and $E(W_i(1)^2) = 1$. Hence, by Corollary 1 of Phillips and Moon (1999), as $N, T \to \infty$

$$I_2^\circ = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=p+2}^{T}g_{it-1}^2 \to_p \frac{1}{6},$$

from which we deduce that

$$I = o_p(1) \quad \text{(A17)}$$

as $N, T \to \infty$ with $\frac{N}{T} \to 0$.

Next, consider II, which we write as

$$II = \frac{\left(\frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} \left(\varepsilon_{it}^2 - w_i^{-1}\right) g_{it-1}^2\right)^2}{\frac{1}{NT^3} \sum_{i=1}^{N} \sum_{t=p+1}^{T} \varepsilon_{it}^2 g_{it-1}^4} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) = \frac{(II_1^\circ)^2}{II_2^\circ} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right).$$

Since $\check{\sigma}_i^2$ is unbiased,

$$II_1^\circ = \frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} E(\varepsilon_{it}^2 - w_i^{-1}) E(g_{it-1}^2) = 0.$$

The variance of II_1° can be computed in the same way as in the proof of Theorem 1. We begin by rewriting II_1° in terms of mean deviations, which gives

$$\begin{aligned} II_1^\circ &= \frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \sum_{t=p+1}^{T} (\varepsilon_{it}^2 - 1) \left(g_{it-1}^2 - \frac{1}{T} \sum_{k=p+1}^{T} g_{ik-1}^2\right) \\ &\to_w \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sqrt{\kappa_i - 1} \int_0^1 \left((W_i(r) - rW_i(1))^2 - \int_0^1 (W_i(u) - uW_i(1))^2 du\right) dV_i(r) \end{aligned}$$

as $T \to \infty$, and therefore

$$\text{var}(II_1^\circ) \to \frac{1}{N} \sum_{i=1}^{N} (\kappa_i - 1) \int_0^1 E\left[\left((W_i(r) - rW_i(1))^2 - \int_0^1 (W_i(u) - uW_i(1))^2 du\right)^2\right] dr,$$

where

$$\begin{aligned} \int_0^1 \left((W_i(r) - rW_i(1))^2 - \int_0^1 (W_i(u) - uW_i(1))^2 du\right)^2 dr &= \int_0^1 (W_i(r) - rW_i(1))^4 dr \\ &- \left(\int_0^1 (W_i(r) - rW_i(1))^2 dr\right)^2. \end{aligned}$$

The expected value of the first term on the right-hand side is given by

$$\begin{aligned} \int_0^1 E\big((W_i(r) - rW_i(1))^4\big) dr &= \int_0^1 E\big(W_i(r)^4 - 4rW_i(1)^3 W_i(r) + 6r^2 W_i(1)^2 W_i(r)^2 \\ &- 4r^3 W_i(1) W_i(r)^3 + r^4 W_i(r)^4\big) dr = \frac{1}{10}, \end{aligned}$$

where we have used that $E(W_i(1)^3 W_i(r)) = 3r^2$, $E(W_i(1) W_i(r)^3) = 3r$, $E(W_i(1)^2 W_i(r)^2) = r + 2r^2$ and $E(W_i(1)^4) = 3$. The second term can be expanded as

$$\begin{aligned} \left(\int_0^1 (W_i(r) - rW_i(1))^2 dr\right)^2 &= \int_0^1 \int_0^1 W_i(r)^2 W_i(u)^2 dr du \\ &- 4W_i(1) \int_0^1 \int_0^1 r W_i(r) W_i(u)^2 dr du + \frac{2}{3} W_i(1)^2 \int_0^1 W_i(r)^2 dr \\ &+ 4W_i(1)^2 \int_0^1 \int_0^1 ru W_i(r) W_i(u) dr du - \frac{4}{3} W_i(1)^3 \int_0^1 r W_i(r) dr \\ &+ \frac{1}{9} W_i(1)^4, \end{aligned}$$

where we know from before that the first term on the right-hand side has expectation $\frac{7}{12}$. Moreover, since

