THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Index theory in geometry and physics

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Abstract

This thesis contains three papers in the area of index theory and its applications in geometry and mathematical physics. These papers deal with the problems of calculating the charge deficiency on the Landau levels and that of finding explicit analytic formulas for mapping degrees of Hölder continuous mappings.

Paper A deals with charge deficiencies on the Landau levels for non-interacting particles in $\mathbb{R}^2$ under a constant magnetic field, or equivalently, one particle moving in a constant magnetic field in even-dimensional Euclidian space. The K-homology class that the charge of a Landau level defines is calculated in two steps. The first step is to show that the charge deficiencies are the same on every particular Landau level. The second step is to show that the lowest Landau level, which is equivalent to the Fock space, defines the same class as the K-homology class on the sphere defined by the Toeplitz operators in the Bergman space of the unit ball.

Paper B and Paper C uses regularization of index formulas in cyclic cohomology to produce analytic formulas for the degree of Hölder continuous mappings. In Paper B Toeplitz operators and Henkin-Ramirez kernels are used to find analytic formulas for the degree of a function $f : \partial\Omega \to Y$, where Ω is a relatively compact strictly pseudo-convex domain in a Stein manifold and Y is a compact connected oriented manifold. In Paper C analytic formulas for Hölder continuous mappings between general even-dimensional manifolds are produced using a pseudo-differential operator associated with the signature operator.

Keywords: Index theory, cyclic cohomology, regularized index formulas, Toeplitz operators, pseudo-differential operators, quantum Hall effect.

2000 *Mathematics Subject Classification*: 19KXX, 46L80, 19L64, 47N50, 58J40.

Preface

The purpose of this thesis is to obtain the degree of Doctor for its author. The work in this thesis is based on three papers written from material gathered by the author under time spent as a PhD-student in Gothenburg and during visits to the University of Copenhagen under the time period June 2007 to May 2011. The thesis is divided up into two parts. The first part is an introduction with a summary of the results. The second part of the thesis consists of the following papers:

A. ”Index formulas and charge deficiencies on the Landau levels”, Journal of Mathematical Physics 51 (2010).

B. ”Analytic formulas for topological degree of non-smooth mappings: the odd-dimensional case”, submitted.

C. ”Analytic formulas for topological degree of non-smooth mappings: the even-dimensional case”, submitted.

Only minor modifications on these papers have been made for this thesis. These minor modifications include correcting typos and changing of notations for a homogeneous notation throughout the thesis.

In addition to the above, there are four other papers by the author. These, however, will not be included in the thesis:

* ”Projective pseudo-differential operators on infinite-dimensional Azumaya bundles”, submitted.
* ”The Pimsner-Voiculescu sequence for coactions of compact Lie groups”, to appear in Mathematica Scandinavica.
* ”A remark on twists and the notion of torsion-free discrete quantum groups”, to appear in Algebras and Representation Theory.
* ”Equivariant extensions of $*$-algebras”, New York Journal of Mathematics 16 (2010), p. 369–385.

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Contents

Part I

Introduction

Introductory remarks

"Where there is matter, there is geometry."
Kepler

The starting point for index theory and K-theory was the Riemann-Roch theorem which originates from Riemann and Roch in the 1850:s, see [73] and [74]. The Riemann-Roch theorem relates an analytic quantity, the holomorphic Euler characteristic, with a topological quantity associated with a surface. The holomorphic Euler characteristic is the index of a twisted Dolbeault operator on the Riemann surface. The generalizations were many and came in different shapes. Hirzebruch made a generalization to complex manifolds allowing a calculation of the holomorphic Euler characteristic in terms of topological quantities in [52] and Grothendieck found the place for Riemann-Roch's theorem in the realm of algebraic geometry. Grothendieck's formulation was made in terms of his K-theory, a group of formal differences of locally free sheafs.

The ideas of Grothendieck were transformed by Atiyah, see [2], into topological K-theory and used in the proof of the Atiyah-Singer index theorem. The Atiyah-Singer index theorem was a large step from the Riemann-Roch theorem in that it gave an explicit method to calculate the index of any elliptic differential operator in terms of topological data from the manifold and the highest order symbol of the differential operator. On a vague level the index theorem related an analytic, or for that matter a global, invariant such as the Fredholm index with a geometric or local invariant such as the topological index. More generally, finding index formulas deals often with going from global to local or from analytic to geometric.

Recall that a Fredholm operator is a closed operator with finite-dimensional kernel and cokernel. The index of a Fredholm operator T is given by $\mathrm{ind}(T) := \dim\ker T - \dim\ker T^*$. An elliptic differential operator D between two smooth vector bundles E_1 and E_2 over a closed smooth n-dimensional manifold X is Fredholm. The principal symbol $\sigma(D)$ is a morphism between the vector bundles

E_1 and E_2 pulled back to the cotangent bundle and if D is elliptic this morphism is an isomorphism outside a compact set. Thus the principal symbol defines a compactly supported K-theory class $[D] \in K_c^0(T^*X)$, where $\pi : T^*X \to X$ denotes the cotangent bundle. With a K-theory class $[D]$ one can associate its Chern character $\mathrm{ch}[D]$ which is an even de Rham cohomology class. The Atiyah-Singer index theorem states that

$$\mathrm{ind}\,(D) = \int_{T^*X} \mathrm{ch}[D] \wedge \pi^* Td(X), \tag{1}$$

where $Td(X)$ denotes the Todd class of the complexified tangent bundle of X. This index theorem, its generalizations and the ideas in and around K-theory are central in this thesis.

As an example of how index theory has applications in classical geometric situations, let us consider the problem of finding the number of holomorphic sections of a holomorphic vector bundle $E \to X$ on a compact complex manifold X. With the vector bundle E there is an associated twisted Dolbeault operator $\bar{\partial}_E$ from $\wedge^{ev} T^{(0,1)} X \otimes_{\mathbb{C}} E$ to $\wedge^{odd} T^{(0,1)} X \otimes_{\mathbb{C}} E$. The twisted Dolbeault operator is elliptic. The quantities involved in the Atiyah-Singer index theorem are of a topological nature, so they can not reproduce $H^0(X,E)$ but the index theorem tells something about the holomorphic Euler characteristic of E:

$$\chi(X,E) := \sum (-1)^k \dim_{\mathbb{C}} H^k(X,E).$$

Sometimes it is possible to find $\dim_{\mathbb{C}} H^0(X,E)$ from the Euler characteristic. Since $\ker \bar{\partial}_E = H^{ev}(X,E)$ and $\ker \bar{\partial}_E^* = H^{odd}(X,E)$, the holomorphic Euler characteristic of E is the index of the Dolbeault operator. As is seen from equation (4.1) in [7] we have that $\pi_* \mathrm{ch}[\bar{\partial}_E] = \mathrm{ch}[E] \wedge Td(T_c^* X)^{-1}$, where $Td(T_c^* X)$ denotes the Todd class of the complex cotangent bundle $T_c^* X \to X$. Thus the Atiyah-Singer index theorem and the identity $Td(X) = Td(T_c X \oplus T_c^* X) = Td(T_c X) \wedge Td(T_c^* X)$ implies the Hirzebruch-Riemann-Roch theorem:

$$\chi(X,E) = \int_X \mathrm{ch}[E] \wedge Td(T_c X).$$

See more in Part 4 of [7].

In this thesis we deal with two problems that have their origin in mathematical physics. The first problem we address in Paper A is that of finding a topological invariant of a system of n particles moving in $\mathbb{C}$ under the influence of a constant magnetic field known as that system's charge deficiency. The charge deficiency of a system is proportional to the system's Hall conductance. The problem of calculating the charge deficiency is an index problem for a class of Toeplitz operators with symbols in $C(S^{2n-1})$ acting on a Hilbert space that is given as higher excitations of the Fock space in $\mathbb{C}^n$. As a special case we can give index formulas for Toeplitz operators acting on the Fock space.

The second problem we adress in this thesis is that of finding analytic formulas for mapping degrees of non-smooth mappings. This question has been originally motivated by problems in non-linear partial differential equations starting with work of Brezis and Nirenberg. One instance of such a problem is the Ginzburg-Landau equation for a superconductor in some domain $G \subseteq \mathbb{R}^2$ who's solutions are pairs (A, Φ), where A is a gauge field and Φ a complex vector field with $|\Phi| = 1$ on ∂G, that minimizes the Ginzburg-Landau functional, see more in [21] and [24]. The behavior of the solutions can change drastically depending on the degree of $\Phi|_{\partial G} : \partial G \to S^1$. What makes matters difficult is that the natural setting to define the partial differential equations in is when $\Phi \in H^1(G, \mathbb{C})$. Hence the function $\Phi|_{\partial G}$ is in general not smooth but rather in the Sobolev space $H^{1/2}(\partial G, S^1)$ where ordinary degree theory breaks down.

With problems like these in mind Brezis and Nirenberg extended degree theory to the setting of VMO-functions in [27]. However, the main argument in the approach used in [27] is in terms of approximations by smooth mappings so it only defines the degree in terms of abstract properties. What we will do is to give integral formulas for degrees of Hölder continuous mappings with Hölder order arbitrarily close to zero. The main technique that we use is the regularization of index formulas in cyclic cohomology, a technique previously used in [34], [48], [76] and [77]. The special case of a mapping $f : \partial\Omega \to Y$, where Ω is a strictly pseudo-convex domain, plays a very interesting role. In this case one can express mapping degrees in terms of the index theory for Toeplitz operators and the quantities involved can be explicitly computed for some examples. This is the setting of Paper B. We treat the general case in paper C by using pseudo-differential operators. These types of results produce certain estimates of mapping degrees.

The first part of the thesis consists of three introductory chapters to describe the framework that we will be working in and the problem setting. The second part consists of research papers. In the introductory part we introduce some concepts relevant for the rest of the thesis. The introductory part is organized as follows; in Chapter 1 we recall some definitions and properties of the basic tool for dealing with index theory, K-theory and its dual homology theory, namely, K-homology. Chapter 2 consists of some motivation from physics stemming from the quantum Hall effect, placing this physical problem in the context of index theory. Chapter 3 is devoted to a short introduction to index theory and generalizations of the Atiyah-Singer index theorem and Boutet de Monvel's index theorem for Toeplitz operators.

Chapter 1

K-theory and K-homology

"Algebra is the offer made by the devil to the mathematician. The devil says: –I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."
Atiyah

In this chapter we recall the basics of K-theory and K-homology, the homological toolbox for dealing with index problems. Both theories can be formulated in many different ways and we refer the reader to [2], [13], [14], [22], [29], [37], [39], [51] and [58] for a more thorough presentation. In the first section we will review the even part of these theories, the even K-theory consists of vector bundles and the even K-homology can be thought of as elliptic differential operators on the space. The second section consists of a short introduction to the odd part; the odd K-theory consists of matrix valued symbols and elements of the odd K-homology are the equivalence classes of Toeplitz quantizations of the space.

1.1 Even K-theory

The even K-theory of a topological space X is a topological invariant of X whose elements are equivalence classes of vector bundles over X. The set of isomorphism classes of vector bundles over X forms an abelian monoid under the direct sum. Following [2], the even K-theory $K^0(X)$ of a compact Hausdorff space X is defined as the Grothendieck group of the abelian monoid of isomorphism classes of vector bundles over X. That is, $K^0(X)$ is the abelian group of formal differences of vector bundles over the topological space X.

Since the pullback of vector bundles is functorial up to an isomorphism and additive, $K^0(X)$ depends contravariantly on the compact Hausdorff space X so one can define $K^0(X)$ for arbitrary locally compact Hausdorff spaces X as the kernel of the mapping induced by the inclusion mapping $\{\infty\} \to \hat{X}$ of the infinite point in the Alexandroff compactification $\hat{X}$ of X. Because of the functoriality of the Alexandroff compactifiation, K-theory depends contravariantly on X with respect to *proper* mappings. The tensor product of vector bundles defines a cup product $K^0(X) \times K^0(X) \to K^0(X)$.

The Serre-Swan theorem establishes a one-to-one correspondence between the isomorphism classes of vector bundles over a compact space X and projection valued continuous functions $p : X \to \mathscr{K}$, see [85]. Here $\mathscr{K}$ denotes the C^*-algebra of compact operators on some separable, infinite dimensional Hilbert space $\mathscr{H}$. This correspondence is given by associating with the projection $p \in C(X) \otimes \mathscr{K}$ the vector bundle $E \to X$ whose $C(X)$-module of sections is $C(X,E) = pC(X,\mathscr{H})$. Any projection in $\mathscr{K}$ is of finite rank, so E has finite-dimensional fibers. Following the Serre-Swan theorem, an equivalent approach to K-theory is to use equivalence classes of projections $p \in C(Y) \otimes \mathscr{K}$. The K-theory is denoted by $K_0(C(Y))$. To read more about K-theory, see [2] and [22].

To give an example of how to associate a projection-valued function with a vector bundle, consider the tautological line bundle $L \to \mathbb{C}P^n$. We define the function $v : \mathbb{C}P^n \to \mathbb{C}^{n+1}$ in complex homogeneous coordinates $[Z_0, Z_1, \dots, Z_n]$ by

$$v(Z_0, Z_1, \dots, Z_n) := \frac{1}{\sqrt{|Z_0|^2 + |Z_1|^2 + \cdots + |Z_n|^2}}(Z_0, Z_1, \dots, Z_n).$$

The fiber of the tautological line bundle over a point of the form $[Z_0, Z_1, \dots, Z_n]$ in homogeneous coordinates is the line spanned by the vector $(Z_0, Z_1, \dots, Z_n)$, or for that matter we can span the fiber by $v(Z_0, Z_1, \dots, Z_n)$. Thus the projection-valued function $p_L : \mathbb{C}P^n \to M_{n+1}(\mathbb{C})$ associated with the tautological line bundle is given by

$$p_L(Z_0, Z_1, \dots, Z_n)w = \langle v(Z_0, Z_1, \dots, Z_n), w\rangle v(Z_0, Z_1, \dots, Z_n), \quad w \in \mathbb{C}^{n+1}.$$

In the matrix form the projection p_L has the form

$$p_L(Z_0, Z_1, \dots, Z_n) = \frac{1}{|Z_0|^2 + \cdots + |Z_n|^2}\begin{pmatrix} |Z_0|^2 & Z_0\overline{Z}_1 & \cdots & \cdots & Z_0\overline{Z}_n \\ Z_1\overline{Z}_0 & \ddots & |Z_1|^2 & \cdots & Z_1\overline{Z}_n \\ \vdots & & \ddots & & \vdots \\ Z_n\overline{Z}_0 & \cdots & & Z_n\overline{Z}_{n-1} & |Z_n|^2 \end{pmatrix}.$$

A rather straight-forward calculation gives that the 2-form $\mathrm{tr}_{\mathbb{C}^{n+1}}(p_L \mathrm{d}p_L \mathrm{d}p_L)$ coincides with the Fubini-Study metric on $\mathbb{C}P^n$. This is not a coincidence; if

X is a smooth manifold and $p \in M_N(C^\infty(X))$ is a self-adjoint projection, the associated vector bundle is a smooth Riemannian bundle in the metric induced from the embedding $pC^\infty(X, \mathbb{C}^N) \subseteq C^\infty(X, \mathbb{C}^N)$. The curvature of the associated Levi-Civita connection is the matrix valued form $p\mathrm{d}p\mathrm{d}p$, see equation (8.33) in [44].

The formulation of K-theory in terms of projections can be defined for any algebra $\mathscr{A}$ as equivalence classes of projections $p \in \mathscr{A} \otimes \mathscr{K}$, see Definition 5.5.1 of [22]. In particular, if X is non-compact, one can define K-theory with compact supports, $K^0_c(X)$, as the K-theory of $C_c(X)$. In fact, K-theory is very stable, so, under mild assumptions, dense embeddings induce isomorphisms on K-theory. A sufficient assumption is that the dense embedding is isoradial. A morphism of bornological algebras is called isoradial if it preserves spectral radius of bounded subsets, see Definition 2.21 and Definition 2.48 of [37]. By Lemma 2.50 of [37] a dense isoradial embedding $\mathscr{A} \subseteq A$ preserves invertibility, i.e. $a \in \mathscr{A}$ is invertible in A if and only if a is invertible in $\mathscr{A}$. Under these assumption $K_0(\mathscr{A})$ is isomorphic to $K_0(A)$ via the embedding mapping, see Theorem 2.60 of [37]. For instance, $C_c(X)$ is a dense isoradial subalgebra of $C_0(X)$. Therefore there are natural isomorphisms $K^0_c(X) \cong K_0(C_0(X)) \cong K^0(X)$.

One can think of K-homology as the homology theory dual to K-theory. This duality is the first instance of a Kasparov product which in this case comes from the index pairing. The Kasparov product is a fundamental tool in constructing a bivariant homology theory for operator algebras, see more in [22] and [58].

The first step in abstracting a homology theory from index theory was made in Atiyah's definition of analytic K-homology, see [3]. The motivation for Atiyah's definition of analytic K-homology comes from the case of an elliptic differential operator D between two vector bundles $E_1 \to E_2$ over the compact manifold X. More generally, one can consider a pseudo-differential operator. If we have a smooth vector bundle $E \to X$ with associated projection valued function $p_E \in M_N(C^\infty(X))$ we can define the twisted operator

$$D_E := (1 \otimes p_E)(D \otimes 1)(1 \otimes p_E) : C^\infty(X, E_1 \otimes E) \to C^\infty(X, E_2 \otimes E).$$

Consider the association $E \mapsto \mathrm{ind}(D_E)$. The number $\mathrm{ind}(D_E)$ clearly only depends on the isomorphism class of E and is additive under direct sums of vector bundles. Therefore we may conclude that any elliptic differential operator D induces a group homomorphism $\mathrm{ind}_D : K^0(X) \to \mathbb{Z}$. This is actually the model case of a K-homology class on a manifold.

To formalize the construction, we change setting to bounded operators on Hilbert spaces and replace the elliptic differential operator by an *abstract elliptic operator*. We define the graded Hilbert space $\mathscr{H} := L^2(X, E_1 \oplus E_2) = L^2(X, E_1) \oplus L^2(X, E_2)$ with grading induced from this decomposition. With the elliptic operator D we associate the zero order elliptic pseudo-differential operator $F := \tilde{D}(1+\tilde{D}^2)^{-1/2}$, where $\tilde{D}$ is the odd operator on $E_1 \oplus E_2$ defined by

$$\tilde{D} := \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}. \tag{1.1}$$

Since D is elliptic, F is an odd self-adjoint bounded operator and since D is of positive order, $F^2 - 1 = -(1+\tilde{D}^2)^{-1}$ is a pseudo-differential operator of negative order, therefore a compact operator. Furthermore, the point-wise action on the vector bundles defines an even representation $\pi : C(X) \to \mathscr{B}(\mathscr{H})$. If $a \in C^\infty(X)$, then $[F, \pi(a)]$ is a negative order pseudo-differential operator and therefore $[F, \pi(a)] \in \mathscr{K}(\mathscr{H})$ for any continuous a. The last property of F is called pseudo-locality with respect to π.

