What is taught and what is learned

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What is taught and what is learned

Professional insights gained and shared by teachers of mathematics

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Abstract

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The aim of the thesis is to contribute to knowledge about relationships between teaching and learning in school. The framework used in this research, variation theory, states that, to improve student learning, attention must be paid to *what* is being learned, the capability that is to be improved and the features (critical features) that it is necessary for the learner to discern. The studies reported here focus on the significance of critical features and are based on two 'learning studies' in mathematics, one of the density of rational numbers and one of the addition and subtraction of negative numbers. In a learning study, teachers work together with design, analysis and revision of their teaching of a single lesson with the aim of enhancing students' learning by gaining insight about features that are assumed to be critical and enacting them in their teaching.

The question answered in this research is whether the insight gained in the learning studies about critical features can be shared by other teachers and used to enhance other students' learning. Two studies based on the previous studies were carried out together with a total of eight teachers and sixteen groups of students. Each teacher enacted two lessons with different conditions in terms of the critical features made use of. The lessons were video recorded and analysed with respect to which critical features were enacted in the lessons and what the students learned as indicated in pre and post tests.

It is suggested that the critical features were transferable in two regards, in terms of student learning and in terms of a means of communication that could be shared among teachers. It was indicated that the critical features that were enacted in the teaching constrained what it was possible to experience in the classroom and what students learned. What was taught seemed to be reflected in what the students learned. Furthermore, the analysis indicates that it was not sufficient to simply name the critical features to the students; it seems that they must be *discerned* in order for learning to take place. It was found that the teachers made use of the critical features that were identified by other teachers in a learning study as a means to plan and teach lessons. This suggests that teachers can make use of the notion of critical features in their own teaching to enhance student learning.

Contents

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Mölnlycke, May 2010

Angelika Kullberg

To Michael, Clara and Oscar

CHAPTER ONE

INTRODUCTION

An interest in the relationship between teaching and learning in classrooms has been the inspiration for this research. Kennedy (1999) argues that the relationship between teaching and learning is the most central issue in teachers' work, but also the most perplexing and least understood. When I worked as a primary school teacher in mathematics I never questioned whether my students would learn from my teaching, since I took for granted that they would. I had an idea that if I only used interesting tasks the students would be motivated to learn by themselves. There are shifting views among teachers about how teaching affects student learning. Research shows, for instance, that some teachers experience the relationship as though there is a transmission of knowledge; therefore, what is taught is learned. If students do not learn, it is because they are not motivated enough or not able to learn. Others teachers describe short-term results of teaching outcomes as unpredictable and mysterious (cf., Kennedy, 1999; Lortie, 1975). How can relationships between teaching and student learning be understood?

According to Nuthall (2004) there have been many attempts to explain teaching–learning relationships; however what has been missing is a theory that could explain the effects of teachers' actions on student learning in a useful way. In this thesis teaching and student learning is explored by means of a framework called variation theory. The framework makes it possible to analyse teaching and learning in commensurable terms, which implies that 'what the teacher intends the students to learn', 'what is made possible to learn in a lesson' and 'what the students learn' are connected and described in a similar way.

The research reported in this thesis has its origins in a research project, *The Pedagogy of Learning*[1], conducted by researchers at the University of Gothenburg

[1] *The Pedagogy of Learning* research project was funded by the Swedish National Council of Educational Research from the year 2003 to 2006. The principle investigator in the project was Mona Holmqvist, Kristianstad University College.

and mathematics teachers on different school levels. It was the first research project that I was involved in not only as a tutor in one of the project groups but also as a mathematics teacher. In this project teachers collaboratively explored their teaching by working in a systematic and iterative way by planning a single lesson together, teaching the lesson and making an analysis of the teaching and student learning. The aim of this work was to enhance student learning. The findings from the research were exciting since students' learning improved considerably and the teachers were able to explain what caused the dramatic change. The teachers could explain what in their teaching made a difference for student learning, and they used a learning theory, variation theory, to explain it.

I would argue that it is not a common thing for teachers in Sweden to explore and improve their teaching in a systematic way. For many years it has been suggested that many other things than teaching affect student learning (cf., Kilpatrick, Swafford, & Findell, 2001). Even researchers have long advocated that other things besides teaching and instruction make a difference for student learning, for instance 'participation in mathematical practices' or the 'development of mathematical discourse' (cf., Lobato, Clarke, & Ellis, 2005).

A Swedish study by the National Agency for Education claims that the teaching is one important factor in students not performing as well as in mathematics as they did ten years ago (Skolverket, 2009). The same is reported for compulsory school, where students' performance in math has declined in international comparisons such as TIMSS[2] and PISA[3] (Skolverket, 2007, 2008). It is argued that students are left to a large extent to work on their own with mathematical exercises in textbooks (cf., Johansson, 2006; Löwing, 2004; Skolverket, 2008). A video study of teaching in some selected countries that participated in TIMSS showed that mathematics was taught in varying ways (Stigler & Hiebert, 1999). Analysis of lessons held in Asian countries that perform well in TIMSS and PISA, such as China (Häggström, 2008; Ma, 1999) and Japan (Stigler & Hiebert, 1999), showed that the teachers enacted features of the content in a systematic way and took the variation in students' ability into account. For example, Hiebert and Handa (2004) analysed lessons taught in Hong Kong and found that there was a carefully chosen sequence of tasks and that basic definitions were presented early in a lesson to provide an anchor for

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[2] TIMSS, Trends In Mathematics and Science Study

[3] PISA, The Programme for International Student Assessment

the discussion while allowing exploration of boundary conditions (ibid, p. 10). Furthermore, repetition with variation was used in a deliberate way.

> Looking at the flow of tasks and discourse across a lesson, the notion of repetition with variation […] seems to be an appropriate description. But it is not just any kind of repetition and variation. The claim we will elaborate here is that the topic around which the repetition was executed and the kind of variations introduced were carefully selected to develop both procedures and concepts simultaneously (Hiebert & Handa, 2004, p. 9).

It is common practice that teachers in China and Japan study their own teaching in collaboration with other teachers (Fernandez & Yoshida, 2004; Lewis, 2002). This form of collaboration conducted by teachers, in Japan called lesson study, was put forward as one explanation for excellent student learning in TIMSS. The researchers believe that it develops the teachers' teaching skills and thereby provides better possibilities for their students to learn (Stigler & Hiebert, 1999).

The following sections describe the model used in the research project called learning study. What has been considered in teaching to make a difference for student learning is then discussed. At the end of the chapter questions regarding teaching and learning are elaborated further. Finally, the aim of this thesis and its research questions are presented.

Learning study: a way to enhance teaching and learning with means of a learning theory

Learning study was developed from the ideas about lesson study of Marton (Marton & Morris, 2002; Marton & Pang, 2003; Marton & Tsui, 2004) together with colleagues in Hong Kong[4] at the beginning of 2000. That project was followed by a Swedish research project[5] carried out 2003 to 2006 with learning studies in mathematics, English and Swedish (Gustavsson, 2008; Holmqvist, 2006; Holmqvist, Gustavsson, & Wernberg, 2008; Kullberg, 2004, 2006; Kullberg & Runesson, 2006; Runesson, 2007, 2008).

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[4] The project Catering for individual differences through learning study.

[5] The research project the *Pedagogy of Learning* was funded by the Swedish National Council of Educational Research from 2003 to 2006.

Lesson study inspired the development of learning study and they share many common features. Learning study and lesson study both have: first, a collaborative and iterative process of planning, analysing and revising lessons; second, an aim to improve students' and teachers' learning; and third, a specific learning goal. In a learning study it is common for a researcher to participate in the study, although this could also be the case in a lesson study. The most important differences between lesson study and learning study are that, in a learning study, a) a learning theory[6] is used to analyse and plan lessons – in this sense learning study can be seen as a hybrid between lesson study and design experiments (Brown, 1992; Cobb, Confrey, di Sessa, Lehrer, & Schauble, 2003; Schoenfeld, 2006); b) the observation is most often a video recording of the lesson; c) specific student learning is in the foreground; and d) pre and post tests/scans are used to explore students' learning and what may be critical for student learning.

The main goal of a learning study is to enhance student learning and, furthermore, explore what it takes to learn something particular but in a generalizable sense as well. The teachers explore their teaching to identify what can be *critical features* for their students' learning. The notion of *critical features* is used to describe features with regard to the *content* and *students' understanding* of what is taught, what it is necessary to be aware of to be able to experience what is to be learned in a certain way.

> In a learning study the aim is to help the students to learn something specific: We ask: What are the necessary conditions for learning something and how can these be met in the learning situation? If students do not learn what we expected, we do not seek the answers to their failure in inadequacy of the student; neither do we seek them in the teaching arrangements or the methods used. Instead we focus on the students' learning – what their difficulties are – and on how the content must be handled in the lesson in order to overcome the learning obstacles (Runesson, 2008, p. 169).

In a learning study, a group of teachers work collaboratively, often together with a researcher, for a whole semester to find out what matters for students to learn the topic and try to improve their teaching in a systematic way. It is a cyclic process of planning and evaluating, in most cases, one lesson (see figure 1). Conducting a learning study is however not only following certain steps in a

[6] Variation theory has commonly been used.

model but rather the quality of analysis and discussions among the teachers in the group.

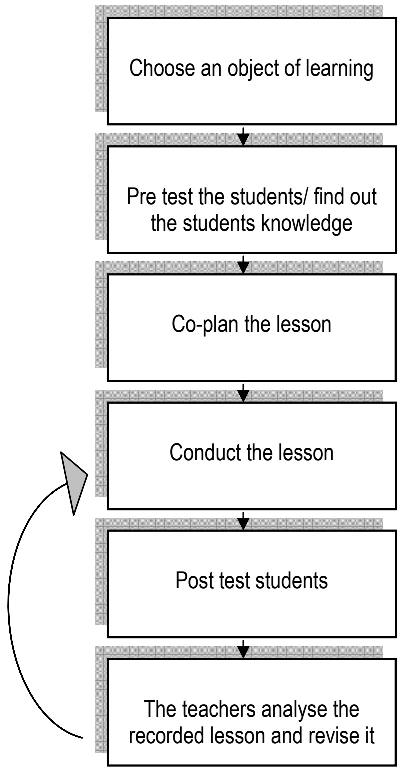

Figure 1. The learning study cycle.

What is 'critical' for student learning is explored through analysis of lessons and students' pre and post tests/scans. The analysis of tests/scans provides the teachers with information about how students experience what is to be learned. The variation in how students experience what is to be learned is a source of planning the lesson and hence contributes to the analysis. Research about the effect of learning study supports the conjecture that student learning increases through the learning study process (see for instance, Lo, Pong, & Chik, 2005; Marton & Pang, 2003; Marton & Pang, 2006; Runesson, 2008). For instance, the learning outcomes of lesson two and three in the learning study cycle show more improvement compared to the first lesson (Runesson & Kullberg, in press).

The teachers investigate in a systematic way what makes a difference for their students' learning by analysing teaching. The findings of a study are what the teachers had found to be critical for learning a specific capability. The teachers contribute in this way to a 'production' of knowledge of teaching and learning (Runesson & Kullberg, in press). In a learning study teachers and researchers are involved in production of knowledge. They start by identifying a certain capability they want the students to develop. They work systematically with a particular model and with a theory as a guiding principle, gathering and evaluating data about learning and teaching in their own practice (Runesson & Kullberg, in press). In the studies reported in this thesis, the transferability of such knowledge is put to the test to explore how other teachers can make use of the findings with their students.

What has been considered to make a difference for student learning?

In the following section an overview of research will set the picture for what has previously been considered to make a difference for student learning. In the research reported, different approaches are shown to explore the phenomenon. In the study of 'excellent teachers' and parts of 'process-product' research, the focus has primarily been on teachers' behaviour, in contrast to 'design experiments' and 'lesson study' where the focus has been on teaching.

One way used to study what made a difference for student learning was through interviews and observations of *excellent teachers* beginning in the 1930s. These teachers were selected by the results that their students showed or by having a good reputation among staff and principals. However, it was already found in the 1930s that ratings of teachers' effectiveness made by school principals and teacher educators do not correlate with student achievement (Nuthall, 2004, p. 283). What the teachers did in the classroom varied with the content taught and, when pre and post tests were used to investigate student learning, the results varied over time and context. "The ratings depend on when, and in what context, the observations were made" (ibid, p. 283).

The findings of a large international study (Hopkins & Stern, 1996) about characteristics of excellent teachers suggested that the teachers had i) a passionate commitment to doing the very best for their students, ii) a love of children enacted in warm caring relationships, iii) pedagogical content knowledge, e.g., knowing how to identify, present and explain key concepts, iv) the use of a variety of models of teaching and learning, v) a collaborative working style with other teachers to plan, observe and discuss each other's work and vi) a constant questioning of, reflecting on, and modifying their own practice (see Nuthall, 2004, p. 282). However, a finding in Haney (2002) was that teachers and schools that were rewarded for making great gains in student achievement one year did no better than average another year. Nuthall (2004) argues that:

> The underlying problem with research on the best teachers is that it is based on a confusion between good teachers and good teaching. It is assumed that anything the good teacher does must be good teaching [...] research about good teachers is primarily about ideal personality: what our culture expects of the behaviour and characteristics of the model teacher (Nuthall, 2004, p. 283).

The assumption taken for granted according to Nuthall (2004) about 'best teachers' and good teaching supports Haney's (2002) finding of a lack of a connection over time between 'best teachers' and students' performance. In this type of research teachers' personal qualities or behaviours was seen to promote student learning. Another somewhat similar approach is the research of 'teachers' beliefs' and views of learning, teaching or the content taught. In this research it is assumed that the teachers' ideas about the teaching or content influence the teaching. For instance, they conjecture that "The […] teachers' view of mathematics is important because it will influence the way they will teach mathematics" (Hannula, Kaasila, Laine, & Pehkonen, 2005, p. 89).

Another type of teaching and learning research is correlational and experimental studies done in the 1970s and onwards called *process-product research* (cf., Floden, 2001; Medley & Mitzel, 1963). The *process* was seen as what the teachers and students did in the classroom and the *product* as what the students learned. A scholar in this research, Gage (1963), attempted to connect variations in teaching and teachers to variation in desired student outcomes. However, Gage (1963) was interested in correlations that were not simply accidental but correlations about when teaching would have a somewhat predictable effect (Floden, 2001, p. 4). An analysis was made of systematic observations or audio and video recordings of teacher or student behaviour, and student learning was directly related to experiences during lessons. Floden writes that "through the mid-1970s, most of research on teaching was based on the model of looking for associations between measure of student learning (most often achievement tests) and variables describing classrooms, including both teacher actions and those of students" (Floden, 2001, p. 5).

Floden (2001) describes the TIMSS video study (Stigler & Hiebert, 1999) to be in the process-product tradition since the aim is to examine connections between teaching and learning.

> The study was designed, however, to explain differences in achievement, not just to document them. To that end, two forms of teacher process data were collected. A survey of teacher practice was developed and pilot tested in several countries. The survey was then given to the teachers whose students participated in the TIMSS student testing, yielding data needed to search for associations between process and product in a variety of countries. In three countries (United States, Japan, and Germany) TIMSS also gathered process data by making videotapes of a nationally representative sample of teachers. The rational for this costly research investment was that it would provide data on teaching process necessary for explaining differences in student achievement, by searching for associations between what teaching is like and what students learn (Floden, 2001, p. 8).

The TIMSS video study explored the influence of *teaching* on student learning. In other studies in this research approach it was commonly assumed that frequencies of a particular behaviour on the part of the teacher or students would promote student learning. However, no such established relationships could be found in a way that provided evidence for a relationship between behaviour and student learning (Nuthall, 2004).

Starting at the beginning of the 1990s, researchers designed lessons, often in collaboration with teachers, to improve student learning (Lagemann, 2000). The *design experiment* and *design-based research* could be described as an iterative process of design, enactment, analysis and redesign of teaching sequences or programs[7] (Brown, 1992; Cobb *et al.*, 2003; Collins, 1992). The aim of a design experiment is to find effective learning opportunities and develop practice-based theories that can be used by teachers and researchers. One of the first attempts was made by Collins (1992) who studied technological innovations in schools and carried out a series of design experiments with the aim of constructing a design theory for technology innovation. This theory would attempt to specify all variables that affect the success or failure of different designs. The educational innovation was in this case the use of technology, a computer program that allowed students in the 4th grade to explore different views of the earth-sun relationship. Collins was interested in how the intervention affected the students' learning and he used

[7] Design experiments share common features with 'teaching learning sequences' (Andersson & Bach, 1996; Olander, Hagman, & Wallin, 2001).

control groups to investigate differences in learning outcomes between the intervention group and other groups.

In mathematics education, Cobb (2000) and his colleges conducted *design experiments*, which varied from a few weeks to an entire school year. One goal was to develop instructional activities for students. The result of a design experiment is some kind of product, for instance a teaching sequence, a teacher manual or a software program. According to Nuthall (2004), a weak point in a design experiment is that "we do not know what aspects of the unit were directly related to student learning" (ibid, p. 289). Furthermore, the way in which a specific unit or program is later used by the teachers is not taken into account. How teachers make use of the designed lesson or program can be different, leading to what the Design-based Research Collective (2003) calls a "lethal mutation". In this research the designed teaching sequence is tested through enactment, analyses and redesign, which develop practice based evidence that it promotes student learning in the researchers' way of enacting the sequence. The theory-driven design improved through empirical study can contribute to knowledge about teaching and learning, in a context with instructional strategies and tools.

Lesson study, another approach, was highlighted in the book *The Teaching Gap* (Stigler & Hiebert, 1999) and was described as playing a major role in the development of teaching practice in Japan (Fernandez & Yoshida, 2009; Lewis, Perry, & Murata, 2006; Yoshida, 1999). Lesson study is a form of action research where teachers explore their teaching. The Japanese word for lesson study, 'jugyou kenkyuu', can be translated as 'research lesson', and this can be seen as research conducted by teachers themselves.

In a lesson study, a group of teachers work together in an iterative process of planning and revising a single lesson. The group has several meetings to plan, observe, analyse and revise the lesson. The group decides a specific part of a unit that the teachers want to focus on in the research lesson and sets a clear learning goal (Hiebert, Gallimore, & Stigler, 2002). A lesson plan is made to ensure that all the features of the content that were decided upon are taken into account in the lesson. Furthermore, expected student responses to questions and events are also anticipated in advance in order to prepare the teachers' response to them. One of the teachers is selected to teach the lesson in his/her class. The other teachers in the group observe the lesson and take detailed notes about students' learning and the teaching. The data collected from the lesson are discussed at a post lesson meeting. During a meeting the teachers share their experiences of the lesson with a focus on evidence of student thinking and analysis of the teacher's instruction. The purpose of the post session meeting is to improve the lesson from the observations made. Another teacher is selected to implement the revised lesson and, again, colleagues observe and take notes. After the second lesson the teachers meet and discuss their findings and the conclusions they have drawn.

Two lessons are usually conducted in a lesson study. After a lesson study, teachers often conduct public research lessons. The findings from a lesson study are local proof of how the topic is taught in a good way. Lesson study contributes to the individual teachers' learning and to the development of commonly shared knowledge among teachers. In Japan, lesson study is used as in-service training to enhance professional development and has been identified to have a major impact on student learning (Stigler & Hiebert, 1999). The quality of how a lesson study is carried out affects the outcome of the study and the conclusions that can be drawn. In this research conducted by teachers, the iterative process of planning, observations of teaching and students, analysis and revision of a lesson is seen to provide evidence for better student learning.

This section has discussed different types of research that have explored teaching–learning relationships. It has been shown that design experiment, lesson study and learning study investigate *teaching* in order to provide better opportunities for student learning. Findings from this research also show that student learning can be affected in a positive way by the systematic investigation of what is taught (see for instance, Lo *et al.*, 2005). This is however not a claim that a certain way of teaching gives rise to certain learning. What is claimed is that the teaching sometimes does not make it possible to learn something in a certain way and, when it is possible to learn something from teaching, it is more likely that students will indeed learn.

Learning study shares some common features with process-product research, since the teaching is connected and explained in relation to student learning, although not as a one-to-one relationship. The following section discusses teaching and learning from the point of view of acts of pedagogy.

The 'what' of teaching

The educational debate sometimes claims that a particular way of teaching and learning would promote student learning, for instance, if the teaching is 'student centered' or 'students discover what is to be learned by themselves' (cf., Marton & Morris, 2002). Marton *et al.* (2004) argue that, if the desire is to improve student learning, attention must be paid to *what* is being learned, the capability that is to be improved. Marton *et al.* (2002) argue that learning always implies the learning of something.

> We firmly believe that teaching and learning cannot be described without reference to what is being taught and learnt. In other words, teaching and learning is always teaching and learning of *something* (Marton *et al*, 2002, p. 3).

Marton (1994) claims that there has been an "erosion of the content" in school and that not enough attention has been paid to what students are supposed to learn. The way in which a particular content should be taught and thus develop students' skills seems to have been taken for granted. Marton *et al.* (2004) argue that "No conditions of learning ever cause learning. They only make it possible for learners to learn certain things" (p. 22). In the following section I show that, over the last 30 years, the focus has shifted away from teaching and moved towards students.

Teaching and learning in classrooms

Teachers create possibilities for student learning in their teaching by acts of pedagogy, for instance through instruction and student activities. Looking back, the view of instruction in teaching has shifted over the years. For example, in Europe in the 1960s and 1970s, experts and educational philosophers replaced 'formal phase teaching' with 'intuitive instruction' (Oser & Baeriswyl, 2001, p. 1032). Intuitive instruction focused to a greater extent on the students than the content. Now, in the 2000s, some teachers may even be reluctant to give instructions. If one way of teaching is not seen as better than another way of teaching, this seems rational. Hiebert *et al.* (1997) state that "Periodically, educational reformers have advocated presenting less information, shifting more responsibility to the students to search for or to invent the information they need" (p. 36). Oser and Baeriswyl (2001) claim that this shift in instruction has weakened the belief that learning can be initiated outside the learner.

> The earlier belief that learning can, in fact, be initiated outside the learner has gradually weakened as teachers focus more on creating a positive learning climate, on enhancing communication, and on stimulating cooperation. Teachers often even shy away from pushing students to learn and sometimes even inhibited in giving minimal directions just to keep the learning in motion (Oser & Baeriswyl, 2001 , p. 1032).

It follows from this that teachers' attention seems to have shifted from teaching to students and from instruction to guidance. A possible explanation may be teachers' and researchers' interpretation of theoretical perspectives on learning into acts of pedagogy (cf., Lobato *et al.*, 2005). Lobato *et al.* suggest that instruction in the form of initiating and eliciting by 'telling' has been downplayed due to the reaction against a transmission model of teaching. The transmission model was based on the assumption that, for instance, mathematics consists of a fixed set of facts and procedures and that teaching is centered on telling students how to carry out those procedures (ibid, p. 103). The shift towards a constructivist theory and its implications for teaching provoked strong criticism of the transmission model. Several researchers have noted the common misconception that constructivism implies a "discovery" view of pedagogy that advises against telling students anything (see for instance Cobb, 1994), even though a constructivist stance sees lectures as also being effective. According to some researchers, this has instead disempowered teachers and prevented them from playing a strong role in the classroom (Chazan & Ball, 1999).

In design experiments, lesson study and learning study, it is believed that teaching by means of instruction and designed activities can promote student learning. Instruction is seen in this thesis in the way that Cohen and Ball (2001) describe it: that it consists of interactions involving teachers, students and content (ibid, p. 75).

Knowledge about teaching and learning

A central question is whether knowledge about teaching and learning relationships can be produced. On the one hand, scholars argue that how students' learning is shaped by teachers' actions is unique and context dependent (cf., Barab & Kirshner, 2001). Classrooms are diverse, a specific class of students is always unique and the teacher works with a class with respect to its needs and demands. Consequently, they argue that no generalizable knowledge about student learning can be established. On the other hand, other scholars, such as

Hattie (2009) and Nuthall (2004), advocate that relationships between teaching and learning can be established.

Hattie (2009) argues that teaching and learning are visible and that "there is no deep secret called teaching and learning" (p. 25). He sees teachers as activators, as deliberate change agents and as directors of learning. Hattie suggests that the feedback in terms of students' questions and statements is the single most important indicator for teachers in terms of shaping and improving students' learning. Furthermore, he suggests that students' learning should be made visible to the teacher. In this way, teachers learn about their students' learning and act accordingly. Hattie argues for an approach to teaching where teachers are more aware of the impact of their teaching on students' learning. He suggests that "Maybe we should constrain our discussion from talking about qualities of teachers to *the quality of the effects of teachers on learning* – so the discussion about teaching is more critical than the discussion about teachers" (ibid, p. 126).

Nuthall (2004) suggests that, to establish relationships between teaching and learning, one needs to explore the experience of individual students and make in-depth analyses of changes that take place in students' knowledge, beliefs and skills, and ways of identifying the real-time interactive relationship between them. He argues for an explanatory theory that lets teachers know what to look for and how to interpret what they see when they are guiding student learning.

To explore this phenomenon, Nuthall set up a study where teachers had to connect the content of their teaching to what the students learned. He found that this way of relating teaching to learning was new to the teachers. Nuthall concludes that the practice of teaching, as we commonly understand and talk about it, is not actually about learning. Furthermore, he state that teachers commonly attribute failure in student learning to the students' lack of ability or motivation, rather than to their own teaching (ibid, p. 293).

Purpose and rationale of the study

The aim of this study is to see how a specific way of enacting *teaching* can make difference in student learning. This research should be seen as one example of a larger approach investigating this matter with a variation theory framework (Al-Murani, 2006, 2007; Al-Murani & Watson, 2009; Pang & Marton, 2005; Runesson, 2009; Runesson & Marton, 2009). This thesis is also further research on learning studies (Cheung, 2005; Chik, 2006; Gustavsson, 2008; Wernberg, 2009).

Research questions

A theoretical framework called variation theory (Marton & Booth, 1997) is used to explore how teaching makes a difference for learning. The thesis explores questions about and tests the usefulness of a model of description that concerns the relation between teaching and learning that emanates from the framework – above all *critical features,* but also patterns of variation and invariance. The concepts are central in variation theory and have been used previously in analyses of lessons (cf., Runesson, Kullberg, & Maunula, in press).

In this thesis, *critical features* that have been identified as findings gained in previous learning studies (Kullberg, 2008; Runesson & Kullberg, in press) are shared and tested with new teachers and students. The aim is to answer the following questions with regard to the two studies:

- What difference does the presence or absence of these critical features make for student learning?
- Can these critical features be used as a description and resource of knowledge between teachers to enhance student learning?
- In what way can the notion of critical features contribute to knowledge about the relationships between teaching and learning?

Structure of the thesis

The second chapter presents the theoretical frame for the study – variation theory. This section presents findings from research using phenomenography or variation theory that have implications for the study. The aim is to place the study in a context and to give the reader an opportunity to become familiar with the variation theory framework and concepts used, for instance *critical features* and patterns of variation.

The third chapter provides the background to the two studies and reports the findings of two learning studies. Here, *critical features* found in learning studies are presented. Furthermore, educational research about the mathematical topics is discussed to give a background to the learning studies.

The fourth chapter presents the procedure used in the research and elaborates on the choices made. Questions about ethics, validity and reliability are also discussed.

The fifth and sixth chapters report the findings and analyses of two studies, one about the teaching and learning of the density of rational numbers in grades 5 and 6 and one about the addition and subtraction of negative numbers in grade 7.

The seventh chapter discusses the findings and implications of the studies. A synthesis of the findings from the two studies is first made. The findings have implications for both teacher practice and for the development of variation theory.

The eighth chapter is a summary in Swedish of the thesis that gives the most important findings.

CHAPTER TWO

VARIATION THEORY

The research presented in this thesis explores teaching and learning using a variation theory framework. Variation theory has proven to be a powerful and useful tool for analysis in several studies of mathematics teaching and learning (see for instance, Häggström, 2008; Olteanu, 2007; Runesson, 2008; Wernberg, 2009). This chapter elaborates the theoretical concepts used in the thesis and discusses findings in previous research that developed or used the framework.

A theory of learning

More than thirty years of phenomenographic research has provided insight into learners' different ways of experiencing and understanding phenomena (cf., Dahlgren, 1975; Marton, 1981; Marton, Dahlgren, Svensson, & Säljö, 1977; Neuman, 1987; Svensson, 1976, 1977; Säljö, 1982). Using that research as a basis, Marton and Booth (1997) devised the theory of variation that was described in *Learning and Awareness*. The theory is based on ideas from Gurwitch (1964), who says that the nature of human awareness is directed towards something [in our focal awareness] while other features are not noticed [not in focal awareness]. First, there is a differentiation in things noticed; everything is not paid attention to at the same time even if it may be possible to notice everything. For example, if you are walking to the bus you might not be aware of the birds singing in the trees even if it is possible for you to hear them. They are not in your focal awareness at that moment, although they could have been. Marton & Booth (1997) say "we cannot be aware of everything at the same time in the same way" hence "we are aware of everything at the same time, albeit not in the same way" (p. 123).

Second, there is a variation as to how things and phenomena are perceived. For instance, two persons discern a car driving towards them in the fog and one of them discerns that the car is dark whereas the other person discerns more features of the car approaching them – it is a black Saab 9-3 Aero. The feature that a person is able to discern depends on previous experiences of discerning differences in the specific object or phenomenon. Experiencing a phenomenon is, according to Marton and Booth, to become capable of discerning certain entities or features[8] and to be simultaneously and focally aware of these (Marton & Booth, 1997, p. 123). In conclusion, what someone discerns depends on previous experiences and what features of the phenomenon have been noticed.

Dimensions of variation

The idea that every concept, situation or phenomenon has particular features and that, if a feature is changed or varied and another remains unchanged, the altered feature is likely to be noticed, is a fundamental point of departure in variation theory. What the students are able to discern is dependent on the variation that is created or present in a learning situation, the different *dimensions of variation (d.o.v.)*. For instance, if a person sees a cup on a table s/he could experience different d.o.v. of the cup. A d.o.v. is in this case different aspects of the cup that could vary. The color, the material and the shape are all examples of *d.o.v.*, which could vary since the cup could have been made of another material, color and/or shape (Marton *et al.*, 2004).

> […] we cannot discern anything without experiencing variation of that object. There would not be any gender if there were only one, no color if there were only one color etc. So we believe that what varies and what is invariant is fundamentally important (Marton & Morris, 2002, p. 20).

Watson and Mason (2006) refer to the values in a variation as the "range of change", for instance, a cup's color: e.g. blue, red, or yellow. Another d.o.v., material, creates another range of change: e.g. glass, ceramic or plastic. In a mathematical context these dimensions of what could vary are connected to the content specific critical features and how a variation for discerning them could be brought about, for instance by making one feature vary while keeping other features invariant.

However, people discern different features and dimensions in different phenomena. Sports could be used as an example to illustrate this. Zlatan Ibrahimovic, a talented Swedish soccer player, once said in an interview for a sports magazine that (Offside, 2007, no 7) "he could see things that other players

[8] In this thesis critical features are used with the same meaning as critical aspects.

couldn't see" on the soccer field. Certainly, he did not mean he had better eyesight than his team mates. Instead he explained that he had visions when he played that made him discern certain dimensions of the game – possibilities when he played that made him a good player. An experienced chess player can similarly discern the most powerful moves on the chessboard. All 'experts' certainly have discerned more features of the object of their expertise than novices (Bransford, Brown, & Cocking, 2000; de Groot, 1965).

The object of learning and its critical features

In variation theory, learning is seen as becoming able to discern critical features of an *object of learning*. The object of learning concerns a capability or understanding of *something*, for example a particular content taught in school. The object of learning is hence not the same as learning objectives; it is not the subject or the content taught and learned but rather the capability connected to that particular knowledge. Pong and Morris (2002) state that "students' achievement must spring from their learning of something, something they have become capable of doing, something that they as a result of their learning they must know or understand" (ibid, p. 16). The object of learning has thus two parts: the *direct* and the *indirect objects of learning*.

> These two aspects are analytically separated, although neither can exist without the other. The *content* (the direct object) can never be the aim or the outcome of learning in itself. It is the *capability* of using that content (the indirect object) that is the target or result (Marton & Pang, 2006, p. 197, my italics).

The indirect object of learning regards the nature of the capability, for instance 'understanding', 'explaining' or 'calculating', whereas the direct object of learning regards the content, for example the density of rational numbers. These two are not separated in a learning situation, since the object of learning in this example would be 'to understand the density of rational numbers'. In this thesis the direct and indirect objects of learning are not separated, since they are not in a learning situation.

The object of learning can be defined by its *critical features* (Marton *et al.*, 2004, p. 22). Marton *et al.* (2004) state that "the critical feature is critical in distinguishing one way of thinking from another, and is relative to the group participating in the study, or to the population represented by the sample" (p. 24). A critical feature can not be derived from content/subject matter only. A critical feature, as the notion is used in this thesis, is related to the content but is connected to the students' experiences of the object of learning.

> The critical features have, at least in part, to be found empirically – for instance through interviews with learners and through the analysis of what is happening in the classroom – and they also have to be found for every object of learning specifically, because the critical features are critical features of specific objects of learning (Marton *et al.*, 2004, p. 24).

However, it is possible that critical features found in one group would apply for similar groups. Critical features can be found for instance during interviews with learners, analyses of lessons together with analyses of what students have learned from the lessons. Even if students have the possibility to experience critical features of an object of learning in a lesson, some students may focus on other features than those intended by the teacher.

> What is of decisive importance for the students, is what actually comes to fore of their attention, i.e., what aspects of the situation they discern and focus on. In the best case they focus on the critical aspects of the object of learning, and by doing so they learn what the teacher intended. But they may also fail to discern and focus on some of the critical aspects, or they may discern and focus on other aspects (Marton *et al.*, 2004, p. 5).

Note that critical features are not the same as students' difficulties with the content taught. Instead it is what they must be able to discern to experience the object of learning in a certain way.

The object of learning in the research reported in this thesis is seen from three different points of view: the teacher's, the student's and the researcher's. These can all be expressed in commensurable terms (Runesson, 1999). First, the object of learning can be seen from the point of view of what the teacher intends the students to learn. The teacher has a particular goal and intention about what the students should learn, namely that the students are able to understand that there is an infinite number of decimals. That is the *intended object of learning*. What the students (the learners) actually learn is the *lived object of learning*. The lived object of learning can be analysed during the learning situation by the students' expressions, or after the lesson in student tests. Thus, the intended and the lived object of learning may not coincide. The object of learning seen and analysed from the researcher's point of view, implying 'what it is made possible to learn', is the *enacted object of learning*. The enacted object of learning describes what features of the content it is possible to experience during a lesson. The variation used to elicit the features, the values within a dimension of variation, is the *range of change* (Watson & Mason, 2006) and the way it is possible to discern the variation is through different *patterns of variation*.

Patterns of variation and the space of learning

It does however not favour learning to explicitly tell students the critical features, since these must be discerned. There is a significant difference between 'being told' something and 'experiencing a variation of different features' of an object of learning (Marton & Tsui, 2004). The teacher could focus students' attention on the critical features: however, if the student does not discern them, s/he will not learn what is intended. Thus, experiencing variation concerning critical features of learning is, in variation theory, essential for learning. A critical feature can be seen as a dimension of variation and varying values within a feature creates a space of learning. "It is this space of learning that either constrains learning or makes learning possible" (Marton *et al.*, 2004, p. 189). This implies that it is possible to analyse the space of learning and that it would be possible to predict, to some extent, what the students will learn. The design of the lesson becomes important since the lesson could be defined in terms of critical features and patterns of variation and invariance constituted in the classroom. The design of a lesson, as in this study, could be defined (and prepared) in terms of a pattern of variance and invariance of the critical features that it contains. Emanuelsson (2001) states that the participants that are involved in an interaction concerning a specific content (e.g. in a classroom) could open up or close for different features of that content, "Dimensions of variation are introduced and are taken away, leading to different patterns of variation being constituted" (Emanuelsson, 2001, p. 36, my translation).

The idea of whether *dimensions of variation within critical features for learning are opened up* is used in this thesis to describe what is made possible to discern in a situation. '*Opening up dimensions of variation*' is used to describe what someone (or a task) makes possible to discern in a learning situation. When features of the object of learning are brought up/talked about in the classroom, the teacher could, for instance by asking for more than one answer to a question, give the students an opportunity to discern a variation of possible answers that could lead to a deeper or more complex understanding of what is to be learned. For instance, if a teacher wants the students to discern that there is an infinite number of numbers between two whole numbers, for examples 1 and 2, opening up a critical feature would entail making it possible for the students to discern the density and continuity of number by facing several numbers in the interval (Kullberg, 2009). Furthermore, seeing decimal numbers as a form of rational numbers (where fractions and percentages are other forms) is also likely to contribute to understanding decimals. The fact that the interval can be divided into smaller and smaller parts (e.g. hundredths, thousandths) is another critical feature that could be brought up. Hence, it is also possible to close for variation within features. In the case described above, this would entail not asking for more numbers in the interval than just a few (e.g. 1.1, 1.2, 1.3 etc.). This could imply that the students learn that there are only ten numbers in the interval[9].

