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ON PERFORMANCE OF METHODS FOR STATISTICAL SURVEILLANCE

by

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SUMMARY.

Statistical surveillance is used to detect a change in a process. It might for example be a change of the level of a characteristic of an economic time series or a change of heart rate in intensive care. An alarm is triggered when there is enough evidence of a change. When surveillance is used in practice it is necessary to know the characteristics of the method, in order to know which action that is appropriate at an alarm.

The average run length, the probability of a false alarm, the probability of successful detection and the predictive value of an alarm are measures that are used when comparing the performance of different methods for statistical surveillance.

In the first paper a detailed comparison between two important methods, the Exponentially Weighted Moving Average and the CUSUM, is made. Some consequences of using only the average run length as the measure of performance are demonstrated. Differences between the methods are discussed in regard to the measures mentioned above.

The second paper is focused on the predictive value of an alarm, that is the relative frequency of motivated alarms among all alarms. The interpretation of an alarm is difficult to make if the predictive value of an alarm varies with time. Thus conditions for a constant predictive value of an alarm are studied. The Shewhart methods and some Moving Average methods are discussed and some general differences in performance are pointed out.

Three different types of Exponentially Weighted Average are discussed and some differences established. It is further stated that if a Fast Initial Response feature is added to a method, this will in general lower the level of the predictive value of an alarm in the beginning of the surveillance. The increased probability of alarm in the beginning might thus be useless.

The thesis consists of this summary and the following two papers;

(1) Frisén, M and Åkermo, G. (1993). Comparisons between two methods of surveillance: Exponentially Weighted Moving Average vs CUSUM. *Research Report 1993:1.* Department of Statistics, Göteborg University. Revised 1994.

(2) Åkermo, G. (1994). Constant predictive value of an alarm. *Research Report 1994:6.* Department of Statistics, Göteborg University.

COMPARISON BETWEEN TWO METHODS OF SURVEILLANCE: CUSUM VERSUS EXPONENTIALLY WEIGHTED MOVING AVERAGE

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When control charts are used in practice it is necessary to know the characteristics of the charts in order to know which action is appropriate at an alarm. The probability of a false alarm, the probability of successful detection and the predictive value are three measures (besides the usual ARL) used for comparing the performance of two methods often used in surveillance systems. One is the "Exponentially weighted moving average" method, EWMA, (with several variants) and the other one is the CUSUM method (V-mask). Illustrations are presented to explain the observed differences. It is demonstrated that methods with high probabilities of alarm at the first time points of the surveillance should be used with care. They tend to have good ARL properties. However their low predicted value makes action redundant at early alarms.

KEY WORDS: Quality control; Control charts; EWMA; FIR; V-mask; Predicted value; Performance;

Methods for continual surveillance to detect some event of interest, usually presented in the form of control-charts, are used in many different areas, e.g. industrial quality control, detection of shifts in economic time series, medical intensive care and environmental control.

A wide variety of methods have been suggested, see e.g. Zacks (1983) and Wetherill and Brown (1990). Some methods (like the Shewhart test) only take the last observation into account. Others (simple sums or averages) give the same weight to all observations. For most applications it is relevant to use something in between. That is, all observations are taken into account but more weight is put on recent observations than on old ones. The CUSUM and the EWMA are such methods. They are much discussed and both are nowadays often recommended. Both these methods include the extremes mentioned above as special cases and the relative weight on recent observations and old ones can be continuously varied by varying their two parameters. A description of the methods is given in Section 2.

Several extensive comparisons of these methods have been done, see e.g. Roberts (1966), Ng and Case (1989), Lucas and Saccucci (1990) and Domangue and Patch (1991). Most comparisons are made for cases where the out-of-control state is present when the surveillance starts. The study by Domangue and Patch includes the case where the out-of-control state is a linearly increasing change, but also this state is assumed to start at the same time as the surveillance starts. Roberts (1966) gives, by technical reasons results for the case where the change appears at time 8. Lucas and Saccucci (1990) give results both for the case where the change appears immediately and the case of "steady state" where the time of the change tends to infinity. The comparisons have not demonstrated any great differences. This is not surprising since by the two parameters the methods can be designed to fulfil two conditions. The methods can thus be designed to have the same average run length, ARL, (see Section 3.2) for both the in-control and the out-of-control state. Nearly all comparisons have been based on the ARL.

Here a study is made of the remaining differences when the methods have the same ARL.

Because of the dependence on the length of the period of the surveillance and on the time of the change, the significance level and the power have to be generalized in some of many possible ways. Other variables such as the rate of change (if the change takes place successively) will also influence the performance of a method of surveillance. However, the following discussion will be restricted to the influence of the first two mentioned variables which always influence the performance. In the examples below the case of a sudden shift in the mean of Gaussian random variables from an acceptable value μ^0 (zero) to an unacceptable value μ^1 (one) is considered.

This paper uses three measurements of performance suggested by Frisén (1992) for the comparison of the two methods in cases where, by the choice of design parameters, the first moment of the run-length distributions are set equal. The main interest is the influence of time and the different risks of false judgements involved when repeated decisions will be made about hypotheses which might successively change.

In Section 1 the two methods are presented. In Section 2 measures to be used in the evaluations are introduced. In Section 3 the results are given and in Section 4 the results are discussed.

1. METHODS

Since repeated decisions are made, the theory of ordinary hypothesis testing does not apply. Two specific methods of surveillance often used in quality control will be described below. For more exhaustive descriptions of methods used in quality control see e.g. Wetherill and Brown (1990). The two methods will be evaluated by the measures suggested in Section 2. Thus their principal differences will be enlightened. However, the two methods are by no means the only ones to be considered. Similar comparisons of other methods were made by Frisén (1992). The EWMA- and the CUSUM-methods both take past observations into account by summation. They also have two parameters each. They can thus have the same ARL both with and without a specific shift. To make the methods comparable the parameters of the methods are set by the requirement that the ARL^0 and ARL^1 (as described in Section 2) are the same. The actual values used in this study are for the in-control-state $ARL^0=330$ and for the out-of-control state of a shift to $\mu^1=1$ at the start of the surveillance, $ARL^1=9.7$. Very extensive simulations were used to find parameter sets which resulted in the same values of ARL and for the figures. Thus only one set of parameters is used. However, this is enough to prove that important differences might exist in spite of equal ARL values. The results will also support the general discussion about which qualities we should require.

Two-sided methods are used in the examples and simulations. The methods are illustrated in Figures 2 - 5 with data (Figure 1) used by Lucas and Crosier (1982) and Lucas and Saccucci (1990). In order to get simulation results which are suitable for comparisons between methods, the up to 1,000,000 sets of random numbers for the control sequences are the same for all methods. The value for the first time point (and in some figures also the second one) is achieved by exact calculation. Although discrete time is considered continuous curves are drawn by linear connections between values to simplify the pictures.

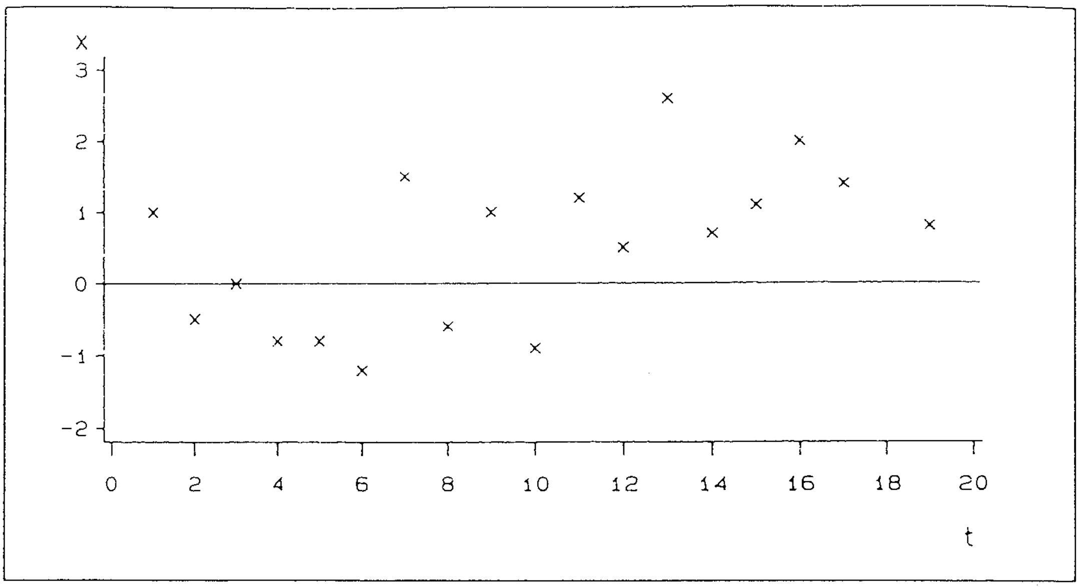

Figure 1. *The observed values X for each time t were generated by Lucas and Crosier (1982) by a process with constant mean (zero) for the first 10 observations and with a shift in mean of one standard deviation (one) for the last observations.*

1.1 *CUSUM*

Page (1954) suggested that the cumulative sums of observed values (x_t t=1,2,...) should be used in a specific way to detect a shift in the mean of a normal distribution. His suggestion was that you calculate C_t=sum(x_i-μ^0), i=1,...,t , and that there will be an alarm for the first t with $|C_t-C_{t-i}|$ is greater than h+ki for some i, and C_0=0. Sometimes (see e.g. Siegmund 1985) the CUSUM test is presented in a more general way by likelihood ratios (which in the normal case reduce to C_t-C_{t-i}). The test might be performed by moving a V-shaped mask over a diagram until any earlier observation is outside the limits of the mask (see Figure 2). Thus the method is often referred to as "the V-mask method". Another name used in some fields of the literature is "Hinkley's method".

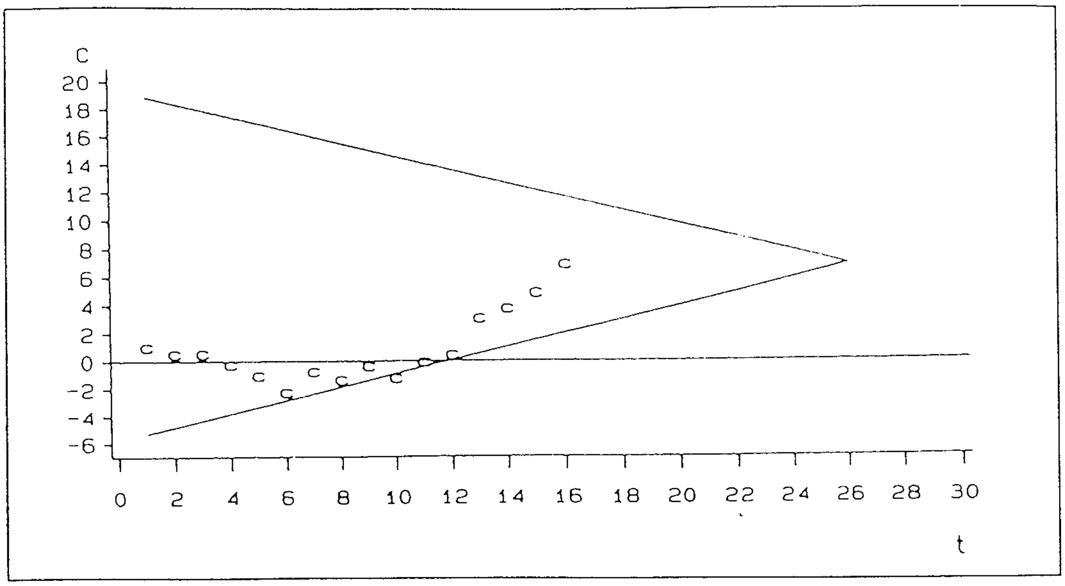

Figure 2. *CUSUM. An alarm occurs at the first time any C_t falls outside the V-shaped mask. In this case the first alarm is at time $t=16$*

Recent observations have more weight than old ones. If h=0, the V-MASK-test degenerates to a SHEWHART-test with the alarm-limit equal to k. With a shift of size μ^1-μ^0 and a constant variance, k=(μ^1-μ^0)/2 is usually recommended (see e.g. Bissel (1969)). This value of k is supposed to give a test having the shortest ARL^1 (for this specific shift) for a given ARL^0. Here the main aim is to demonstrate that important differences exist in spite of equal ARL.

The examples are very similar to those in Lucas and Saccucci (1990)). The average run lengths have been fixed at ARL^0=330 and ARL^1=9.7. The parameter h, which determines the distance between the last observation and the apex of the "V" is set to 4.73 and the parameter k, which determines the slopes of the legs is set to 0.49.

Several variants of the method have been suggested. Lucas (1982) suggested a combination with the Shewhart method. Observe that a CUSUM will always give an alarm if any observation deviates more than h+k from the target value. Also the standard version of CUSUM can thus be regarded as a combination with a Shewhart test with the limit h+k. Yashchin (1989) has suggested that the weights of different observations should be separately chosen to meet some specific purposes. Here the original version of the method by Page (1954) is studied. The method has certain optimality properties as described in Moustakides (1986), Pollak (1987) and Frisén and de Maré (1991).

1.2 *EXPONENTIALLY WEIGHTED MOVING AVERAGE*

Exponentially weighted forecasts have been advocated by e.g. Muth (1960). A method for surveillance based on exponentially weighted moving averages, here called EWMA, was introduced in the quality control literature by Roberts (1959) but has for a long time been rarely used. Recently it has got more attention as a process monitoring and control tool. This may be due to papers by Robinson and Ho (1978), Crowder (1987), Lucas and Saccucci (1990), Ng and Chase (1989) and Domange and Patch (1991) in which techniques to study the properties of the method and also positive reports of the quality of the method are given.

The statistic is

$$Z_i = (1-\lambda)Z_{i-1}+\lambda X_i, \ i=1,2,..$$

where $0<\lambda<1$ and in the standard version of the method $Z_0 = \mu^0$.

EWMA gives the most recent observation the greatest weight, and gives all previous observations geometrically decreasing weights. If λ is equal to one only the last observation is considered and the resulting test is a Shewhart test. If λ is near zero all observations have approximately the same weight.

If the observations are independent and have a common standard deviation σ_X, the standard deviation of Z_i is

$$\sigma_{z_i}=\sigma_z\sqrt{1-(1-\lambda)^{2i}}$$

where σ_Z is the limiting value for large i.

An out-of-control alarm is given if the statistic $|Z_i|$ exceeds an alarm limit, usually chosen as $L\sigma_Z$, where L is a constant. It might seem natural (and is sometimes advocated) to use the actual value of the standard deviation of Zi. However, usually the limiting value σ_Z rather than σ_{Zi} is used in the alarm-limits for EWMA control charts (see e.g. Roberts (1959), Robinson and Ho (1978), Crowder (1989) and Lucas and Saccucci (1990)). For a two-sided control chart this results in two straight warning-limits, one on each side of the nominal level of Z. This variant is therefore called the "straight EWMA" henceforth.
See Figure 3.