More generally, if X is a compact Hausdorff space, a pair (π, F) consisting of a graded representation $\pi : C(X) \to \mathscr{B}(\mathscr{H})$ and a pseudo-local, odd, self-adjoint operator F with $F^2 - 1 \in \mathscr{K}(\mathscr{H})$ is called an analytic K-cycle, or, sometimes an even Fredholm module. The operator F was in [3] called an abstract elliptic operator on X. The analytic K-cycle is called degenerate if $F^2 = 1$ and $[F, \pi(a)] = 0$ for all $a \in A$. The quotient of the semigroup of homotopy classes of analytic K-cycles by the degenerate K-cycles forms an abelian group under the direct sum operation; it is called the analytic K-homology of X and is denoted by $K_0(X)$, or $K^0(C(X))$ to denote its dependence on the C^*-algebra $C(X)$. The analytic K-homology $K^0(A)$ for a general unital C^*-algebra A is constructed in the same way as for $C(X)$, see Definition 8.1.1 of [51].

Before we describe the pairing of the analytic K-homology with the even K-theory, let us make an interlude with some theory of the Fredholm index. For proofs of the statements we refer the reader to section 1.4 of [66]. As was previously mentioned, the index of a Fredholm operator T is defined as

$$\operatorname{ind}(T) = \dim\ker T - \dim\ker T^*.$$

The index of Fredholm operators is very stable in the sense that if K is compact then $\operatorname{ind}(T+K) = \operatorname{ind}(T)$. Furthermore the index is homotopy invariant, so if $(T_t)_{t\in[0,1]}$ is a norm continuous path of Fredholm operators, $\operatorname{ind}(T_1) = \operatorname{ind}(T_0)$. Also if T and T' are Fredholm then

$$\operatorname{ind}(TT') = \operatorname{ind}(T \oplus T') = \operatorname{ind}(T) + \operatorname{ind}(T'). \tag{1.2}$$

The first equality follows from the homotopy invariance of the index, since $TT' \oplus 1 \sim_h T \oplus T'$, and the second is a straight-forward calculation. By Atkinson's theorem an operator $T \in \mathscr{B}(\mathscr{H})$ is Fredholm if and only if the class of T in the Calkin algebra $\mathscr{C}(\mathscr{H}) := \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ is an invertible element. Therefore the index induces a group homomorphism

$$\widetilde{\operatorname{ind}} : \mathscr{C}(\mathscr{H})^{-1} \to \mathbb{Z}.$$

In fact, this homomorphism indexes the connected components of the topological group $\mathscr{C}(\mathscr{H})^{-1}$.

The analytic K-homology forms a generalized homology theory and pairs with the even K-theory via the index pairing. This relation is described in Proposition 8.7.2 of [51]. Let us concretize this index pairing for a C^*-algebra A. We can represent a K-homology class $x \in K^0(A)$ by an analytic K-cycle (π, F). Since F is assumed to be odd and π to be even, we can decompose

$$F = \begin{pmatrix} 0 & F_+ \\ F_- & 0 \end{pmatrix} \quad \text{and} \quad \pi = \begin{pmatrix} \pi_+ & 0 \\ 0 & \pi_- \end{pmatrix}. \tag{1.3}$$

Any element $[p] \in K_0(A)$ can be represented by a projection $p \in M_N(A) := A \otimes M_N(\mathbb{C})$ for some large matrix algebra $M_N(\mathbb{C})$. Let us use the notation

$$p_+ := (\pi_+ \otimes \mathrm{id})(p) \quad \text{and} \quad p_- := (\pi_- \otimes \mathrm{id})(p),$$

which are operators on $\mathscr{H}_+ \otimes \mathbb{C}^N$ respectively $\mathscr{H}_- \otimes \mathbb{C}^N$. We also define the Hilbert spaces $\mathscr{H}_+^p := p_+(\mathscr{H}_+ \otimes \mathbb{C}^N)$ and $\mathscr{H}_-^p := p_-(\mathscr{H}_- \otimes \mathbb{C}^N)$. The operator

$$p_- F_+ p_+ : \mathscr{H}_+^p \to \mathscr{H}_-^p$$

is Fredholm since F commutes with $\pi(A)$ up to compact operators so Atkinson's theorem implies that $p_+ F_- p_- : \mathscr{H}_-^p \to \mathscr{H}_+^p$ is an inverse to $p_- F_+ p_+$ modulo compact operators. Therefore we may define the bilinear pairing $K_0(A) \times K^0(A) \to \mathbb{Z}$ by

$$([p], x) \mapsto \mathrm{ind}(p_- F_+ p_+), \tag{1.4}$$

which is well defined due to the stability and homotopy invariance of the index.

In general, this pairing is very hard to calculate and this is what index theory is about. The Atiyah-Singer index theorem describes this pairing for the case when A is the algebra of continuous functions on a closed manifold explicitly in terms of de Rham cohomology of the manifold. The problem in general is to find a concrete realization of the index pairing in terms of some "local" homology theory.

For a topological space X, Baum-Douglas, p. 154 of [14], defined *the index problem* as the problem of representing an analytic K-homology class on X by a *geometric* K-homology class, i.e. the representative of the K-homology class defined as the push-forward of the Dirac operator on a vector bundle E over a spin^c manifold M. Then the Atiyah-Singer index theorem will produce a local index formula by pulling back to M. It is impossible in practice to solve or even to define the index problem for general C^*-algebras, compare with [54], one rather needs to look at dense isoradial subalgebras which admit well behaving homology theories. But there is no free lunch, the rigidity of C^*-algebras is lost. In theory, as mentioned above, the index pairing is a Kasparov product and is described by KK-theory.

1.2 Odd K-theory

So far we have only discussed the even K-homology and the even K-theory. The odd versions can be defined in many ways, for example, using Clifford algebras or the suspension functor, but the way we choose is a more straight-forward one that fits better with index theory. The odd K-theory forms a useful homological tool for dealing with symbols of Toeplitz or pseudo-differential operators. We will use the notation $\tilde{A}$ for the unitalization of a C^*-algebra A, see section 1.2 of [66]. In the setting where $A = C_0(X)$ the unitalization can be described as $\tilde{A} = C(\hat{X})$ where $\hat{X}$ is the Alexandroff compactification of X.

The group $\mathrm{GL}_N(A)$ is defined as consisting of invertible matrices in $\tilde{A} \otimes M_N(\mathbb{C})$ and $\mathrm{GL}_\infty(A) := \varinjlim \mathrm{GL}_N(A)$, where we embed $\mathrm{GL}_N(A) \to \mathrm{GL}_{N+1}(A)$ by $x \mapsto x \oplus 1$. The group $\mathrm{GL}_\infty(A)$ becomes a topological group in the inductive limit topology. We denote the identity component of $\mathrm{GL}_\infty(A)$ by $\mathrm{GL}_\infty(A)_0$ which by standard theory is a normal subgroup. The odd K-theory is defined as in Definition 8.1.1 of [22] as

$$K_1(A) := \mathrm{GL}_\infty(A)/\mathrm{GL}_\infty(A)_0.$$

So the invariant $K_1(A)$ is a group of equivalence classes of invertible matrices over $\tilde{A}$, the equivalence relation involves stable homotopy. The class of a matrix $u \in GL_N(A)$ in $K_1(A)$ is denoted by $[u]$. By Proposition 8.1.3 of [22] the group $K_1(A)$ is abelian so the odd K-theory can be viewed as a covariant functor on the category of C^*-algebras to the category of abelian groups. This statement follows from that if $u, v \in GL_N(A)$ for some large N then

$$[u] + [v] = [uv] = [u \oplus v] = [v \oplus u] \quad \text{in } K_1(A).$$

This situation is to be compared with the properties of the index in equation (1.2). The odd K-theory can be calculated from the even K-theory by $K_1(A) \cong K_0(C_0(\mathbb{R}) \otimes A)$ by Theorem 8.2.2 of [22]. If we try to define higher K-theory groups $K_i(A) := K_0(C_0(\mathbb{R}^i) \otimes A)$, the Bott periodicity implies that there is a natural isomorphism $K_{i+2}(A) \cong K_i(A)$, see Theorem 9.2.1 of [22]. Thus odd and even K-theory contains all information that topological K-theory sees, contrasting the situation in algebraic K-theory.

As it was mentioned previously, K-theory is merely half-exact. This deficiency of exactness is exactly what gives rise to index theory. So, let $0 \to I \to A \to A/I \to 0$ be a short exact sequence of C^*-algebras. With the short exact sequence there is an associated mapping $\partial : K_1(A/I) \to K_0(I)$, known as the index mapping.

The index mapping can be constructed rather explicitly. This construction can be found in Definition 8.3.1 of [22]. Represent a class $x \in K_1(A/I)$ by the matrix $u \in GL_N(A/I)$. Since u is invertible, there is an inverse $v \in GL_N(A/I)$ to

u. Let $U,V\in M_N(A)$ be pre-images of u respectively v. We define the matrix

$$W := \begin{pmatrix} (1-UV)U+U & UV-1 \\ 1-UV & V \end{pmatrix} \in M_{2N}(A).$$

Observe that $1-UV, 1-VU \in M_N(I)$ since the image of V in the quotient is the inverse of the image of U. Thus the image of W under the quotient mapping is $u\oplus v$. Furthermore, $W\in GL_{2N}(A)$ since an inverse is given by

$$W^{-1} := \begin{pmatrix} V & 1-VU \\ UV-1 & (1-UV)U+U \end{pmatrix}.$$

The index mapping of x is defined by

$$\partial x := [W^{-1}(p_N\oplus 0)W]-[p_N\oplus 0]\in K_0(I) \tag{1.5}$$

where $p_N\in M_N(A)$ denotes the identity. That ∂x is well defined follows from that w commutes with p_N up to an element of I. Furthermore, the element ∂x does not depend on our particular choice $W\in GL_{2N}(A)$ that lifts $u\oplus v$. The index mapping is clearly additive since $\partial([u]+[u'])=\partial[u\oplus u']$ and we can lift $u\oplus u'\oplus(u\oplus u')^{-1}$ by means of lifts of $u\oplus u^{-1}$ and $u'\oplus(u')^{-1}$. For future reference we observe that

$$W^{-1}(p_N\oplus 0)W = \begin{pmatrix} -(1-UV)^2+1 & U(1-VU)^2 \\ (1-VU)V & (1-VU)^2 \end{pmatrix}. \tag{1.6}$$

The index mapping is natural with respect to short exact sequences. By Theorem 9.3.1 of [22], the index mapping makes the following diagram exact under the Bott periodicity:

$$\begin{tikzcd} K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[d] \\ K_1(A/I) \arrow[u] & K_1(A) \arrow[l] & K_1(I) \arrow[l] \end{tikzcd} . \tag{1.7}$$

To show some calculations of K-theory groups let us find the K-groups of the n-sphere S^n and its cosphere bundle. To calculate $K^*(S^n)$ we fix a point $\infty\in S^n$ and define the $*$-homomorphism $C(S^n)\to\mathbb{C}$ as the point evaluation in ∞. Since $S^n\setminus\{\infty\}\cong\mathbb{R}^n$ we have a short exact sequence of C^*-algebras

$$0\to C_0(\mathbb{R}^n)\to C(S^n)\to\mathbb{C}\to 0.$$

Considering the associated six-term exact sequence we have the following diagram:

$$\begin{tikzcd} K^0(\mathbb{R}^n) \arrow[r] & K^0(S^n) \arrow[r] & \mathbb{Z} \arrow[d] \\ 0 \arrow[u] & K^1(S^n) \arrow[l] & K^1(\mathbb{R}^n) \arrow[l] \end{tikzcd} .$$

By the Bott periodicity $K^i(\mathbb{R}^n)$ is $\mathbb{Z}$ when $i-n$ is even and 0 when $i-n$ is odd. Furthermore, since there is a splitting to $C(S^n)\to\mathbb{C}$, which simply maps a constant to a constant function, it follows that the index mapping is 0. Therefore $K^*(S^n)=\mathbb{Z}\oplus\mathbb{Z}$ with two of the summands being of even grading if n is even and one summand of each parity if n is odd. Using that the Chern character is an isomorphism of rings it follows that $K^*(S^n)\cong\mathbb{Z}[x]/x^2$ as rings where x is a formal variable of degree n whose Chern character is the volume form on S^n. For an explicit construction of the class x in K-theory see below in Paper B for n odd and Paper C for n even.

To calculate the K-groups of S^*S^n we perform a similar trick. We let $\pi: S^*S^n\to S^n$ denote the projection. There are diffeomorphisms $\pi^{-1}(S^n\setminus\{\infty\})\cong\mathbb{R}^n\times S^{n-1}$ and $\pi^{-1}(\infty)\cong S^{n-1}$. So there is a short exact sequence of C^*-algebras $0\to C_0(\mathbb{R}^n\times S^{n-1})\to C(S^*S^n)\to C(S^{n-1})\to 0$. Taking the K-theory of this short exact sequence gives

$$\begin{tikzcd}
K^n(S^{n-1}) \arrow[r] & K^0(S^*S^n) \arrow[r] & K^0(S^{n-1}) \arrow[d] \\
K^1(S^{n-1}) \arrow[u] & K^1(S^*S^n) \arrow[l] & K^{n+1}(S^{n-1}) \arrow[l]
\end{tikzcd}.$$

The index mappings happens to be 0 also in this case, so

$$K^*(S^*S^n)=K^*(S^{n-1})\oplus K^{n+*}(S^{n-1})=\mathbb{Z}^2\oplus\mathbb{Z}^2. \tag{1.8}$$

As an example of how we can use the index mapping on K-theory, let us consider Toeplitz operators. Assume that Ω is a strictly pseudo-convex domain in some complex manifold with smooth compact boundary, in complex dimension 2 we also assume that Ω is relatively compact. We denote the Hardy space by $H^2(\partial\Omega)$, the closed subspace of $L^2(\partial\Omega)$ consisting of functions with a holomorphic extension to Ω. More generally, if Ω is a strictly pseudo-convex domain in some complex space such that there are no singularities on $\partial\Omega$, one can consider the Hilbert space of functions in $L^2(\partial\Omega)$ with a holomorphic extension in a neighborhood of $\partial\Omega$. Let $P: L^2(\partial\Omega)\to H^2(\partial\Omega)$ denote the orthogonal projection, P is called the Szegö projection of $\partial\Omega$. The operator P is pseudo-local with respect to the pointwise action of $C(\partial\Omega)$ on $L^2(\partial\Omega)$.

A Toeplitz operator with symbol $u\in C(\partial\Omega)$ is an operator of the form $T=PuP+K$ on $H^2(\partial\Omega)$, where $K\in\mathcal{K}(H^2(\partial\Omega))$. The C^*-algebra $\mathcal{T}$ generated by all Toeplitz operators contains the ideal of compact operators. Since P is pseudo-local and $PuP\in\mathcal{K}(H^2(\partial\Omega))$ if and only if $u=0$, the symbol mapping $PuP+K\mapsto u$ is a $*$-homomorphism whose kernel is $\mathcal{K}(H^2(\partial\Omega))$. Therefore, the symbol mapping induces an isomorphism $\mathcal{T}/\mathcal{K}\cong C(\partial\Omega)$. We can hence fit the symbol mapping into a short exact sequence of C^*-algebras

$$0\to\mathcal{K}\to\mathcal{T}\to C(\partial\Omega)\to 0.$$

Observe that these properties and Atkinson's theorem imply that if we have a matrix-valued symbol $u \in M_N(C(\partial\Omega))$, the associated Toeplitz operator acting on the vector-valued Hardy space $PuP \in \mathscr{B}(H^2(\partial\Omega) \otimes \mathbb{C}^N)$ is Fredholm if and only if $u \in GL_N(C(\partial\Omega))$.

The associated index mapping $\partial : K_1(C(\partial\Omega)) \to K_0(\mathscr{K}) = \mathbb{Z}$ does in fact map a class $[u]$ to the index of the Toeplitz operator

$$PuP : H^2(\partial\Omega) \otimes \mathbb{C}^N \to H^2(\partial\Omega) \otimes \mathbb{C}^N$$

whenever we can represent $[u] \in K_1(C(\partial\Omega))$ by $u \in GL_N(C(\partial\Omega))$. The property that the index mapping produces the index is correct in general for semi-split short exact sequences $0 \to \mathscr{K} \to E \to A \to 0$ whenever the $*$-monomorphism $\mathscr{K} \to E$ is non-degenerate. This is the motivation for the name index mapping. The index mapping in fact maps the K-theory class of a symbol $u \in GL_N(C(\partial\Omega))$ to the index of PuP, which can be seen from the following reasoning. The operator PuP is Fredholm so by Fredholm theory there is an operator $R \in \mathscr{B}(H^2(\partial\Omega) \otimes \mathbb{C}^N)$ such that $1 - PuPR = P_0$ and $1 - RPuP = P_1$, where P_0 and P_1 denotes the orthogonal projections onto the finite-dimensional spaces $\mathrm{coker}\, PuP$ respectively $\ker PuP$. In fact, since the class of R in the Calkin algebra is an inverse of the class of PuP, the operator R is a Toeplitz operator with symbol u^{-1}. Hence the invertible operator

$$\tilde{T} := \begin{pmatrix} PuP & P_0 \\ P_1 & R \end{pmatrix}$$

provides a lift of $u \oplus u^{-1}$. Furthermore, a direct calculation using (1.5) and (1.6) gives that

$$\partial[u] = \left[\tilde{T}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{T}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = [P_1] - [P_0] \in K_0(\mathscr{K}), \tag{1.9}$$

which under the isomorphism $K_0(\mathscr{K}) \cong \mathbb{Z}$ corresponds to the index of PuP.

The index formula of Boutet de Monvel from [23] enables us to calculate this index in terms of de Rham cohomology in the case when u is smooth. If $u : \partial\Omega \to GL_N(\mathbb{C})$ is smooth, we define the Chern character of the class $[u]$ as the closed differential form

$$\mathrm{ch}[u] := \sum_{j=1}^{\infty} \frac{(j-1)!}{(2\pi i)^j (2j-1)!} \mathrm{tr}(u^{-1}\mathrm{d}u)^{2j-1}.$$

The de Rham class of $\mathrm{ch}[u]$ is in fact independent of the choice of representative for $[u]$ (for a proof of this see for instance section 1.8 of [91]). The Boutet de Monvel index formula states that

$$\mathrm{ind}\,(PuP) = -\int_{\partial\Omega} \mathrm{ch}[u] \wedge Td(\Omega).$$

Let us consider another example of a short exact sequence. Assume that X is a compact manifold. Let $\Psi(X)$ denote the C^*-algebra generated by the zero order classical pseudo-differential operators on X. The C^*-algebra $\Psi(X)$ contains the C^*-algebra generated by the negative order pseudo-differential operators on X, which is the algebra of compact operators on $L^2(X)$. Therefore the principal symbol mapping defines a short exact sequence

$$0 \to \mathscr{K} \to \Psi(X) \to C(S^*X) \to 0,$$

and the index mapping $K_1(C(S^*X)) \to \mathbb{Z}$ coincides with the index of pseudo-differential operators which on smooth symbols can be calculated by the Atiyah-Singer index theorem. Compare this example with Section $2.8(b)$ of [51]. As shown in Theorem 5.2 of [47], if X is real analytic, S^*X can be considered as the boundary of a strictly pseudo-convex Grauert tube and a pseudo-differential operator is thus a Toeplitz operator. In [14] the more general statement that the pseudo-differential extension on a manifold is the extension of $C(S^*X)$ associated with the spin^c-Dirac operator on S^*X was proven. The manifold S^*X has a canonical spin^c-structure induced from the almost complex structure on T^*X. Therefore, pseudo-differential operators are really Toeplitz operators in disguise, as it is expressed in [14]. For manifolds that are not real analytic, pseudo-differential operators are Toeplitz operators with respect to an almost complex manifold rather than a complex structure.

