UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

A Further Look at Short –Term Interest Rate Dynamics

Syed Fazal Hussain Shah

Graduate School

Master of Science in Finance
Master Degree Project No. 2009:98
Supervisor: Rolf Poulsen

A FURTHER LOOK AT SHORT-TERM INTEREST RATE DYNAMICS

Short-term interest rate analysis is one of the most important topics in finance and economics. This paper looks at short-term interest rate dynamics using three different interest rate proxies with different maturities, one-month Eurodollar deposit rate, overnight Federal Funds rate, and Three-month Treasury bill yield under one flexible parametric specifications. This flexible parametric specification is one-factor diffusion model with several nested cases. The analyzed data series and used flexible parametric specification encompasses enormous literature in the area, used in books and analyzed in articles. The result evaluates the nonlinear drift specification and linear drift specifications. The name of the paper is inspired by its guiding article by Turan G. Bali et al. (2006). Their theoretical framework is base of this paper.

Table of Contents:

Introduction

Having strong implications on interest rate derivatives and fixed income securities' pricing, short-term interest rate is one of the core concepts of asset pricing theory. Therefore there is a great interest for derivation of an effective model for its dynamics. This leads it to one of most important subject in theoretical as well as empirical finance. Diffusion process is used by most of the researchers, using continuous time arbitrage arguments, for derivative pricing. One might feel inevitability of high research demand on stochastic behavior of short-term interest rate. A huge range of study and effort has been put for the understanding of movement of short-term interest rate. Different data sets, unalike range of times series, and various parametric or nonparametric specifications were used for understanding this phenomenon almost 40 years. Results were confusing, conflicting, contradictory and inconsistent with same range of data sets, and specifications. This paper is influenced by one paper of Turan G. Bali and L. Wu (2006) named, "A comprehensive analysis of the interest-rate dynamics". As described by them in their paper and I quote, "Fundamental questions remain unanswered: (i) Is the short-rate drift linear or nonlinear? (ii) How sensitive is the conclusion to the choice of interest-rate series and parametric specifications? " Turan G. Bali and L. Wu (2006, p. 1270). I will follow their study to past year's interest-rate series with same parametric specification. The reasons and needs for that follow up are simple and straight. First and most important reason is that their interest-rate series is up till 1999. Second reason is the economic dynamics, which I have been noticed past years.

Large range of studies to understand short-term interest rate dynamics using diverse historical interest rate data as proxy for short-term interest rate are focusing on parametric specifications. Lack of adequacy to fit all interest rate data gave urge to researchers to use non-parametric or semi parametric specifications for more flexibility. In this paper I will use one flexible parametric specification as proposed by Turan G. Bali and L. Wu (2006) with various constrained versions. The parametric specification encircles most of the short-term dynamics used in literature.

First parametric specification is one-factor diffusion framework. I will go to follow rather simple single-factor diffusion process. It has been noticed that the significance

of nonlinearity depends upon selection a certain specification and preferred proxy for interest rate series. Significance of nonlinearity depending on the data series shows inconsistent result when conditional variable moves from affine function to fifth-order parametric function and then general form. Nonlinearity declines as the maturity of an instrument increases for example Eurodollar deposit rate to yield of Treasury bull.

I have organized my thesis by explaining preliminary and necessary theory for interest rate dynamics with focus to one-factor diffusion model. I will rise from minor details, which built up interest rate theory or so-called asset pricing theory. This background study will help me, and reader to understand the passage to sophisticated knowledge of asset pricing. This section will also explain the thesis with general level of interest. Next section will be based on the literature review, which is self-explaining. I will concentrate on important authors in the area of my study. Since vast majority of authors have written their papers using different parametric of non-parametric specification using different data series, but it seems difficult to differentiate or given importance due to their method selection of estimation differences. Empirical model will finally introduce the one factor model, which I chose to study short-term interest rate dynamics. I am following the article of T.G. Bali and L. Wu (2006), which proposes a flexible parametric specification to simulate short-term interest rate dynamics. I will explain the model selected and estimation technique used by me in this section. Showing result will be the next section for this paper. Here I will explain the statistics and results approximated by model. Explain and reason the arguments to justify the study. Conclusion will be final section of my paper where I will explain and wrap up the thesis paper with future interest and verdict.

Theoretical Background

Stochastic Process

Starting from the basics let me introduce the well-known concept of stochastic process. On a same probability space $(\Omega, \mathbb{F}, P)$ a collection of random variable $X_t, t \in T$ is called stochastic process. In other words, collection of random variables which lies on same probability space from time t to T is stochastic process. Stochastic process can be further explained by discrete parameter process and

continuous parameter process depending on T, time, is in discrete time steps i.e. T= 1, 2, 3, 4, 5 … T.   Continuous parameter process occurs when T is not countable like $T = [0, \infty]$.   To prove that certain stochastic process (X_t) is stationary one assumes that for any $n \geq 1$ and $t_1, t_2, t_3, t_4 .. t_n \in T$, where the finite-dimensional distribution $F_{s+t_1, s+t_2, ts+3, ts+4 ..., s+t_n}, s + t \in T$ is independent of s, which is defined as follows:

Let $X_{t1}, ..., X_{tn}$ and $t_1, t_2, t_3, t_4 .. t_n \in T$ has a specific joint distribution function of random variables:

$$F_{t_1, t_2, t_3, t_4 ..., t_n}(x_1, x_2, x_3, x_4, ..., x_n) = P(X_{t1} \leq x_1, X_{t2} \leq x_2, X_{t3} \leq x_3, X_{t4} \leq x_4, ..., X_{tn} \leq x_n)$$

Finite-dimensional distribution is the collection of all disjoint distributions $F_{t_1, t_2, t_3, t_4 ..., t_n}(x_1, x_2, x_3, x_4, ..., x_n)$ of random variables $P(X_{t1}, X_{t2}, X_{t3}, X_{t4}, ..., X_{tn}), n \geq 1, t_1, t_2, t_3, t_4, ..., t_n \in T$.

Brownian Motion

I am specifically interested and concentrate on Brownian motion W_t which is a process if it has following properties:

- Process starts at 0 or more precisely, $W_0 = 0$ with probability 1.
- W_t is almost surely continuous for every $t \geq 0$
- W_t has independent increment and has normal distribution with mean 0 and variance Δt. One can say that for all $0 \leq s \leq t$, the increment $W_t - W_s \sim N(0, t - s)$

Discretization Using Euler Scheme

Now I will discuss the discretization of almost any continuous time model using Euler approximation. This technique is significant due the fact that one can only monitor data in discrete time. For discretization will concentrate only on Euler scheme, after all most of the literature use Euler discretization or Milstein scheme but I will use the same technique for my further investigation into short-term interest rate dynamics. Euler Scheme is mostly used for discretization of continuous process and easy to handle form of Taylor approximations.