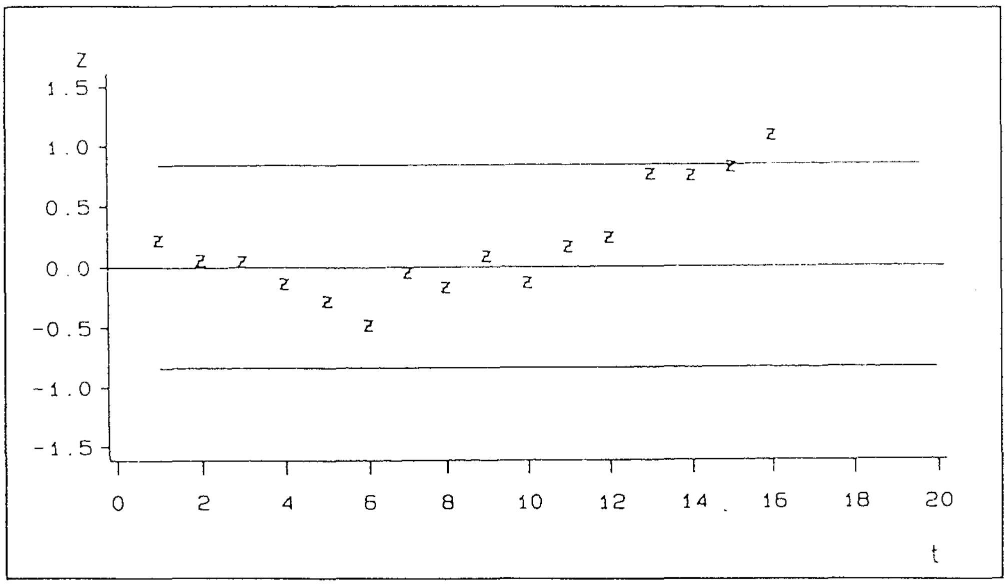

Figure 3. *Straight EWMA. Z denotes the exponentially weighted sum of the observations X. The straight alarm limits are at a distance $L\sigma_Z$ from the target value.*

By using σ_{Zi} the alarm limits start at a distance of $L\lambda\sigma_X$ from the target value and increases to $L\sigma_Z$. This variant is called "variance corrected EWMA" henceforth. See Figure 4.

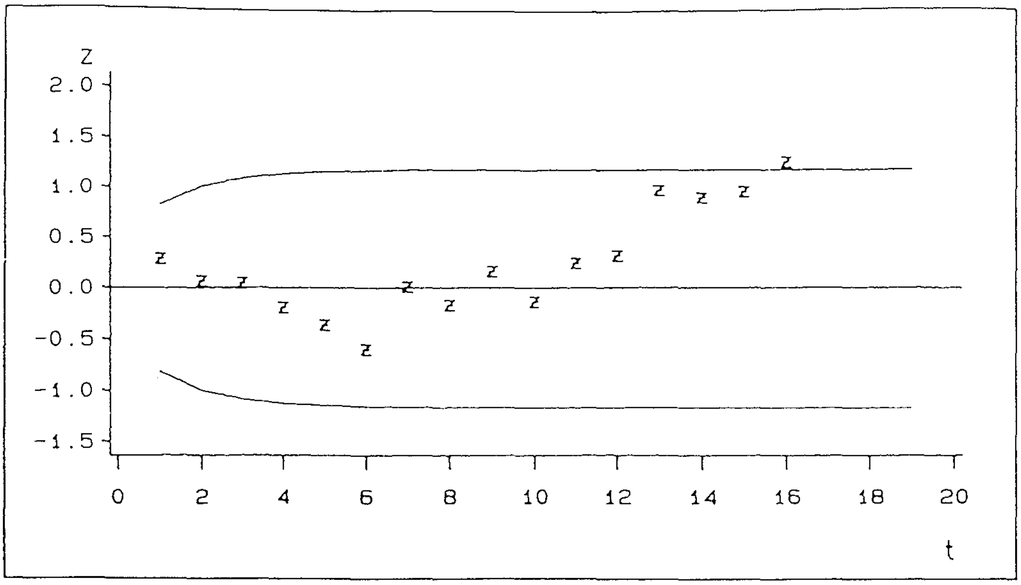

Figure 4. *Variance corrected EWMA. The alarm limits are based on the actual values of the variance of Z for each time point.*

Lucas and Saccucci (1990) recommend that instead of the standard starting value $Z_0 = \mu^0 = 0$, another value should be used to achieve a "Fast Initial Response", FIR. Two one-sided EWMA control schemes are simultaneously implemented. One is implemented with $Z_0 = a$ and one with $Z_0 = -a$. There is an alarm if any of the one-sided schemes exceeds its constant limit. We will now study the relation between the different variants of EWMA more closely and concentrate on the one-sided upper limits for simplicity.

Let

$$c = L\sigma_Z = L\sqrt{\frac{\lambda}{(2-\lambda)}}\sigma_X$$

The straight EWMA gives alarm for

$$Z_i > c.$$

The variance corrected EWMA gives alarm for

$$Z_i > c\sqrt{1-(1-\lambda)^{2i}}$$

The FIR have the same alarm value c as the straight EWMA but because of the starting value we have

$$Z_i' = Z_i + a(1 - \lambda)^i > c$$

that is

$$Z_i > c - a(1 - \lambda)^i$$

If

$$a = L\sigma_X\{(\lambda/(2 - \lambda))^{1/2} - \lambda\}/(1 - \lambda)$$

then the upper limit for the first observation will be the same as for the variance corrected EWMA which has the limit

$$L\sigma_{Z_i}$$

Both the FIR and the variance corrected EWMA have the same alarm limit as the straight EWMA for late observations. However the limits will converge faster to the constant limit for the last mentioned method than for the FIR method for all values of λ as can be proved by direct evaluation of the difference between the limits. See Figure 5 where the three variants (with the same λ and L as used for the variance corrected method in the other figures) are compared. In this figure the parameters (λ=0.283 and L=2.858) are not chosen to give the same ARL but to give the alarm limit the same asymptotic value and to give the FIR and the variance corrected variants the same limit at time t=1.

Also other variants of EWMA have been proposed, e.g. for multivariate problems (Lowry et.al. 1992). In the present study the characteristics of the straight and the variance corrected EWMA as described above are studied in detail. The parameter values are chosen to give the same average run lengths (see below) as the CUSUM both when there is no shift and when there is a shift to $\mu^1 = 1$, ARL^0=330 and ARL^1=9.7. The parameter values are for the straight EWMA L=2.385 and λ=.220 and for the variance corrected EWMA L=2.858 and λ=.283. Except for Figure 5 these parameter values are used in all figures and simulations.

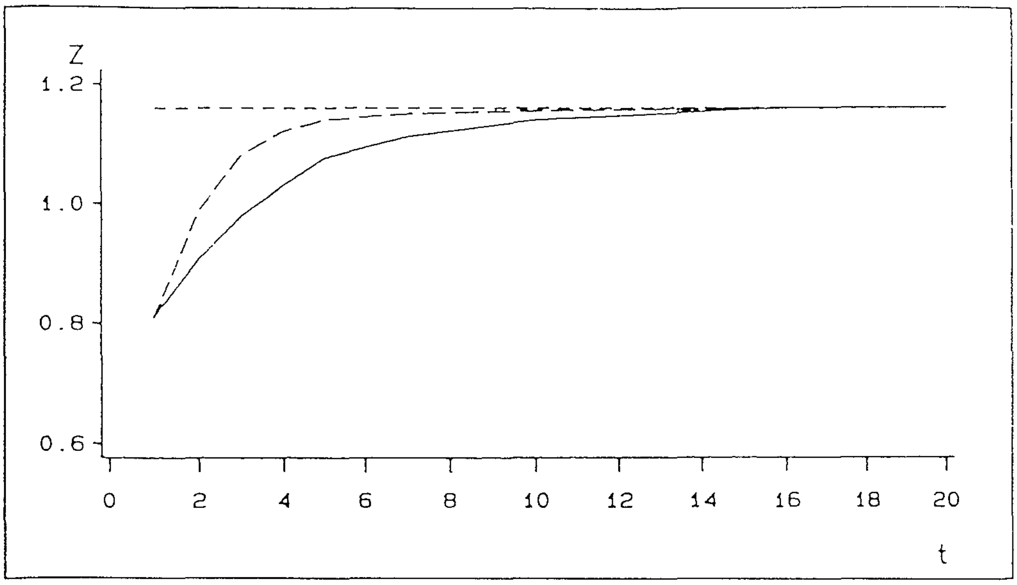

Figure 5. *Straight EWMA------, Variance corrected EWMA— — —, FIR——————*

The alarm region at the first time point is simple. As soon as the first observation exceeds a limit there is an alarm. At the second time point the alarm region is more complicated. The combinations of observations at the first and second time point which would result in an alarm not later than at the second time point are illustrated for CUSUM, straight EWMA and variance corrected EWMA in Figure 6.

The alarm region for the first three steps is illustrated for CUSUM and the variance corrected EWMA in Figure 7. In this three-dimensional figure, the alarm region for the Shewhart method is given to add reference lines in the figure. The implications of Figures 6 and 7 are further discussed in Sections 3 and 4.

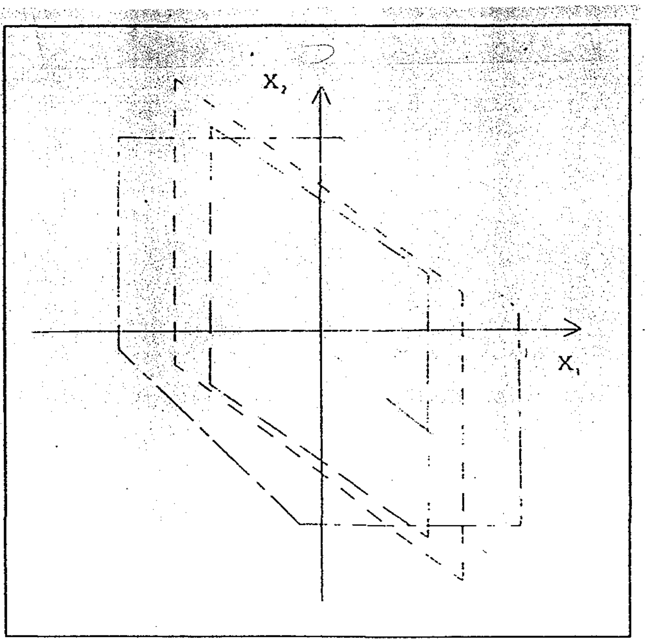

Figure 6. Detailed comparison between CUSUM and EWMA for the first two observations. The parameters in this and the following figures are the same as in Figure 2 - 4. Limits for alarm not later than at the second observation.

CUSUM —— - ——

Straight EWMA- - - - - -

Variance corrected EWMA— — — —

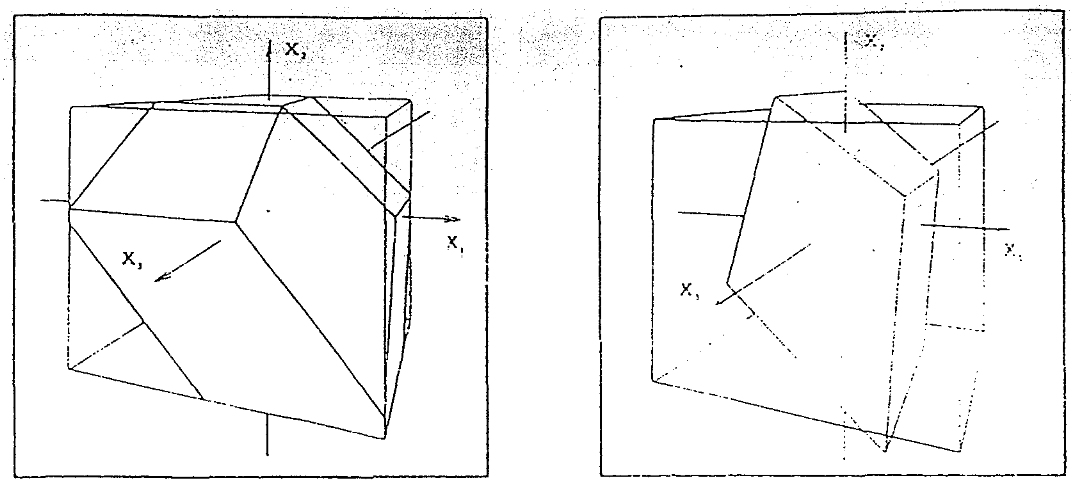

Figure 7.Limits for alarm not later than at the third observation. For reference the cube that is the limit for the Shewhart method with alarm limit 5.22 for each time point is included.
a.CUSUM b.Variance corrected EWMA

2. MEASURES OF THE PERFORMANCE

2.1 *RUN LENGTH DISTRIBUTION*

The run-length distributions for all interesting cases (also those where the change appears after the start of the surveillance) contains the information necessary for an evaluation of a method or a comparison between some methods. The actual comparison is usually based on some of the run-length distributions characteristics, mostly the average run length, but also the median or some other percentile could be considered. Several authors e.g. Zacks (1980), Crowder (1987) and Yashchin (1989) have pointed out that only one summarizing measure of the distribution is not enough. Run-length distributions are usually skew, especially those connected to the alternative hypotheses (see Figures 8 - 11).

2.2 *ARL*

A measure which is often used in quality control is the average run length (ARL) until an alarm e.g. Wetherill and Brown (1990). It was suggested already by Page (1954). The average run length under the hypothesis of a stable process, ARL^0, is the average number of runs before an alarm when there is no change in the system under surveillance. The average run length under the alternative hypothesis, ARL^1, is the mean number of decisions that must be taken to detect a true level change that occurred at the same time as the inspection started.

Values of the ARL are much used information for the design of control charts for specific applications. Roberts (1966) has given very useful diagrams of the ARL. Later several authors e.g. Saccucci and Lucas (1990), Champ and Rigdon (1991), Champ et.al. (1991), Yashchin (1992) and Yashchin (1993) have studied the ARL of specific methods and models. The distribution of the "run length" is markedly skew at the out-of-control case. The skewness differs between methods. The ARL will thus not give full information. This has been pointed out by e.g. Woodall (1983).

Since both the EWMA and the CUSUM methods have two parameters they can be constructed to give the same ARL both for the null- and for an alternative situation (here $\mu^1=1$). By the choice of design parameters ARL^0 is set to 330 and ARL^1 to 9.7 for the methods compared below. Here the remaining differences are of main interest.