Before we move on to the definition of the odd analytic K-homology, let us digest on the theory of Toeplitz operators with abstract symbol. Assume that A is a unital C^*-algebra, $\pi : A \to \mathscr{B}(\mathscr{H})$ a unital representation and P an orthogonal pseudo-local projection with respect to π. We will call an operator of the form $P\pi(a)P$ the Toeplitz operator with symbol a. Because of the similarity with pseudo-differential operators, a pair (π, P) of this form is sometimes called a Toeplitz quantization. Since P is pseudo-local, Atkinson's theorem implies that $P\pi(a)P$ is a Fredholm operator on $P\mathscr{H}$ whenever a is invertible. Let $\mathscr{T}$ denote the C^*-algebra generated by all these Toeplitz operators and the compact operators on $P\mathscr{H}$. Under the further assumption that $P\pi(a)P \in \mathscr{K}$ if and only if $a = 0$ the symbol mapping $P\pi(a)P \mapsto a$ is well defined. We obtain the short exact sequence of C^*-algebras

$$0 \to \mathscr{K} \to \mathscr{T} \to A \to 0. \tag{1.10}$$

So a pair (π, P) like above defines an index mapping $K_1(A) \to \mathbb{Z}$ as in equation (1.9). Pairs like this will form the odd analytic K-homology.

The definition of odd K-homology is quite similar to the even case. An odd analytic K-cycle of a C^*-algebra A is again a pair (π, F) consisting of a representation $\pi : A \to \mathscr{B}(\mathscr{H})$ and an operator $F \in \mathscr{B}(\mathscr{H})$ such that $F^2 - 1$ and

$F^* - F$ are compact and F is pseudo-local in the sense that $[F, \pi(a)]$ is a compact operator for all $a \in A$. As in the even case, a cycle (π, F) is called degenerate if $F^2 - 1 = F^* - F = [F, \pi(a)] = 0$. The quotient of set of homotopy classes of odd analytic K-cycles by the degenerate cycles form an abelian group under the direct sum operation; it is denoted by $K^1(A)$ and is called the odd analytic K-homology of A. Sometimes, we will use the notation $K_1(X)$ for the group $K^1(C(X))$ to indicate the dependence on the topological space. The difference in definitions between the odd and even analytic K-homology lies in the grading, this changes the cycles drastically. For instance, if one considers an even K-cycle as an odd by just forgetting the grading then its class in odd K-homology is 0.

Such a pair (π, P) defining the short exact sequence (1.10) determines the odd analytic K-cycle $(\mathscr{H}, 2P+1)$. In fact, short exact sequences like the one in equation (1.10) have many similarities with K-homology and KK-theory. This relation is described by the theory of extensions, see more in section 16.3 of [22].

The index pairing $K_1(A) \times K^1(A) \to \mathbb{Z}$ between the odd K-theory and the odd K-homology is much like the index of a Toeplitz operator. This index pairing is described in Proposition 8.7.1 of [51]. If we represent an element $x \in K^1(A)$ by an extension $0 \to \mathscr{K} \to E_x \to A \to 0$, the six term exact sequence (1.7) defines the index mapping $\mathrm{ind}_x : K_1(A) \to \mathbb{Z}$, and for $u \in K_1(A)$ the index pairing is given by

$$(u, x) \mapsto \mathrm{ind}_x(u).$$

To be a bit more precise, if (π, F) is an odd analytic K-cycle representing x such that $F^2 = 1$ we define the projection $P_F := (F+1)/2$. The condition $F^2 = 1$ is not restrictive since any odd K-homology class can be represented by such a cycle, see Lemma 8.3.5 of [51] or below in Paper C. For an odd K-theory class represented by the matrix $u \in GL_N(A)$, the abstract Toeplitz operator

$$P_F \pi(u) P_F : P_F \mathscr{H} \otimes \mathbb{C}^N \to P_F \mathscr{H} \otimes \mathbb{C}^N$$

is Fredholm if u is invertible because of Atkinson's theorem. The index pairing of x with an odd K-theory class u is given by the index $\mathrm{ind}(P_F \pi(u) P_F)$. In general, there is no obvious way to calculate the index pairing. But in concrete applications there is the tool of cyclic cohomology that we recall below in Chapter 3.

Chapter 2

Magnetism and K-theory

"Not everything that can be counted counts, and not everything that counts can be counted."
Einstein

One of the most concrete applications of K-theory and index theory is in mathematical physics, for systems containing a magnetic field. In this section we will give a brief review of the quantum Hall effect. To motivate the nature of charge as a K-theoretic object, consider a real life example of a closed four manifold M and a magnetic field F on M. The magnetic field is given as the curvature of a connection ∇ on a vector bundle $E \to M$. Thus $F = \nabla^2$ is a section of $\wedge^2 T^*M \otimes \mathrm{End}(E)$. The magnetic field should minimize the energy of the system due to Fermat's principle, thus the connection should minimize the Yang-Mills functional:

$$YM(\nabla) := \frac{1}{2g^2}\int_M \mathrm{tr}(F \wedge *F) + \frac{\theta}{8\pi^2}\int_M \mathrm{tr}(F \wedge F)$$

The first term is a dynamic term whose Euler-Lagrange equation is known as the Yang-Mills equation. The second term is the charge of F and its variation is given by ∇F, so it is locally constant and a topological invariant due to the Bianchi identity. In fact the second term is given by $\frac{\theta}{2}\int_M ch(E) \in \frac{\theta}{2}\mathbb{Z}$, the integral of the Chern character of E which by the Atiyah-Singer index theorem is the Euler characteristic of E. Thus all information about the charge of a magnetic field on the bundle E is given by the K-theory class $[E] \in K^0(M)$. Therefore, the K-theory class of a vector bundle is the topological invariant classifying its charge.

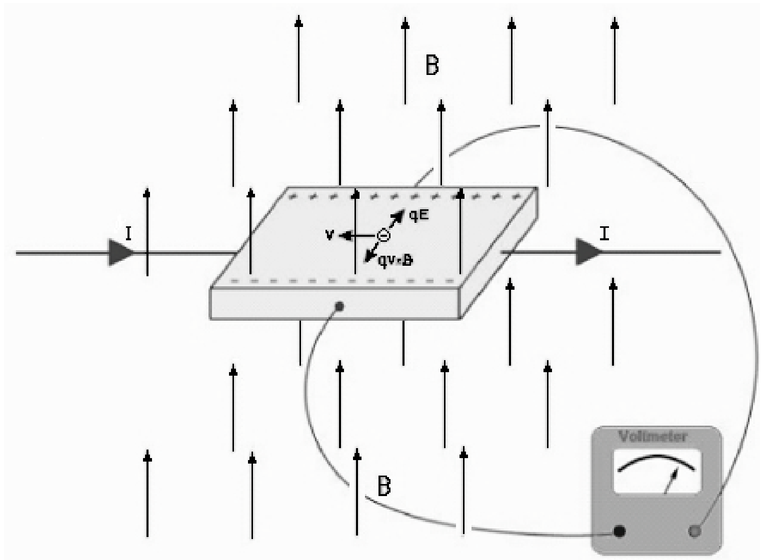

Figure 2.1: The classical Hall effect in electromagnetism.

2.1 Quantum Hall effect

The quantum Hall effect was discovered 1980 by von Klitzing, Dorda and Dr. Pepper who were investigating the scattering of electrons in the interface of two thin layers of silicon and silicon oxide, see [57]. Von Klitzing was awarded the Nobel prize in physics 1985 for the discovery of the quantum Hall effect. This effect is of interest to us because of a rather intrinsic relation with index theory due to an argument of Laughlin.

The name quantum Hall effect originates from the similarity with the classical counterpart in electromagnetism. The Hall effect in electromagnetism was discovered by Hall in 1879 and occurs when a constant magnetic field goes perpendicularly through a flat conductor in which a current flows which induce a current perpendicular to the original current, see Figure 2.1. The induced current is called the Hall current. If the strength of the magnetic field is given by B, the conductor is of width d and the current is I the Hall current can be expressed as $I_H = R_H \cdot B \cdot I/d$ where R_H denotes the Hall coefficient which can be calculated from the formula $R_H = -1/ne$ and e denotes the charge of an electron and n is the charge carrier density.

The quantum Hall effect works in a similar fashion as the classical Hall effect except the unexpected phenomena that the Hall coefficient R_H quantizes

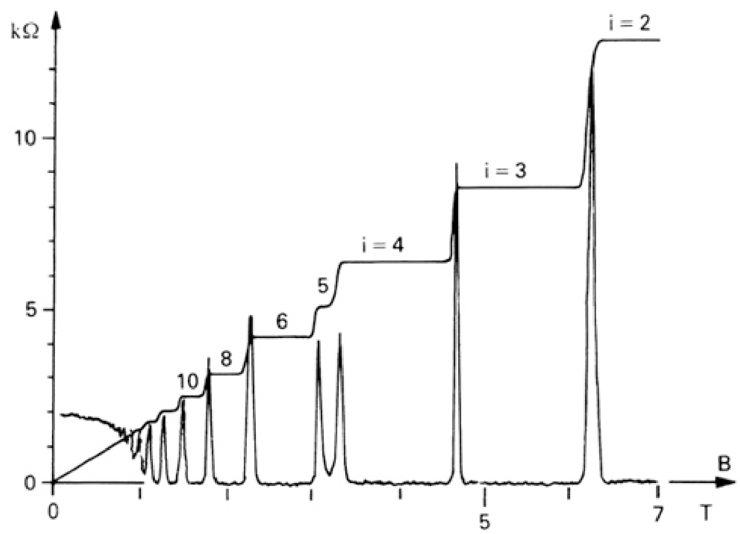

Figure 2.2: The integer quantum Hall effect.

and can only take certain values. The number $\sigma := 1/R_H$ is called the Hall conductance and $\sigma = \nu e^2/h$ where ν is called the filling factor and h denotes Planck's constant. In certain geometries, for a more thorough presentation see [60], the filling factor is quantized to integer values, see Figure 2.2. This effect is called the integer quantum Hall effect ($IQHE$). The measurements of the $IQHE$ is *extremely* accurate, up to parts in 10^5, see [57]. The theoretical explanation for this accuracy is due to Laughlin's gauge argument. The gauge argument together with results of Avron-Seiler-Simon relates the $IQHE$ to index theory of Toeplitz operators.

A much more complicated situation can occur when ν is quantized to fractional values, which is called the fractional quantum Hall effect ($FQHE$). The simple heuristic explanation for this is that a complicated sample can contain many type of composite fermions, see [20] and [67], and the filling factor is an average over the different composite particles. The $FQHE$ is not as well understood as $IQHE$ and some theories are still rather speculative.

Laughlin's "gauge argument", for which we refer the reader to [59] and [60], reduces the $IQHE$ to considering the case of a pure sample, with no impurities. So the situation can be seen in figure 2.3 where the current through a periodic sample and the magnetic field makes one particle "jump" from one side of the sample to the other. The number of jumping particles will be the relative

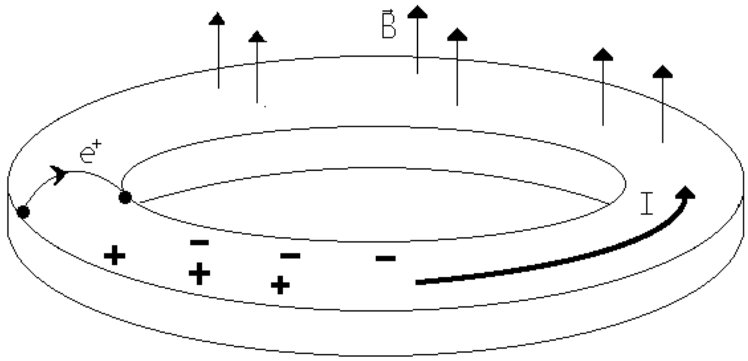

Figure 2.3: An electron jumping over the periodic rod.

index of the projection onto the state space before (respectively after) the gauge transformation, for the definition of a relative index see below in equation (3.5). The exactness of the $IQHE$ is described by Laughlin in [60] as:

> "*The quantum Hall effect does not measure any quantum of surface charge density, because there is no such quantum. It measures instead the number of electrons transferred in a thought experiment. It measures* e. That *is why it is so accurate.*"

The ideas of Laughlin were put onto firm mathematical ground by Bellissard, see [15] and [16] or [34] for a survey of Bellissard's results. Bellissard used Connes' framework of non-commutative geometry and showed that Kubo's formula, which physicists use to calculate the Hall conductance, defines a cyclic cohomology class on a non-commutative Brillouin zone and the Hall conductance was simply an integral pairing with K-theory.

There is a profound relation between the $IQHE$ and index theory. Even though the gauge argument was not stated as an index problem by Laughlin, the view on Laughlin's ideas as an operator index was further developed in [11]. In the paper [11], the idea that the Hall conductance can be calculated as a "jump" in Laughlin's thought experiment was given mathematical meaning. Let U be the unitary operator implementing the gauge transformation in Laughlin's thought experiment and let P denote the projection onto the subspace with energy below the Fermi energy, which we assume to lie in a spectral gap. If it is the case that P commutes with U up to a compact operator the relative index of P to U^*PU, i.e. the number of particles after applying U relative to the number of particles before, is finite and equals the index of the Toeplitz operator PUP. In a pure sample the system is modeled by the Landau Hamiltonian which describes the energy of particles moving in a constant magnetic field.

2.2 Hall conductance

Below, in Paper A, we consider the case of n types of non-interacting particles. This is described by the Landau hamiltonian in $\mathbb{C}^n$, an elliptic second order operator with discrete spectrum and lowest level being the Fock space. Its eigenspaces are called Landau levels. We use the notation $\mathscr{L}^\ell$ for the ℓ:th Landau level and P_ℓ denotes the corresponding orthogonal projection. We show that the projection P_ℓ commutes up to compact operators with continuous functions that have radial limits. A Toeplitz operator whose symbol has compact support is shown to be compact so we obtain an odd K-homology class on S^{2n-1} for each Landau level.

The charge deficiency of such a projection is defined to be the K-homology class $[P_\ell] \in K_1(S^{2n-1})$. Originally, the charge deficiency of a projection P was defined in [12] in dimension $n = 1$ to be the integer $\mathrm{ind}\,(PuP)$, where $u \in C(S^1, S^1)$, defined as $u(z) = z$, generates $K^1(S^1)$. Because of the universal coefficient theorem for KK-theory of [75], the class $[P] \in K_1(S^1)$ is determined by the integer $\mathrm{ind}\,(PuP)$. Thus we can equally well speak of the charge deficiency as the more geometrically significant K-homology class that the projection defines.

For higher dimensions, the group $K_1(S^{2n-1})$ is again free of rank 1, this follows from the calculation $K^1(S^{2n-1}) = \mathbb{Z}$ above and the universal coefficient theorem. To calculate the K-homology class that a Landau level defines, it is really sufficient to calculate one integer, namely the index of a Toeplitz operator with a K-theoretically non-trivial symbol. This integer will be the charge deficiency of [12] for higher dimensions. The calculation of the charge deficiency immediately gives the Hall conductance of the system. We obtain the following index theorem:

Theorem 1. *If $a : \mathbb{C}^n \to M_N(\mathbb{C})$ has a smooth radial limit function $a_\partial : S^{2n-1} \to GL_N(\mathbb{C})$, the index of $P_\ell a|_{\mathscr{L}^\ell \otimes \mathbb{C}^N}$ can be expressed as*

$$\mathrm{ind}\,(P_\ell a|_{\mathscr{L}^\ell \otimes \mathbb{C}^N}) = \frac{-(\ell + n - 1)!}{\ell!(2n-1)!(2\pi i)^n} \int_{S^{2n-1}} \mathrm{tr}((a_\partial^{-1} \mathrm{d} a_\partial)^{2n-1}).$$

The charge deficiency $[P_\ell] \in K_1(S^{2n-1})$ may be expressed in terms of the Bergman projection P_B on the unit ball, the generator of $K_1(S^{2n-1})$, as

$$[P_\ell] = \frac{(\ell + n - 1)!}{\ell!(n-1)!}[P_B]. \tag{2.1}$$

For a symbol a whose radial limit a_∂ is not smooth, but merely Hölder continuous, we can use the techniques of Paper B to write down explicit index formulas for the operator $P_\ell a P_\ell$. The explicit index formula can be found in equation (3.15) below. This fact is due to (2.1) which implies that the Toeplitz quantization of the Landau levels is equivalent to a multiple of the Bergman quantization on the ball which in turn is equivalent to the quantization on the Szegö space of the sphere. On the latter quantization the techniques of Paper B can be applied directly to Hölder continuous symbols.

An interesting corollary of this index theorem is an index theorem for the Fock space. It has quite recently come to the author's attention that the index theorem for Toeplitz operators on the Fock space was already proven in [31]. Recall that the Fock space $\mathscr{F}(\mathbb{C}^n)$ is the Hilbert space of holomorphic functions in $L^2(\mathbb{C}^n, \mathrm{e}^{-|z|^2/2})$. Multiplication by $\mathrm{e}^{-|z|^2/4}$ gives an isomorphism $\mathscr{F}(\mathbb{C}^n) \cong \mathscr{L}^0$ that commutes with the $C_b(\mathbb{C}^n)$-action. If we let $P_{\mathscr{F}} : L^2(\mathbb{C}^n, \mathrm{e}^{-|z|^2/2}) \to \mathscr{F}(\mathbb{C}^n)$ denote the orthogonal projection, then the Theorem above, or Corollary 1 of [31], implies that $[P_{\mathscr{F}}] = [P_B]$ in $K_1(S^{2n-1})$.

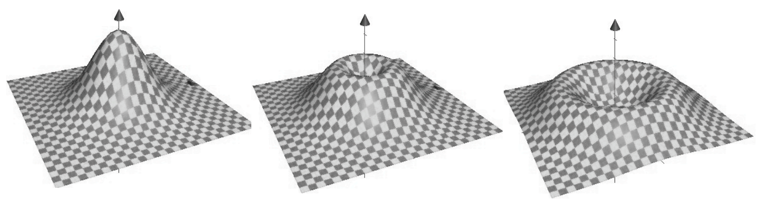

Figure 2.4: The absolute value of some states in the first Landau level.

The case of interacting particles is rather complicated and is expected to be an explanation for the fractional quantum Hall effect ($FQHE$). The $FQHE$ occurs in more complicated materials, for instance it has been shown to occur in graphene [88]. Stormer, Laughlin and Tsui were awarded the Nobel prize in physics 1998 for work on the $FQHE$. See [84] for a nice review of the subject.

We will not discuss the $FQHE$ in that much detail but just mention that there is some work going on, trying to explain the $FQHE$ as an index. The $FQHE$ has been verified by Karlhede and Soursa [55] as a topological effect, in fact a kind of average. Being a topological effect, it is to be expected that it is an index. Marcolli and Mathai proposed a model for the $FQHE$ in [61] and [62] based on a 2-dimensional hyperbolic geometry. They worked on an orbifold and the fractional effect came from cuspidal points. Surprisingly their model made the prediction that the the fractional Hall conductance always is bounded from below by $1/42$, a statement that has yet to be experimentally verified.

