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ABSTRACT BOOTSTRAP CONFIDENCE INTERVALS IN LINEAR MODELS

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A bootstrap method for generating confidence intervals in linear models is suggested. The method is motivated by an abstract nonobservable bootstrap sample of true residuals leading to an observable final result. This means that the only error in the method is the pure bootstrap error obtained by replacing the true residual distribution by the empirical one. It is shown that the method is valid, having the same asymptotic conditional distribution as the ordinary bootstrap method. Simulations indicate clearly that the abstract bootstrap method works better than the ordinary bootstrap method for small samples.

1. INTRODUCTION

Already in the very beginning of the development of bootstrap methods, Efron (1979) discussed the possibility to use bootstrap methods for treating regression problems. He suggested the simple method of using the distribution of the empirical residuals as a bootstrap distribution for the error term.

In formulas this means the following. The general regression model used is

$$Y_i = g_i(\beta) + \varepsilon_i \qquad i = 1, 2, ..., n$$

where $g_i(.)$ are known functions of the unknown parameter β and ε_i $i=1,2,...,n$ are i.i.d. with some unknown c.d.f. F(.). The bootstrap sample is then constructed as

$$Y^*_i = g_i(\hat{\beta}) + e^*_i \qquad i=1,2,...,n$$

where β is a least square estimate of β and the e^*_i:s are independently drawn with probabilty 1/n for each empirical residual

$$e_i = y_i - g_i(\hat{\beta}) \qquad i=1,2,...,n$$

These empirical residuals are however not independent in the original problem and neither do they have exactly the same distribution there. Thus they can at most serve as an approximation of a sample of true i.i.d. residuals. A bootstrap method using these empirical residuals thus has a further approximation beside the bootstrap approximation itself.

In the book Efron(1982) and in several papers is also mentioned another method. For the simple linear regression model

$$Y_i = \alpha + \beta\,(x_i - \bar{x}) + \varepsilon_i \qquad i=1,2,...,n$$

with i.i.d. ε_i:s, the method consists in choosing randomly n times among the pairs (x_i,y_i) to get a bootstrap sample. This is sometimes called the paired bootstrap method.

The two methods are in fact related to two different designs of the original problem. The first one corresponds to x values at the disposal and choice of the experimenter, while the second method corresponds to random x values, which are not possible for the experimenter to choose. This is so because the aim of the bootstrap procedure is to depicture the original experiment, which must also include the deterministic or random mechanism for choosing the x values. The second method is 'clean' for the situation of random x values, since there are no further approximation beside the one imposed by the bootstrap itself, but the situation with randomly choosen x values has restricted applicability.

In this paper we will study bootstrap methods for the first type of situation, where the design in a linear model can be choosen by the experimenter. We will suggest a 'clean' method with no approximations beside the one imposed by the bootstrap itself. We will study some of its properties and compare it to the originally proposed method.

2. ABSTRACT BOOTSTRAPPING IN SIMPLE LINEAR REGRESSION

Suppose we have a simple linear regression model

$$Y_i = \alpha + \beta\,(x_i - \overline{x}) + \varepsilon_i$$

where independent observations $Y_1, Y_2,..., Y_n$ are obtained for the regressor values $x_1, x_2,...., x_n$. The true residuals ε_i are supposed to have any continuous distribution with expectation 0 and to be independent, and the x values are at the disposal of the experimenter.

The abstract bootstrap method for this problem means the following.Imagine bootstrap samples from the set of true residuals ε_i $i=1, 2,...., n$. Neither the true residuals nor the bootstrap samples generated by these are observable, since they involve unknown parameters. Calculate theoretically what would happen, if these bootstrap samples were used for some statistical metod e.g. creating a confidence interval for β. It then might happen (and in this case it does happen) that the final result involves only observable variables. The only approximation in the method would thus be the pure bootstrap error imposed by using the empirical c.d.f. $\hat{F}(.)$ instead of the true c.d.f. $F(.)$.