$$\begin{aligned} E(W_i(1)W_i(r)W_i(u)^2) &= E(W_i(r)W_i(u)^3) + E((W_i(u) - W_i(r))W_i(r)W_i(u)^2) \\ &+ E((W_i(1) - W_i(u))W_i(r)W_i(u)^2) \\ &= E(W_i(u)^4) + E((W_i(r) - W_i(u))^2 W_i(u)^2) = ru + 2u^2 \end{aligned}$$

if $u < r$ and

$$E(W_i(1)W_i(r)W_i(u)^2) = E(W_i(r)^4) + 3E((W_i(u) - W_i(r))^2 W_i(r)^2) = 3ru$$

if $r < u$, we obtain

$$\int_0^1 rE(W_i(1)W_i(r)W_i(u)^2)dr = \int_0^1 r\left(\int_0^r (ru + 2u^2)du + 3\int_r^1 rudu\right) dr = \frac{13}{30},$$

and by a similar calculation, $\int_0^1 \int_0^1 ruE(W_i(1)^2 W_i(r)W_i(u))drdu = \frac{16}{45}$. But we also have

$$\begin{aligned} \int_0^1 E(W_i(1)^2 W_i(r)^2)dr &= \int_0^1 E\big((W_i(1) - W_i(r))^2 W_i(r)^2 + W_i(r)^4\big)dr \\ &= \int_0^1 (r + 2r^2)dr = \frac{7}{6}, \\ \int_0^1 rE(W_i(1)^3 W_i(r))dr &= \int_0^1 rE(W_i(1)^2 W_i(r)^2)dr = \int_0^1 r(r + 2r^2)dr = 1, \end{aligned}$$

from which we obtain

$$\int_0^1 E\left[\left((W_i(r) - rW_i(1))^2 - \int_0^1 (W_i(u) - uW_i(1))^2 du\right)^2\right] dr = \frac{1}{10} - \frac{1}{20} = \frac{1}{20}.$$

It follows that as $N, T \to \infty$

$$\text{var}(II_1^\circ) \to \frac{1}{20}(\kappa - 1),$$

which, together with Theorem 2 in Phillips and Moon (1999), yields

$$II_1 \to_d \frac{1}{20}(\kappa - 1)\, Y^2$$

as $N, T \to \infty$ with $\frac{N}{T} \to \infty$.

Also,

$$\begin{aligned} E\left(\frac{1}{T^3}\sum_{t=p+2}^{T} \varepsilon_{it}^2 g_{it-1}^4\right) &= \frac{1}{T^3}\sum_{t=p+2}^{T} E(\varepsilon_{it}^2)E(g_{it-1}^4) = \frac{1}{T^3}\sum_{t=p+2}^{T} E(g_{it-1}^4) \\ &\to \int_0^1 E\big((W_i(r) - rW_i(1))^4\big)dr = \frac{1}{10}. \end{aligned}$$

Thus, since the conditions of Corollary 1 in Phillips and Moon (1999) are satisfied, $II_2^\circ \to_p \frac{1}{10}$ as N, $T \to \infty$ with $\frac{N}{T} \to 0$, and so

$$II \ \to_d \ \frac{1}{2}(\kappa - 1)\, Y^2, \tag{A18}$$

which establishes the required result under H_0.

In order to isolate the effect of the trend under H_1 note that from Lemma C.1,

$$\begin{aligned}
\Delta\hat{w}_{it} &= \Delta y_{it} - \hat{\Phi}_i' \Delta\mathbf{y}_{it} - \hat{\lambda}_i \ = \ \Delta y_{it} - \Phi_i' \Delta\mathbf{y}_{it} - \hat{\lambda}_i + O_p\left(\frac{1}{\sqrt{T}}\right) \\
&= \Delta y_{it} - \Delta\overline{y}_i - \Phi_i'(\Delta\mathbf{y}_{it} - \Delta\overline{\mathbf{y}}_i) + O_p\left(\frac{1}{\sqrt{T}}\right) \ = \ \phi_i(L)(\Delta y_{it} - \Delta\overline{y}_i) + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