Four different patterns of variation have been found in analyses of lessons, *contrast*, *separation*, *fusion* and *generalisation* (Marton *et al.,* 2004). Runesson and Mok (2004) found in their study patterns of variation within features concerning mathematical concepts. They point out that in order to learn, for instance, what a square is, the students must discern what it is not, in order to distinguish what it is – *contrast.* Runesson and Mok also point to critical features that the student "must recognize" such as the size of angles and number of sides of a square. These features are handled one at a time, separate from other features. This is described by another pattern of variation, *separation.*

> Learning what a square is – and what it is not – for instance, takes the discernment of critical features of that geometrical shape. Consequently, if the teacher aims at making the learner understand what a square is, he or she must start with what it takes to know and recognize the critical features of a square, in order to make learning possible. In this case, the critical features are the size of angles, the number of sides, and the relations between them, and it must be possible for the learner to discern these features. However, simply pointing out these critical features to the learners is not enough […] According to the theoretical framework […], that which varies is likely to be discerned. For instance, it is necessary to know what a right angle is not, in order to learn what it actually is (Runesson & Mok, 2004, p. 64).

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[9] This line of reasoning is similar to Gibson's (1979) concept of affordances. Gibson called 'possibilities for action' an affordance, although what is an affordance is dependent on the individual's ability to recognize what is afforded. Greeno (1994) describes Gibson's affordance as it "refers to whatever it is about the environment that contributes to the kind of interaction that occurs" (p. 338). The concept affordance can be used both to describe possible affordances in general and affordances recognized by the individual. Analysis with variation theory describes what is possible to discern, what is afforded in a situation on a group level.

Fusion describes a state in which several features are simultaneously possible to discern that from the beginning have been separated. Finally, the most commonly used pattern of variation, *generalization*, describes the state in which the concept is invariant and instances of the concept vary (Marton, 2005). For instance, using the example of squares above, generalization would imply experiencing a variation of squares of different sizes in order to grasp the commonalities of squares.

Learning as differentiation

Variation theory is influenced by ideas of perceptual learning as differentiation (Gibson & Gibson, 1955). Gibson and Gibson raised in their work the question of whether perception is a creative or a discriminative process and distinguished between what they called the *enrichment* and the *specificity theory* of perceptual learning. They argued that learning is a matter of differentiation of information and specific features related to what is learned. Over time, people discern more and more features and qualities of the world, leading to the objects in the world becoming more and more distinctive.

> [...] perception changes over time by progressive elaboration of qualities, features, and dimensions of variation; that perceptual experience even at the outset consist of a world, not of sensation, and that the world gets more and more properties as the objects in it get more and more distinctive (Gibson & Gibson, 1955, p. 34).

Gibson and Gibson argued that learning is to a large extent perceptual. Gibson & Levin (1975) make the following distinction between enrichment and specificity [differentiation] and describe perceptual learning as:

> [...] it is not *adding on* anything. It is, rather, an increase of specificity of discrimination to stimulus input, an increase in differentiation of stimulus information. It is extraction or "pulling out" rather than adding on. The modification is in *what* is perceived (Gibson & Levin, 1975, p. 13).

Gibson and Gibson (1955) say "for a child to identify an object, he must be able to identify the differences between it and other objects, or at least that when he can identify an object he also can identify its properties" (p. 39). Marton (2006) also argues that, to be able to discern a phenomenon in a specific way, the learner must differentiate and make finer and finer discrimination.

> This differentiation amounts to becoming attuned to distinguishing features – or critical differences – that can be used for making distinctions. These distinctive features are simply respects, dimensions, in which things vary and they can be used for telling apart instances from non-instances. So perceptual learning amounts very much to discerning "distinctive features", *i.e.* critical dimensions of variation (Marton, 2006, p. 28).

Other scholars have had similar ideas about learning. For example, Dewey wrote in *Democracy and Education* (1916) about learning as differentiation. He stated that how a phenomenon differs [its difference] from other phenomena is of importance for learning.

> We do not really know a chair or have an idea of it by inventorying and enumerating its various isolated qualities, but only by bringing these qualities in connection with something else – the purpose which makes it a chair and not a table: or its difference from the kind of chair we are accustomed to, or the 'period' which it represents, and so on (Dewey, 1916, p. 168).

Another scholar in the field of mathematics education is Dienes (1960), who suggested a theory of mathematics learning that was built on the importance of variation and invariance for learning. In his work, he was influenced by gestalt psychology (Werner, 1957; Wertheimer, 1945) in a similar way as Gibson and Gibson. In this theory Dienes regarded mathematics as a structure of relationships, where the formal symbolism was merely a communicating part of the structure. He suggested that variation should be applicable to the variables [features that could vary] concerning concepts [invariant].

> A mathematical concept usually contains a certain number of variables and it is the constancy of the relationship between these, while the variables themselves vary, that constitutes the mathematical concept. To give the maximum amount of experience, structured so as to encourage the growth of the concept, it seems *a priori* desirable that *all* possible variables should be made to vary while keeping the concept intact (Dienes, 1960, p. 42).

The theory includes four principles: i) dynamic principle, ii) constructivity principle, iii) mathematical variability principle and iv) perceptual variability principle. Whereas the first two principles focus on the process of student learning and activity, the last two focus on teacher instruction. In the last two, there is an emphasis on variation. The 'mathematical variability principle' he refers to is variation of features within the mathematical concepts. This is close to the idea of variation of features concerning objects of learning. Dienes argues foremost for variation in perceptual representation, keeping the conceptual structure constant. "To allow as much scope as possible for individual variation in concept formation, as well as to introduce children to gather the mathematical essence of an abstraction, the same conceptual structure should be presented in the form of as many perceptual equivalents as possible" (Dienes, 1960, p. 44).

Phenomenography and variation theory - what has been learnt from research?

The following section uses some examples to show what has been learnt from research during the development of variation theory from phenomenographic research. It first treats research on how learners experience phenomenon, then research about possibilities for learners to learn in classrooms and finally how variation theory is used to enhance learning. The aim is to provide an overview of research carried out with this framework and further to elicit what it is the learner need to experience to be able to learn.

How learners experience phenomena

The roots of variation theory can be found in findings from phenomenographic research[10]. Phenomenography and variation theory share the same non-dualistic ontological position, namely that the world – as seen – is an experienced world. Although there is an existing world independent of the human mind, the world as we see it prevails only through our experience of it. Since people experience things in different ways, the experienced world varies between people. Phenomenographic research explored how different learners experienced the same phenomenon, for instance force or speed. The phenomenographic approach was developed in the INOM group[11] at the Department of Education at the University of Gothenburg in the early 1970s (Dahlgren, 1975; Marton, 1981; Marton *et al.*, 1977; Svensson, 1976, 1977; Säljö, 1982). Ference Marton, the leader of the group, also later developed the theory of variation together with colleagues (Emanuelsson, 2001; Marton & Booth, 1997; Marton & Morris, 2002; Marton & Pang, 2006; Marton & Tsui, 2004; Rovio-Johansson, 1999; Runesson, 1999). Phenomenograhic studies explored *students' learning* and *learners' ways of experiencing a specific phenomenon* (Ahlberg, 1992; Alexandersson, 1994; Lybeck,

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[10] Phenomenography has its roots in phenomenology (Husserl, 1995).

[11] INOM Inlärning och Omvärlsuppfattning. In English 'Learning and experience of the world'.

1981; Lybeck, Marton, Strömdahl, & Tullberg, 1988; Neuman, 1987). A finding from these studies was that a limited number of ways of experiencing/ understanding a phenomenon can be found in a group. Other phenomenographic studies (meta-cognitive studies) investigated how children experience learning (Ekeblad, 1996; Pramling, 1983) and a more recent study used Phenomenography as a base for exploring teachers' conceptual models in mathematics (Bentley, 2008).

In phenomenographic studies, the unit of research is "a way of experiencing something" (Marton & Booth, 1997, p. 111). The method used is interviews in which qualitatively different ways of experiencing a phenomenon are categorized and described. Categories of description are then ordered in a hierarchy. I use Neuman's (1987) study to illustrate this. Neuman (1987) studied different strategies that young children used in early arithmetic in grade 1. She had previously noticed that some students had the same difficulties in arithmetic as younger pupils. She sought explanations for why some children were more successful in arithmetic than others. The study showed that children who had identified the part–whole relationship in arithmetic were more successful than those who had not yet discerned this relationship. The part–whole relationship refers to, for example, the fact that two and four are parts (2+4=6) of six (the whole). Moreover, a number can be both a sum and a part of a larger number. For example, six is the sum of two and four, but six is also a part of ten, since six and four are parts of ten (see figure 2).

●● + ○○○○ = ●●○○○○	●●○○○○ + ○○○○ = ●●○○○○○○○○
2 + 4 = 6	6 + 4 = 10

Figure 2. An example of a part – whole relationship of numbers.

Neuman found five different ways of experiencing numbers among the pupils: i) "numbers as names" (children are aware of part-whole relationships, only ordinal aspect), ii) "number as extents" (children experience wholes and parts without the units within the parts, only cardinal aspect), iii) "counted numbers" (ordinal and cardinal aspects of numbers are present in parallel but separately), iv) "finger numbers" (fingers are used, making cardinal and ordinal aspects visual simultaneously, as well as parts and whole) and v) "number facts" (a simultaneous experience of cardinal and ordinal aspects). She found that the pupils experienced different aspects of the numbers. A finding this study gives us is that pupils' difficulties with early arithmetic could, according to Neuman, be traced to their ways of experiencing numbers since it seems to impact on the capability of solving mathematical problems. To be able to develop powerful concepts of numbers, the learner has to structure the numbers and understand their part–whole relationship. Neuman suggested that experiencing the structure of numbers, rather than counting, developed good number sense and arithmetic skills.

Neuman's finding is interesting, considering that there is extensive literature on children's learning of simple arithmetic and there is to a great extent a consensus view that modelling and counting strategies are the developmental path to learning arithmetic. It may be concluded from Neuman's study that the aspects that are simultaneously focused on are important for how something is experienced or understood. The categories found may be seen as necessary for the learner to discern and are similar to the notion of *critical features*. A difference between them is that the phenomenographic outcome space is ordered in a hierarchy. The critical features, as I use the notion, are related to the content, but build on the students' experiences in relation to the content. On the one hand, in Neuman's study, a category, for instance "counted numbers", refers to the capability of discerning ordinal and cardinal features separately, not simultaneously, and describes a stage in the development of arithmetic skill. A critical feature on the other hand describes features necessary to develop the capability for that particular group: what have the learners not yet discerned? In Neuman's example this would imply that critical features for learning counting and basic number sense would be for example i) discerning ordinal and cardinal features of a number simultaneously and ii) discerning part-whole relationships of numbers.

Looking at the phenomenon from the learners' point of view is to see it from a *second order perspective*. When a researcher is describing, for instance, what is happening in a classroom, it is from a *first order perspective*. The *second order perspective* would entail analysis and description of how the students experience what is happening in the classroom. In the studies reported in this thesis, analyses are made from both a first and a second order perspective.

Possibilities for learners to learn

Several scholars have contributed to the development of variation theory by making empirical studies in classrooms (see for instance, Emanuelsson, 2001; Marton & Morris, 2002; Marton & Pang, 2006; Rovio-Johansson, 1999; Runesson, 1999). In these studies variation theory was used to analyse teaching in classrooms in which the teachers were teaching the same content. The teaching seemed to offer different possibilities for learners to learn. Rovio-Johansson (1999) describes the relationship between teaching and learning as:

> […] teaching constitute different learning conditions, which the teacher and the students constitute in cooperation. Depending on what the teacher chooses to do with the subject matter (the object) in the lecture, in cooperation with the students, differing teaching objects as well as learning conditions are offered to the students for experience (Rovio-Johansson, 1999, p. 54).

The development of variation theory made it possible to analyse classrooms and teacher practices with new analytical tools. For example, Runesson (1999) studied five experienced teachers teaching the same content, fractions and percentages, in grades 6 and 7 in Swedish compulsory school. The aim was to reveal differences in how the teachers handled the content in classrooms. This was done by analysing what features the teacher and the learners open up as dimensions of variation in relation to the content and what learners' attention is drawn to. Runesson (1999) showed that, even if the teachers taught the same topic, used the same mathematical textbooks and arranged the lessons in the same way, there was still a difference between the lessons in terms of the different patterns of variation that were brought up during the lessons. She found different patterns of variation and invariance in the classrooms. For instance, one of the teachers [teacher 1] let the students solve many mathematical tasks [variation] with the same method [invariant]. One teacher [teacher 2] used the same task [invariant] but encouraged the students to come up with many different solutions [varied] for that task (see table 1).

Table 1. A pattern of variation and invariance. v= variation i=invariant

	task	method
Teacher 1	v	i
Teacher 2	i	v

How the content was handled in terms of patterns of variation and invariance was not in the teachers' awareness. In interviews the teachers focused instead on teaching arrangements, organisation or material used in the lessons and not the content taught (cf., Alexandersson, 1994). Although organisation, teaching arrangements, tasks and communication in classrooms influence learning, what is communicated about, the content, is of great importance from the point of view of variation theory. Furthermore, Runesson (1999) found that differences in teaching could be described as variation in the enacted object of learning (E) introduced by the teacher and students. When the teacher handled the mathematical content by introducing the mathematical principles, rules and single techniques for solving, a low variation was introduced by both the teacher and the students (see figure 3, E1).

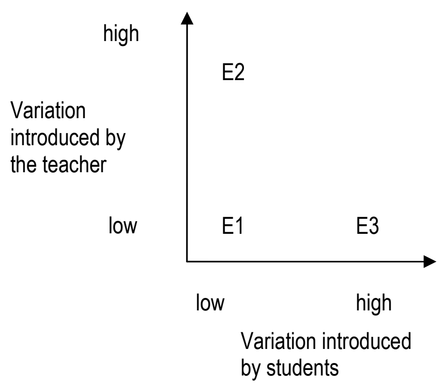

Figure 3. The degree of variation and who introduced the variation (modified from Runesson, 1999, p. 290). E in the figure is the 'enacted object of learning'.

Further, in E2 and E3, the content taught is problemetized as compared to E1, in which it is not. In E2, it is the teacher's view of student difficulties that makes the teacher open up for more variation with regard to the content taught. For instance, if the teacher is aware that the sign for a negative number and the subtraction sign are often a problem for student learning, then the teacher tries to show the difference between them. Significant for E3 is that students open up for variation, for instance when different students present how they solved a particular task. The variation in methods for solving could then be experienced by other students. The teacher role in E3 is to make it possible for students to experience this by asking for different ways of solving or different ways of thinking.

In another study conducted by Häggström (2008) differences were found between Swedish and Chinese teaching strategies in lessons about systems of linear equations. He found that even if Chinese students were already familiar with the topic and the teacher could assume that the students already knew much of the content taught and could take many features of the content "for granted" in his/her teaching, this was not the case. On the contrary, the opposite was found: many dimensions of variation were opened and few features were taken for granted. This was not found in the Swedish lessons. The question of whether the effect on student learning was greater in the Chinese teacher's lessons compared to the Swedish lessons was not explored. Do subtle differences in how the content is handled make differences for student learning? This question was explored by Olteanu (2007).

In a similar way as Häggström (2008), Olteanu (2007) studied differences between how teachers in upper secondary school handled the same content; second-degree equations and functions. To be able to draw conclusions about how the content was handled and how the students learnt, she also collected data from four tests given to all the students that participated. In the study, Olteanu first observed two teachers [video recorded lessons and individual sessions in the lesson] and then interviewed the teachers and eight students. The teachers used the same mathematics textbook. In the analysis, Olteanu focused on which features of the content the teacher, the students and the textbook brought up during the lessons and compared them with what the students focused on when they solved exercises. Features ignored (or taken for granted) by the teacher and the textbook (cf. Häggström, 2008) were also reported. She found that one of the teachers brought up more features of second-degree equations and functions than the other teacher (and the textbook) in both implicit and explicit ways. This had an impact on students' ways of solving tasks. The way the content was handled in terms of features focused on was reflected in the focus of the students when they solved the corresponding tasks.

Several classroom studies in different subject areas have been conducted using the variation theory framework, for example: in natural science education (Ingerman, Linder, & Marshall, 2009; Vikström, 2005), religious education (Hella, 2007), history (Lilliestam, 2009) and mathematics education (see for instance, Kullberg, in press; Liljestrand & Runesson, 2006; Reis, 2010; Runesson, 2007; Runesson & Mok, 2004; Watson & Mason, 2006). Other studies have for instance studied the effect on student learning when teachers use systematic variation in teaching mathematics (Al-Murani, 2007) or Maria Montessori's use of variation in pedagogical tasks and material (Marton & Signert, 2005).

Using variation theory to enhance learning

Questions have been raised as to the way in which variation theory can be used to enhance learning and be used by teachers to plan and analyse their practice. To answer these questions a special form of intervention study was designed. The inspiration came from results reported in the international TIMSS (Trends In Mathematics and Science Study) video study presented in the book *The Teaching Gap* (Stigler & Hiebert, 1999). This study found differences between the ways teachers in mathematics taught in different countries. Students in Asian countries were more successful in their mathematical performance, and one explanation given was the Japanese model for teachers' in-service training and development called lesson study described in chapter 1 (Fernandez & Yoshida, 2004; Lewis, 2002; Yoshida, 1999). Marton developed the *learning study* model together with researchers at Hong Kong University and the Hong Kong Institute of Education in early 2000. Learning study (LS) was inspired by the lesson study model, but the teachers used variation theory to plan and analyse their lessons. About three hundred learning studies have been conducted in projects (e.g. the VITAL project) in Hong Kong. Research on learning (e.g., Cheung, 2005; Chik, 2006; Lo *et al.*, 2005). LS was introduced in Sweden by a research project[12] in the year 2003 (Gustavsson, 2008; Holmqvist, 2006; Holmqvist *et al.*, 2008; Kullberg, 2006; Kullberg & Runesson, 2006; Runesson, 2008). Studies in mathematics, Swedish and English were conducted in the Swedish project.

The foremost aim of a learning study, as stated earlier, is to improve student learning. Participation in a learning study also enhances teachers' professional development (Gustavsson, 2008). LS can further be used as a way for researchers to explore practice and therefore also to improve academic learning and the production of knowledge. In LS, teachers try to find the critical features of an object of learning. For example, if a teacher wants the students to learn what an angle is, one could ask: *What* features of angles are important that the students discern? What must they see and understand in order to grasp what an angle is?

[12] The research project, The Pedagogy of Learning, was funded by the Swedish National Council of Educational Research from 2003 to 2006.

In a learning study about angles (Runesson, 2007) the teachers, on the basis of iterative analysis, evaluation and re-designing of a lesson, found critical features for student learning about angles. The object of learning in this study was to be able to recognise the 180 degree angle, to differentiate an angle from a non-angle, and to see that the size of the angle is independent of the length of its 'arms'. For example, to be able to see that the length of the 'arms' is not part of the actual size of the angle, the teachers and researcher planned for this feature to be brought up in the lesson 1. During the analysis of the first (video recorded) lesson in the cycle and the pre and post test, the teachers found that they had not succeeded in making the students discern this feature since only 33 percent of the pupils showed on the post test that 'arms' are not part of the angle itself. Consequently, pupils in their answers to the question of which of two angles is biggest (see figure 4) gave the answer b.

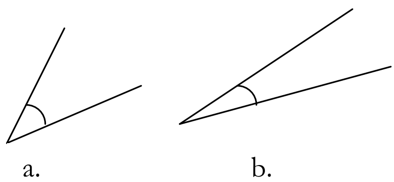

Figure 4. Compare the angles. Which one is the biggest?

During the analysis of lesson 1 the teachers experienced that they needed to make the feature of the angle as a turn more explicit in lesson 2. In lesson 1 the feature of the length of the 'arms' was brought up in terms of the teacher repeating to the pupils statements like "an angle is two lines (sides) and a point" and "the lengths of the arms do not have to be equal in size".

> For example, they came to the conclusion that the pupils see the sides as constituent part of the angle, and this way of understanding is reasonable and logical from the point of view of how the definition of an angle was given in the lesson. So, they decided to come up with another definition in the next lesson and bring out the idea of angle as a turning more clearly (Runesson, 2007, p. 12).

It was found that that this decision to bring out the concept of angle as a turn (dynamic) was more successful than the definition used in lesson 1 (static). The result of the post test for lesson 2 showed that 95 percent of the pupils could differentiate between angles with different sizes and different lengths in 'arms' (see figure 4). 'Comparing' angles (on different sizes of clock faces) and making the angles smaller and bigger by 'turning' the arms, as in lesson 2, were necessary conditions for learning what was intended. A critical feature for learning was, for this group of students, to see that the size of the angle is dependent on the turning of the 'arms'. This feature made it possible to separate the 'arms' from the turning. Hence the 'arms' were constant and did not vary; instead it was the 'turning' that made the angle vary in size. Variation theory says it is likely to discern that which varies; the varied feature was made possible for the pupils to notice. This feature was seen as critical for learning and was, with similar results in student learning, also brought out in lesson 3, where 78 percent of the pupils answered correctly to the items about comparing angle size.

Are findings from learning studies transferable to other teachers and students?

Variation theory has been used as a framework in a number of classroom studies. The use of variation theory to explain differences in students' possibilities to learn is therefore nothing new. The next step and what is new in the present study is that variation theory is used to test findings from learning studies. In this study, lesson designs with critical features found in learning studies are tested with new teachers and students (Kullberg, 2007a; Runesson & Marton, 2009). In these studies several teachers implement the same critical features with new groups of students. The lessons are video recorded and analysed together with pre and post tests. In the analyses the researchers try to relate how the content is handled in the lessons with student learning. Variation theory is used in the analysis as a means to describe what made it possible for the students to discern the critical features of the object of learning.

CHAPTER THREE

TWO LEARNING STUDIES IN MATHEMATICS

The research in this thesis is based on findings from two learning studies in mathematics, conducted in a research project at the University of Gothenburg in collaboration with teachers. The findings from these two studies are critical features that have been identified for students' learning. The research reported in this thesis explores whether these critical features can be communicated to other teachers and used to enhance other students' learning. This chapter provides a description of the critical features and how they emerged in a systematic investigation by the teachers in the learning study and from their practice.

As stated earlier, a learning study is a systematic and iterative process with the aim of exploring students' learning and can be seen as research done by teachers to produce knowledge about this phenomenon (see chapter 1, figure 1). In a research like process the teachers in the present case gathered information and analysed students' learning from the point of view of what it was possible to experience in a lesson. During the analysis the teachers became aware of what contributed to student learning and what did not. How the teachers gained knowledge about what was critical for student learning is described in the following section.

The same teachers participated in both learning studies reported, and a fourth teacher also participated in one of the studies. The teachers were experienced mathematics teachers that had volunteered to participate in the project. The participating students had written consent from their parents. In the learning studies the teachers had ownership of the study and the decisions made, and hence the role of the researcher in these studies was to support the teachers during the course of the study. Each learning study lasted for almost a whole semester and the team had six meetings to plan and revise the lesson. There were also introductory and final meetings to prepare the study and later discuss the findings. Every meeting lasted for about one and a half hours. In each study three or four lessons were video recorded and later transcribed verbatim for research purposes.

A learning study about the density of rational numbers

A learning study about rational numbers was conducted during the spring of 2003 with three teachers, a researcher and three classes in grade 6 (12-year-old students). It was the team's first learning study and they made the decision to work with student learning of decimal numbers. Decimal numbers are usually taught in Sweden from grades 4 to 9 in compulsory school. Decimal numbers are, as are fractions and percentages, a type of rational numbers. At the start of the learning study process a pre test was used to give the team information about the students' pre knowledge about decimal numbers. The team found that students had the greatest difficulty with a specific item on the test:

Ann says there is a number between 0.97 and 0.98. John says there is no such number. Who is right and why?

The question about the amount of numbers in a given interval [0.97; 0.98] was designed to explore students' experience of the density of rational numbers. The density of rational numbers implies that there is an infinite amount of rational numbers between any two non equal numbers. From this follows that it is impossible to say what the next number after for example 1, since the number could have an infinite number of digits. The rationale for choosing 'understanding the density of rational numbers' as an object of learning in the learning study was that the teachers found that most of their students thought that there were no numbers between a pair of rational numbers (cf., Hart, 1981; Vosniadou, Vamvakoussi, & Skopeliti, 2008). The teachers had the idea that this object of learning would also make the students become aware of more general features of rational numbers.

The test item about the amount of numbers in the interval [0.97; 0.98] generated many qualitatively different answers. In the analysis of the pre test the following categories of answers were found in the three classes: i) there is one number 0.97.5, ii) there are many numbers between 0.97 and 0.98, iii) there are ten numbers between 0.97 and 0.98, iv) there is one number, 0.975, and v) there are no numbers between 0.97 and 0.98. The number of students in each category varied somewhat between the three classes; in class A (lesson 1) only one student showed in the pre test that there were many or infinitely many numbers (ii), as compared to four students in class B (lesson 2) and two in class C (lesson 3).

Lesson 1

In lesson 1, the team wanted the students to understand that rational numbers are dense by letting the students discuss how many numbers there were in an interval [0.97; 0.98]. The task about the amount of numbers in an interval was taken from the pre test as the teachers found it very useful because of the diversity of answers it generated. The lesson plan for lesson 1 was to let the students discuss in small groups the different answers that the students had given in the pre test.

After the discussion the groups would present their answers to the task on a large poster, as a number line with numbers in the interval [0.97; 0.98], for the whole class. The rationale was that the students themselves would choose the correct answer and develop an understanding of density of decimal numbers through the discussion. However, the post test showed that few students in lesson 1 answered that there were many or infinitely many numbers in a given interval. Analysis of the lesson and tests showed that many students answered ten or nine numbers in a given interval. This was interesting since none of the students in lesson 1 showed this on the pre test. Several students answered that there were no numbers in a given interval.

The team analysed the video recorded lesson, which was enacted by one of the teachers in his/her class, together with the post tests. It is important to keep in mind that the lesson was carried out according to the plan that the team had made. It was therefore the whole team's lesson that was evaluated and analysed and not an individual teacher's lesson. The team found that the teaching did not contribute very much to help the students to enhance their understanding before the group work and discussion. The solutions to the task that came out of the group work and that were presented by the different groups to the class showed that an understanding of density of rational numbers did not develop 'automatically'. It was clear that the team had not planned for how the teacher would handle the discussion with the whole class at the end of the lesson. The group with the most advanced solution to the task showed that there were nine numbers in the interval (see figure 5).

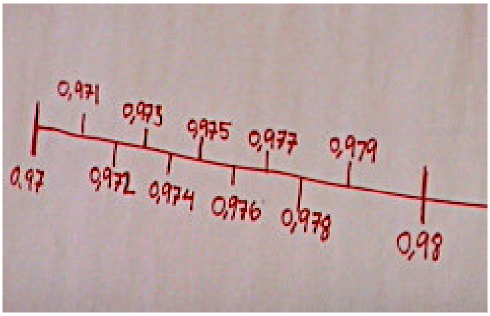

Figure 5. Picture of a poster from lesson 1 illustrating nine numbers in the interval [0.97; 0.98] (Kullberg, 2004, p. 17)

Another student group argued that there was one number and three groups that said there were no numbers in an interval between the two decimal numbers. What the teacher and the students contributed during the lesson was apparently not enough to develop the students' understanding.

During lesson 1 different examples of decimal numbers (decimal numbers as points on the number line) in intervals were discussed. The video recording of the lesson showed, for instance, how one student tried to make sense of numbers with different amounts of digits. Does the number of digits in a decimal number make a number smaller or larger? The student expressed that numbers in between two decimal numbers [e.g. 0.97; 0.98] has more digits [e.g. 0.975]; however, this contradicted her view of "when it is more digits it [the number] becomes smaller" although it should be a bigger number [0.97< 0.975]. This was not further elaborated in the lesson.

The excerpt shows that the teacher asked the students to name different numbers in the interval; the teacher said, "And zero point nine seven two [0.972] and what would come after that then?" (line 1). One student (Jennifer) said that the interval between the numbers, for example 0.971 and 0.972, could be split into smaller and smaller parts and that "it will never end" (line 10).

1. Teacher: And zero point nine seven two [0.972] and what would come after that then?
2. Erika: (…)
3. Teacher: One, should we put one there [0.9721]?
4. Erika: It will never end.
5. Teacher: It will never end, see now something has happened in this discussion, from that we had, do you realise this now, from the beginning we had three groups that said that there were no numbers and we

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                had one group that said there were nine numbers and
                one group that said that there was one number and
                now Erika says something very exciting she says it
                can never end, do I understand you correctly now?
6.  Erika:      Mm
7.  Teacher:    Okay Jennifer
8.  Jennifer:   But, it, I mean what is it called, we have split
                them, zero point ninety-seven [0.97]/
9.  Teacher:    /Aha
10. Jennifer:   To (0.975), so they are split in nine pieces and
                then you take every part and split it all the time
                so it will never end.
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Hence ideas about density were brought up during the lesson, but most likely in a way that most students did not make sense of. In fact, about half of the class answered, as mentioned earlier, that there were only ten (or nine) numbers in the interval on the post test. Being told by classmates or the teacher was evidently not enough to develop an understanding of the density of rational numbers since most students did not discern that which was intended. A conclusion made by the teachers from the analysis was that naming numbers in the interval and thereby treating them as countable seemed not to be sufficient to develop an understanding of the density of rational numbers. The teachers thus changed their plan for teaching in lesson 2 in the learning study cycle from the point of view of eliciting other features of rational numbers that they thought would promote learning about density.

Lesson 2

Lesson 2, implemented in a new group of students and with another teacher, was planned in order to bring out the connection between rational numbers in a more explicit way. With an awareness of students' learning from lesson 1 and analyses of the lesson, the teachers were more prepared and focused on what they wanted to achieve in the classroom. Lesson 2 was designed to give the students a better opportunity to discern features of decimal numbers. The plan for the lesson was to introduce assumed critical features, the *different representations of a rational number* such as fractions and percentage and the *part-whole relationship of a rational number* at the beginning of the lesson. The rationale for this was that the team thought that this would have an impact on the discussion of the group assignment and students' experience of the density of rational numbers.

The assumed critical feature of different representations of rational numbers was introduced through a question asked by the teacher; "Could 0.97 be said and interpreted in another way?" To this question the students gave many different answers such as 97 hundredths, 970 thousandths and 97 percent. The question opened up a variation in the representation of rational numbers that had not be made possible to experience in lesson 1. The other critical feature, the part-whole relationship of a rational number, was elicited by fractions and percentage. The teacher asked the students; "Is it possible to take 97 hundredths of something?" In this case the students gave suggestions like a ruler, a person, a pen and a human being. It was possible to experience 0.97 of a whole, for instance a ruler, as 0.97 was seen as 97 centimetres on the one-metre ruler. These two features were also elicited for another decimal number, 0.98, and hence the difference between 0.97 and 0.98 was discussed (1/100 or 0.01). After this introduction the students worked in groups with the same assignment as in lesson 1. In the oral account of the assignment the teacher made the students explain and reflect upon features of rational numbers. For instance, the teacher elicited the relationship between the numbers of parts in a number with the place value of a decimal number.

At the end of the lesson the teacher made it possible to experience another assumed critical feature, *divisibility,* by showing the same number (0.975) on several rulers with different numbers of parts (ten parts/tenths, hundred parts/hundredths, thousand parts/thousandths etc.). By doing this, the teacher showed the students how the same number could be expressed in different number of parts and that this implied that the ruler could be 'cut up' into smaller and smaller parts (see figure 6).

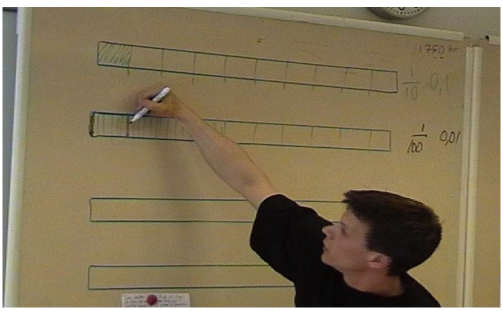

Figure 6. The picture shows how the teacher split several rulers in different numbers of parts (tenths, hundredths, thousandths and ten-thousandths).

The team was pleased with the result of lesson 2 since all the students except one answered correctly in the post test (see table 2). The assumed critical features were therefore identified as being critical for student learning. The team decided to implement the same critical features in lesson 3 as were used in lesson 2.

Lesson 3

A new teacher with a new class conducted lesson 3 enacting the four critical features that were identified: i) *decimal numbers as points on a line*, ii) *the interchangeable representation* of rational numbers, iii) *a rational number as a part of a whole* and iv) *divisibility*. Lessons 2 and 3 used a similar introduction before group work and the same student task as in lessons 1 and 2.

A difference between lessons 2 and 3 was the way the teacher enacted the feature of *divisibility*. This feature was enacted in lesson 3 by letting the students give examples of different numbers of parts in an interval, such as hundredths, thousandths, ten thousandths, ten billionths, infinitely many etc., compared to lesson 2 where a ruler was split into smaller and smaller parts. The post test results, presented in the next section, showed that students in lesson 3 performed as well as students in lesson 2.

Learning outcomes and identified critical features

In the learning study only one item was used in pre and post tests to analyse student learning. This item was different[13] in the pre and post tests but was designed to investigate the same thing – students' understanding of the density of rational numbers. The tasks were the same for all the classes that participated. Analysis of the whole post test[14] shows that there is a significant difference at the 0.05 level between the learning outcomes from lesson 1 and the other two lessons (2 and 3) whereas there is no significant difference between the groups on the pre test. Table 2 shows a small/medium effect size[15] for lesson 1 (0.35) compared to a medium/high effect size for lessons 2 and 3 (1.98) on the task concerning the density of rational numbers.

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13 Pre test question: "Anne claims that there is a number between 0.97 and 0.98.

14 Maximum points on the pre test were 7 and on the post test 3.

15 Cohens d= $\dfrac{M_1 - M_2}{\sqrt{\dfrac{SD_1^2 + SD_2^2}{2}}}$

Table 2. Re-analysis of the pre and post tests from the learning study first analysed by Kullberg (2004). Task concerning density, maximal points= 1.

	Lesson 1 (N=19)			Lesson 2 (N=17)			Lesson 3 (N=17)		
	Mean	SD	Effect size	Mean	SD	Effect size	Mean	SD	Effect size
Pre test	0.05	0.23		0.24	0.44		0.06	0.24	
Post test	0.16	0.38	0.35	0.94	0.24	1.98	0.76	0.44	1.98

On the basis of the differences in learning outcomes, the team identified four features that were assumed to be critical for learning the density of rational numbers.

- Decimal numbers as points on a line
- The interchangeable representation of rational numbers
- A rational number as a part of a whole
- Divisibility of rational numbers

Critical features are found in students' experience of what is taught and therefore could not be derived from the subject matter only. It is possible that more features could be identified as being critical for student learning. In lesson 1, it was not possible to experience the necessary conditions for student learning of the object of learning. However, the high result in student learning in lessons 2 and 3 suggests that the necessary features of the object of learning were possible to experience during the lessons.

In conclusion, when decimal numbers varied in the interval and the numbers were treated as countable (0.971, 0.972, 0.973 etc.) the impact on student learning was small. However, when the numbers were seen as different amounts of parts in an interval, and the parts in the interval varied, the impact on student learning was significant. It was made possible in lesson 1 to experience numbers in the interval [0.97; 0.98] and hence it was possible to experience a dimension of variation concerning *numbers as points*. In lessons 2 and 3 another dimension of variation, concerning *divisibility,* was possible to experience since it was made possible to experience different amounts of parts in the same interval (see figure 7).

In lesson 1 the numbers varied

In lesson 2 the number of parts in the interval varied

Figure 7. Two dimensions of variation for experiencing rational numbers in decimal form (Runesson & Kullberg, in press).

The findings in the light of previous research

The following section discusses the critical features in the light of previous research. The findings from the learning study can be seen as knowledge produced by teachers through a research-like process and an empirical investigation of practice. A common way among educational researchers to do research about the topic is through interviews or surveys with students. What similarities and differences could be noted between the findings from the learning study compared to previous research? What support is there for the conjectures made about the critical features, which are based on teachers' investigation of their own practice?