As I have a stochastic process $X = \{X_{t,t_0} \leq t \leq T\}$, which satisfies following simple stochastic differential equation:

$$dX_t = a(t,X_t)dt + b(t,X_t)dW_t$$

the initial value at time $t_0 \leq t \leq T$ is

$$X_{t0} = X_0$$

Divining the time interval from t_0 to T into N intervals with $t_0 < t_1 < t_2 < t_3 ..... < t_n = T$, Euler scheme will have following expression

$$Y_{n+1} = Y_n + a(t_n,Y_n)(t_{n+1} - t_n) + b(t_n,Y_n)(W_{tn+1} - W_{tn})$$

for n=0,1,2,3,….,N-1 the initial value will be

$$Y_0 = X_0$$

In case the intervals are equal between two observations the Euler Scheme will not look so messy. So I can write Euler scheme

$$Y_{n+1} = Y_n + a\Delta + b\Delta W_n$$

Moments and Moments Generating Function

Now I am going to dig little deeper in the theory and go forward to random variable moments and moment generating functions. The first moment of the random variable about the origin is conditional mean of random variable. Starting by letting X as random variable, discrete and probability density function be f (x), the kth moment about the origin of X is

$$\mu'_r = E(x^k) = \sum_x x^k p(x)$$

Where $k = 1,2,3,....,n$ and $\sum_x | x^k | p(x) < \infty$

Now letting X be a random variable, continuous and f(x) be its probability density function. The kth moment about the origin of X is

$$\mu'_r = E(x^k) = \int_{-\infty}^{\infty} x^k f(x) dx$$

Where $\int_{-\infty}^{\infty} |x^k| f(x) dx < \infty$

These two versions, continuous and discrete moment about the origin of a random variable is self-explanatory and moment about the origin's value depends only on the probability distribution.

One knows the value of mean for a certain random variable by calculating the first moment about its origin. Variance of the random variable can be calculated but with extra calculations. Even though the calculations seem messy but the fundamental principal used for a general variance calculation.

Again I let X be random variable and its mean calculated as per moment about its origin formulae depending upon if continuous or discrete variable. Probability density function is given, kth moment about its mean, and it can be showing as follows, if X is discrete:

$$\mu_r = E[(X-\mu)^k] = \sum_x (x-\mu)^k p(x)$$

Here power k is used symbolically this shows the kth moment about the mean. Where $k = 1, 2, 3, ..., n$.

For continuous X random variable the moment about its mean is:

$$\mu_r = E[(X-\mu)^k] = \int_{-\infty}^{\infty} (x-\mu)^k f(x) dx$$

Assuming that $\int_{-\infty}^{\infty} |(x-\mu)^k| f(x) dx < \infty$

The 2nd moment about the random variables mean is variance. Now it is known that for any random variable, continuous or discrete, the mean and variable exists, emancipating through first moment about its origin and second moments about its mean respectively.

Sometimes one has to use moment generating functions, as calculating moments of random variable using direct can be difficult.

Moment generating function for a random variable X, given probability density functions, is shown as follows:

$$M_x(t) = E(e^{tx}) = \begin{cases} \sum_{allx} e^{tx} p(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx \end{cases}$$

Assuming the existence of expected values at all x, and for discrete random variable X $\sum_{allx} |e^{tx}| p(x) < \infty$ and for continuous random variable X $\int_{-\infty}^{\infty} e^{tx} f(x) dx < \infty$.

Using moment generating functions one can easily find moments about the random variable's origin. After finding moment generating functions under above certain mention assumptions then kth moment about random variable is:

$$M_x^k(0) = E(X^k) = \mu'_r$$

Maximum Likelihood Method Parameter Estimation

Parameter estimation is no doubt the most popular procedures during almost all the econometric studies. Parameter estimation is the major part of model estimation and using it. Parameters are the building blocks of model. The parameter estimation can be done in various manners. Taking random observations sample from the population and using these observations for parameter estimation is most common practice. I am going to use maximum likelihood due to flexibility and effectiveness or perhaps its most commonly used method in given literature. This helped me understand the concept at wide spectrum and facilitated the drawing connections between other papers and my paper. For the sake of coherence, cross reference and verification Maximum likelihood method was a definitive choice for me.

Vasicek Model and Cox, Ingersoll, and Ross Model

I am going to show the parameter estimation of CIR model and Vasicek model. These models are ineffective to show the dynamics of short-term interest rate but these are commonly used due to their mathematical tractability. The process and behavior of

short-term interest rate dynamics may be shown by following stochastic differential equation (SDE):

$$dr(t) = \kappa(\theta - r(t))dt + \sigma r(t)^{\gamma} dW(t)$$

Where $\kappa, \theta, \gamma, \sigma$ are model parameters, and Z_t are Brownian motion. When applied different restriction to the generic stochastic differential equation, come up with different single factor models; this can be defined as follows:

Model	SDE form	Restriction
Vasicek	$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t)$	$\gamma = 0$
CIR	$dr_t = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t)$	$\gamma = \dfrac{1}{2}$

Vasicek Model and Parameter Estimation

I will consider general form continuous time process, which satisfies following stochastic differential equation:

$$dr = \mu(r,t)dt + \sigma(r,t)dW$$

Where W is standard Brownian motion

To be precise for Vasicek model introduced by Vasicek, it can be showed that instantaneous spot rate follows an Ornstein-Uhlenbeck process with positive constant coefficients (Vasicek, 1977):

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t)$$

Where $W(t) = B(t) + \displaystyle\int_0^t \phi(s)\,ds$

The model shows mean-reversion with kappa is speed of reversion to mean, if rate is greater than theta which is long term mean value kappa will make drift negative resulting in rates moving to long term mean value. In case rate is smaller than long term mean value the kappa makes drift positive resulting an increase in rates. This phenomenon makes sense after all in case of high interest rate market tends to slow down resulting in a decrease in interest rate and vice versa. Even though model has

10

disadvantages, the plus point is its explicit solution. Here I am showing solution to Vasicek model.

Letting $X(t) = r(t) - \theta$

Now the model is $dX(t) = -\kappa X(t)dt + \sigma dB(t)$;Ornstein Uhlenbeck process.

Now Let

$$Y(t) = e^{\kappa t} X(t)$$

which will result in

$$dY(t) = \kappa e^{\kappa t} X(t)dt + e^{\kappa t} dX(t)$$

substituting $dX(t) = -\kappa X(t)dt + \sigma dB(t)$ in above expression one gets

$$dY(t) = e^{\kappa t} \sigma dB(t)$$

and

$$Y(t) = Y(0) + \int_0^t e^{\kappa s} \sigma \, dB(s)$$

Using above expression one can show

$$Y(0) = X(0)$$

and that

$$e^{\kappa t} X(t) = X(0) + \int_0^t e^{\kappa s} \sigma \, dB(s)$$

$$X(t) = e^{\kappa t}[X(0) + \int_0^t e^{\kappa s} \sigma \, dB(s)]$$

Now putting value of X(t) into above expression r(t) can be calculated as follows, one gets following form with each $t \le s$:

$$r(t) = r(s)e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) + \sigma \int_s^t e^{-\kappa(t-s)} dB(s)$$

so that $r(t)$ is normally distributed with conditional mean given as

$$E(r(t)|\mathbb{F}_s) = r(s)e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)})$$

and conditional variance is

$$Var(r(t)|\mathbb{F}_s) = E((\sigma \int_s^t e^{-\kappa(t-s)} dB(s))^2 |\mathbb{F}_s)$$

and

$$Var(r(t)|\mathbb{F}_s) = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(t-s)})$$

The major drawback of Vasicek model is undesirable negative interest rate possibility, which is certain impossibility during course of life of one interest rate. The reason of the popularity is its mathematical tractability and easy implementation.

