UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

WORKING PAPERS IN ECONOMICS

No 396

On the Law of Demand

A mathematically simple descriptive approach for general probability density functions

Lars-Göran Larsson

November 2009

ISSN 1403-2473 (print)
ISSN 1403-2465 (online)

Department of Economics
School of Business, Economics and Law at University of Gothenburg
Vasagatan 1, PO Box 640, SE 405 30 Göteborg, Sweden
+46 31 786 0000, +46 31 786 1326 (fax)
www.handels.gu.se   info@handels.gu.se

On the Law of Demand.
A mathematically simple descriptive approach for general probability density functions.

by

Lars-Göran Larsson

Department of Economics and Statistics,
School of Business, Economics and Law
University of Gothenburg
P.O Box 640
SE 450 30 Goteborg
Sweden

E-mail lars-goeran.larsson@economics.gu.se

November 2009

Abstract.

In this paper we assume that choice of commodities at the individual (household) level is made inside the budget set and that the choice can be described by a probability density function. We prove that law of demand $\frac{\partial E(x)}{\partial p_x} < 0$ *is valid for one(x) or two choice variables (x, y)*.*
The law of demand at the market level is valid by summation.
We use general probabilistic density functions p(x), p(x, y) defined over the bounded budget set to calculate E(x) and prove law of demand. The expected demand functions are homogeneous of degree zero in prices and income (p_x, p_y, m) *.The commodities x and y are normal goods**.*

The present approach is less complex in a mathematical sense compared to other approaches and is descriptive in its nature.

Why not keep descriptions as simple as possible?

Entia non sunt multiplicanda praetor necessitatem
Beings ought not to be multiplied except out of necessity

“Occam´s razor”

Encyclopedia Brittannica

Keywords: Law of Demand and other properties of consumer demand, Microeconomics, Consumer theory , Consumer behaviour, Choice described in random terms, Expected individual and market demand.

JEL classification: C60, D01, D11

***** The proof can be used in higher dimensions
**In this text the words referring to traditional theory like normal goods or own price negativity etc should be seen as average (expected) properties.

Introduction

We here present the neoclassical theory and some later modifications in attempts to prove law of demand.
In section 1 we give our proof and in the appendix a few examples

Traditional theory and law of demand

The traditional neoclassical theory assumes a utility description of consumer preferences and that

the consumer makes his choice of commodities (x,y)

by maximizing his utility function u(x,y) subject to a budget constraint.

$u(x,u)$ is maximized over the budget set D= $\left((x,y)\text{: } x \geq 0, y \geq 0,\ p_x x + p_y y \leq m\right)$

where p_x, p_y are prices of (x,y) and m is income(budget)

The result of this maximization gives the **individual** demand functions

$x(p_x, p_y, m), y(p_x, p_y, m)$

Assuming the utility function u(x, y) is an increasing quasi concave function it is proven that the demand function has the following property

$$\frac{\partial x}{\partial p_x} + x(p_x, p_y, m)\frac{\partial x}{\partial m} \leq 0$$

For more details see Jehle and Reny (2001 pp 82-83)

It is not possible to exclude the Giffen case where

$\frac{\partial x}{\partial p_x} > 0$ since $\frac{\partial x}{\partial m} < 0$ is possible in the theory (inferior good)

The law of demand in this context is formulated

"If the demand for a goods increases when income increases, then the demand for that goods must decrease when its price increases" Varian (2006 p 147).

$\frac{\partial x}{\partial p_x} < 0$ since $\frac{\partial x}{\partial m} > 0$ (normal good)

The normal goods assumption is sufficient to give law of demand at the individual and **market level (sum of individual demands) and is used in economic literature.**

Some other approaches to obtain law of demand.

In the literature Quah (2000), Hildenbrand (1983) and Härdle; Hildenbrand , and Jerison (1991) are different examples how to prove law of demand.