Previous research shows that students have difficulty with rational numbers in general and that students at all levels, even at the secondary and university levels, have difficulty realising that a set of rational numbers is dense as compared to discrete whole numbers (e.g., Fischbein, Tirosch, & Hess, 1979; Hart, 1981; Stacey, Helme, & Steinle, 2001; Steinle, 2004; Vamvakoussi & Vosniadou, 2007; Vosniadou *et al.*, 2008). It was reported by Hart (1981) that few students could solve the task "how many numbers could you write down which lie between 0.41 and 0.42?". Only 10 to 22 percent of the students at age 12 to 15 answered the question adequately. A more recent study by Vosniadou *et al.* (2008) also showed that students in grades 7, 9 and 11 frequently answered that there is a finite number of numbers in a given interval. The study confirmed that this assumption remains robust through the different grades.

> The results confirmed that the presupposition of discreteness is strong for younger students (seventh graders) and remains robust even for older students (nine and eleventh graders), despite noticeable developmental differences. Students from all age groups answer frequently that there is a finite number of numbers in a given interval, regardless of whether they were asked in an interview (Vamvakoussi & Vosniadou, 2004), in an open-ended questionnaire (Vamvakoussi & Vosniadou, 2007), or in a forced choice questionnaire (Vamvakoussi & Vosniadou, in preparation) (Vosniadou *et al.*, 2008, p. 12).

Another study by Pehkonen *et al.* (2006) showed that few students in grades 5 and 7 had any or little understanding of the density of rational numbers. To the survey question, "Which is the largest number still smaller than one?" they identified three levels of understanding: i) *finite*, ii) *potential infinity* and iii) *actual infinity*[16]. At the lowest level (i) students showed only finite answers, for example, "the students at this level answered with one or two decimals or other numbers with a fixed set of digits (e.g. 0.9999999999999). In the intermediate level (ii), potential infinity, students answered with an unending number, e.g. 0.9999… At the highest level (iii), actual infinity, students would reply that there is no such number; it is not possible to say since there are infinitely many.

While students' difficulties with rational numbers as dense are recognized in previous research, little is reported about what can be done about it and what students need to experience in order to learn. One explanation given for students' difficulties with rational numbers is their previous experiences of whole numbers. For instance, Ball (1993) argues that students say and experience things that are "true in their current frame of reference, in relation to what they currently know, but that will be wrong in other contexts" (ibid, p. 391).

> When a first grader announces that 3 is the next number after 2, he is right – in his domain, which is the counting numbers. But, for a sixth grader considering rational numbers, there is no next number after 2, for rational numbers are "infinitely dense", which means that between any two rational number, there is another rational number. Between 2 and 2.1 are 2.01, 2.02, and so on. Between 2 and 2.01 are 2.001, 2.002, and so on. Consequently, there is no "next number" unless you specify a context – e.g., the next hundredth (Ball, 1993, p. 391).

[16] The distinction between potential and actual infinity was used by Aristotle.

This is also argued by Vosniadou *et al.* (2008) who point to differences between students' initial whole number concept and the rational number concept. Whereas children's concept of numbers entails that i) numbers are countable, ii) there are no numbers in between numbers, iii) there is a smallest number, iv) numbers are ordered by means of their position in the counting list, v) longer numbers are larger, vi) addition and multiplication make a number larger and division and subtraction smaller, and vii) a number has only one representation, while rational numbers have opposite features. Consequently, there are many features that the learners must differentiate between in the case of whole numbers and rational numbers.

From the findings of the learning study it is argued that one feature that students need to discern is that there are numbers in between numbers - *a number as a point on a number line*. Previous research has shown that this is an important feature for experiencing the density of rational numbers. Why? Scholars argue that the representation of whole numbers made on the number line from an early age become problematic further on (Dufour-Janvier, Bednarz, & Belanger, 1987, p. 117). They argue that seeing whole numbers as "stepping stones" on a number line with nothing in between, as is often done in primary grades, is the opposite of a view of density later on.

> First, when children use the number line during the learning of positive integers, they develop the notion of the number line as a series of "stepping stones." Each step is conceived as a rock, and between two successive rocks there is a hole! [...] It is hardly surprising that at the secondary so many students say that between two whole numbers there are no numbers, or at most one. (Dufour-Janvier *et al.*, 1987, p. 117).

The *interchangeable representation* of rational numbers was also found in the learning study to be a critical feature for student learning. The *interchangeable representation* implies that the learners "must realise that fractions and decimals are alternative representations of rational numbers (and not different kinds of number)" (Vosniadou *et al.*, 2008, p. 11).

> […] rational numbers do not have only one symbolic representation. Rather they can be represented symbolically either as decimals, or as fractions. For example, the number one half can be presented as 0.5 and also as ½. To make things more complicated, the one half can be represented also as 0.50, 0.500, 2/4, 4/8 etc. This presents the learner with yet another difficulty: one must realise that fractions and decimals are alternative representations of rational numbers (and not

different kinds of numbers), despite their differences in notation, ordering, operations and contexts of use (Vosniadou *et al.*, 2008, p. 11).

Furthermore, previous research has found that an understanding of fractions as numbers is an important predictor for understanding the density of rational numbers (Pehkonen *et al.*, 2006). A proper understanding of fractions entails distinguishing several features of fractions, for instance a rational number as *a part – whole relationship*.

The divisibility of parts and numbers in an interval was seen in the learning study to be of decisive importance for students' experience of rational numbers as dense. Smith *et al.* (2005) found that "children's spontaneous acknowledgement of the existence of numbers between 0 and 1 was strongly related to their induction that numbers are infinitely divisible in the sense that they can be repeatedly divided without ever getting to zero" (Smith *et al.*, 2005, p. 101). There are *other features* of rational numbers identified in previous research that have not been identified as critical for students' learning in the specific learning study. Several of these features are concerned with the representation of rational numbers. For instance, Janvier (1987) describes differences and similarities in symbolic representation between decimal numbers and whole numbers that students might not have experienced. Similarities are for instance that the column place values decrease by a magnitude of ten moving from left to right on a number line. For example, 4 is ten times less than 40 and 0.04 is ten times less than 0.4. Another common feature is the function of zero as a place holder.

Differences in symbolic representation between decimals are for instance that a zero in the right most column within a decimal number does not make the number ten times larger, e.g. 0.40 is not ten times greater than 0.4, as it does for whole numbers since 40 is ten times greater than 4. For whole numbers, it is true that the further away the digit is from the decimal point the larger is the value of that number, 400 is larger than 4, while this does not apply for rational numbers since 0.004 is not a larger number than 0.4. Moskal & Magone (2000) found that students sometimes ignore symbols in a number, for instance the zeros in front of eight in 0.008000, but make sense of zeros behind the number, as with whole numbers. They also found that differences in representation within rational numbers were difficult for students, for instance some students interpret 3.8 as $3/8$ or 0.8 as $1/8$. These students have not discerned differences in notation and "treat the decimal value as if it was a fraction with a decimal point replacing the fraction bar" (p. 317). The difference in notation is that "the denominator explicitly indicates the number of partitions that are made of the unit. For decimals, the number of partitions is implied through place value and is limited to factors of ten" (p. 317). Even negative numbers are confused with decimal representation by some students, for example 0.9 with -9 (Stacey *et al.*, 2001). Steinle and Stacey (1998) found student strategies that suggested that the length of the decimal numbers, in terms of amount of symbols (digits), "longer is larger" or "shorter is larger", or the opposite, determined the size of the number.

Kullberg, Watson & Mason (2009) suggest that some of the difficulties experienced by students' could be explained by that fact that it is difficult to coordinate symbolic representation and representation on the number line (as a point). They argue "that changes in decimal and fraction representations of very close numbers can be visually dramatic, while the shift on the line is tiny" (ibid, p. 435). For example, the difference between the number 0.97 and 0.971 or 97/100 and 971/1000 could be experienced as large while, on a number line, the numbers could be very close. In their study they found that students have difficulties with coordination of positional variation on the number line and digital variation in the number. For instance when students were supposed to represent the numbers 1.7, 1.71, 1.701, 1.7001 (see figure 8), half who attempted the task placed them as being equally spaced along the number line. Only 14 of 100 students in years 7 and 9 (11/12 and 13/14-year-olds) completed the task correctly.

Task a: Represent these numbers on this line: 1.7, 1.71, 1.701, 1.7001

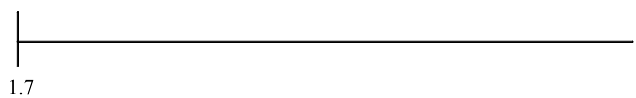

Figure 8. Task used to investigate students' understanding of decimal numbers (Kullberg *et al.*, 2009, p. 348).

In conclusion, previous research confirms that there are many features of rational numbers that students ought to be aware of that they sometimes are not, for example differences between rational and whole numbers. A difference between the difficulties and critical features is that critical features are instead related to how students experience what is taught. There is not much said in previous research however about how teachers could make it possible for students to overcome the difficulties. In the learning study, the teachers worked in collaboration in exploring this question. The teachers worked for a long period of time, a whole semester, to find out what students needed to discern in order to learn and understand the object of learning in a certain way. The teachers were not aware of the previous research reported in this chapter and yet identified similar features as has been pointed out by research as being difficulties for students' learning. Furthermore, in the learning study lesson, it was possible to experience the critical features simultaneously and not on different occasions, and this could have contributed largely to student learning. Experiencing one critical feature at a time in different lessons might not have been sufficient for student learning.

A learning study about addition and subtraction of negative numbers

The learning study reported was conducted by four teachers, a researcher and four classes with grade 7 and 8 (13 and 14-year-olds) students during the spring of 2004 (Kullberg, 2007b; Maunula, 2006). Addition and subtraction of negative numbers are usually taught in grade 8 in the Swedish compulsory school, and the topic was therefore new to the students. It was the team's third learning study. In the teachers' collaborative planning of the lessons, the teachers used their own previous experience, literature on teaching and learning the topic and analysis of their own students' learning of the topic (analysis of their teaching and student test) as resources in the learning study process. The object of learning in the study was for students to be able to calculate addition and subtraction with negative numbers, for example being able to solve such tasks as 5 - (-3) and (-5) + (-3).

Negative numbers were originally conceived through algebraic calculation and as an extension of whole numbers. They were first seen as a contradiction to the concept of number as a quantity. Numbers were used for counting, for instance sheep or silver coins. A number less than zero would entail that you had no sheep or no coins. Even students in secondary school may argue that a negative quantity would be the same as zero. It was not until the 19th century that directed numbers – numbers with magnitude (absolute value) and direction (positive or negative) – were accepted (Fischbein, 1987). It is a fact that many students have difficulty learning to operate with negative numbers and that it is a hard topic to teach (e.g., Ball, 1993; Carraher & Schliemann, 2002; Gallardo, 1995). A rule – 'two minuses make a plus' – is commonly used as a means in teaching and learning negative numbers. It has been shown that some students apply this rule without sufficient understanding, for instance as illustrated in the excerpt below (see Vlassis, 2004, p. 480).

1.	L1:	−9y − 4y … mm … minus by minus (in a very low voice). That makes 13.
2.	INT:	Plus or minus?
3.	L1:	Plus because minus by minus gives plus.

Students can easily experience negative numbers as mysterious and incomprehensible. Teachers strive to make it easier for students to learn by applying different representations and metaphors such as the number line, thermometer and magic peanuts in a pocket or elevators going up and down. However, the metaphors have limits in what they can represent, which causes confusion about when to use which metaphor (Kilhamn, in press). The teachers in the learning study wanted to investigate their teaching of negative numbers and find a more powerful way of teaching than applying the rule 'two minuses make a plus'. During the study the teachers became aware of critical features for their students' learning, which was reflected in an improvement in student learning outcomes.

The pre test taken by all classes in the learning study gave the teachers valuable information about which tasks were most difficult for students and how they answered. "They found, for example, that many students could solve some tasks with problem solving skills or by using the rule 'two minus signs make plus' and without an understanding of addition and subtraction with negative numbers. This 'rule' was often of no meaning to the students and was used as a method of a procedure" (Runesson *et al.*, in press). During the LS meetings the team discussed metaphors and representations of negative numbers that they had found in mathematics textbooks and come upon through experience. The team came to the conclusion that it was difficult to find one metaphor that worked in all situations (e.g., a + (- b), (- a) + (- b), (-b) + a, (- b) + (- a), a - (- b), (-a) - b, - a - (- b), (-b) - a, (-b) - (-a)).

The first critical feature found during the analysis of students' answers in the pre test and the literature was the different meanings of the operational sign for subtraction and the sign for a negative number (cf., Ball, 1993; Vlassis, 2004). The team observed that the students did not separate the two signs and hence treated the symbols as though they had the same meaning. The teachers considered: how could teaching make it possible for the students to discern the difference between the signs?

> The teachers considered possible solutions to this problem: different words for the number (e.g., 'negative three') and the operation (subtraction or minus) could be used, the two signs could be separated by putting the sign for the negative number 'higher up' than the operational sign. They discussed whether concrete representations and metaphors, for example temperature (-3 as three degrees below zero) and debt, would be a limitation for a deeper understanding. These representations seemed to be of use for solving some of the tasks but not all. How could, for instance subtraction task '5- (-3)=', be represented in a good way? And how could the different tasks '5-3=' and '5- (-3)=' be told apart (Runesson *et al.*, in press).

The team decided to use the expression 'negative five' for (-5) to signify the difference between the signs, instead of 'minus five'. Furthermore, the team decided to use patterns in lesson 1 to make it possible for the students to discover connections between addition and subtraction of negative numbers and thereby discover the rule 'two minuses make a plus'.

Lesson 1

During lesson 1 in the LS cycle the teacher discussed the different meanings of the signs with the students. The teacher also introduced the notion of opposite numbers, that for example an addition of +1 and -1 equals zero. The rationale was that the students would discover the rule namely – that 'adding (subtracting) a negative number is the same as subtracting (adding) its opposite' (cf., Freudental, 1983, p. 437), e.g., 5-(-5) = 5+5 (see figure 10). This was done by using patterns with negative and positive numbers in both addition and subtraction and later combining the patterns so that the students were able to discover the 'rule'. The students first worked in pairs to discover patterns in addition and were told to explore what happened in addition (5+2=7, 5+1=6, 5+0=5) when a negative number was subtracted (5+(-1)=?). Different possible solutions to the task 5+(-1)= were compared, and students were asked to argue for the correct pattern. Note that patterns A and B were introduced by students and C by the teacher (see figure 9).

Pattern A	Pattern B	Pattern C
5 + 5 = 10	5 + 5 = 10	5 + 5 = 10
5 + 4 = 9	5 + 4 = 9	5 + 4 = 9
5 + 3 = 8	5 + 3 = 8	5 + 3 = 8
5 + 2 = 7	5 + 2 = 7	5 + 2 = 7
5 + 1 = 6	5 + 1 = 6	5 + 1 = 6
5 + 0 = 5	5 + 0 = 5	5 + 0 = 5
5 + (-1) = 4	5 + (-1) = -4	5 + (-1) = 6
5 + (-2) = 3	5 + (-2) = -3	5 + (-2) = 7
5 + (-3) = 2	5 + (-3) = -2	5 + (-3) = 8
5 + (-4) = 1	5 + (-4) = -1	5 + (-4) = 9
5 + (-5) = 0	5 + (-5) = 0	5 + (-5) = 10

Figure 9. Three patterns used in lesson 1 for the students to explore "What happens in addition when we go below zero?" (Note that patterns B and C are incorrect) (Runesson *et al.*, in press).

Later, a pattern in addition was connected to a pattern in subtraction and the students were supposed to see how the 'rule' worked and how an addition could be substituted by a subtraction and vice versa, for instance 5-(-5)=10 to 5+5=10.

5 + 5 = 10	5-5=0
5 + 4 = 9	5-4=1
5 + 3 = 8	5-3=2
5 + 2 = 7	5-2=3
5 + 1 = 6	5-1=4
5 + 0 = 5	5-0=5
5 + (-1) = 4	5-(-1)=6
5 + (-2) = 3	5-(-2)=7
5 + (-3) = 2	5-(-3)=8
5 + (-4) = 1	5-(-4)=9
5 + (-5) = 0	5-(-5)=10

Figure 10. Using patterns to discover the rule 'two minuses make a plus' in lesson 1.

The results of the post test showed that students improved in tasks with addition but not in tasks with subtraction. Tasks with subtraction with two negative numbers showed an even poorer result in the post test than in the pre test. The team realised that this way of teaching did not offer students much possibility to discern critical features for understanding negative numbers. During the analyses of the video recorded lesson the team came to the conclusion that they had to make possible two different ways of experiencing subtraction, as a 'difference' and as 'take away'. Although the teachers were aware of both subtraction as a difference and a take away they chose not to bring up this difference in lesson 1 since they thought it would confuse the students. However, a student said during lesson 1, "isn't it possible to see it [subtraction] as a difference?" and the teachers realised that it would be powerful to see subtraction this way. The teachers thought that seeing subtraction as a difference, a comparison between numbers, would develop students' understanding of operating with negative numbers.

Lesson 2

In lesson 2, a new teacher in a new class implemented the revised lesson plan with the critical feature found in the analyses of lesson 1. The main theme of the lesson was 'subtraction as a difference' between numbers on a number line. The teacher used many examples of subtraction as a 'difference' compared to 'take away' in order to make the students experience subtraction as a comparison between numbers. The teacher said about the subtraction sign: "Today I want you to think about the minus sign as something, something new, something different (…) for example a difference, a difference between two different things, between two numbers" [writes 'difference' on the whiteboard]. The feature of the sign, as the distinction between the operation and a negative number, was not possible to experience in lesson 2. The teacher probably forgot about it.

The lesson was enacted according to the planning. However in all examples given, the differences between the numbers were always a positive number. e.g. 5 – 3 = 2, 6 - (-3) = 9 or (-3) - (-5) = 2. Lesson 2 paid little attention to addition and the results in those tasks thus did not improve very much. The post test showed an increase in correct answers to tasks about subtraction, especially subtraction of a negative number from a positive number. There was a smaller increase in tasks with subtraction of two negative numbers. One explanation for this result was that, during the lesson, only positive differences, for instance (-3) - (-5) = 2, were discussed but not (-5) - (-3) = (-2). The reason for only dealing with positive differences was that the team had at this point not found a powerful way to deal with negative difference.

The meeting after lesson 2 was a 'turn in view' for the teachers. The teachers discovered a metaphor that worked for 'all cases' of addition and subtractions – money/debt as a financial state (cf., Ball, 1993 p. 382). For example, if Lisa has 7 kronor and Bo has a debt of 4 kronor, their 'shared economy/financial state' would be 3 kronor (7 + (-4) = 3). If Lisa has 7 kronor and Bo -4 kronor, the 'difference' between their economy/financial state (7 - (-4)

= 11) would be 11 kronor, seen from Lisa's point of view. The team had previously take for granted that the students were aware of the feature that subtraction could be seen a difference from the first term and that commutative law does not apply to subtraction. The team decided to make the direction of the difference more explicit by first introducing comparisons of age and lengths in lesson 3.

> For example, they suggested to compare the age of two persons differing nine years in age, one will be the younger and the other the older. How you represent that (positive ore negative difference) depends on whose perspective you take. For instance: John is 12 and David is 9 years old. Starting with the oldest (12-9=3) you say: "John is 3 years older". If you do it the opposite way (9-12=-3) you say "David is 3 years younger than John". The possibility of using different metaphors like 'longer/shorter', 'smaller/bigger' and so on were discussed (Runesson *et al.*, in press).

Later in lesson 3 the teacher would introduce debt, in terms of financial state/economy (cf., Ball, 1993). A negative numbers as a debt was a possible way to show subtraction as a difference between two numbers, for instance to give a scenario of two persons sharing their economies (addition) and comparing them (subtraction).

Lesson 3

A new teacher tried to implement the features that were found in a lesson with a new class of students. Although the intention was to implement three critical features in lesson 3, i) *the sign*, ii) *subtraction as a difference* and iii) *the perspective* – the commutative law does not apply in subtraction, this lesson did not turn out as expected because of an event that happened outside the classroom[17]. For this reason, the analysis and learning outcomes of this lesson is not presented here. However, the team was convinced that they had found critical features for student learning and a powerful way of teaching the topic and therefore decided to create lesson 4.

Lesson 4

In lesson 4 the teacher tried to implement the three critical features identified as planned by the fourth teacher in a new class. During the lesson the teacher brought up the difference between *the signs*, *subtraction as a difference* and *the*

[17] Lightning struck a field nearby the school.

perspective taken from the first term in subtraction. However, during the lesson yet another critical feature was found. This became evident when the teacher asked the student where the two numbers (-2) and (-1) would be placed in the expression (___- ___ = 1). The teacher's idea was that the students would place the largest number on the first line and the difference between the numbers seen as a comparison from the first term would give a positive difference, in this case 1. Some students were not sure about which was the largest number and were hence not in agreement about how the numerical system worked. The teacher said:

> I know what your problem is, and it was stupid of me not having considered this before. We have to find out which of the two numbers (-1) and (-2) is the biggest number (Runesson *et al.*, in press).

To the question of which number is the largest number, -1 or -2, many students in lesson 4 answered (-2). Why? Those students probably believed that, starting from the point zero on the number line, the positive numbers 'become larger the farther to the right hand side you get' and the negative numbers get larger 'the farther to the left hand side from zero you get'. The teacher was first not aware that all the students agreed that the number gets larger the farther to the right side of a number line they are, hence (-2) and 1 are larger numbers than (-18). This feature of the *numerical system* was identified to be a fourth critical feature for student learning.

Learning outcomes and critical features identified

Analysis of the pre and post test showed that students participating in lesson 4 were more successful (see table 3). Note that the students in lesson 4 were in grade 8 compared to students in lessons 1 and 2, who were in grade 7. However, the pre test show similar results for lessons 1 and 4 compared to lesson 2, which had the highest pre test score. The analysis of tests show that the mean gain (the difference between post test and pre test) for the students in lesson 4 was much higher (+1.43) compared to lessons 1 (+0.53) and 2 (+0.71). The effect size[18] for lesson 4 was higher (1.57) than for lessons 1 (0.56) and 2 (0.66). It was

[18] Cohens d= $\dfrac{M_1 - M_2}{\sqrt{\dfrac{SD_1^2 + SD_2^2}{2}}}$

concluded from this study that, since student learning increased, the critical features implemented in the lesson had an impact on student learning.

Table 3. Re-analysis of the pre and post tests[19] from the learning study (Kullberg, in press; Maunula, 2006) from lessons 1, 2 and 4. Maximal points=3.

	Lesson 1 (N=17)			Lesson 2 (N=17)			Lesson 4 (N=21)		
	Mean	SD	Effect size	Mean	SD	Effect size	Mean	SD	Effect size
Pre test	0.82	0.88		1.35	1.22		0.81	0.98	
Post test	1.35	1.00	0.56	2.06	0.90	0.66	2.24	0.83	1.57

On the basis of differences in learning outcomes, the team identified four features that were assumed to be critical for learning addition and subtraction of negative numbers:

- The difference between the signs
- Seeing subtraction as a difference
- The perspective/commutative law does not apply in subtraction
- The numerical system

It is possible that more features could be found if the team had continued the study or if another study had been made. The features found in this learning study are to a larger extent connected to subtraction than addition. Considering variation theory, it is possible to experience the features of subtraction when they are contrasted with features of addition and it is therefore necessary to experience both simultaneously.

The findings in the light of previous research

In this section the critical findings are discussed in the light of previous research. Although students' difficulties with negative numbers are well documented

[19] Tasks (5-(-2)=, -5+(-2)=,-5-(-2)=)

(Kuchemann, 1981; Vlassis, 2004) not much is said about what is taught and what is learned about negative numbers in classrooms. What similarities and differences could be noticed between the findings of the learning study compared to other previous research? What support is there for the conjectures made about the critical features?

A study that distinguishes itself from the previous research reported in this chapter is Ball's (1993) study of her own practice. Ball investigated, posing similar questions as in a learning study, her own teaching of negative numbers to eight-year-old students. Metaphors are commonly used in teaching about negative numbers. Ball systematically reflected on the consequences of metaphors used for teaching about negative numbers for her students' learning. Through her investigations she came to conclusions about the implications of different metaphors, and found some metaphors more useful than others. She started with commonly used metaphors for teaching found in the CSMP material[20], i) magic peanuts, ii) money – debt, iii) frog on a number line, iv) game scoring and v) building floors. The first one she did not find appealing – "Whenever a magic peanut and a regular peanut are in Eli's pocket at the same time, they both disappear" (i.e. -1+1=0). Ball thought that the idea of opposite numbers cancelling out each other would contribute to a view of mathematics as mysterious and beyond reason. About using money or game metaphors she wrote at one point in her teacher journal:

> I don't like the money or game models right now because they both seem to fail to challenge kids' tendency to believe that negatives are the same as zero (owing someone five dollars – i.e., -5 – seem the same as having no money (p. 380) [...] With money, they seemed to avoid using negative numbers – maybe precisely because the representation entails quantity, not position (Ball, 1993, p. 382).

Ball found that some students thought the "lowest number" was zero and that some students focused only on the magnitude of a number, for instance 2 in -2 seemed less than 4 in -4. She made the conclusion that "Simultaneously understanding that -5 is, in one sense, more than -1 and, in another sense, less than -1 is at the heart of understanding negative numbers" (p. 379). Furthermore, Ball separated the *different meanings of signs* in her teaching by using different symbols for a negative number and subtraction. "The rationale for substituting the circumflex for the minus sign is to focus children on the idea of

[20] CSMP, The Comprehensive School Mathematics Program (Remillard, 1990).

a negative number as a *number*, not as an *operation* (i.e., subtraction) on a positive number" (p. 380). Ball found that "For subtraction, we could model its comparison sense, but not the sense in which subtraction is about 'taking away' " (p. 381) and, even if money and debt at first seem to be problematic, at some point it later turned out to be useful.

> I decided that I needed an 8-yearold's version of 'net worth' so as to focus the children on the inverse relationship between debt and money, on financial *state* rather than on *actions* of spending or getting money (Ball, 1993, p. 382).

Interestingly, the features that Ball found in her study of her own teaching about negative numbers to eight-year-old students were similar to the content specific critical features found in the learning study. She identified *the sign*, *subtraction as a difference* – a comparison between numbers, and *the numerical system* when she made the conclusion that students need to experience "that -5 is, in one sense, more than -1 and, in another sense, less than -1 is at the heart of understanding negative numbers" (p. 379). Despite the fact that the teachers in the learning study were not aware of Ball's findings, they found similar features to be critical for student understanding of negative numbers. One thing that differed between Ball's study and the learning study findings was that the teachers in the learning study also pointed out the feature of *the perspective* in subtraction to be critical for student learning.

A study by Altipmak and Ösdogan (2009) identified three difficulties experienced in teaching negative numbers in a study of previous research. They found that: i) the meaning of the numerical system and the direction and multitude of the number, ii) the meaning of arithmetic operations and iii) the meaning of the minus sign as important for teaching and students' learning. First, research shows that the distinction between the signs has been found to be a key to understanding negative numbers in a proper way. Historically, the symbol used for representing a negative number has shifted during different times; for example, in Hindu mathematics in about the year 1500, a dot was placed over the number to indicate negativity (Freudental, 1983). Today the symbol is a minus, and hence the sign for the operation subtraction and a negative number are the same. In Sweden it is common to use parentheses to point out that it is a negative number instead of the operation. However, research shows that students interpret the sign in their own ways or ignore the signs. Vlassis (2004) showed that some students in grade 8 in algebra did not separate the sign for subtraction from the sign for a negative number. "Based on the students' difficulties that are reported [...], we suggest that the minus sign plays a major role in the development of understanding and using negative numbers" (ibid, p. 471). Herscovics & Linchevski (1994) found a similar use of signs in a study of algebra and that some students even ignored the operation sign preceding the number in a subtraction (ibid, p. 73).

Kuchemann (1981) found that students that had not yet mastered negative numbers had the greatest difficulties with subtraction of two negative numbers, e.g. (-2) - (-5)=, which has three 'minus signs'. Most students tried to solve these tasks by making use of or inventing a rule, 'two minus make a plus'. Students were more successful in tasks where the rule could be applied easily, for instance when the signs come after each other, 8 – (-6). Other tasks, for instance (-6) – 3 and (-2) – (-5), were more difficult for the students, where only about 40 percent of 302 14-year-old students answered correctly in the two items[21]. Dominant wrong answers were to the first item, 3 or -3, and to the second item, 7 or -7. Kuchemann states that results of a fairly simple subtraction item, such as (+6 – +8 =), indicates that many secondary school children have a very limited understanding of subtraction (Kuchemann, 1981, p. 86). For instance, some students may not be aware of the fact that subtraction does not admit the commutative law. In conclusion, several studies support that students have difficulty separating the different meanings of the signs for subtraction and negative numbers and with subtraction as a mathematical operation.

It has been found that some students confuse decimal numbers and also fractions with negative numbers. This was shown by the fact that students placed decimal numbers to the left of zero on the number line (Stacey *et al.*, 2001). For instance, Stacey *et al.* (2001) found in a decimal comparison test that 0.22 was seen by some students as a smaller number than 0. The rational for this according to Stacey *et al.* is the conceptual metaphors used – students do not recognize the difference between positive and negative numbers being mirrored around 0, but decimals and fractions around 1 (a whole). For instance, 0.6 is less than 1 (whole) but a larger number than 0. The following excerpt shows an example of this confusion of a student of 'the whole' and negative numbers.

> I know 0.6 is a portion of one. I may have been thinking along the lines of 0.6 is less than the whole number zero. Is zero a whole number? I don't even know….

[21] In Kuchemann's text the sign for a negative number is not in parentheses but instead placed a little higher than the operational sign for subtraction.

> I'm looking at whole numbers as being positives and decimals as being negatives …. decimals aren't, they are just fraction amounts (Stacey *et al.*, 2001, p. 221).

This was also found in the study reported in chapter 6, where one student wanted to place 0.5 on the left hand side of the zero (lesson 2B). The fact that, the farther on the right hand side along the number line a number is the larger the number, is a feature of the number line that students seem to experience differently (Kullberg, in press). A feature that is not described in the research reported here but was identified in the learning study is *the perspective* in subtraction. A deep understanding of subtraction that some researchers suggest could entail this feature but it is not made explicit. Another important fact is that, in the learning study, the students had the opportunity to experience the critical features simultaneously and not over a period of several lessons.

In conclusion, it is important to note that the critical features in the two learning studies were seen as critical for a particular group of students and that there could be other features not identified by the teachers. Four critical features were identified in the learning studies reported, although there is no fixed number of critical features to be identified, since critical features may vary between groups of students (Marton *et al.*, 2004).

CHAPTER FOUR

FROM DATA PRODUCTION TO RESULTS

As stated in chapter 1, the aim of the research reported here is to answer questions relating to a model of description for teaching and learning, above all *critical features*. An empirical study was thus set up to explore and analyse how teachers make use of critical features in lessons to achieve student learning. In this case the critical features were derived in advance, from completed learning studies in mathematics. The critical features are explored in two regards, first as a resource for teachers and second with respect to student learning. Chapter 2 described the theoretical base of the thesis, variation theory, which was used as a tool to analyse teaching and learning. Chapter 3 illustrated the background to the studies, two learning studies and the findings. In this chapter I show the design of the studies conducted and discuss choices made concerning selection, data collection and analysis. Note that the design of these studies differs from the learning studies. Validity and reliability of the study as well as ethical considerations are elaborated at the end of the chapter.

Design of the study

To explore how teachers make use of *critical features* in lessons two teaching experiments with a quasi-experimental design were set up to investigate what difference critical features made. The rationale for making two studies was that these would contribute to a specific as well as a more general analysis of how teachers make use of critical features and its implications. The studies were designed with the intention to test and compare lessons with different conditions in terms of the critical features implemented. In this sense the studies have a deductive approach (Bryman, 2004, p. 216). What difference critical features made for student learning and whether these could be used by other teachers to promote student learning were tested.

Two learning studies in mathematics were chosen, one about addition and subtraction of negative numbers and one about the density of rational numbers to be replicated, in terms of critical features found. The choice of mathematical topic depended on the fact that these learning studies had shown particularly good results in terms of student learning outcomes (Kullberg, 2004; Kullberg, 2007). The replication involved implementation of the same critical features in several lessons, but with new teachers and students. The studies were carried out during the period 2006 to 2008 together with a total of eight teachers and 16 groups of students. The total amount of empirical data consists of sixteen video recorded lessons (about 16 hours), eight video recorded meetings with teachers (about 16 hours), five stimulated recall interviews with teachers (about eight hours)[22] and student written pre and post tests.

As explained earlier the two studies explored how teachers made use of critical features and what difference different conditions, in terms of critical features, made in the lessons. In each of the two studies, four teachers implemented two lesson designs (see figure 11).

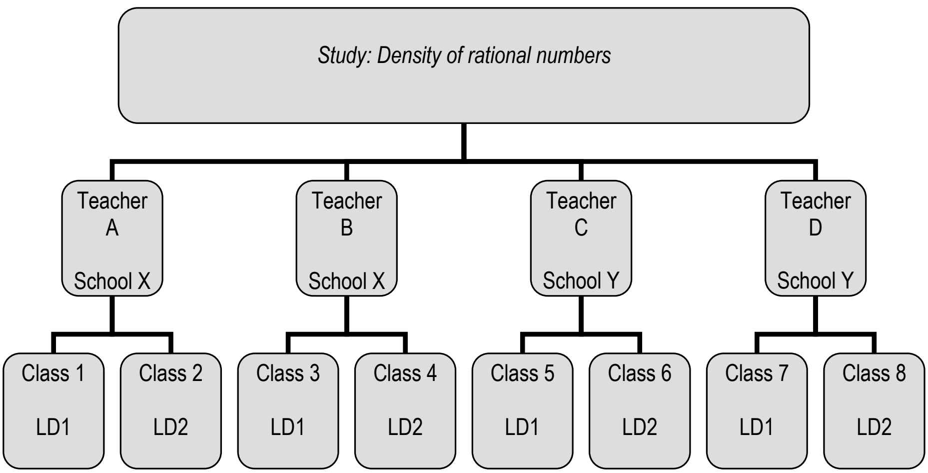

Figure 11. The eight lessons implemented and lesson designs 1 and 2 (LD1 and LD2) in the study of rational numbers.

This implied that the same teacher implemented two different lesson designs in order to make it possible to 'separate the teacher from the teaching'. Lesson design 1 (LD1) and lesson design 2 (LD2) differed in terms of which critical

[22] Stimulated recall interviews were used as a background and provided information to the analyses made.

features were planned to be implemented in the lessons. Furthermore, the lessons were planned jointly; two teachers from the same school planned the lessons in detail with the aim to implement the critical features in a lesson design in a similar way. For instance, teachers A and B first planned a lesson with lesson design 1 together with the researcher and later a lesson with lesson design 2.

A point of departure in this thesis is that teaching could systematically be tried out in practice. In this way the studies share similarities with design experiments (see for instance Brown, 1992; Cobb *et al.*, 2003; Collins, 1992). Schoenfeld (2006) defines design experiments in a broad sense. He writes "there are times when one has to create something to explore its properties. The act of creation is one of design. If the creation is done with an eye toward the systematic generation and examination of data and refinement of theory, the result may be considered a design experiment" (p. 193). In this thesis lesson designs with specific critical features are tested to explore how teachers used and implemented critical features in lessons and the implications for student learning. The studies differ from design experiments since it is not an ongoing revision of lessons but instead a quasi-experimental test of lesson designs implemented.

However, a qualitative analysis of lessons is also made in these studies and what actually happened in the classrooms has been taken into account. This implies in some cases a breakdown in the experimental design since what was intended was not enacted. The qualitative analysis thus made it necessary for some lessons to be moved to another category (lesson design).

Setting and selection

The selection of teachers was made on the rationale that they were somewhat familiar with variation theory. The teachers' previous experiences of learning study and variation theory contributed to the researcher and the teachers sharing a 'common language' and the meetings with the teachers therefore became focused and explicit in terms of planning the lessons. Seven of the eight teachers had worked in a learning study project. The teachers each had about ten years of working experience and they had voluntarily chosen to participate in the study. The two compulsory schools in which the teachers worked were located in different garden suburban areas of a large Swedish city. The students had similar socio-economic and socio-culture backgrounds, which implies in both schools few students with other ethnic backgrounds than Swedish. Both schools are similar in size and have the same grades, 0 to 9 (six year old to 15 year old students). The principals of both schools had a positive view of the study and supported their teachers with compensation in working hours for the time that the teachers participated in the study.