Now I can go further with it and with discrete time observations of steps 1, 2,… ,T, I can determine the analytic form of likelihood function.

$$L = (r_1, r_2, ..., r_T; \phi)$$

Now I can transform original continuous time process by Euler approximation, which will have following equation form:

$$r_t = r_{t+1} + \mu(r_{t+1}; \phi) + \sigma(r_{t+1}; \phi)\varepsilon$$

where ε is Gaussian white noise. Now I must calculate parameter set $\phi = \{\theta, \kappa, \alpha\}$ using maximum likelihood method for Vasicek model.

$$L = \prod_{i=1}^n (2\Pi \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t}))^{\frac{1}{2}} \exp(-\frac{1}{2}v^2(r_t, r_{t+1}, \Delta t))$$

where $v^2(r_t, r_{t+1}, \Delta t) = \dfrac{r_{ti+1} - (\theta + (r_{ti} - \theta)e^{-k\Delta t}}{\sqrt{var_{ti}}}$

and $var_{ti} = \frac{\sigma^2}{2\kappa}(1 - e^{-2k\Delta ti})$

This may become little complicated but I can simplify the likelihood function as our observations have equal interval. So using time homogeneity property:

$$L = 2\Pi \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t})^{\frac{N-1}{2}} \exp(-\frac{1}{2}\sum_{i=1}^{n-1} v^2(r_t, r_{t+1}, \Delta t))$$

Furthermore its easier to maximized log likelihood function as compare to likelihood function so I will take log of likelihood function for ease of use.

$$\ln(L) = \frac{N-1}{2}\ln 2\Pi - \frac{N-1}{2}\ln(\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t}))^2 - \frac{1}{2}\sum_{i=1}^{n-1} v^2(r_t, r_{t+1}, \Delta t)$$

Now I can maximize log likelihood function using with parameter set $\phi = \{\theta, \kappa, \alpha\}$. which can found using following equation:

$$\hat{\phi} = \arg\max(\ln(L))$$

Vasicek Closed form Maximum Likelihood

Rewriting the Vasicek model interest rate dynamics as

$$dr(t) = (b - ar(t))dt + \sigma dW(t)$$

Integrating the stochastic differential equation for s<t one gets

$$r(t) = r(s)e^{-a(t-s)} + \frac{b}{a}(1 - e^{-a(t-s)}) + \sigma\int_s^t e^{-a(t-u)}dW(u)$$

with condition on information at s the r(t) is normally distributed with mean and variance as follows:

$$E(r(t)) = r(s)e^{-a(t-s)} + \frac{b}{a}(1 - e^{-a(t-s)})$$

$$Var(r(t)) = \frac{\sigma^2}{2a}(1 - e^{-2a(t-s)})$$

Which demands to estimate $\beta = \frac{b}{a}, \alpha = e^{-a\Delta t} and V^2 = \frac{\sigma^2}{2a}(1 - e^{-2a\Delta t})$, the maximum likelihood estimators can be solved into following:

$$\hat{\alpha} = \frac{n\sum_{i=1}^{n} r_i r_{i-1} - \sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_{i-1}}{n\sum_{i=1}^{n} r_{i-1}^2 - (\sum_{i=1}^{n} r_{i-1})^2}$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} (r_i - \hat{\alpha} r_{i-1})}{n(1 - \hat{\alpha})}$$

$$\hat{V}^2 = \frac{1}{n} \sum_{i=1}^{n} (r_i - \hat{\alpha} r_{i-1} - \hat{\beta}(1 - \hat{\alpha}))^2$$

These estimated numbers give full information of delta transition probability for process r under objective measure allowing single day discrete time steps, which makes estimation robust and effective.

Cox, Ingersoll, and Ross Model and Parameter Estimation

Cox Ingersoll and Ross model can be shown as following stochastic differential equation with nonnegative constants (Cox et all, 1985):

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

Cox et all.(1985) introduced square root to original Vasicek(1977). This model has also been widely used due to its tractability along with Vasicek model. After integrating the stochastic differential equation one gets:

$$r(t) = r(s) + \kappa \int_s^t (\theta - r(s))ds + \sigma \int_s^t \sqrt{r(s)}dW(s)$$

applying Ito's Formula one gets

$$r(t)^2 = r(s)^2 + 2\kappa \int_s^t (\theta - r(s))ds + 2\sigma \int_s^t \sqrt[2/3]{r(s)}dW(s) + \sigma^2 \int_s^t r(s)ds$$

Rearranging above expression one gets

$$r(t)^2 = r(s)^2 + (2\kappa\theta + \sigma^2)\int_s^t r(s)ds - 2\kappa \int_s^t r(s)^2 ds + 2\sigma \int_s^t \sqrt[2/3]{r(s)}dW(s)$$

Writing r(t) in terms of its initial value:

14

$$r(t) = r(0) + \kappa \int_0^t (\theta - r(s))ds + \sigma \int_s^t \sqrt{r(s)}dW(s)$$

the unconditional mean using above expression is

$$E(r(t)) = r(0) + \kappa(\theta t - \int_0^t E(r(s)ds)$$

solving equation one gets

$$E(r(t)) = \theta + (r(0) - \theta)e^{-\kappa t}$$

Rearrange above expression one gets final conditional mean

$$E(r(t)|\mathbb{F}_s) = r(s)e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)})$$

similarly one can use $r(t)^2$ expression which I got after using Ito's formula to calculate unconditional variance

$$E(r(t)^2) = r(0)^2 + (2\kappa\theta + \sigma^2)\int_0^t E(r(s)ds) - 2\kappa \int_s^t E(r(s)^2 ds)$$

substituting the value of mean into above expression and using second moment one gets conditional variance:

$$var(r(t)) = \frac{\sigma^2}{\kappa}(1 - e^{-\theta t})[r(0)e^{-\theta t} + \frac{\kappa}{2}(1 - e^{-\theta t})]$$

and conditional variance will be as follows:

$$var(r(t)|\mathbb{F}_s) = r(s)\frac{\sigma^2}{\kappa}(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}) + \theta\frac{\sigma^2}{2\kappa}(1 - e^{-\kappa(t-s)})^2$$

In order to develop closed-form likelihood function again I will consider general form continuous time process, which satisfies following stochastic differential equation:

$$dr = \mu(r,t)dt + \sigma(r,t)dW$$

Where W is standard Brownian motion and with discrete time observations of steps 1, 2,…, T, I can determine the analytic form of likelihood function.

$$L = (r_1, r_2, \ldots, r_T; \phi)$$

Now I can transform original continuous time process by Euler approximation, which will have following equation form:

$$r_t = r_{t+1} + \mu(r_{t+1}; \phi) + \sigma(r_{t+1}; \phi)\varepsilon$$

where ε is Gaussian white noise. Now I must calculate parameter set $\phi = \{\theta, \kappa, \alpha\}$ using maximum likelihood method for CIR model.

$$L = \prod_{i=1}^{n} (2\Pi \frac{\sigma^2}{2\kappa}(e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}) + \theta\frac{\sigma^2}{2\kappa}(1 - e^{-\kappa\Delta t})^{\frac{1}{2}} \exp(-\frac{1}{2}v^2(r_t, r_{t+1}, \Delta t))$$

Where $v^2(r_t, r_{t+1}, \Delta t) = \dfrac{r_{ti+1} - (\theta + (r_{ti} - \theta)e^{-k\Delta t}}{\sqrt{\text{var}_{ti}}}$

and $\text{var}_{ti} = \dfrac{\sigma^2}{\kappa}(e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}) + \theta\dfrac{\sigma^2}{2\kappa}(1 - e^{-\kappa\Delta t})^2$