Let $G_{it} = \sum_{k=p+1}^{t} s_{ik} - (t-p-1)\bar{s}_i$, where $\bar{s}_i = \frac{1}{T-p-1}\sum_{k=p+1}^{T-1} s_{ik}$. By using (1) and then (2),

$$\begin{aligned}
\phi_i(L)(\Delta y_{it} - \Delta\overline{y}_i) &= \phi_i(L)\left(\Delta z_{it} - \frac{1}{T-p-1}\sum_{k=p+2}^{T}\Delta z_{ik}\right) \\
&= (\rho_i - 1)\phi_i(L)\left(z_{it-1} - \frac{1}{T-p-1}\sum_{k=p+1}^{T-1} z_{ik}\right) + (\epsilon_{it} - \overline{\epsilon}_i), \\
&= \sigma_i((\rho_i - 1)(s_{it-1} - \bar{s}_i) + \Delta g_{it}) + O_p\left(\frac{1}{\sqrt{NT}}\right) \\
&= \sigma_i((\rho_i - 1)\Delta G_{it-1} + \Delta g_{it}) + O_p\left(\frac{1}{\sqrt{NT}}\right),
\end{aligned}$$

where the third equality uses that $\phi_i(L)z_{it} = \sum_{k=p+1}^{t}\epsilon_{ik} + O_p(1)$. It follows that

$$\Delta\hat{w}_{it} \ = \ \sigma_i((\rho_i - 1)\Delta G_{it-1} + \Delta g_{it}) + O_p\left(\frac{1}{\sqrt{T}}\right). \tag{A19}$$

Similarly,

$$\begin{aligned}
\hat{w}_{it-1} &= (\rho_i - 1)\phi_i(L)\left(\sum_{k=p+1}^{t-2} z_{ik} - \frac{t-p-1}{T-p-1}\sum_{k=p+1}^{T-1} z_{ik}\right) + \sum_{k=p+1}^{t-1}(\epsilon_{ik} - \overline{\epsilon}_i) + O_p(1) \\
&= \sigma_i((\rho_i - 1)G_{it-2} + g_{it-1}) + O_p(1). \qquad \text{(A20)}
\end{aligned}$$

These results, together with the consistency of $\hat{\sigma}_i^2$, imply

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}\Delta\hat{e}_{it}\hat{e}_{it-1} + \frac{\sqrt{N}}{2} \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}((\rho_i - 1)\Delta G_{it-1} + \Delta g_{it})((\rho_i - 1)G_{it-2} + g_{it-1}) + \frac{\sqrt{N}}{2} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=p+2}^{T}((\rho_i - 1)^2\Delta G_{it-1}G_{it-2} + (\rho_i - 1)(\Delta g_{it}G_{it-2} + \Delta G_{it-1}g_{it-1}) \\
&+ \ \Delta g_{it}g_{it-1}) + \frac{\sqrt{N}}{2} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) \ = \ R_1 + R_2 + I_1^\circ + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right),
\end{aligned}$$

where $I_1^\circ = o_p(1)$ as under H_0 and $R_1 = O_p(1/\sqrt{N})$.

Consider R_2. Note first that by Corollary 1 of Phillips and Moon (1999), as $N, T \to \infty$

$$\begin{aligned} R_2 &= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=p+2}^{T}(\rho_i - 1)(\Delta g_{it}G_{it-2} + \Delta G_{it-1}g_{it-1}) \\ &= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=p+2}^{T}c_i(\Delta g_{it}G_{it-2} + \Delta G_{it-1}g_{it-1}) \\ &\to_p \mu_c \lim_{T\to\infty}\frac{1}{T^2}\sum_{t=p+2}^{T}E(\Delta g_{it}G_{it-2} + \Delta G_{it-1}g_{it-1}). \end{aligned} \tag{A21}$$