A total of 247 students participated in the studies. The studies were conducted in the same grade as the learning studies; in the study about negative numbers students attended 7th grade (13 year old students) and in the study about rational numbers students were in 5th and 6th grades (11 and 12 years old). The students had written consent from their parents to participate. In most classes there were approximately one or two students who were not allowed to participate. In these cases the students took part in the lesson but were not video recorded and their tests were not used. Only students who had been present in all parts of the study; pre test, video recorded lesson and the post test, were part of the data. In this study the students in school Y were ordinary classes compared to the students in school X where two (of three) classes were used to make three groups. The reason for this was that the research design needed four groups or classes for each study. The grouping of students was not made by random. As Bryman (2004) points out, "if the method used to select the sample is not random, there is a possibility that human judgement will affect the selection process, making some members of the population more likely to be selected than others" (ibid, 2004, p. 88). The rationale behind the grouping of students was that the groups should be as equal as possible, and the score from the pre test was therefore used to group students. In these groups there were also equal numbers of girls and boys. Statistical analyses of mean differences, ANOVA, show no significant difference between the classes/groups in the pre test results.

Procedure

The meetings

Before each study started the researcher met the teachers to give information about the study. The researcher thus met with the teachers for two hour meetings before the implementation of each lesson design. During these meetings video recorded lessons from the learning studies were watched and critical features in those lessons were discussed. This was done for the purpose of giving the teachers a better understanding of the critical features, since in most cases teachers were not aware of them. The lessons were carefully planned in collaboration during the meetings, so that the two teachers would handle the object of learning in a similar way in the classroom. The rationale for this was that a specific method of teaching or organising teaching should not differ within the teacher teams. They thus had a joint lesson plan and used the same tasks and implemented the critical features in the same order. The role of the researcher in the meetings was to highlight and discuss the critical features and to give support in the planning of the lessons. The meetings were video recorded to give support to analysis of the lessons.

At the end of each study the researcher met the teachers[23] individually for stimulated recall interviews. The interviews with each teacher lasted about one and a half hours. These interviews were conducted for two reasons, for research purposes and for ethical reasons. After the first study conducted (about negative numbers, school X) the researcher found it ethical to give both teachers an opportunity to explain their actions and to offer an analysis of the lesson.

Video recordings of the lessons

The observations made in the lessons were documented by video recordings. The choice of method was made by the rationale that it would generate reliable data that could be analysed repeatedly and by other researchers as well. The recordings were made with a digital video camera (Canon pal MV 500i), an external microphone (Sony ECM-ZS90) and a tripod. In all cases except one the sound and picture from the classroom were of good quality. In one case there was a technical error (Class G) for about ten minutes when no sound was picked up. Fortunately most of that period was during the students' individual work in their seats and the researcher's handwritten notes could be used to support the analysis of this part of the lesson. The duration of the lessons was from 40 to 70 minutes (see table 4).

[23] The stimulated recall interviews were offered to the teachers voluntarily.

Table 4. The duration of the video recorded lessons.

Study	Teacher	School	Lesson	Lesson duration
Rational numbers	A	x	1A	about 47 minutes
	B	x	1B	about 45 minutes
	C	y	1C	about 49 minutes
	D	y	1D	about 35 minutes
	A	x	1E	about 49 minutes
	B	x	1F	about 44 minutes
	C	y	1G	about 47 minutes
	D	y	1H	about 39 minutes
Negative numbers	E	x	2A	about 45 minutes
	F	x	2B	about 55 minutes
	G	y	2C	about 53 minutes
	H	y	2D	about 45 minutes
	E	x	2E	about 58 minutes
	F	x	2F	about 51 minutes
	G	y	2G	about 55 minutes
	H	y	2H	about 70 minutes

The camera was positioned at the back of the classroom and pointed primarily to the front of the classroom and the teacher. This choice was based on the interest of the study, namely how teachers enacted the critical features in the lesson and hence was in the students' attention. During the time students worked in groups or alone with an exercise, the camera was directed at some of the students and not at the teacher. In a similar way, the Third International Mathematics and Science Study, TIMSS, used one camera to study teaching in classrooms in different countries (Stigler & Hiebert, 1999). However, with a one-camera approach it is not possible to capture the full complexity of a classroom, for example the interaction between students. However, it was possible to analyse the teaching as it was enacted in whole class discussions. The use of one camera in the TIMSS study was criticized by researchers in another international comparative study, The Learners' Perspective Study (Clarke, Emanuelsson, Jablonka, & Mok, 2006; Clarke, Keitel, & Shimizu, 2006). In the LPS project the researchers systematically video recorded the students, *the learners*, instead of only the teachers or the teaching. In the LPS project three cameras were used to capture the teacher, the whole class and pairs of students. Using three cameras gives the researcher more information from different perspectives, but in fact all classrooms studies suffer from not being able to give a fair picture of the complexity of classroom practice. For instance, one difficulty is to get information about, to observe and to notice in classrooms the teachers' (as well as the students') intentions. Lortie (1975) writes that this phenomenon makes the difference between being a teacher and seeing a teacher act. On what grounds does the teacher act in a particular way in the classroom? It might thus not be the number of cameras in a classroom that is of the greatest importance, but instead gaining more information about the intentions, what was intended and what was enacted in the classroom. Nevertheless, it must be recognised that important information and data from the classroom could in fact be lost by using a one-camera approach. The use of a camera could also affect the teacher's and the students' behaviour in the classroom.

Transcriptions

Large parts of the video recorded lessons were transcribed verbatim in Swedish. The transcriptions were used as a means to analyse the lessons together with video recorded lessons. The selection of parts to be transcribed was made with regard to the significance of analyses; hence parts when students worked individually, in pairs or in groups were not transcribed owing to the focus and limitation of the present study. The parts used in the excerpts in the thesis have been translated to English. Other notions of implicit meaning such as gestures, facial expressions, tone of voice and posture are generally not transcribed, but only occasionally when it appeared to contribute to the interpretation. The recordings of the meetings were not transcribed. In the transcriptions the symbol (…) is used to show a three second pause while the symbol / is used for interrupted talk and // for overlapping talk (see figure 12). Cases of inaudible sound are also explicit in the excerpts. Ericsson (2006) writes that no transcript is ever complete and the transcription is equally influenced by theoretical presuppositions as by other choices made in the research process. The reliability of the transcriptions has been promoted by special software (Inqscribe) that facilitated the repetition and the speed of the sound.

(…) pause 3 seconds

/ interrupted talk

// overlapping talk

/…/ removed talk

[ ] clarifications

text on whiteboard $\boxed{}$

Figure 12. Symbols used in transcriptions.

Written tests

In this study tests were used to compare student learning before and after the lessons. The tests consisted of pre and post tests and, in one of the two studies, a delayed post test[24]. The pre test was conducted a week before the first meeting with the teachers and the post test was conducted two days after the video recorded lesson. The delayed post test was conducted seven to eight weeks after the lesson. It is important to note that only the results of the students that participated in pre test, the lesson and the post test are part of the collected data. The number of students who participated in the study is thus lower than the actual number of students participating in the lessons. The test took about 30 minutes to complete, although the students were allowed to work as long as needed. There was no teaching about the specific topic between test occasions, except for the video recorded lesson.

The tests were compared to previous results of learning studies[25] (about the same objects of learning) and some of the tasks were similar. The test items were designed to investigate the learning of the object of learning and used several tasks to look into the same capability to increase reliability. The tests were discussed and revised with the help of peers through face validity. The test that dealt with rational numbers was tested before it was used in one class in grade 6, which did not participate in the study, and was revised after analyses. The pre, post and delayed post tests in this study consisted of the same tests (see appendix). The researcher conducted the tests with the students in their schools

24 Delayed post test was only used in the study about negative numbers.

25 Learning studies conducted in the project The Pedagogy of Learning, from 2003 to 2006.

to make sure that all students were given the same instructions and help while taking the test. In a few cases a teacher managed the test the following day for a single student who was sick the day of the test.

The test items were selected to cover more features, for instance about rational numbers, than the specific object of learning. In the study of rational numbers three items in particular were designed to test students' experience of the density of rational numbers, 3b, 5 and 6b[26]. It should be recognized that the test items could have affected how the students answered. The question used affects how the students' answers. Therefore, different intervals, e.g. [0.5; 0.6 and 0.97; 0.98], and different types of questions were used (5 and 6b). Asking repeatedly for the amount of numbers in an interval could also affect students. However, the test was the same for all student groups and the repeated questioning for the same capability would affect all groups in the same way. In the study about negative numbers, the test was also designed to have several items for the same type of task but the test only treated addition and subtraction of negative numbers. However, some tasks required a careful reading, e.g. items 3 and 4 (see appendix 4).

It should be recognised that students' ability to understand different tasks and their skills in writing and articulating their answers are different and that this could have an impact on the result. Students' motivation in taking the test can also differ between students and classes. In this study the tests are used as indicators of which features of the learning objective have been noticed in the lesson. If many students have changed their answers in a particular task from the pre to post test, this could be an indicator of what it was possible for them to discern.

Data analysis

Analysis of lessons

The unit of analysis in the studies is single lessons. Hiebert & Grouws (2007, p. 377) argue that a lesson has the advantage of being large enough to include key

26 3b) Are there numbers between 0.5 and 0.6? 5) Ann says there is a number between 0.97 and 0.98. John says there is no such number. Who is right and why? Explain 6) Write four numbers that are larger than 0.99 but smaller than 1.1. Are there more numbers in between? If so, how many?

interactions among teaching features and small enough to be thoroughly analysable. The microanalysis of lessons in this thesis focused on how the object of learning was handled. This is expressed in terms of patterns of variation of critical features for each object of learning, although features of the content that were not identified as critical are reported as well.

The tool for analysis is variation theory described in chapter 2 and the theoretical concepts of discernment, object of learning, critical features and dimensions of variation. The lessons are analysed according to three commensurable terms: *the intended*, *enacted* and *lived object* of learning (see chapter 2). The *intended object of learning* was known since it is the capability that the students are intended to learn. The analysis shows i) what it was possible to learn during a particular lesson, ii) how the teacher handled the mathematical content in the lesson and iii) what the students contributed in terms of questions asked and statements concerning the content. The analyses were made by watching the video recorded lessons, repeatedly focusing on what it was possible for students to discern in the view of the researcher – *the enacted object of learning*. An analysis of what the students actually learned – *the lived object of learning* – is presented for each study and is described in the following section.

Analysis of tests

In addition to a qualitative analysis of the tests, SPSS was used to arrange and analyse the data. Statistical measure, ANOVA (analysis of variance), was used to explore significant mean differences between groups, for instance in the pre test. Descriptive measures, standard deviation, mean and effect size (Cohens d),[27] were used to describe differences between groups (see Hattie, 2009). Effect size is not dependent on a specific number of students (n) and can be used to compare classes/groups, taking into account the variance within the groups. Hattie (1999) studied the general effects of schooling in meta-analyses and found that the average effect size of interventions in schools improves achievement by about 0.4 of a standard deviation. Hattie (1999) states that the typical effect size of 0.4 does not imply placing a teacher in front of a class, since an attempt to deliberately change, improve, plan modify or innovate is needed. An effect size of approximately 0.20 is considered small, whereas 0.50 is medium and 0.80 large.

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[27] Cohens d= $\dfrac{M_1 - M_2}{\sqrt{\dfrac{SD_1^2 + SD_2^2}{2}}}$

Ethics

This study followed the ethics guidelines of the Swedish Council of Scientific Research. The participants voluntary chose to take part in the study and students had written consent from their guardians. All participants are kept anonymous, and the names of persons in this study were changed to protect individuals. This study would not have been possible to carry out without these teachers, who invested both time and effort. An ethical issue is that the teachers could in fact recognise the analysis of their own teaching. It is not possible, to give a fair picture of these teachers' ordinary teaching or teaching style from a single lesson. However, this is not the purpose of the study. The interest is instead in how the teachers made use of critical features and what the students learned.

In this study two lesson designs were implemented in different classes. The question of whether it is ethical with respect to students to test two designs that had shown different learning outcomes in previous studies was discussed by email with a regional member of an ethics committee. This member was positive to the study and could not see an ethical problem since it was an empirical question not yet fully explored. The record of video recorded lessons has been reported to the University of Gothenburg.

Validity and reliability

In terms of the quality of the research it is essential to raise questions about validity and reliability. A method or procedure for conducting research is never valid in itself (Maxwell, 2002). According to Brinberg and Mc Grath (1985) "Validity is not a commodity that can be purchased with techniques […] Rather, validity is like integrity, character and quality, to be assessed relative to purpose and circumstances (ibid, p.13)". Can the research questions raised in this thesis be answered with this choice of method and the way in which the study was carried out? I would argue that to be able to carry out research on teaching it is necessary to analyse teaching as it is enacted in classrooms. In this study the research was set up to explore how teachers make use of critical features in lessons. The research design was quasi-experimental and systematic to bring the impact and use of the critical features to the fore of both the researcher's and the teachers' attention.

With regard to reliability the researcher aimed to be as thorough as possible in the collection of data, interviews and test analyses. Different sources of data, e.g. video recorded lessons and tests, also contributed to deeper descriptions and analyses. Repeated analyses of tests and lessons contributed to a higher accuracy. Excerpts were selected from 16 hours of video recorded lessons to illustrate how the teacher together with the students enacted critical features in the lessons. This selection was made by the researcher and represents only parts of the lessons. However, the intention has been to provide the reader with many excerpts to present as whole a picture as possible of how the object of learning was handled in the classroom and how the critical features were enacted. The rationale is to 'follow data' and show the outcomes of a lesson design as it was enacted. It has also been important to analyse both the lesson as a whole and details to provide a reliable analysis. In terms of the student tests it is important to note that the pre and post tests were the same and that the results could be affected by repeated testing. However, it is not the test results *per se* that are of the greatest importance in this study but rather the differences between groups.

Kilpatrick (1993) argues for the following criteria for quality of research in mathematics education: relevance, validity, objectivity, originality, rigor and precision, predictability and reproducibility. With regard to the relevance of research I would argue that the present research has what Nuthall (2004) refers to as pragmatic validity, since it is aimed to have implications for teachers and teaching. If research is to have credibility for teachers, a teacher should be able to trust, with some certainty, that the results will hold up in different contexts. "It would be unethical to expect a teacher to use a new method if the results of research on that method had never been independently replicated. It would be similarly unethical to expect a teacher to believe a theoretical explanation of the teaching process if that explanation had been based on results of a single, unreplicated study" (Nuthall, 2005, p. 900).

Another criterion of quality brought up by Kilpatrick is objectivity while conducting research. To what extent the researcher is objective, and for instance does not favour a certain outcome and implication is up to the judgement of the readers. Moreover, with respect to research results being predictable, Kilpatrick states that, even if not possible to fully predict, it is reasonable that within certain limits it would be likely that a similar approach would generate similar results.

> Control of the phenomena of teaching and learning mathematics, in a way the growth of bacteria are controlled in a biological laboratory, was perhaps never a sensible goal for research, but prediction remains important. One would like to be able to predict, within limits, how children will respond to a task or what

> difficulties teachers will encounter when explaining a concept (Kilpatrick, 1993, p. 27).

This study tested how teachers make use of previously identified critical features in lessons and what the implications for student learning were. In this sense the predictability of the research results is both tested and generated through the research.

CHAPTER FIVE

A STUDY OF TEACHING AND LEARNING OF THE DENSITY OF RATIONAL NUMBERS

This chapter reports results of an intervention, a teaching experiment, about the teaching and learning of density of rational numbers. The background is a research project on learning studies conducted at the University of Gothenburg. The study is a follow–up of the project and is designed to address questions about the extent to which teachers can make use of findings produced in new contexts and situations and whether there is support for the conjectures made in the learning study. The study also tests the mode of communication used, the *critical features* and the theory of variation.

The intervention

The methods used to produce data in the study were described and choices made for the research were elaborated in chapter 4. The present chapter reports the results of the teaching experiment concerning learning about density of rational numbers. In the study, two lesson designs were implemented by four experienced mathematics teachers from two different schools, in eight different classes in grades 5 and 6 with a total of 113 students (see figure 11 chapter 4). Two teachers from the same school worked in a team with the researcher and developed and implemented lesson plans based on two lesson designs[28]. Data from six video recorded meetings[29], eight video recorded lessons, 262 student tests and four video recorded stimulated recall interviews[30] were used to analyse the intended, the enacted and the lived object of learning. The analysis of the

[28] School X participated in the study in the autumn of 2006 and school Y in the spring of 2008.

[29] The video recordings of the meetings were used in this study as background material and not explicitly used in the analysis.

[30] The video recorded stimulated recall interviews were used to explore how the teachers experienced the enacted object of learning and the implemented critical features. During the interviews, which lasted about one and a half hours, the teachers commented on the two lessons.

lessons was made on a micro level and focused on what it was possible to discern during the implemented lessons.

The lessons were organised in the same way and lasted about 50 minutes. They started with an introduction, continued with a group activity and ended with an oral account and discussion of the activity. The tasks and activities used by the two teacher teams were somewhat different, however. Teachers A and B chose a group activity, while teachers C and D selected an individual worksheet followed by a comparison and discussion of the worksheet in pairs. The analysis focused on how the teacher handled the content in the classroom. The planned difference between lesson designs was content specific critical features that were identified in a previous empirical study as being critical for student learning about density of decimal numbers described in chapter 3 (cf., Kullberg, 2004; Kullberg, 2007a).

- Lesson design 1 (LD1) included the critical feature of 'decimal numbers as points on a line'.
- Lesson design 2 (LD2) included the critical features 'decimal numbers as points on a line', the 'interchangeable representation', a number as 'a part of a whole' and 'divisibility'.

It is important to note that the lesson designs only imply an implementation of critical features and do not include organisation, tasks and activities for teaching.

Findings

Intended, enacted and lived object of learning

This section discusses the analysis of how the object of learning, the understanding of density of rational numbers, was handled during the lessons. Lessons 1A, 1B, 1C and 1D were intended to implement the critical feature in lesson design 1 (LD1). Lessons 1E, 1F, 1G and 1H were intended to implement the critical features in lesson design 2 (LD2). The analyses show that LD1 was implemented as planned and that the critical feature of *numbers as a point on a line* was enacted (see table 5). However, in the following analysis, I argue that the intended and the enacted lesson designs did not coincide in some cases. In consequence, the intended object of learning differed from the enacted object of learning. As shown in table 5, LD2 was enacted in the classroom in two of the lessons, 1E and 1F, but not in full in 1G and 1H. One explanation might be that the teacher forgot a particular feature intended to be enacted in the lesson[31]. Another explanation is that the researcher did not communicate a particular critical feature well to the teacher, which could have had an impact on the critical features that were implemented in the lesson. Whatever the explanation, the lessons were analysed as enacted in the classroom. Consequently, lessons 1G and 1H are reported under a third heading, "lesson design 3, LD3" and lessons 1E and 1F under LD2 (see table 5).

Table 5. The enacted lesson design and the critical features implemented in lessons 1A to 1H. Note that the lessons were implemented in grade 5 in the Y school due to the ceiling effect on pre test in grade 6.

Lesson Design	Lesson	Teacher	N	Numbers as points on a line	Interchange-able representation	Part of a whole	Divisibility
LD1	1A	A	19	✓			
LD1	1B	B	13	✓			
LD1	1C	C	16	✓			
LD1	1D	D	15	✓			
LD2	1E	B	12	✓	✓	✓	✓
LD2	1F	C	12	✓	✓	✓	✓
LD3	1G	A	13	✓	✓	✓	
LD3	1H	D	13	✓	✓	✓	

[31] The critical features of "part of a whole" and "divisibility" were presented to the teachers as two parts of one critical feature (part of a whole). This could have had an impact since both teachers did not implement one of these parts, "divisibility". The stimulated recall interviews held about three weeks after the lessons did not show that the teachers thought that they had missed a feature. This suggests that the teachers had not discerned the feature of divisibility, probably due to miscommunication. Note that these two teachers worked in different schools.

An important finding in this study is the connections that are seen between learning and teaching (cf., Chik, 2006). The analysis showed that the enacted object of learning is reflected in student learning, in the lived object of learning. The analysis also shows that, when all critical features were not enacted in full, as in LD1 and LD3, the effect on students' learning concerning density of rational numbers was small. However, when all critical features were enacted in the classroom, as in LD2, the effect on student learning was significant. It is shown that the divisibility of numbers had a decisive difference for student learning about density of rational numbers.

The difference in learning outcomes between the classes is only shown in tasks related to the object of learning. The statistical analysis, ANOVA, showed no significant mean difference in the whole test between the enacted lesson designs. However, a significant mean difference was found between LD2 and LD1 and LD3 at the 0.01 level regarding tasks about density of rational numbers. The results of student pre and post tests (whole test) showed a medium effect size (Cohens d) for LD1, d= 0.40, LD2, d= 0.62 and LD3, d= 0.38 (see table 6). The effect size of LD2 is higher as compared to LD1 and LD3. However, the largest difference in results is found in the analysis of the tasks concerning density. The effect size from these test items shows that LD2 had a high level, d= 1.55, as compared to LD1, d= 0.59, and LD3, d= 0.53.

Table 6. Results of the pre and post test. Whole test (W), maximal points= 14, tasks concerning density (D), maximal points= 3.

	Lesson design1 (N=63)			Lesson design 2 (N=24)			Lesson design 3 (N=26)		
Test	Mean	SD	Effect size	Mean	SD	Effect size	Mean	SD	Effect size
W Pretest	8.92	2.80		8.42	3.36		9.19	2.98	
W Posttest	10.06	2.94	0.40	10.46	3.27	0.62	10.35	3.06	0.38
D Pretest	1.02	1.06		1.00	1.25		1.38	1.17	
D Posttest	1.70	1.25	0.59	2.58	0.72	1.55	1.98	1.09	0.53

There were three tasks: i) Anne claims that there is a number between 0.97 and 0.98, John says there is no such number; who is right and why?"; ii) Are there numbers between 0.5 and 0.6? and a follow-up question to "Write 4 numbers that are larger than 0.99 but smaller than 1.1"; iii) Are there more numbers in between? If so, how many? Explain your reasoning. This tested students' knowledge about density of rational numbers. The rationale for using only three tasks to explore student learning was that the test itself should not be a learning situation. It was possible to analyse in the students' written explanations whether they thought that there was an infinite number, expressed by the students as "numerous", "many" or "infinitely many", of decimal numbers[32]. The students were asked to explain their answers and give examples. Students who did not answer correctly in these items answered for example that there were no numbers or ten numbers in the intervals. The test was used as an indicator of student learning of the lived object of learning. The results of the pre and post tests show that classes with LD2 improved the most. In these lessons the teachers succeeded in bringing out the critical features in the lesson.

Analysis of lessons

Lesson design 1

Lesson design 1 was used in four classes to explore the impact of one feature, 'decimal numbers as points on a line'. The following section shows how the teachers, together with the students, enacted the critical feature in LD1 in lessons 1A, 1B, 1C and 1D.

Critical feature 1: Decimal numbers as points on a number line

Seeing *a number as a point on a line* entails becoming aware that a number has a specific place, a point in relation to other numbers. Between two numbers, for instance 0.17 and 0.18, there are other numbers, other points. To elicit this feature, the teachers planned to make numbers in intervals explicit. During these lessons, different numbers in intervals between decimal numbers were therefore identified, for instance 0.171, 0.172, 0.173 etc.

Lesson 1A implemented LD1 and the feature of *decimal numbers as points on a number line*. At the beginning of the lesson, the teacher asked the students whether there are any numbers in the interval [2; 3]. The students gave many

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[32] Analysis of test question 3b was made in combination with task 5 in a few cases. This was done when a student had answered only yes in 3b but had given a fuller explanation in task 5.

answers, 2.3, 2.5, 2.7, 2.01, 2.6, 2.007 and 2.1 (see figure 13). Even though seven examples were given, it was possible to experience that there could be more numbers than nine[33] in the interval [2; 3] since decimal numbers with different amounts of digits were presented at the same time, for instance 2.3 and 2.007. On the other hand, the fact that there were infinitely many numbers was not made explicit.

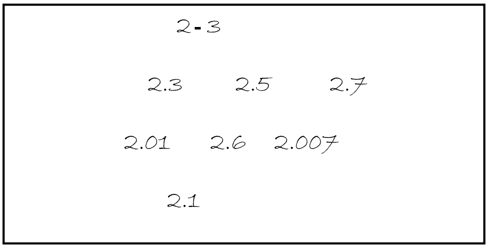

Figure 13. Examples (*range of change*) of numbers in the interval [2; 3] presented on the whiteboard in lesson 1A.

The next question the teacher asked was whether there are numbers in another interval [0.17; 0.18.]. The students named three numbers, 0.177, 0.1701 and 0.1702 (see figure 14). The 'range of change' in the three examples opened a possibility to experience that the decimal numbers could have at least four decimal digits. Note that the number line was not used to illustrate the order of numbers, although this was later done by the teacher and students during a group activity.

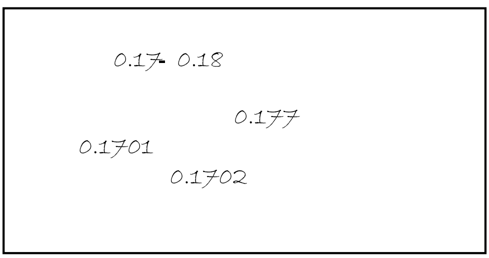

Figure 14. Examples (*range of change*) of numbers in the interval [0.17; 0.18] presented on the whiteboard in lesson 1A.

[33] 2.1, 2.2, 2.3 etc. to 2.9.

After the introduction, the students were asked to discuss two questions in groups: "How many numbers are there between 0.17 and 0.18?" and "Are there more numbers between 1 and 2 than between 0.17 and 0.18?". The teacher gave three answers to consider: there are A) fewer numbers between 0.17 and 0.18 than between 1 and 2, B) more numbers between 0.17 and 0.18 than between 1 and 2, and C) an equal amount of numbers between 0.17 and 0.18 as there is between 1 and 2. The task was designed to enhance a discussion about density and the fact that there are infinitely many numbers in any interval of rational numbers.

During the oral discussion of the activity, the answers were presented on posters. The following excerpts show how one student (Matilda) claims that there are equally many numbers in the intervals since it is possible to put more digits after the last digit and that this makes it a larger number (excerpt 1, line 1). In this case, the number of digits placed after one another 'horizontally' is most likely seen as an indicator of a higher value. Putting more digits after the last seems not to be sufficient to experience that there are infinitely many numbers in intervals, and the question remains unclear throughout the lesson. Another student (Ivan) argues that there are more numbers between 2 and 3 than between 0.17 and 0.18, even though he says there are "tremendously many" or "numerous" numbers between 0.17 and 0.18 (excerpt 1, line 5). The fact that there should be the same amount of numbers in a smaller interval seems puzzling to Ivan at this point. Another student says that "it must end sometime", a finite view that contradicts the idea that "you can go on as long as you want" (line 1) with the numbers expressed earlier (by Matilda).

Excerpt 1. [00:44:17 to 00:45:30]

1.	Matilda:	We believe it's C because there is an equal amount of numbers. Since you could always insert a number, you could always put a zero, you could always put one here at the end [pointing to the last digit in 0.1701 on the board] and it [the number] becomes bigger. You can go on as long as you want.
2.	Teacher:	There is no limit.
3.	Matilda:	No
		Some lines later.
4.	Teacher:	How many are there then? If I ask you, Ivan, how many are there between 0.17 and 0.18?
5.	Ivan:	Tremendously, tremendously many. There are numerous.

6. Teacher: Numerous

7. Ivan: Yes

8. Teacher: If there are numerously many between 0.17 and 0.18 then my question is: are there more between 2 and 3? Because I was thinking like this, how much is more than numerous.

9. Student: It must end sometime.

Even though it was said that there are infinitely many numbers in intervals between numbers, the students did not have the possibility to experience that rational numbers are dense. The post test showed that few students improved in tasks having to do with density of rational numbers. Although the students had the possibility to experience decimal numbers in intervals and the numbers as points on a number line, this was not sufficient to experience density.

Lesson 1B implemented LD1 with the feature of seeing *a number as a point on a line* at the beginning of the lesson when the teacher together with the students discussed numbers in intervals and placed them on the number line. In the interval [1; 2] (see figure 15), it was possible to experience the following numbers: 1.25, 1.5, 1.75 and 1.99. In this case, it was possible to discern four numbers in the interval, although it is possible that students could experience that there were more numbers with two decimals in the interval [1; 2].

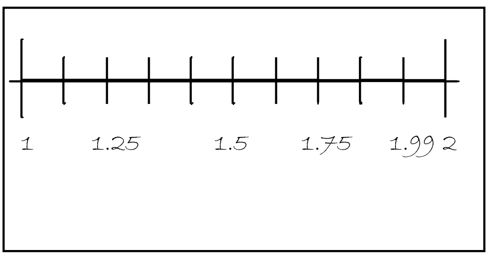

Figure 15. Examples (*range of change*) of numbers in the interval [1; 2] presented on the whiteboard in lesson 1B.

The posters from the student group activity showed that all students wrote examples of numbers in the interval [0.17; 0.18]. The following excerpt shows on the one hand how a student (Lisa) tries to explain that there are millions of "things" in the interval and that there are numbers "between" (line 1) numbers. On the other hand, the teacher promotes examples of numbers in the interval, as though the numbers were countable (lines 2, 5 and 8).

Excerpt 2 [00:45:59 to 00:47:24]

1.	Lisa:	Ho ho (laughs) There are a lot, millions of ”things” between this number and that number like this (points and draws a line between 0.17 and 0.18), but we have written some.
2.	Teacher:	You have written some, can you give some examples? It’s hard to see from a distance.
3.	Lisa:	A lot
4.	Student:	Say one
5.	Teacher:	Zero comma[34] something or
6.	Student:	Say the first
7.	Lisa:	Zero comma thousand (…) I’m not finished [referring to the task] (laughs).
8.	Teacher:	But you can read two or three [numbers] that you have written.
9.	Lisa:	Zero one seven hundred and five [refers to 0.1705]
10.	Teacher:	Ah
11.	Student:	Ah and then zero comma
12.	Student:	Zero comma a hundred seventy one [0.171]
13.	Student:	Like this
14.	Teacher:	Come on
15.	Lisa:	Ah and then there are between. After that here there are zero comma one two three comma one two three comma one two three comma (laughing)
16.	Teacher:	So one can continue a while [with the numbers]

At the end of the lesson the teacher summarised the groups’ oral account on the whiteboard (figure 16). It was possible to discern from this potential infinity (group C) or a finite number of numbers (groups A and B) in the interval between 0.17 and 0.18.

[34] The Swedish word for decimal point is ‘decimal comma’. In the excerpts I will therefore use the word decimal comma.

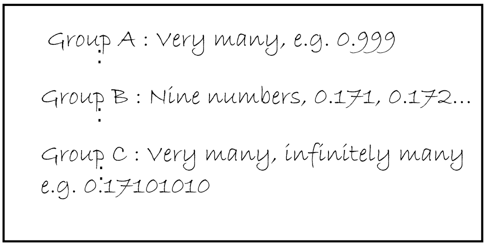

Figure 16. Three solutions (*range of change*) for the amount of numbers there are in the interval [0.17; 0.18] presented on the whiteboard in lesson 1B.

The teacher did not discuss which of the three solutions was more correct than the others; this was left for the students to discern themselves. Even if density is talked about, it is not possible for most students to discern that there are infinitely many numbers in an interval, and this is shown in the post test results. While the students had the possibility to experience numbers in intervals, this did not make it possible for most students to experience density.

Lesson 1C implemented LD1. In lesson 1C, the teacher discussed numbers in intervals, first [0; 100] and later [5; 6]. Students said (excerpt 3, line 2) that there were "hundredths", "millionths" (line 6) and/or "thousandths" (line 8) in the interval [5; 6]. The parts that the students suggested related to fractions and not to decimal numbers as points. However, the teacher responded that it would take too much time to split the number line into that many parts and s/he continued to ask for examples of numbers in the interval (line 12). The examples given were a finite number of numbers, 5.5, 5.1, 5.2, 5.3 and 5.4. The teacher continued by asking about numbers mid between 5.3 and 5.4 (line 18), and one number, 5.35, was presented. This opened a possibility to experience 'halves' between the numbers on the number line (see figure 17).

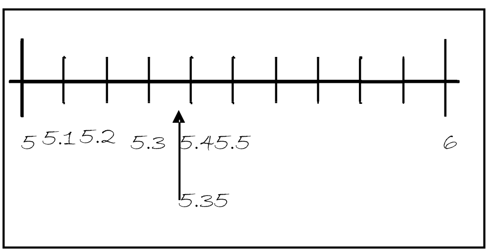

Figure 17. Examples (*range of change*) of numbers in the interval [5; 6] presented on the whiteboard in lesson 1C.

Excerpt 3 [00:03:22.01 to 00:05:17.01]

1.	Teacher:	What is there in between then? Victor
2.	Victor:	There are both (…) Hundredths are there.
3.	Teacher:	Hundredths are there, so you mean that I could split this [number line], put hundredths, wow then I'll be busy for a while.
4.	Victor:	Ah [Yes]
5.	Teacher:	What do you say Darin?
6.	Darin:	Millionths
7.	Teacher:	Millionths, wow
8.	Student:	Thousandths
9.	Student:	Then you're busy for a while.
10.	Teacher:	Ah then I am busy for a while. What do you say Kate?
11.	Kate:	One could split in half.
12.	Teacher:	One could split in half. What do we have here then? What do we have here then?
13.	Kate:	Five comma five [5.5]
		Some lines later.
14.	Kate:	One can [put] five comma one, five comma and so.
15.	Teacher:	Five comma one, five comma two, five comma wow now the space between them was not so good, five comma three and five comma four (…) If we look here then. What is (…) in between [5.3 and 5.4]?
16.	Student:	Five comma three and a half
17.	Teacher:	Five comma three and a half, how do you write that then. Five comma three (…)
18.	Student:	A little five like that [5.35].

After the introduction, the students first worked individually and then in pairs with tasks on a worksheet (see appendix 2). There was a discussion about the answers to the tasks at the end of the lesson. One student said that 5.5 and 5.50 are the same and the teacher wrote zeros behind the numbers on the number line representing numbers in the interval [5; 6], for instance 5.1 as 5.10. The reason for this was that a student said this would help to determine the size of the number, for example whether 5.40 is a bigger number than 5.35 (figure 18).

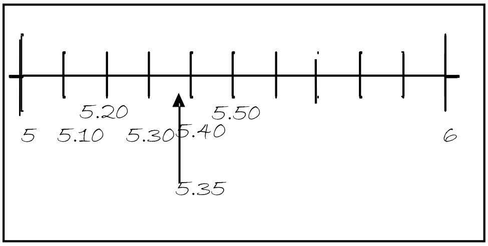

Figure 18. Examples (*range of change*) of numbers with two decimals in the interval [5; 6] presented on the whiteboard in lesson 1C.

At the end of the lesson, the teacher again discussed the number of numbers in the interval [5; 6]. The following excerpt shows one student who said there were billions of numbers in the interval and another (Jimmy) who said there were an infinite number. Jimmy's explanation for this line of reasoning was that it is possible to put digits at the end of the number and that this makes the number smaller. An alternative explanation could be that he is aware that the parts become smaller in terms of smaller parts the further away from the decimal point one goes. He said "the more like that you have, the smaller they are, there are numerous" (excerpt 4, line 6) and "the smaller it is, sometimes it is more parts, there are numerous amounts of numbers" (line 10).

Excerpt 4 [00:35:44 to 00:37:14]

1.	Teacher:	How many is it here then? Have we reached an answer?
2.	Student:	Billions it is then.
3.	Teacher:	Okay Jimmy
4.	Jimmy:	Infinite
5.	Teacher:	Infinite, okay. Can you prove it in any way or explain your thinking.
6.	Jimmy:	Ah, the more [digits] like that you have, the smaller they are, there are numerous.
7.	Teacher:	Say that aloud one more time so we can hear you (…) The
8.	Jimmy:	The more numbers there are like that decimal
9.	Teacher:	Ah more digits
10.	Jimmy:	The smaller it is, sometimes there are more parts, there are numerous amounts of numbers
11.	Teacher:	Okay, there are more parts you say, mm (…) mm. We'll move on to the next task, order these decimals by size.

During the whole class discussion, there was in most cases a consensus view of the results of the tasks from the point of view of the teacher. The excerpts from this lesson show that some students, for instance Jimmy (excerpt 4), has a notion about density although the discussion did not make it possible for other students to experience density. In summary, as in lessons 1A and 1B, density is discussed but the argument that it is possible to put zeros or other digits behind the last digit seems not to be sufficient to be able to experience that there would be an infinite number in an interval of decimal numbers. The post tests show that few students were aware that rational numbers are dense.

Lesson 1D, which was similar in many ways to lesson 1C, started with an introduction about numbers in intervals, [0; 1000], [0; 500], [0; 100], [0; 10] and [5; 6]. In the interval [5; 6] the students gave examples of numbers, 5.1, 5.5, 5.9, 5.4, 5.2 and 5.6 (see figure 19). It was possible to discern from these examples that there could be nine numbers in the interval [5; 6].