Stronger assumptions on utility functions to obtain law of demand at the individual level. Since law of demand is valid at the individual level it is **valid at the market level by summation.**

Quah (2000) identifies sufficient conditions on an agents indirect utility function $v(p_x, p_y, m)$ which guarantees law of demand at the individual level. They are **convexity** in prices of the indirect utility function. and a numerical condition see Quah (2000 p 916).Earlier studies used assumptions to guarantee **strict inequality** as

concave utility functions and a numerical condition see Quah (2000 p 912).Utility maximization is a maintained hypothesis.

Income (expenditure) distributional assumptions to obtain law of demand at the market level

Hildenbrand (1983) takes a different route. Referring to Hicks " A study of individual demand is only a means to the study of market demand" Hildenbrand (1983 p 997) proves that the "law of demand" holds for **the market(mean) demand function**, i..e.,

$\frac{\partial F_h(p)}{\partial p_h} < 0$ where $\mathrm{F_h}(p) = \int_0^1 f_h(p,w)dw,\ 1 \le \mathrm{h} \le l$ where h is commodity h.

Hence, the partial market demand curve for every commodity is strictly decreasing. Hildenbrand (1983 p 998). The paper extends the result for more general distribution functions $\rho(w)$ than the uniform used in the text above. Hildenbrand also points out: "This remarkably simple result shows clearly that aggregating individual demand over a large group of individuals can lead to properties of the market demand function F which, in general, individual demand functions f do not posses. There is a *qualitative difference* in market and individual demand functions. This observation shows that the concept of a "representative consumer", which is often used in the literature, does not simplify the analysis ; on the contrary, it might be misleading."

Härdle; Hildenbrand , and Jerison (1991) takes a more general approach in proving Law of demand at the market level:

"In conclusion, assuming that the mean Slutsky matrix S(p) is negative semidefinite, a sufficient condition for monotonicity of F is that the mean income effect matrix M(p) is positive definite. **This property does not follow from an assumption on "rational" individual behaviour."**

Summarising the referred literature most of it tries to obtain law of demand by maintaining the utility maximization hypotheses. Only Härdle; Hildenbrand , and Jerison (1991) leaves it by concentrating on income distribution and the aggregate level.

All results however are obtained by using considerable mathematical complexity in the analysis.

The present approach is less complex in a mathematical sense and is descriptive in its nature.

We assume choice is made inside the budget set and that the choice can be described by a probability density function.

1 Proof of law of demand for positive density functions p(x) and p(x,y)

One dimensional frequency function of choice p(x).

To find the frequency function p(x) we start by assuming a positive continuous function f(x)>0 defined on the interval I = (0, c) Let a<c.

Define

$F(a)=\int_0^a f(x)dx$ and G(a)=$\int_0^a xf(x)dx$. We then have

$E(x)=\frac{G(a)}{F(a)}<a$. 0<G(a)<aF(a), a=$\frac{\mathrm{m}}{\mathrm{p_x}}$, $f(x)>0$

F (a) is the area below the positive function f(x) and the frequency function p(x, a)=f(x)/F(a). Note that the **parameter** a is part of the frequency function and that a is a **variable** in the expected value function.

Properties of E(x)

Next we want to find some properties of E(x). **Homogeneity of degree zero in price and income is obvious since a is homogeneous of degree zero in price and income.**

To find sensitivity to price and income we can use the chain rule of differentiation $\frac{\partial E(x)}{\partial m}=\frac{dE(x)}{da}\frac{\partial a}{\partial m}$ and $\frac{\partial E(x)}{\partial p_x}=\frac{dE(x)}{da}\frac{\partial a}{\partial p_x}$

We take

$$\frac{dE(x)}{da}=\frac{1}{F(a)^2}\left(G'(a)F(a)-F'(a)G(a)\right)=\frac{1}{F(a)^2}(af(a)F(a)-f(a)G(a))$$

$$=\frac{f(a)}{F(a)^2}\left(a\int_0^a f(x)dx-\int_0^a xf(x)dx\right)>0$$

(1.1) $\quad \frac{dE(x)}{da}>0$

We have found that the expected demand function **follows the law of demand and is a normal good for all continuous choice frequency functions.**