Consider $E(\Delta g_{it}G_{it-2})$, which can be expanded as

$$\begin{aligned} E(\Delta g_{it}G_{it-2}) &= E\left[(\varepsilon_{it} - \bar{\varepsilon}_i)\left(\sum_{k=p+1}^{t-2}s_{ik} - (t-p-1)\bar{s}_i\right)\right] = E\left(\varepsilon_{it}\sum_{k=p+1}^{t-2}s_{ik}\right) \\ &- (t-p-1)E(\varepsilon_{it}\bar{s}_i) - E\left(\bar{\varepsilon}_i\sum_{k=p+1}^{t-2}s_{ik}\right) + (t-p-1)E(\bar{\varepsilon}_i\bar{s}_i), \end{aligned}$$

where the first term on the right-hand side is zero, while as for the second,

$$E(\varepsilon_{it}\bar{s}_i) = \frac{1}{T-p-1}E\left(\varepsilon_{it}\sum_{k=t}^{T-1}s_{ik}\right) = \frac{T-t}{T-p-1}.$$

Similarly,

$$\begin{aligned} E\left(\bar{\varepsilon}_i\sum_{k=p+1}^{t-2}s_{ik}\right) &= \frac{1}{T-p-1}E\left(\sum_{t=p+2}^{T-1}\varepsilon_{it}\sum_{k=p+2}^{t-2}s_{ik}\right) = \frac{(t-p-2)(t-p-3)}{2(T-p-1)}, \\ E(\bar{\varepsilon}_i\bar{s}_i) &= \frac{1}{(T-p-1)^2}E\left(\sum_{t=p+2}^{T-1}\varepsilon_{it}\sum_{k=p+2}^{T-1}s_{ik}\right) = \frac{T-p-2}{2(T-p-1)}, \end{aligned}$$

which yields

$$\begin{aligned} E(\Delta g_{it}G_{it-2}) &= -\frac{1}{2(T-p-1)}\Big((t-p-2)(t-p-3) + 2(T-t)(t-p-1) \\ &- (T-p-2)(t-p-1)\Big). \end{aligned} \tag{A22}$$

Next, consider $E(\Delta G_{it-1}g_{it-1})$. It holds that

$$\begin{aligned} E(\Delta G_{it-1}g_{it-1}) &= E\big((s_{it-1} - \bar{s}_i)(s_{it-1} - (t-p-1)\bar{\varepsilon}_i)\big) \\ &= E(s_{it-1}^2) - (t-p-1)E(s_{it-1}\bar{\varepsilon}_i) - E(s_{it-1}\bar{s}_i) + (t-p-1)E(\bar{\varepsilon}_i\bar{s}_i), \end{aligned}$$

where $E(s_{it-1}^2) = t-p-1$, implying

$$E(s_{it-1}\bar{\varepsilon}_i) = \frac{1}{T-p-1}E\left(s_{it-1}\sum_{k=p+1}^{T-1}\varepsilon_{ik}\right) = \frac{1}{T-p-1}E(s_{it-1}^2) = \frac{t-p-1}{T-p-1}.$$

Also,

$$
\begin{aligned}
E(s_{it-1}\bar{s}_i) &= \frac{1}{T-p-1}E\left[s_{it-1}\left(\sum_{k=p+1}^{t-1} s_{ik} + \sum_{k=t}^{T-1} s_{ik}\right)\right] \\
&= \frac{t-p-1}{2(T-p-1)}((t-p-2)+2(T-t)), \\
E(\bar{\varepsilon}_i\bar{s}_i) &= \frac{1}{(T-p-1)^2}E\left(\sum_{k=p+1}^{T-1} \varepsilon_{ik} \sum_{k=p+1}^{T-1} s_{ik}\right) = \frac{T-p-2}{2(T-p-1)},
\end{aligned}
$$

from which we deduce that

$$
E(\Delta G_{it} g_{it-1}) = \frac{(t-p-1)(T-t)}{2(T-p-1)}. \tag{A23}
$$

Equations (A21) to (A23) imply

$$
R_2 \to_p \mu_c \lim_{T\to\infty} \frac{1}{T^2} \sum_{t=p+2}^{T} \frac{t-p-2}{T-p-1} = O\left(\frac{1}{T}\right).
$$

Hence, $I_1 = o_p(1)$ as N, $T \to \infty$ with $\frac{N}{T} \to 0$, and we already know from before that $I_2 \to_p \frac{1}{6}$. Therefore, $I = o_p(1)$. But it also holds that $II \Rightarrow \frac{1}{2}(\kappa - 1)\, Y^2$, and so the proof is complete.■

Table 1: Size and size-adjusted power at the 5% level for model 1.