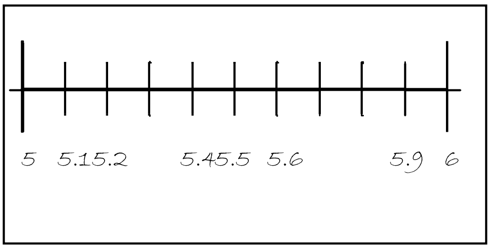

Figure 19. Examples (*range of change*) of numbers in the interval [5; 6] presented on the whiteboard in lesson 1D.

The teacher handed out a worksheet (see appendix 2) after the introduction. The students worked individually and later discussed their answers in pairs and with the whole class. In a similar way as in lesson 1C, some students said that 5.5, 5.6 and 5.4 were the same as 5.50, 5.60 and 5.40. The following excerpt illustrates the discussion of the number of numbers in the interval [5.55; 5.6]. A diversity of different answers (there are a million, numerous, one or 14 numbers in the interval) were discussed, but most likely without a possibility to experience density of rational numbers (lines 1 to 18).

Excerpt 5 [00:33:55 to 00:35:04]

1. Teacher: Now when we have been working a little while with decimals, how many decimals are there here between

for instance 5.55 and 5.6? How many can there be in between? Michael

2. Michael: A million

3. Teacher: A million. What else do we have? Andy

4. Andy: Infinite

5. Teacher: Infinite. What does John say?

6. John: Numerous.

7. Teacher: Numerous, Peter?

A few lines later

8. Peter: You don't have [enough] paper to be able to write all of them down.

9. Teacher: Ben

10. Ben: There could be one

11. Teacher: Could there be one, ah Clint

12. Clint: Numerous

13. Teacher: Numerous, Mike

14. Mike: Numerous

15. Teacher: Numerous. As a matter of fact there are numerous (…) numerous here in between because one can do as I did there, enlarging and enlarging and enlarging, and there are more and more decimals.

16. Student: We know exactly how many there are, 14.

17. Teacher: How did you come up with 14?

18. Student: I have no idea. I only guessed.

Although the teacher states that there are numerous amounts of numbers in the interval (line 15) and the explanation for this is that it is possible to "enlarge" the number line and that "there are more and more decimals", this was not sufficient to enhance student learning about the density of rational numbers. The post test shows that few students learned that a set of rational numbers is dense.

Other features of decimal numbers

It was possible in lessons 1C and 1D to experience other features of rational numbers than was intended when the worksheet was discussed (see appendix 2), for instance placing a value in ordering different decimal numbers with different amounts of digits (task 3). Although it was possible to see that 0.605 was placed before 0.61, the rationale for placing the decimal numbers in a certain order was not discussed and not likely possible to discern.

Lesson design 2

Lessons 1E and 1F used lesson design 2 (LD2), which included all of the critical features identified. Besides the feature of decimal *numbers as points on a line* presented in LD1, i) *interchangeable representation*, ii) *part of a whole* iii) and *divisibility*, were also enacted. The teachers planned to implement the critical features in the lesson and hence were sensitive to students' questions with regard to them. It was found that the first three features were enacted in similar ways in lessons 1E and 1F. However, the feature of divisibility was enacted in somewhat different ways. The interval between two decimal numbers was shown in lesson 1E to be repeatedly divisible, while the whole was repeatedly divisible in parts in lesson 1F. Even though the ways of experiencing the feature of divisibility were different, the analysis shows that the effect on student learning was similar. Note that the same teachers are teaching both lesson designs 1 and 2.

Critical feature 1: Decimal numbers as points on a number line

The critical feature was enacted in a similar way as in lessons 1A, 1B, 1C and 1D. All lessons in this study started with a question about numbers in different intervals.

Lesson 1E started with the question "Are there numbers between 0 and 1?" This question opened a variation in the number of numbers in the interval (see figure 20). One student (Henry) said that there were numerous amounts of numbers in the interval. His line of reasoning was that you could always put an extra digit after the comma. Another student (Ella) said that "you can put anything there as long as there is a comma" (excerpt 6, line 1).

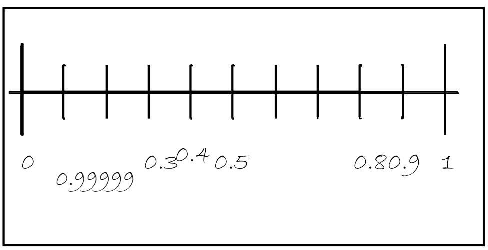

Figure 20. Examples (*range of change)* of numbers in the interval [0; 1] presented on the whiteboard in lesson 1E.

Even though the students talk about infinitely many numbers in the interval, several students are not sure at this point about exactly how many numbers there could be there. This became evident when one student asked “How many digits like that do you have to have to be able to reach one?” (excerpt 6, line 6). This question, it seemed, reoriented the students’ attention from the question about the amount of numbers in the interval. Many of them were not sure about which was the last number before reaching 1 – could it be 0.99 or 0.99999? The students focused on a finite set of digits, thus contradictory to reasoning about density. Furthermore, the questions “are there numbers in the interval” and “how many digits must one have before reaching one” created a variation of different possible solutions. Note that the one solution or answer that was more correct than the other was not determined.

Excerpt 6 [00:04:19 to 00:06:13]

1.	Ella:	But, you can put anything there as long as there is a comma [decimal point] and then when one has ninety-nine nine there as when.
		A line later.
2.	Ella:	They are wholes [whole numbers] and you can put as many numbers as you like [inaudible] between them.
3.	Teacher:	Okay between zero and one there are in fact numerous.
4.	Student:	// Ah because it is a comma. You put a comma.
		Two lines later.
5.	Teacher:	As long as you have a comma then you can fill up as many digits as you like [summarises the student’s answer].
6.	Student:	How many digits like that do you have to have to be able to reach one?
7.	Teacher:	Exciting (…) exactly how many of these ’comma digits’ is needed to reach one?
		Some lines later.
8.	Teacher:	Then after ninety-nine [refers to the student’s answer] you jump to one. Or do you ever reach one if you go on like that [putting decimals in the end]. Because you said that it was just to put more decimals all the time.

It could be seen from the discussion that, in this case, the students keep putting the digit nine after the decimal point to make it a larger number and closer to one. The excerpt shows that the numbers are treated as countable. On the one hand, one student (Ella) distinguishes between wholes and the digits on the other side of the decimal point (line 2) and says “that you could put as many numbers as you like between them" (the whole numbers). On the other hand, some students see a finite number of numbers in the interval.

Lesson 1F started with examples of numbers (0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and 0.9) in the interval [0; 1] from which it was possible to experience nine numbers in the interval. The teacher also asked about numbers in another interval [0.5; 0.6]. One student (line 1) said there are numerous numbers. The following excerpt shows that one student reasoned that "one can put as many zeros as one wants, it is still worth half (the same)". This statement shows an understanding that it is possible to add zeros, not explicitly other digits. When the teacher wanted to reason about it, s/he asked another student (Kajsa) and there was a shift of attention. Now, instead of talking about numerosity, examples of numbers in the interval were given, 0.51 and later 0.52, and 0.53 (line 3).

Excerpt 7 [00:08:58 to 00:09:11]

1.	Student:	There are numerous. One can put as many zeros as one wants, it [0.5] is still worth half.
2.	Teacher:	Okay, how do we know that what you are saying is true. Now we must reason a little. What do you say Kajsa.
3.	Kajsa:	One can add zero comma fifty-one [put 0.51 in between].

In this part of lesson 1F, it was possible to experience numbers in the interval [0.5; 0.6] with two decimals and the numbers are at this point treated as countable.

Critical feature 2: Interchangeable representation

The critical features were implemented in a sequence, starting with experiencing numbers as points followed by the second feature of *interchangeable representation*. Vosniadou *et al.* (2008) point out that, when students meet different representations of rational numbers, it differs from their experience of whole numbers, in which numbers had only one notation. The following parts show how the feature of interchangeable representation was enacted.

Lesson 1E enacted the feature of *interchangeable representation* by representing 0.30 as fractions, 3/10 and 30/100. Another decimal number, 0.29, was also represented by different fractions, as 29/100 and 2/10 + 9/100, and compared with 0.30. The following excerpt (lines 1 to 13) shows how the teacher asks for different examples of representations of the same number and compares the representation of two decimal numbers, 0.29 and 0.30 (see figure 21).

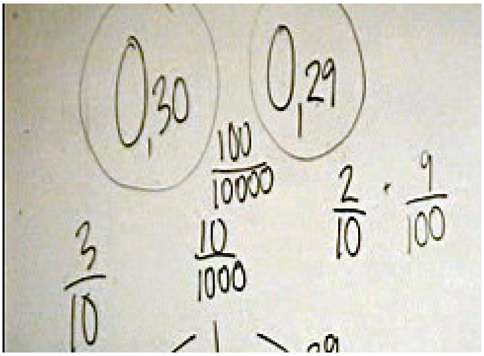

Figure 21. Representation of 0.29 and 0.30 by fractions: interval between the numbers expressed by fractions in lesson 1E.

Excerpt 8 [00:06:30 to 00:11:02]

1.	Teacher:	We can take these two numbers, 0.30 and 0.29; is it possible to say them in another way (…) instead of zero comma xxx [nonsense word] all the time. Can one express, if one should say (…), now I have this much. Can it [the numbers] be said in another language in mathematics or in another way in mathematics? Anne
2.	Anne:	Three tenths
3.	Teacher:	Three tenths would work. That you recognise right.
4.	Student:	Yes
5.	Student:	But three (…).
6.	Teacher:	I write it [the number] here. Are there more? Linda? (…) Anne has found a way I can see. Sara has not said so much (…) ah (…) Ella said that we could write three zero there and then it is thirty something. What do you think (…) Linda.
7.	Linda:	Thirty hundredths [30/100]
8.	Teacher:	Thirty hundredths, does that sound familiar to you?
		Some lines later.
9.	Heidi:	Maybe twenty-nine hundredths
10.	Teacher:	If that was thirty hundredths, then that is twenty-nine hundredths, are you with me.
		Some lines later.

11. Teacher: (…) Can we write it [0.29] in another way? I want to try this. You can check if it seems okay. If I say that it is 2 tenths and 9, what should we call/

12. Student: //Raise your hand [to another student]

13. Teacher: // hundredths (…) So could that [2/10 and 9/100] be the same as that one [29/100].

The fact that it is the same number represented by different notations was made explicit by the teacher's comments, for instance "So could that [2/10 and 9/100] be the same as that one [29/100]" (line 13).

Lesson 1F enacted the feature of *interchangeable representation* in a sequence when the teacher asked about the meaning of the digits in decimal numbers. "These numbers, zero comma one (0.1), zero comma two (0.2), what does it actually mean, zero comma one, do you know what these digits stand for?". The students gave two examples of fraction representation, one tenth and five tenths. The teacher followed this up by making a generalisation and said that, if 0.1 is one tenth and 0.5 is five tenths, then 0.6 would be six tenths. Later in the lesson, in a discussion of the worksheet, other numbers, for instance 0.7, 0.94 and 0.25, were written as different fractions on the whiteboard.

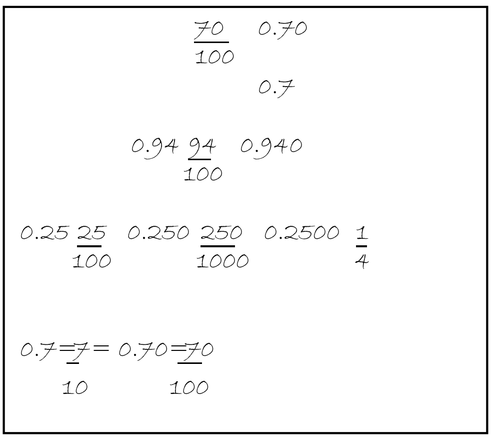

Figure 22. Examples (range of change) of representations of 0.7, 0.94 and 0.25 as fractions in lesson 1F.

It was possible to experience variations in representation, as at the beginning of the lesson, but this time with a wider 'range of change' (figure 22), including thousandths and ten thousandths, which most likely contributed to students' experience of divisibility.

Critical feature 3: Part of a whole

This section shows how the teachers together with the students in lessons 1E and 1F enacted the critical feature of *part of a whole*. The feature entails that a rational number, for instance 0.17 or 17/100, could be seen as a part of a whole – as 17 centimetres of a one-metre ruler. This feature is, as in the interchangeable representation, also related to the representation of rational numbers.

Lesson 1E enacted the feature in which the teacher in lesson 1E used a cucumber to illustrate that 0.29 and 0.30 could be seen as *parts of a whole*. The teacher asked, "Can we take away 0.30 from something?" (excerpt 9, line 1). The cucumber (the whole) was divided into ten parts, and 0.30 was seen as three parts of the cucumber. The teacher consistently made connections to fractions during the lesson, for instance when s/he said, "Okay, and because we also said it was three tenths and even said that one could say 30 hundredths, so we then have 30 tiny hundredths here in this piece of cucumber that I have also cut off" (line 5). The teacher talked simultaneously about 0.30 in fractions, as tenths, as hundredths and as a part of a whole. Even different parts, 0.29 and 0.30, were compared when the teacher said "If I take 0.29 of the cucumber, do I get a larger or smaller piece?" (line 7).

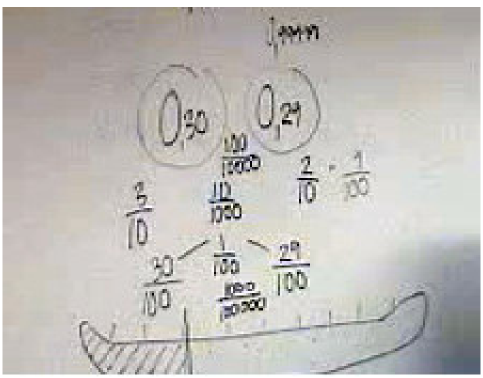

Figure 23. A cucumber is used to show 0.29 and 0.30 as parts of a whole. The cucumber is split into ten parts, and three of these parts together show 0.30 (shaded area) in lesson 1E.

Excerpt 9 [00:11:02 to 00:14:00]

1.	Teacher:	You said one fifth and so, can one take three tenths of something. Yes, what do I have here (…) I have my giant cucumber here. Could I take (…) 0.30 from that [the cucumber] Can we take away 0.30 from something? What do you think?
		Two lines later.
2.	Student:	You divide it [the cucumber] into ten pieces and take away three of them.
3.	Teacher:	Then, let’s see ten pieces (…) one, two (…) something like that (…) are there ten pieces now (*counts the pieces aloud*) (…) there are only nine (*draws one extra piece*) so then I take three of them (…) I cut it, with the knife. I cut about here, could that be zero comma thirty of my cucumber?
4.	Student:	Ah [Yes]
		Some lines later.
5.	Teacher:	Or three tenths as we said. Okay, and because we also said it was three tenths and even said that one could say thirty hundredths, so we then have thirty tiny hundredths here in this piece of cucumber that I have also cut off or (…)
6.	Student:	Aa [Yes]
7.	Teacher:	Nice. If I take 0.29 of the cucumber, do I get a larger or smaller piece?
8.	Students:	Smaller piece
9.	Teacher:	Leonie
10.	Leonie:	Smaller
11.	Teacher:	About how much smaller?
12.	Student:	Like, a hundredth
13.	Teacher:	Like a hundredth. Is that a big or small bit? Like that [showing with her hands]. I cut about there and get 0.29, a little less cucumber. Okay, then we’ve looked at these two numbers 0.29 and 0.30, or twenty-nine hundredths and thirty hundredths. How, how, big a difference is it between these? How much is it, we said that we get less cucumber if we take (…) How big is the difference between them? How much is in between these two?
14.	Heidi:	One
15.	Teacher:	One something (…) one percent (…) one litre (…) one kilo
16.	Heidi:	One hundred [1/100] (…) a hundredth
17.	Teacher:	Ah that might be right. There we have twenty-nine hundredths and there thirty hundredths, between

twenty-nine and thirty there is in most cases one, then we could say that there is one hundredth [1/100].

From the excerpt above it is noticeable that the teacher elicited the interval between 0.29 and 0.30 by asking "How big is the difference between them?" (line 13). The discussion about the difference between the numbers was later developed into a discussion about divisibility.

Lesson 1F, as in lesson 1E, enacted the feature of *part of a whole*. The teacher said "What can one take five tenths of? What could that be, could one take five tenths of this table for instance? Or if we took a pizza, maybe, could we do that?" The teacher showed five times one tenth of a 'pizza'. It was made possible to experience a variation in terms of different wholes was opened when examples of parts and wholes was introduced, 0.5 of a table, a pizza, a bag of candy or of anything.

Critical feature 4: Divisibility

This section shows how the teachers together with the students enacted the critical feature of *divisibility* in lessons 1E and 1F. The feature entails the notion that a set of rational numbers is dense and hence repeatedly divisible by ten. From this it follows that, in the interval [0.17; 0.18], there could be 1/100, 10/1000 or 100/10000 etc. Another way to experience 'the same thing' is to experience the divisibility of a whole. A whole, for example a ruler, could be repeatedly divisible, in ten, a thousand and ten thousand parts and further.

Lesson 1E made it possible to experience the feature of *divisibility* when the 'space' between 0.29 and 0.30 was expressed by the students as different amounts of parts in the interval, such as 1/100, 10/1000 and 100/10000 (excerpt 10). One hundredth was in this case expressed in smaller and smaller parts of the cucumber. The following excerpt shows that some students are aware that there could be even smaller parts. One student says that "You can keep on forever, like a million, you can continue as long as you want" (line 14).

Excerpt 10 [00:14:08 to 00:15:55]

1. Student: Or ten thousandths [10/1000]
2. Teacher: Or ten thousandths, she says. That sounds cool. Is it the same (as much)?
3. Student: Ah [Yes]

4.	Teacher:	What does Linda say. Can we write a number with so many zeros? Does it work? Can one say ten, ten thousandths [10/1000]. You look a little (…) sceptical. Ah, it was okay. Ella
5.	Ella:	Isn’t it possible with hundred ten thousandths [100/10000] also?
6.	Teacher:	Okay, one hundredth [1/100] is as much as a hundred ten-thousandths [100/10000] says (…)
7.	Student:	/it works with zero zero comma one tenths [the teacher does not respond or does not hear this].
8.	Teacher:	Ten thousandths, now we are talking about not large pieces at all. I if you get one bite like that in your mouth you’ll hardly feel any cucumber at all if you take a bite that small. Can one continue even more?
9.	Student:	Thousand hundred-thousandths [1000/100000]
10.	Teacher:	Schoo [Oh boy] (…) Now it gets sweaty when I’m writing this under thousand hundred-thousandths.
11.	Student:	Angela has one more
12.	Teacher:	Is this alright with you, John, are you following us so far? But Angela you’ve said so much, do you have even more?
13.	Angela:	Ten-thousand millionths [10000/1000000]
14.	Student:	//You can keep on forever, like a million, you can continue as long as you want. Do we have to continue [joking]?
15.	Teacher:	//Aaa Ten-thousand millionths [10000/1000000]. And all this is as much as there is between 0.29 and 0.30.
16.	Student:	You can split it into smaller pieces.
17.	Teacher:	Have we split it all the time now or could it be that these ten-thousand millionths [10000/1000000] is as much as one hundredths [1/100]?

After this the teacher introduced a group activity. The students were asked to answer the question “How many numbers can you find between 0.17 and 0.18?” and to draw a picture to show their answer. In the oral account of the task all three groups explained their answers to the class. All groups answered that there were infinitely many numbers in the interval. During the discussion the teacher raised a question about the difference between 0.170 and 0.17. One student (Ella) said that the numbers are equal but another student (Hanna) said that 0.170 is a smaller number than 0.17. The teacher wrote 0.171 and 0.17 and explained that the “bold **1**” in 0.17**1** had another value than the first 1 after the decimal point.

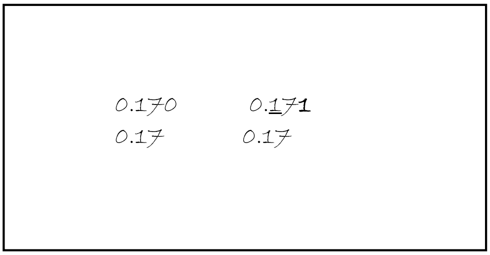

Figure 24. Numbers 0.17, 0.170 and 0.171 were used in lesson 1E to show similarities and differences in numbers and to place value.

The teacher said ”This one is less than that one. We talked about that before (during group work). How many are sharing the cucumber now?” (figure 24). A student answered “a thousand” (referring to the last 1) and the teacher said “Yes, that part tells us how many are sharing the cucumber, the first one after the decimal point, how many are sharing the cucumber?” At this point the teacher connected different place values in the decimal number with how many people that “share the pieces”, in tenths, hundredths or thousandths. The teacher did not however discuss the issue of whether or not the number of digits in a decimal number makes the number smaller, as was raised by Hanna earlier in this part. At the end of the lesson the teacher again raised the question of whether one could ever reach 0.18. One student (Mia) answered “No, one never does, if one keeps on putting nines like that”. Mia’s answer that 0.18 can never be reached could be interpreted as an expression of an idea of actual infinity (cf., Pehkonen *et al.*, 2006).

Lesson 1F enacted the feature in a somewhat different way than in lesson 1E. At the beginning of lesson 1F, in the discussion about the relationship of parts of a whole, the teacher asked about the amount of parts on a one-metre ruler, “How many small little parts can fit into the whole?”. One student (Frida) answered “hundred” and the teacher continued “A hundred. So, that we can call a hundredth (0.01) or how (changes her question) and that we can write as (…) Jonathan” and Jonathan replied “zero comma zero one (0.01)”. At this point one student (Kajsa) said “but there are thousandths also (…) zero comma zero zero one (0.001)” and the teacher answered, “You mean that one can split every little hundredth into ten. Ah that is very, very fiddly [to draw]. We’ll wait a little with that, but I agree it can be done”. Kajsa’s statement made it possible to experience that there could be smaller parts in the interval. This elicited both the feature of a part of a whole and a divisibility of rational numbers. Even if the teacher at this point chose not to develop this further, it was brought up and developed later on. The introductory discussion offered the students a possibility to discern numbers and different amounts of parts in the interval, although not explicitly how many there were.

Later, in the discussion of the worksheet and tasks about interchangeable representation, it was made possible to discern repeated *divisibility* because of the variation in representation (0.25, 25/100, 0.250, 250/1000, 0.2500 and 1/4). At the end of the lesson, the teacher once again elicited the feature of a decimal number as a part of a whole and connected decimal numbers with the number of parts, hundredths and thousandths in fractions. 0.70 was first represented by 0.7 and 70/100. As a contrast, a smaller number, 0.550, was used to show *divisibility* in thousandths. Thus, even if the number after the decimal point could be experienced as a larger number, since '550' is larger than '70', it was possible to discern the divisibility of the whole determined size of the number. The teacher said "If it were the case that we had split it in hundredths, the whole ruler, and started counting here and came all the way up to here, then we would have found 70 such hundredths". The same whole, the ruler, is shown to be divisible even in thousandths. "We could have split it into a thousand parts. If we had split the ruler in a thousand parts. That we had found 550 thousandths if we had split the ruler into thousandths we would have found 550 thousandths at this part between the zero and this part then" (excerpt 11, line 1).

Excerpt 11 [00:38:31 to 00:40:43]

1. Teacher: It is just zero comma seventy you say. What does it say here then? Could we say it in another way (…) Do you agree with me if I say it says seventy hundredths [70/100] (…) some of you nod, because that one is the tenth and that one the hundredth, mm here we have zero comma seven [0.7] on the number line and that we could write like zero comma seventy [0.70] or seventy hundredths [0.70] (*writes the numbers on the board*). If it were the case that we had split it into hundredths, the whole ruler, and started counting here and came all the way up to here, then we would have found 70 such hundredths [70/100]. Do you agree with me? That's why it says so. We could have split it into a thousand parts. If we had split the ruler into a thousand parts (…) That we had found 550 thousandths [550/1000] (…) if we had split the ruler in thousandths (.) would we have found 550 thousandths [550/1000] at this part between the zero and this part then John?

2. John: What did you say?

3. Teacher: If we had split this piece into thousandths (…) then we would have found five hundred fifty thousandths [550/1000] on this piece between the zero and zero comma five five zero [0.550].

4. Student: Yes

5. Teacher: Yes, that is why it is called that.

Other features of decimal numbers

In Lesson 1F it was possible to experience other features of decimal numbers going through the worksheet (see appendix 3), for instance placing value in the ordering of different decimal numbers with different amounts of digits (task 4).

Lesson design 3

Lessons 1G and 1H were intended to implement lesson design 2 but did not succeed in full, since only three of four critical features were implemented in the lessons. In both lessons, it was not made possible to discern the feature of *divisibility*.

Critical feature 1: Decimal numbers as points on a number line

Lesson 1G started with a discussion of whether there are numbers between two whole numbers, 2 and 3. The students contributed by naming many numbers in the interval, for instance 2.2, 2.3, 2.5 and 2.99 (see figure 25). In lesson 1G the feature of *numbers as points on a line* was enacted by naming numbers in intervals.

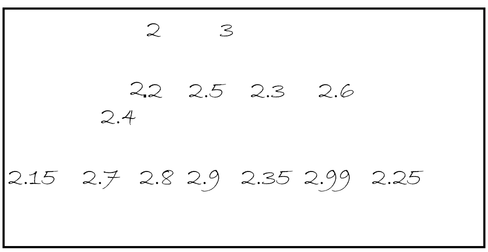

Figure 25. Examples of numbers in the interval [2; 3] presented on the whiteboard in lesson 1G.

It was possible by the examples of numbers for the students to experience a variation of numbers in the interval and that these numbers could have one or two decimals. The students gave examples of numbers in another interval [0.17;

0.18]. The following excerpt comes from a discussion about numbers in the interval, e.g. 0.171 and 0.172. The students also discussed the space between 0.17 and 0.18 and expressed it as 1/100 on a number line or a ruler (excerpt 12, line 1).

Excerpt 12 [00:10:47 to 00:12:35]

1.	Teacher:	A hundredth [1/100] there in between (the teacher points to the ruler) okay (…) if we look at these two (…) is there something in between (…) are there numbers here in between? Or are there not? Or is it limited? What do you say? (…) Josef
		Four lines later.
2.	Josef:	Zero comma a hundred and seventy-one [0.171].
3.	Teacher:	How do you know it is in between?
4.	Josef:	How? (…)
5.	Teacher:	How do you know (…) you said that there is a number in between (…) how do you know that?
6.	Josef:	Because, it is before as, no but (laughter).
7.	Teacher:	Is it a hard question? Malte, please help.
8.	Malte:	Zero comma a hundred and seventy-one [0.171] is bigger than zero comma seventy [0.170] (inaudible).
9.	Teacher:	Is it possible to write zero comma seventeen [0.17] in another way (…) so that you could see it if you were unsure (…) Ah Josef.
10.	Josef:	Zero comma a hundred and seventy [0.170].
11.	Teacher:	Zero comma hundred and seventy (…) so if I have seventeen hundredths [17/100] or zero comma seventeen [0.17] is it the same thing as zero comma a hundred and seventy [0.170] (…) is that (…) okay. Could one get more numbers in there in between? Is it difficult? (…) You can guess, try, test. What do you say, Felicia?
12.	Felicia:	Zero comma a hundred and seventy-two [0.172].

The discussion continued with one more example, 0.173 of a number in the interval [0.17; 0.18]. From the examples given it is possible to discern that there could be numbers with three decimals. However, the numbers were treated as countable.

After this introduction the students worked with a group assignment. The students' oral account of the group assignment showed that there were shifting views about the amount of numbers in the two intervals [1; 2] and [0.17; 0.18].

Decimal numbers in intervals were presented by all four groups. The students made no connections to other forms of rational numbers such as fractions and percents. Three groups argued that there are more numbers between 1 and 2 than between 0.17 and 0.18. One of these groups reasoned that, since 1 and 2 are not 'split numbers' as in decimal numbers, there must be more numbers in between two whole numbers. One group tried to show that there is an equal amount of numbers between 0.17 and 0.18. They said that there were 14 numbers in each interval. The reason for that is that "you just put a digit behind" and "you can put as many as you want".

Lesson 1H started with a discussion about whether there are numbers in the interval [0; 1]. The students named numbers in the interval: 0.7, 0.01, 0.1, 0.4 and 0.9. The number 0.01 was given as an example but was changed to 0.1 because it was considered to be too close to zero. It was therefore possible to discern four numbers in the interval, which was divided into ten parts (see figure 26).

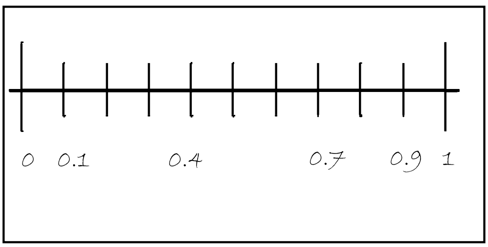

Figure 26. Examples (*range of change*) of numbers in the interval [0; 1] presented on the whiteboard in lesson 1H.

The teacher continued the discussion by exploring numbers in another interval, [0.4; 0.5]. The students contributed by naming several numbers in the interval: 0.45, 0.49, 0.41 and 0.43. Note that the interval is again divided into ten parts, and nine numbers could be experienced to be in the interval (figure 27).

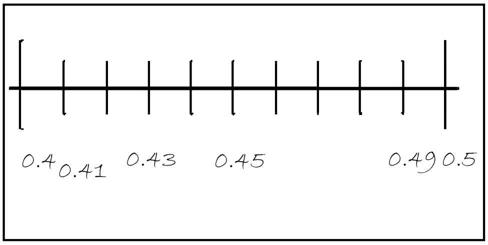

Figure 27. Examples (*range of change*) of numbers in the interval [0.4; 0.5] presented on the whiteboard in lesson 1H.

Critical feature 2: Interchangeable representation

Lesson 1G opened the possibility to discern the feature of *interchangeable representation* when the teacher asked whether the students knew "other names" for the numbers 0.17 and 0.18. The students contributed by naming the decimal numbers with fractions, as seventeen and eighteen hundredths. The teacher raised the question of whether 17/100 and 0.17 were numbers for the same "thing" (same value). When the teacher was not clear about whether the student understood the interchangeable representation s/he focused instead on an easier fraction, 1/100.

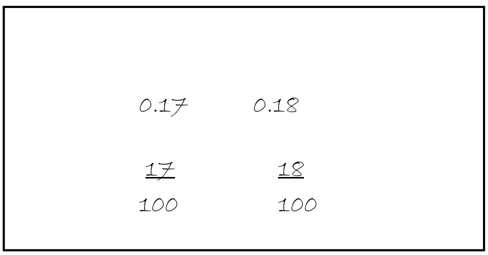

Figure 28. Representations of 0.17 and 0.18 by fractions on the whiteboard in lesson 1G.

Lesson 1H, as in lesson 1G, implemented the feature of *interchangeable representation* of rational numbers. It was possible in the examples given by the students to experience that 0.90, 0.900, 9/10, 90/100 and 900/1000 all represented the same number (value). Later, another number, 0.43, was represented by 0.430 and 43/100. At the end of the lesson, several different decimal numbers (0.7, 0.74, 0.25 and 0.112) were represented by fractions. In this case the teacher restricted the task somewhat by the instruction "we'll only write two ways" - "we'll run out of time otherwise", hence generating only one or two examples of other representations of the decimal numbers.

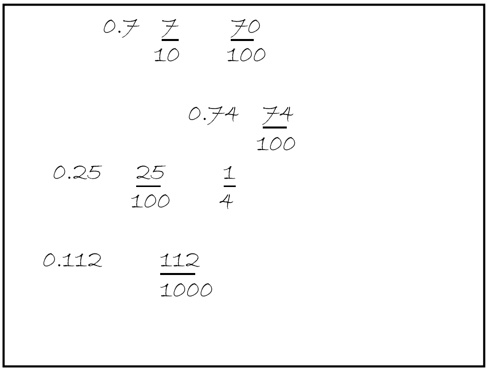

Figure 29. Representations of 0.7, 0.74, 0.25 and 0.112 as fractions. The examples (the *range of change*) of the task of representing decimal numbers by fractions in lesson 1H.

Critical feature 3: Part of a whole

Lesson 1G shows that the teacher used 0.17 and 0.18 to represent the relationship between a part and the whole of a rational number. S/he asked where the numbers could be seen on the ruler and used the human body as another example of a whole. The teacher had previously asked about where one hundredth [1/100] of the ruler was.

Lesson 1H shows that the teacher used a ruler to show that 0.9 could be seen as a part of a whole. It was possible to experience a variation of wholes, a chocolate bar, a dog and a human being. The teacher also showed on a ruler (a whole) that, between 0.44 and 0.45, there was a centimetre (part) and that was as much as one hundredth (1/100).

Critical feature 4: Divisibility

This part will show how the feature of divisibility was not enacted in lessons 1G and 1H.

Lesson 1G shows that, even if it was said in the lesson that there are infinitely many numbers in the interval, it seems not to have been possible to discern other ways than verbally. The following excerpt shows that students say that "it is ten-thousandths and a hundred-thousandths and millionths" (line 5) and that "One can still go as far back as one wants on the numbers [digits]" (line

9). This seems to relate to an extension of digits horizontally at the end of the right-most column and not to the fact that one can split the interval between numbers in infinitely many parts. Hence, it is not made possible to experience the divisibility of numbers.

Excerpt 13 [00:40:39 to 00:41:42]

1.	Student:	One can put as many (digits) as one wants behind.
2.	Teacher:	Okay I have a question (…). Before you said that zero comma seventeen [0.17] was as much as seventeen hundreds [17/100]. There you have written zero comma one seven one [0.171], is that the same as then?
3.	Student:	(laughter) That (…) no it is a hundred and seventy-one thousandths.
4.	Teacher:	A hundred and seventy-one thousandths right.
5.	Student:	Then it is ten thousandths and a hundred thousandths and millionths.
6.	Teacher:	Then we are talking about small parts, aren't we the longer away one gets. How far can one get? How far is it possible?
7.	Student:	(inaudible)
8.	Teacher:	There is no limit. So, if we compare what you have done with what the girls have done, the girls said that there were equal amounts of numbers, consider that?
9.	Student:	One can still go as far back as one wants on the numbers [digits].
10.	Teacher:	Okay

In conclusion, fractions were used in lesson 1G to represent decimal numbers, but the meaning of fractions in terms of the amount of parts was not made explicit. It is not clear whether the students see the digits after the decimal point as more than different names of place value.

Lesson 1H did not bring out the feature of divisibility at all. The question about the amount of numbers in intervals was not discussed, although an understanding of density of rational numbers in decimal form was the object of learning in the lesson.

Other features of decimal numbers

At the end of lesson 1H there was a discussion about sorting numbers by value, in a similar way as in lessons 1C, 1D and 1F. The correct answers, 1/10, 0.6, 0.605, 0.61, 0.75, 1.2, were presented on the whiteboard but there was no discussion about the task. Hence, on the one hand, students could experience features of place value, but, on the other, the students' attention was not directed towards explicit features.

Conclusions

Critical features for student learning

One of the aims of this study was to explore the extent to which other teachers can make use of findings gained in a learning study and whether there is support for conjectures made about the critical features. The analysis of the study shows that teachers can make use of the findings by enacting critical features in lessons and that these features have an impact on student learning. However, it was shown that only students in lessons 1E and 1F were given the possibility to discern that rational numbers are dense. Students in lessons 1E and 1F outperformed their peers in tasks on density of rational numbers. Even though density of rational numbers was discussed in all lessons (except in lesson 1H), it was not sufficient for the possibilities of students to experience that rational numbers are dense. Why?

Putting more digits at the end of a decimal number or telling the students that there are infinitely many numbers was not sufficient. If critical features of rational numbers do not come to the fore in students' focal awareness, they are not given the possibility to discern them. Therefore, even if all teachers had the same intended object of learning and similar tasks and organised their work in a similar manner, this did not have the same effect on the lived object of learning – the students' learning. Features of the object of learning are dimensions of (possible) variation. Some of these features are critical for specific learning while others are not.

One finding in this study is that critical features implemented in this study had an impact on student learning. As shown in the analysis of the learning outcomes in LD3, including more critical features did not generate better results than LD1, which included one critical feature. One interpretation could be that some of the features were not critical for student learning. However, these features are intertwined with *divisibility,* and the features could be a prerequisite for understanding density (cf., Vosniadou *et al.*, 2008). This is an empirical question to be further explored in the future.

Furthermore, a potential critical feature of rational numbers was found in the student test and lesson analysis – the difference between the concept of number and digit. For instance, to the question of whether there are numbers in an interval, some students answered "There are no numbers, there are only digits". This suggests that students do not see a decimal number as a number and hence see it as a line of digits. This is a question to be explored in the future.