As one can assume time-homogeneity of model, I can simplify it further as follows:

$$L = 2\Pi\frac{\sigma^2}{\kappa}((e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}) + \theta\frac{\sigma^2}{2\kappa}(1 - e^{-\kappa\Delta t}))^{\frac{N-1}{2}} \exp(-\frac{1}{2}\sum_{i=1}^{n-1} v^2(r_t, r_{t+1}, \Delta t))$$

Now I will take log of likelihood function will make calculation easier. So I will transform our likelihood function into log likelihood function as follows:

$$\ln(L) = \frac{N-1}{2}\ln 2\Pi - \frac{N-1}{2}\ln(\frac{\sigma^2}{\kappa}((e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}) + \theta\frac{\sigma^2}{2\kappa}(1 - e^{-\kappa\Delta t})))^2 - \frac{1}{2}\sum_{i=1}^{n-1} v^2(r_t, r_{t+1}, \Delta t))$$

Now I can maximize log likelihood function using with parameter set $\phi = \{\theta, \kappa, \alpha\}$. which can found using following equation:

$$\hat{\phi} = \arg\max(\ln(L))$$

Standard Error Estimation

Standard error for parameters estimated by maximum likelihood method can be calculated by using numerical Hessian (2nd derivative). The inverse of hessian matrix can be seen as Covariance matrix. Let me define theoretical how to estimate Standard and Covariance Matrix.

As I am using and explaining theory which involved more than one parameter let me define:

$\phi = (\phi_1, \phi_2, ..., \phi_k)$ are parameters and $\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2, ..., \hat{\phi}_k)$ are parameters estimated by using Maximum likelihood

Our log likelihood function can be defined generically:

$$L_m = \sum_{i=1}^{n} \log f(X_i; \phi)$$

the 2nd derivatives of likelihood using all parameters will give us Fisher Information Matrix

$$I_n(\phi) = - \begin{bmatrix} E_\phi(H_{11}) & E_\phi(H_{12}) & ... & E_\phi(H_{1k}) \\ E_\phi(H_{21}) & E_\phi(H_{13}) & ... & E_\phi(H_{21}) \\ . & . & . & . \\ E_\phi(H_{k1}) & E_\phi(H_{k2}) & .... & E_\phi(H_{kk}) \end{bmatrix}$$

Now taking inverse of Fisher Information Matrix, Let following be inverse of FIM:

$$J_n(\phi) = I_n^{-1}(\phi)$$

Standard Errors for estimated parameters are the diagonal values of Inverse Fisher information matrix.

Literature Review

Several attempts have been taken by researchers in the past to probe the behavior of short-term interest rate dynamics, which resulted in conflicting results. There have been developed and used throughout the time. Some of the most important models are Black-Scholes (1973), Merton (1973), Vasicek (1977), Cox, Ingersoll, Ross (1985). Chan, Karolyi, Longstaff, and Sanders (1992) used a common framework and compared different single-factor specification short-term interest models. K. B. Nowman (1997) used Chan, Karolyi, Longstaff, and Sanders (1992) techniques and analyzed for short-term interest rate for British data. They came up with contradictory results from US that volatility of interest rate does not depend on high interest rates. Treepongkaruna, Sirimon (2003) compared different widely accepted short-term

interest rate models for Australian data and came up contradictory results as compared to K. B. Nowman (1997). This may give an idea on selection of proxies for interest rate series. These studies differ in different data set and various specifications. Yacine Ait-Sahalia (1996) proposed a nonparametric (semi-nonparametric) estimation procedure to examine stochastic model. Yacine Ait-Sahalia (1996) used revised nonlinear specification and come up with interesting results. Further Yacine Ait-Sahalia (2002) solved explicitly the parametric specification using Hermite expansion; this made maximum likelihood estimation effective for discrete time observation of interest rate. Richard Stanton (1997) studied continuous time diffusion process using a nonparametric specification technique on daily observations of three and six months Treasury bill. Their study resulted in significant non-linearity in drift function. The guiding article of Chan et al. (1992) almost same results with parametric specification, this increases inconsistency of results in different studies using different specifications. George J. Jiang (1998) used a flexible nonparametric model of interest rate term and showed that nonparametric models gives more than satisfactory results and traditionally used models are developed on wrong grounds such as Chan, Karolyi, Longstaff, and Sanders (1992) and Yacine Ait-Sahalia (1996). Matt Pritsker (1998) worked on nonparametric tests of Yacine Ait-Sahalia (1996) and showed that its asymptotic distribution of kernel density estimators was ineffective, urged using the combination of parametric and nonparametric model for analyzing real time data (Matt Pritsker, 1998). He further notices that non-linearity in drift function is not statistically significant. David A. Chapman and Pearson (2000) worked on model of Yacine Ait-Sahalia (1996). The conclusion of David A. Chapman and Pearson (2000) paper is rather of skeptical nature. In other word they concluded that nonparametric or semi-nonparametric models tested in the paper are no efficient to show nonlinearity in drift of short-term interest rate (David A. Chapman and Pearson, 2000). Another attempt to correct boundary bias of the kernel estimator was introduced Yongmiao Hong and Haitao Li (2005). They rejected a series of one-factor diffusion process models on daily Eurodollar rate data and some multi-factor affine models on monthly US Treasury yields. They concluded that nonparametric models could be reliable tool for financial data (Yongmiao Hong and Haitao Li, 2005).

18

Christopher S. Jones (2003) used a new Bayesian method to examine diffusion process. He described that nonlinear drift in interest rates can be confirmed only under prior informative distribution and rejects nonlinearity of drift; results differ on the choice of sampling procedure, model and frequency (Christopher S. Jones, 2003). Christopher S. Jones (2003) urges the need and use of other factors effecting nonlinear drift, for example long-term yields and interest rate options, which can be challenging.

Garland B. Durham (2002) came up with conclusion that volatility is important factor in short-term interest rate analysis. Garland B. Durham (2002) found that CEV model might not fully explain the short-term interest rate dynamics, highly flexible volatility models.

Use of proxy for short-term interest rate is highly controversial. The study of short-term interest rate dynamic is mostly based on three or six-month Treasury bills, federal funds rate and Eurodollar rates. Christopher S. Jones (2003) argue against use of seven-day Eurodollar deposit rate data as proxy for short-term interest rate due to noise. On the other hand David A. Chapman et al. (1999) is not satisfied with longer time maturity proxies. According to David A. Chapman et al. (1999), for nonlinear model using three-month Treasury bill yield is not smart.

T.G. Bali and L. Wu (2006) used three widely used interest rate series and tested on them with flexible parametric specifications; single factor with nonlinear parametric drift and variance functions, and GARCH volatility with non-normal innovation. (T.G. Bali and L. Wu, 2006) It feels like that T.G. Bali and L. Wu (2006) have same conclusion about proxies as David A. Chapman et al. (1999) that longer maturities proxies do not proved to be best choice for short-term interest rate dynamics. In this paper I am going to follow footsteps of T.G. Bali and L. Wu (2006). I will study the consistency of results with updated and new data.

So question unanswered or made rather more confused is parametric specification or nonparametric specification and longer maturity proxies or short-term maturities proxies?