		$\phi = 0$			$\phi = 0.5$			$\phi = -0.5$		
T	N	FLM_1	t^*_δ	Z_{tbar}	FLM_1	t^*_δ	Z_{tbar}	FLM_1	t^*_δ	Z_{tbar}
					Case 1: $\rho_i = 1$ for all i					
50	10	8.1	14.7	7.5	7.5	21.9	8.0	8.3	7.3	7.3
	20	10.0	21.2	8.8	8.9	33.3	8.9	10.3	9.0	8.6
100	10	6.5	10.3	6.2	6.0	15.2	6.4	6.7	6.1	6.0
	20	6.9	13.3	6.0	6.8	21.7	6.4	7.4	6.3	5.9
200	10	5.9	5.3	4.8	6.0	8.3	5.1	6.1	3.5	4.5
	20	6.2	6.7	5.4	5.9	11.5	5.5	6.4	3.6	5.4
					Case 2: $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$ with $c_i = -10$ for all i					
50	10	38.2	9.2	33.7	28.1	8.4	30.3	42.7	6.9	35.2
	20	43.9	9.6	35.3	32.5	7.9	33.6	48.2	7.2	36.1
100	10	50.3	6.1	37.0	43.5	5.4	34.8	52.7	4.7	38.2
	20	57.7	5.6	39.4	50.1	5.5	36.3	59.7	4.3	40.5
200	10	56.3	4.0	41.8	52.6	4.1	38.6	57.9	3.2	41.8
	20	67.7	2.9	38.8	66.0	3.4	38.5	68.0	2.2	39.8
					Case 3: $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$ with $c_i \sim U(-20, 0)$					
50	10	30.8	10.2	34.1	23.5	8.8	29.3	33.5	8.2	35.7
	20	34.9	10.1	35.3	27.1	8.7	32.5	38.2	8.1	36.5
100	10	40.4	7.1	37.2	35.0	7.1	34.4	41.2	5.8	37.9
	20	46.9	6.5	40.1	41.7	6.1	36.5	48.9	5.4	41.5
200	10	46.2	5.2	42.1	43.4	5.1	39.2	47.1	4.3	42.0
	20	56.2	3.9	39.9	54.7	4.1	38.9	55.7	3.0	41.2
					Case 4: $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$ with $c_i \sim U(-40, 20)$					
50	10	36.6	0.8	2.9	49.3	0.5	2.4	21.7	0.9	3.0
	20	17.5	0.0	0.3	32.9	0.0	0.3	7.4	0.0	0.2
100	10	43.0	0.7	3.2	35.5	0.6	3.1	43.4	0.6	3.3
	20	36.5	0.1	0.5	24.5	0.1	0.6	57.4	0.1	0.5
200	10	61.9	0.3	4.2	39.8	0.3	4.0	64.8	0.3	4.4
	20	72.3	0.0	0.9	37.3	0.0	1.0	77.5	0.0	1.0

Continued overleaf

Table 1: Continued.