An interesting problem to study further is if there is some more general theory for topological invariants associated with the spectral projection of Schrödinger operators. The problem has been studied from a more spectral theoretic approach in [63] and [81]. The general formulation of the question is that if H_E is the Laplacian associated to a connection on a vector bundle $E \to M$, is it possible to find some non-trivial topological characteristics associated with the spectral projections of H_E?

The theorem above from Paper A implies that this question has a positive answer even for a compact subset of the spectrum of a Schrödinger operator on the trivial bundle over a euclidian space. This indicates both that the answers can be very interesting and that it is a hard problem. In the general setting there is a problem already on operator level since the spectrum of a differential operator on a non-compact manifold does not need to have spectral gaps. Another problem is geometric and can be seen from the euclidian case, all topological properties of the spectral projections might not be seen inside the manifold but rather on a compactification.

Chapter 3

Index formulas

"Think globally, act locally."
Lennon

As we have seen above, it is of interest to calculate the index pairing. It can be calculated in applications if one knows some additional structure, but there is no general solution for how to calculate the index pairing. Some formulas hold in a rather general setting but are in reality much harder to calculate. To quote Erdös: *"Problems worthy of attack prove their worth by fighting back."* The earliest example of an index theorem was of course the Fredholm alternative theorem from 1903, see [42], stating that the index of a compact perturbation of the identity operator is a Fredholm operator of index 0. The first index theorem that related the index with some geometric properties came two decades later in [68], where the index of a singular integral operator on S^1 was calculated as the symbol's winding number. It was in this paper that the term index was coined.

In the setting of pseudo-differential operators, or more generally Toeplitz operators, there are some further geometric structures allowing for index theorems such as Boutet de Monvel's index theorem and the Atiyah-Singer index theorem. Let $\mathscr{T}$ denote the C^*-algebra closure of the algebra of classical pseudo-differential operators on a manifold M or the C^*-algebra of Toeplitz operators on the boundary $\partial\Omega$ of a strictly pseudo-convex domain Ω. The C^*-algebra $\mathscr{T}$ contains the ideal of compact operators and the quotient $\mathscr{T}/\mathscr{K}$ is the commutative C^*-algebra $C(X)$, where X is the cotangent sphere of M in the case of pseudo-differential operators and $\partial\Omega$ in the case of Toeplitz operators. While the Fredholm property and the index are determined by the image in the Calkin algebra, an operator $T \in \mathscr{T}$ is Fredholm if and only if the symbol $\sigma(T)$ is invertible and the index of T ought to be calculatable from the continuous function $\sigma(T)$. The question of how to perform this calculation was posed by Gel'fand [43].

Only years after Gel'fand's question, Atiyah and Singer announced their celebrated index theorem in [4]–[9], opening for generalizations in many directions. The first statement of the Atiyah-Singer index theorem holds for an elliptic differential operator between smooth vector bundles with smooth symbol on a smooth closed manifold. There has been a considerable work done since the first statement to loosen every restriction, some of them where successful and some problems remain.

In this thesis we mainly focus on generalizations of Atiyah-Singer type index theorems that loosens the regularity conditions on the symbols to Hölder continuity. In Paper B we loosen the regularity condition on the symbols of Toeplitz operators on the boundary of a strictly pseudo-convex domain in a Stein manifold. We obtain analytic index formulas and apply them to calculations of mapping degree. We will also loosen the regularity on the vector bundle in the calculation of the index pairing (1.4) in Paper C. We obtain analytic index formulas for classical zero order pseudo-differential operators twisted by a Hölder continuous vector bundle. Again we apply this index formula to the problem of calculating degrees of non-smooth mappings.

3.1 Atiyah-Singer's index theorem

The first proof of the Atiyah-Singer index theorem was by means of a rather topological approach using K-theory in [5]. There has appeared a large variety of proofs since, using for instance heat kernel methods, E-theory or groupoids. Below we sketch the argument of the K-theoretical proof of the index theorem from [5] and how this produces the index formula (1) in de Rham cohomology. The full proof of this cohomological form can be found in [7]. In the end of this section we will also recall the proof of Hirzebruch's signature theorem using the Atiyah-Singer index theorem.

Assume that X is a closed manifold and let $\pi : T^*X \to X$ denote the cotangent bundle. An elliptic pseudo-differential operator D on X between two vector bundles E_1 and E_2 has a symbol $\sigma(D) \in C^\infty(T^*X, \mathrm{Hom}(\pi^*E_1, \pi^*E_2))$ that takes values in the bundle of invertible morphisms $\pi^*E_1 \to \pi^*E_2$ outside a compact subset of T^*X. So there is an associated difference class $[D] \in K^0_c(T^*X)$. While D is elliptic the Fredholm index $\mathrm{ind}(D)$ is well defined and due to homotopy invariance of the index it does in fact only depend on $[D]$. The symbol mapping is a surjection, at least on K-theory, so the analytic index $[D] \mapsto \mathrm{ind}(D)$ defines a mapping $\mathrm{ind}_a : K^0_c(T^*X) \to \mathbb{Z}$.

On the other hand if we choose a smooth embedding $X \to \mathbb{R}^N$ we obtain a smooth embedding $i : T^*X \to \mathbb{R}^{2N}$ that defines a push-forward $i_! : K_c^0(T^*X) \to K_c^0(\mathbb{R}^{2N})$. The push-forward is well defined since T^*X and $\mathbb{R}^{2N}$ have canonical spin^c-structures. Using the Bott periodicity $K_c^0(\mathbb{R}^{2N}) \cong K^0(pt) = \mathbb{Z}$, we obtain a mapping $K_c^0(T^*X) \to \mathbb{Z}$. The topological index ind_t is defined as $i_!$ and can be shown to be independent of the choice of i. In particular, since the push-forward is functorial, the Riemann-Roch theorem for push-forwards implies that

$$\begin{aligned}\mathrm{ind}_t[D] &= \int_{\mathbb{R}^{2N}} \mathrm{ch}(i_![D]) \wedge \pi^* Td(\mathbb{R}^N) = \\ &= \int_{\mathbb{R}^{2N}} i_*(\mathrm{ch}[D] \wedge \pi^* Td(X)) = \int_{T^*X} \mathrm{ch}[D] \wedge \pi^* Td(X).\end{aligned}$$

For a more detailed description see [7].

The K-theoretical version of the Atiyah-Singer index theorem states that for any manifold X, $\mathrm{ind}_a = \mathrm{ind}_t$. The main idea in the proof in [5] consists in using the concept of an index mapping. An index mapping is a natural mapping $\mathrm{ind}_X : K_c^0(T^*X) \to \mathbb{Z}$ with respect to push-forward, normalized in such a way that $\mathrm{ind}_{\mathbb{R}^n}$ coincides with Bott periodicity. The proof in [5] is completed by proving the three statements that the topological index mapping is an index mapping, the analytic index mapping is an index mapping and finally that an index mapping is unique. This statement is of topological nature since it also holds for abstract elliptic operators on topological manifolds, see [87].

As an example of an application of the Atiyah-Singer index theorem, let us consider a standard example of a differential operator; the signature operator on a closed $2n$-dimensional manifold X taken from section 6 of [7]. We will use the signature operator for deriving degree formulas for Hölder continuous functions between even-dimensional manifolds in Paper C. Equip X with a Riemannian metric. The Hodge grading τ is an involution on the exterior algebra of the complexified cotangent bundle $\bigwedge^*(T^*X \otimes \mathbb{C}) \to X$ defined on a p-form ω by

$$\tau\omega = i^{p(p-1)+n} * \omega,$$

where $*$ denotes the Hodge duality. That τ is an involution follows from that $*^2\omega = (-1)^p\omega$. We let E_+ denote the sub-bundle of $\bigwedge^*(T^*X \otimes \mathbb{C})$ consisting of even vectors with respect to the Hodge grading and E_- the sub-bundle of odd vectors with respect to the Hodge grading. The operator τ anti-commutes with the Hodge-de Rham operator $\mathrm{d} + \mathrm{d}^*$ so $D = \mathrm{d} + \mathrm{d}^*$ is a well defined operator from E_+ to E_-. The symbol $\sigma(D) \in C^\infty(T^*X, \pi^*\mathrm{Hom}(E_+, E_-))$ is given by $\sigma(D)(x,\xi) = \xi \wedge + \xi \neg$ in orthonormal coordinates. As is calculated in section 6 of [7] cohomology class $\mathrm{ch}[D]$ is mapped to $2^n \pi^*(L(TX) \wedge Td(X)^{-1})$ under the Thom isomorphism, here L is the genus associated with the function $\frac{x/2}{\tanh(x/2)}$.

The operator $\tilde{D}$ associated with D as in equation (1.1) is an odd self-adjoint first-order differential operator on the graded vector bundle $E_+ \oplus E_-$. The operator $\tilde{D}$ coincides with the usual Hodge-de Rham operator on $\bigwedge^*(T^*X \otimes \mathbb{C}) \to X$. By Hodge's theorem there is a graded isomorphism

$$\ker \tilde{D} \cong H^*_{dR}(X) = H^+_{dR}(X) \oplus H^-_{dR}(X).$$

Note that the grading we are using on $H^*_{dR}(X)$ is that induced from τ. Define the τ-invariant linear space $\mathfrak{H}^k := H^{n+k}_{dR}(X) \oplus H^{n-k}_{dR}(X)$ and the operator $\varepsilon_k := 1 \oplus (-1)$ on $\mathfrak{H}^k$. It is straight-forward to verify that $\tau\varepsilon_k + \varepsilon_k\tau = 0$ when $k > 0$ and therefore

$$\dim(\mathfrak{H}^k \cap H^+_{dR}(X)) = \dim(\mathfrak{H}^k \cap H^-_{dR}(X)), \quad k > 0.$$

Hence, the index of the signature operator is given by

$$\operatorname{ind}(D) = \dim_{\mathbb{C}}(H^n_{dR}(X) \cap H^+_{dR}(X)) - \dim_{\mathbb{C}}(H^n_{dR}(X) \cap H^-_{dR}(X)).$$

But Hodge's theorem implies that we can represent elements of $H^*_{dR}(X)$ by harmonic forms so we have that $\ker \tilde{D} \cong \left(H^+_{dR}(X, \mathbb{R}) \oplus H^-_{dR}(X, \mathbb{R})\right) \otimes \mathbb{C}$. Let us denote

$$\mathfrak{H}^+ := H^n_{dR}(X, \mathbb{R}) \cap H^+_{dR}(X, \mathbb{R}) \quad \text{and} \quad \mathfrak{H}^- := H^n_{dR}(X, \mathbb{R}) \cap H^-_{dR}(X, \mathbb{R}).$$

In this notation, the index of D can be written as

$$\operatorname{ind}(D) = \dim_{\mathbb{R}} \mathfrak{H}^+ - \dim_{\mathbb{R}} \mathfrak{H}^-,$$

This later quantity is in fact the signature of the integral pairing on the real vector space $H^n_{dR}(X, \mathbb{R})$. The integral pairing is the bilinear mapping defined by

$$(\alpha, \beta) := \int_X \alpha \wedge \beta,$$

and its signature $\operatorname{sign}(X)$ is known as the signature of X. Thus we come to the well-known conclusion that $\operatorname{ind}(D) = \operatorname{sign}(X)$. The Atiyah-Singer theorem implies Hirzebruch's signature formula

$$\operatorname{sign}(X) = 2^n \int_X L(TX).$$

In particular, if $E \to X$ is a real Riemannian vector bundle and we set $\mathfrak{H}(E) := H^n_{dR}(X, E)$, the Atiyah-Singer index theorem for the twisted signature operator on $\bigwedge^*(T^*X \otimes \mathbb{C}) \otimes_{\mathbb{R}} E$ implies that

$$\operatorname{sign}(\mathfrak{H}(E) \times \mathfrak{H}(E) \to \mathbb{R}) = 2^n \int_X L(TX) \wedge \operatorname{ch}[E \otimes \mathbb{C}].$$

3.2 Cyclic homology

There are some rather general index formulas, even though they are in general quite hard to make useful. We will start with reviewing the most general formula, the Calderon index formula, which was the starting point for Connes' index formula in cyclic cohomology. Due to the non-degeneracy of the index pairing, abstract isomorphisms reduce the index formula of Connes to the manageable index formula of Atiyah-Singer in the special case of pseudo-differential operators and the index formula of Boutet de Monvel in the case of Toeplitz operators.

Let $T : \mathscr{H} \to \mathscr{H}$ be a bounded linear operator. We observe that if T has finite-dimensional kernel, then T is Fredholm if and only if the vector space quotient $\mathscr{H}/T\mathscr{H}$ is finite-dimensional. A Fredholm operator is closed so the vector space $\mathscr{H}/T\mathscr{H} \cong \ker T^*$ is of finite dimension if T is Fredholm. Conversely, if $\mathscr{H}/T\mathscr{H}$ is of finite dimension, let us say N-dimensional, then there is a linear mapping $Y : \mathbb{C}^N \to \mathscr{H}$ such that $T \oplus Y : \mathscr{H} \oplus \mathbb{C}^N \to \mathscr{H}$ is surjective and the open mapping theorem implies that T is closed since $\mathscr{H}$ is of finite codimension in $\mathscr{H} \oplus \mathbb{C}^N$. Therefore T is closed and $\ker T^* \cong \mathscr{H}/T\mathscr{H}$ is of finite dimension.

Henceforth, we assume that T is Fredholm and let P_T and P_T' denote the orthogonal projections onto the kernel of T respectively the cokernel of T. The simplest form of an index formula for T is that $\mathrm{ind}(T) = \mathrm{tr}(P_T) - \mathrm{tr}(P_T')$. In general it is very hard to find P_T and P_T' but it is in applications possible to find good approximations. We will call an operator $T_0 \in \mathscr{B}(\mathscr{H})$ such that $1 - TT_0, 1 - T_0T \in \mathscr{K}(\mathscr{H})$ a parametrix for T. If T is a pseudo-differential operator, a parametrix in the usual sense will also be a parametrix in this abstract operator setting. There is in fact a parametrix T_0 such that $1-TT_0 = P_T'$ and $1 - T_0T = P_T$. In this case we have that

$$\mathrm{ind}(T) = \mathrm{tr}(1 - T_0T)^k - \mathrm{tr}(1 - TT_0)^k \tag{3.1}$$

for any positive integer k. The content of the Calderon index formula is that (3.1) holds for *any* parametrix T_0 such that $1 - T_0T, 1 - TT_0 \in \mathscr{L}^q(\mathscr{H})$ where $q \leq k$. Here $\mathscr{L}^q(\mathscr{H})$ denotes the dense ideal in $\mathscr{K}(\mathscr{H})$ of Schatten class operators of order q. Let us shortly recall the proof of this formula from [39]. If R_0 is a parametrix for T with $1 - TR_0, 1 - R_0T \in \mathscr{L}^1(\mathscr{H})$ then $R_0 - T_0 \in \mathscr{L}^1(\mathscr{H})$ since the classes of R_0 and T_0 in the algebra $\mathscr{B}(\mathscr{H})/\mathscr{L}^1(\mathscr{H})$ are both multiplicative inverses to the class of T. The trace is cyclic so the statement holds for $q = 1$. In general, if R_0 is a parametrix for T with $1 - TR_0, 1 - R_0T \in \mathscr{L}^q(\mathscr{H})$ we set

$$R := \sum_{j=0}^{k} (1 - R_0T)^j R_0. \tag{3.2}$$

In this case $1 - RT = (1 - R_0T)^k \in \mathscr{L}^1(\mathscr{H})$ and similarly $1 - TR = (1 - TR_0)^k \in \mathscr{L}^1(\mathscr{H})$ which imply that (3.1) holds for any summable parametrix.

Suppose that $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ is a graded Hilbert space, F is an odd operator satisfying that $F^2 = 1$ and p is an even projection such that $[F, p]$ is Schatten class for some $q > 1$. We will use the standard notation str for the supertrace, that is $\mathrm{str}(K) = \mathrm{tr}(\gamma K)$ where γ denotes the grading. Just as in the index pairing we consider the operator $p_+F_+p_- : p_-\mathscr{H}_- \to p_+\mathscr{H}_+$. Observe that

$$[F,p] = \begin{pmatrix} 0 & F_+p_- - p_+F_+ \\ F_-p_+ - p_-F_- & 0 \end{pmatrix}.$$

Since $[F, p]$ is compact, being in the Schatten class, $p_+F_+p_-$ is Fredholm and using the assumption $[F, p] \in \mathscr{L}^q(\mathscr{H})$ the Calderon index formula implies that

$$\mathrm{ind}\,(p_+F_+p_- : p_-\mathscr{H}_- \to p_+\mathscr{H}_+) = (-1)^k\, \mathrm{str}_{\mathscr{H}}(p[F,p]^{2k}), \tag{3.3}$$

when $2k \geq q$. The last equality is due to Connes and is proved purely algebraically, see Proposition 4 in Chapter IV.1.γ of [34]. This situation allows for calculations of the index pairing between the even K-theory and the even K-homology *if* we can represent the K-homology class by an analytic K-cycle (π, F) and the K-theory class by a p such that $[F, \pi(p)]$ is in the Schatten class.

The analogous setting in the odd case is an operator of the form PUP where P is an orthogonal projection and U is an invertible operator such that $[P, U]$ is Schatten class of some order q. The operator PUP is Fredholm because the commutator $[P, U]$ is compact, so $PU^{-1}P$ is a parametrix to PUP. The assumption $[P, U] \in \mathscr{L}^q(\mathscr{H})$ and the Calderon index formula imply that

$$\mathrm{ind}\,(PUP : P\mathscr{H} \to P\mathscr{H}) = \mathrm{tr}_{\mathscr{H}}(U^{-1}\underbrace{[P,U][P,U^{-1}]\cdots[P,U^{-1}][P,U]}_{2k+1 \quad \text{factors}}), \tag{3.4}$$

when $2k+1 \geq q$. Observe that $[P, U^{-1}] = -U^{-1}[P, U]U^{-1}$ so the right-hand side is well defined. The formula (3.4) is closely related to the formula from [12] for the relative index of two projections. The relative dimension between two projections $P, Q \in \mathscr{B}(\mathscr{H})$ satisfying that $P - Q \in \mathscr{K}(\mathscr{H})$ is defined as

$$\mathrm{ind}\,(P, Q) := \dim(\ker(1-(P-Q))) - \dim(\ker(1+(P-Q))). \tag{3.5}$$

Since $P-Q$ is compact, the operators $1 \pm (P-Q)$ are Fredholm and therefore the right-hand side is well defined. If $P - Q$ is in some Schatten class, $\mathrm{ind}\,(P, Q) = \mathrm{tr}(P-Q)^{2k+1}$ whenever the right-hand side is finite by Proposition 2.2 of [12]. Furthermore, if $Q = U^*PU$ then $\mathrm{ind}\,(P, Q) = \mathrm{ind}\,(PUP)$ by Proposition 2.4 of [12].

The index formulas (3.3) and (3.4) can be used to describe the index pairing. The correct setting for this is to consider dense isoradial $*$-subalgebras $\mathscr{A} \subseteq A$, so $K_*(\mathscr{A}) \cong K_*(A)$ via the embedding. Furthermore, we want to represent our K-homology classes by analytic K-cycles (π, F) such that

$$F^2 - 1, F^* - F \in \mathscr{L}^q(\mathscr{H}) \quad \text{and} \quad [F, \pi(a)] \in \mathscr{L}^q(\mathscr{H})\ \forall a \in \mathscr{A}.$$

Such an analytic K-cycle (π, F) is called a q-summable Fredholm module over $\mathscr{A}$. There are many regularity notions on a Fredholm module, such as θ-summability and analagous notions over general semifinite von Neumann algebras, see Chapter IV of [34]. We will focus on the simplest case of q-summability.