Theoretical model

The empirical side of my study will lead us to model which I m going to use for short-term interest rate dynamics. I have chosen one-factor model introduced by T.G. Bali and L. Wu (2006). The model is used without and with various restrictions. Showing short-term interest rate process by following stochastic differential equation

$$dr_t = \mu(r_t)dt + \sqrt{v(r_t)}dW_t$$

Where

r_t = Interest rate at time t

W_t =Wiener process or Standard Brownian motion

$\mu(r_t)$ = Flexible drift function

$v(r_t)$ =Flexible variance function

There are various authors, which have used different nonparametric, parametric or semi-parametric specifications for drift and variance function. Yacine Ait-Sahalia (1996) has introduced the famous nonlinear specification for drift function. Brenner, Harjes, and Kroner (1996) have also included nonlinear drift function in their study. I will stick to T.G. Bali and L. Wu (2006) flexible specification for it covers almost all of the one-factor models in past years. The drift, in Laurent series expansion with degree of five and negative one, and variance, combination of Constant Elasticity volatility (CEV) and an affine specification, except that I increased another parameter β_2, which is an addition to all these given models. Here functions are shown as follows:

$$\mu(r_t) = \alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5 + \alpha_6 r_t^{-1}$$

$$v(r_t) = \beta_0 + \beta_1 r_t + \beta_2 r_t^2 + \beta_3 r_t^{\beta_4}$$

here r_t is interest rate at time t, $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4$ are unknown parameters.

This model is very generic in nature, encompasses different alternative short-term interest rate models, which is a reasonable and heuristically and with mathematical

logic. Here are some of the important alternative models which has been develop with course of time using different restriction on above-mentioned generic model.

Models	α_0	α_1	α_2	α_3	α_4	α_5	α_6	β_0	β_1	β_3	β_4
Ait-Sahalia					0	0	0				
CKLS			0	0	0	0	0	0	0		
Vasicek			0	0	0	0	0		0	0	0
BS			0	0	0	0	0	0		0	1
CIR			0	0	0	0	0	0		0	.5
CEV		0	0	0	0	0	0	0	0		
Merton		0	0	0	0	0	0		0	0	0
Duffie et al.			0	0	0	0	0			0	
Ahn et al.	0		0	0	0	0	0	0	0		3
Jones				0	0	0		0	0		

In order to put the theory into practice I face some limitations, big one is continuous process cannot be empirically tested. In order to overcome or honestly speaking escaping from this problem, I need to transform the continuous time model into discrete time form. I can use first order Euler scheme for discretization. After applying Euler approximation on model, I can show our model as follows:

$$r_t - r_{t-\Delta} = \mu(r_t)\Delta + \sqrt{v(r_t)}\Delta\varepsilon_t$$

or

$$r_t - r_{t-\Delta} = (\alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5 + \alpha_6 r_t^{-1})\Delta + \sqrt{(\beta_0 + \beta_1 r_t + \beta_2 r_t^2 + \beta_3 r_t^{\beta_4})}\Delta\varepsilon_t$$

Here $\Delta = 1/252$, denotes time interval between two consecutive observations, ε_t shows independently identically normally distributed error terms or

$\varepsilon_t \sim N(0,1)$.

I am going to use most common quasi-maximum likelihood estimation for parameter estimation under assumption that transition density is Gaussian. Because probabilities function of most of above mentioned models are not developed or known, especially

our model. Euler-based Gaussian maximum likelihood is an approximation, which works fine if the time steps are small. For that I used smallest possible time interval. For that I assume that the distribution of our finite sample with interval delta is normal, which is not the case in reality. But this will make our model into practice. Using the assumptions and Euler approximation the likelihood function is:

$$L = \frac{T-1}{2}\ln(2\Pi) - .5\sum_{t=2}^{T}\left[\ln(r_{t-\Delta})\Delta + \frac{(r_t - r_t - \mu(r_{t-\Delta})\Delta)^2}{v(r_{t-\Delta})\Delta}\right]$$

Now I can maximize log likelihood function using with parameter set $\phi = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4\}$, which can found using following equation:

$$\hat{\phi} = \arg\max(\ln(L))$$

Data and Estimation

Before I start with estimation I want to introduce the relationship between different parameters. I checked different studies which used same or similar kind of specifications but with conflicting results depending on the interest rate series selection. Why so much inconsistencies? The probable answer can be due to estimation bias. Let me present a table which shows different parameters relationship with each other under somehow similar specification used by Ait-Sahalia (1996b). The specification used is:

$$r_t - r_{t-\Delta} = (\alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_{-1} r_t^{-1})\Delta + \sqrt{(\beta_0 + \beta_1 r_t + \beta_2 r_t^{\beta_3})\Delta}\varepsilon_t$$

Here are some conditions applied on these parameters:

| Condition	Occurrence
$\beta_0 \geq 0$	Always
$\beta_1 > 0$	$\beta_0 = 0$ and $\beta_3 > 0$
$\beta_1 > 0$	$0 < \beta_3 < 1$ and $\beta_2 = 0$
$\beta_2 > 0$	$\beta_0 = 0$ and $0 < \beta_3 < 1$
$\beta_2 > 0$	$\beta_3 > 0$ or $\beta_1 = 0$

$\beta_3 > 1$	$\beta_2 > 0$

I am using three different interest-rate data series; overnight federal funds rate, 3-month US Treasury Bill yield and one-month Eurodollar deposit rate. These series comprises almost all of the literature used to understand short-term interest-rate dynamics. There have been different choices for data series with in short-term interest rate dynamics articles. Using one particular data series as proxy for analyzing short-term interest rate has been criticized. There is not consensus on whether to use or not a certain data series, which can fully encompass all the significant factors influencing short-term interest rate diffusion process. But there has been some consistency in choosing three interest rate series. Most of the authors used either one of them or combined them in their studies. They used different observation frequency and different sizes of sample. I am going to use all three-interest rate series with daily frequencies except weekends. These interest rate series are Federal Funds rate, US Treasury bill, and Eurodollar Deposit rate. These series were acquired from Board of Governors of the Federal Reserve System website. These series are available to everyone on their Statistics and Historical Data segment. Federal Funds are short-term borrowing dealt between financial institutions. Financial institutions required to hold reserves with Federal Reserve Banks can only borrow them. Federal Funds are quintessential part of overnight credit market in US with their current and expected interest rates are basis of money market in US. Federal funds rate used in the series is weighted average of rates on brokered trades. Eurodollar Deposits are bank deposits denominated in US dollars located outside US. Eurodollar Deposits are located in London in the series. US Treasury bills are short-term Securities issued by US Treasury. US Treasury bill yield is quoted on a discount of 360 days per year. I chose recent sample of these interest rate series. Total number of daily observations is 3545 for each interest rate series starting from January 1996 to July 2009. One reason of not choosing nearest observations could be due to Current Credit crunch, which may or may not influence the prices of these instruments. Giving benefit of doubt to that factor delimited me to use most recent observations. If I use these observations, I have to make necessary adjustments and assumptions that do not come under given topics and would make this paper messy. I am going to assume no exogenous factor influencing especially current credit crunch. Given are three graphs of these interest