$\frac{\partial E(x)}{\partial p_x}<0$ and $\frac{\partial E(x)}{\partial m}>0$

The same result is obtained if we change variables. Let x=au. We then find

$E(x)=\frac{a^2\int_0^1 uf(a,u)du}{a\int_0^1 f(a,u)du}=\frac{a\int_0^1 uf(a,u)du}{\int_0^1 f(a,u)du}=\frac{G(a)}{F(a)}$ where a is the Jacobian of the transformation.

For the change of variable formula see Buck(1956) p244.

We differentiate

(1.2($\quad \frac{dE(x)}{da}=\frac{1}{F(a)^2}\left(G'(a)F(a)-F'(a)G(a)\right)>0$

following the result (1.1)) above.

The latter result (1.2) will be useful when we turn to two dimensional choices.

Choice in two dimensions

To find the frequency function p(x, y) we start by assuming a positive continuous function f (x, y)>0 defined in a set E, where $D \subset E$.
For the function f(x, y) we first integrate over the budget set
D= $\left\{(x, y) \text{ in } \mathrm{R}^2 : p_x x + p_y x_2 \le m \text{ and } x \ge 0, y \ge 0\right\}$
to find the constant K. $(\iint_D Kf(x, y)dxdy = 1)$.
The function f(x, y) is now turned into a frequency function p(x, y) = f(x, y)/K

To make the calculations somewhat simpler we do some change of variables.
We put x=u $\frac{\mathrm{m}}{\mathrm{p_x}}$ =au and y=v $\frac{\mathrm{m}}{\mathrm{p_y}}$ =bv to integrate over the set
$D' = \left((u,v) \in R^2 : u \ge 0, v \ge 0, u+v \le 1\right)$
We then have $\iint_D f(x, y)dxdy = ab \iint_{D'} f(x(u), y(v))dudv$
where the Jacobian (= ab)of the transformation is taken outside the integral on the right side.
For change of variable formulas see Buck (1956 p 244).

Homogeneity of degree zero a property of frequency functions (p(x, y))

The change of variables where
we put x=u $\frac{\mathrm{m}}{\mathrm{p_x}}$ =au and y=v $\frac{\mathrm{m}}{\mathrm{p_y}}$ =bv
shows how the transformation connecting (x, y) space and (u, v) space is homogeneous in degree zero in prices and budget. . We then have

$$\iint f(x, y)dxdy = \frac{m^2}{p_x p_y} \iint_{D'} f(x(u), y(v))dudv = ab \iint_{D'} f(x(u), y(v))dudv$$

homogeneous of degree zero and $\iint_D xf(x)dx$ as well.

Calculating E(x)

To find E(x) we first integrate f(x, y) over the area D'

$$ab \iint_{D'} f(x(u), y(v))dudv = ab \iint_{D'} f(u, v, a, b)dudv =$$

$$ab \int_0^1 du \int_0^{1-u} f(u, v.a, b)dv.$$

We now put the last integral $\int_0^{1-u} f(u, v, a, b)dv = F(u, a, b)$ and the volume below f(x,y) is

$ab\int_0^1 F(u,a,b)du$ .The next integral to be calculated is

$$aba\iint_{D'} uf(u,v,a,b)dudv =$$

$$aba\int_0^1 udu \int_0^{1-u} f(u,v.a,b)dv = aba\int_0^1 uF(u,a,b)du$$

We now find $E(x) = \dfrac{aba\int_0^1 uF(u,a,b)du}{ab\int_0^1 F(u,a,b)du} = \dfrac{a\int_0^1 uF(u,a,b)du}{\int_0^1 F(u,a,b)du} = \dfrac{G(a,b)}{F(a,b)}$

Finding the properties of E(x)

Next we want to find other properties of E(x). The chain rule of differentiation help us to find sensitivity in relation to income (m) and prices $(p_x, p_{y)})$.