It is in general unclear whether a K-homology class can be represented by a K-cycle that is q-summable over a dense subalgebra. For instance, for a q-summable *unbounded* Fredholm module to exist over A there must be a tracial state on A by [33]. However, for a q-summable Fredholm module over a dense isoradial subalgebra the isomorphism $K_*(\mathscr{A}) = K_*(A)$ allows us to calculate the index pairing between $K_*(A)$ and such an analytic K-cycle. The classical setting of this construction is that $\mathscr{A} = C^\infty(X)$ and the analytic K-cycle comes from some elliptic pseudo-differential operator, in this case *any* $q > \dim(X)$ will make the associated Fredholm module q-summable. The corresponding index formula reduces to a calculation in de Rham cohomology via the Chern character of the symbol. In this geometric setting, the q-summability condition is much weaker than most smoothness conditions and often requires less regularity.

With a q-summable Fredholm module (π, F) and $k \geq q$ we can associate a linear functional $\mathrm{cc}_k(\pi, F)$ on $\mathscr{A}^{\otimes k+1}$ where we require k to have the same parity as (π, F). Here we use choose the projective tensor product. Following Definition 3 of Chapter IV.1 of [34] we define the linear functional $\mathrm{cc}_k(\pi, F)$ as

$$\mathrm{cc}_k(\pi, F)(a_0, a_1, \ldots, a_k) := \begin{cases} c_k \mathrm{str}_{\mathscr{H}}(a_0[F, a_1][F, a_1]\cdots[F, a_k]) & k \text{ even}, \\ c_k \mathrm{tr}_{\mathscr{H}}(a_0[F, a_1][F, a_2]\cdots[F, a_k]) & k \text{ odd}. \end{cases} \tag{3.6}$$

The constants c_k are introduced as a certain dimensional normalization which play a certain role that will be explained below in equation (3.13). The constants c_k are defined as

$$c_k := \begin{cases} (-1)^{k(k-1)/2}\left(\frac{k}{2}\right)! & k \text{ even}, \\ (-1)^{k(k-1)/2}\sqrt{2i}2^k\Gamma\left(\frac{k}{2}+1\right) & k \text{ odd}. \end{cases} \tag{3.7}$$

With the index formulas (3.3), (3.4) and (3.6) in mind we define a "Chern character" by constructing $\mathrm{ch}_k : \{p \in M_\infty(\mathscr{A}) : p^2 = p\} \to \mathscr{A}^{\otimes k+1}$ for k even respectively $\mathrm{ch}_k : GL_\infty(\mathscr{A}) \to \mathscr{A}^{\otimes k+1}$ for k odd as follows

$$\mathrm{ch}_k[p] := d_k \mathrm{tr}_{M_\infty(\mathbb{C})}\left(p^{\otimes_{M_\infty(\mathbb{C})} k+1}\right), \tag{3.8}$$

$$\mathrm{ch}_k[u] := d_k \mathrm{tr}_{M_\infty(\mathbb{C})}\left(\left((u^{-1}-1)\otimes_{M_\infty(\mathbb{C})}(u-1)\right)^{\otimes_{M_\infty(\mathbb{C})}(k+1)/2}\right). \tag{3.9}$$

Here $d_k := (-1)^{k(k-1)/2}c_k^{-1}$. In this formalism we can calculate the index pairing

$$x \circ [\pi, F] = \mathrm{cc}_k(\pi, F)(\mathrm{ch}_k(x)) \tag{3.10}$$

whenever (π, F) is q-summable over $\mathscr{A}$. The element $\mathrm{ch}_k(x) \in \mathscr{A}^{\otimes k+1}$ depends a priori on the choice of representative for $x \in K_*(\mathscr{A})$. To solve this problem and set the stage for index formulas, Connes introduced cyclic homology.

We will recall Connes' original definition of cyclic homology following Chapter III of [34]. The construction above of the Chern character and the Chern-Connes character (3.8) and (3.9) fits directly into this formulation of the cyclic theories. After that we will give the definition of cyclic homology in terms of the universal differential algebra following [36]. In this formulation one can rather easily define the periodicity operator, a vital tool in the *regularization* of index formulas that leads to unusual index formulas.

We will let $\mathscr{A}$ denote a unital topological algebra and as above we denote the projective tensor product by $\otimes$. The Hochschild differential $b : \mathscr{A}^{\otimes k} \to \mathscr{A}^{\otimes k-1}$ is defined by

$$b(x_0 \otimes x_1 \otimes \cdots \otimes x_k \otimes x_{k+1}) := (-1)^{k+1} x_{k+1} x_0 \otimes x_1 \otimes \cdots \otimes x_k + \\ + \sum_{j=0}^{k} (-1)^j x_0 \otimes \cdots \otimes x_{j-1} \otimes x_j x_{j+1} \otimes x_{j+2} \otimes \cdots \otimes x_{k+1}.$$

The cyclic permutation operator $\lambda : \mathscr{A}^{\otimes k} \to \mathscr{A}^{\otimes k}$ is defined as

$$\lambda(x_0 \otimes x_1 \otimes \cdot \otimes x_k) = (-1)^k x_k \otimes x_0 \otimes \cdots \otimes x_{k-1}.$$

We define a complex of $\mathbb{C}$-vector spaces $C_*^\lambda(\mathscr{A})$ by

$$C_k^\lambda(\mathscr{A}) := \mathscr{A}^{\otimes k+1}/(1-\lambda)\mathscr{A}^{\otimes k+1},$$

with differential given by b. The homology of the complex $C_*^\lambda(\mathscr{A})$ is called the cyclic homology of $\mathscr{A}$ and will be denoted by $HC_*(\mathscr{A})$. A cycle in $C_\lambda^k(\mathscr{A})$ will be called a cyclic k-cycle. It can be verified that for any representative of x and any k of the right parity, the element $\mathrm{ch}_k(x)$ defines a cyclic k-cycle. As it is shown in [32], the Chern character $\mathrm{ch}_k : K_0(\mathscr{A}) \to HC_k(\mathscr{A})$ and $\mathrm{ch}_k : K_1(\mathscr{A}) \to HC_k(\mathscr{A})$ are well defined whenever k is of right parity.

The complex $C_\lambda^k(\mathscr{A})$ is defined as the space of continuous linear functionals μ on $\mathscr{A}^{\otimes k+1}$ such that $\mu \circ \lambda = \mu$. The Hochschild coboundary operator $\mu \mapsto \mu \circ b$ makes $C_\lambda^*(\mathscr{A})$ into a complex. The cohomology of the complex $C_\lambda^*(\mathscr{A})$ will be denoted by $HC^*(\mathscr{A})$ and is called the cyclic cohomology of $\mathscr{A}$. For a q-summable Fredholm module (π, F) the linear functional $\mathrm{cc}_k(\pi, F)$ is a well defined cyclic k-cocycle for any $k > q$.

Let us return to the abstract K-theory setting for dealing with index problems for abstract Toeplitz operators. So we assume that (π, F) is an odd analytic K-cycle (π, F) on the C^*-algebra A. We will also make the assumptions that $F^2 = 1$ and that there is a dense isoradial subalgebra $\mathscr{A} \subseteq A$ for which (π, F) is finitely summable. If $u \in GL_N(\mathscr{A})$ we do on the one hand have the index formula (3.10) for $P_F \pi(u) P_F$ but on the other hand the index coming from the abstract formula (1.9). These coincide, not merely because they both are equal to the same index, but because of the relations between the Calderon index formula (3.1) and the form of the index mapping in (1.5). We can by equation (1.5) and (1.6) represent the class $\partial[u] \in K_0(\mathscr{K})$ by the formal difference

$$\begin{bmatrix} -(P_F - P_F u P_F R)^2 + 1 & P_F u P_F (P_F - R P_F u P_F)^2 \\ (P_F - R P_F u P_F) R & (P_F - R P_F u P_F)^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where R is constructed from $P_F u^{-1} P_F$ as in equation (3.2) for some large k. Since the isomorphism $K_0(\mathscr{K}) \cong \mathbb{Z}$ is defined from the trace, the class $\partial[u]$ is mapped to the integer

$$\mathrm{tr}(\partial[u]) = \mathrm{tr}\left(\begin{pmatrix} -(P_F - P_F u P_F R)^2 + 1 & P_F u P_F (P_F - R P_F u P_F)^2 \\ (P_F - R P_F u P_F) R & (P_F - R P_F u P_F)^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) =$$

$$= \mathrm{tr}(P_F - P_F u^{-1} P_F u P_F)^{2k} - \mathrm{tr}(P_F - P_F u P_F u^{-1} P_F)^{2k} = \mathrm{cc}_k(\pi, F)(\mathrm{ch}_k(u)).$$

By a result of Connes-Cuntz [35] any cyclic cocycle arise as the Connes-Chern character of a summable *semi-finite* Fredholm module, so the algebra of bounded operators is replaced by an arbitrary semi-finite von Neumann algebra and the summability is defined with respect to a trace in the von Neumann algebra. Even though cyclic homology and cyclic cohomology are algebraically defined, these theories really consists of analytic structures. In general, cyclic homology behaves badly if we take $\mathscr{A} \subseteq A$ too big since most cohomology theories behave badly on C^*-algebras. It is due to cohomological reasons that one needs to look at dense subalgebras. See more in [54].

On the other hand, on manifolds there is a standard example of a cyclic cocycle. Any closed k-form ω on an n-dimensional manifold X defines a cyclic $n-k$-cocycle $\widetilde{\omega}$ on $C^\infty(X)$ by

$$\widetilde{\omega}(f_0, f_1, \ldots, f_{n-k}) := \int_X f_0 \mathrm{d} f_1 \wedge \cdots \mathrm{d} f_{n-k} \wedge \omega. \tag{3.11}$$

Motivated by this example, cyclic cohomology can be viewed as an algebraic generalization of de Rham homology. The main difference lies in that the dimension defines a grading on the de Rham theories, while the dimension defines a filtration on the cyclic theories. This difference can be explained by a theorem of Connes [34] stating that if X is a compact manifold, there is an isomorphism

$$HC^k(C^\infty(X)) \cong Z_k(X) \oplus \bigoplus_{j>0} H^{dR}_{k-2j}(X), \tag{3.12}$$

where $Z_k(X)$ denotes the space of closed k-currents on X. A similar statement also holds for cyclic homology and de Rham cohomology. The isomorphism is constructed via a certain modification of the association $\omega \mapsto \widetilde{\omega}$ in equation (3.11). Without going into details about the modification, let us mention that it is exactly what makes cyclic homology filtered rather than graded. This fits well with a description of cyclic homology in terms of a differential algebra.

Let us first recall the notion of the universal differential algebra. We let $\tilde{\mathscr{A}}$ denote the unitalization of $\mathscr{A}$. We define $\Omega^1\mathscr{A} := \tilde{\mathscr{A}} \otimes \mathscr{A}$ and $\mathrm{d} : \mathscr{A} \to \Omega^1\mathscr{A}$ by $\mathrm{d}a := 1 \otimes a$, so any element of $\Omega^1\mathscr{A}$ is of the form $(a_0 + \lambda)\mathrm{d}a_1$. Let us equip $\Omega^1\mathscr{A}$ with the $\mathscr{A}$-bimodule structure

$$x((a_0 + \lambda)\mathrm{d}a_1)y := (xa_0 + \lambda x)\mathrm{d}(a_1 y) - (xa_0a_1 + \lambda xa_1)\mathrm{d}y$$

The derivation $\mathrm{d} : \mathscr{A} \to \Omega^1\mathscr{A}$ is universal in the sense that any other $\mathscr{A}$-bimodule $\mathscr{E}$ and derivation $\nabla : \mathscr{A} \to \mathscr{E}$ must be of the form $\nabla = \rho\mathrm{d}$ for a bimodule morphism $\rho : \Omega^1\mathscr{A} \to \mathscr{E}$. We also define

$$\Omega^k\mathscr{A} := \Omega^1\mathscr{A} \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} \Omega^1\mathscr{A}.$$

Observe that as a vector space, $\Omega^k\mathscr{A} = \tilde{\mathscr{A}} \otimes \mathscr{A}^{\otimes k}$. The universal differential algebra of $\mathscr{A}$ is given by $\Omega^*\mathscr{A} := \oplus_{k\in\mathbb{N}}\Omega^k\mathscr{A}$.

The cyclic homology of an algebra can be constructed from its universal differential algebra. We set

$$D^n(\mathscr{A}) := \bigoplus_{k\in\mathbb{N}} \Omega^{n-2k}\mathscr{A},$$

with the convention $\Omega^{-k}\mathscr{A} = 0$ for all $k > 0$. Projection onto the first $n-2$ coordinates defines an operator $\tilde{S} : D^n(\mathscr{A}) \to D^{n-1}(\mathscr{A})$. The operator $\tilde{S}$ will induce the periodicity operator on cyclic homology. The cyclic homology of $\mathscr{A}$ is by Proposition 2.14 of [36] isomorphic to the homology of the complex

$$\cdots \xrightarrow{B-b} D^n(\mathscr{A}) \xrightarrow{B-b} D^{n-1}(\mathscr{A}) \xrightarrow{B-b} \cdots \xrightarrow{B-b} D^1(\mathscr{A}) \xrightarrow{B-b} D^0(\mathscr{A}) \to 0,$$

where b is the Hochschild differential defined as a linear mapping $D^n(\mathscr{A}) \to D^{n-1}(\mathscr{A})$ using the linear decomposition $D^n(\mathscr{A}) = \oplus_k \tilde{\mathscr{A}} \otimes \mathscr{A}^{n-2k}$ and B is defined as a cyclic symmetrization of $d : D^n(\mathscr{A}) \to D^{n+1}(\mathscr{A})$ composed with $\tilde{S}$. The operator $\tilde{S}$ commutes with the differential $B-b$ and therefore induces an operator $S : HC_n(\mathscr{A}) \to HC_{n-2}(\mathscr{A})$ and dually an operator $S : HC^n(\mathscr{A}) \to HC^{n+2}(\mathscr{A})$. The normalization in equation (3.7) is choosen so that

$$S\mathrm{ch}_k = \mathrm{ch}_{k-2} \quad \text{and} \quad S\mathrm{cc}_k = \mathrm{cc}_{k+2}. \tag{3.13}$$

The periodic cyclic (co-)homology $HP_*(\mathscr{A})$ is a $\mathbb{Z}/2\mathbb{Z}$-graded (co-)homology theory defined as the projective (inductive) limit of the cyclic (co-)homology using the periodicity operator. Since the Chern character commutes with S by equation (3.13), the Chern character induces a natural transformation $\mathrm{ch} : K_*(\mathscr{A}) \to HP_*(\mathscr{A})$.

3.3 Hölder continuous symbols

One of the interesting applications of the periodicity operator is constructing index formulas in the case when there is no classical differential geometry or when the regularity of data is insufficient. This procedure is inspired by Proposition 3 in Chapter III.2.α of [34] where the cyclic 1-cocycle on $C^\infty(S^1)$ defined by

$$\mu(f_0 \otimes f_1) := \frac{1}{2\pi i}\int_{S^1} f_0 \mathrm{d}f_1, \tag{3.14}$$

is regularized and extended to the algebra of Hölder continuous functions. This idea was later used in [76] and [77] to regularize index formulas for operator-valued pseudo-differential operators and in [48] to extend the Mickelsson-Faddeev cocycle from the loop group to a fractional loop group.

Let us start by describing how to regularize the cyclic cocycle (3.14). The Cauchy operator $F \in \mathscr{B}(L^2(S^1))$ is defined form the principal value integral

$$Ff(z) = \frac{1}{\pi i}\int_{S^1} \frac{f(w)\mathrm{d}w}{z-w}.$$

By evaluating F on the orthonormal basis $e_k(z) := (2\pi)^{-1}z^k$ of $L^2(S^1)$ we obtain $Fe_k = \mathrm{sign}(k)e_k$ in the convention $\mathrm{sign}(0) = 1$. The cyclic cocycle $S\mu$ is cohomologous to $\mathrm{cc}_3(\pi, F)$, where $\pi : C^\infty(S^1) \to \mathscr{B}(L^2(S^1))$ is defined via point-wise multiplication. The Toeplitz quantization associated with this odd Fredholm module is the ordinary Toeplitz quantization of S^1 on the Hardy space. This fact follows from that the Hardy space, consisting of functions in $L^2(S^1)$ with a holomorphic extension to the interior of S^1, is spanned by $\{e_k\}_{k\geq 0}$ which implies that $(F+1)/2$ is the Szegö projection onto the Hardy space. Since the odd Chern character is an isomorphism, it is sufficient to verify this statement on the cyclic 3-cycle

$$(z^{-1}-1)\otimes(z-1)\otimes(z^{-1}-1)\otimes(z-1) \in C^\infty((S^1)^4).$$

A straight-forward integral estimate shows that $cc_{2k+1}(\pi, F)$ is in the image of $HC^{2k+1}(C^\alpha(S^1)) \to HC^{2k+1}(C^\infty(S^1))$ whenever $\alpha(2k+1) > 1$. In particular, using the index formula (3.4) we obtain that if T is a Toeplitz operator on S^1 with symbol u Hölder continuous symbol of exponent α and k satisfies $\alpha(2k+1) > 1$ then $\mathrm{ind}(T) = -\deg(u)$ so

$$\deg(u) = -\frac{1}{(2\pi i)^{2k}}\int u(z_0)^{-1}\frac{u(z_1)-u(z_0)}{z_1-z_0}\frac{u(z_2)^{-1}-u(z_1)^{-1}}{z_2-z_1}\cdots\frac{u(z_0)-u(z_{2k})}{z_0-z_{2k}}\mathrm{d}z,$$

where we set $\mathrm{d}z := \mathrm{d}z_0\ldots\mathrm{d}z_{2k}$, see Proposition 3 in Chapter III.2.α of [34].

In general, the problem that we addressed in the beginning of this chapter, to express the index of a pseudo-differential or a Toeplitz operator when the symbol is not smooth enough and when the underlying space is non-smooth can be dealt with by using the Connes-Chern character of some suitable Fredholm module. The cyclic cocycle that we want to regularize is defined from the de Rham cohomology class $Td(X)$ under the isomorphism (3.12). Just as it was done above for S^1 we will use the kitchen door by constructing the index pairing first and use the Connes-Chern character to construct a regularizing cyclic cohomology class.

In Paper B we will study Toeplitz operators on the Hardy space of the boundary $\partial\Omega$ of a strictly pseudo-convex domain Ω in an n-dimensional Stein manifold and generalize the index formula of Connes for Hölder continuous symbols to the higher-dimensional setting. As mentioned above, this problem is motivated by non-linear partial differential equations and degree theory for VMO-mappings developed by Brezis-Nirenberg [27]. The degree theory for VMO-mappings is based upon the fact that continuous functions are dense in the space of VMO-functions. By the result of Brezis-Nirenberg the degree of a continuous mapping is continuous in the VMO-topology so the degree of a VMO-mapping can be defined by means of continuity. Using Boutet de Monvel's index formula for Toeplitz operators with smooth symbol we provide an explicit formula for the degree of a Hölder continuous function in Theorem B.4.4.