Although the critical features enacted seem to have made a great impact on student learning in terms of test results, some issues about decimal numbers remain unclear. To elicit issues of place value, students' generalisation to whole numbers and the number of decimals, other features also need to be addressed. Data show, for instance, that students seem to see that the amount of digits affects the value of the number – more digits make the number smaller. A contrasting example shows how students generalise knowledge about whole numbers to decimal numbers and say 0.1718 is a larger number than 0.172 (Moskal & Magone, 2000). These are important features for understanding decimal numbers in an adequate way. In conclusion, naming numbers in intervals without making it possible for students to discern the divisibility of numbers and hence treating the numbers as countable does not promote student learning about density.

The range of change

Another finding in this study is that the enacted examples – the 'range of change' – *affected* the variation that was possible to discern in the critical features. My conjecture is that the 'range of change' offered different possibilities to discern features of rational numbers. For instance, representing 0.17 and 0.18 by 17/100 and 18/100 made it possible to discern *interchangeable representation* but that a wider range of change that also included 170/1000 and 180/1000 made it possible to see that there could be even more ways of representing the numbers.

What is made possible to discern is of the utmost importance for learning from the theoretical standpoint taken in this study, and a wider range of change is therefore suggested: in this case, to offer students a better opportunity to discern the feature of *interchangeable representation*. For instance, experiencing the same number in hundredths, thousandths and ten thousandths (0.25, 25/100, 0.250, 250/1000, 0.2500) makes it more likely to be able to discern *divisibility* compared to the case in which it is only possible to experience hundredths (e.g., 0.17 as 17/100). The range of change could on the one hand open a larger variation within a dimension of variation or on the other hand close up variation by offering only few examples within a small range. However, whether or not the range of change in a lesson enhances learning is an empirical question that depends on the object of learning.

CHAPTER SIX

A STUDY OF TEACHING AND LEARNING OF ADDITION AND SUBTRACTION OF NEGATIVE NUMBERS

This chapter describes the results of an intervention, a teaching experiment, concerning the teaching and learning of operating with negative numbers. The background to the study is a research project on learning studies at the University of Gothenburg. Four critical features for student learning were identified in a learning study of addition and subtraction of negative numbers. The aim of this study is to explore how other teachers in new contexts can make use of the present findings. The study furthermore tests the conjecture of critical features found as well as the theory of variation.

The intervention

The results of a teaching experiment on addition and subtraction of negative numbers are reported here. Two lesson designs were implemented by four teachers from two different schools, in eight different classes in grade 7 with a total of 134 students. As in the study reported in chapter 5, two teachers at a time worked together with the researcher to develop lesson plans based on lesson designs (critical features). The meetings and lessons implemented were video recorded by the researcher. Data from a total of six meetings, eight lessons and 380 tests were used as empirical material to analyse the intended, enacted and lived object of learning. A pre test was conducted about a week before the lesson, a post test two days after the lesson and a delayed post test seven weeks after the post test. The analysis of the lessons was made on a micro level and the object of research was what it was possible for students to discern during these lessons. The lessons lasted for about 50 to 70 minutes. They started with an introduction followed by group or pair activities and ended with a final discussion of the activities. The lessons in the same lesson design were planned to handle the content in the same way – in terms of critical features to be enacted in the lessons – although the tasks differed somewhat between the teacher teams. For example, teachers E and F used hands on material and small group discussions while teachers G and H used a worksheet and pair activities. However, the planned difference between the lesson designs was content specific features that have been identified as critical for student learning (Kullberg, 2007b).

- Lesson design 1 (LD1) included 'subtraction as a difference' and the 'perspective: commutative law does not apply in subtraction'
- Lesson design 2 (LD2) included 'the sign', 'the number system', 'subtraction as a difference' and the 'perspective: commutative law does not apply in subtraction'

Findings

Intended, enacted and lived object of learning

The intended object of learning in all eight lessons was for the students to be able to calculate addition and subtraction tasks with negative numbers. The lessons with the same lesson design were intended to implement the same critical features in the lessons. Lessons 2A, 2B, 2C and 2D were intended to implement two critical features, *the sign* and *subtraction as a difference*. However, in work with the first teacher team, the teachers insisted on implementing two other features, *subtraction as a difference* and *perspective: commutative law does not apply in subtraction* in lesson design 1 (LD1). LD1 was therefore changed to include the features that the teachers wanted to implement and hence did not follow the order in which the critical features were found in the learning study. Consequently, this change affected the design of the study in terms of critical features enacted in the lessons and hence the comparisons with previous results from the learning study. Despite this, analyses of the lessons can provide evidence for conjectures made in the previous study and may even broaden the analysis.

Lessons 2E, 2F, 2G and 2H were intended to implement lesson design 2 (LD2) which included the four critical features identified, *the sign, the number system, subtraction as a difference* and *the perspective*. In all the lessons parentheses and a minus sign in front of the number were used to mark a negative number and the metaphor used for a negative number was debt. Economy/financial state was used in all the lessons as an example of operating with negative numbers.

For instance, an addition was seen as two people's shared economy/financial state and a subtraction as a comparison between economies/financial state. In the following analysis I argue that, in one lesson with LD1, lesson 2B, more features than intended were enacted in the lesson (see table 7). Besides the features of subtraction as difference, other features of the numerical system and the sign could be experienced.

Table 7. The lesson design and the enacted critical features enacted in the lessons.

Lesson Design	Lesson	Teacher	N	The sign for a negative number	The numerical system	Subtraction as a difference	Perspective, commutative law does not apply in subtraction
LD1	2A	E	21			✓	✓
LD2	2B	F	11	✓	✓	✓	✓
LD1	2C	G	30			✓	✓
LD1	2D	H	14			✓	✓
LD2	2E	E	16	✓	✓	✓	✓
LD2	2F	F	12	✓	✓	✓	✓
LD2	2G	G	16	✓	✓	✓	✓
LD2	2H	H	14	✓	✓	✓	✓

The analysis shows that a particular student's questions in lesson 2B made an impact on the critical features brought up during the lesson and hence what it was possible to learn. Further, even more distinctions concerning features of the object of learning were made in this lesson. In lesson 2B it was also made possible to experience the fact that decimal numbers, such as 0.5, are not placed on the left hand side of zero on the number line and hence not together with the negative numbers. Moreover, one student (Felicia) made an explicit distinction between magnitude (the difference or absolute value) and magnitude with direction (the answer) that was not present in the other lessons. The student made a fusion of two critical features, difference and that the perspective: commutative law does not apply in subtraction.

An important finding in this study is that connections between learning and teaching are shown. The analysis shows that a positive effect on student learning was shown when all critical features derived from the learning study were enacted with new teachers and students. Consequently, when it was possible for students to experience more features of the object of learning simultaneously, there was a noticeable difference in students' learning outcomes compared to when it was possible to experience fewer features of the object of learning. Hence, the analysis of pre and post tests (see appendix 4) shows that students that enacted LD2 had better results in post tests than LD1 students (see table 8). The effect size on the whole test for lesson design 1 is above medium, d=0.65 compared to highest level d=0.90, for lesson design 2.

Table 8. Results from pre and post tests. Whole test (W) maximal points=31.

	Lesson design 1 (N=65)			Lesson design 2 (N=69)		
	Mean	SD	Effect size	Mean	SD	Effect size
Pretest	9.26	6.24		10.45	6.64	
Posttest	13.65	7.19	0.65	17.12	8.07	0.90

Analysis of the post tests (table 9) showed that there was a significant difference between LD 1 and LD2 at the 0.01 level (t=0.004) and almost on the 0.05 level (t=0.057) in the delayed post test (seven weeks after the lesson). However, no significant difference was found between LD1 and LD2 in the pre test (t=0.588).

Table 9. The table shows the results of the pre, post- and delayed post tests. Note that n in this table is smaller than previous table; this is because only students who participated in the delayed post test are counted.

	Lesson design 1 (N=53)		Lesson design 2 (N=59)	
	Mean	SD	Mean	SD
Pretest	9.55	6.40	10.19	6.06
Posttest	13.51	6.67	17.49	7.51
Delayed test	12.23	6.49	14.98	8.36

The next section shows the analysis and implementation of the critical features in the lessons.

Analysis of lessons

Lesson design 1

Lesson design 1 (LD1) included the features of *subtraction as a difference* and *the perspective*. Subtraction as a difference entails that it can be seen both as a difference and a take away. 'Take away' is the most common metaphor used to represent subtraction but seems not to be sufficient when operating with negative numbers (cf., Ball, 1993). The teachers in this study introduced the idea of subtraction as a difference by comparing numbers on the number line, for instance that the difference (the counting steps) between (-3) and 2 is five. The difference is the magnitude, in this case with a negative direction, (-3) –2 = (-5). The perspective entails seeing subtraction as a comparison, a difference from the perspective of the first term (-3), which gives a negative difference if it is a smaller number. The perspective entails the fact that the commutative law does not apply in subtraction.

Critical feature 1: Subtraction as a difference

Experiencing *subtraction as a difference* is one way of seeing subtraction. With an awareness of the fact that 'take away' is more often used as a model of thought, both ways are compared in some of these lessons to make them more easily noticeable. In this sense, it is made possible to experience a variation in ways of seeing subtraction. The example of two persons comparing their economies and the difference between them is used in all lessons. The examples (*range of change*) given in all lessons were numbers from the positive side, the negative side and both sides of zero on the number line.

In ***Lesson 2A*** it was made possible to experience two ways of seeing subtraction. The teacher compared the same subtraction, 5 - 3, as a *difference* "I have five kronor. You have three kronor. How much more do I have compared to you?" with a 'take away' thought "I have 5 kronor, I give away 3 kronor. What do I have left?" The rationale was that the teacher wanted to emphasise that seeing subtraction as a take away is "really hard" when counting with negative numbers. During the lesson the students worked with a task about finding number pairs with the difference 4 and the students contributed some examples in the oral account of the task: 1 and 5, 2 and (-2), (-1) and (-5). In a contrasting example, the difference between (-4) and 8 was discussed and used to illustrate that, even if it looks like 4 is the difference (if you ignore the signs in front of the numbers) it is actually 12. In this part of the lesson it was possible to experience the magnitude (absolute difference) of the difference between negative and positive numbers.

Lesson 2C showed that *subtraction as a difference* was enacted by number pairs with the difference of two (5 - 3= 2, 4 – 2 = 2, 3 – 1 = 2, 2 – 0 = 2, 1 - (-1) = 2, (-3) - (-5) = 2). In a similar way as in lesson 2A, students also worked with and gave examples of number pairs with the difference of four. The teacher wrote the number pairs as subtractions: 10 - 6 = 4, 9 – 5 = 4, (-3) - (-7) = 4 and 1 - (-3) = 4. At the end of the lesson the teacher emphasised that "When we are supposed to look at difference /.../, it is important that you keep apart whether it is plus with negative numbers or it is minus, will we count subtraction and then look at difference /.../". The statement clarified that the students must pay attention to the operational sign. If it is addition, it is not the difference between the numbers, as it is with subtraction.

Lesson 2D made it possible to experience *difference* between numbers but it was not pointed out that 'difference' is another way to see subtraction compared to 'take away'. The difference between the numbers was expressed by number pairs, for instance 5 – 3 was talked about as 5 *to* 3 or 5 *and* 3. Later in the lesson when the example of economies/financial state was discussed, the subtraction sign was used to represent the difference between numbers. In lesson 2D the teacher asked which of the following, 10, (-10) or 6, is the correct answer to 8 - (-2). This question challenged the students to see *subtraction as a difference* (magnitude) and to take the sign of number (direction) into consideration. The answers were ruled out one by one. If a student answered 6, for instance, that student had not taken into account the magnitude (8 - 2), or answered (-10), s/he had not taken into account the direction of magnitude.

Critical feature 2: Perspective - the commutative law does not apply to subtraction

The notion of the *perspective* in subtraction of negative numbers entails the direction of the magnitude. The feature explicitly points to the fact that the subtraction is regarded from the first term (a – b = c). All lessons with LD1 made it possible to experience both the direction and magnitude of negative numbers. The following excerpts show examples of how the feature was enacted in the lessons.

Lesson 2A enacted the feature that difference is seen from the first term when the teacher pointed out that "it is always the one that is first that I look at, that one that has plus five is the one that is in the first position, it is the first digit one writes, it is at that advantage one checks, that one was two more than that one, but this three is still in the first position /.../". Another example of how the *perspective* was elicited is shown in the following excerpt where the class discussed (-5) - 3. The teacher said "But why it is a negative eight, that I'm not so sure about everybody is following" (excerpt 1, line 1) and thus wanted the student to be able to explain the line of reasoning. One student's statement, "Probably because you started counting with the five first" (line 3) and "you started with that" (line 7), indicates that the student takes the perspective of the first term into consideration in the calculation. This is also emphasised by the teacher, "It is always the first, it is always from that perspective" (line 8).

Excerpt 1 [00:41:13.21 to 00:41:31]

1.	Teacher:	But why it is a negative eight, that I'm not so sure about everybody is following.
2.	Student:	(inaudible)
3.	Pat:	Probably because you started counting with the five first.
4.	Teacher:	I started counting with the five first. Elaborate on that Pat.
5.	Pat:	I don't know exactly
6.	Teacher:	You're on the right track

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7.  Pat:       But you started with that [the first number]
8.  Teacher:   It is always the first, it is always from that
               perspective /…/
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Lesson 2C enacted the feature in a similar way as was done in lesson 2A, when the teacher pointed out that "it is important that we compare from the one that comes first" (excerpt 2) when looking at the difference between numbers from the *perspective* of the first term. The contrast between the first and second terms gave an opening for experiencing variation within the feature.

Excerpt 2 [00:23:05]

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1.  Teacher:   Now they are on bad terms[35] with each other, being
               on bad terms is negative, now we are going to count
               subtraction, minus, so here they are on bad terms,
               we are going to count minus with negative numbers,
               subtraction and to understand this then, exactly as
               it was on the test, we will try to implement this
               thought, we will look at the difference between
               their economies, we will compare them, and then it
               is important that we compare from the one that
               comes first.
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Later, dealing with another example, 5 – (-3) = 8, a student said that Åsa, whose money is represented by the first term, had 8 kronor more compared to Tina, whose money was represented by the second term. When the teacher changed it, and Åsa had less than Tina, (-3) – 5 = (-8), the teacher compared the answers, "minus eight, that is why it is important that we start looking at the first number here, minus eight". In this case the difference (magnitude) is the same but the order of the terms determines the direction of the magnitude.

Lesson 2D, as in lessons 2A and 2C, enacted the feature of *perspective* in the lesson, although the feature in lesson 2D was elicited by a student's question. The student said "why is it just when Åsa has the most and minus" (excerpt 3, line 1). The teacher responded "she is the one that is here first, then she is the one we compare against" (line 3). Another student concluded that "It is always the first number" (line 4) that one compares against.

[35] The teacher used the expression "on bad terms" to direct students' awareness towards subtraction of negative numbers (the friends have moved apart and look at the difference between their economies and "friends" with regard to addition of negative numbers (the friends share a common economy).

Excerpt 3 [00:24:38 to 00:24:52]

1.	Student1:	What matters or why is it just when Åsa has the most and minus/
2.	Student2:	/Now we count how much Åsa has or has less, not the other way around
3.	Teacher:	Well, it is about this, it is Åsa that we are working with now right so she is the one that is first here then it is her we compare against.
4.	Student1:	It is always the first number.
5.	Teacher:	Ah, exactly, Good /…/

In conclusion, the feature was possible to experience in lesson 2A, 2C and 2D owing to a contrast made between the first and second terms. In all three lessons a distinction was made that it is "always from the first".

Other features of negative numbers

In lesson 2D a student's question about the task (-3) – (-5) = 2 made it possible to experience that operating with only negative numbers could generate a positive number as an answer. The student asked whether the answer is positive "even if it is on the negative side?" The teacher replied, "Yes, even if it is on the minus side [the numbers in the task], yes it could be like that, that you could get a number where the answer is not a negative number". This was not elaborated upon in the other lessons in this way. Although similar examples were discussed, this particular question was not made explicit in the other lessons. This feature relates to the feature of the numerical system.

Lesson design 2

Besides the implemented critical features in LD1, two more features were enacted in LD2, the sign and the numerical system. These features were implicit in LD1 and not elaborated upon further. Under this heading, lesson 2B, which was intended to implement LD1 but enacted LD2, is also analysed. Note that the same teachers are teaching both lesson designs 1 and 2.

Critical feature 3: The sign

The feature of the sign entails becoming aware of the difference between *the sign* for a negative number and for subtraction (cf., Ball, 1993). The fact that students have difficulty differentiating between the signs for subtraction and a negative number has been identified as a major problem for student learning of negative numbers (cf., Herscovics & Linchevski, 1994; Vlassis, 2009). The feature was enacted in all lessons by comparing the function of the subtraction sign with the negative number sign. Furthermore, the notion of positive numbers was introduced to enhance the understanding of the sign for a negative number.

Lesson 2E made it possible to experience features of *the sign* when the class compared 400 - 317 with 400 - (-317) and discussed the different signs used in the expressions (excerpt 4). The teacher asked "are these the same, what is the difference between the minus signs, are there some that are the same and some that are different, does this signify subtraction in all three cases?" (line 1). Furthermore, the excerpt shows how the teacher distinguished between a negative number and the subtraction sign by calling a number "negative 317", for instance, instead of "minus 317", which is common in Swedish classrooms (line 9). Comparing the signs for subtraction and a negative number and naming them differently, negative 317 for the number and minus 317 for the operation, are two examples of contrasts given to make features of the sign more explicit to students. The variation that was possible for the students to experience in the examples consists of the contrast made.

Excerpt 4 [00:04:49 to 00:08:26]

1.	Teacher:	I've written two expressions on the board [400-317, 400-(-317)]. On the one on this side it says, here, it says four hundred minus three hundred and seventeen and there it says four hundred minus one more minus like that three hundred and seventeen. Then I wonder a little, are these the same, what is the difference between the minus signs, are there some that are the same and some that are different, does this signify subtraction in all three cases? Eva what do you think? Following your feeling here, guess, what do you think this means?
2.	Student:	(inaudible)
3.	Teacher:	That is minus, subtraction minus. Does that one also signify that Adam.
4.	Adam:	Isn't it so that two minus signs that are after each other are plus.
5.	Teacher:	Ah you bring up a lovely old rule. It sounds a little hocus pocus really, I leave that for a while, maybe we can come back to that. But what do you think, if one does not add them and make a plus [refers to the rule]. What does each and everyone signify? Is it the same sign, that one and that one. What do you think?
6.	Adam:	One of them is ah (inaudible) and the other one is ah how much one should reduce it.

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7.   Teacher:   You mean that it tells that it is negative and that
                then
8.   Student:   It tells that one should take away (inaudible)
9.   Teacher:   Make minus subtraction minus. (…) Okay this is a
                little tricky, you see that it is actually two
                signs that or one sign that signifies different
                things. And this we should hold on to a little
                while. That –317 [400–317], that is nothing
                strange, that is a subtraction sign, which you have
                encountered many times. But this 400 subtraction
                minus huhu (makes a noise) negative 317 [400–(-
                317)], these signify different things. And it is
                actually like this. It really should be like this
                (…) in front of positive numbers (a plus sign), we
                should put out a plus also, plus one, plus three,
                ah that is plus four. We really should have it
                there too but we don't, we don't bother to, maybe
                we are a little bit lazy we just do it when the
                numbers are on this side of zero [the left side].
                Then we show it by saying that it is a negative
                four or something else that is negative.
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The excerpt also shows how one student referred to the rule, "Isn't it so that two minus signs that are after each other are plus", but this is not elaborated further in the lesson (line 4). It is made possible to experience positive numbers when the teacher stated that in front of positive numbers there is a plus sign that is most often not made explicit (line 9).

Lesson 2F made it possible for students to experience the difference between addition and *the sign* for a positive number (+5) + (+5) = (+10), the difference between the subtraction sign and a positive number in the expression (+5) - (+3) = (+2) and the difference between the subtraction sign and a positive number in 5 - 3 when compared to (-3). The latter is brought up but not elaborated upon further. In the lesson the teaching made it possible to experience the difference between subtraction and a positive number. The teacher said "What is the difference between these two signs (- and +3 in (+5) - (+3) = (+2)). Aa what can one say about these two signs?". A student (Eric) answered "that is subtr (..) or what it is called and the other is positive". Moreover, the teacher in lesson 2F made a distinction between negative numbers and the subtraction sign by calling negative numbers, for instance (-3), negative three in a similar way as in lesson 2E.

In ***Lesson 2G*** it was made possible to experience the different meanings of *the sign* for subtraction and a negative number when the meanings of the signs were compared, in 5 - 2 = 3 and 5 - (-2) = 7. In a similar way as in lessons 2E and 2F, positive numbers, marked with a plus in front of them, were introduced as a comparison to the minus in front of negative numbers.

Lesson 2H made it possible to discern the two different meanings of the minus *sign*. The following excerpt shows how the teacher differentiates between the meanings, between the number and the operation (line 1). The teacher said "There is a lot of information here that is not visible or that one does not even think about is there". The excerpt shows how the teacher made the students consider which of the numbers is positive or negative and which sign is the operation in the expressions 5 – 3 = 2 and 5 – (-3) = 8 (see excerpt 5).

Excerpt 5 [00:13:14 to 00:16:35]

1.	Teacher:	So, here I have a minus sign [5 - 3= 2]. Does it say minus three, the number minus three or is it minus three the 'counting part'. What is it in this case? What does Sara say?
2.	Sara:	The counting part
3.	Teacher:	The counting part, I have no good word for that, one usually says operation /.../
		A line later.
4.	Teacher:	If we take five minus minus three [5-(-3)=]. (inaudible)(…) If we look at that calculation. We see we have the numbers five, and which number also, what do you say Philip.
5.	Philip:	Minus three
6.	Teacher:	Minus three. What is it that says it is minus three?
7.	Student:	(inaudible)
8.	Teacher:	I have chosen to write it with parentheses and we said before that we would use this thorough way to mark that it is a negative number so in this case the minus sign is a negative number because that what it signifies in this case and what does that minus sign signify?
9.	Student:	Minus
10.	Teacher:	That signifies minus so what is it I shall do? And this calculation describes then (…)
11.	Student:	The answer
12.	Teacher:	The difference isn't it, the difference there, and the question is what is the difference between five and minus three.

The teacher introduced the notion of positive numbers in comparison to negative numbers on the number line and in numerical expressions. In a similar way as in lessons 2E, 2F and 2G, the teacher wrote a small plus in front of positive numbers to bring out the fact that numbers have both magnitude and direction.

Lesson 2B made it possible for the students to experience that *the sign* of positive and negative numbers is not the same as the sign for the operation (addition). The teacher said "Plus (addition), one adds, we say (…) that Bengt has (…) I put a little plus sign here in front to show that it is a positive number, he has eight (+8)" and marked the positive number with parentheses. The teacher also showed an example with a negative number "I put parentheses around this (points to (-7)) to show that it is a negative 7". The teacher concluded that "we usually don't write that out, the plus sign in front (+8), but it is actually there because it is a positive eight, but now we are talking about positive and negative numbers so just to know which of these it is (points to +8 and -8 on the number line). That it is this instead of that I put parentheses around it".

Critical feature 4: The numerical system

The feature of the *numerical system* entails understanding that the numbers are larger the farther on the right hand side of the number line they are. For instance, (-15) is a smaller number than (-1), albeit that 15 is a larger number than 1 on the positive side of the number line. Ball (1993) advocates that "Simultaneously understanding that -5 is, in one sense, more than -1 and, in another sense, less than -1 is at the heart of understanding negative numbers" (ibid, p. 379).

Lesson 2E made it possible to experience the *numerical system* and how the size of numbers is determined. First, a comparison between a positive (8) and a negative number (-19) was made. It was concluded that even (-1000 000) was a smaller number than 3. The size of negative numbers was also later discussed, for instance (-4) and (-1). One student said that a "negative four" is smaller than a "negative one", with the explanation that you have less if you have negative four. Dealing with subtraction tasks later in the lesson also made it possible to experience which of two negative numbers was the smallest.

In ***lesson 2F*** the feature of the *numerical system* was implemented in a similar way as in lesson 2E, by discussing what determines the size of a number. The excerpt shows a discussion about which was the largest number, 6 or (-15). The teacher posed a question about the distinction between magnitude (absolute value) and direction (positive or negative) of numbers. The teacher challenged the students “But, if you look at this one it is actually fifteen, which is the biggest, isn’t fifteen bigger than six, then shouldn’t it be biggest?” (excerpt 6, line 2). Later, two negative numbers were compared, (-1) and (-11) (line 14). The number line was used to illustrate how it is possible to determine which number is larger than the other. Two views of the number line were presented and the incorrect way was rejected (lines 8 to 14).

Excerpt 6 [00:09:50 to 00:11:30]

1. Sara: Minus fifteen is below zero.
2. Teacher: Below zero, here somewhere, does that mean that is smaller then? But, if you look at this one it is actually fifteen, which is the biggest, isn’t fifteen bigger than six, then shouldn’t it be biggest?
3. Student: Minus
4. Teacher: I beg your pardon
5. Student: There is a minus sign in front
6. Teacher: Ah, what does that stand for then?
7. Student: That it is negative below zero.
8. Teacher: Is it like this, that if one would draw it that it more should be that the numbers get bigger and bigger and bigger like that [the more to the right hand side of zero], or?
9. Student: Ah
10. Teacher: It is not like this that they go from here aops [makes a sound] they get bigger and bigger and from here bigger and bigger [the numbers get bigger on each side of zero]
11. Student: No
12. Teacher: What did you say?
13. Student2: Maybe
14. Teacher: Ah I think it is like that, that they go this way [the numbers get bigger the farther to the right hand side they are] and the numbers over there get smaller and smaller and smaller when you have past zero they shrink. /.../ This then, minus one and minus eleven. Which would be the biggest or smallest of them?
15. Student: Minus one is the biggest.

Lesson 2G made it possible to experience which of a set of numbers is the largest. At the beginning of the lesson the teacher asked which of the two numbers "minus 10000" or zero was largest. The teacher concluded that the numbers get larger the farther to the right they are and smaller the farther to the left they are and hence zero is the larger number. The teacher used more examples of numbers, (-1), 2, (-7) and 0, to discuss which is largest and smallest and also compared different negative numbers.

Lesson 2H discussed which of the following numbers is the largest, (-1), (-2), 7 or (-9). One student suggested that "positive numbers are always bigger than a negative" (excerpt 7, line 4). The teacher continued the discussion by comparing two positive numbers, 6 and 7, and asked "How do you know that seven is bigger than six? (line 9). This part of the discussion ended with the conclusion "That the number that is biggest is the one that is most to the right on the number line" (line 11).

Excerpt 7 [00:07:12 to 00:07:54]

1.	Teacher:	Okay. I write another number [-9], which is the biggest now? Let's see, Mike
2.	Mike:	Ah still seven
3.	Teacher:	Seven still. Why?
4.	Mike:	Because it is positive and positive numbers are always bigger than negative.
5.	Teacher:	Ha okay if I write like this then (writes 6). Which is the biggest now Sara?
6.	Sara:	Seven
7.	Teacher:	Is it still seven, but six is also positive.
8.	Sara:	But seven is bigger than six.
9.	Teacher:	How do you know seven is bigger than six?
10.	Sara:	Because seven is higher on the scale.
11.	Teacher:	Seven is higher on the scale. There you said something fine. That the number that is biggest is the one that is most to the right on the number line. When you place different numbers then you can with the help of this [the number line] determine which is the biggest because it is the one furthest away on the number line.

Lesson 2B showed that the teacher and a student (Felicia) opened the possibility to discern features of the *numerical system*. Note that this way of enacting the feature differs from the other lessons with LD2 in which the feature was enacted through the explicit comparison of the value of numbers, for instance which is the largest number of (-1) and (-5), leading to the conclusion that numbers become larger the farther to the right on the number line they are. In lesson 2B the teacher posed a question about whether 500 and (-500) are the same number and this started a discussion about the difference between positive and negative numbers (see excerpt 8) and about what in this thesis is called the numerical system. The following excerpt shows how a student (Felicia) reasoned with the teacher and says that (-500) and 500 are not the same number because if they were the same number and you add seven to both numbers the answer would be the same (see excerpt 8 lines 6 and 8). In the lesson it was made possible for the students to discern that when you add a positive number to a positive or a negative number, the number becomes larger. This is shown by both answers to the tasks (500 + 7 = 507 and -500 + 7 = -493) that the numbers are farther on the right hand side of the number line (lines 11-21).

Excerpt 8 [00:03:59 to 00:06:04]

1.	Teacher:	Is there a difference between these two numbers, we settle with these [-500 and 500] (…) or is it the same number (…) what do you say. Any thought about that. Hmm
2.	Boy:	The difference is thousand.
3.	Teacher:	The difference is a thousand between them. The difference between these numbers five hundred this way and five hundred that way. Is it the same number? (...) Felicia
4.	Felicia:	(inaudible)
5.	Teacher:	Not (…) why?
6.	Felicia:	Because if you add seven for example on both.
7.	Teacher:	Ahaa
8.	Felicia:	If it were the same number it would be the same.
9.	Teacher:	So it wouldn’t be the same.
10.	Felicia:	If it were the same number it would be the same.
11.	Teacher:	Aha (…) are you following her? She said that if I add seven on that side and then you think you have a number line that is so long, then we have the zero there and we have minus five hundred there and plus five hundred over there (…) and then I add seven on that side (…) [writes -500+7] what do I get?
12.	Felicia:	Do I get more minus then?
13.	Teacher:	What did you say?
14.	Felicia:	Do you get more minus then?

15. Teacher: Yes what do I get then? (…) what do you think?

16. Boy: On the positive side you get five hundred and seven (inaudible)

17. Teacher: Then I jump to five hundred and seven on that side (…) and here you say (…) that I have, I will come to get (…) four hundred and ninety three. Minus in front of?

18. Boy: Ah

19. Teacher: How do you think when you think like that?

20. Boy: (inaudible) lies on the minus side (…) so one comes seven plus (inaudible).

21. Teacher: Mmm. So you go closer that way and go seven steps so (…) so they are different numbers here then. [The teacher shows, on the number line, that adding seven means that one goes seven steps to the right on both in the positive and the negative number.]

From this it follows that the students had the possibility to experience that the numbers become larger the further to the right on the number line they are.

Critical feature 1: Subtraction as a difference

In ***lesson 2E*** it was made possible to experience the feature of *difference* when the teacher made a contrast between subtraction as "take away" and as a "difference" (comparison). The excerpt shows how the teacher explicitly pointed out that there are two ways of thinking about subtraction. The teacher said "I think you are used to thinking about subtraction as one kind of thought. I am not sure, but I think so (…)" (excerpt 9, line 1). It was possible to simultaneously experience examples of 5 – 3 = 2 as a "difference" (comparison) – "How much more do I have compared to Lisa here? /…./ I have five you have three, the difference was evidently two" and as "take away" – "I still have five kronor then I give away three kronor of what I have".

Excerpt 9 [00:12:49 to 00:16:50]

1. Teacher: I think you are used to thinking about subtraction as one kind of thought. I am not sure, but I think so /…/ [a line later] (…) I have five kronor and you have three kronor. I have five and you have three. How much more do I have compared to Lisa here?

2. Student: Plus two kronor [+2]

3. Teacher: I have plus two kronor. How can you write that, what we just said, how can one write it? Exactly

what we said there. I have five you have three, the difference was evidently two.

4. Daniel: (Inaudible)

5. Teacher: We write this five minus three is two (5–3 =2). But now it is like this that I still have five kronor, then I give away three kronor of what I have /…/ two same ways of writing two different thoughts /…/ one thing minus the other is a take away thought. Is this a take away thought when I had five kronor and Lisa had three and I wondered how much more I had?

6. Student: A comparison

7. Teacher: It is a comparison thought, we are looking at the difference, and difference and take away are two different ways of looking at subtraction and we think that most often you are so used to the idea of take away that you don't see it as a difference and when you are working with negative numbers it is (…) damn bad so to speak to think take away because it is really hard then, it is much better to think about the minus sign as a difference.

The teacher pointed out that when you are working with negative numbers "it is (...) damn bad so to speak to think take away because it is really hard then" (line 7). The last statement makes clear that subtraction as a "take away" is not appropriate when dealing with negative numbers. Later in the lesson the students were given the task to find pairs of numbers that had a difference of four. The numbers were supposed to be positive, negative or one on each side of the zero on the number line.

In ***lesson 2F*** the teaching made it possible to, instead of seeing subtraction as a "take away", to see it as a *difference* between numbers. The teacher used the number line, debts and different lengths to show differences between numbers. The teacher showed a difference in the amount of money two persons have, "I have five kronor and Cissi has three kronor, what is the difference between what we have?". This was also compared with the same numbers on the number line: "If I would look at the number line, I would have three here and five kronor that I have. Between these it differs two". The students were, as in lesson 2E, encouraged to see subtraction as a difference instead of take away. In the same way as in lesson 2E, the student worked with a task of finding pairs of numbers where the difference is four.

Lesson 2G, compared to 2E and 2F, enacted the feature of *difference* in the lesson in a somewhat different way. In lessons 2E and 2F, it was pointed out that ‘take away’ and ‘difference’ were two ways of thinking about subtraction. In lesson 2G it was treated as different names for the same sign, “this sign that you call take away (subtraction sign) means the difference between two numbers” (excerpt 10, line 1).

Excerpt 10 [00:07:04 to 00:08:56]

1.	Teacher:	One way of understanding this is to understand that this sign that you call take away means the difference between two numbers. So, now we look at our number line again /…/ (a line later). The signs can actually have two meanings, it could mean this subtraction and take away and it could also indicate a negative number. Now find two numbers on the number line where the difference between them is two. Ahh
2.	Student:	Minus one and one
3.	Teacher:	Minus one and one, yes [strong voice] but, the difference between minus one and one because here there are two steps in between, the difference between minus one and one is two. Now let’s see mmm. It is not really right here because it should actually be minus two here then [(−1) − 1 =]. But, mm a negative two in difference. If we start with saying the biggest number first all the time we make it a little bit easier for ourselves.

The excerpt shows that the teacher points out that it is easier to put the largest number first when the difference between two numbers is determined, probably in order to avoid a negative difference (line 3). However, it was later made possible to experience other examples that generate negative numbers as answers (magnitude with negative direction) in subtraction tasks.

Lesson 2H also made it possible to experience *difference*, although in a somewhat different way. When the teacher introduced the notion of difference on a number line, one student interpreted the difference as the numbers in between the numbers and not as the steps in between. The teacher expected the answer to be 2 as the difference between 3 and 1 since it is two steps between the numbers. The student answered 1 since it is one number, namely the number 2, in between. After a discussion about difference as steps between numbers, several examples of number pairs with the difference of 2 were presented (2 and 0, 1 and (-1), (-1) and (-3)). The subtraction sign was later connected to the notion of difference.

In ***lesson 2B*** *subtraction as a difference* was introduced first as a difference between two persons comparing economies or lengths. The difference between numbers on the number line was then discussed. There was talk of subtraction as a difference between two persons' economies/financial state and the students finally produced examples themselves for practice.

Critical feature 2: Perspective – the commutative law does not apply in subtraction

Lesson 2E made it possible to experience that subtraction could be seen as a difference in the second term from the first term. For instance, when the class discussed (-4) - (-1), it was made clear that the term from which the difference between the numbers is seen is not arbitrary. The teacher said "it could be a new thought for you, one usually looks at the one that is in the first position" (line 1) and later repeated "one always looks at the first" (line 5).

Excerpt 11 [00:51:53 to 00:52:50]

1. Teacher: Okay, now I am going to say a new thing, it could be a new thought for you, one usually looks at the one that is in the first position, in this case it is a negative four [(-4)-(-1)], one takes that perspective for just that number and one checks, is that one in a worse position (is less) than that one, which number is biggest and which is smallest, one compares, is it bigger or smaller than that.
2. Student: Less
3. Teacher: It is less yes, I call it "worse factor". It is, if one thinks about money right, this could have been Holly's money and Jasmine's money then Holly would have much worse economy than Jasmine. From her perspective she is in a worse position and then it is negative. So it is negative three. But, we turn it around if it would look like this instead (…) now it is this one [(-1)] that is in the first position like here, that is minus one and a negative four in the second place [(-1)-(-4)], what is the difference between them, (…) Eva what do you say is the difference between these two, in numbers on the number line. Negative one and negative four, I put negative one here and what is it? How many steps are there?
4. Eva: (inaudible)
5. Teacher: Three steps. Is it in a better or a worse position then that one then it is a positive three, perspective, one always look at the first.

The teacher implemented a new concept “worse factor” to indicate which number was the smallest in the expression. In this case, (-4) – (-1), (-4) is less and in a worse position compared to (-1). The answer is therefore (-3) since the difference between the numbers is seen from the position of the first term. The contrast was a variation of meaning in the order of terms.