As an example $\dfrac{\partial E(x)}{\partial p_x} = \dfrac{\partial E(x)}{\partial a}\dfrac{\partial a}{\partial p_x}$

We first differentiate wrt a. and use the result (1.2) in one dimension

$$\frac{\partial E(x)}{\partial a} > 0$$

This in turn gives us

$\dfrac{\partial E(x)}{\partial m} > 0$ and $\dfrac{\partial E(x)}{\partial p_x} < 0$

Law of demand is valid for all continuous frequency functions p (x. y).
The property is additive so the law of demand is valid at the aggregate (market) level as well. As noted x (and y) are normal goods.

References

Buck, R. C. (1956). *Advanced calculus*, McGraw-Hill, New York.
Hildenbrand,W. (1983) “ On the Law of Demand,” *Econometrica*, 51, 997-1019.
Härdle, W, Hildenbrand,W and Jerison, M. (1991) “Empirical Evidence on the Law of Demand,” *Econometrica*, 59, 1525-1549.
Jehle, G. A and Reny, P, J (2001) *Advanced microeconomic theory*, Addison Wesley, Boston.
Krugman, P (2008) http://www.princeton.edu/ ~pkrugman/howiwork.html
Larsson, L-G (2008) Non-Utility Maximizing Behavior, Scandinavian Working Papers in Economics S-WoPEc, nr 293 2008. http://swopec.hhs.se/
Quah, John K.-H.(2000)The monotonicity of individual and market demand, *Econometrica*, Vol 68,No4,911-930.
Shone, R (1975) *Microeconomics: a modern treatment*, MacMillan, London
Varian, H. (2006). *Intermediate microeconomics,* Norton, New York.

Appendix – a few examples

In an earlier working paper Larsson (2008) several examples are given. We therefore only give a small sample.

Even if law of demand is valid for all density functions we also illustrate the stochastic approach by calculating some examples to make it more practical.

Traditional theory in expected form

To facilitate a comparison with traditional theory we note that it can be expressed in expected values as well since estimation and/or testing the theory mostly handle the same properties to E(x) as to x. In testing and/or estimation we have

E(x) = $x(p_x, p_y, m) + E(\nu) = x(p_x, p_y, m)$

where ν is a random error term with E(ν)=0 . This assumption gives the result that the expected demand function has the same slope as the ordinary demand function $Ex_{p_x} = x_{p_x}$, and so on for the other properties.

The independent properties of demand functions in traditional theory are

Budgetbalancedness E ($p_x x(p_x, p_y, m) + p_y y(p_x, p_y, m)) = m$

and negative semidefiniteness of the symmetric substitution matrix.

$$\begin{bmatrix} Ex_{p_x} & Ex_{p_y} \\ Ey_{p_x} & Ey_{p_y} \end{bmatrix} + \begin{bmatrix} Ex \\ Ey \end{bmatrix} \begin{bmatrix} Ex_m & Ey_m \end{bmatrix}.$$

For details see Jehle and Reny (1991 pp 82-83)

Expected choice is confined to stay **inside the budget set, not on the budget line as in traditional theory. This implies that budgetbalancedness is not a property in the present approach**

A first example makes a comparison: For f(x, y) = $x^2 y$, p(x,y)= $\dfrac{x^2 y}{\dfrac{a^2 bab}{60}}$

corresponds to a "Cobb-Douglas" utility function we have

$$E(x) = \frac{m}{2p_x}$$

$$E(y) = \frac{m}{3p_y}$$

The utility maximizing solution in expected form for the function f(x, y) = $x^2 y$

E(x) = x= $\dfrac{2m}{3p_x}, E(y) = y = \dfrac{m}{3p_y}$

We also have expected expenditure $E(p_x x + p_y y) = p_x \dfrac{m}{2p_x} + p_y \dfrac{m}{3p_y} = \dfrac{5m}{6} < m$

We are relatively close to the budget constraint. But **the budgetbalancedness property is not valid.**

The expected demand functions were obtained by integrating a **"Cobb-Douglas" function** over a bounded set (bounded by the budget constraint) .The parameters (m, p_x, p_y) in the integration now are variables in the expected demand functions.