The reason that we consider domains in Stein manifolds is that we can use a certain projection approximating the Szegö projection $P_{\partial\Omega}$ of $\partial\Omega$, known as the Henkin-Ramirez projection, a non-orthogonal projection onto the Hardy space that differs from the Szegö projection by a Schatten class operator. We will denote the Henkin-Ramirez projection by P_{HR}. To prove that the Toeplitz operators with Hölder continuous symbols stem from a finitely summable Fredholm module we will use a theorem by Russo, see [80], giving a sufficient condition for the finite summability of an integral operator in the terms of its integral kernel. The integral kernel of the Henkin-Ramirez projection is well studied and using some known integral estimates for this integral kernel and Russo's theorem we show that the Henkin-Ramirez projection commutes up to q-summable operators with Hölder continuous functions of exponent α for any $q > 2n/\alpha$. Therefore, the Toeplitz operators on the Hardy space defines a q-summable odd Fredholm module over the Hölder continuous functions $C^\alpha(\partial\Omega)$ of exponent α for any $q > 2n/\alpha$. Therefore there is a well defined associated Connes-Chern character on the algebra of Hölder continuous functions. In particular, we derive the following index formula:

Theorem 2. *Suppose that Ω is a relatively compact strictly pseudo-convex domain with smooth boundary in a Stein manifold of complex dimension n. Denote the corresponding Henkin-Ramirez kernel by $H_{\partial\Omega}$ and the Szegö kernel by $C_{\partial\Omega}$.*

If $a : \partial\Omega \to \mathrm{GL}_N$ is Hölder continuous of exponent α, then for $2k+1 > 2n/\alpha$ the index formulas hold

$$\mathrm{ind}\,(P_{\partial\Omega}\pi(a)P_{\partial\Omega}) = -\int_{\partial\Omega^{2k+1}} \mathrm{tr}\left(\prod_{j=0}^{2k}(1-a(z_{j-1})^{-1}a(z_j))H_{\partial\Omega}(z_{j-1},z_j)\right)\mathrm{d}V =$$

$$= \mathrm{ind}\,(P_{HR}\pi(a)P_{HR}) = -\int_{\partial\Omega^{2k+1}} \mathrm{tr}\left(\prod_{j=0}^{2k}(1-a(z_{j-1})^{-1}a(z_j))C_{\partial\Omega}(z_{j-1},z_j)\right)\mathrm{d}V,$$

where the integrals converge absolutely.

As an example of this index theorem, consider a Toeplitz operator T_a on S^{2n-1} with Hölder continuous symbol $a : S^{2n-1} \to GL_N(\mathbb{C})$. Since S^{2n-1} is convex, the Henkin-Ramirez projection coincides with the Szegö projection. The calculation can be found in Theorem IV.3.4 of [71]. The integral kernel is

$$H_{S^{2n-1}}(z,w) = C_{S^{2n-1}}(z,w) = c_n(1-z\bar{w})^{-n},$$

where $c_n = (n-1)!/(2\pi i)^n$. Therefore, we have the explicit index formula for Toeplitz operators on the sphere with Hölder continuous symbols:

$$\mathrm{ind}\,(T_a) = \frac{((n-1)!)^{2k+1}}{(2\pi i)^{n(2k+1)}}\int_{(S^{2n-1})^{2k+1}} \mathrm{tr}\left(\prod_{j=0}^{2k}\frac{1-a(z_{j-1})a(z_j)^{-1}}{(1-z_{j-1}\bar{z}_j)^n}\right)\mathrm{d}V \tag{3.15}$$

Let us make a remark on the choice of symbols being Hölder continuous. The proof of Theorem 2 goes through word by word if we replace the Hölder continuous functions by bounded functions f satisfying a kind of integral continuity depending on one more parameter r. The formulas and restrictions become slightly more complicated with more parameters. The restriction we must put on f is that

$$\int_{\partial\Omega}\left(\int_{\partial\Omega}\frac{|f(z)-f(w)|^r}{d(z,w)^{\alpha r}}\mathrm{d}V_{\partial\Omega}\right)^{q/r}\mathrm{d}V_{\partial\Omega} < \infty, \tag{3.16}$$

where d denotes the Euclidean metric or more generally the Koranyi metric on $\partial\Omega$. The Koranyi metric is the pseudo-metric on $\partial\Omega$ associated with the contact structure. It is closely related to the Szegö kernel and very directly related to certain estimates of the integral kernel of the Henkin-Ramirez projection, see for instance Proposition 3.1 of [71] or in section B.2 below. Equation (3.16) is equivalent to the condition that

$$J^d_\alpha(f) : (z,w) \mapsto |f(z)-f(w)|/d(z,w)^\alpha$$

is in the mixed L^p-space $L^{(r,q)}(\partial\Omega\times\partial\Omega)$. If f satisfies (3.16) for some parameters α, q, r such that

$$\alpha \in]0,1], \; r,q > 2n/\alpha \quad \text{and} \quad rq/(rq-r-q) < 2n/(2n-\alpha), \tag{3.17}$$

then $[P_{HR}, f] \in \mathscr{L}^q(L^2(\partial\Omega))$. This commutation relation is the necessary tool for producing the index formulas of Theorem 2. Denote the space of bounded functions satisfying (3.16) with d being the Euclidean metric by $F^{euc}(\partial\Omega, \alpha, q, r)$ and analogously with $F^{kor}(\partial\Omega, \alpha, q, r)$. These function spaces become Banach spaces in the norm $f \mapsto \|f\|_{L^\infty} + \|J^d_\alpha(f)\|_{L^{(r,q)}}$, with the corresponding choices of d. Since the Koranyi metric bounds the Euclidean metric on small distances there is a bounded embedding

$$F^{euc}(\partial\Omega, \alpha, q, r) \subseteq F^{kor}(\partial\Omega, \alpha, q, r).$$

A straight-forward estimate shows that there is a bounded embedding

$$F^{euc}(\partial\Omega, \alpha, q, r) \subseteq VMO(\partial\Omega)$$

if $\alpha r q \geq n(r+q)$. The Gagliardo characterization of Sobolev spaces, see [26], implies that whenever $0 < s < 1$ and $1 < p < \infty$ then

$$W^{s,p}(\partial\Omega) = F^{euc}\left(\partial\Omega, \frac{2n-1}{q} + s, q, q\right).$$

The spaces $F^{euc}(\partial\Omega, \alpha, q, r)$ do not give any new type of degree formulas for Sobolev spaces. Set $s = (2n-1)/q - \alpha$; then (3.17) is true if

$$q^2/(q^2 - 2q) < 2n/(2n - \alpha) \quad \Longleftrightarrow \quad sq > 2n + 1.$$

This means that for any parameters s, q making the degree formula well defined, the Sobolev embedding implies that $W^{s,q}(\partial\Omega) \subseteq C^\alpha(\partial\Omega)$. It is not clear how much more exotic functions can occur for $r \neq q$ and how low regularity the space $F^{kor}(\partial\Omega, \alpha, q, r)$ admits. An interesting question that deserve some attention is how does the structure of the spaces $F^{kor}(\partial\Omega, \alpha, q, r)$ look? Are there any embeddings into standard spaces for $r \neq q$?

As mentioned previously, pseudo-differential operators are Toeplitz operators in disguise. To illustrate the scheme of representing the pseudo-differential quantization by a Toeplitz quantization we consider the case of a sphere and calculate explicit integral kernels for the associated Toeplitz quantization. It is sufficient to calculate the Henkin-Ramirez kernel of the cosphere bundle of a sphere S^n in the Riemannian metric induced from the embedding $S^n \subseteq \mathbb{R}^{n+1}$. For $\mu \in [0, 1[$ we define the analytic subvariety $Z_\mu \subseteq \mathbb{C}^{n+1}$ as the zero set of the polynomial $z \cdot z - \mu$, where $\cdot$ denotes the bilinear product in $\mathbb{C}^{n+1}$. If $\mu > 0$, the variety Z_μ is smooth but Z_0 has an isolated singularity in $z = 0$. We consider the strictly pseudo-convex domain $\Omega_\mu := B_{2n+2}(2-\mu, 0) \cap Z_\mu \subseteq Z_\mu$. Observe that

$$S^*S^n = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |x| = |y| = 1, x \cdot y = 0\}.$$

If we set $z = x + iy\sqrt{1-\mu}$, then $z \cdot z = |x|^2 - |y|^2(1-\mu) + 2ix \cdot y = \mu$. This construction defines a diffeomorphism $S^*S^n \cong \partial\Omega_\mu$ for all $\mu \in]0,1[$. When $\mu = 0$, the corresponding mapping defines a diffeomorphism away from the singularity. Observe that the calculation (1.8) implies that there is a graded isomorphism $K^*(\partial\Omega_\mu) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2$ for any $\mu \in [0,1[$.

We are now going to calculate the Henkin-Ramirez kernel for $\mu \in [0,1[$ using the method of [1]. We will denote the Henkin-Ramirez kernel of $\partial\Omega_\mu$ by P^μ_{HR}. If we partial integrate the example in equation 6.2 of [1] we obtain that P^μ_{HR} is defined on $f \in L^2(\partial\Omega)$ by

$$P^\mu_{HR} f(z) = \int_{\partial\Omega_\mu} f(w) \cdot \gamma(w) \neg \left(h(z,w) \wedge \frac{\bar{w} \cdot \mathrm{d}w \wedge (\mathrm{d}\bar{w} \cdot \mathrm{d}w)^{n-1}}{(2\pi i)^n (2-\mu-z\bar{w})^n} \right),$$

where γ is a $(1,0)$-vector field that is smooth, except in $w = 0$ when $\mu = 0$, satisfying that $\gamma(w)\neg(2z \cdot \mathrm{d}z) = 2\pi i$ and h is a Hefer form with respect to $f(w) - f(z)$. In our case we may take

$$\gamma(w) = \frac{\pi i}{|w|^2}\, \bar{w} \cdot \frac{\partial}{\partial w} \quad \text{and} \quad h(z,w) = (w+z) \cdot (\mathrm{d}\bar{w} + \mathrm{d}\bar{z}).$$

After some calculations we come to the identity for the scalar kernel $H_{\partial\Omega_\mu}$:

$$H_{\partial\Omega_\mu}(z,w)\mathrm{d}V_{\partial\Omega_\mu} = \frac{(n-2)! \sum_{j,k,l=1}^{n+1} (w_j + z_j)\bar{w}_k \bar{w}_l * (\mathrm{d}w_l \wedge \mathrm{d}\bar{w}_k \wedge \mathrm{d}w_l)}{(2\pi i)^{n-1}(2-\mu)^2(2-\mu-z\cdot\bar{w})^n}. \quad (3.18)$$

Using Guillemin's theorem, Theorem 5.2 of [47], the pseudo-differential quantization of S^n is equivalent to the Szegö quantization on $\partial\Omega_\mu$ for $\mu > 0$ since $T^*S^n \cong \Omega_\mu$. As a corollary of the construction (3.18) we obtain an explicit splitting $T_0 : C^\infty(S^*X) \to \Psi_{cl}(S^n)$ of the pseudo-differential extension:

$$0 \to \Psi^{-1}_{cl}(S^n) \to \Psi^0_{cl}(S^n) \xrightarrow{\sigma} C^\infty(S^*S^n) \to 0,$$

that extends to a completely positive mapping $T : C(S^*S^n) \to \Psi(S^*S^n)$, where $\Psi(S^*S^n)$ denote the C^*-algebra generated by the classical zero order pseudo-differential operators on S^n.

Another observation is that the formalism for integral representations of [1] should allow one to generalize the index theorem for Hölder continuous symbols from Stein manifolds to Stein varieties as long as there are no singularities on $\partial\Omega$. This is the setting in which Boutet de Monvel's index formula for Toeplitz operators works in its full generality. When there are singularities on $\partial\Omega$, problems arise already on an analytic level even for smooth functions.

We perform a similar calculation as Theorem 2 in Paper C for pseudo-differential operators twisted by Hölder continuous vector bundles. We do this by showing that any zero order pseudo-differential operator on a closed n-dimensional manifold X defines a q-summable even Fredholm module over the Hölder continuous functions $C^\alpha(X)$ of exponent α for any $q > n/\alpha$. The method of proof is just as in the the proof of Theorem 2 based upon Russo's theorem. Using the Connes-Chern formula we obtain analytic formulas for the index pairing between elliptic pseudo-differential operators with Hölder continuous vector bundles.

The initial motivation for these calculations was to find an explicit degree for Hölder continuous mappings. Theorem 2 can only be used to derive degree formulas for Hölder continuous mappings under the topological and analytical requirement that the domain is the boundary of a strictly pseudo-convex domain. If we want to calculate the mapping degree of a function $f : X \to Y$ we can in fact restrict our attention to manifolds of a specified dimension parity, since we can always replace f by $f \times \mathrm{id}_{S^1} : X \times S^1 \to Y \times S^1$ without changing the degree. The idea for finding a general formula is to restrict to even-dimensional manifolds and combine the Atiyah-Singer index formula with the q-summable Fredholm that an elliptic pseudo-differential operator defines on the Hölder continuous functions.

To be more precise, if A is an elliptic pseudo-differential operator and $E \to X$ is a smooth vector bundle, the Atiyah-Singer index theorem implies that

$$\mathrm{ind}\,(A_E) = \int_{T^*X} \mathrm{ch}[A] \wedge \pi^* \mathrm{ch}[E] \wedge \pi^* Td(X).$$

If we can make the right-hand side well defined for Hölder continuous bundles we can take a smooth line bundle $L \to Y$ and consider the index of A twisted with the Hölder continuous bundle $f^*L \to X$. By the remarks above we can make everything well-defined by only considering zero order operators A. Naturality of the Chern character implies that

$$\mathrm{ind}\,(A_{f^*L}) = \int_X \mathrm{ch}[A] \wedge \pi^* f^* \mathrm{ch}[L] \wedge \pi^* Td(X).$$

If we can construct L such that $\mathrm{ch}[L] = 1 + \mathrm{d}V_Y$, we do in fact arise at the conclusion that

$$\mathrm{ind}\,(A_{f^*L}) = \int_X \mathrm{ch}[A] \wedge \pi^* f^*(1 + \mathrm{d}V_Y) \wedge \pi^* Td(X) = \mathrm{ch}_0[A]\deg(f) + \mathrm{ind}\,(A).$$

This construction can be done for even-dimensional manifolds, see more in Theorem C.3.2. The left-hand side of can be calculated by means of cyclic cohomology. We perform this scheme in Paper C by taking $A := D(1 + D^2)^{-1/2}$, where D denotes the signature operator on X discussed in section 3.1. The main result of Paper C is that:

Theorem 3. *Suppose that X and Y are smooth connected closed $2n$-dimensional manifolds and $f : X \to Y$ is Hölder continuous of exponent α. When $k > n/\alpha$ the following integral formula holds:*

$$\begin{aligned}\deg(f) &= 2^{-n}\langle \mathrm{cc}_k(\tilde{\pi}, \tilde{F}_D), f^*([L_Y]-1)\rangle_{2k} = \\ &= -2^{-n}\mathrm{sign}(X) + 2^{-n}(-1)^k \int_{X^{2k}} \tilde{f}_k(x_1, \ldots, x_{2k})\mathrm{d}V_{X^{2k}}\end{aligned}$$

where $(\tilde{\pi}, \tilde{F}_D)$ is the even analytic K-cycle associated with the signature operator as in Theorem C.4.2, $L_Y \to Y$ is the line bundle described in Theorem C.2.2 and $\tilde{f}_k$ is the integrable function defined explicitly from the function f as below in the equation (C.20).

One of the more direct applications of Theorem 2 and Theorem 3 are Hölder norm estimates of mapping degrees. We will let

$$|f|_\alpha := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

First of all, Theorem 3 implies the general estimate

$$|\deg(f)| \leq |\mathrm{sign}(X)| + c_{X,Y}^{2k}|f|_\alpha^{2k},$$

for $f : X \to Y$ Hölder continuous mapping of exponent larger than $n/2k$ between n-dimensional manifolds. Secondly, the multiplicative nature of the index (1.2) and Theorem 2 implies that if $f, g : \partial\Omega \to Y$ are two Hölder continuous mapping of exponent larger than n/k and Ω is a strictly pseudo-convex domain in an n-dimensional Stein manifold then

$$|\deg(f) - \deg(g)| \leq c_{\Omega,Y}^{2k+1}\left(\|g\|_{C(\partial\Omega,Y)}|f-g|_\alpha + |g|_\alpha\|f-g\|_{C(\partial\Omega,Y)}\right)^{2k+1}.$$

Part II

Research papers

Paper A

Index formulas and charge deficiencies on the Landau levels

Abstract

The notion of charge deficiency from Avron-Seiler-Simon [12] is studied from the view of K-theory of operator algebras and is applied to the Landau levels in $\mathbb{R}^{2n}$. We calculate the charge deficiencies at higher Landau levels in $\mathbb{R}^{2n}$ by means of an Atiyah-Singer type index theorem.

Introduction

The paper is a study of the charge deficiencies at the Landau levels in $\mathbb{R}^{2n}$. The Landau levels are the eigenspaces of the Landau Hamiltonian which is the energy operator for a quantum particle moving in $\mathbb{R}^{2n}$ under the influence of a constant magnetic field of full rank.

In [12], the notion of charge deficiency was introduced as a measure of how much does a flux tube change a fermionic system in $\mathbb{R}^2$. The setting of [12] is a quantum system where the Fermi energy is in a gap and the question is what happens when the system is taken trough a non-trivial cycle. Letting P denote the projection onto the state space and U the unitary transformation representing the cycle, the projection Q onto the new state space after it had been taken through a cycle can be expressed as $Q = UPU^*$. The relative index $\text{ind}\,(Q,P)$ is defined as an infinite dimensional analogue of $\dim Q - \dim P$ and is well defined whenever $Q - P$ is a compact operator. The condition that $Q - P$ is compact is equivalent to that $[P,U]$ is compact. In the setting of [12] the relative index represents the change in the number of fermions that U produces. In [12] the following formula was proven:

$$\text{ind}\,(Q,P) = \text{ind}\,(PUP).$$

For sufficiently nice systems in $\mathbb{R}^2$ one can choose the particular unitary given by multiplication by the bounded function $U := z/|z|$. The condition on the system that is needed is that P commutes with U up to a compact operator. The charge deficiency of a projection P in the sense of [12] is then defined using U as

$$c(P) := \text{ind}\,(PUP).$$

The viewpoint we will have in this paper is that the charge deficiency is a K-homology class. This viewpoint lies in line with the view on D-brane charges in string theory, see more in [28], [72]. In the case studied in [12] the charge deficiency is realized as an odd K-homology class on the circle S^1. The unitary U define a representation of $C(S^1)$ and using the fact that P commutes with U up to a compact operator we get a K-homology class. Let us denote this K-homology class by $[P]$ and by u we will denote the generator of $C(S^1)$. In this notation, the charge deficiency is given by $c(P) = [u] \circ [P] \in KK(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$, the Kasparov product between $[P] \in K^1(C(S^1))$ and $[u] \in K_1(C(S^1))$. Thus the charge deficiency is the image of $[P]$ under the isomorphism

$$K^1(C(S^1)) = KK_1(C(S^1),\mathbb{C}) \cong Hom(K_1(C(S^1)), K_0(\mathbb{C})) \cong \mathbb{Z},$$

where the first isomorphism is the natural mapping coming from the Universal Coefficient Theorem for KK-theory and the second isomorphism comes from choosing $[u]$ as a generator for $K_1(C(S^1))$. So a better picture is that the K-homology class $[P] \in K^1(C(S^1))$ is the charge deficiency of P.