Because of a complicated time dependence, and the dependence of the incidence of the change to be detected, other measures (Frisén 1986, 1992) than the average run length should be considered in the evaluation of different methods. Beckman et al. (1990) advocate similar measures as those in Sections 2.4 and 2.5 for the case of flood warning systems.

2.3 *THE PROBABILITY OF FALSE ALARM*

The distribution when the process is under control is described by a measure α_t which corresponds to the probability of erroneous rejection of the null hypothesis, the level of significance, but is a function of the time t. α_t is the probability of an alarm no later than at t given that no change has occurred. It is also the cumulative distribution function of the run length when the process is in control. Computer programs for the calculation has been given by Gan (1991) for EWMA and by Gan (1993) for CUSUM.

$$\alpha_t = P(RL \leq t \mid \mu_t = \mu^0)$$

2.4 *THE PROBABILITY OF SUCCESSFUL DETECTION*

The distance between the change and the alarm, sometimes called "residual RL" (RRL) is of interest in many cases. The optimality conditions by Girshick and Rubin (1952) and Shiryaev (1963) are based on this distance. One characterization of the distribution of the RRL is the probability that the RRL is less than a certain constant d (the time limit for successful rescuing action). This measure, PSD(d), the probability of successful detection, is the probability to get an alarm within d time units after the change has occurred, conditioned that there was no alarm before the change. The PSD is a function of the time distance d, the time of the change t' and the size of the shift μ^1.

$$PSD(d, t', \mu^1) = P(RL < t' + d \mid RL \geq t')$$

2.5 *PREDICTIVE VALUE*

The predictive value of an alarm is the probability that a change has occurred given that there is an alarm. Here, the time point τ where the change occurs is regarded as a random variable. The incidence of a change, inc(t'), is the probability that the stochastic time τ of the change takes the value t', given that there has been no change before t'. In the following examples the incidence is assumed constant. That is, τ has a geometric distribution.

The predictive value, PV, depends on the incidence inc, the size of the shift μ^1 and the time t" of the alarm. It gives information on whether an alarm is a strong indication of a change or not.

$$PV(t'', inc, \mu^1) = P(\tau \leq t'' \mid RL = t'').$$

Sometimes a late alarm is regarded with some doubt (e.g. Johnson 1961). This might be for the same reason as a significant result at a very big sample size is considered less impressing than a significant result at a small sample size. However there is no analogy here unless you only consider cases where the change appears at the same time as the surveillance starts. The trust you should have in an alarm is measured by the predictive value.

3. RESULTS

The alarm regions up to the first two observations are given in Figure 6. Considering the first and second observation the CUSUM has an "acceptance region" which contains that of the straight EWMA-method, except the extreme situation with two observations on the boundary, one in each direction. This "worst case" was discussed by Yashchin (1987) and Lucas and Saccucci (1990). The differences in size of the areas illustrate the different alarm probabilities at the first time points. Notable is also the shape of the regions, determined by the choice of the weight parameter λ and the reference value k.

In Figure 7 above, the three-dimensional regions of alarm at any of the runs 1, 2 or 3 are given.

In Figures 8 and 9 below, the cumulative probabilities of false alarms illustrate the differences (in spite of equal ARL) between the methods. The probabilities are estimated by simulation of at least 100,000 replicates of each situation.

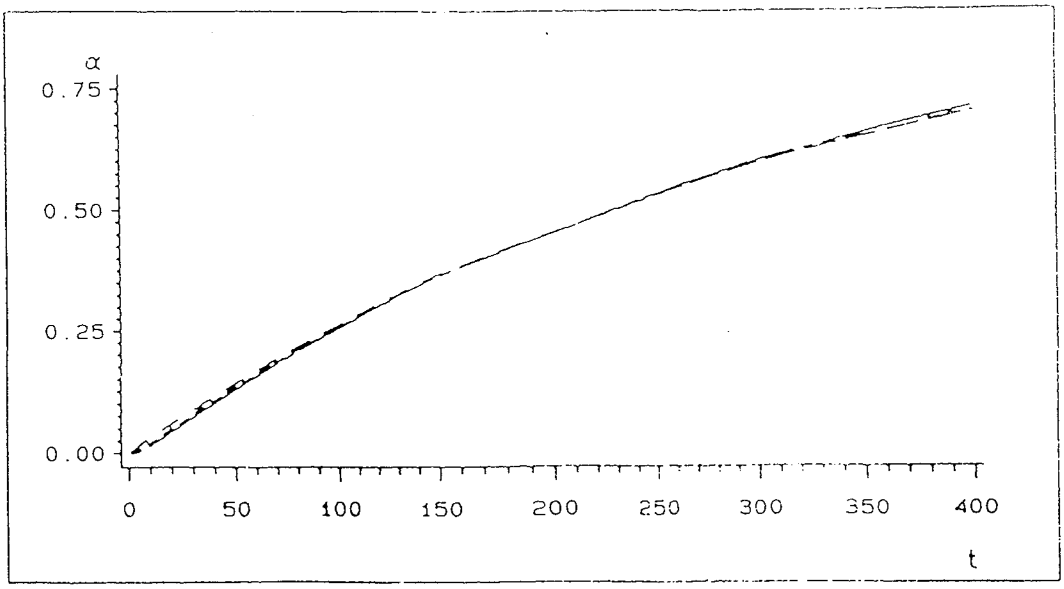

Figure 8. *The probability α of an alarm not later than at time t given that no change has occurred. The lines are linear connections between the values for each time point. Overview up to t=400.*

CUSUM —— - ——
Straight EWMA - - - - - -
Variance corrected EWMA — — — —

The variance corrected EWMA has a greater probability (about 1%) of false alarm in a great part of the beginning than the straight EWMA which in turn has a slightly greater false alarm rate at the start than the CUSUM.

The median is much smaller (about 230) than the ARL (330) which illustrates the skewness of the distribution. The probability to exceed the ARL is about 30%.

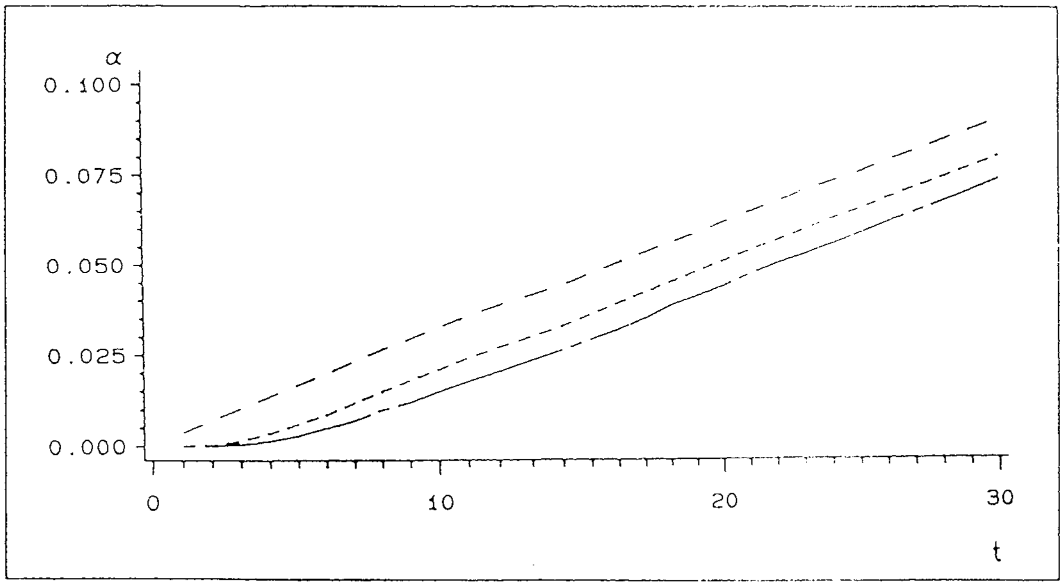

Figure 9. *As Figure 8, but detailed picture up to t= 30*

In Figures 10 and 11 the probability distributions of the residual run length are given for different times of shift. The results are based on at least 40,000 replicates. In Figure 10 it is given for the case where the shift occurs at the same time as the surveillance starts. ARL^1 is the expected value under this assumption. It is the same 9.7 for all methods. The median is however one unit less (equal to 7) for the variance corrected EWMA than for the CUSUM (equal to 8) which is an indication of the different shapes of the distributions, as is also seen in the figure. The EWMA has a higher probability for small run lengths. In Figure 11 the distributions are given for the case where the shift occurs at the 9th run. The positions of the curves are now interchanged. In the figure, the median is the RRL that corresponds to F_{RRL}=0.5. Now, the CUSUM has the least median.

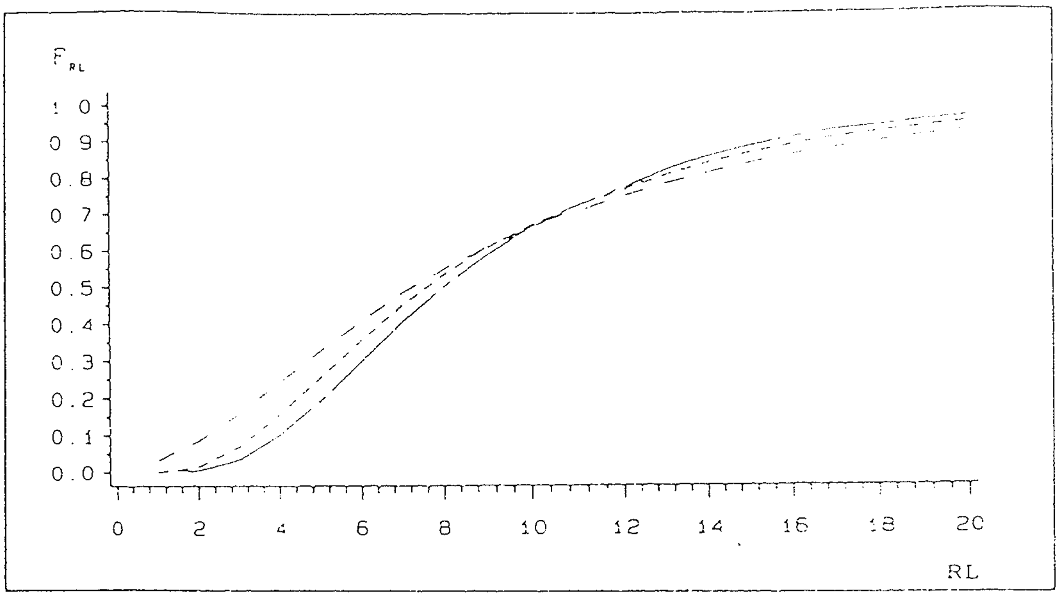

Figure 10. *The distribution of run lengths after a shift to $\mu'=1$ at different times t'. The run length distribution F_{RL} when $t'=1$.*

CUSUM	—— - ——
Straight EWMA	- - - - - -
Variance corrected EWMA	— — — —

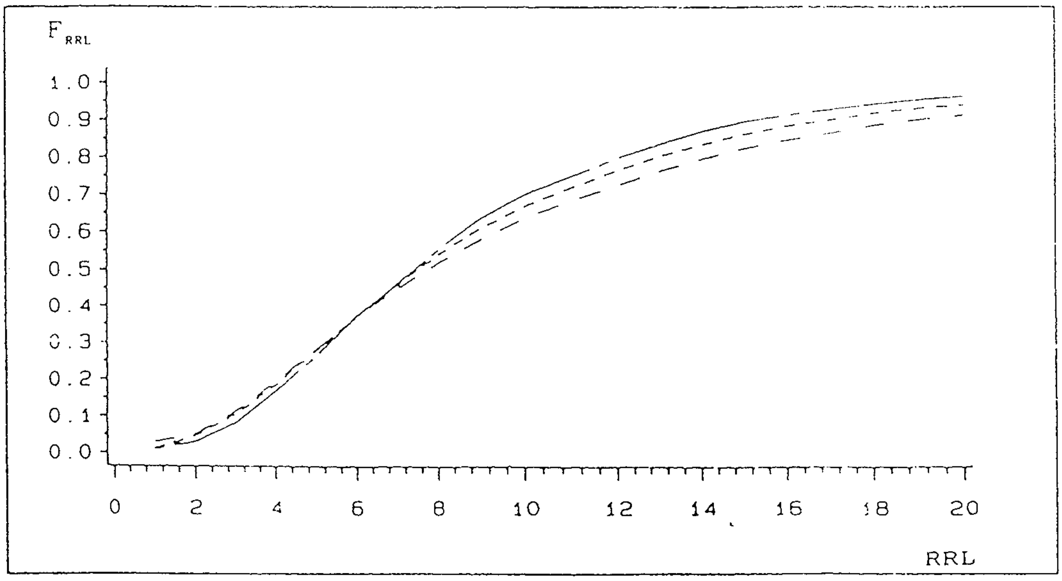

Figure 11. *As Figure 10, but the residual run length distribution F_{RRL} when $t'=9$.*

In Figure 12 it is demonstrated that the probability of successful detection within d=1 unit, that is immediately after the shift, is best for the variance corrected EWMA and better for the straight EWMA than for the CUSUM. The differences are most pronounced if the shift occurs soon after the surveillance has started (small t'). This is a case where fast initial response is desired and the FIR variant of EWMA by Lucas and Saccucci (1990) would be relevant. Each point in Figures 12 an 13 is based on at least 40,000 replicates.In Figure 13 it is demonstrated that the differences are in the opposite direction for detection within d=10 units. Then the differences are least pronounced soon after the start of the surveillance.