Let us now study the simple regression problem in some more detail. The true residuals are

$$\varepsilon_i = Y_i - \alpha - \beta (x_i - \bar{x}) \qquad i = 1, 2, ..., n.$$

From these we take an abstract sample ε_i^* $i = 1, 2, ..., n$, where we denote the number of the ε_j choosen in the i:th place by $j(i)$. This means that we have

$$\varepsilon_i^* = \varepsilon_{j(i)}.$$

With this sample of residuals we get the (abstract !) bootstrap observations

$$Y_i^* = \alpha + \beta (x_i - \bar{x}) + \varepsilon_{j(i)} \qquad i = 1, 2, ..., n.$$

When substituting $\varepsilon_{j(i)}$ from its defining equation, we get the bootstrap Y observations

$$Y_i^* = \alpha + \beta (x_i - \bar{x}) + Y_{j(i)} - \alpha - \beta (x_{j(i)} - x) =$$

$$= Y_{j(i)} - \beta (x_{j(i)} - x_i)$$

which yields the bootstrap estimate

$$\hat{\beta}^* = (W_Y^* - \beta W_x^*)/Q_x + \beta$$

where

$$W_x^* = \sum_{i=1}^{n} x_{j(i)} (x_i - \bar{x})$$

and

$$W_Y^* = \sum_{i=1}^{n} Y_{j(i)} (x_i - \bar{x}) .$$

Now a confidence interval with confidence level q for β is obtained (as always) by checking for each hypothetical β, if it is rejected or not in a test with significance level 1 - q. If we want a two-sided confidence interval, we have to make one-sided tests at level (1 - q)/2 in each direction. This means that we here use the observable two-dimensional bootstrap variable

$$(W_Y^*,W_x^*) = \left(\sum_{i=1}^{n} Y_{j(i)} (x_i - \bar{x}) , \sum_{i=1}^{n} x_{j(i)} (x_i - \bar{x})) \right).$$

to check the extremeness of the outcome β in the bootstrap distribution of $\hat{\beta}^*$ for different hypothetical β:s.

In order to obtain an upper 1 - (1-q)/2 confidence limit, consider some β' and a level (1-q)/2 test of the hypothesis β = β' against the alternative β < β'. Now the formula

$$\hat{\beta}^* = \beta + (W_Y^* - \beta W_x^*)/ Q_x \tag{1}$$

can be written

$$\hat{\beta}^* = W_Y^*/Q_x + \beta (1 - W_x^*/Q_x). \tag{2}$$

Here usually $W_x^* < Q_x$, since in the bootstrap distribution $E^*(W_x^*) = 0$ and $Var^*(W_x^*) = Q_x^2/n$. Even by the rough Chebyshev inequality