rate series. The red line shows Eurodollar deposit rate, green is for US Treasury bill and blue is for Federal funds rate. It look obvious that their movement is quite similar over period of time, keeping in mind that these rates sample of taken from same period of time. The only major shock which can not be seen in other series is sudden upward movement of Federal funds rate and sudden collapse probably within the week. These sudden movements were possibly due to panic in the market. Other than that I see gradual decline in interest rates in all the series, reaching to nearest of zero in contemporary times. All series are currently at their lowest in past almost fifteen years time span.
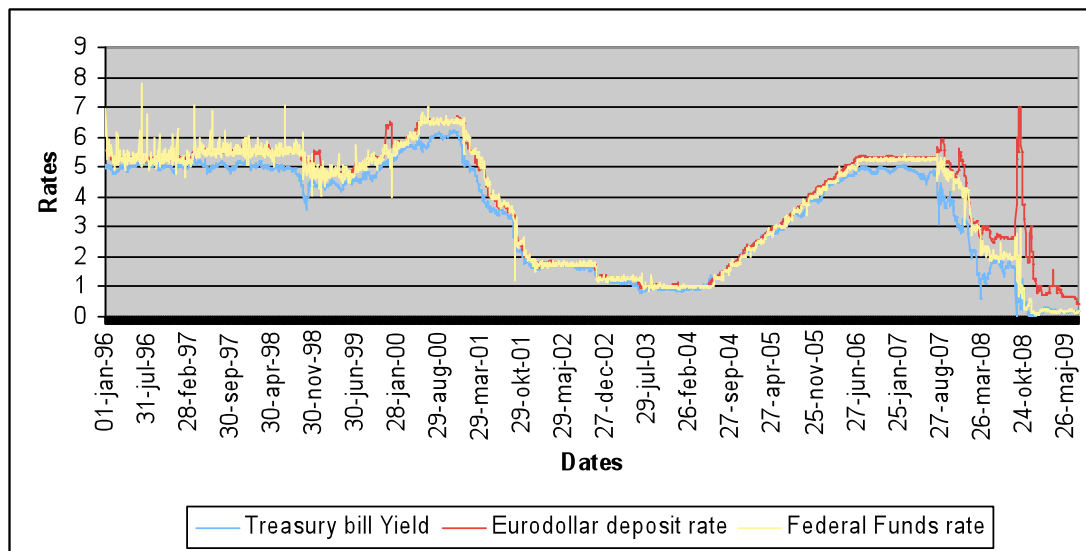
All three-interest rate series show extreme kurtosis and negative skewness. For normal distribution, kurtosis has to be 3 and skewness has to be zero. Our data is mostly based on the left side of mean and kurtosis is less than 3, which means that our interest rate series are less outlier-prone. The Unconditional means of Federal Funds rate, Eurodollar deposit rates and US Treasury bill Yield are 3.7322, 3.9022, and 3.41 respectively.

Interest rate series	Mean	Variance	Std. Dev.	Skewness	Kurtosis
Fed. Funds Rate	3.7322	3.7715	1.9420	-0.3989	1.6734
Eurodollar Dep. Rate	3.9022	3.4535	1.8584	-0.4128	1.6248
US Treasury bill	3.4147	3.2157	1.7932	-0.3820	1.6539

Results

Before going into my general analysis of nonlinearity of interest rate diffusion process, I am going present working explicitly solved closed form vasicek parameter estimation for Maximum likelihood. For each interest rate series the result are as follows:

Interest rate series	Kappa	Theta	sigma
Federal Funds rate	0.9475	0.0329	0.0277
Eurodollar deposit rate	0.1358	0.0123	0.0126
US Treasury bill yield	0.0463	-0.0396	0.0093

One knows that theta is level of drift in a long term where interest rate is pulled with speed of kappa. This seems implausible when one looks at theta value of US Treasury bill yield. When using almost same kind of interest rate series with similar movement over the time with almost similar shocks, the level of interest rate over long run is near, the theta should lay near each other. Federal funds rate and Eurodollar has positive values with 3.29 and 1.22 respectively. US Treasury bill yield, according to vasicek model, has long term mean of -3.94 with speed 0.05. This inconsistency creates confusion in my mind over effectiveness of Vasicek model. This may give strong support for claiming that interest rate is highly nonlinear and using a linear model like vasicek may result in miscalculation and hence inference on interest rate dynamics.

Nonlinearity in Drift Function of Specification

For analysis I have chosen different nested specification to my original specification to estimated original results for parameter estimation. Here are details to my nested tests with same variance for all the cases:

Specification	Drift
Affine	$\alpha_0 + \alpha_1 r_t$
Fifth-order polynomial	$\alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5$

Following graph shows the movements of drift function of the specification after model parameters were estimated. As one can see that there is positive drift in Federal Funds rate when interest rates are over 3, In Eurodollar deposit rate and US Treasury

bill yield, the drift is negative when Eurodollar deposit rate is over 2.5 and US Treasury bill yield is over 2. Less than this rate there is no drift.
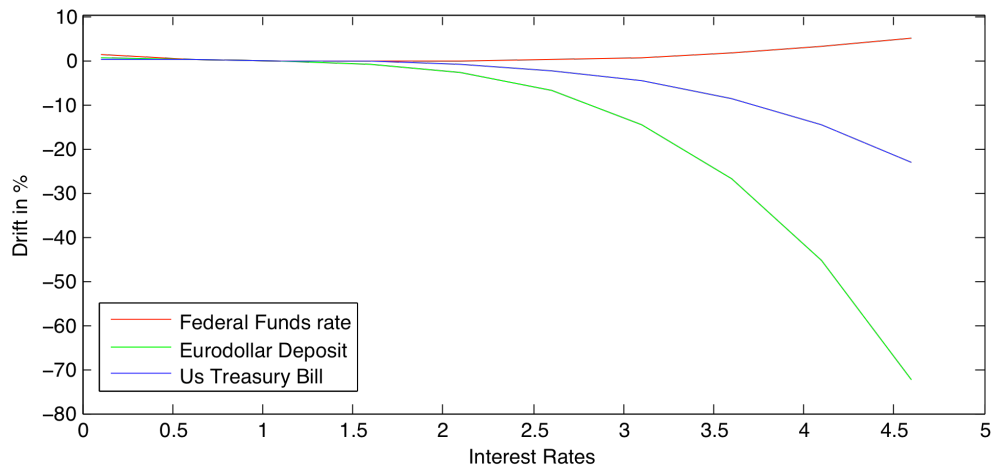
Following table shows Log likelihood parameters and concerning standard errors for general specification, fifth-order polynomial, and affine specifications for daily observations of Eurodollar deposit rate. All parameter except $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta_4$ have small t-statistics. There is identification problem with general specification used by me. There have been inconsistencies between different parameter estimation among different studies. The t test statistics shows that $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta_4$ may be significantly different from zero. When I ran test with fifth-order polynomial shows rather different t test significance test. Here most of the parameters have smaller values than in general specification. $\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \beta_4$ has t-test statistics values which may be explained as these parameters may be significantly different from zero. Now I ran test with affine specification and inference. All t-test statistics have smaller values expect $\beta_2, \beta_3, \beta_4$, which may be significantly different from zero. The log likelihood values increased from general to fifth-order polynomial and again decreased in affine case slightly.