A second example has no unique solution in utility theory (a constant utility) and the interior solutions are **independent** of price and income.
The following example shows

$$E(x)=\frac{1}{3p_x}(m-p_y y_{_0})+\frac{2}{3}x_0$$

$$E(y)=\frac{1}{3p_y}(m-p_x x_{_0})+\frac{2}{3}y_0$$

the expected demand functions obtained by integrating **a uniform distribution** over a bounded set. The set is bounded by the budget constraint and lower bounds (x_0,y_0) on consumption. Given the expected demand we can study properties such as own price and cross price derivatives, income derivatives, homogeneity and symmetry properties as in ordinary theory. In the example given we have $E(x)'_{p_x}<0, E(x)'_m>0, E(x)'_{p_y}<0, E(y)'_{p_x}<0, E(x)'_{p_y}\neq E(y)'_{p_x}$
In words: Own prise derivative negative (law of demand), cross price derivatives negative(x and y are gross complementary commodities), income derivative positive (normal good) . The own price elasticity is inelastic $El_{Expx}=\dfrac{\partial E(x)p_x}{\partial p_x E(x)}>-1$.

The expected substitution matrix.is not symmetric in general as seen next.

Numerical example.

Numerical values for m=100, $\mathrm{p_x}=10,\ \mathrm{p_y}=5,\ \mathrm{x_0}$=5 and $\mathrm{y_0}$=6 are

$\mathrm{E(x)}=5\frac{2}{3},\ \mathrm{E(y)}=7\frac{1}{3}$

Average expenditure 0,933m < m.
Note the relative closeness to m due to the tight lower border. **Identify lower bounds and the choice and/or preferences inside the set matter less.** The substitution matrix in the numerical example is **nonsymmetric and negative definite.**

$$\begin{bmatrix}\frac{-7}{30} & \frac{-1}{5}\\ \frac{-1}{3} & \frac{-2}{3}\end{bmatrix}+\begin{bmatrix}\frac{17}{3}\\ \frac{22}{3}\end{bmatrix}\begin{bmatrix}\frac{1}{30} & \frac{1}{15}\end{bmatrix}=\begin{bmatrix}\frac{-4}{90} & \frac{8}{45}\\ \frac{-8}{90} & \frac{-8}{45}\end{bmatrix}.$$

The own price elasticity $\mathrm{El_{Exp_x}}=\dfrac{\partial E(x)p_x}{\partial p_x E(x)}=\dfrac{-7}{17}$

The cross price elasticity $\mathrm{El_{Exp_y}}=\dfrac{\partial E(x)p_y}{\partial p_y E(x)}=\dfrac{-3}{17}$

The income elasticity $\mathrm{El_{Exm}}=\dfrac{\partial E(x)m}{\partial m E(x)}=\dfrac{10}{17}$

Note the homogeneity condition in elasticities hold (sum of elasticities =0).

For elasticity forms in traditional theory see Shone (1975 p 91).

Why use of the uniform choice frequency functions might be reasonable.
As a start we assume **limited knowledge of preferences**. The uniform distribution assumption might reflect our **ignorance** of what determines the choice inside the bounded budget set. If we know that choice is taken place in the set we know that expected choice changes when the "walls" (budget line and lower bounds) of the set changes. A uniform distribution can also approximate a more complex function if we use a lower bound as in the example.

The assumption also makes for simple calculations (used before in economics) and produces some results. To quote Krugman (2008): "express your ideas in **the simplest possible model**."