The system we will consider in this paper consists of a particle moving in $\mathbb{R}^{2n}$ under the influence of a constant magnetic field B of full rank. If we choose a linear vector potential A satisfying $\mathrm{d}A = B$ the Hamiltonian of this system is given by

$$H_A := (-i\nabla - A)^2,$$

This *Landau Hamiltonian* should be viewed as a densely defined operator in the Hilbert space $L^2(\mathbb{R}^{2n})$. Taking $\mathscr{D}(H_A) = C_c^\infty(\mathbb{R}^{2n})$, the operator H_A becomes essentially self-adjoint, see more in [64]. Due to the identification $\mathbb{R}^{2n} = \mathbb{C}^n$ we will use the complex structure and we will assume that $B = \frac{i}{2}\sum \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$.

The Landau Hamiltonian has a discrete spectrum with eigenvalues $\Lambda_\ell = 2\ell + n$ for $\ell \in \mathbb{N}$ and the eigenspaces $\mathscr{L}^\ell$ are infinite dimensional. Let

$$P_\ell : L^2(\mathbb{R}^{2n}) \to \mathscr{L}^\ell$$

denote the orthogonal projection to the ℓ:th eigenspace. Our point of view on the charge deficiencies for the Landau levels is that they are K-homology classes of the sphere S^{2n-1}. For a bounded continuous function $a : \mathbb{R}^{2n} \to M_N(\mathbb{C})$ we define the continuous function $a_r \in C(S^{2n-1})$ as

$$a_r(v) := a(rv).$$

We let A_N be the subalgebra of $C_b(\mathbb{R}^{2n}) \otimes M_N(\mathbb{C})$ such that a_r converges uniformly in v to a continuous function a_∂ on S^{2n-1}. The mapping $a \mapsto a_\partial$ defines a $*$-homomorphism $A_N \to C(S^{2n-1}) \otimes M_N(\mathbb{C})$. The projection P_ℓ commutes up to a compact operator with $a \in A_N$ (see below in Theorem A.2.2) and

$$P_\ell a|_{\mathscr{L}^\ell \otimes \mathbb{C}^N} : \mathscr{L}^\ell \otimes \mathbb{C}^N \to \mathscr{L}^\ell \otimes \mathbb{C}^N$$

is Fredholm if and only if a_∂ is invertible (see Proposition A.2.6). Now we may present the main theorem of this paper:

Theorem 4. *If a_∂ is smooth and invertible, the index of $P_\ell a|_{\mathscr{L}^\ell \otimes \mathbb{C}^N}$ can be expressed as*

$$\operatorname{ind}(P_\ell a|_{\mathscr{L}^\ell \otimes \mathbb{C}^N}) = \frac{-(\ell + n - 1)!}{\ell!(2n-1)!(2\pi i)^n} \int_{S^{2n-1}} \operatorname{tr}((a_\partial^{-1}\mathrm{d}a_\partial)^{2n-1}).$$

The charge deficiency $[P_\ell] \in K^1(C(S^{2n-1}))$ may be expressed in terms of the Bergman projection P_B on the unit ball in $\mathbb{C}^n$ as

$$[P_\ell] = \frac{(\ell + n - 1)!}{\ell!(n-1)!}[P_B].$$

A.1 Particular Landau levels

The spectral theory of the Landau Hamiltonian is well known and we will review it briefly. See more in [79]. We will let $\varphi := \frac{|z|^2}{4}$ and assume that the magnetic field B is of the form $B = i\partial\bar{\partial}\varphi$. Here ∂ is the complex linear part of the exterior differential d. Define the annihilation operators as

$$q_j := 2\frac{\partial}{\partial \bar{z}_j} + z_j \quad \text{for} \quad j = 1, \ldots, n.$$

The adjoints are given by the creation operators $q_j^* := -2\frac{\partial}{\partial z_j} + \bar{z}_j$. The annihilation and creation operators satisfy the following formulas:

$$[q_j, q_i] = [q_j^*, q_i^*] = 0, \quad [q_i, q_j^*] = 2\delta_{ij} \quad \text{and} \quad H_A = \sum_{j=1}^{n} q_j^* q_j + n = \sum_{j=1}^{n} q_j q_j^* - n.$$

Here we view H_A as a densely defined operator in $L^2(\mathbb{C}^n)$. Thus the lowest eigenvalue is n with corresponding eigenspace $\mathscr{L}_0 = \mathsf{e}^{-\varphi}\mathscr{F}(\mathbb{C}^n)$ where $\mathscr{F}(\mathbb{C}^n) := L^2(\mathbb{C}^n, \mathsf{e}^{-2\varphi}) \cap \mathscr{O}(\mathbb{C}^n)$ denotes the Fock space. Here $\mathscr{O}(\mathbb{C}^n)$ denotes the space of holomorphic functions in $\mathbb{C}^n$. In one complex dimension there is only one creation operator q^* and the eigenspaces are given by $\mathscr{L}_k = (q^*)^k \mathscr{L}_0$. Using multi-index notation, for $\mathbb{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ we define $q_{\mathbb{k}} := q_1^{k_1} \cdots q_n^{k_n}$ and

$$\mathscr{L}_{\mathbb{k}} := q_{\mathbb{k}}^* \mathscr{L}_0 = \mathscr{L}_{k_1} \otimes \mathscr{L}_{k_2} \otimes \cdots \otimes \mathscr{L}_{k_n}.$$

We will call this space for the *particular* Landau level of height $\mathbb{k}$. Using that q_j and q_j^* define a representation of the Heisenberg algebra in n dimension we obtain the eigenvalues of H_A as $\Lambda_\ell = 2\ell + n$ with the corresponding eigenspaces

$$\mathscr{L}^\ell := \bigoplus_{|\mathbb{k}|=\ell} \mathscr{L}_{\mathbb{k}} = \bigoplus_{|\mathbb{k}|=\ell} \mathscr{L}_{k_1} \otimes \mathscr{L}_{k_2} \otimes \cdots \otimes \mathscr{L}_{k_n}.$$

The ℓ:th eigenspace $\mathscr{L}^\ell$ is called the Landau level of height ℓ. Since the Hamiltonian commutes with the representation of $SU(n)$ on $\mathbb{C}^n$, its eigenspaces are $SU(n)$-invariant. Also the orthogonal projections $P_\ell : L^2(\mathbb{C}^n) \to \mathscr{L}^\ell$ are invariant under the $SU(n)$-action.

Recall that the vacuum subspace $\mathscr{L}_0 \subseteq L^2(\mathbb{C}^n)$ has a reproducing kernel induced by the reproducing kernel on the Fock space. The reproducing kernel of $\mathscr{F}(\mathbb{C}^n)$ is given by $K(z, w) = \mathsf{e}^{\frac{w\cdot\bar{z}}{4}}$. So the reproducing kernel of $\mathscr{L}_0$ is given by

$$K_0(z, w) := \mathsf{e}^{\frac{1}{4}(w\cdot\bar{z} - |z|^2 - |w|^2)}.$$

This expression for the reproducing kernel implies that the orthogonal projection $P_0 : L^2(\mathbb{C}^n) \to \mathscr{L}_0$ is given by

$$P_0 f(z) = \int_{\mathbb{C}^n} f(w)\overline{K_0(z,w)}\mathrm{d}V.$$

By [78] the orthogonal projection $P_{\Bbbk} : L^2(\mathbb{C}^n) \to \mathscr{L}_{\Bbbk}$ onto the particular Landau level of height $\Bbbk$ is also an integral operator with kernel

$$K_{\Bbbk}(z,w) = \mathrm{e}^{\frac{1}{4}(w\cdot\bar{z}-|z|^2-|w|^2)} \prod_{j=1}^{n} L_{k_j}\left(\frac{1}{2}|z_j - w_j|^2\right). \tag{A.1}$$

Here L_k is the Laguerre polynomial of order k. Notice that the projections $P_{\Bbbk}$ are not $SU(n)$-invariant in general.

A.2 Toeplitz operators on the Landau levels

We want to study topological properties of the particular Landau levels using Toeplitz operators. The symbols will be taken from a suitable subalgebra of $C_b(\mathbb{C}^n)$, the bounded functions on $\mathbb{C}^n$. The standard notation $\mathscr{B}(\mathscr{H})$ will be used for the bounded operators on a separable Hilbert space $\mathscr{H}$ and the compact operators will be denoted by $\mathscr{K}(\mathscr{H})$. We will let $\pi : C_b(\mathbb{C}^n) \to \mathscr{B}(L^2(\mathbb{C}^n))$ denote the representation given by pointwise multiplication. This is clearly an $SU(n)$-equivariant mapping. Define the linear mapping $T_{\Bbbk} : C_b(\mathbb{C}^n) \to \mathscr{B}(\mathscr{L}_{\Bbbk})$ by $T_{\Bbbk}(a) := P_{\Bbbk}\pi(a)|_{\mathscr{L}_{\Bbbk}}$.

Lemma A.2.1. *If $a \in C_0(\mathbb{C}^n)$ then $T_{\Bbbk}(a) \in \mathscr{K}(\mathscr{L}_{\Bbbk})$ for all $\Bbbk \in \mathbb{N}^n$.*

The proof of this lemma is analogous to the proof for the same statement for Toeplitz operators on a pseudoconvex domain from [89].

Proof. It is sufficient to prove the claim for $a \in C_c(\mathbb{C}^n)$, since $T_{\Bbbk}$ is continuous and $C_c(\mathbb{C}^n) \subseteq C_0(\mathbb{C}^n)$ is dense. Define the compact set $K := \mathsf{supp}(a)$. Let $R : \mathscr{L}_{\Bbbk} \to L^2(\mathbb{C}^n)$ denote the operator given by multiplication by χ_K, the characteristic function of K. We have $T_{\Bbbk}(a) = P_{\Bbbk}\pi(a)R$ so the Lemma holds if R is compact. That R is compact follows from Cauchy estimates of holomorphic functions on a compact set. □

Define the $SU(n)$-invariant C^*-subalgebra $A \subseteq C_b(\mathbb{C}^n)$ as consisting of functions a such that $a(rv)$ converges uniformly in v as $r \to \infty$ to a continuous function $a_\partial : S^{2n-1} \to \mathbb{C}$ when $r \to \infty$. Thus we obtain a surjective $SU(n)$-equivariant $*$-homomorphism $\pi_\partial : A \to C(S^{2n-1})$ given by

$$\pi_\partial(a)(v) := \lim_{r\to\infty} a(rv).$$

The mapping π_∂ satisfies $\ker \pi_\partial = C_0(\mathbb{C}^n)$. We will henceforth consider $T_\Bbbk$ as a mapping from A to $\mathscr{B}(\mathscr{L}_\Bbbk)$.

If we let B_{2n} denote the open unit ball in $\mathbb{C}^n$, another view on A is as the image of the $SU(n)$-equivariant $*$-monomorphism $C(\overline{B_{2n}}) \to C_b(B_{2n}) \cong C_b(\mathbb{C}^n)$ where the last isomorphism comes from an $SU(n)$-equivariant homeomorphism $B_{2n} \cong \mathbb{C}^n$.

Theorem A.2.2. *The projection $P_\Bbbk$ satisfies $[P_\Bbbk, \pi(a)] \in \mathscr{K}(L^2(\mathbb{C}^n))$ for all $a \in A$. Therefore the $*$-linear mapping $T_\Bbbk : A \to \mathscr{B}(\mathscr{L}_\Bbbk)$ satisfies*

$$T_\Bbbk(ab) - T_\Bbbk(a)T_\Bbbk(b) \in \mathscr{K}(\mathscr{L}_\Bbbk).$$

The proof is based on a similar result from [19] where the Fock space was used to define a Toeplitz quantization of a certain subalgebra of $L^\infty(\mathbb{C}^n)$. The case of the Fock space is more or less the same as the case $\Bbbk = 0$ for Landau quantization. To prove the Theorem we need a lemma similar to part (iv) of Theorem 5 of [19]. Using the isomorphism $A \cong C(\overline{B_{2n}})$ we define the dense subalgebra $A_1 \subseteq A$ as the inverse image of the Lipschitz continuous functions in $C(\overline{B_{2n}})$.

Lemma A.2.3. *For $a \in A_1$ then for any $\varepsilon > 0$ we may write $a = g_\varepsilon + h_\varepsilon$ where $h_\varepsilon \in C_0(\mathbb{C}^n)$ and $g_\varepsilon \in A$ satisfies*

$$|g_\varepsilon(z) - g_\varepsilon(w)| \leq \varepsilon |z - w| \quad \forall z, w \in \mathbb{C}^n. \tag{A.2}$$

Proof. Let C denote the Lipschitz constant of $\pi_\partial(a)$. Take an $\varepsilon > 0$ and let χ_ε be a Lipshitz continuous $SU(n)$-invariant cutoff such that $\chi_\varepsilon(z) = 0$ for $|z| \leq R$ and $\chi_\varepsilon(z) = 0$ for $|z| \geq 2R$ where $R = R(\varepsilon, C)$ is to be defined later. To shorten notation, define $a_\partial := \pi_\partial(a)$. Let

$$g_\varepsilon(z) := \chi_\varepsilon(z) \cdot a_\partial\left(z/|z|\right)$$

and $h_\varepsilon := a - g_\varepsilon$. Clearly $h_\varepsilon \in C_0(\mathbb{C}^n)$ and $g_\varepsilon \in A$ so what remains to be proven is that R can be chosen in such a way that g_ε satisfies equation (A.2).

We have elementary estimates

$$\left|\frac{z}{|z|} - \frac{w}{|w|}\right| \leq \frac{|z-w|}{|z|} + \left|\frac{w}{|z|} - \frac{w}{|w|}\right| \leq 2\frac{|z-w|}{|w|}.$$

Thus for $z, w \neq 0$ the function a_∂ satisfies

$$\left|a_\partial\left(\frac{z}{|z|}\right) - a_\partial\left(\frac{w}{|w|}\right)\right| \leq \frac{2C}{|w|}|z-w|.$$

The function χ_ε has Lipschitz coefficient $1/R$ so if we take $R > 2C/\varepsilon$ then g_ε satisfies equation (A.2). $\square$

Let $\mathscr{C}(L^2(\mathbb{C}^n)) := \mathscr{B}(L^2(\mathbb{C}^n))/\mathscr{K}(L^2(\mathbb{C}^n))$ denote the Calkin algebra and $\mathfrak{q}$ the quotient mapping.

Proof of Theorem A.2.2. Since Lipschitz continuous functions are dense in A we may assume that $a \in A_1$, so by Lemma A.2.3 we can for any $\varepsilon > 0$ write $a = g_\varepsilon + h_\varepsilon$. In this case we have for $f \in L^2(\mathbb{C}^n)$

$$[P_{\Bbbk}, \pi(g_\varepsilon)]f(z) = \int (g_\varepsilon(z) - g_\varepsilon(w))K_{\Bbbk}(z,w)f(w)\mathrm{d}w.$$

Define the operator

$$Bf(z) := \int |z-w|K_{\Bbbk}(z,w)f(w)\mathrm{d}w.$$

By equation (A.1) we have that for some C the integral kernel of B is bounded by

$$|z-w||K_{\Bbbk}(z,w)| \le C|z-w|^{|\Bbbk|+1}\mathrm{e}^{-\frac{1}{8}|z-w|^2}.$$

Therefore the kernel of B is dominated by the kernel of a bounded convolution operator and $\|B\| < \infty$. The estimate (A.2) for g_ε implies that

$$\|[P_{\Bbbk}, \pi(g_\varepsilon)]\| \le \varepsilon\|B\|.$$

Using that $[P_{\Bbbk}, \pi(g_\varepsilon)] = [P_{\Bbbk}, \pi(a)]$ modulo compact operators, by Lemma A.2.1, we have the inequality

$$\|\mathfrak{q}([P_{\Bbbk}, a])\|_{\mathscr{C}(L^2(\mathbb{C}^n))} \le \varepsilon\|B\| \quad \forall \varepsilon > 0.$$

Therefore $\mathfrak{q}([P_{\Bbbk}, a]) = 0$ and $[P_{\Bbbk}, a]$ is compact. $\square$

Theorem A.2.2 implies that the mapping $\tilde{\beta}_{\Bbbk} := \mathfrak{q} \circ T_{\Bbbk} : A \to \mathscr{C}(\mathscr{L}_{\Bbbk})$ is a well defined $*$-homomorphism. Define the C^*-algebra

$$\tilde{\mathscr{T}}_{\Bbbk} := \{a \oplus x \in A \oplus \mathscr{B}(\mathscr{L}_{\Bbbk}) : \ \tilde{\beta}_{\Bbbk}(a) = \mathfrak{q}(x)\}.$$

This C^*-algebra contains $\mathscr{K}$ as an ideal via the embedding $k \mapsto 0 \oplus k$ and we obtain a short exact sequence

$$0 \to \mathscr{K} \to \tilde{\mathscr{T}}_{\Bbbk} \to A \to 0. \tag{A.3}$$

Lemma A.2.4. *Let* $(\Bbbk_p)_{p=1}^N \subseteq \mathbb{N}^n$ *be a finite collection of distinct* n*-tuples of integers. Then the mapping*

$$A \ni a \mapsto \mathfrak{q}\left(\left(\sum_{p=1}^N P_{\Bbbk_p}\right)\pi(a)\left(\sum_{p=1}^N P_{\Bbbk_p}\right)\right) \in \mathscr{C}\left(\bigoplus_{p=1}^N \mathscr{L}_{\Bbbk_p}\right)$$

coincides with the mapping

$$A \ni a \mapsto \oplus_{p=1}^{N} \tilde{\beta}_{\Bbbk_p}(a) \in \mathscr{C}\left(\bigoplus_{p=1}^{N} \mathscr{L}_{\Bbbk_p}\right).$$

Proof. The Lemma follows if we show that $P_{\Bbbk}\pi(a)P_{\Bbbk'} \in \mathscr{K}(L^2(\mathbb{C}^n))$ for $\Bbbk \neq \Bbbk'$. But Theorem A.2.2 implies that $P_{\Bbbk}\pi(a)(1-P_{\Bbbk}) \in \mathscr{K}(L^2(\mathbb{C}^n))$. So the Lemma follows from

$$P_{\Bbbk}\pi(a)P_{\Bbbk'} = P_{\Bbbk}\pi(a)(1-P_{\Bbbk})P_{\Bbbk'}.$$

□

In particular we can look at the collection of all $\Bbbk$:s such that $|\Bbbk| = \ell$. We will define the $SU(n)$-equivariant mapping $\tilde{\beta}_\ell : A \to \mathscr{C}(\mathscr{L}^\ell)$ as

$$a \mapsto \oplus_{|\Bbbk|=\ell} \tilde{\beta}_{\Bbbk}(a).$$

Just as for the particular Landau levels we define

$$\tilde{\mathscr{T}}^\ell := \{a \oplus x \in A \oplus \mathscr{B}(\mathscr{L}^\ell) : \tilde{\beta}_\ell(a) = \mathfrak{q}(x)\}.$$

The projection mapping $\tilde{\mathscr{T}}^\ell \to A$ given by $a \oplus x \mapsto a$ defines an $SU(n)$-equivariant extension

$$0 \to \mathscr{K} \to \tilde{\mathscr{T}}^\ell \to A \to 0.$$

Lemma A.2.5. *The kernel of $\tilde{\beta}_\ell$ is $C_0(\mathbb{C}^n)$.*

Proof. Lemma A.2.1 implies that $C_0(\mathbb{C}^n) \subseteq \ker \tilde{\beta}_\ell$. To prove the reverse inclusion we observe that the mapping $\tilde{\beta}_\ell$ is a unital $SU(n)$-equivariant $*$-homomorphism. Since $\tilde{\beta}_\ell$ is equivariant, the ideal $\ker \tilde{\beta}_\ell \subseteq A$ is $SU(n)$-invariant. The inclusion $C_0(\mathbb{C}^n) \subseteq \ker \tilde{\beta}_{\Bbbk}$ implies that there is an equivariant surjection $C(S^{2n-1}) \to A/\ker \tilde{\beta}_\ell$ which must be an isomorphism since $C(S^{2n-1})$ is $SU(n)$-simple and $\tilde{\beta}_\ell$ is unital. It follows that $\ker \tilde{\beta}_\ell = C_0(\mathbb{C}^n)$. □

It is interesting that although the statement of Lemma A.2.5 sounds algebraic, it is really the analytic statement that $T_\ell(a)$ is compact if and only if a vanishes at infinity. And this is proven with algebraic methods!