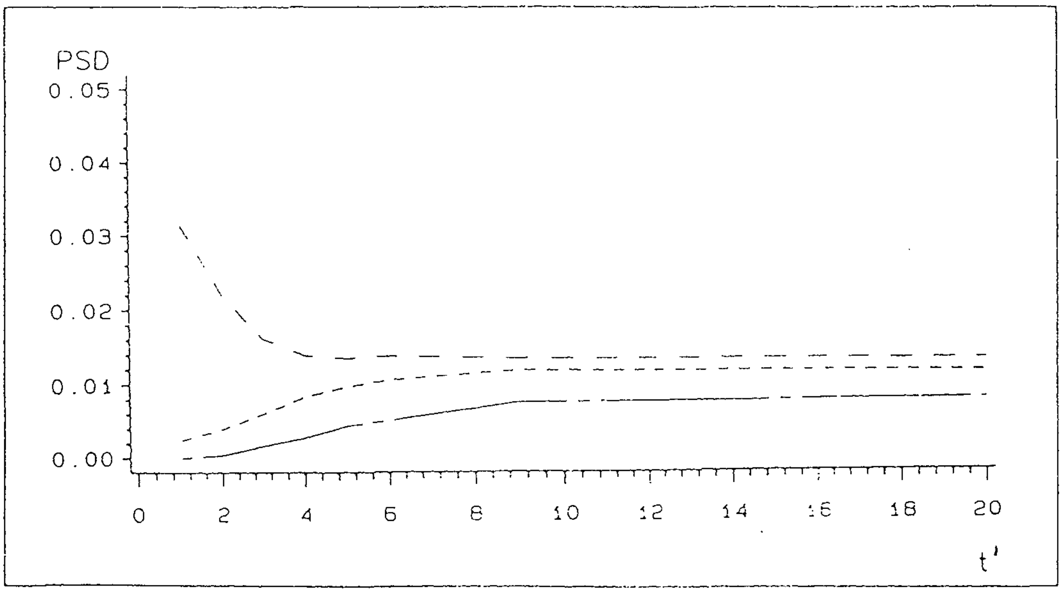

Figure 12. *Probability of successful detection, PSD. The probability of an alarm within d time units after the time t' of a shift to $\mu^1=1$, given that there was no alarm before t'. d=1.*

CUSUM —— - ——

Straight EWMA - - - - - -

Variance corrected EWMA — — — —

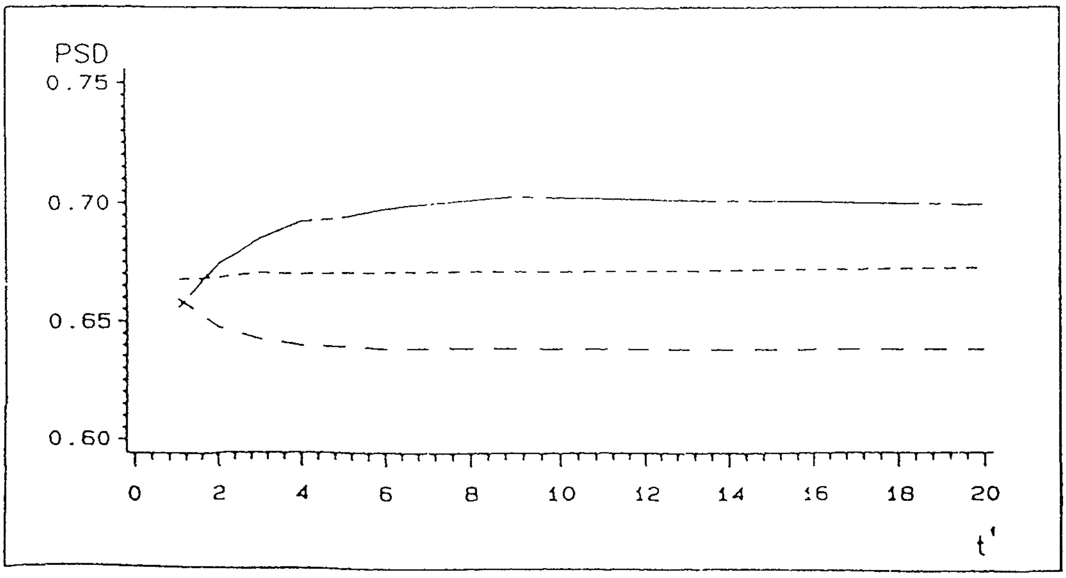

Figure 13. *As Figure 12, but d=10.*

In Figures 14 and 15 it is seen that at early time points the predicted value is low and varying for the EWMA methods (specially the variance corrected one). This implies that early alarms for the EWMA are very hard to interpret.

Each point in the figures is calculated as a function of the probabilities of false alarms and motivated alarms. The probabilities of false alarms are estimated by simulations of at least 100,000 replicates while the estimates of the probabilities of motivated alarms are based on at least 40,000 replicates. For t''=1 the probabilities are calculated exactly and for t''= 2, 3 and 4 the probabilities of motivated alarms are based on 1,000,000 replicates. The fact that the curve (for small values of t'') in Figure 15 is not a smooth one is thus not due to uncertainty in the simulations. In fact, the predicted value is not always an increasing function of t'' (Frisén (1992)).

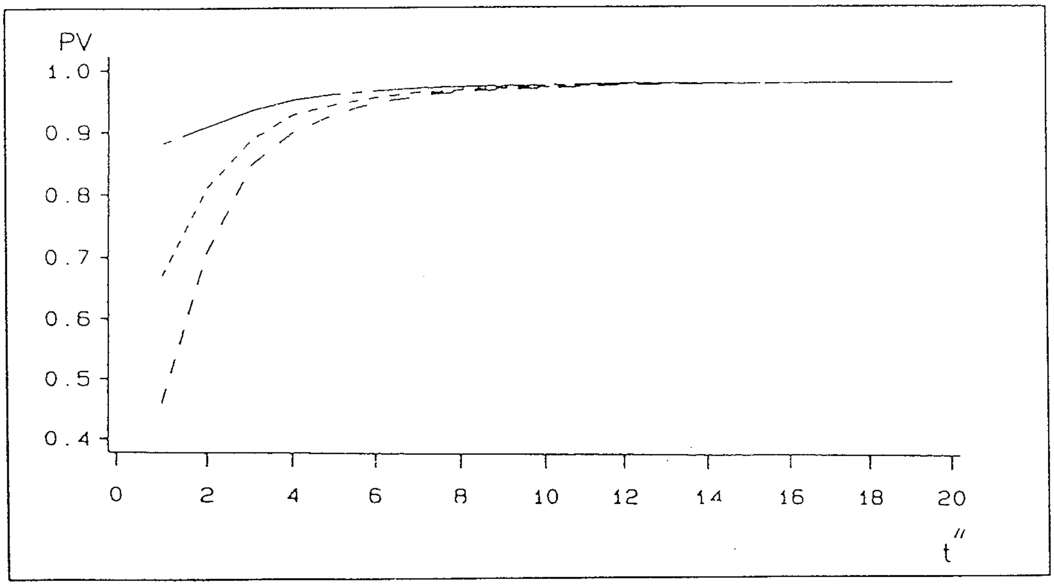

Figure 14. *Predicted value of an alarm, PV. The incidence of a shift to $\mu^1=1$ is 0.1.*

CUSUM	—— - ——
Straight EWMA	- - - - - -
Variance corrected EWMA	— — — —

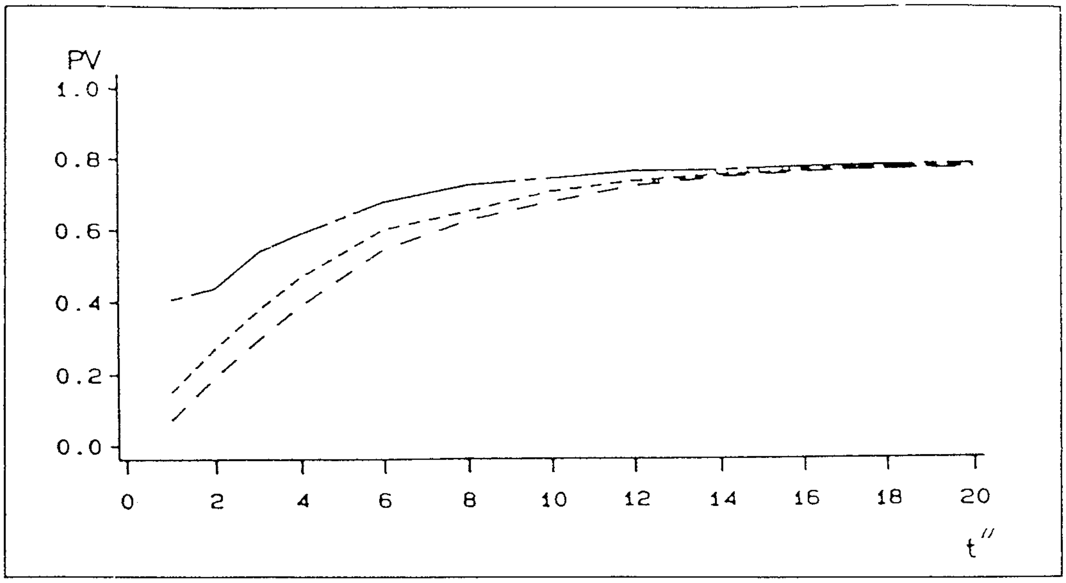

Figure 15. *As Figure 14 but the incidence of a shift to $\mu'=1$ is 0.01*

In Figures 14 and 15 it is seen that the predicted value is low and

4. DISCUSSION

As was also commented in the results, Figures 6 and 7 illustrate a difference in shape of the alarm region between the EWMA and the CUSUM which is general and which explains why the EWMA has bad "worst possible" properties (Yashchin 1987) while the CUSUM has minimax optimality (Moustakides 1986).

In Figures 6 and 7 interesting differences in symmetry are also illustrated. The alarm area is symmetrical for the CUSUM but not for the EWMA methods. That is for the probability of an alarm not later than at t all observations up to X_t have the same weight for the CUSUM. For the EWMA methods the older ones have more weight. However for the probability of an alarm at time t the last observations have the greatest weight both for CUSUM and EWMA methods.

In Figures 10 and 11 it is also demonstrated that for changes which occurred at the same time as the surveillance started the probability of a detection within a short time (shorter than 10) is better for the examined EWMA methods than for the CUSUM, while the opposite is true for times longer than 10. If the shift occurs some time after the start the short time is less than 10. In most studies only the case of a shift at the same time as the start of the surveillance is studied. As was seen above CUSUM compares more favourable with EWMA in other cases.

As is illustrated by the relative size of the rejection areas in Figures 6 and 7, and more generally seen by the formulas for the methods, the examined EWMA methods have a higher probability than the CUSUM for alarms shortly after the surveillance has started - both false and motivated ones. This does not mean that the probability is higher shortly after the shift has appeared, if the shift occurs later, as is seen in Figures 10-13.

One relation between the false and motivated alarms is given by the predictive value. In the simulations the predicted value is never better for the EWMA than for the CUSUM. In Figures 14 and 15 it is seen that the low and variable predicted value for the EWMA methods (specially the variance corrected one) at early time-points makes the early alarms for the EWMA methods very hard to interpret. This may make the variance corrected EWMA worthless shortly after the start. In the beginning when the predicted value of an alarm is very low and varying no alarm could be trusted. In the example with $ARL_1=9.7$ the alarms by the EWMA before the 9th run have such a low predicted value that for most applications they must be disregarded. Thus the benefit of a higher probability of an alarm in the beginning cannot be taken advantage of.

The general conclusion from the comparisons is that there might be important differences in characteristics in spite of equal ARL^0 and ARL^1. Even though only one set of parameters were examined for each method this is enough to demonstrate that differences exist.

In this paper only constant incidences are considered and the above discussion is relevant for this case. However, in some applications a higher incidence at the start of the surveillance might be relevant. The properties of the EWMA methods (especially the FIR variant) will then be more favourable. Only an approximately constant predictive value makes the method easily usable since only then it is possible to have the same kind of action independently of how far from the start the alarm is.

ACKNOWLEDGEMENT

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CONSTANT PREDICTIVE VALUE OF AN ALARM

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SUMMARY

One main purpose of statistical surveillance is to detect a change in a process, often expressed as a shift from one level to another. When a sequence of decisions is made, measures, like the number of decisions that have to be taken before an alarm are of interest. In many situations a shift might occur any time after the surveillance was initiated.

Prior knowledge of the probability of a change, the incidence, can become crucial when a method is selected and the parameter values of the method are set. The predictive value of an alarm is a measure of performance that takes this information into consideration and is an important tool for evaluating methods.

Mostly an alarm is useful only if its predictive value is large. The predictive value of an alarm is the probability that a change has occurred given an alarm. In this paper it is demonstrated that the incidence in the first point has to be relatively high, or the alarm limits very wide, in order to achieve a predictive value greater than, say 0.5.

The interpretation of an alarm is difficult to make if the predictive value of an alarm varies with time. For the ordinary Shewhart method and a selection of Moving Average Methods it is demonstrated how the predictive value increases with time if the incidence is constant. The incidence which would give the methods a constant predictive value are determined. The methods are thus demonstrated to give easily interpreted alarms only if the values of the incidence are strongly decreasing with time.

Since in most applications a constant incidence is assumed a modification of the ordinary Shewhart method is suggested. With this modification it is possible to obtain a constant predictive value in the whole range of observations or in some interesting interval.

KEY WORDS: Predictive Value; Shewhart; Moving Average.

CONTENTS

1 INTRODUCTION

This paper deals with the situation where the number of observations is successively increasing and successive decisions are required. The goal is to detect an important change in an underlying process as soon as possible after the change has occurred. The time point when the change occurs is here regarded as a random variable.

The predictive value is a measure of how strong an indication of a critical event an alarm is. A constant predictive value is desired if the same action is supposed to be taken whether the alarm occurs late or early.

In Section 2 the situation is formally described with the notation introduced by Frisén and de Maré (1991)[1]. In Section 3 some measures of performance are discussed and in Section 4 some general aspects of the predictive value are given.

In the following section a selection of statistical standard methods are studied. The conditions necessary for a constant level of the predictive value of an alarm are examined. The methods used represent different ways to invoke the history of the process in the test procedure.

Finally in Section 6 some concluding remarks are made about how the methods discussed in this paper behave in some different situations and also the possibilities of modifications are further discussed.

2 SPECIFICATIONS.

The random process that determines the state of a system is denoted $\mu=\{\mu(u):u\in T\}$. In the following examples there is a shift in the mean value of Gassuian random variables. The shift occurs as a sudden jump from an acceptable value μ^0 to an unacceptable value μ^1 ($\mu^1=1$ in examples). $\mu(u)=\mu^0$ for $u=1,\ldots,\tau\text{-}1$ and $\mu(u)=\mu^1$ for $u=\tau,\tau+1,\ldots$. Both μ^0 and μ^1 are assumed to be known values and τ is a generalized random variable with density $P(\tau=k)=\pi_k$ ($k=1,2,\ldots$) and $\Sigma\pi_k=1\text{-}\pi_\infty$. The incidence of a shift is $p_k=P(\tau=k|\tau\geq k)$.

At decision time point s the aim is to discriminate between the critical event C(s) and some other set of events, in this case its complement, D(s). C(s) is a set of realizations of the μ-process. The critical event under consideration is $C(s)=\{\tau\leq s\}=\{\mu(s)=\mu^1\}$, the event that a shift occurs at s or earlier with the complement $D(s)=\{\tau>s\}=\{\mu(s)=\mu^0\}$. All changes before s are of interest, and hypotheses involved are such that they change successively.

The test procedures are based on observation $X_s=\{X(u):u\in T,u\leq s\}$ and in the following it is assumed that $X(1)\text{-}\mu(1),X(2)\text{-}\mu(2),\ldots$ are independent Gaussian random variables with expected value zero and the same and known variance ($\sigma^2=1$ in examples). A(s) is the alarm set given by the method under consideration. It is a set of events with the property that when X_s belongs to A(s) it is an indication that C(s) occurs and a hypothesis stating a stable system is rejected.