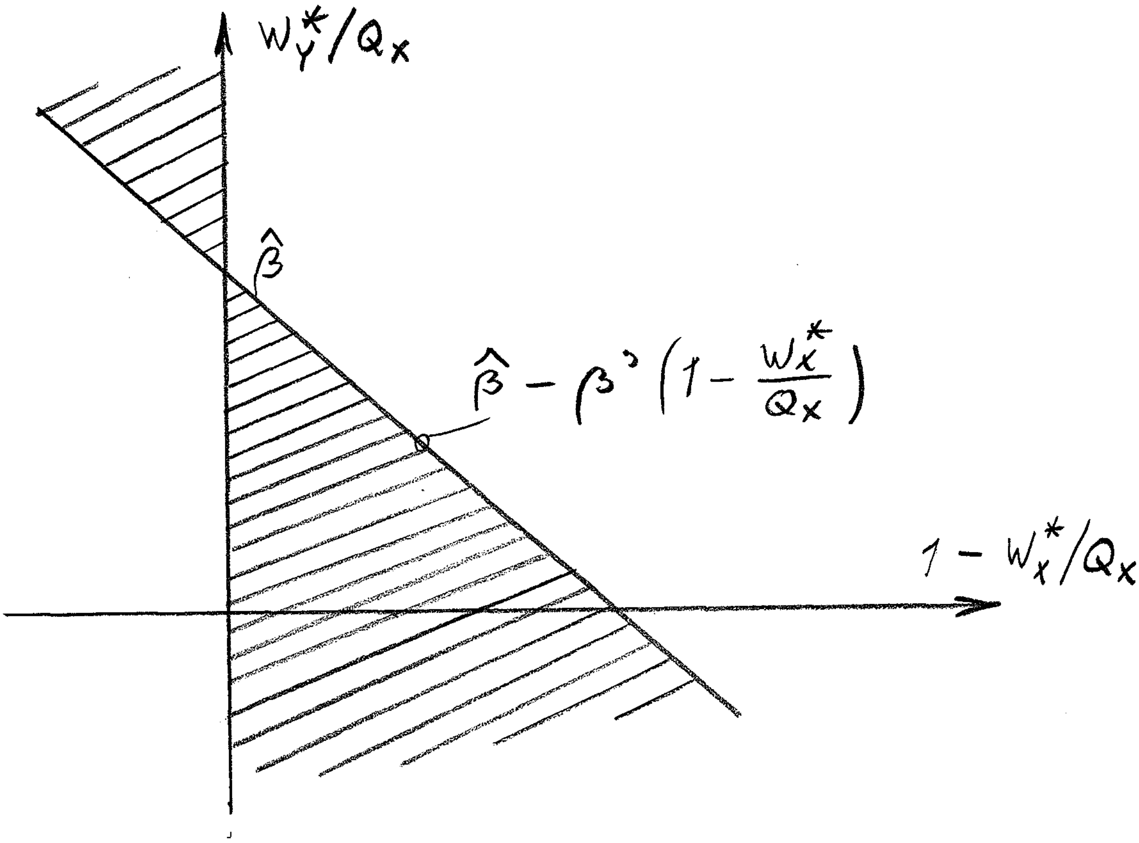
$$P^*(1 - W_x^*/Q_x < 0) < 1/n.$$

Thus there is an indication that β is smaller than β' when $\hat{\beta}^*$ is small if $W_x^* < Q_x$. If occationally $W_x^* > Q_x$, the sign of $1 - W_x^*/Q_x$ is reversed, and there is an indication that β is smaller than β' if $\hat{\beta}^*$ is large. The limits at which the hypothesis $\beta = \beta'$ is rejected in favour of $\beta < \beta'$ are determined in the bootstrap distribution under the assumption that $\beta = \beta'$, one limit for each hypothetical β'.

In particular we are interested in obtaining those β', for which we reject $\beta = \beta'$ in favour of $\beta < \beta'$, when the outcome of $\hat{\beta}^*$ equals $\hat{\beta}$, since it gives us the bootstrap symmetric upper confidence limit for β, when the outcome is $\hat{\beta}$. According to formula (2) and the comments following it, the bootstrap estimate of the probability to get a more extreme outcome than $\hat{\beta}$ for $\hat{\beta}^*$ equals

$$P^*((\hat{\beta}^* < \hat{\beta} \text{ and } W_X^*/Q_X < 1) \text{ or } (\hat{\beta}^* > \hat{\beta} \text{ and } W_X^*/Q_X > 1)) =$$
$$= P^*((W_Y^*/Q_X + \beta' (1 - W_X^*/Q_X) < \hat{\beta} \text{ and } W_X^*/Q_X < 1) \text{ or }$$
$$(W_Y^*/Q_X + \beta' (1 - W_X^*/Q_X) < \hat{\beta} \text{ and } W_X^*/Q_X > 1)).$$

This probability equals the probability of the shadowed area in the following figure for the bootstrap variables Q_X-W_X^* and W_Y^*.