In ***lesson 2F***, as in lesson 2E, experience was offered that the order of the terms matters in subtraction. The teacher used an easy example to discuss that subtraction does not apply to commutative law. The teacher asked, “If I turn them. Is it [+5 - +3=] the same thing as that above [+3 - +5=]?”. In the same way as in lesson 2E, the contrast in terms made it possible to experience the feature.

In ***lesson 2G*** the teaching made it possible to experience that subtraction is seen from the *perspective* of the first term. In a first step s/he asked, “How big is the difference” between the terms and, in a second, “how is [the first term] Stina then in relation to [the second term] Åsa?”. The number line was used in the first step (see figure 30). The teacher made a distinction between the fact that addition is commutative and subtraction is not, “So, here it mattered now when we counted with subtraction and I changed the order (strong voice). It made the same answer here (points at the addition tasks) but it is not here (points to the subtraction tasks) therefore we must always think from the one that is first” (excerpt 12, line 5).

Excerpt 12 [00:29:59 to 00:31:40]

1.	Teacher:	How big is the difference between the money they have? Ah
2.	Student:	Two
3.	Teacher:	It is two kronor then. Åsa, or Stina has two kronor more than Åsa [5 – 3]. Does it matter if we change here now. Now we say that Stina has three kronor and Åsa has five kronor [3–5] . And we check, what has, how is Stina then in relation to Åsa? Ah
4.	Student:	Minus two
5.	Teacher:	She has minus two. She is 2 kronor worse than than Åsa /…/ Mm so, here it mattered now when we counted with subtraction and I changed the order (strong voice). It made the same answer here (points at the addition tasks) but it is not here (points to the subtraction tasks) therefore we must always think from the one that is first. So, Stina how is she in relation to Åsa oh the difference there is minus two when I look at what Åsa has.

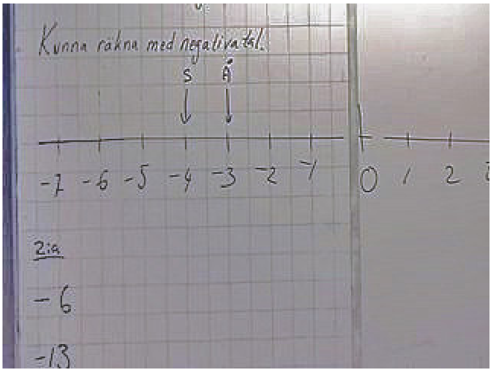

Figure 30. The difference between Stina's (S) and Åsa's (Å) economies, for instance (-4) - (-3)= (-1) as shown on the number line.

Lesson 2H made it possible to experience subtraction as a difference seen from the first term. The teacher said "The thing is that when one looks at difference one always compares with the number that is first". Another way the feature was elicited was through a discussion of the commutative law. The teacher said "does it matter who has the most debt when one looks up on their shared economy?" (excerpt 13, line 1). The excerpt shows that a student mixes up addition and subtraction and that it could be seen as difference between the numbers (lines 3, 5 and 7).

Excerpt 13 [00:32:41 to 00:33:19)

1.	Teacher:	/…/ does it matter who has the most debt when one looks up on their shared economy [addition]?
2.	Student:	No
3.	Student:	Yes
4.	Teacher:	Does it matter?
5.	Student:	Ah [Yes]
6.	Teacher:	In what way?
7.	Student:	Three is on the right
8.	Teacher:	You are thinking about this when we worked with difference.
9.	Student:	Ah [Yes]
10.	Teacher:	But in this case it is not the difference we are looking at, is it. Now they are friends and we add. So here one can say the counting sign [operation] stands for adding together. Before it was the difference. It is a good observation you make.

Later in the lesson, when they discussed subtraction, the teacher asked again "does it matter if Åsa has a debt of three (-3) and Tina has a debt of five (-5)?". The teacher concluded, "yes it does". It was therefore possible for the students to discern that the order of the terms does not matter in addition but that it does in subtraction.

In ***lesson 2B*** the feature of *perspective* was enacted when the teacher and a student pointed out that the difference between the numbers should always be compared from the first term. A student (Felicia) said "But it is always from the first" (excerpt 14, line 2) and the teacher responded "Ah it is always from the first you look at" (line 3). The student (Felicia) also made a distinction between the answer (direction and magnitude) and the difference (magnitude). She argued that the answer could be either positive or negative but that the difference is always positive (line 8). This distinction was not made in the other lessons.

Excerpt 14 [00:37:41 to 00:38:14]

1.	Teacher:	Ah the difference is three between these. But you must see from which point of view from his point of view from Bo's point of view/
2.	Felicia:	/But it is always from the first.
3.	Teacher:	Ah it is always from the first you look at.
4.	Felicia:	Then you can't say that the answer is difference.
5.	Teacher:	(…) Ohh
6.	Felicia:	It is (becomes) the answer (…)
7.	Teacher:	Ah it is true (…)
8.	Felicia:	The answer is the answer. The answer is minus three the difference is three. It is not the same thing.

Other features of negative numbers

In lesson 2F it was possible to experience that a decimal number, for instance 0.5, is not to be placed together with negative numbers (cf., Stacey *et al.*, 2001). The excerpt below shows how one student (Therese) at the beginning of the lesson placed 0.5 on the left side of zero on the number line (line 1) indicating that it is a negative number. The teacher introduced (-0.5) and asked where it should be placed (line 13). During the discussion it became possible to discern that 0.5 and (-0.5) were different numbers placed on each side of zero on the number line.

Excerpt 15 [00:02:32 to 00:03:00]

1.	Therese:	Zero comma five.
2.	Teacher:	Zero comma five. (Writes 0.5 on the left hand side of zero). Is it on this side of zero (the left side) just by the zero here. Okay, what do others say. Any one that thinks something else?
3.	Boy:	Minus five
4.	Teacher:	Minus five. Hm here we have one with a minus in front. Where should it be, here or somewhere, but this then (0.5)?
5.	Boy:	A little in front.
6.	Teacher:	A little in front you say (writes 0.5 to the right hand side of zero).
7.	Boy:	Yes
8.	Teacher:	What determines whether it is on this side or that side? What do you say Therese, are you with us on that one. They want a minus in front there. Hm is there anyone that can
9.	Boy:	Ah that is a minus number (inaudible) on the left side (inaudible).
10.	Teacher:	Ah yes. If it is a minus sign it goes on that side and if there is a plus sign (points to the right of zero). Is there a number that looks like this (0.5)? If you look at that one (0.5).
11.	Student:	Ah
12.	Student:	No, it doesn't.
13.	Teacher:	Where should it (-0.5) be then?
14.	Students:	(inaudible)
15.	Teacher:	Are you with us there Therese. That this is positive, is, it has squeezed itself in between one, but has not reached all the way down to zero. It is on this side (the positive side) /…/

Previous research (Stacey *et al.*, 2001) and this excerpt indicate that the feature of the distinction between negative and decimal numbers could be critical for some students that have not yet discerned the placement of decimal numbers below 1, for instance 0.5, in the numerical system.

Conclusions

Critical features for student learning

One of the aims of this study was to explore the extent to which other teachers could make use of findings of a learning study. The study was designed to compare lessons in which the same teacher used different lesson designs. The difference between the lesson designs was that only some of the critical features identified were implemented in LD1 while all features were implemented in LD2. The analysis of the eight lessons in this study shows that all teachers enacted the critical features as intended and in a systematic way. In lesson 2B a student's questions opened further features of the object of learning to be enacted in the lesson. The teachers were aware of the sequencing of the implementation and used patterns of variation to make the feature more likely for the students to notice. For example, a pattern of variation, contrast, was used to make the different meanings of the minus sign more explicit, for instance by comparing the functions of the signs in 5 – 3 with 5 – (-3), as in lesson 2F.

The analysis of lessons and student tests indicates that LD2 had a greater impact on student learning than LD1. In LD2, all features identified from the learning study were implemented in the lessons, and this had a greater impact on student learning than lessons in which some of the features were missing. If it is possible to experience several features simultaneously, it is more likely that the student will develop a more complex understanding. It is plausible that some students in LD1 and LD2 were already aware of some features of negative numbers – especially the feature of *sign*. Although researchers point out that this is a very important feature, it was the feature that teachers seemed to be aware of (Ball, 1993; Vlassis, 2009).

In the learning study, this was the first feature that was identified that was found in both the literature and practice. The feature of the numerical system was another feature that was more likely discerned by the students in advance as compared to the other two features of difference and perspective. I argue that it is more likely that students had previous experience of these two features. This could explain why the difference between LD1 and LD2 in terms of learning outcomes is smaller than in the study of density of rational numbers, where the different lesson designs more closely followed the order in which the features were discovered in the learning study.

The range of change

The examples that it was possible to discern in the dimension of variation, the 'range of change', in the lessons analysed in this study were more similar in the different lessons than in the study of rational numbers. For instance, the different 'range of change' in examples of numbers in intervals made it possible for students to experience density of rational numbers differently. In the study discussed in this chapter, all lessons had a systematic approach and a similar range of change, going from using only positive numbers to negative numbers in the examples and tasks given. I use lesson 2D to illustrate a typical range of change used that implements the feature of subtraction as difference (see figure 31). In this example, the difference is constant and the numbers vary.

9 - 5 = 4

8 – 4 = 4

2 - (-2)= 4

(-2) - (-6)= 4

Figure 31. The examples (*range of change*) implementing subtraction as difference in lesson 2D.

The 'range of change' used in addition tasks was constant, allowing only the terms to vary in place, in order to make it possible to experience commutative law, 5 + 3 = 8, 3 + 5 = 8, and later the sign in front of the term, 5 + (-3) = 2, (-3) + 5 = 2, (-5) + (-3) = (-8) and (-3) + (-5) = (-8). Using the same 'range of change' (examples/values) as a contrast to addition, it was made possible to experience the perspective that commutative law does not apply in subtraction, 5 - 3= 2, 3 - 5 = (-2), 5 – (-3) = 8, (-3) – 5= (-8), (-5) – (-3) = (-2) and (-3) – (-5) = 2.

However, there was a noticeable difference in the patterns of variation used. I use the implementation of *subtraction as difference* as an example to illustrate this. In some lessons (e.g., 2A, 2B) the contrast between two views of experiencing subtraction as a 'take away' or 'difference' was brought up and discussed. In other lessons (e.g., 2D, 2G), however, this was taken for granted to a greater extent and seen as one view with different names or one view without introducing the other. The same critical feature was enacted in the lessons in one sense, but there were different possibilities to notice the feature. Patterns of variation, such as contrast, separation, fusion and generalisations, could be used as a means to make features noticeable to students and thereby make it more likely that the intended object of learning would coincide with the lived object of learning.

CHAPTER SEVEN

CONCLUSIONS AND DISCUSSION

The overall aim of this thesis is to contribute to research on teaching and learning. The relationship between teaching and learning has been the subject of a substantial amount of research (cf., Nuthall, 2005; Oser & Baeriswyl, 2001). A fundamental distinction that is made between some of that research is whether it is the *teacher* or the *teaching* that makes the difference for student learning in classrooms. In design research and lesson studies, for instance, teaching is suggested to play a major role for student learning. In a similar approach from which the research in this thesis emanates, learning study, the teaching of a particular topic is investigated by teachers with the purpose of enhancing student learning. In a learning study, teachers collaborate in an iterative process to explore their students' learning; they plan a lesson, enact the lesson, analyse the lesson and what the students have learned, and revise the same lesson plan several times. In this way, the teachers make an in depth analysis of what is taught and what is learned from their teaching. However, this does not imply a one to one relationship between teaching and learning. It is rather argued that, when something is made possible to discern from teaching, it is more likely that students experience and learn.

This is not to say that teachers are not of importance. Certainly they are, although Nuthall (2004) argues that there has been an emphasis on teachers' personal qualities instead of how they act in the classroom. Nuthall argues for an approach in research that will produce a practical understanding of how teachers' actions contribute to student learning. This thesis can be seen as such an attempt and to make a contribution in this spirit. It takes as its point of departure findings gained from learning studies in mathematics and explores whether these findings can be of use to other teachers in other contexts to promote student learning. A quasi-experimental design was set up to explore how these findings – *critical features* identified for student learning – implemented in different lesson designs were made use of by other teachers and enacted in classrooms and what their implications were.

What is taught and what is learned

Questions the thesis was set to answer concern the usefulness of the model of the description of teaching and learning used – the notion of *critical features* – for the understanding of the nature of this relationship and teachers' use of them to enhance student learning. Analyses were made of how the teachers implemented the critical features in new lessons, what was made possible for the students to experience, and what students learned. The questions that the research has explored are:

- What difference does the presence or absence of these critical features make for student learning?
- Can these critical features be used as a description and resource of knowledge between teachers to enhance student learning?
- In what way can the notion of critical features contribute to knowledge about the relationships between teaching and learning?

This chapter presents a synthesis of two studies and conclusions drawn from the research are presented and discusses the research questions. Theoretical and practical implications of the research are elaborated. In the final section, conclusions drawn from the research and limitations are discussed and suggestions for future research are presented.

The significance of critical features

The thesis elicits teaching – learning relationships by exploring what was made possible for students to learn from teaching. A point of departure in this research was that, if critical features of an object of learning are made salient for students to discern, it is then possible and more likely for them to learn. To test the implementation of critical features, two lesson designs with different features were developed. The rationale for two lesson designs was that they would reflect the order in which the features were found in the learning study. The lesson designs also reflected an implementation *of all* or *some* identified critical features in the lessons. An important part of the research design was that the same teacher implemented both lesson designs 1 and 2. The idea in the quasi-experimental design was 'methodologically' to separate the teacher from the 'teaching'.

What difference did the presence or absence of the critical features make for student learning?

From the analyses made of the data in this research I suggest that the critical features identified in the two learning studies were transferable to the investigated groups. Both studies showed that the critical features enacted in lessons most likely promoted student learning. The grounds for this claim are based on the analysis of student tests and the video recorded lessons. Analysis showed that, when it was possible for students to discern all the critical features that were identified in a lesson, the student learning outcome was high. When it was possible to experience only some critical features, student learning according to the post test was smaller. Subsequently, the effect size was higher for lesson design 2 (LD2) compared to lesson design 1 (LD1) in both studies. Additional support for the finding that critical features can be made use of to enhance student learning in other groups has been suggested by Runesson and Marton (2009). In their study, critical features from two learning studies in Hong Kong, one about fractions and one about writing skills, were made use of and found to enhance student learning when taught by teachers in lessons in Sweden.

Further support for the analysis of the significance of critical features explored in this thesis was that students performed better only on test items directly related to the capability that the students were supposed to learn – the object of learning. This was shown for instance in the study of rational numbers, where three particular tasks on 'density' indicated significantly improved results for lessons 1E and 1F. This supports the conjecture about critical features of an object of learning. Hence, what is critical is dependent on the capability to be learned. This was not possible to distinguish in the same way in the study concerning negative numbers since the whole test was about the object of learning, being able to calculate addition and subtraction of negative numbers, with the exception of one item about different meanings of the signs (appendix 4, item 7). This item showed a significant difference between the lesson designs in favour of LD2. It is plausible that there are more critical features that have not been identified in the learning studies or in these studies. This is more likely in the study of negative numbers since the effect on student learning in this case was smaller than in the study about the density of rational numbers. However, this needs further empirical investigation.

The findings contribute to mathematics education research by suggesting how content specific features made a difference for student learning in the investigated groups. It is plausible that critical features could be identified through learning studies in topics other than mathematics as well and most likely be used in the teaching of other groups of students to enhance student learning. However, there must be a sensitivity to students' previous experience of what is taught with regard to their pre knowledge and how they experience what is taught. The critical features are a result of the investigation of how students experience what is taught and could therefore not be derived only from the contents and topic taught. I use an example of a critical feature found in the study on negative numbers to show this. The feature of the *numerical system*, that the numbers on the number line become larger the farther on the right hand side you go, could not been identified from the mathematics itself. It was instead identified when the teacher realised that the students experienced the numerical system in different ways. It was found that some students experienced that the numbers become larger from both sides of zero and some that the numbers become larger the farther on the right hand side of zero they are. Nevertheless, research about students' difficulties in mathematics and student learning of mathematics could contribute to identify critical features for student learning. The way in which teachers made use of the knowledge about critical features is elaborated in the next section.

Critical features, a common resource of knowledge for teachers to enhance student learning?

The findings of these studies suggest that critical features could be made use of by the teachers who participated. It is important to note that these teachers were somewhat familiar with learning study and variation theory. It is possible that other teachers would not have succeeded with the implementation of critical features in the same way as these teachers since some pre knowledge about the theory can be necessary. Analysis of the video recorded meetings with the teachers suggests that critical features were used by the teachers as a means to promote student learning. For instance, discussions about critical features raised the question of what the students were supposed to discern and hence provided a clear view of what they aimed to bring to the fore of their students' awareness. Before the teachers planned and enacted the lessons, they watched a video recording of another teacher implementing the feature in the learning study lesson. This provided an opportunity to discuss the features, although the teachers did not mimic that teacher's teaching. Instead they tried to adapt the teaching by implementing the critical features with their own tasks and organisation, taking their own students into account.

The teachers' different experience in conducting learning studies was not found to be of the utmost importance, however. This was shown by the fact that one of the teachers with the least experience and one of the teachers with the most experience succeeded in bringing out a wide range of change within the critical features (e.g. teachers in lessons 1E and 2B). It was rather the way the teacher and students opened up dimensions of variation in the critical features that seemed to matter.

Another finding with regard to critical features was that it seemed not to be sufficient to tell the students the critical features. For example, in the study about rational numbers (LD1), some teachers explicitly told the students that there were an infinite number of decimal numbers in an interval. However, this did not contribute to student learning as shown on the post tests. This suggests that these students did not have the possibility to experience that rational numbers are dense. This supports conjectures made in variation theory that it is probably not sufficient simply to express the critical features. Rather, these must likely be discerned by the learner. To be able to experience the density of rational numbers, there are features of that object of learning that must be made possible to discern.

The critical features were implemented in somewhat different ways with different possibilities to experience variation within the features. On a detailed level this implies differences in patterns of variation used in the lessons by the teachers and students. The way in which the features were enacted in the lessons was shown in excerpts. All teachers except two implemented the features as planned. In two cases the implications of the features was not fully communicated to the teachers or was not in the teachers' awareness. Neither of these two teachers commented that they missed a feature in the stimulated recall interviews.

On a more general level it was found that how variation within dimensions of variation or range of change (values e.g., examples, counter examples, numbers etc.) was introduced in a lesson, by teachers or students, also differed. Runesson (1999) states that the variation present in a lesson could be different, since these are dynamic and exist to various extents. She states that, on the one hand, if a teacher singly and alone introduces variation (see figure 32, E1 and E2), there is a risk that unrecognised features emanating from students' experience of what is learned are not made possible to experience. Then the teacher does not open up other possible features coming from students' questions in relation to the content. On the other hand, if students are relied upon to introduce and contribute to variation within dimensions of variation of the object of learning, this might not be sufficient (E3). Runesson did not find examples in her study of high levels of variation being introduced by both the teacher and students (E4).

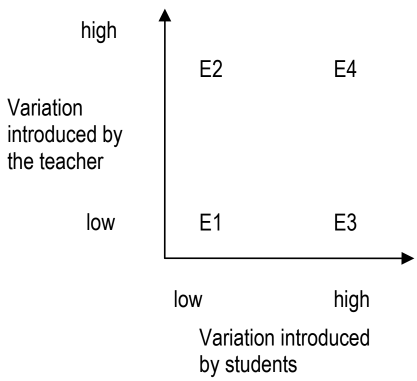

Figure 32. The variation introduced in the enacted object of learning. The figure is based on Runesson's (1999, p. 290) model in which E4 was not found in her empirical data. E is the variation in the enacted object of learning.

In this thesis an example where both the teacher and students opened up a high level of variation (E4) was found. In the study about negative numbers (in lesson 2B) the students and the teacher expanded the space of learning and hence what was possible to learn by opening the possibility to experience more features than planned by the teacher. In this lesson the students, and in particular one student, raised questions about the object of learning, possibly due to the need to establish meaning and understanding about negative numbers. In this case this opened yet another *possible critical feature* to be discerned, namely the differentiation between what the students call 'the difference' ('the answer', a positive or negative number) and the 'absolute difference' (a positive number).

The importance of variation generated by students claimed by Runesson (1999) is supported by Al-Murani and Watson (2009), who suggest that the variation generated by the learner is a partial articulation of the lived object of learning (OOL). Variation generated by learners provides a window into students' awareness and contributes to the public domain while being possible for other learners to experience.

> The process of articulation moves the lived OOL from the private domain into the public domain, contributing to the potential development of the enacted OOL by making it available for all learners. The enacted OOL is therefore influenced not only buy the intended OOL but also the expressed lived OOL of learners (Al-Murani & Watson, 2009, p. 4).

In some lessons it was the quite the opposite, for instance lessons 2C and 2G, where the teaching did not open as much for variation with regard to students' questions and comments in relation to the content (E1 and E2) and a correspondingly lower variation in students' understanding of the object of learning was visible. Even if critical features and variation with regard to them was present, it was more restricted. From a theoretical point of view, one can advocate that this can matter for students' possibilities to learn. The studies indicate that some teachers seemed more aware of creating variation in their teaching of critical features by systematically engaging in variation from students' questions and students' feedback. Of the 16 lessons analysed in this study, most of them show that the teacher introduced a high level of variation (E2), but students did not contribute as much to the variation introduced. Although it is common that students introduce variation in some parts of a lesson, the teacher is not always open or sensitive to that variation.

In summary, it is suggested that critical features can be gained and shared by teachers. The variation that it is possible to experience with regard to the critical features can make the features more or less noticeable to students. The next section discusses critical features as a means to produce knowledge about teaching and learning.

Can the notion of 'critical features' contribute to knowledge about the relationships between teaching and learning?

In the spirit of Nuthall (2004; 2005) and Hattie (1992; 2009), a conjecture made from these studies is that what was taught was reflected in what was learned. This is not however a claim that the relationship between teaching and learning should be seen as deterministic and as a one to one relationship. Teaching and learning are complex enterprises that include many different factors. As has been stated earlier, Nuthall argues for a theory that could be used to explain and establish teaching – learning relationships. Variation theory was used in this thesis as such a tool. The analysis made with variation theory indicates that implementation of content specific critical features in lessons had an effect on learning. For example, when learning about the density of rational numbers, the feature of *divisibility* seemed to be of decisive importance (see lessons 1E and 1F). A conclusion drawn was that, if it was possible to discern the feature in a lesson, it was more likely that students learned; hence this indicates possibilities for learning the density of rational numbers. The other critical features identified seemed to have contributed to a deeper understanding of rational numbers and to be a prerequisite for divisibility. In the study about negative numbers the features of *the sign* and *the numerical system* contributing to student learning (see LD2). In this case it was not taken for granted that the students were already aware of these features (cf., Häggström, 2008) and they were therefore elicited by the teachers. Even if this example is different from the example of density, it shows that teachers can make use of features that 'should not be taken for granted' of the object of learning.

Theoretical implications of the study

The theoretical contributions of the findings of this research are in relation to variation theory and to research about educational practice in general. As pointed out earlier, variation theory offers tools to analyse teaching and learning in commensurable terms. This implies that the intended, enacted and lived object of learning can be analysed in a similar way with the same analytic tools. At the same time, variation theory is also a learning theory that focuses on conditions for learning and says that variation is necessary to differentiate and notice what is to be learned. The presence or absence of critical features has been indicated to influence student learning. As stated previously, even if a critical feature is made possible to experience in a lesson, this does not imply that all students learn. For instance, in mathematics, students might focus on dimensions of variation (d.o.v.) offered by the modelled situation (e.g., weather or shopping) instead of the d.o.v. offered by generalities of mathematics (Watson & Mason, 2006). A conjecture made is that if it is not possible to experience critical features of an object of learning in any way during a lesson, it is not likely that students learn what is intended.

Critical features and range of change

Critical features are claimed to be critical in relation to a specific group of learners (Marton & Tsui, 2004). This study has shown that critical features assumed for one group of students identified in a learning study were indicated to be critical for other groups of students of a similar age, too. This suggests that critical features in these two studies can to some extent be seen to be ‘transferable’ to similar age groups. This finding is supported by similar research regarding critical features of fractions (Runesson & Marton, 2009). Furthermore, the finding supports the claim that there are certain features of the content that students need to experience to be able to learn (cf., Neuman, 1987; Runesson, 2009). This finding strengthens conjectures made in variation theory about critical features.

The studies have shown that the way in which critical features are enacted in a lesson seems to be important. It has been suggested that the variation that is possible to discern with regard to critical features is important for being able to discern the feature. The variation of ‘values’ in a dimension of variation present in a lesson is found in the examples used, the *range of change*. It was found that the teacher, students and the tasks used contribute to the range of change (cf., Watson & Mason, 2006). With Cohen and Ball’s (2001) definition of instruction, that it consists of interactions involving teachers, students and content it is possible to say that dimensions of variation and range of change are instruction in a broad sense. What is made possible to experience creates the space of learning. It was found in the study about rational numbers that the range of change opened up a smaller or wider space of learning. For example, experiencing only one way of interchangeable representation of 0.17, by 17/100, does not open as wide a space of learning as 17/100, 170/1000, 1700/10000 would. If it is only possible to experience 17/100, it is in fact possible for a student to experience that there is only one representation, i.e. by fractions. A question for future research is how the range of change affects students’ possibilities to discern the object of learning.

Practical implications of the study

While the findings of this study have been described on a detailed level, what can be learned from this study on a more general level? How can teachers benefit from this research? On a general level, the findings with respect to critical features indicate that learning can be initiated outside the learner (cf., Lobato *et al.*, 2005). For instance, a teacher can elicit critical features for an object of learning during a lesson to enhance student learning. An observation in the research is that acts of teaching are significant for student learning. As stated earlier in this chapter, learning is promoted when students have the possibility to experience content specific critical features. This has also been indicated in previous research (see for instance, Kullberg, in press; Marton & Pang, 2006; Runesson, 2007). One interpretation of the results can be that the implementation of several critical features opened a larger complexity of what was learned and thereby expanded the space of learning (cf., Runesson, 2009; Runesson *et al.*, in press). Runesson *et al.* (in press) showed that teachers in the beginning of a learning study wanted to reduce complexity in order to promote student learning. They wanted to make it easier for the students and thereby reduced the possibility for the students to make important distinctions. In the process of the learning study reported by Runesson *et al.*, the teachers discovered that it was necessary for student learning to make the object of learning appear in a certain way and opened for complexity in terms of more critical features and distinctions made in the lessons. Enacting several critical features simultaneously in a lesson could be experienced by teachers as making it too complicated for students. However, this thesis and other studies suggest that it can be necessary for student learning to experience several features simultaneously.

The studies showed that it was not the tasks used in different lessons that were of decisive importance but rather the dimensions of variation that it was possible to experience from the tasks and the lessons. This was shown for instance in lessons 1E and 1F, where different tasks were used. The teacher in lesson 1E used group discussions followed by a whole class discussion to elaborate the density of rational numbers, and lesson 1F used an individual worksheet and a whole class discussion. Despite the different tasks and organisation of learning, it was possible to experience the same critical features in the classroom, and these two classes were the most successful in terms of student learning. These findings concerning the role of critical features may contribute to an understanding of teacher practice and what teachers can address in order to promote their students' learning. In this way, the notion of critical features can be used as a tool for teachers to communicate about practice and student learning. The critical features explicitly point out elements in the teaching – learning process that teachers can use, observe and evaluate.

Critical features a source of knowledge among teachers?

The notion of critical features meets the criteria for what Hiebert *et al.* (2002) call public professional knowledge for teachers that can be communicated among colleagues and examined publicly. Hence the knowledge 'produced' in terms of critical features is both storable and sharable and contributes to teachers' awareness of what students need to discern to be able to learn a particular object of learning. The implication of this is that teachers can in this way explore their practice in a systematic way and share their findings. Therefore, to enhance student learning, teachers should have the possibility to explore their teaching in collaboration with other teachers. It has been indicated that this enhances both teachers' (cf., Gustavsson, 2008; Ma, 1999) and students' learning (cf., Runesson, 2009). This study suggests that other teachers and students may also benefit from teachers being involved in in-service training that enhances students' chances for learning. There are several ways for teachers to work in collaboration to enhance student learning, for instance in a learning study, lesson study (Fernandez & Yoshida, 2009), teaching learning sequences (Olander *et al.*, 2001) or design experiments (Cobb *et al.*, 2003). The knowledge produced may benefit other teachers and learners and provide a more 'evidence' based teaching approach. Teachers can benefit from analysis of teaching and reflection over the implications for students' learning. Pre service teachers should also be able to participate in this kind of work during their training, and it could be seen as an attempt to bridge the gap between theory and practice (Lagemann, 2000). This could bring about a more detailed discussion about mathematics content knowledge, pedagogical content knowledge (Shulman, 1987) and students' experience of what is to be learned. In this way teachers could become more aware of their own teaching and how it affects students' learning.

Final conclusions

The findings from the studies suggest that what it was possible to experience in terms of *critical features* and a certain *range of change* made a difference for students' learning. In this way a relationship between teaching and students' learning was indicated. The students' learning (the lived object of learning) was reflected in what it was possible to experience in the classroom (the enacted object of learning). The findings suggest that teachers could improve their practice by gaining or sharing knowledge in terms of critical features for their students' learning.

A conclusion drawn from the research is that the presence or absence of critical features in the two studies made a difference for students' possibilities to learn. This finding is supported by previous research using the same framework (cf., Kullberg & Runesson, 2006; Pang & Marton, 2005). It was shown that the *teaching* in these lessons had an impact on what it was possible to experience in the classroom and what students learned.

A second conclusion drawn was that it seemed not to be sufficient to tell the students the critical features. Analysis showed that, even if the teacher said that for instance there are an infinite number of rational numbers in an interval, most students did not take that into account when they answered the post test questions on the density of rational numbers. This suggests that it should be made possible for learners to experience critical features.

A third conclusion is that teachers and students together constructed the space of learning and what was possible to experience during the lessons. What the teacher and the students contributed in terms of questions and examples given opened for different possibilities for learning. The range of change is likely to be of importance for being able to discern critical features.

A fourth conclusion is that there are indications that critical features identified from learning studies are usable for other teachers in order to enhance their students' learning (cf., Runesson & Marton, 2009). A suggestion is that critical features can be used as a knowledge base for teachers that can contribute to discussions about teaching and what students have the possibility to experience from teaching. The final conclusion is that the theoretical framework used, variation theory, is a powerful tool for analysis of lessons that contributed to the analysis of teaching and learning in classrooms (cf., Häggström, 2008; Marton & Pang, 2006; Olteanu, 2007; Runesson & Marton, 2009).

Critical reflection on the studies

With what confidence can these conclusions from the studies be drawn? One could argue that some of these conclusions and claims are unwarranted. On the one hand it is important to note that the analysis is made from the theoretical framework used. On the other hand, there are limitations to the study that must be addressed. What do the studies not give an answer to? Some limitations in the studies are discussed in the following section.

The findings

With regard to the internal validity of the studies, it is sensible to question whether the conclusions made about the significance of the critical features in teaching for student learning are reasonable. Could something else explain student learning from the lessons? Is it possible to interpret the data in another way? Even though the analysis was regularly discussed with other scholars in different forums, it is possible that alternative interpretations could have been made of the video data. For instance, it is a matter of judgement as to whether a critical feature is present or not. In the studies reported, the grounds for this judgement have been described by excerpts to make it possible for the reader to judge the credibility. The study analysed critical features to have been implemented when the teacher or a student made the features noticeable by a pattern of variation, for instance through contrast or by creating variation with regard to the critical features within a dimension of variation. However, in some cases, the variation that it was possible to discern in terms of the critical feature was small and was still analysed as being present.

One important question to consider with respect to the reliability of test results, which in these studies are described on a group level, is group size. The groups in these studies must be considered small, and some scholars might argue that it is not reasonable to draw conclusions about student learning on tests between groups. Therefore, in these studies, the test results should be interpreted with caution. Furthermore, the teachers and students participating in these studies were not chosen at random. This also contributes to the fact that the results must be interpreted with a more modest certainty. Could something that happened inside or outside the classroom have affected the results? This can not be ruled out. It is possible that some students discussed the questions from the pre test with their parents or peers and that this would have affected the results.

It is important to note that the tests used were constructed by the researcher and are not standardized tests. A question to consider is whether the test items used explored the required capability. Can the questions asked have affected how the students answered? Moreover, it is important to note that the test items reflected the object of learning and therefore what was taught during the lessons. In the study about negative numbers, there were several items that accounted for different types of tasks, for instance items with two or three 'minus signs'. In this way there were many items that could be used for analysis. It is important to note that it is not possible to rule out that some students used the rule 'two minuses make a plus' or other ways of thinking in their answers to different items, even if some items called for written explanations. Three tasks were used in the study about rational numbers, where the students had to answer whether there were numbers in different intervals, [0.5; 0.6], [0.97; 0.98] and [0.99; 1.1]. It is possible that repeated testing in the same test of these items could influence student learning. However, examples of this were not observed in students' answers in the tests. The analysis of student answers was made on a qualitative, single-item basis and by looking at several items to analyse what a student tried to express and communicate in their answers. The use of words in tasks may also affect how students answer. For example, in the task, *Anne says there is a number between 0.97 and 0.98. John says there is no such number. Who is right and why?,* "a number" could have been interpreted as one number. However, in these cases, the researcher looked at the other items to see how the students answered questions about the density of rational numbers. A question to consider is also whether the pre test influenced the results on the post test. The same items were used on both tests and there is thus a risk of effects of repeated testing. This affected all groups, however, and therefore not the comparison between groups.

One can only hypothesise about whether the teachers in this study could have identified the critical features themselves by exploring their own students' learning. It is plausible that they would have identified at least some of them and perhaps other critical features as well. In this study they used critical features found by other teachers and other students. It is possible that the learning outcomes would have been even better if the teachers had identified them on the basis of their own students' experience of the object of learning. It should also be noted that the mathematical topic chosen and the decisions made for teaching affected the findings of the learning study. With respect to the mathematical content taught it is possible that, if the teachers had chosen to use other metaphors than, for instance, debt in their teaching about negative numbers, other critical features would have been found. However, this is uncertain and not within the scope of this study.

Video recorded lessons (observations) and tests were used to generate the data used in this research. In relation to the tests used, some scholars might argue that it is not possible at all to explore student learning by means of tests. There are many factors that can affect the outcomes of student tests (Bryman, 2004). Some scholars might advocate that other methods could be used, for instance interviews or stimulated recall interviews of learners. Video recordings of lessons is often used to analyse lessons (Clarke, Keitel *et al.*, 2006). However, *what* you analyse from the rich material that a video recorded lesson produces depends on choice. For instance, what was not explored in this research and could have made a difference for student learning, was the fact that some students at the back of the classroom did not participate as wholeheartedly as other classmates in other places in the classroom in the lessons that were analysed. This was noticed in every class and could have affected the learning outcomes shown in tests. It should be noted that, in the analysis made, many other things were left unexplored, for instance, the climate between the teacher and the students in the classroom, the use of artefacts or verbal language in relation to student learning, interaction and individual students' learning, and teacher and student behaviour. However, this was not within the scope and specific interest of the thesis.

A weak point in this research is that it is possible that the critical features that were identified and which form the grounds for this research might have been made from an incorrect analysis on the part of the teachers in the learning study. Whether the findings in the learning studies were usable for other teachers and learners was however explored in this research. With regard to external validity and whether the findings can apply to other teachers and students in other situations, these findings relate to specific studies and are not general. Even if it is possible to make connections with other subjects in school or other topics of mathematics, for instance, this is not within the scope of this thesis. It is important to take into consideration that teachers that try to make use of the critical features that are identified might not succeed. Why? It is possible that teachers do not succeed in making it possible to experience the features in the classroom in a way that is sufficient for student learning.

The theory

Variation theory has primarily been used to explore teaching and learning in school practice (cf., Lo *et al.*, 2005). Variation theory states that one way of teaching could be better than another way, since one way may provide different possibilities for the learner to experience the object of learning. Analyses made with variation theory focus on the content taught and what is made possible to experience. In the present research, variation theory was used as an analytic tool to analyse teaching and learning in commensurable terms. Variation theory was used in the learning studies that are the grounds for this research, in the design of the lessons reported, and for analysis of the lessons. Using the same theory to both design and analyse lessons may lead to circularity. However, the qualitative analyses of lessons showed that the intended, enacted and lived object of learning sometimes did not coincide, since in some cases what was intended to be enacted was not, and furthermore what was not intended was enacted (Wernberg, 2009). This suggests that the analysis followed the data. A theoretical perspective orients the researcher towards specific aspects of the research. In the analysis presented in this thesis, variation theory made it possible to analyse and systematise critical features and dimensions of variation that were opened for students to experience. On the one hand, it could be argued that this influenced the analysis. On the other hand, the deductive and quasi-experimental design could instead contribute to keeping certain variables such as the teacher and the critical features constant in order to make the connection to student learning more noticeable.