The surveillance is active in the sense that the procedure is stopped if A(1) occurs. Otherwise the complement $A^c(1)$ occurs, the procedure continues for $s=2,3,\ldots$ as long as $A^c(s\text{-}1)$ occurs.

3 MEASURES OF PERFORMANCE

In most application areas averages from the run length distributions, the time to an alarm, are used both for evaluating methods and comparisons between competing methods. ARL^0, the average run length under the hypothesis of a stable process, is defined as the average number of decisions taken until an alarm occurs when there is no change in the process under surveillance. The average run length under the alternative hypothesis, ARL^1 is the average number of decisions that must be taken to detect a true level change that occurred at the same time as the inspection started.

In this paper the ARL is used as the basis for a comparison between some test procedures. In the following examples all methods are assigned the same ARL^0. The average run length profiles for the test procedures are presented in figures. This measure has the obvious disadvantage of being sensitive to skew run length distributions. And furthermore when this measure is used the test procedure is restricted to taking only a single alternative hypothesis into consideration, which in most cases of continual surveillance is an unrealistic simplification.

The false alarm probability, conditioned on no alarm before s, for each time s is defined as

$$\alpha(s)=P(A(s)\,|\,D(s),A^C_{s-1}),$$

where

$$A^C_{s-1}=A^C(1)\cap\ldots\cap A^C(s-1)$$

The cumulative false alarm probability is here defined as

$$\alpha_s = P(A_s | D_s) =$$

$$1-P(A^C(1) | D_s, A_0^C) \cdots P(A^C(s) | D_s, A_{s-1}^C)$$

where

$$D_s = D(1) \cap \ldots \cap D(s),$$

which in the present case is D(s). Also the probability of a false alarm at s is used in this paper and it is written as

$$\alpha^{\cdot}(s) = \alpha_s - \alpha_{s-1}$$

For each method discussed, α_s is shown in a figure.

The probability to neglect to stop the process when the critical event occurs is defined as

$$\beta(1)=P(A^c(1) | C(1)), \quad \beta(s)=P(A^c(s) | C(s), A^c_{s-1}) \quad (s \geq 2)$$

The predictive value of an alarm is a measure where the time point of the change is considered as a random variable, this measure will be defined in the next section.

4 GENERAL CHARACTERISTICS OF THE PREDICTIVE VALUE

The measure of performance that this paper is focused on is the predictive value of an alarm. The predictive value is the relative frequency of motivated alarms among all alarms at s,

$$PV(s)=P(C(s)\mid A(s),A^{C}_{s-1})=\frac{PMA(s)}{PMA(s)+PFA(s)}$$

where the unconditional probability of a false alarm at t^A is

$$PFA(t^A)=[\prod_{t=1}^{t^A}(1-p_t)]\ \alpha^*(t^A)$$

where

$$\alpha^*(t^A)\ =\alpha_{t^A}-\alpha_{t^A-1}$$

If the incidence of a shift is constant, $p_t=p$, PFA is reduced to

$$PFA(t^A)\ =\ (1-p)^{t^A}\alpha^*(t^A).$$

The probability of a motivated alarm at t^A is defined as

$$PMA(t^A)\ =\ \sum_{t^C=1}^{t^A}[\prod_{i=1}^{t^C}(1-p_{i-1})]p_{t^C}P(RL=t^A\mid \tau=t^C)\ ,$$

where $p_0=0$. With a constant incidence of a shift, PMA becomes

$$PMA(t^A)=\sum_{t^C=1}^{t^A}(1-p)^{t^C-1}p\ P(RL=t^A\mid \tau=t^C)$$

Contrary to passive surveillance, with active surveillance the predictive value typically has an asymptote below one, Frisén (1994)[2], but with a constant incidence the function is not necessarily monotonously increasing for all methods. If it is possible to obtain a constant level of the predictive value of an alarm, this would be a desirable property of a method in those applications where it is considered important to take the same action whenever an alarm occurs.

A constant level of the predicted value implies

$$\frac{PMA(1)}{PFA(1)} = \frac{PMA(t)}{PFA(t)}, \quad t > 1.$$

And furthermore

$$\frac{PV(t)}{1-PV(t)} = \frac{PMA(t)}{PFA(t)} < 1.0 \quad \Leftrightarrow \quad PV(t) < 0.5$$

Also,

$$\frac{p_1}{1-p_1} < \frac{P(A(1)|D(1))}{P(A(1)|C(1))} \qquad (i)$$

$$\Leftrightarrow \quad PV(1) < 0.5,$$

since

$$\frac{PV(1)}{1-PV(1)} = \frac{PMA(1)}{PFA(1)} = \frac{p_1 P(A(1)|C(1))}{(1-p_1)P(A(1)|D(1))} < 1.0,$$

With the requirement of a constant level of the predictive value the condition (i) implies PV(t)<0.5.

5 THE PREDICTIVE VALUE OF SOME METHODS.

In this section a selection of standard methods representing different approaches in statistical surveillance are briefly described. The observation X_s, together with some rules for rejecting the hypothesis of a stable process constitutes a method. Some characteristic differences between methods are pointed out, particulary in regard to different structures in shift probabilities. The methods to be mentioned represent different ways to include the history of the process in the test procedure.

The two main approaches discussed in this paper are rules based on the last observation, $X_s=\{X(t):t\in T,t=s\}=X(s)$, that is no consideration is taken to earlier observations. This approach will henceforth be referred to as the Shewhart method.

The second approach is rules based on the history of the process, $X_s=\{X(t):t\in T,t\leq s\}$. Some methods based on moving averages will be discussed.

For methods where it is possible to obtain a constant, or almost constant level of the predictive value, the corresponding shift probabilities are illustrated in figures.

5.1 SHEWHART.

The Shewhart method is widely used in different application areas of statistical surveillance. For a situation formulated by Frisén and de Maré(1992)[1] the method is optimal in an extended Neyman-Pearson sense. This is the situation when the only interesting time point is the present, s=t. It is assumed that no previous observations include any valuable information about the state of the underlying process.

The performance of the Shewhart method in different situations often stands as a standard for comparisons when other methods, maybe competing ones, are evaluated. There are probably several reasons for this. One obvious reason is the fact that the method is a standard norm since it is used in a wide range of areas, i.e. quality control, where the Shewhart method often has the alarm limits 3 σ away from the target value, μ^0, Bergman and Klevsjö (1994)[3], which in the literature concerning quality control is referred to as the natural variation of the process. The illustrations in this paper are based on the Shewhart method with alarm limits closer to μ^0 and ARL^0 equals 15. For comparison with other methods, the corresponding sequence from the Shewhart method is added in each figure.

Another reason is that the method has a simple structure. This is achieved through the restrictive assumptions about the underlying process. If the successive observations are independent, successive values of the statistic used also becomes independent and by that measures of performance can be reached with straightforward calculations.

Also, the Shewhart method turns out to be a special case of many other methods, i.e. EWMA, MA and Hinkley's method. In more complicated test procedures, like Hinkley's method, the Shewhart test is usually one component in that procedure. Another example of the latter is Shewhart control charts with additional warning limits, or/and some other decision rules added to it. Notable is also that at t=1 almost any method has the same properties as the corresponding Shewhart test. Methods excluded are for example, tests based on window techniques with a window size greater than one.

Often the ordinary theory of testing hypotheses is applied to the problem, every time t the hypothesis is challenged by the alternatives it is done with identical probabilities of rejection for each test.

In the one-sided case the method only has one parameter G, the alarm limit. This makes the Shewhart method easy to work with and the interpretation of the parameter setting and the outcome at t becomes straightforward.

With Normally distributed observations it is possible to find distributions of the incidence that gives a constant level of the predicted value of an alarm. To obtain a constant and preferable high level of the predictive value, the probability that a shift in the underlying process has occurred at, or before the first observation, has to be rather high compared to the probability for the following observations. This will be further discussed in Section 5.1.2.

The method is defined by the quantity X(t), which might be the observation itself or some other quantity derived from a set of observations obtained in the choosen interval. The usual choice of quantity is an average calculated over equally spaced intervals and the method is also referred to as a Xbar Chart.

Constant Predictive Value of an Alarm

The hypothesis stating a stable process is rejected when, for the first time X(t) exceeds the alarm limit G. Usually G is expressed as

$$L\sigma_{X(t)},$$

where L is a chosen constant, and

$$\sigma_{X(t)},$$

the standard deviation of X(t). With independent observations and a common standard deviation the unconditional probability of a false alarm is $\alpha(t)=\alpha$, and the expected value in the RL distribution is α^{-1} when the hypothesis of no change in the process is true. Considering the sequential procedure the probability of a false alarm at or before a certain time point t, Fig.1, is α_t, where

$$\alpha_t-\alpha_{t-1} = \alpha^*(t) = (1-\alpha)^{t-1}\alpha.$$

In Fig. 1 and the following figures the twosided case is considered where the alarm limits G_u and G_l are located on each side of, and at the same distance from, μ^0.

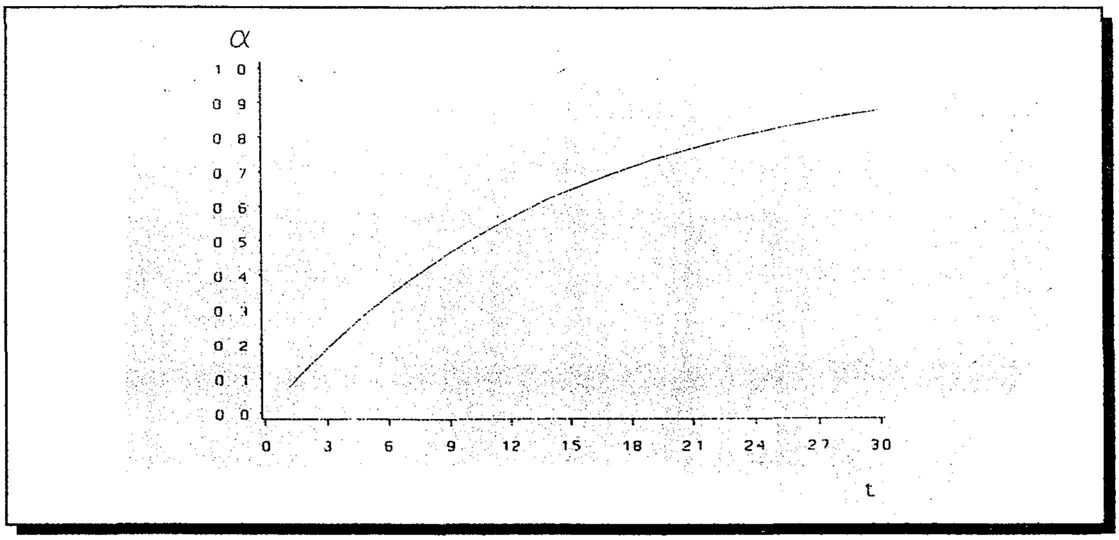


Figure 1. *The Shewhart method. The probability of a false alarm at time point t or earlier.*

In Fig. 1 the constant L is equals 1.83 and σ_X is unity.

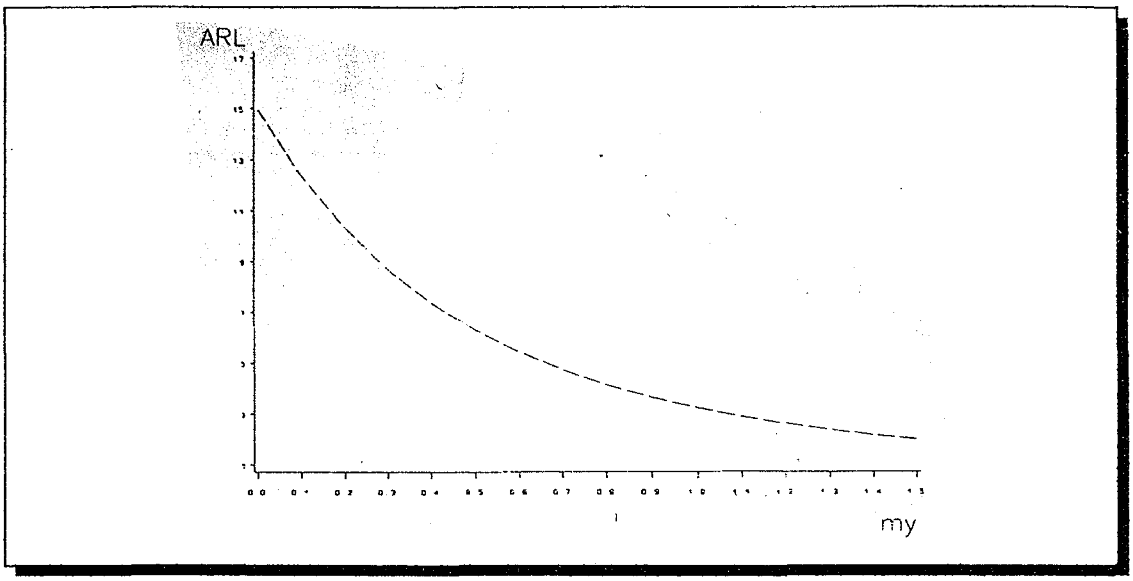


Figure 2 *The average run length profile for the Shewhart method, $ARL^0=15$.*

This gives a series, with an expected value of RL equal to 15, if the process is stable. All methods discussed in this paper have an average run length, ARL^0, equal to 15.

In Fig.2 the average run length profile is calculated with the usual assumption that the shift occurred at the same time as the inspection started, $t^C=1$. If the time point for the shift is described by a stochastic variable, the probability of a false alarm at t^A is

$$PFA(t^A)=[\prod_{t=1}^{t^A}(1-p_t)](1-\alpha)^{t^A-1}\alpha,$$

and with a constant incidence of a shift, $p_t=p$, the probability of a false alarm at t^A becomes,

$$PFA(t^A)=(1-p)^{t^A}(1-\alpha)^{t^A-1}\alpha.$$

Constant Predictive Value of an Alarm

In this model the probability of not stopping at t when there has been a shift is $\beta(t)=\beta$, and the probability of a motivated alarm becomes

$$PMA(t^A)=\sum_{t^C=1}^{t^A}\left[\prod_{i=1}^{t^C}(1-p_{i-1})\right]p_{t^C}P(RL=t^A|\tau=t^C)$$

where

$$P(RL=t^A|\tau=t^C)=(1-\alpha)^{t^C-1}\beta^{t^A-t^C}(1-\beta)$$

With the same incidence of a shift at each time the probability of a motivated alarm becomes

$$PMA(t^A) = \sum_{t^C=1}^{t^A}(1-p)^{t^C-1}p\,P(RL=t^A|\tau=t^C)$$

Usually the alarm limits G_u and G_l are located at the same distance from μ^0, and are also considered to be constants. If any assumptions are made about the incidence of a shift it is mostly stated that it is constant with time. In the following section some consequences of this approach will be discussed, particulary when requirements on the predictive value are present.