The bootstrap probability of this shadowed area is a monotone function of β'. Computationally the confidence limit can most easily be found by a further simple construction. Starting with the extreme ' β' = and decreasing β' , one point after the other will be included in the area. A point ($Q_x - W_x^*$, W_Y^*) will be included at a β' determined by

$$W_Y^* = \hat{\beta} Q_x - \beta' (Q_x - W_x^*)$$

which means

$$\beta' = (\hat{\beta} Q_x - W_Y^*) / (Q_x - W_x^*)$$
$$= \hat{\beta} + (\hat{\beta} W_x^* - W_Y^*) / (Q_x - W_x^*). \quad (3)$$

The computation of the confidence limits can be administrated by calculating for each simulated bootstrap sample the value $(\hat{\beta} Q_x - W_Y^*)/(Q_x - W_x^*)$ and then after the whole simulation sorting those values and finding the $(1-q)/2$ and $1 - (1-q)/2$ quantiles.

Let us now compare this abstract bootstrap procedure to the earlier used percentile method. In that method a bootstrap sample is taken from the empirical residuals

$$e_i = Y_i - \hat{\alpha} - \hat{\beta} (x_i - x) .$$

With the same notations as before the random choice of residual to add to the linear function $\hat{\alpha} + \hat{\beta} (x - x)$ in the point $x = x_i$ is

$$e_{j(i)} = Y_{j(i)} - \hat{\alpha} - \hat{\beta} (x_{j(i)} - x)$$

and we get the bootstrap sample of Y values

$$Y_{j(i)} - \hat{\beta} (x_{j(i)} - x_i) \qquad i = 1, 2, ..., n$$

for the design points x_i $i = 1, 2, ..., n$.

The bootstrap estimate of β becomes

$$\hat{\beta}^* = (W_y^* + \hat{\beta} (Q_x - W_x^*)) / Q_x .$$

The implicit model is here that $\beta - \beta$ has a fixed distribution approximated by that of $\hat{\beta}^* - \hat{\beta}$. Thus we find the upper limit in a symmetric confidence interval with confidence level $1 - q$ by finding the β' making

$$P^*(\hat{\beta} - \beta' > (W_Y^* - \hat{\beta} W_x^*) / Q_x) = (1-q)/2$$

i.e. the upper confidence limit is obtained as the $1 - (1-q)/2$ fractile in the bootstrap distribution of

$$(\hat{\beta} (Q_x + W_x^*) - W_Y^*) / Q_x =$$
$$= \hat{\beta} + (\hat{\beta} W_x^* - W_Y^*)/Q_x. \qquad (4)$$

In the same way we get the lower confidence limit as the $(1-q)/2$ quantile in the same distribution. This is to be compared to the more accurate abstract bootstrap method, where we use the same quantiles in the distribution of

$$(\hat{\beta} Q_x - W_Y^*) / (Q_x - W_x^*) =$$
$$= \hat{\beta} + (\hat{\beta} W_x^* - W_Y^*)/(Q_x - W_x^*).$$

Observe that the 'bootstrap random parts' in the two cases have a common factor, including the main variation, and different dividends Q_x and $Q_x - W_x^*$. The difference of the bootstrap variables for the abstract method and the ordinary percentile method equals

$$(\hat{\beta} W_x^* - W_Y^*) W_x^* / (Q_x (Q_x - W_x^*)). \qquad (5)$$

For the ordinary bootstrap method the expectation in the bootstrap distribution equals

$$E^*(\hat{\beta} + (\hat{\beta} W_x^* - W_Y^*)/Q_x) = \hat{\beta}$$

and the variance equals

$$Var^*(\hat{\beta} + (\hat{\beta} W_x^* - W_Y^*)/Q_x) = \sum_{i=1}^{n} e_i^2 / n Q_x$$

where

$$e_i = Y_i - \hat{\alpha} - \hat{\beta} (x_i - x).$$

Thus the unconditional variance equals

$$E(Var^*(\hat{\beta} + (\hat{\beta} W_x^* - W_Y^*)) = (n-2) \sigma^2 / n Q_x$$

which is smaller than the variance of $\hat{\beta}$ by a factor (n-2)/n. This illustrates the wellknown fact, pointed out already by Efron (1982), that the ordinary bootstrap method has a tendency of underestimating the variation. For the abstract bootstrap method the expectation not even exists, since with positive probability $W_x^* = Q_x$. Nevertheless the bootstrap distribution can be a very good approximation of the true distribution.