Parameters	General	Fifth	Affine
α_0	0.0077	-0.0280	-0.0002
	(0.0392)	(0.0208)	(0.0448)
α_1	-0.0072	-0.0434	-0.0010
	(0.0142)	(0.0142)	(0.0137)

	0.1078	0.0596	
α_2	(0.0030)	(0.0044)	-
	-0.0183	-0.0106	
α_3	(0.0006)	(0.0009)	-
	-0.1631	-0.0044	
α_4	(0.0142)	(0.0142)	-
	0.0001	4.9e-05	
α_5	(1.8e-05)	(3.0e-05)	-
	0.0636		
α_6	(0.0515)	-	-
	0.7528	0.0056	0.0889
β_0	(0.4455)	(0.0270)	(0.9321)
	-0.0936	0.0779	0.7907
β_1	(0.2369)	(0.0294)	(0.3972)
	0.01374	0.0077	0.5174
β_2	(0.0850)	(0.0281)	(0.0844)
	0.0120	0.0383	-0.2838
β_3	(0.0182)	(0.0158)	(0.0493)
	0.0332	0.0362	0.0231
β_4	(0.0075)	(0.0040)	(0.0010)
Log L	3315.11	3313.95	3304.03

Following table shows Log likelihood parameters and concerning standard errors for general specification, fifth-order polynomial, and affine specifications for one month Federal funds rate.   When I ran my general specification, almost all parameter $\alpha_0,\alpha_2,\alpha_3,\alpha_5,\alpha_6,\beta_0,\beta_1,\beta_2,\beta_3,\beta_4$ have larger t-statistics, only one parameter has smaller t-statistics. There are inconsistencies more or less with same kind of specification or same short-term interest rate series. The t test statistics shows that $\alpha_0,\alpha_2,\alpha_3,\alpha_5,\alpha_6,\beta_0,\beta_1,\beta_2,\beta_3,\beta_4$ may be significantly different from zero. This fact shows that nonlinearity is caused by selection of proxy for analyzing short-term interest rate dynamics. When I ran test with fifth-order polynomial shows rather different t test significance test.   Here β_0 has smaller value than in general specification and shows that parameter may not be significantly different from zero. $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6,\beta_0,\beta_2,\beta_3,\beta_4$ has t-test statistics values which may be explained as these parameters may be significantly different from zero.   This shows nearly

consistent results theoretical but notice the difference of parameter estimated. Now I ran test with affine specification and results are more confusing. All t-test statistics have smaller values expect β_4 which may be significantly different from zero. The log likelihood values decreased from general to fifth-order polynomial and affine slightly.

Parameters	General	Fifth	Affine
α_0	-0.0623	0.4625	0.0097
	(0.0267)	(0.0785)	(0.0604)
α_1	-0.0040	0.3628	-0.0037
	(0.0058)	(0.0330)	(0.0313)
α_2	-0.1426	0.03039	
	(0.0011)	(0.0079)	-
α_3	0.0412	-0.0391	
	(0.0002)	(0.0016)	-
α_4	0.0113	-1.0583	
	(0.0058)	(0.0330)	-
α_5	-0.0005	0.0002	
	(6.0e-06)	(0.0001)	-
α_6	0.1454		
	(0.0215)	-	-
β_0	1.4558	4.7250	-0.2430
	(0.0440)	(1.1807)	(0.3765)
β_1	-0.0368	0.6995	2.5427
	(0.0075)	(0.6479)	(1.7379)
β_2	-0.0072	-3.3488	-0.0864
	(0.0012)	(0.3016)	(0.4562)
β_3	-0.0136	1.8217	2.3657
	(0.0024)	(0.1863)	(1.5106)
β_4	0.0161	0.0257	0.0118
	(0.0007)	(0.0011)	(0.0043)
Log L	3350.81	3317.42	3313.83

Following table shows Log likelihood parameters and concerning standard errors for general specification, fifth-order polynomial, and affine specifications for US

Treasury bill yield daily observations. When I ran my general specification all parameter except $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ have smaller t-statistics. There are inconsistencies more or less with same kind of specification or same short-term interest rate series. The t test statistics shows that $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ may be significantly different from zero. This fact shows that nonlinearity is caused by selection of proxy for analyzing short-term interest rate dynamics. When I ran test with fifth-order polynomial shows rather different t test significance test. I see another inconsistency, here all parameters except $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_3$ has smaller value than in general specification hence $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_3$ can only be shown as may be significantly different from zero. As I wanted to conclude as US Treasury bill to be almost perfect proxy for my general specification, the fifth-order polynomial result dazzled me. Let us see what happens when I run test with affine specification. Results are way more satisfying than fifth-order polynomials. All t-test statistics have larger values expect a_0, a_1. All other parameter may be significantly different from zero. The log likelihood values decreased from general to fifth-order polynomial and affine slightly.

Parameters	General	Fifth	Affine
α_0	-0.0146	-0.0249	-0.0012
	(0.0444)	(0.0383)	(0.0370)
α_1	-0.0166	0.1862	-0.0006
	(0.0204)	(0.0092)	(0.0085)
α_2	0.0618	-0.0166	
	(0.0050)	(0.0019)	
α_3	-0.0130	0.0028	
	(0.0100)	(0.0004)	
α_4	-0.0523	-0.1540	
	(0.0204)	(0.0092)	
α_5	0.0001	-2.5e-05	
	(3.9e-05)	(1.3e-05)	
α_6	0.0158		
	(0.0462)	-	
β_0	-0.0409	0.0094	2.9678
	(0.2038)	(0.7779)	(0.6679)

β_1	0.1582 (0.2477)	0.0087 (0.1918)	-0.4580 (0.1369)
β_2	0.4891 (0.4133)	-0.0163 (0.4190)	-0.2739 (0.0255)
β_3	0.0833 (0.0668)	2.8612 (0.9819)	0.6377 (0.0541)
β_4	-0.0321 (0.0166)	-0.0020 (0.0026)	0.0156 (0.0005)
Log L	3308.13	3300.49	3300.33

Now from our Log likelihood values I can test parameters with likelihood ratio test. I used affine drift specifications and fifth-order polynomial specifications for Federal funds rate, US Treasury bill yield, and Eurodollar deposit rate, as used by T.G. Bali and L. Wu (2006).

Goodness of fit using Likelihood Ratio Test

The likelihood ratio test for three short-term interest rate series is:

$$LR = 2(\Delta LogL) \sim \chi^2(4)$$

I am going to run test for $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, which mean that all given parameters are jointly zero. The results from likelihood ratio are 0.3316, 2.2300, and 7.1815, respectively for US Treasury bill yield, Eurodollar deposit rate, and Federal funds rate. When comparing test-statistics to Critical value with degree of freedom of 4 and 95% confidence interval; 9.49. I cannot reject null hypothesis of parameter jointly zero for US Treasury bill yield and Federal funds rates and Eurodollar deposit rate with degree of freedom of 4 and 95% confidence interval.

These conflicting result shows that diffusion process of short-term interest rate is highly nonlinear depending upon the selection of proxy for short-term interest rate diffusion process. These results are somewhat consistent in some areas with previous studies and inconsistent at the same time.

Nonlinearity in Variance Function of Specification

I analyzed the presence of nonlinearity in drift function in previous part and came up with confusing results. I noticed that sometimes affine specification or Constant elasticity volatility specification does not explain the nonlinearity in short-term

interest rate diffusion. Therefore one might think to compare the general variance specification with affine and Constant Elasticity Volatility variance specification.