T	N	$\phi = 0$			$\phi = 0.5$			$\phi = -0.5$		
		FLM_1	t_δ^*	Z_{tbar}	FLM_1	t_δ^*	Z_{tbar}	FLM_1	t_δ^*	Z_{tbar}
Case 5: $\rho_i = 1 + c_i$ with $c_i = -0.05$ for all i										
50	10	26.3	7.2	23.5	19.8	7.2	22.3	28.9	5.5	24.6
	20	52.7	11.0	41.6	39.1	8.7	39.2	57.6	8.2	43.0
100	10	83.9	13.4	72.7	75.7	11.4	67.7	86.6	9.9	75.1
	20	99.5	26.1	96.7	98.0	21.6	93.5	99.6	19.6	97.5
200	10	99.9	50.2	100.0	99.8	43.3	99.9	99.9	41.2	100.0
	20	100.0	84.7	100.0	100.0	78.4	100.0	100.0	76.0	100.0
Case 6: $\rho_i = 1 + c_i$ with $c_i \sim U(-0.1, 0)$										
50	10	22.7	7.9	23.7	17.1	7.1	21.6	24.3	6.2	24.6
	20	41.2	11.6	41.5	31.5	9.5	37.5	44.8	9.2	43.0
100	10	62.8	14.8	68.4	56.6	12.4	62.2	64.4	11.7	70.6
	20	88.0	25.8	93.4	84.2	19.4	89.3	88.9	21.0	94.6
200	10	90.9	40.8	98.9	89.2	34.5	98.2	91.4	35.9	99.0
	20	98.7	64.6	100.0	98.6	55.8	100.0	98.7	57.7	100.0
Case 7: $\rho_i = 1 + c_i$ with $c_i \sim U(-0.15, 0.05)$										
50	10	19.7	2.0	6.2	19.4	1.5	5.3	25.8	1.7	6.8
	20	17.2	0.4	2.2	24.5	0.3	2.1	23.3	0.4	2.1
100	10	48.0	3.2	11.6	41.5	2.2	11.0	59.4	2.8	11.9
	20	43.4	0.5	4.8	48.3	0.4	6.0	58.2	0.5	3.7
200	10	77.2	5.3	11.8	62.0	4.7	15.4	76.8	5.2	11.5
	20	71.6	0.7	5.9	63.2	0.6	9.6	78.0	0.7	2.7

Notes: The parameter ϕ refers to the autoregressive coefficient, while t_δ^* and Z_{tbar} refer to the tests of Levin *et al.* (2002) and Im *et al.* (2003), respectively.

Table 2: Size and size-adjusted power at the 5% level for model 2.

		$\phi = 0$			$\phi = 0.5$			$\phi = -0.5$		
T	N	FLM_1	t^*_δ	Z_{tbar}	FLM_1	t^*_δ	Z_{tbar}	FLM_1	t^*_δ	Z_{tbar}
Case 1: $\rho_i = 1$ for all i										
50	10	6.9	20.0	11.2	7.8	31.4	11.7	5.5	5.4	10.6
	20	8.2	31.7	13.9	9.8	50.4	15.0	6.5	6.2	13.4
100	10	6.0	14.9	8.2	6.5	23.4	8.5	5.2	5.3	7.8
	20	7.5	20.4	9.6	8.1	35.7	10.2	7.3	5.6	8.9
200	10	6.0	2.7	6.4	6.0	5.7	6.6	5.6	0.9	6.3
	20	6.5	2.6	6.8	6.5	6.2	7.4	6.3	0.4	6.6
Case 2: $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$ with $c_i = -10$ for all i										
50	10	6.2	9.3	11.8	6.2	7.8	11.1	6.5	8.1	12.7
	20	6.9	8.2	10.9	6.3	6.9	9.7	7.0	7.5	11.5
100	10	6.4	8.1	13.8	6.1	7.4	12.3	6.7	7.4	14.6
	20	6.4	6.0	11.1	6.0	5.7	9.8	6.3	6.0	11.4
200	10	5.6	5.9	13.7	5.7	5.8	12.8	5.5	5.6	14.0
	20	5.9	5.3	11.1	6.2	5.3	10.3	6.0	5.1	11.6
Case 3: $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$ with $c_i \sim U(-20, 0)$										
50	10	6.3	10.1	13.2	5.8	8.3	11.7	6.2	8.6	14.2
	20	7.4	8.7	12.2	6.7	6.7	10.3	7.3	8.1	13.3
100	10	6.4	9.6	14.9	5.8	8.3	13.3	6.4	8.5	15.7
	20	5.9	7.0	12.5	5.8	6.5	10.8	5.9	7.1	12.9
200	10	5.3	6.8	15.2	5.4	6.3	14.1	5.4	6.4	15.6
	20	5.8	6.0	12.4	5.9	5.8	11.4	5.6	5.7	12.9
Case 4: $\rho_i = 1 + \frac{c_i}{\sqrt{NT}}$ with $c_i \sim U(-40, 20)$										
50	10	16.4	1.2	1.7	22.3	0.7	1.3	8.8	1.0	2.1
	20	2.2	0.0	0.1	5.1	0.0	0.0	1.2	0.0	0.1
100	10	6.1	1.2	2.4	5.0	0.9	1.9	5.5	1.2	2.5
	20	4.0	0.1	0.3	3.6	0.1	0.3	7.5	0.1	0.3
200	10	6.6	1.0	2.9	4.8	0.8	2.6	8.6	0.9	3.0
	20	7.6	0.2	0.7	3.4	0.2	0.6	14.0	0.2	0.8