3. GENERAL LINEAR MODELS

Now let us consider more general linear models like linear regresion models with more than one regressor or analysis of variance models. We write the model in the form

$$Y = \alpha + X' \beta + \varepsilon$$

where we have singled out the general mean vector α (with all component values equal to some common α_0) and use a reduced design matrix X. If the number of observations is n, and the number of components in β is p, the X matrix is of type pxn. The error vector ε is supposed to consist of i.i.d. components having some unknown continuous distribution with expectation 0. We further suppose the design to be orthogonal between the α component and the individual β components i.e.

$$X \alpha' = 0 \qquad \text{(p-vector).}$$

No orthogonality between β components is required. The orthogonality condition means that the LS estimate of β equals

$$\hat{\beta} = S^{-1} X Y$$

where $S = X X'$.

The components of the unobservable error vector

$$\varepsilon = Y - \alpha - X' \beta$$

are 'used' in an abstract bootstrap procedure, where we imagine ε^* having components choosen randomly with replacement from the components of ε. Let j(i) be the number of the ε component choosen to be the i:th component of ε^*. Then this component is

$$\varepsilon_{j(i)} = Y_{j(i)} - \alpha_0 - X_{j(i)}' \beta$$

where $X_{j(i)}$ is the j(i):th column vector of X. For the whole vector of abstract bootstrap observations we use the notation

$$\varepsilon^* = Y^* - \alpha X^{*\prime} \beta .$$

The bootsstrap observation vector would now be

$$\alpha + X' \beta + Y^* - \alpha - X^{*\prime} \beta = Y^* - (X^{*\prime} - X') \beta.$$

Observe that we have the same design in the bootstrap experiment as in the original one. The bootstrap estimate of β thus equals

$$S^{-1} (S \beta + X Y^* - X X^{*\prime} \beta) .$$

Like in the simple case with one regressor we consider the possibility of a more extreme observation than β for different β:s. This means that a crucial point is when

$$\hat{\beta} = S^{-1} (S \beta + X Y^* - X X^{*\prime} \beta)$$

i.e. when

$$\beta = (S - X X^{*\prime})^{-1} (S \hat{\beta} - X Y^*)$$

if the inverse exists. The bootstrap distribution of the variable

$$(S - X X^{*\prime})^{-1} (S \hat{\beta} - X Y^*) \qquad (6)$$

should be studied in order to obtain confidence sets of different types for β or to test different hypotheses.

The inverse does not exist if X^* happens to be equal to X. The probability of this is n^{-n} . The nonexistance of the inverse, which might occur also in other cases, ought to be a rare event. It has however to be taken into account in the registration procedure for the bootstrap simulation results as well as in calculation of the risk probabilities in the bootstrap distribution.

Like in the case of simple linear regression it might illuminate the method to compare it to the ordinary percentile method based on Efron (1982). After some elementary calculation we find in this case that the confidence interval with confidence coefficient q is given by the (1-q)/2 and (1+q)/2 fractiles of the bootstrap distribution of

$$S^{-1}\left(\left(S + X X^{*\prime}\right)\hat{\beta} - X Y^{*}\right) =$$
$$= \hat{\beta} + S^{-1}\left(XX^{*\prime}\hat{\beta} - X Y^{*}\right).$$

The difference between the two bootstrap distributions equals

$$\left(\left(S - X X^{*\prime}\right)^{-1} - S^{-1}\right)\left(XX^{*\prime}\hat{\beta} - XY^{*}\right) =$$
$$= S^{-1} XX^{*\prime}\left(S - XX^{*\prime}\right)^{-1}\left(XX^{*\prime}\hat{\beta} - XY^{*}\right), \qquad (7)$$

which could be compared to the special case of simple linear regression in formula (5).