Is it possible to reach these conclusions with other theories? Other theories, for instance socio culture theory (Lave & Wenger, 1991), individual constructivism (Piaget & Inhelder, 1956) and distributed cognition (Hutchins, 1995), do not argue for the significance of how content is handled for students' possibilities to learn. Instead they focus on interactions (distributed cognition, socio culture theory) or constructive actions of the learner (individual constructivism). What does variation theory not do? In my experience, variation theory has not been used to study an individual student's learning but is instead most often used on a group level. Furthermore, the analysis made is very specific and one could argue that studies using a variation theory framework can be difficult to generalise since what is particular to the study in question is explored in great detail.

The researcher's influence on the studies

The researcher in the present case was involved in the studies and worked together with the teachers in meetings and discussions of how to implement the critical features in lessons. The role of the researcher was to communicate the critical features and support and encourage the teachers to follow their way of thinking about the planning of the lessons. Could the teachers have interpreted and implemented the critical features without the help of the researcher? Considering that a conjecture made in this thesis is that teachers can make use of the critical features in their practice, this is an essential question. It is possible that the collaboration with the teachers affected the study and the lessons in some way, even though the researcher tried to take a neutral position. The researcher's knowledge about learning study and variation theory could have affected and contributed to the planning of the lessons. It is possible that another tutor who was not knowledgeable in the theory and in learning studies would have influenced the studies in another way.

Another question is whether the researcher influenced the analysis of the lessons and tests by being biased with regard to the theory used. Research is a theory-laden practice. Research questions, choice of methods and analysis are influenced by theory. The analysis of video data required some kind of data selection that was influenced by the researcher's interpretation and analysis of what happened in the classrooms and in this way was also influenced by theory.

The analysis of lessons and tests was made with an effort to be as objective as possible.

Suggestions for future research

New questions arising from this research concern the use of variation theory to further explore how teachers' actions contribute to student learning. The thesis explored how eight teachers made use of *critical features*, a central concept in variation theory. The teachers in question were somewhat familiar with the theory of variation and made use of it in planning their teaching. The use of variation in teaching is nothing new. Variation is commonly used to tell things apart – to make contrasts between features of an object of learning that the teacher wants the student to distinguish (cf., Gibson & Gibson, 1955). What is new is that the teachers were aware of the critical features and used variation in a deliberate manner and planned ahead for the variation they would bring into the lesson. Studying how teachers used variation theory has not been the explicit focus of this study. Even so, the analysis showed that the teachers made use of variation in their teaching. Three suggestions for further research concerning critical features and teaching are suggested in the following section from the point of view of variation theory.

A first suggestion for further research would be to make a follow-up study to explore the variation used by teachers in more detail, particularly the *range of change* introduced in various lessons. This thesis showed that, even when the teachers implemented the same features, the students did not have the same possibilities to experience a similar range of change in the lessons. Studying the range of change may contribute to deepen an analysis using a variation theory framework with regard to the enacted object of learning. The implications of different ranges of change in relation to an object of learning in mathematics or in other subjects would shed further light on the relationship between teaching and learning (cf., Kullberg *et al.*, 2009; Watson & Mason, 2006). The range of change specifically points to differences in the variation that it is possible to discern in relation to a dimension of variation or specific tasks or concepts. To be able to study the effect of different ranges of change, it is necessary to conduct a micro analysis of teaching. This can be done with a quasi-experimental design such as the one used in this thesis.

A second suggestion for future research would be to further investigate critical features using interviews with students. Taking into account students' experiences of the object of learning before and after lessons in greater detail would contribute to deeper analyses of lessons in a learning study, for instance, or other research on critical features. An analysis of student interviews would add to a broader understanding of what is taught and what is learned. What does the individual student need to discern in order to experience an object of learning in a certain way? What critical features have not yet been identified? This has previously mainly been explored on a group level in learning studies and in the studies presented in this thesis. Investigations in this direction may be a step towards phenomenographic research, although analysed with a variation theory framework.

A third suggestion would be to explore how these teachers made use of variation theory to plan and implement lessons on regular basis in their ordinary work (Al-Murani, 2007; Al-Murani & Watson, 2009). Variation theory has been used in learning studies as a tool for teachers to plan and analyse their teaching. In this thesis, the teachers tried to use variation when they enacted the critical features in the lessons and, in this sense, the theory was separated from the learning study. To further explore how variation theory is used by teachers not in a learning study this would entail, for instance, following a group of teachers that deliberately use variation theory in their teaching over a longer period of time, collecting data from the teachers' daily planning or planning meetings together with analyses of video recordings from sequences of lessons. The sometimes justified methodological critique of drawing conclusions from analyses of single lessons may be met by using longitudinal empirical investigations of teaching and learning in this way.

SUMMARY IN SWEDISH

Vad lär eleverna och vad görs möjligt för dem att lära? - Professionella insikter om undervisning och lärande funna och delade av lärare i matematik

Inledning

Denna avhandling har tillkommit utifrån ett intresse av att bättre förstå relationer mellan undervisning i matematik och elevers lärande. Marton *et al.* (2004) menar att om man har för avsikt att förbättra elevernas möjlighet att lära, så måste man ta hänsyn till *vad* eleverna skall lära och den förmåga de skall utveckla. Flera forskare anser att det som eleverna skall lära – innehållet i undervisningen – har åsidosatts inom forskning och skolutveckling på senare år och istället har elevens lärande och sökande efter kunskap på egen hand betonats (Hiebert *et al.*, 1997; Lobato *et al.*, 2005). Lärarens betydelse för elevernas lärande har tonats ned, vilket enligt Chazan och Ball (1999) kan ha hindrat lärare från att ta en aktiv roll i elevernas lärande.

I avhandlingen görs antagandet att lärarens undervisning påverkar elevernas lärande (Nuthall, 2004). Avhandlingen undersöker vad eleverna lär utifrån den undervisning som eleverna erbjuds. Utgångspunkten för avhandlingen är tidigare studier, genomförda av lärare som arbetat tillsammans med att systematiskt studera sin undervisning, genom att försöka ta reda på vad eleverna behöver få möjlighet att erfara för att lära sig det som lärarna avsett. Studierna prövar om denna kunskap, funnen av lärare i en situation, kan kommuniceras och användas av andra lärare. I avhandlingen analyseras två studier om lärande i matematik, en om rationella tal och en om addition och subtraktion av negativa tal.

Bakgrund

Bakgrunden till avhandlingens båda studier är ett forskningsprojekt om *learning studies* (lärstudier)[36]. Learning study utvecklades i början på 2000-talet utifrån en japansk modell, *lesson study* (Fernandez & Yoshida, 2009; Lewis, Perry & Murata, 2006; Yoshida, 1999). Lesson study blev uppmärksammat i samband med jämförande studier av undervisning i olika länder och gavs som en förklaring och bidragande orsak till de japanska elevernas goda kunskaper i matematik (Stigler & Hiebert, 1999). I Japan och andra asiatiska länder är det vanligt att lärare arbetar tillsammans med att systematiskt planera och analysera undervisning. Forskarna menar att det bidrar till att utveckla lärarnas kunnande och därmed bidrar till bättre möjligheter för eleverna att lära (Ma, 1999; Stigler & Hiebert, 1999).

I en *learning study* arbetar lärare tillsammans med att planera, analysera och revidera sin undervisning om ett avgränsat kunskapsinnehåll (Marton & Pang, 2003; Marton & Tsui, 2004). Lärarna arbetar i en mindre grupp och ofta med stöd av en forskare, under en längre period, med att systematiskt undersöka vad som har betydelse för elevernas lärande av en avgränsad förmåga. Undervisning av en lektion förbättras, i en iterativ cyklisk process, genom att lärarna identifierar *kritiska aspekter* som de antar bidrar till att elevernas möjlighet att lära (se exempelvis Gustavsson, 2008; Holmqvist, 2006; Kullberg, in press; Lo, Chik, & Pang, 2006; Runesson, 2009). En skillnad mellan *learning study* och *lesson study* är att i en learning study så använder lärarna en teori om lärande, till exempel variationsteorin, när de planerar och analyserar undervisning. En ytterligare skillnad är att i en learning study så analyseras elevernas kunskaper både före och efter den genomförda lektionen, med hjälp av intervjuer eller tester för att kunna bedöma om undervisningen bidragit till elevernas lärande.

Resultaten från de genomförda learning studies, som studierna i denna avhandling har sin bakgrund i, visade att elevernas resultat på testen förbättrades genom lärarnas ändringar av undervisningen och att lärarna kunde förklara vad i undervisningen som gjorde att eleverna lärde sig. Lärarna hade identifierat *kritiska aspekter*, som i förhållande till elevernas förståelse, ansågs vara nödvändiga för eleverna att erfara. Denna avhandling studerar hur andra lärare använder sig av kunskapen om de kritiska aspekter som dessa lärare identifierat och ifall samma aspekter kan ha betydelse även för andra elevers lärande.

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Avhandlingens syfte och forskningsfrågor

Avhandlingen syftar till att ge ett kunskapsbidrag till forskning om undervisning och lärande i matematik. Inom ramen för avhandlingen genomfördes två studier, en om elevers lärande om negativa tal och en om decimaltal. I studierna deltog sammanlagt åtta lärare och sexton elevgrupper. Det undersöktes hur *kritiska aspekter* identifierade av lärare i learning studies, används i undervisning av andra lärare och betydelsen av dessa för elevers lärande i matematik. Följande frågor belyses i förhållande till de två studierna:

- Vilken betydelse har närvaro respektive frånvaro av de kritiska aspekterna i undervisningen för elevernas lärande?
- Kan insikter om kritiska aspekter förmedlas mellan lärare och användas för att bidra till andra elevers möjlighet att lära?
- På vilket sätt kan insikter om kritiska aspekter bidra till kunskap om undervisning och lärande?

Teoretiska utgångspunkter

Variationsteorin används i denna avhandling som teoretisk ram (Marton & Booth, 1997; Marton & Tsui, 2004). Teorin har använts i ett flertal klassrumsstudier för att analysera elevers möjlighet att lära utifrån den undervisning som erbjuds (Emanuelsson, 2001; Marton & Morris, 2002; Marton & Pang, 2006; Rovio-Johansson, 1999; Runesson, 1999). Variationsteorin är en teori om lärande som utvecklats utifrån fenomenografisk forskning (Marton *et al.*, 1977; Marton, Hounsell, & Entwistle, 1986; Marton & Neuman, 1996) och idéer om lärande som differentiering (Gibson & Gibson, 1955). Lärande beskrivs inom variationsteorin som att erfara något på ett nytt sätt, närmare bestämt, man lär något när man urskiljer aspekter som inte urskiljts tidigare. Marton och Booth skriver ”Att lära sig erfara olika fenomen, som ur vår synvinkel är den mest grundläggande formen av lärande, innebär att bli förmögen att urskilja vissa enheter eller aspekter, och att ha förmågan att vara samtidigt och fokuserat medveten om dessa enheter eller aspekter” (Marton & Booth, 2000, s. 161).

En central utgångspunkt inom variationsteorin är att man sig alltid *något*. Det kan vara en förmåga att göra något, som till exempel en addition av två tal, eller en förståelse av något, exempelvis att det finns oändligt många decimaltal mellan två olika tal. Det som man vill att eleverna skall lära benämns inom variationsteorin som *lärandets objekt*. Lärandeobjektet har *kritiska aspekter* som den lärande bör urskilja för att uppfatta lärandeobjektet på ett visst sätt. Vad som är en kritisk aspekt är beroende på vad som skall läras och ses i förhållande till dem som skall lära. Ett exempel på en kritisk aspekt som identifierats i en learning study om addition och subtraktion av negativa tal, i skolår 7 och 8, var att eleverna behöver erfara att talens värde blir större ju längre åt höger på tallinjen man kommer, exempelvis att 3 är ett större tal än -11 eller att -3 är större än -11. Denna aspekt identifierades när lärarna upptäckte att eleverna hade olika uppfattningar av talen, en del elever hade uppfattningen att talens värde blev större åt båda sidor om nollan medan andra elever hade uppfattningen att talen blev större ju längre åt höger på tallinjen talen var. En kritisk aspekt kan inte bestämmas i förväg enbart utifrån ämnesinnehållet, utan vad som är en kritisk aspekt blir synlig först i ljuset av elevernas förståelse av det som skall läras. En aspekt blir möjlig att urskilja om den varierar. I ovanstående exempel var den variation som erbjöds en kontrast mellan olika sätt att se talen. Variation kan exempelvis bestå av olika exempel som görs möjliga att erfara inom en aspekt genom variation och invarians.

Metod

Studierna har en kvasiexperimentell design. Genom två olika lektionsdesigner var det möjligt att jämföra lektioner i termer av olika kritiska aspekter. Lektionsdesignerna bestod enbart av kritiska aspekter och var ingen lektionsplan med uppgifter eller arbetsmetoder som lärarna förmodades följa. Samma lärare genomförde både lektionsdesign 1 och 2 (LD1 och LD2), vilket gjorde det möjligt att separera 'lärarens person' från undervisningen. Lärarna som deltog kom från två olika 0-9 skolor och var bekanta med variationsteorin och learning study. Lärare från samma skola arbetade tillsammans med forskaren med att planera lektionerna. Lärarna fick tillsammans med forskaren titta på videoinspelade lektioner (från learning study) där andra lärare använde de kritiska aspekterna i undervisning. Utifrån de kritiska aspekterna planerade lärarna tillsammans nya lektioner och elevuppgifter. I både LD1 och LD2 använde samma lärare likadana elevuppgifter och organiserade undervisningen på liknande sätt.

Lektionerna analyserades tillsammans med elevtester utifrån vad variationsteorin beskriver, det intentionella (vad läraren avsåg att eleverna skulle lära), det iscensatta (vad eleverna hade möjlighet att lära), och det levda lärandeobjektet (vad de lärde sig) (Marton & Tsui, 2004). Vad eleverna lärde sig från lektionen studerades med ett test, som genomfördes en vecka före och två dagar efter lektionen. Förtest och eftertest var identiska och analyserades både kvalitativt och med deskriptiva statistiska mått. Det empiriska materialet i de två studierna bestod sammanlagt av sexton videofilmade lektioner (ca 16h), åtta videoinspelade planeringsmöten med lärarna (ca 16h, bakgrundsmaterial), intervjuer med lärarna (ca 8h, bakgrundsmaterial), och elevtester. Elevgrupperna var inte slumpmässigt valda, utan lärarna undervisade i första hand i sina ordinarie elevgrupper. De deltagande eleverna hade skriftligt tillstånd från sina målsmän att medverka.

Analys och resultat

Studie 1: Elevers lärande om rationella tal

Bakgrunden till studien är en learning study om rationella tal i skolår 6 (Kullberg, 2004, 2007a; Runesson & Kullberg, in press). Det som lärarna arbetade med och eleverna förväntades lära var att förstå att det finns oändligt många decimaltal mellan två olika tal (Vosniadou *et al.*, 2008). Lärarna identifierade följande aspekter som nödvändiga för att eleverna skulle lära det som avsågs:

1. *Decimaltal som punkter på en tallinje*, denna aspekt innebär att se att ett decimaltal har en plats på tallinjen i förhållande till andra tal. Det innebär även att mellan två olika decimaltal finns det många andra tal.
2. *Rationella tals olika representationsformer*, innebär att se att ett rationellt tal kan ha flera olika representationer. Exempelvis kan 0.97 ses som 97/100, 97% eller 970/1000.
3. *Rationella tal som del av en helhet*, innebär att se att ett decimaltal kan ses som en del av en helhet, exempelvis 0.97 kan ses som 97 centimeter av en linjal på 1 meter, liksom 97/100 som 97 delar av hundra på linjalen.
4. *Rationella tals delbarhet*, medför att se att ett intervall, exempelvis mellan två decimaltal kan delas oändligt många gånger. Samma intervall kan delas i tio delar (tiondelar), tusen delar (tusendelar), tio tusen delar (tiotusendelar) etc.

Lärarna i avhandlingsstudien genomförde inte en learning study, utan använde enbart de identifierade kritiska aspekterna. Studien designades, som tidigare beskrivits, att fyra erfarna matematiklärare från två olika skolor genomförde två lektioner var med olika lektionsdesigner (LD1 och LD2). Lektionerna genomfördes i åtta olika elevgrupper med elever från skolår 5 och 6. Lärandeobjektet, den förmåga som eleverna skulle utveckla från undervisningen, var den samma för både LD1 och LD2, nämligen att eleverna skulle förstå att det finns oändligt många tal i ett intervall mellan två decimaltal. Lektionsdesign 1 (LD1) innehöll *en* kritisk aspekt och innebar att *decimaltal som punkter på en tallinje* skulle göras möjlig att erfara under lektionen. Lektionsdesign 2 (LD2) innehöll alla *fyra* kritiska aspekter, *decimaltal som punkter på en tallinje*, *rationella tals olika representationsformer, rationella tal som del av en helhet,* och *rationella tals delbarhet.*

Två lärare från samma skola arbetade tillsammans med att planera lektionerna för att de skulle bli så identiska som möjligt och använde samma elevuppgifter[37] och arbetssätt. Däremot arbetade lärarna från olika skolor på olika sätt under lektionerna. Två lärare hade en inledande diskussion om det fanns tal mellan 2 och 3 och lät eleverna därefter diskutera i grupp hur många tal det var mellan 0.17 och 0.18. Denna lektion avslutades med en diskussion i helklass där eleverna fick delge sina idéer om hur många tal det fanns i intervallet. De andra två lärarna hade en introduktion som handlade om det fanns tal mellan 5 och 6, följt av ett arbetsblad som eleverna arbetade med parvis, och som senare diskuterades i helklass. Alla elevgrupper som deltog i studien gav exempel på olika tal inom intervallen, exempelvis mellan 5 och 6 fanns 5.1, 5.2 osv.

Analysen av de videoinspelade lektionerna visade att LD1 genomfördes som planerat i fyra grupper (1A, 1B, 1C, 1D) medan LD2 enbart i två (1E, 1F) av fyra grupper. I två grupper (1G, 1H) var det enbart möjligt att erfara tre av de planerade fyra kritiska aspekterna. *Rationella tals delbarhet* var inte möjlig för eleverna att erfara under lektion 1G och 1H. Lärarna i dessa klasser fick helt enkelt inte med alla de kritiska aspekterna de skulle ha med och därför analyserades de lektionerna som en tredje lektionsdesign (LD3). Analysen av testen[38] och de videofilmade lektionerna pekar på undervisningen reflekteras i elevernas lärande. Ett exempel är att de elever som haft möjlighet att erfara aspekterna i LD1, *decimaltal som punkter på en tallinje* hade ett lägre resultat på eftertestet jämfört med de elever som fick möjlighet att erfara alla kritiska aspekter. Resultatet från förtest och eftertest visar att det var främst de uppgifter som handlade om lärandeobjektet som visade ett bättre resultat för elevernas lärande om de fått möjlighet att erfara de kritiska aspekterna i LD2. Uppgifterna; i) *Finns det tal mellan 0,5 och 0,6?*; ii) *Jonna påstår att det finns ett tal mellan 0,97 och 0,98. Pelle säger att det inte finns något sådant tal. Vem har rätt och varför?* och iii) *Skriv fyra tal som*

[37] Se bilaga 2 och 3.
[38] Se bilaga 1, uppgift 3b, 5 och 6.

är större än 0,99 men mindre än 1,1. Finns det fler tal mellan? Hur många i så fall?, avsåg att belysa elevernas förståelse för antal tal inom olika intervall. Elever som svarade inkorrekt på uppgifterna svarade exempelvis att det fanns, inga tal, ett tal eller tio tal inom ett intervall. Resultat från eftertestet (se bilaga 1) visar att de elever som haft LD2 hade ett högre resultat än de elever som haft LD1. Medelvärdet för eleverna med LD2 ökade mellan förtest och eftertest med +1.58 (effektstorlek[39], d=1.55) jämfört med +0.68 (effektstorlek, d=0.59) för elever med LD1. LD3 visar också ett lägre resultat än LD2, med en ökning på +0.60 mellan förtest och eftertest (effektstorlek d=0.53).

Tabell 10. Resultat av förtest och eftertest, uppgift 3b, 5 och 6. Maxpoäng = 3.

	LD1 (N=63)			LD2 (N=24)			LD3 (N=26)		
	Medel värde	SD	Effekt-storlek	Medel värde	SD	Effekt-storlek	Medel värde	SD	Effekt-storlek
Förtest	1.02	1.06		1.00	1.25		1.38	1.17	
Eftertest	1.70	1.25	0.59	2.58	0.72	1.55	1.98	1.09	0.53

Den kritiska aspekten om talens *delbarhet,* som endast var möjlig att erfara i LD2, visade sig i analysen av lektionerna ha stor betydelse för elevernas resultat på eftertestet. De kritiska aspekterna, i LD1 och LD3, kan dock vara en nödvändig förutsättning för att erfara talens *delbarhet* och därmed talens 'täthet'. Att namnge och räkna upp olika tal i ett intervall mellan två tal, som genomfördes under alla lektioner, var inte tillräckligt för att förstå att det finns oändligt många decimaltal i ett intervall. En slutsats är, att även om alla lektioner hade samma avsedda lärandeobjekt, så hade eleverna olika möjligheter att lära det som avsågs. Elevernas resultat på testen var inte beroende av elevernas ålder och de skolår (5 eller 6) eleverna gick. Det visade sig att det eleverna hade möjlighet att erfara under en lektion sammanföll med vad eleverna lärde sig.

39 Cohens d= $\frac{M_1 - M_2}{\sqrt{\frac{SD_1^2 + SD_2^2}{2}}}$

Studie 2: Elevers lärande av addition och subtraktion med negativa tal

Bakgrunden till studien är en learning study om negativa tal i skolår 7 och 8 (Kullberg, 2007; Maunula, 2006). Lärarna i studien hade som mål att eleverna skulle kunna utföra beräkningar i addition och subtraktion med negativa tal, som exempelvis 5-(-3) och (-5)+(-3). Under en learning study, som varade under nästan en hel termin, identifierade lärarna följande fyra aspekter som kritiska för elevernas lärande:

1. *Minustecknets olika betydelser*, innebär att eleverna behöver urskilja skillnaden mellan subtraktionstecknet och tecknet för ett negativt tal.
2. *Subtraktion som skillnad*, är en aspekt som betonar att subtraktion kan ses både som en skillnad mellan tal och som 'ta bort'. Subtraktion kan ses som skillnaden mellan två tal på en tallinje, exempelvis skillnaden mellan (-2) och (-3) är 1.
3. *Perspektivet*, medförde att eleverna behövde urskilja att kommutativa lagen inte gäller för subtraktion, utan att en subtraktion kan ses som en jämförelse (skillnad) sedd från den första termen.
4. *Talsystemets uppbyggnad*, innebär att eleverna behöver urskilja att talens värde blir större ju längre åt höger på tallinjen man kommer.

I avhandlingsstudien använde fyra erfarna matematiklärare sig av kunskapen om de identifierade kritiska aspekterna i åtta elevgrupper i skolår 7. På samma sätt som i studien om rationella tal arbetade två lärare från samma skola tillsammans med att planera lektioner för att få fram de kritiska aspekterna i undervisningen och använde samma elevuppgifter[40] och arbetssätt. I alla åtta lektioner användes metaforen skuld i samband med addition och subtraktion av negativa tal (se Ball, 1993). Addition med negativa tal beskrevs av lärarna som två personers sammanlagda ekonomi och subtraktion som skillnaden mellan två personers ekonomier.

Lärarna från olika skolor arbetade på olika sätt under lektionerna. Två av lärarna hade en inledande diskussion med de kritiska aspekterna, följt av olika gruppuppgifter där eleverna bl.a. skulle göra egna additions- och subtraktionsuppgifter. De andra två lärarna hade en inledande diskussion med de kritiska aspekterna följt av en uppgift där eleverna arbetade i par och senare med ett arbetsblad (se bilaga 5 och 6). Varje lärare genomförde både lektionsdesign 1 och

[40] Se bilaga 5 och 6.

2, men i olika elevgrupper. Lektionsdesign 1 (LD1) innebar att *subtraktion som skillnad* och *perspektivet* skulle göras möjliga för eleverna att urskilja i lektionen och i lektioner med lektionsdesign 2 (LD2) var alla identifierade kritiska aspekterna; *minustecknet, talsystemets uppbyggnad, subtraktion som skillnad* och *perspektivet.*

Analysen av lektionerna visade att de kritiska aspekterna blev implementerade som planerat, förutom i en lektion (lektion 2B) där en elevs frågor om innehållet bidrog till att fler kritiska aspekter var möjliga att erfara. Även om tanken var att endast de kritiska aspekterna i LD1 skulle göras möjliga att erfara, i lektion 2B, så blev det möjligt att erfara alla kritiska aspekterna i LD2. Förutom *subtraktion som skillnad* och *perspektivet,* så hade eleverna i lektion 2B även möjlighet att erfara '*minustecknets' olika betydelser* samt *talsystemet.* Lektion 2B redovisas därför i tabell 11 under LD2. Resultat från eftertestet (se bilaga 4) visar att de elever som haft LD2 hade ett högre resultat på eftertestet än de elever som haft LD1. Medelvärdet för eleverna som haft LD2 ökade mellan förtest och eftertest med +6.7 (effektstorlek, d=0.90) jämfört med +4.4 (effektstorlek, d=0.65) för elever med LD1.

Tabell 11. Resultat från för- och eftertest, maxpoäng = 31.

	LD1 (N=65)			LD2 (N=69)		
	Medelvärde	SD	Effektstorlek	Medelvärde	SD	Effektstorlek
Förtest	9.26	6.24		10.45	6.64	
Eftertest	13.65	7.19	0.65	17.12	8.07	0.90

I den här studien så var sig skillnaden mellan LD1 och LD2 något mindre jämfört med studien om rationella tal. En tolkning är att det är möjligt att en del elever redan hade urskiljt skillnaden på *tecknet* mellan subtraktion och ett negativt tal och *talsystemet.* Det är även möjligt att det finns fler kritiska aspekter som inte har identifierats eller att de kritiska aspekterna skiljer sig åt mellan grupperna. Analysen av lektionerna tyder på att vad eleverna har möjlighet att erfara under lektionerna reflekteras i vad de lär.

Diskussion

Avhandlingen belyser vad elever lär i förhållande till vad som görs möjligt för dem att lära. Det finns emellertid inte en direkt relation mellan den undervisning elever får och vad de lär, ingen 'ett till ett' relation, mellan undervisning och lärande. Det går därför inte att förutse vad en elev kommer att lära från en undervisningssituation. Elever kan rikta sin uppmärksamhet mot annat än det som läraren avser (Marton & Booth, 1997; Watson & Mason, 2006). Om det är möjligt för en elev att erfara kritiska aspekter av det som eleverna förväntas lära är det däremot mer troligt att eleven gör det än när det inte är möjligt (Marton & Tsui, 2004). Studierna i denna avhandling visar på skillnader i elevers möjlighet att lära utifrån den undervisning de får, men även på lärares möjlighet att påverka vad eleverna kan lära sig.

Betydelsen av de kritiska aspekterna för elevernas lärande

Analysen av studierna tyder på att de kritiska aspekterna bidrog till elevernas lärande. Det framgick att när enbart några av de kritiska aspekterna var närvarande under en lektion (med LD1), så var resultatet för elevernas lärande på eftertestet lägre, än när alla de identifierade kritiska aspekterna fanns med (LD2). Elevtesten visar att det var en skillnad i avseende på elevernas lärande mellan de kritiska aspekter eleverna fått möjlighet att erfara. Ytterligare argument som ger stöd för denna tolkning är att det endast var de testuppgifter som var direkt knutna till lärandeobjektet som förbättrades. De kritiska aspekterna i LD2 var möjliga att erfara under en och samma lektion och det kan ha bidragit till elevernas möjlighet att erfara lärandeobjektet på ett visst sätt, jämfört med om de kritiska aspekterna hade varit möjliga att erfara vid olika tillfällen. Variationsteorin lyfter fram samtidighetens betydelse för lärandet och menar att det är av betydelse om den lärande har möjlighet att urskilja flera aspekter av ett lärandeobjekt samtidigt (Marton & Tsui, 2004).

Analysen visade att lärare och elever tillsammans bidrog till vad som var möjlig för eleverna att lära under lektionerna. Elevernas frågor, olika exempel och svar bidrog till att aspekter av innehållet konstituerades och blev möjliga att erfara på ett visst sätt. Analysen tyder på att det var otillräckligt att bara omnämna kritiska aspekter av det som skall läras. Om den lärande inte har möjlighet att urskilja, exempelvis rationella tals delbarhet, så visade det sig att det inte räckte med att berätta att det finns oändligt många decimaltal. Med andra ord så samvarierade det *levda lärandeobjektet* med det *iscensatta lärandeobjektet* i de genomförda studierna.

De kritiska aspekterna som kunskapsresurs för lärare

Analys av de videoinspelade planeringsmötena visar att lärarna använde sig av de, av andra lärare identifierade, kritiska aspekterna när de planerade och genomförde undervisning. Lärarna i de två studierna var bekanta med learning study och variationsteorin vilket troligen bidrog till deras förståelse för de kritiska aspekterna och hur de använde dem i undervisningen. Det är osäkert om lärare utan denna erfarenhet hade lyckats genomföra en lektion på liknande sätt. Lärarna använde de kritiska aspekterna när de planerade lektionerna och det blev på så sätt tydligt vad lärarna behövde lyfta fram i undervisningen och göra möjligt för eleverna att erfara. Studiens resultat antyder att det inte var 'läraren i sig' som i dessa studier var av störst betydelse för elevernas lärande, utan istället det som läraren gjorde det möjligt för eleverna att erfara. Detta visade sig genom att alla lektioner med LD2 hade ett bättre resultat på eftertestet jämfört med LD1 även om samma lärare genomfört både LD1 och LD2.

Betydelsen av kritiska aspekter för förståelse för undervisning och lärande

Studierna berör i hög grad lärares arbete genom att de fokuserar vad elever skall lära, den undervisning de får, och vilken betydelse den har för elevernas lärande. Studierna har även visat att lärare kan använda sig av information om vad som antas vara kritiska aspekter när de undervisar. Det som är en kritisk aspekt identifieras i den praktik lärarna arbetar i. Lärare och lärarstuderande bör därför ges tid att systematiskt undersöka sin undervisning och elevernas lärande. Ett sätt för lärare att göra detta är att genomföra en learning study i sin praktik tillsammans med andra lärare. Genom att på djupet studera relationer mellan lärande och undervisning om ett speciellt lärandeobjekt kan lärare öka sin professionella kunskap och därmed också elevernas möjlighet att lära (Gustavsson, 2008; Runesson & Kullberg, in press; Stigler & Hiebert, 1999). På detta sätt kan lärarna bidra till en kollektiv kunskapsproduktion både lokalt, nationellt och internationellt, som kan gynna andra lärare och deras elever.

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Appendix 1: Pre and post tests in the study of rational numbers (translated into English)

Pre-test   name: class:

1. Which number is largest in each pair?

1.973 19.73	1.45 1.46	0 0.6
0.567 0.3	0.088 0.87	0.7 0.71

2. Sort by size. Put one under the lowest number, two under the next lowest number etc.

0.109	0.19	0.9	0.09	0.019

3a Are there numbers between 0.5 and 0.9?
3b Are there numbers between 0.5 and 0.6?
Explain your reasoning

4 Which number is closest to 0.18? Circle your answer and explain your reasoning.

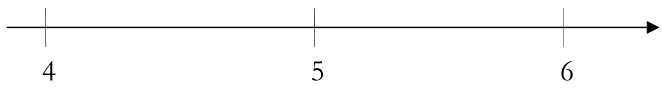
0.1	0.2	17	0.15	2

5. Anne says there is a number between 0.97 and 0.98. John says there is no such number. Who is right and why?

6. Write four numbers that are larger than 0.99 but smaller than 1.1. Are there more numbers in between? If so, how many?

7. 4.2+2=____

Appendix 2: Worksheet lessons 1C and 1D (translated into English)



1. Place the numbers on the number line:
5.45   5.55   5.41   5.51   5.50

2. Do the numbers 5.451 and 5.405 exist?
If yes, where are they on the number line?

3. Sort the following numbers by size. Start with the smallest.

0.6   0.75   0.61   0.605   1.2   0.7410

4a. 5.2 + 0.2 = _____   b. 0.52 + 0.2 = _____

c. 5.2 + 0.02 = _____   d. 5.2 - 0.2 = _____

e. 0.52 - 0.2 = _____

Appendix 3: Worksheet lessons 1F and 1H (translated into English)

1. Place the numbers on the number line: 0.51 0.55 0.45 0.50

2. Do the numbers 0.550 and 0.472 exist?
   If yes, where are they on the number line?

3. Write these numbers as fractions, in as many ways as you can.

0.7…………………………………………………………………..

0.74………………………………………………………………….

0.25………………………………………………………………….

0.112…………………………………………………………………

4. Sort the following numbers by size. Start with the smallest.

0.6 0.75 0.61 0.605 1.2 0.7410 $\frac{1}{10}$

……………………………………………………………………………

Appendix 4: Pre, post and delayed post tests in the study concerning negative numbers (translated into English)

Name:

1. Write the numbers in order of size. Start with the largest number.

-3 15 -5 4 -18

2.	5 + -2=___	-4 - -3=___	-4 + -3=___
	5 - -2=___	-3 - -4=___	-3 - 4=___
	-5 + -2=___	4 - -3=___	-2 - -5=___
	-5 - -2=___	3 + -4=___	-2 - 5=___

3. Write one negative number and one positive number to make the task correct.

___+___=1 Motivate your answer:____________________

___+___= -1 Motivate your answer:____________________

4. Write two negative numbers to make the task correct.

___-___=1 Motivate your answer:____________________

___-___= -1 Motivate your answer:____________________

5. 5- - 2= _____ Which alternative(s) is(are) correct. Circle the correct answer(s)!

a) 7 b) -7 c) 3 d) -3

Motivate your answer! ____

6.

-10 +___= -7 ___ -1 = -5

___ + -5 = 9 ___ - -2 = -4

1- ___ = 2 -6 + ___= -8

7. What do the signs mean? Explain.

- 17 – 6 = - 23

a.)

b.)

c.)

d.)

8.

12 - -15=___ , which alternative(s) is(are) correct? Circle the answer(s)!

a) 3 b) -3 c) 27 d) -27

9.

-12 – 15=___ , which alternative(s) is(are) correct? Circle the answer(s)!

a) 3 b) -3 c)27 d) -27

10.
How would you explain to a friend how to solve
-2 - -3= ?

Appendix 5: Worksheet used in lessons 2C and 2D (translated into English)

Addition (+) of negative numbers

5+(-2)=___	(-5)+1=___	(-4)+(-3)=___
8+(-4)=___	(-6)+7=___	(-1)+(-2)=___
(-7)+3=___	4+(-6)=___	(-5)+(-3)=___
(-2)+9=___	(-6)+4=___	(-3)+(-5)=___

Subtraction (-) of negative numbers (the difference between the numbers)

5-(-2)=___	(-5)-1=___	(-4)-(-3)=___
8-(-4)=___	(-6)-7=___	(-1)-(-2)=___
(-7)-3=___	4-(-6)=___	(-5)-(-3)=___
(-2)-9=___	(-6)-4=___	(-3)-(-5)=___

(-2)-5=___	1-(-5)=___	(-3)-(-4)=___
(-4)-8=___	7-(-6)=___	(-2)-(-1)=___
3-(-7)=___	9-(-9)=___	(-4)-(-4)=___
9-(-2)=___	(-9)-9=___	4-(-4)=___

Put one positive and one negative number on the line to make the expression correct!

___+___=3	___+___=(-3)	___-___=3

Put two negative numbers on the line to make the expression correct!

___+___=(-3)	___-___=3	___-___=(-3)

Appendix 6: Worksheet used in lessons 2G and 2H (translated into English)

Addition (+) of negative numbers

Think “friends”: two economies added with each other

5+(-2)=___	(-5)+1=___	(-4)+(-3)=___
8+(-4)=___	(-6)+7=___	(-1)+(-2)=___
(-7)+3=___	4+(-6)=___	(-5)+(-3)=___
(-2)+9=___	(-6)+4=___	(-3)+5=___

Subtraction (-) of negative numbers (the difference between the numbers).

Think ”not friends”: how much is the difference between their economies?

5-(-2)=___	(-5)-1=___	(-4)-(-3)=___
8-(-4)=___	(-6)-7=___	(-1)-(-2)=___
(-7)-3=___	4-(-6)=___	(-5)-(-3)=___
(-2)-9=___	(-6)-4)=___	(-3)-(-5)=___

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