The Shewhart method always has its highest probability of first detection at the time point of the shift, which is seen in that

$$P(RL=t^C|\tau=t^C) \le P(RL=t^C+n|\tau=t^C), \quad n=1,2,...$$

$$\Leftrightarrow \quad (1-\beta) \le \beta^n(1-\beta).$$

Actually, given the same probability of a false alarm, α, no other method exists that has a higher detection probability in the present time point than the Shewhart method, Frisén and de Maré (1992)[1]. That is, if, according to the previous notation, s=t then the best choice of method is the Shewhart method.

5.1.1 CONSTANT INCIDENCE.

With the same incidence of a shift at each time t, and if $\alpha(t)=\alpha$, the predictive value can not be the same in the whole range of time points. This is true except for a test where the hypothesis is always rejected at $t=1$. This is seen in that,

$$PV(t) = PV(t+1) \quad \Rightarrow$$

$$PMA(t+1) = \frac{PFA(t+1)}{PFA(t)} PMA(t), \quad \{t=1,2,\ldots\},$$

$$p\beta^{t}(1-\beta)+(1-p)p(1-\alpha)\beta^{t-1}(1-\beta)+\ldots+(1-p)^{t}p(1-\alpha)^{t}(1-\beta) =$$

$$(1-p)(1-\alpha)[p\beta^{t-1}(1-\beta)+(1-p)p(1-\alpha)\beta^{t-2}(1-\beta)+\ldots+(1-p)^{t-1}p(1-\alpha)^{t-1}(1-\beta)],$$

$$\beta^{t}+(1-p)(1-\alpha)\beta^{t-1}+\ldots+(1-p)^{t}(1-\alpha)^{t} =$$

$$(1-p)(1-\alpha)[\beta^{t-1}+(1-p)(1-\alpha)\beta^{t-2}+\ldots+(1-p)^{t-1}(1-\alpha)^{t-1}] \quad \Leftrightarrow$$

$$\beta^{t} = 0 \quad \Rightarrow \quad \beta = 0.$$

Furthermore, with the same assumptions as above, the predictive value of an alarm is a monotonously increasing function in t. Suppose

$$PV(t) > PV(t+1),$$

$$\frac{PMA(t)}{PFA(t)} > \frac{PMA(t+1)}{PFA(t+1)} \quad \Rightarrow$$

$$\beta^{t}+(1-p)(1-\alpha)\beta^{t-1}+\ldots+(1-p)^{t}(1-\alpha)^{t} <$$

$$(1-p)(1-\alpha)[\beta^{t-1}+(1-p)(1-\alpha)\beta^{t-2}+\ldots+(1-p)^{t-1}(1-\alpha)^{t-1}] \quad \Rightarrow$$

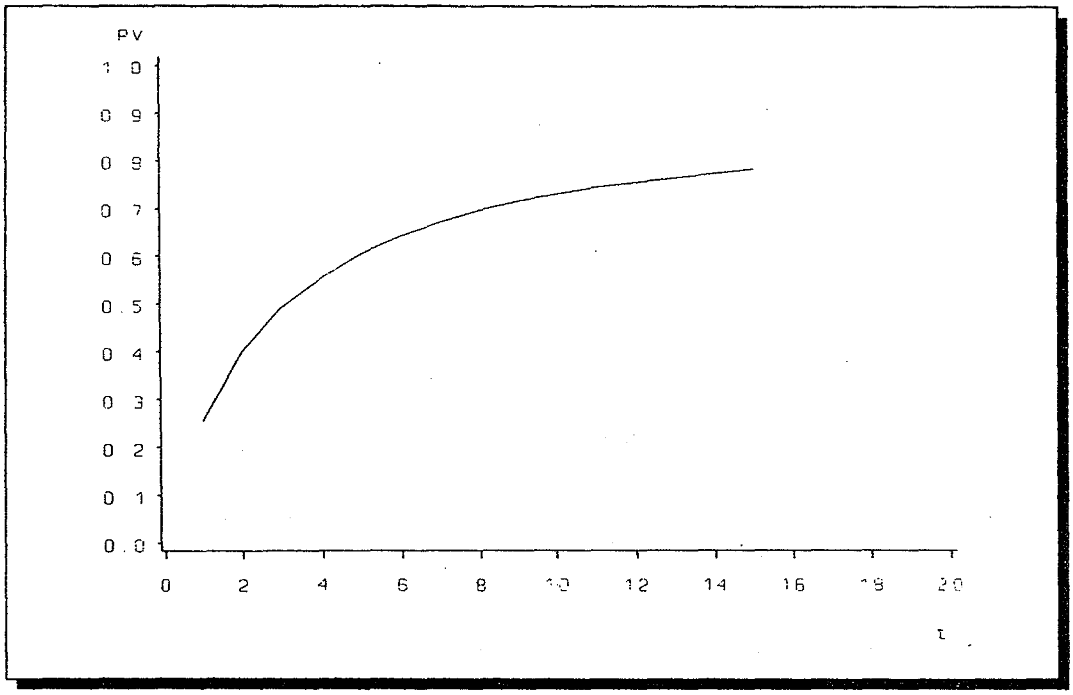
$$\beta^{t} < 0.$$

When t becomes large and the surveillance is active, the predictive value will increase to its limiting value, Frisén(1991)[4],

$$\lim_{t\to\infty} PV_t = \frac{p}{p+\alpha c},$$

where,

$$c = \frac{|(1-p)(1-\alpha)-1|}{(1-\alpha)(1-\beta)} + \frac{1}{1-\alpha}.$$



Figur 3 *Predictive value of an alarm in the case of the same probability of a shift at each time point. inc=0.1*

In Figure 3, p is set to 0.1. The low predictive value of an alarm for early observations is explained by the high false alarm probabilities compared to the probability of a shift. The probability of a false alarm, PFA, is a monotonously decreasing function in t. Suppose

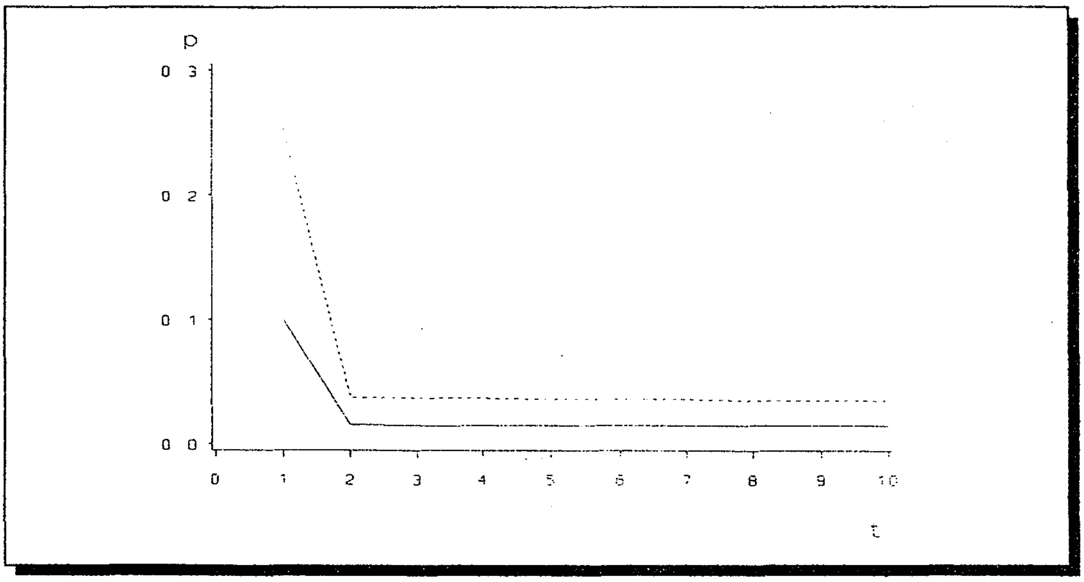
$$PFA(t) \leq PFA(t+1) \quad \Rightarrow$$

$$(1-p)^{t}(1-\alpha)^{t-1}\alpha \leq (1-p)^{t+1}(1-\alpha)^{t}\alpha$$

$$1-p \geq \frac{1}{1-\alpha} > 1.$$

5.1.2 VARYING INCIDENCE.

If, as usual, we have the Shewhart method with constant limits, that is $\alpha(t)=\alpha$, it is possible to obtain a constant level of the predictive value for specific series of decreasing incidence.



Figur 4 *The probability structure that gives a constant predictive value level. The dotted curve corresponds to PV=0.50*

In Figure 4, the lower curve, p_1 is set equal to 0.1 and p_t, $i>1$, are chosen in such a way that at a constant predictive value is obtained.

$$\frac{PMA(t)}{PFA(t)} = \frac{PMA(t+1)}{PFA(t+1)}$$

$$p_1\frac{\beta^t}{(1-\alpha)} =$$

$$(1-p_{t+1})[p_1\beta^{t-1}+(1-p_1)p_2(1-\alpha)\beta^{t-2}+\ldots+(1-p_1)\cdots(1-p)_{t-1})p_t(1-\alpha)^{t-1}]$$

$$-(1-p_1)[p_2\beta^{t-1}+(1-p_2)p_3(1-\alpha)\beta^{t-2}+\ldots+(1-p_2)\cdots(1-p_t)p_{t-1}(1-\alpha)^{t-1}]$$

The required property is satisfied for the first step if $1-\alpha>\beta$ and

$$p_2 = p_1 \frac{(1-\alpha-\beta)}{(1-\alpha)}$$

For greater values of t the formula is more complicated, but it is no problem to compute the values. In figure 4 p_1 is set equal to 0.1 and $\mu^1=1$, the lower curve. In this case the predictive value is just above 0.25 at t=1, which means that on average only one out of four alarms at t=1 will actually be a motivated one. The curve representing the higher shift probabilities in figure 4 corresponds to the situation where the predictive value of an alarm at t^A is 0.50. Both curves indicate that a constant level of the predictive value for the Shewhart method sometimes requires a rather high shift probability at the first time point compared to the second, and later observations.

5.1.3 CONSTANT INCIDENCE AND VARYING ALARM LIMITS.

By letting the alarm limits of the Shewhart method change over time the predictive value can be made constant.

First, consider the first and second observation from a Normal distribution. If $\alpha(1) < \alpha(2)$ this implies that

$$\frac{\alpha(1)}{1-\beta(1)} < \frac{\alpha(2)}{1-\beta(2)} .$$

With a constant incidence of a shift it is possible to obtain the same predictive value of an alarm for both points.

$$PV(2) = PV(1) \quad \Rightarrow$$

$$\frac{PMA(2)}{PFA(2)} = \frac{PMA(1)}{PFA(1)} \quad \Rightarrow$$

$$\frac{\alpha(2)}{1-\beta(2)} = \frac{\alpha(1)}{1-\beta(1)} \cdot k,$$

where

$$k = \frac{\beta(1)}{(1-p)(1-\alpha(1)} + 1 > 1 .$$

This means that, with normally distributed observations, the alarm limits for the second observation should be closer to μ^0 than for the first one if a test procedure with the same predictive value of an alarm is required.

Constant Predictive Value of an Alarm

It is possible to choose the limit g(t+1) in such a way that a constant level of the predictive value is obtained.

$$PV(t) = \frac{PMA(t)}{PMA(t)+PFA(t)} \quad \Rightarrow$$

$$PV(t) = (1+\frac{PFA(t)}{PMA(t)})^{-1} \quad \Rightarrow$$

$$PV(t) = \frac{1}{1+\frac{(1-p)^{t}\prod_{i=1}^{t-1}(1-\alpha(i))\alpha(t)}{\sum_{j=1}^{t}P(\tau=j)P(RL=t \mid \tau=j)}}$$

where the sum in the numerator can be written,

$$p(1-\beta(t))[\beta(1)\cdots\beta(t-1)+(1-p)(1-\alpha(1))\beta(2)\cdots\beta(t-1)+\ldots+$$
$$(1-p)^{t-2}(1-\alpha(1))\cdots(1-\alpha(t-2))\beta(t-1)+(1-p)^{t-1}(1-\alpha(t-1))].$$

This implies that

$$\frac{PV(t)}{1-PV(t)} = \frac{1-\beta(t)}{\alpha(t)}\cdot f_{t-1},$$

where f_{t-1} only depends on the characteristics of the surveillance before time point t.

$$\frac{1-\beta(t)}{\alpha(t)} = \frac{1-\Phi(g-\mu^{1})}{1-\Phi(g-\mu^{0})}$$

increase continuously when the limit, g(t), increase.

Choose g(t+1) in such a way that,

$$\frac{1-\beta(t+1)}{\alpha(t+1)} = \frac{f_{t-1}}{f_t} \cdot \frac{1-\beta(t)}{\alpha(t)} .$$

It can be shown that,

$$g(t) \geq g(t+1) \quad \Rightarrow$$

$$PV(t) \leq PV(t+1) \quad \Rightarrow$$

$$f_{t-1} < f_t .$$

A constant level of the predictive value of an alarm requires that g(t) > g(t+1).

For example, a one sided 3σ control chart (Xbar Chart) is used on Normal distributed observations, $\mu^0=0$ and $\sigma_X=1$, and the aim is to detect a shift of size σ. If the incidence is constant and equals 0.1, this test procedure will give a predictive value of an alarm at t=1 that is just above 0.65. To obtain the same predictive value of an alarm at time point two the alarm limit has to be moved to a 2.1σ distance from μ_0.

5.2 MOVING AVERAGES

The results in the following sections are mainly obtained through computer simulations.

Surveillance methods based on moving averages dates back to about forty years ago, and since then, the most investigated and discussed method in this family has been the ones with weights exponentially decreasing in time, Exponentially Weighted Moving Average. The EWMA methods are developments of the Moving Average, MA, method based on k observations, Roberts(1959)[6]. Some consequences of choosing these type of methods, or methods with similar properties, are discussed. In this paper the MA method based on k observations is modified to a Expanding Average, k=s, to illustrate some effects on the ARL and the predictive value properties of a method if the alarm probabilities in the initial face of the series are not constant.

5.2.1 EXPANDING AVERAGE.

Moving Windows are techniques to describe the influence of earlier observations in the process. The quantity are usually averages calculated from the last k observations at decision time point s. Observations obtained before s-k+1 are considered of no interest at all, while the last k observations usually are considered equally important. By using this techniques to modify methods improved predicted value properties can be achieved since this technique demands k-1 auxiliary points before the first test can be made. If the parameter k is one in this model the Shewhart method is obtained.