4. SOME SPECIAL CASES

In this section we will study the abstract bootstrap method for some other special cases than the simple linear regression, which was used as an introduction in section 2.

EXAMPLE 1. The simplest of all linear models is a two sample case, where the interesting parameter is the translation between the means of the samples from two distributions of the same form. In our model we can formally use design points -1/2 and 1/2 for the two samples, which means that the β parameter is just the translation between the samples. We denote the sample sizes by n_1 and n_2 and the means by $\bar{Y}_1$ and $\bar{Y}_2$. By simple calculations it now follows that

$$\hat{\beta} = \bar{Y}_2 - \bar{Y}_1$$

and

$$Q_x = n_1 n_2 / (n_1 + n_2).$$

The bootstrap random quantities W_x* and W_Y* can be written

$$W_x* = \frac{1}{n_1+n_2} [N_{11}* n_2 + N_{22}* n_1 - n_1 n_2]$$

and

$$W_Y* = \frac{n_1 n_2}{n_1+n_2} [\bar{Y}_2* - \bar{Y}_1*],$$

where

$\bar{Y}_1*$ and $\bar{Y}_2*$ are the means of the Y values of random samples (among all Y:s !) corresponding to the points x_1 and x_2,

$N_{11}*$ is the number of Y values in the bootstrap sample corresponding to x_1, which come from the original sample 1 and

$N_{22}*$ is the number of Y values in the bootstrap sample corresponding to x_2, which come from the original sample 2.

From this it easily follows that the confidence interval generating bootstrap random variable in this case equals

$$(S - XX*')^{-1} (S \hat{\beta} - X Y*) =$$

$$= [(\bar{Y}_2 - \bar{Y}_2*) - (\bar{Y}_1 - \bar{Y}_1*)] / [2 - N_{11}*/n_1 - N_{22}*/n_2] \qquad (8)$$

The method needs at least 5 observations in each original sample, otherwise the probability of getting the nominator equal to 0 will be too large.

Formula (8) gives a simple illustration of the basic behaviour of the abstract bootstrap method. It is seen that the bootstrap random choice of units gives a denominator, which estimates varying factors times the interesting parameter. This factor is compensated by the nominator.

EXAMPLE 2. Another very simple special case is the 'simple analysis of variance situation' with a number of observation series, which can be supposed to have the same unknown distributional form and to differ only in location. For simplicity we consider the same number m of observations in each series. The number of series is denoted by k. The parameter β with k components has the restriction

$$\sum_{i=1}^{k} \beta_i = 0$$

in order to give ortogonality to the α component. In a bootstrap sample we denote the mean of all Y:s choosen in creation of series i by $Y_i{}^*$. Further we denote the number of Y:s choosen from series j in the bootstrap creation of series i by $N_{i,j}{}^*$ and the total number of Y:s choosen from series j by $N_j{}^*$. Now it is easily seen that the confidence interval determining bootstrap random variable equals

$$\left(I_k - \frac{1}{m}[\, N_{i,j}{}^* \,] + \frac{1}{m\,k}\, e_k\, [\, N_j{}^*]' \right)^{-1} [\, \bar{Y}_i{}^* - \bar{\bar{Y}}{}^* - \bar{Y}_i + \bar{\bar{Y}} \,] \qquad (9)$$

where I_k is a unit k×k matrix, e_k is a unit k vector, $\bar{\bar{Y}}$ is the grand mean of the Y:s for the original observations and $\bar{\bar{Y}}{}^*$ is the grand mean for the Y:s in the bootstrap sample. The brackets [and] are used to denote vectors or matrices with the elements written inside the brackets.

It is neccessary to make an inversion in each bootstrap sample although the estimation in the original problem involves no inversion. This is due to the 'problem variation' inherent in the abstract bootstrap method, mentioned also in the previous example.