Continued overleaf

Table 2: Continued.

		$\phi = 0$			$\phi = 0.5$			$\phi = -0.5$		
T	N	FLM_1	t_δ^*	Z_{tbar}	FLM_1	t_δ^*	Z_{tbar}	FLM_1	t_δ^*	Z_{tbar}
Case 5: $\rho_i = 1 + c_i$ with $c_i = -0.05$ for all i										
50	10	5.7	7.6	9.1	6.2	6.7	8.8	5.8	6.7	9.2
	20	7.1	9.1	12.5	6.8	7.3	10.8	7.4	8.1	13.9
100	10	8.4	15.3	29.5	7.3	12.2	24.5	8.2	13.3	31.5
	20	9.6	20.6	47.1	8.6	15.9	39.1	9.5	19.5	49.7
200	10	11.6	45.9	93.1	9.9	36.5	88.1	10.7	41.6	94.3
	20	17.1	74.5	99.7	16.3	64.2	99.2	16.8	68.5	99.8
Case 6: $\rho_i = 1 + c_i$ with $c_i \sim U(-0.1, 0)$										
50	10	5.9	7.8	9.8	5.7	7.1	9.1	5.8	7.0	10.2
	20	7.7	9.8	14.5	6.9	7.3	11.7	7.5	9.1	15.5
100	10	7.6	17.7	33.1	6.8	14.6	27.0	7.5	15.9	35.7
	20	8.7	23.8	51.5	8.1	17.9	41.6	8.4	22.8	54.8
200	10	8.8	44.6	87.4	8.3	35.4	82.4	8.8	42.5	89.2
	20	12.0	67.6	98.5	12.2	57.4	97.2	11.9	63.9	98.9
Case 7: $\rho_i = 1 + c_i$ with $c_i \sim U(-0.15, 0.05)$										
50	10	4.8	3.2	3.7	5.0	2.5	3.2	6.1	2.8	4.1
	20	4.0	0.9	1.5	5.2	0.5	1.0	4.8	0.9	1.8
100	10	7.0	5.1	8.7	6.0	3.7	7.4	8.3	4.6	9.7
	20	5.8	1.2	3.5	5.1	0.8	2.7	7.2	1.2	3.6
200	10	10.3	7.0	12.1	8.5	5.9	11.6	9.0	7.0	12.2
	20	10.0	1.2	3.3	9.0	1.0	5.0	9.3	1.2	2.4

Notes: See Table 1.

Table 3: Empirical results from the feasible Lagrange multiplier test.

Factor treatment	Model	Prices		Income		Price-to-income	
		Test	p-value	Test	p-value	Test	p-value
		Levels					
No factors allowed	1	42.58	0.00	154.85	0.00	170.92	0.00
	2	13.66	0.00	36.62	0.00	17.36	0.00
Principal components	1	10.05	0.01	3.48	0.18	4.39	0.11
	2	3.20	0.07	0.27	0.61	5.20	0.02
		First-differences					
No factors allowed	1	91.89	0.00	248.79	0.00	100.61	0.00
	2	20.40	0.00	71.33	0.00	25.31	0.00
Principal components	1	36.62	0.00	29.98	0.00	21.37	0.00
	2	7.50	0.01	8.78	0.00	1.63	0.20

Notes: The principal components method was implemented with the number of factors estimated using the IC_2 criterion of Bai and Ng (2002). The order of the lag augmentation in the tests was estimated by using the Schwarz Bayesian information criterion.