Following table shows the different variance specification used for likelihood ratio test to check the nonlinearity of interest rate diffusion:

Specification	Fed. funds	Eu/dollar dep.	US Treasury bill
$\beta_0 + \beta_1 r_t$	3307.19	3314.39	3301.64
$\beta_3 r_t^{\beta_4}$	3311.64	3307.09	3300.64
$\beta_0 + \beta_1 r_t + \beta_2 r_t^2 + \beta_3 r_t^{\beta_4}$	3350.81	3315.11	3308.13

Now I am going to test for likelihood ratio for general variance specification to affine specification and Constant elasticity volatility specification. The likelihood ratio test for three short-term interest rate series is, if I jointly check that $\beta_2 = \beta_3 = \beta_4 = 0$ and that $\beta_0 = \beta_1 = \beta_2 = 0$:

$$LR = 2(\Delta LogL) \sim \chi^2(3)$$

Following table shows the likelihood values of Federal funds rate, Eurodollar deposit rates and US Treasury bill yield:

Likelihood ratio test	Federal funds	Eurodollar deposit	US Treasury bill
$LR = 2(LogL_{general} - LogL_{affine})$	87.24	1.44	12.98
$LR = 2(LogL_{general} - LogL_{CEV})$	78.74	16.04	14.98

The critical value of chi-square with degree of freedom of 3 and confidence interval of 95% is 7.815. Comparing the critical value with test-statistics, I may reject the hypothesis that $\beta_2 = \beta_3 = \beta_4 = 0$ for Federal funds rate, and US Treasury bill. This shows that affine specification may not explain the diffusion process of short-term interest rate model using these interest rate series as proxies. Likelihood test on Eurodollar deposit rate shows that null hypothesis of $\beta_2 = \beta_3 = \beta_4 = 0$ may not be rejected. Comparing the test statistics of $\beta_0 = \beta_1 = \beta_2 = 0$, I may reject the hypothesis for all three interest rate series, which means that Constant elasticity volatility specification may not be able to explain interest rate diffusion process using these

interest rate series. I may conclude that diffusion is highly nonlinear or following a random path.

Conclusion

There has not been any agreement on the effective models for analyzing short-term interest rate dynamics. This paper studied a variety of short-rate interest rate models for different widely used interest rate series. I took a step further from a flexible specification for the analysis of short-term interest rate model presented by Turan G. Bali, Liuren Wu (2006), which in fact was influenced by originally given Ait-Sahalia (1996) and further improved in Ait-Sahalia (2002). For the estimation of parameters I used Quasi-maximum likelihood method. Started my analysis from the traditional, easily tractable model, such as Vasicek model and Cox, Ingersoll, Ross model to more modern models, such as Turan G. Bali, Liuren Wu, 2006's specification enabled me to understand the diffusion process dilemma. I ran nested test under general specification from affine to fifth-order polynomial specification. My results showed that nonlinearity of short-term interest rate diffusion process depends upon data series as interest rate proxy. Result showed inconsistencies, but one can examine that nonlinearity is more visible when one runs test under different variance specifications. For that I tested CEV variance specification and affine specification. My result showed that CEV and affine specifications are unable to tackle with diffusion process of short-term interest rate. More attention to be made on variance specification rather than drift specification, Variance specification played a vital role in my study. One can thrive for a better specification to understand short-term interest rate dynamics. Currently used specifications are ineffective but widely used for a better approximation. Interest rate series did not show same kind of likelihood statistics for that one can object on the usage of these specifications, but these inconsistencies have been a part of interest rate analysis throughout the life of the subject. I have to formally acknowledge the existence of nonlinearity which actually will have a great impact on high interest rate levels.

References

Ahn, D.-H., Gao, B., 1999. A parametric nonlinear model of term structure dynamics. Review of Financial Studies 12 (4), 721–762.

Ait-Sahalia, Y., 1996. Nonparametric pricing of interest rate derivatives. Econometrica 64 (3), 527–560.

Ait-Sahalia, Y., 1996. Testing continuous-time models of the spot interest rate. Review of Financial Studies 9 (2), 385–426.

Ait-Sahalia, Y., 2002. Maximum likelihood estimation of discretely sampled diffusions: a closed form approximation approach. Econometrica 70, 223–262.

Bali, T., 2000. Testing the empirical performance of stochastic volatility models of the short term interest rates. Journal of Financial and Quantitative Analysis 35 (2), 191–215.

Black, E., and Scholes, M. 1973. The Pricing of Options and Corporate Liabilities. Journal of Political Economy 81, 637-659.

Chan, K.C., Karolyi, G.A., Longstaff, F.A., Sanders, A.B. 1992. An empirical comparison of alternative models of the short-term interest rate. Journal of Finance 47 (3), 1209–1227.

Chapman, D.A., Pearson, N.D., 2000. Is the short rate drift actually nonlinear? Journal of Finance 55 (1), 355–388.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, 1985. A theory of the term structure of interest rates. Econometrica 53, 385-407.

Damiano Brigo and Fabio Mercurio, 2007. Interest Rate Models Theory and Practice With Smile, Inflation and Credit, Second Edition. Heidelberg, Springer.

Duffee, G.R., 1996. Idiosyncratic variation of treasury bill yields. Journal of Finance 51 (2), 527–551.

Duffie, D., Kan, R., 1996. A yield-factor model of interest rates. Mathematical Finance 6 (4), 379–406.

Durham, G.B., 2003. Likelihood-based specification analysis of continuous-time models of the short term interest rates. Journal of Financial Economics 70 (3), 463–487.

Durham, G.B., Gallant, A.R., 2002. Numerical techniques for simulated maximum likelihood estimation of stochastic differential equations. Journal of Business and Statistics 20, 297–316.

Hong, Y., Li, H., 2005. Nonparametric specification testing for continuous-time models with applications to spot interest rates. Review of Financial Studies 18 (1), 37–84.

Jiang, G.J., 1998. Nonparametric modeling of US interest rate term structure

dynamics and implications on the prices of derivative securities. Journal of Financial and Quantitative Analysis 33 (4), 465–497.

Jones, C.S., 2003. Nonlinear mean reversion in the short-term interest rate. Review of Financial Studies 16 (3), 793–843.

Koedijk, Kees G., Francois G. J. A. Nissen, Peter C. Schotman, and Christian C. P. Wolff, 1997. The dynamics of short term interest rate volatility reconsidered. European Finance Review 1, 105-130.

Longstaff, F.A., Schwartz, E., 1992. Interest-rate volatility and the term structure: A two factor general equilibrium model. Journal of Finance 47 (4), 1259–1282.

Merton, Robert C., 1973. Theory of rational option pricing. Bell Journal of Economics and Management Science 4, 141-183.

Nowman, K.B., 1997. Gaussian Estimation of Single-Factor Continuous Time Models of the Term Structure of Interest Rates. The Journal of Finance 52 (4), 1695-1706

Pritsker, M., 1998. Nonparametric density estimation and tests of continuous time interest rate models. Review of Financial Studies 11 (3), 449–487.

Brenner, R. J.; R. H. Harjes, and K. F. Kroner, 1996. Another Look at Models of the Short Term Interest Rate. Journal of Financial and Quantitative Analysis 31, 85-107.

Sirimon Treepongkaruna, 2003. Quasi-maximum Likelihood estimates of Kiwi short-term interest rate. Applied Economics Letters 10, 937-942

Spencer, P.D., 1999. An arbitrage-free model of the yield gap. The Manchester School Supplement, 116– 133.

Stanton, R., 1997. A nonparametric model of term structure dynamics and the market price of interest rate risk. Journal of Finance 52 (5), 1973–2002.

Turan G. Bali, Liuren Wu, 2006. A comprehensive analysis of the short-term interest-rate dynamics. Journal of Banking & Finance 30, 1269-1290

Vasicek, Oldrich, 1977. An equilibrium characterization of the term structure, Journal of Financial Economics 5, 177-188.

Board of Governors of the Federal Reserve System, (2009). http://www.federalreserve.gov/DataDownload/Choose.aspx?rel=H.15