5. ASYMPTOTIC VALIDITY

A fundamental paper on the asymptotics for ordinary bootstrap in regression models is Freedman (1981). Theorem 2.2 in his paper gives the asymptotic normality for the conditional distribution of the bootstrap estimate in a general case with possibly unequal sample sizes of the original sample (n) and the bootstrap sample (m) under some mild conditions. We will use his results here, but since we discuss only the case of same size of the original sample and the bootstrap sample, we will specialize to the standard case m = n, when we use his results. It will be proved here that the abstract bootstrap method has the same asymptotic distributional properties as the ordinary bootstrap method, i.e. that the abstract bootstrap method is also asymptotically valid.

Freedmans (1981) conditions for the regression problem include the conditions that the model is of the type we study, i.e. that the design is non-random and that the errors are i.i.d. Beside these conditions there is a design convergence condition, which in our notation means that $\lim_{n\to\infty} S/n = \lim_{n\to\infty} XX'/n = V$, for some positive definite V.

THEOREM 1. Suppose that the design in the linear model $Y = \alpha + X'\beta + \varepsilon$ is nonrandom, the components of the error vector ε are i.i.d. and the design matrix X satisfies

$$\lim_{n\to\infty} S = \lim_{n\to\infty} XX' = V$$

where V is positive definite. Then with probability 1, the conditional distribution of the normalized confidence set determining random vector

$$n^{1/2} (S - X X^{*\prime})^{-1} (S \hat{\beta} - X Y^*)$$

converges weakly to a normal distribution with expectation 0 and covariance matrix $\sigma^2 V^{-1}$.

PROOF. The conditions of Theorem 2.2 in Freedman (1981) are satisfied. Thus by that Theorem the conditional distribution of the normalized confidence set determining random vector

$$n^{1/2} S^{-1} [(S + X X^{*\prime}) \hat{\beta} - X Y^*]$$

for the ordinary bootstrap method converges weakly to a normal distribution with expectation 0 and covariance matrix $\sigma^2 V^{-1}$. According to formula (7) the difference between the normalized confidence set determining random vectors for the ordinary bootstrap method and the abstract bootstrap method equals

$$n^{1/2} S^{-1} XX^{*\prime} (S - XX^{*\prime})^{-1} X (X^{*\prime} \hat{\beta} - Y^*) =$$

$$= n S^{-1} \; n^{-1/2} XX^{*\prime} \; n(S - XX^{*\prime})^{-1} \; n^{-1}X(X^{*\prime}\beta - Y^*).$$

Here $n^{-1/2} XX^{*\prime}$ converges in distribution and $n S^{-1}$ as well as $n (S - XX^{*\prime})^{-1}$ converge to V^{-1}. Finally $n^{-1} X (X^{*\prime} \hat{\beta} - Y^*)$ converges in probability to 0, and so does the whole product. Thus by a multidimensional Cramer-Slutsky theorem, the conditional distributions of the normalized confidence set determining random vectors for the ordinary bootstrap method and the abstract bootstrap method converge to the same limit. See e.g. Gänssler & Stute (1977) Korrolar 8.6.6. page 354.

Q.E.D.

5. SIMULATION COMPLEMENTS

The theoretical results and discussions in the previous sections will here be illustrated by simulations for the case of simple linear regression.

Suppose first that we have equidistant x values and normal distributions. For this case we have compared the confidence coefficient and the lengths of the confidence intervals for the slope based on the ordinary t method, the common bootstrap percentile method and the abstract percentile method. The intended confidence coefficient were in all cases equal to 95 %. A simulation with 1000 regression observation sets were generated for each of the sample sizes n = 10, 20, 40. They were treated by the three confidence interval methods for the same samples. Bootstrap sample size for the two bootstrap methods were 1000. The same bootstrap samples were used in both cases. The missing probabilities for the different regression sample sizes obtained in the simulation were

Sample size	t method	Ordinary bootstrap	Abstract bootstrap
10	4.6 %	10.9 %	4.8 %
20	4.8 %	6.6 %	4.6 %
40	5.5 %	6.6 %	5.4 %

The mean length of of the ordinary bootstrap method interval and the mean length the abstract bootstrap method interval had the following changes in relations to the mean length of the t method interval.