The Expanding Average is a special case of a Moving Average. If at decision time point s all available observations are considered equally important, k=s, a Moving Average with equal weights, k^{-1}, assigned to the last k observations, is obtained. This model is mainly discussed because of its remarkably good ARL properties and the general effect on the ARL measure if a skewed RL-distribution is present, Fig 5.

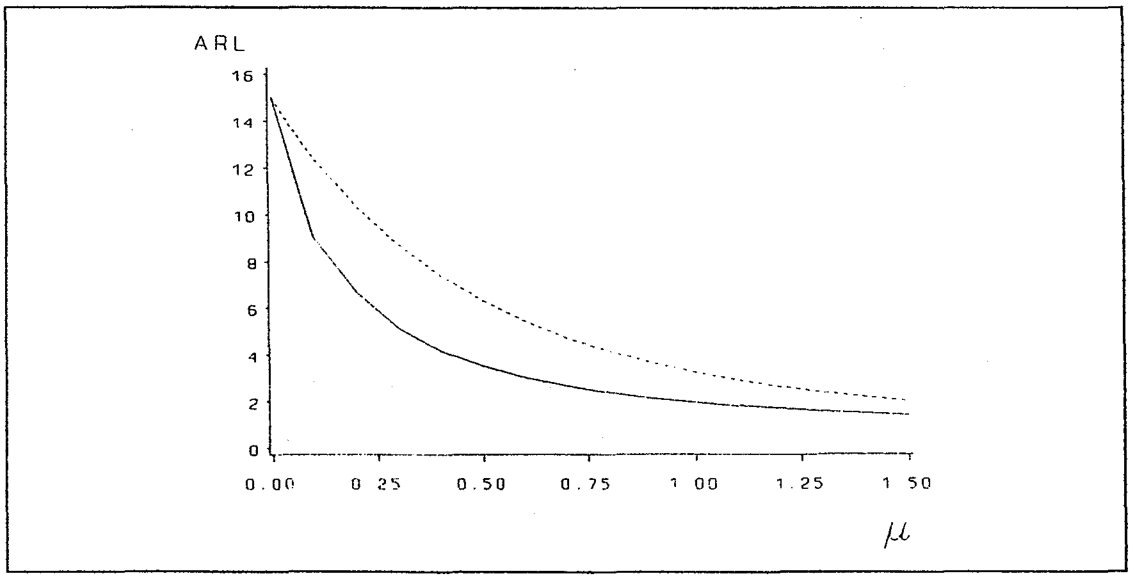


Figure 5. *The ARL profile of the Expanding Average and the Shewhart method (dotted line).*

For each new observation an average is calculated with the actual s as denominator,

$$M_s = \frac{\sum_{k=1}^{k=s} x_k}{s},$$

The quantity M_s converges to μ^0 as s becomes large, and the variance of M_s becomes small.

The process under surveillance is considered to be in balance as long as the outcome on M_s stays within the alarm limits

$$g_s = L\sigma_X \frac{1}{\sqrt{s}},$$

where σ_X is the same known standard deviation of the observation X_t.

In the Figures 5-8 L is 0.924 and $\sigma_X = 1$, this gives an average run length of 15 if $D(t) \equiv D$, the process is in control. In Fig. 6 the probability of a false alarm, α_t, is compared with the Shewhart method.

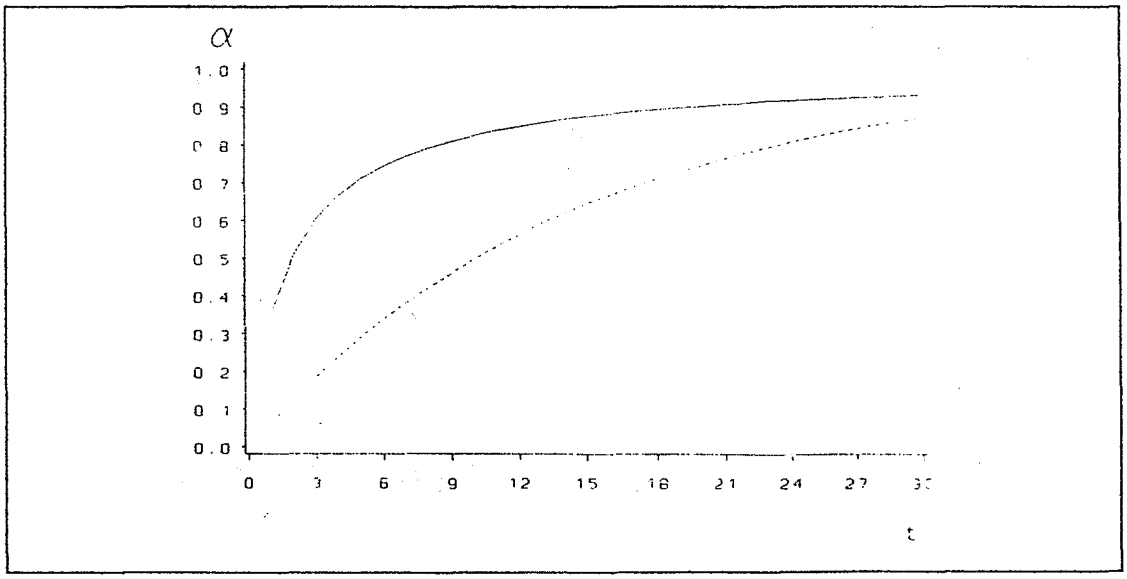


Figure 6. *The probability of a false alarm at or before t for the Expanding Average and the Shewhart method (dotted line).*

This method is characterised by a initially high alarm probability. With this choice of parameter values the false alarm probability at $t=1$ is 0.36. This makes the run length distribution markedly skew to generate an average of 15. Given a shift in the process, this method will most of the times generate an early alarm, but in a few trials it will have difficulties to detect that change. This gives a rather impressive ARL-profile for the method, but it is only suitable in situations where late alarms are not crucial. Given a shift at t^C, the method has its highest probability of detection at t^C at the beginning of the series, just like the Shewhart method. But if there is a late shift the method have its highest probability of detection at some time point later then t^C, and also, the probability of an immediate detection is decreasing with t. For the case illustrated (Fig.5-7) the probability of an immediate detection if $t^C=1$ is 0.55, while if $t^C=20$, the probability falls below 0.1.

In the following sections the predictive value properties of the Expanding Average will briefly be discussed.

5.2.1.1 CONSTANT INCIDENCE.

With this choice of L and a constant incidence of a shift it is not possible to obtain a constant level of the predictive value of an alarm. In the Normal distribution the probability of a motivated alarm, PMA, and the probability of a false alarm, PFA, are decreasing functions in t. The predictive value of an alarm is an increasing function in t which means that an application where this method is suitable has to be such that a low predictive value at the beginning of the sequence is acceptable, Fig. 7.

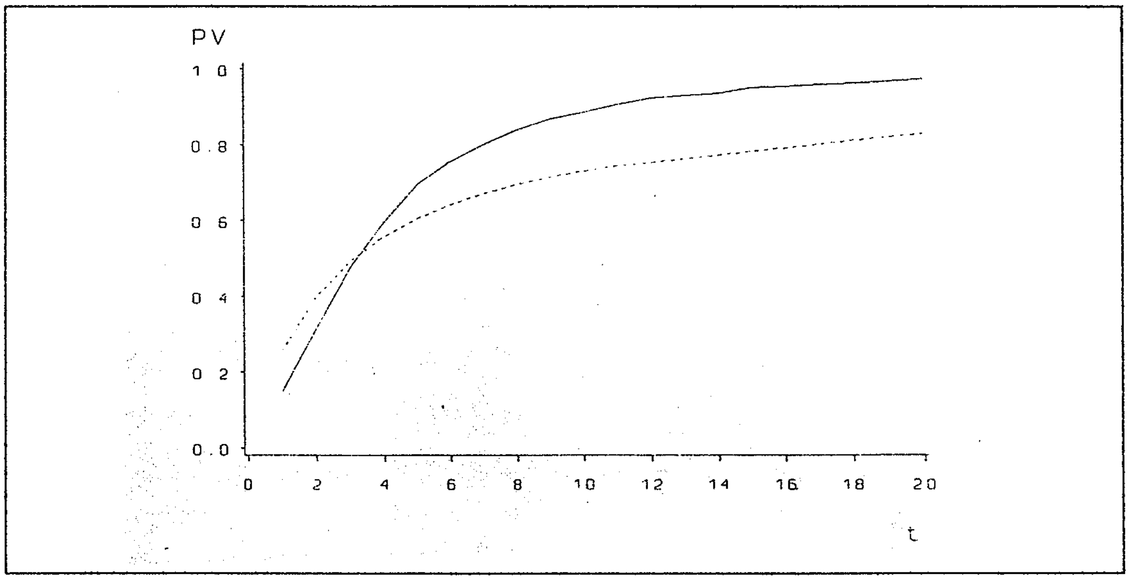


Figure 7. *Predictive value of an alarm for Expanded Average and the Shewhart method, the lower curve. The probability of a shift is 0.1 at each time point.*

5.2.1.2 VARYING INCIDENCE.

In a Normal process it is possible to obtain a constant predictive value of an alarm with this method. Though the incidence of a shift has to be rather high at the first time point relatively to the following points.

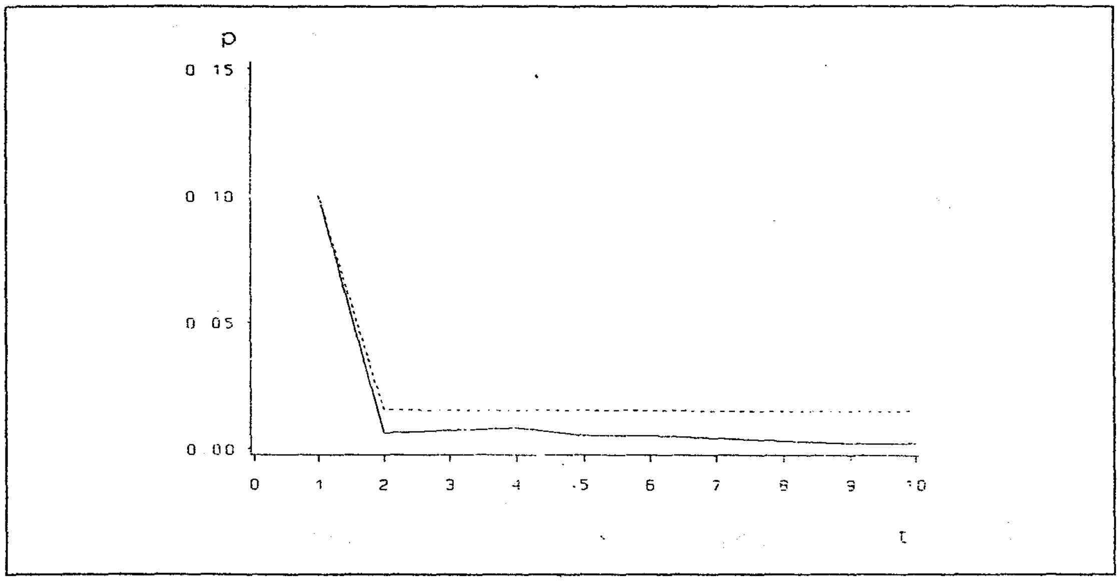


Figure 8. *The incidence of a shift that gives a constant predictive value of an alarm for the Expanding Average method and the Shewhart method (the lower curve).*

In the example the probability of a shift, the incidence, at t=1 is 0.1 and then decreases to give an overall predictive value of an alarm of 0.26 for the Shewhart method and 0.15 for the Expanding Average method. The 50 percent level of the predictive value of an alarm for the Expanding Average corresponds to p_1 equals 0.39.

5.3.2. EXPONENTIALLY WEIGHTED MOVING AVERAGE

The EWMA methods are a group of methods with a particular weight pattern that assigns the history of the underlying process weights in an infinite, geometrically decreasing sequence from the most recent back to the first observation.

If the parameter λ is equal to one in the EWMA model the Shewhart method is obtained, and if λ tends to zero the method degenerates towards a test with the present observation excluded. EWMA(g) methods where the alarm limits, g, are based on the standard deviations of the statistic usually (unimodally distributed statistic) have higher alarm probabilities than the standard version with straight alarm limits, G, when the limiting value of g is G. This generates alarm probabilities, and predictive value properties, closer to the Shewhart method.

If a Fast Initial Response feature, Lucas and Crosier(1990)[6], is added to the standard EWMA(G) the alarm probabilities in the initial points are increased in a way similar to the EWMA(g) methods.

$$Z_0 = \mu^0,$$

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t,\ t > 0.$$

The EWMA is sometimes referred to as a geometric moving average since it can be written as a moving average of the current and the past observations,

$$Z_t=\lambda\sum_{j=0}^{t-1}(1-\lambda)^j X_{t-j}+(1-\lambda)^t Z_0 \,.$$

If the X's are independent and have a common standard deviation σ_X, the standard deviation of Z_i is

$$\sigma_{z_t}=\sqrt{\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2t}\ )}\sigma_x$$

For the first observation σ_Z takes the value $\lambda\sigma_X$, and as i increases σ_Z increases to its limiting value

$$\sigma_z=\sqrt{\frac{\lambda}{(2-\lambda)}}\sigma_x$$

An out of-control alarm is triggered if the estimate $|Z_t|$ falls outside the limits

$$g=L\sigma_{z_t}\ .$$

Usually the limiting value σ_Z rather than σ_{Zt} is used in constructing the warning-limits for EWMA control charts, e.g. Roberts(1959)[5], Robinson and Ho(1978)[7], Crowder(1987)[8] and Lucas and Saccucci(1990)[9]. For a twosided control chart this results in two straight alarm limits, G_u and G_l. They are usually placed at equal distance, on each side, from the nominal level, μ^0. As shown in Fig. 9 the EWMA(G) probability of a false alarm at or before t has a structure similar to the Shewhart method. In this example λ is arbitrarily set to 0.5, any other choice would also have been possible. Actually, for a given shift, μ, the parameter λ and L can be chosen in such a way that both ARL^0 and ARL^1 are exactly the same as for any other method, specified by one or two parameters.