Sample size	Ordinary bootstrap	Abstract bootstrap
10	-23.1 %	-1.6 %
20	-10.2 %	+0.1 %
40	-4.6 %	+0.5 %

This is also what would be expected. The decrease in confidence coefficient for the ordinary bootstrap method ought to be reflected in a corresponding smaller length of the interval. The simulation is general in the sense that it is invariant in changes of the true slope and the error variance and linear transformation of the x scale. There are no essential differences between the intervals obtained with the t method and the abstract bootstrap method for these sample sizes.

We have also made the same type of simulation for a skew long-tailed error distribution with density

$$f(x) = 3 / (x + 1.5)^4 \qquad \text{for } x > -0.5$$

$$f(x) = 0 \qquad \text{for } x < -0.5$$

In a situation where the x values are equally spaced, the distribution is symmetrizised in the estimation of β, since the coefficients of the error contributions appear in pairs with the same absolute value and different signs. The long tails remain however. For this distribution we have obtained by simulation the distribution of the common t statistic used for making tests and confidence intervals for β. The sample size in the simulation is 50000 and the standard deviations in the estimates of the percentiles are less than 0.01. The simulation gave the following 97.5 percentiles.

Sample size	10	20	40
97.5 % percentile	2.18	1.99	1.96

Observe that the limits are more narrow in this case than in the case of normal distribution, which could in fact be supposed.

In a simulation with sample size 1000 three methods for creating 95 % confidence intervals for β were compared, the t method with the table above, the ordinary bootstrap method and the abstract bootstrap method. For sample sizes 10, 20 and 40 the obtained missing probabilities were

Sample size	t-method	Ordinary bootstrap	Abstract bootstrap
10	5.1 %	10.7 %	4.8 %
20	6.0 %	8.0 %	4.7 %
40	6.9 %	7.5 %	6.0 %

The mean lengths of the intervals obtained with the ordinary bootstrap method and the mean lengths of the intervals obtained with the abstract bootstrap method got the following relations to the mean length of the intervals obtained with the t method.

Sample size	Ordinary bootstrap	Abstract bootstrap
10	-18.2 %	+3.8 %
20	-4.1 %	+6.4 %
40	-4.6 %	+0.5 %

The simulations indicate that there is a noticable difference between the common bootstrap method on one side and the abstract bootstrap method and the t method on the other side. At least in the studied cases the abstract method adapts well to the prerequired level of significance for small sample sizes and there are no essential differences between the 'tabel corrected t method' and the abstract bootstrap method.

6. REFERENCES

DiCiccio T. J. & Romano J. P. (1988). A review of bootstrap confidence intervals. J. Roy. Statist. Soc. Ser. B, 50, 338-354.

Efron B. (1979). Bootstrap methods : another look at the jackknife. Ann. Statist. , 7, 1-26

Efron B. (1982). The jackknife, the bootstrap and other resampling plans. Soc. Ind. Appl. Math. CBMS-Natl. Sci. Found. Monogr. 38.

Freedman D. A. (1981) Bootstrapping regression models. Ann. Statist. 9, 1218-1228.

Gänssler P. & Stute W. (1977). Wahrscheinlichkeitsteorie. Springer-Verlag.

1989:1	Johnsson, T.	A procedure for stepwise regression analysis.
1989:2	Johnsson, T.	On the closure of a bootstrap multiple comparison test.
1989:3	Frisén, M. & de Maré, J.	Optimal surveillance
1989:4	Palaszewski, B.	A stepwise test procedure for finding groups with non minimal parameters.
1990:1	Holm, S.	Abstract bootstrap confidence intervals in linear models.