The parameter L is set to 0.924 to satisfy the ARL requirement of 15. The low initial alarm probabilities are consequences of using G_u and G_l as alarm limits. By choosing G_u and G_l as alarm limits the lowest alarm probabilities are obtained at $t=1$. This fact is sometimes considered as a

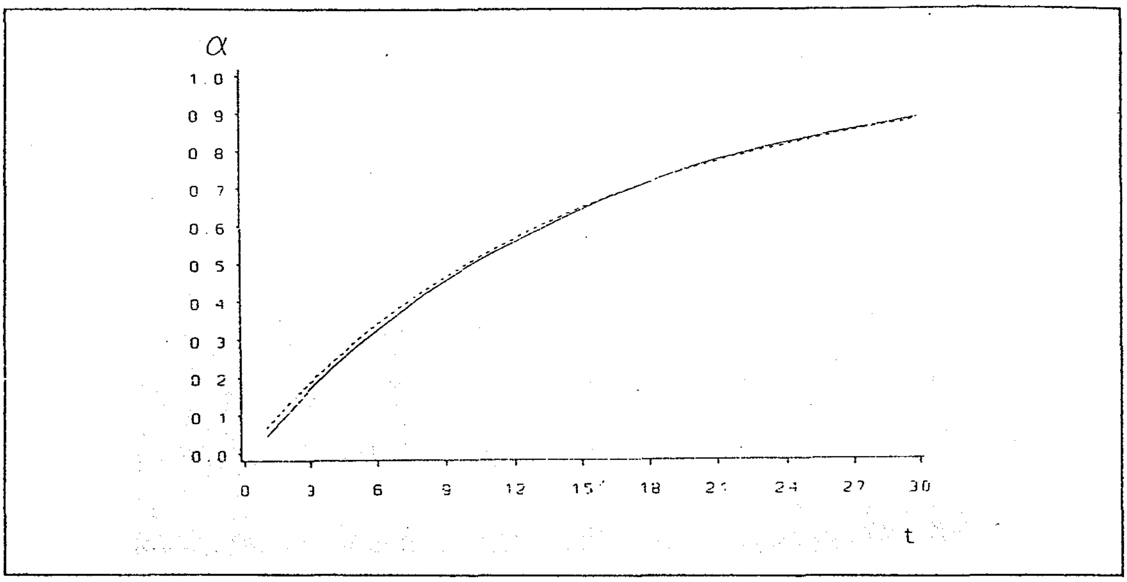


Figure 9 *The probability of a false alarm at time point t or earlier for the EWMA method and the Shewhart method (the dotted line).*

disadvantage of the EWMA(G) method, and other methods with the same structure. To overcome this a FIR feature can be applied to the method. A desirable change of the probability pattern can be accomplished in several different ways. Two different approaches using the same quantity and with the same purpose are discussed and their relation to the EWMA(G) is established.

First, by using σ_{Zt}, alarm limits are created that start at a distance of $L\lambda\sigma_X$ from the nominal level and increase to $L\sigma_Z$. By using the variance of the statistic a EWMA(g_t) is created. This gives faster detection for early shifts, but also higher probabilities of early false alarms are obtained. If the EWMA(G) method, with $G=L\sigma_Z$, is transformed to a EWMA(g_t) in such a way that,

$$\lim_{t\to\infty} L\sigma_{z_t} = G$$

then,

$$\alpha^{EWMA(G)} < \alpha^{EWMA(g_t)} \quad \forall \lambda, t,$$

that is given the same value on $L\sigma_Z$ for both methods, the early alarm probabilities that are obtained after a modification, are higher than they would have been without the modification. The difference decreases as t gets larger. How fast this goes depends on the choice of λ which also determines the magnitude of the difference.

If the FIR technique is applied to the EWMA(G) method a similar false alarm pattern is recognized as for the variance corrected EWMA(g_t). This way to accomplish higher alarm probabilities are of a more general kind than the previous and can be applied to almost any other method. The idea is to assign a starting value, Z_0, that is not equal to the expected value in the process. By this a faster detection is obtained if there is an early shift in the process. If no shift occurs the statistic, Z_t, will find its way back to the expected level. The hypothesis is rejected for the first time z_t exceeds the limit of type g_t,

$$g_t = G - (1-\lambda)^t A,$$

When the constant A is written as,

$$A = L\sigma_x \frac{\sqrt{\dfrac{\lambda}{2-\lambda}} - \lambda}{1-\lambda},$$

it is the same as if Z_0 is selected in such a way that, at $t=1$, the same probabilities of alarm as for the EWMA(g_t) are obtained, and

$$\lim_{t\to\infty} G - (1-\lambda)^t A = L\sigma_z$$

With the same way of reasoning as above, given the same value on $L\sigma_Z$

$$\alpha_t^{\mathrm{EWMA}(g_t)} \leq \alpha_t^{\mathrm{EWMA}(G)^{FIR}} \qquad \forall \lambda, t$$

with equality at $t=1$. This is seen in that

$$L\sigma_x \sqrt{\frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2t}} \geq L\sigma_x - (1-\lambda)^{t-1} L\sigma_x \sqrt{\frac{\lambda}{2-\lambda}-\lambda}$$

for all λ and t. The probability of not rejecting the null hypothesis when one of the alternatives is true is affected in an analogous way for both modification procedures.

A modification of a method might have an influence on all the measures involved in an analysis. But if, like here, the procedures involved are specified by some known functions, the directions of changes are known.

In figure 10 the ARL profile for the EWMA(G) is compared to the Shewhart method.

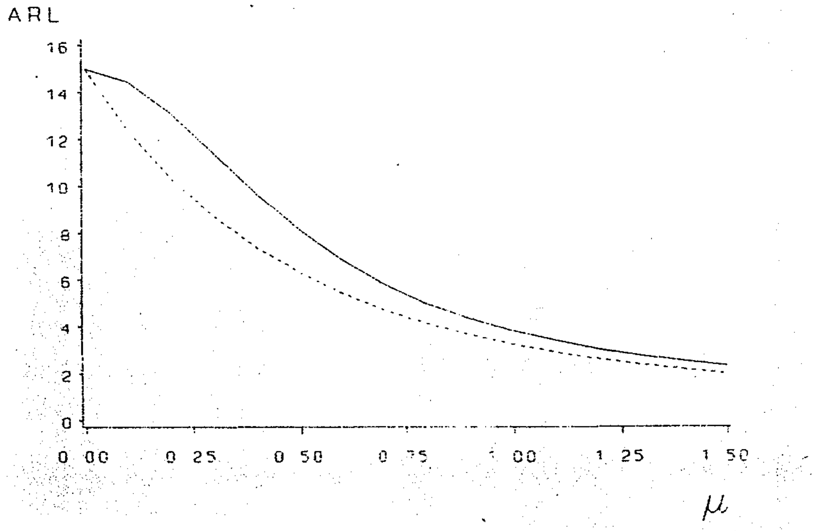


Figure 10 *The ARL profile for the EWMA(G) and the Shewhart method, the lower curve.*

Its relatively poor ARL profile is explained by low alarm probabilities at the beginning of the run, and the way the ARL[1] measure is defined.

If the EWMA(G) is modified according to the previous, both the ARL[0] and the ARL[1] decrease. The only possibility to equality among these methods is if the rejection rule is to always reject at t=1. The magnitudes of the changes are determined by the choice of λ and L. The relationship between the average run length properties given by these examples can be summarized as follows.

Given the same choice of $L\sigma_Z$ for all three methods, and Z_0 in EWMA(FIR) are such that, the probability of a false alarm at the first time point is the same as for EWMA(g_t), then

$$ARL^{0}_{\mathrm{EWMA(G)}^{FIR}} \leq ARL^{0}_{\mathrm{EWMA}(g_t)} < ARL^{0}_{\mathrm{EWMA(G)}},$$

and

$$ARL^{1}_{\mathrm{EWMA(G)}^{FIR}} < ARL^{1}_{\mathrm{EWMA}(g_t)} < ARL^{1}_{\mathrm{EWMA(G)}}.$$

When a method is modified in the way described above its seems reasonable to do so, only if there is a high probability of a shift at the beginning of the series. Using the ARL measure as a guideline might be misleading since crucial information about later observations are lost. The run length distribution at t=1 can become rather extreme compared to distributions obtained at a later shift. A consequence of this is that, in the light of other measures the ARL^1 alone becomes a poor measure of performance when the effect of the FIR features are evaluated. In particular if the assumption of a high initial shift probability is not satisfied.

5.3.2.1 CONSTANT INCIDENCE.

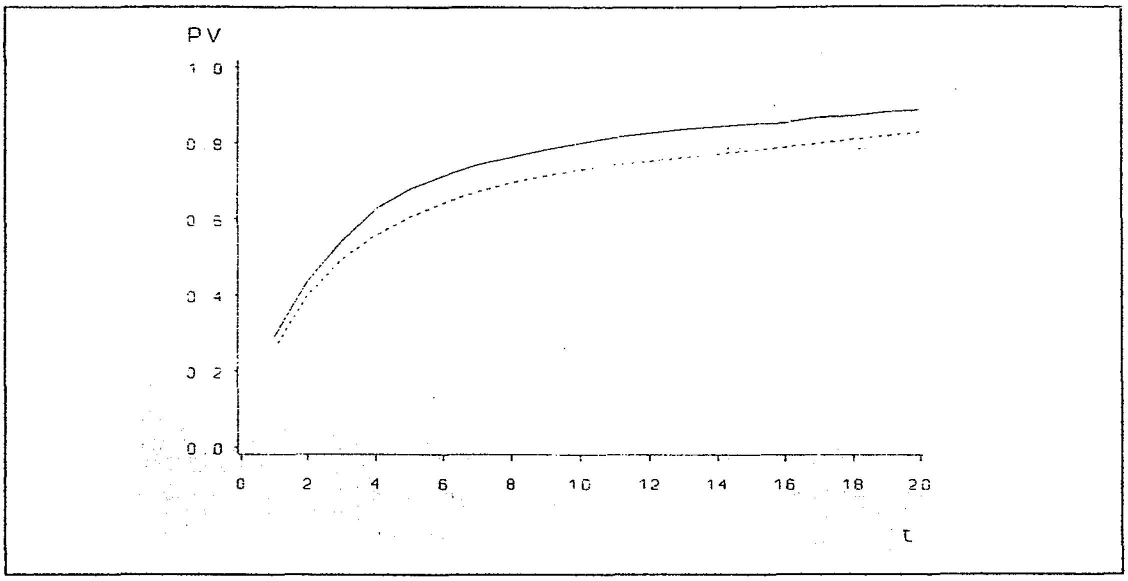


Figure 11 *The predictive value of an alarm with a constant probability of a shift. The EWMA(G), the upper curve, and the Shewhart method.*

The predictive value of an alarm curve for the EWMA(G) has a shape similar to the Shewhart tests, which was indicated by the α-structure, Fig. 6. Consider the case where a Shewhart test is replaced by a EWMA(G) in such a way that $G=L\sigma_Z$. With this restriction the EWMA(G) has a higher PV than the Shewhart, figure 10, and equality only for λ equals one. The differences due to the choice of statistic is mainly stated at the behaviour at the first two observations after a shift. The Shewhart method always has its highest detection probability at the time point of the shift, t^C. With this choice of λ, the detection probability for the EWMA(G) has its peak at t^C+1, and then decreasing faster than for the Shewhart method. In case of a shift, the EWMA(G) is most sensitive to that shift at t^C+1 or higher, depending on the choice of λ. The EWMA(G) might be a candidate if $s>t$, and in particular if a specific time point after the shift is of main interest.

5.3.2.2 VARYING INCIDENCE.

With the parameter setting given in Section 5.3.2 it is not possible to obtain an overall constant level of the predictive value of an alarm.

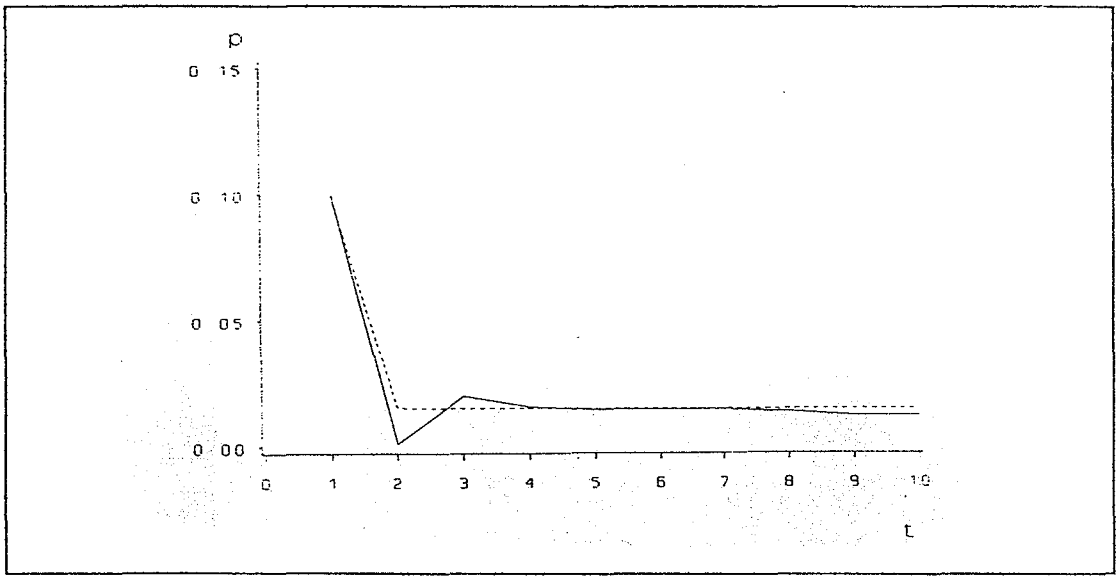


Figure 12. *The probabilities of a shift that generate a constant level of the predictive value of an alarm. The EWMA(G) and the Shewhart method.*

The shift probability for the EWMA(G) has to be negative at t=2 to obtain the desired property.

6 CONCLUDING REMARKS.

A constant level of the predictive value is a reasonable requirement in a situation where an alarm is considered equally important regardless of when the alarm occurs. That is, the surveillance model has to be designed in such a way that the proportionality between the probability of a false alarm, PFA and the probability of a motivated alarm, PMA, is constant and preferably in such a way that the predictive value is greater than 0.5.

If the predictive value is used as a design criterion when a surveillance system is designed knowledge or assumptions about the shift probability structure is crucial, both with regard to the choice of method as well as for the parameter setting. Since all the methods considered in this paper have the properties of the ordinary Shewhart method at $t=1$ and the predictive value functions are increasing it is possible to calculate the minimum predictive value for each method. The rather low values of the predicted value of an alarm is partially explained by the choice of two-sided tests.

If a constant incidence of a shift is present the predictive value can not be constant for the ordinary Shewhart method. If the method is modified in such a way that the alarm limits are moved away from μ^0 for early observations it is possible to obtain a constant level of the predictive value in the whole range of observations. Generally, for a method characterized by a monotonously increasing predictive value function the method has to be modified in such a way that the probabilities of an alarm are decreasing for early observations to reach a constant predictive value. This also means that methods designed to detect early shifts can be applied when initially there is a rather high probability of a shift. If, for instance , the incidence of a shift is constant, the system under surveillance has to be such that the consequences of a false alarm is not crucial.

ACKNOWLEDGEMENT.

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