Proposition A.2.6. *If $u \in A \otimes M_N(\mathbb{C})$, the operator $T_\ell(u)$ is Fredholm if and only if $\pi_\partial(u)$ is invertible.*

Proof. By Atkinson's Theorem $T_\ell(u)$ is Fredholm if and only if $\tilde{\beta}_\ell(u)$ is invertible. Lemma A.2.5 implies that $\ker \pi_\partial = \ker \tilde{\beta}_\ell$ so $\tilde{\beta}_\ell(u)$ is invertible if and only if $\pi_\partial(u)$ is invertible. □

A.3 Pulling symbols back from S^{2n-1}

To put the Toeplitz operators on a Landau level in a suitable homological picture, we must pass from A to $C(S^{2n-1})$. This is a consequence of the circumstance that A is homotopy equivalent to $\mathbb{C}$, so A does not contain any relevant topological information. With Lemma A.2.5 in mind we define the Toeplitz algebra $\mathscr{T}_{\Bbbk}$ for $C(S^{2n-1})$ as if $\tilde{\beta}_{\Bbbk}$ were injective. So let $\lambda : C(S^{2n-1}) \to \mathscr{B}(L^2(\mathbb{C}^n))$ denote the $*$-representation defined by

$$\lambda(a)f(z) = a\left(\frac{z}{|z|}\right)f(z). \tag{A.4}$$

Take $\chi_0 \in C^\infty(\mathbb{R})$ to be a smooth function such that $\chi_0(x) = 0$ for $|x| \leq 1$ and $1 - \chi_0 \in C_c^\infty(\mathbb{R})$. We define the cut-off $\chi(z) := \chi_0(|z|)$ and the operator

$$\tilde{P}_{\Bbbk} := P_{\Bbbk}\chi. \tag{A.5}$$

For the operator $\tilde{P}_{\Bbbk}$, $\mathfrak{q}(\tilde{P}_{\Bbbk})$ is a projection by Lemma A.2.1. We let $\mathscr{T}_{\Bbbk}$ be the C^*-algebra generated by $\tilde{P}_{\Bbbk}\lambda(C(S^{2n-1}))\tilde{P}_{\Bbbk}^*$.

Theorem A.3.1. *For any $\Bbbk, \Bbbk' \in \mathbb{N}^n$ there exists a unitary*

$$Q_{\Bbbk,\Bbbk'} : \mathscr{L}_{\Bbbk'} \to \mathscr{L}_{\Bbbk}$$

such that $Ad(Q_{\Bbbk,\Bbbk'}) : \mathscr{T}_{\Bbbk} \to \mathscr{T}_{\Bbbk'}$ is an isomorphism satisfying

$$\mathfrak{q}(\tilde{P}_{\Bbbk'}\lambda(a)\tilde{P}_{\Bbbk'}^*) = \mathfrak{q} \circ Ad(Q_{\Bbbk,\Bbbk'})(\tilde{P}_{\Bbbk}\lambda(a)\tilde{P}_{\Bbbk}^*). \tag{A.6}$$

Furthermore, for any $\Bbbk \in \mathbb{N}^n$, the representation of $\mathscr{T}_{\Bbbk}$ on $\mathscr{L}_{\Bbbk}$ given by the inclusion $\mathscr{T}_{\Bbbk} \subseteq \mathscr{B}(\mathscr{L}_{\Bbbk})$ is irreducible and has the cyclic vector $\xi_{\Bbbk}$ defined by

$$\xi_{\Bbbk}(z) := q_{\Bbbk}^*(\mathrm{e}^{-|z|^2/4}).$$

Up to normalization the cyclic vectors satisfy

$$Q_{\Bbbk,\Bbbk'}\xi_{\Bbbk'} = \xi_{\Bbbk}.$$

Proof. Let us start with observing that for any $a, b \in C(S^{2n-1})$ we have

$$\tilde{P}_{\Bbbk}\lambda(ab)\tilde{P}_{\Bbbk}^* - \tilde{P}_{\Bbbk}\lambda(a)\tilde{P}_{\Bbbk}^*P_{\Bbbk}\lambda(b)\tilde{P}_{\Bbbk}^* \in \mathscr{K}.$$

So if $\mathscr{T}_{\Bbbk}$ acts irreducibly on $\mathscr{L}_{\Bbbk}$, then $\mathscr{K} \subseteq \mathscr{T}_{\Bbbk}$.

First we will construct a cyclic vector for the $\mathscr{T}_{\Bbbk}$-action on $\mathscr{L}_{\Bbbk}$ and use the cyclic vector in $\mathscr{L}_0$ to show that $\mathscr{T}_0$ acts irreducibly on $\mathscr{L}_0$. Then we will show that for $\Bbbk$ such that $\mathscr{T}_{\Bbbk}$ acts irreducibly on $\mathscr{L}_{\Bbbk}$ and $1 \leq j \leq n$ there is an isomorphism $\mathscr{T}_{\Bbbk} \cong \mathscr{T}_{\Bbbk+e_j}$ induced by a unitary intertwining the $\mathscr{T}_{\Bbbk}$-action on $L_{\Bbbk}$ with the $\mathscr{T}_{\Bbbk+e_j}$-action on $\mathscr{L}_{\Bbbk+e_j}$.

Consider the elements $\xi_{\mathbb{m},\mathbb{k}} \in \mathscr{L}_\mathbb{k}$ for $\mathbb{m} \in \mathbb{N}^n$ defined by

$$\xi_{\mathbb{m},\mathbb{k}}(z) := q_\mathbb{k}^*(z^\mathbb{m} \mathrm{e}^{-|z|^2/4}).$$

The elements $\xi_{\mathbb{m},\mathbb{k}}$ form an orthogonal basis for $\mathscr{L}_\mathbb{k}$. As in the statement of the theorem, we define $\xi_\mathbb{k} := \xi_{0,\mathbb{k}}$. For $a \in C(S^{2n-1})$ we have

$$\langle \xi_{\mathbb{m},\mathbb{k}}, \tilde{P}_\mathbb{k} a \tilde{P}_\mathbb{k}^* \xi_\mathbb{k} \rangle = \langle \xi_{\mathbb{m},\mathbb{k}}, \chi^2 a \xi_\mathbb{k} \rangle =$$
$$\int_{\mathbb{C}^n} \bar{q}_\mathbb{k}^*(\bar{z}^\mathbb{m} \mathrm{e}^{-|z|^2/4}) q_\mathbb{k}^*(\mathrm{e}^{-|z|^2/4}) \chi^2(z) a\left(\frac{z}{|z|}\right) \mathrm{d}V = \int_{S^{2n-1}} p_\mathbb{m}(\bar{z}) a(z) \mathrm{d}S,$$

for some polynomials $p_\mathbb{m}$ of degree at most $2|\mathbb{k}| + |\mathbb{m}|$. It follows that $\mathscr{T}_\mathbb{k} \xi_\mathbb{k}$ span $\mathscr{L}_\mathbb{k}$ and therefore $\overline{\mathscr{T}_\mathbb{k} \xi_\mathbb{k}} = \mathscr{L}_\mathbb{k}$. Thus $\xi_\mathbb{k}$ is a cyclic vector for the $\mathscr{T}_\mathbb{k}$-action.

By standard theory $\mathscr{T}_0$ acts irreducibly on $\mathscr{L}_0$ if and only if there are no non-zero $\xi_0', \xi_0'' \in \mathscr{L}_0$ such that $\xi_0 = \xi_0' + \xi_0''$ and $\mathscr{T}_0 \xi_0' \perp \mathscr{T}_0 \xi_0''$. Assume that for some $\xi_0' \in \mathscr{L}_0$ we have $\mathscr{T}_0 \xi_0' \perp \mathscr{T}_0(\xi_0 - \xi_0')$. The orthogonality condition implies that $\langle \tilde{P}_0 a \tilde{P}_0^*(\xi_0 - \xi_0'), \xi_0' \rangle = 0$ for all $a \in C(S^{2n-1})$ and P_0 is self-adjoint so this relation is equivalent to $\langle \chi^2 a \xi_0, \xi_0' \rangle = \langle \chi^2 a \xi_0', \xi_0' \rangle$ for all $a \in C(S^{2n-1})$. There exist a holomorphic function f_0 such that $\xi_0'(z) = f_0(z) \mathrm{e}^{-|z|^2/4}$ and the equation $\langle \chi^2 a \xi_0, \xi_0' \rangle = \langle \chi^2 a \xi_0', \xi_0' \rangle$ implies

$$\int_{\mathbb{C}^n} \overline{f_0(z)} \mathrm{e}^{-|z|^2/2} \chi^2(z) a\left(\frac{z}{|z|}\right) \mathrm{d}V = \int_{\mathbb{C}^n} |f_0(z)|^2 \mathrm{e}^{-|z|^2/2} \chi^2(z) a\left(\frac{z}{|z|}\right) \mathrm{d}V.$$

Hence f_0 must be real, and since it is holomorphic it must be constant. Thus ξ_0' is in the linear span of ξ_0 and ξ_0 defines a pure state. Since the $\mathscr{T}_0$-action on $\mathscr{L}_0$ has a pure state, it is irreducible.

Assume that $\mathscr{T}_\mathbb{k}$ acts irreducibly on $\mathscr{L}_\mathbb{k}$. Consider the polar decomposition of the unbounded operator q_j on $L^2(\mathbb{C}^n)$, that is $q_j^* = E_j Q_j$ where Q_j is a coisometry and E_j is a positive unbounded operator that is strictly positive on the image of Q_j. Clearly E_j is diagonal on the energy levels and

$$E_j = \bigoplus_{\mathbb{k}' \in \mathbb{N}^n} \sqrt{k_j'}\, P_{\mathbb{k}'}.$$

We define the $*$-homomorphism $\rho_j : \mathscr{T}_{\mathbb{k}+e_j} \to \mathscr{B}(\mathscr{L}_\mathbb{k})$ by $\rho_j(T) := Q_j^* T Q_j|_{\mathscr{L}_\mathbb{k}}$. Since Q_j is a coisometry this is clearly a $*$-monomorphism. It follows from the fact that $q_j^*| : \mathscr{L}_\mathbb{k} \to \mathscr{L}_{\mathbb{k}+e_j}$ is an isomorphism, that $Q_j| : \mathscr{L}_\mathbb{k} \to \mathscr{L}_{\mathbb{k}+e_j}$ is unitary, so ρ_j is unital. If $a \in C^\infty(S^{2n-1})$ then for some non-zero constant c we have

$$\rho_j(\tilde{P}_{\mathbb{k}+e_j} \lambda(a) \tilde{P}_{\mathbb{k}+e_j}^*) = c q_j \tilde{P}_{\mathbb{k}+e_j} \lambda(a) \tilde{P}_{\mathbb{k}+e_j}^* q_j^*|_{\mathscr{L}_\mathbb{k}} =$$
$$= c P_\mathbb{k} \left[\frac{\partial}{\partial \bar{z}_j}, \chi^2 \lambda(a)\right] P_{\mathbb{k}+e_j} q_j^*|_{\mathscr{L}_\mathbb{k}} + \tilde{P}_\mathbb{k} \lambda(a) \tilde{P}_\mathbb{k}^* \in \mathscr{T}_\mathbb{k},$$

because Theorem A.2.2 implies $P_{\Bbbk} b P_{\Bbbk+e_j} \in \mathscr{K}(L^2(\mathbb{C}^n))$ for $b \in A$ and by the induction assumption $\mathscr{K} \subseteq \mathscr{T}_{\Bbbk}$. So we obtain a $*$-monomorphism $\rho_j : \mathscr{T}_{\Bbbk+e_j} \to \mathscr{T}_{\Bbbk}$. However, we have cyclic vectors $\xi_{\Bbbk}$ and $\xi_{\Bbbk+e_j}$ for $\mathscr{T}_{\Bbbk}$ respectively $\mathscr{T}_{\Bbbk+e_j}$. For these vectors, $Q_j \xi_{\Bbbk}$ is a multiple of $\xi_{\Bbbk+e_j}$ so

$$\mathscr{L}_{\Bbbk+e_j} = \overline{\mathscr{T}_{\Bbbk+e_j} \xi_{\Bbbk+e_j}} \xrightarrow{Q_j^*} \overline{\mathscr{T}_{\Bbbk} \xi_{\Bbbk}}.$$

Therefore ρ_j is surjective and an isomorphism. We conclude that $\mathscr{T}_{\Bbbk}$ is independent of $\Bbbk$ and the representations on $\mathscr{L}_{\Bbbk}$ are irreducible since ξ_0 is pure and the $\mathscr{T}_{\Bbbk}$-actions are all equivalent. □

In [30] a weaker, but more explicit, statement was proven in complex dimension 1. Lemma 9.2 of [30] gives an explicit expression of $Q_{k,0}^* T_k(a) Q_{k,0}$ if $a \in A$ is smooth as

$$Q_{k,0}^* T_k(a) Q_{k,0} = T_0(\mathscr{D}_k(a)),$$

where $\mathscr{D}_k := \mathrm{id} + \sum_{j=1}^k d_{j,k} \Delta^j$, for some explicit constants $d_{j,k}$ and Δ is the Laplacian on $\mathbb{C}$.

For $i = 1, \dots, n$ we let $z_i : S^{2n-1} \to \mathbb{C}$ denote the coordinate functions of the embedding $S^{2n-1} \subseteq \mathbb{C}^n$. Clearly $z_i \in C(S^{2n-1})$.

Corollary A.3.2. *The operators $P_{\Bbbk} \lambda(z_i) P_{\Bbbk}$ together with $\mathscr{K}$ generate $\mathscr{T}_{\Bbbk}$ as a C^*-algebra.*

Proof. Let U denote the C^*-algebra generated by $P_{\Bbbk} \lambda(z_i) P_{\Bbbk}$ and $\mathscr{K}$. The C^*-algebra $\mathscr{T}_{\Bbbk}$ is the C^*-algebra generated by the linear space $P_{\Bbbk} \lambda(C(S^{2n-1})) P_{\Bbbk}$ because $P_{\Bbbk} \lambda(a) P_{\Bbbk} - \tilde{P}_{\Bbbk} \lambda(a) \tilde{P}_{\Bbbk}^* \in \mathscr{K}$. So it is sufficient to prove $P_{\Bbbk} \lambda(C(S^{2n-1})) P_{\Bbbk} \subseteq U$. Given a function $a \in C(S^{2n-1})$ the Stone-Weierstrass theorem implies that there is a sequence of polynomials $R_j = R_j(z, \bar{z})$ such that $R_j \to a$ in $C(S^{2n-1})$. The functions R_j are polynomials so it follows that

$$P_{\Bbbk} \lambda(R_j) P_{\Bbbk} - R_j(P_{\Bbbk} \lambda(z) P_{\Bbbk}, P_{\Bbbk} \lambda(z^*) P_{\Bbbk}) \in \mathscr{K}$$

and $P_{\Bbbk} \lambda(R_j) P_{\Bbbk} \in U$. Finally $\|P_{\Bbbk} \lambda(R_j) P_{\Bbbk} - P_{\Bbbk} \lambda(a) P_{\Bbbk}\|_{\mathscr{B}(\mathscr{L}_{\Bbbk})} \leq \|R_j - a\|_{C(S^{2n-1})}$ which implies $P_{\Bbbk} \lambda(a) P_{\Bbbk} \in U$. □

Corollary A.3.3. *The mapping $\beta_{\Bbbk} : C(S^{2n-1}) \to \mathscr{C}(\mathscr{L}_{\Bbbk})$ induced from $\tilde{\beta}_{\Bbbk}$ is injective, so if $u \in A \otimes M_N(\mathbb{C})$ the operator $T_{\Bbbk}(u)$ is Fredholm if and only if $\pi_\partial(u)$ is invertible.*

Proof. Due to equation (A.6) in Theorem A.3.1, the Corollary follows from Lemma A.2.5. The proof of the second statement of the Corollary is proven in the same fashion as Proposition A.2.6. □

From the fact that the mapping $\beta_{\Bbbk}$ is injective it follows that the symbol mapping $\tilde{P}_{\Bbbk}\lambda(a)\tilde{P}_{\Bbbk}^* \mapsto a$ gives a well defined surjection $\sigma_{\Bbbk} : \mathscr{T}_{\Bbbk} \to C(S^{2n-1})$. Clearly the kernel of $\sigma_{\Bbbk}$ is non-zero and $\ker\sigma_{\Bbbk} \subseteq \mathscr{K}$, so by Theorem A.3.1 $\ker\sigma_{\Bbbk} = \mathscr{K}$. Therefore we can construct the exact sequence

$$0 \to \mathscr{K} \to \mathscr{T}_{\Bbbk} \xrightarrow{\sigma_{\Bbbk}} C(S^{2n-1}) \to 0. \tag{A.7}$$

A completely positive splitting of the symbol mapping $\sigma_{\Bbbk} : \mathscr{T}_{\Bbbk} \to C(S^{2n-1})$ is given by $a \mapsto \tilde{P}_{\Bbbk}\lambda(a)\tilde{P}_{\Bbbk}^*$.

The exact sequence (A.7) defines an extension class $[\mathscr{T}_{\Bbbk}] \in Ext(C(S^{2n-1}))$. Since $C(S^{2n-1})$ is a nuclear C^*-algebra there is an isomorphism $Ext(C(S^{2n-1})) \cong K^1(C(S^{2n-1}))$ and we can describe the K-homology class of $[\mathscr{T}_{\Bbbk}]$ explicitly by a Fredholm module as follows; we let $\lambda : C(S^{2n-1}) \to \mathscr{B}(L^2(\mathbb{C}^n))$ be as in equation (A.4) and define the operator

$$F_{\Bbbk} = \frac{(1+\tilde{P}_{\Bbbk})}{2}$$

where $\tilde{P}_{\Bbbk}$ is as in equation (A.5). Clearly, $(L^2(\mathbb{C}^n), \lambda, F_{\Bbbk})$ defines a Fredholm module which represents the image of $[\mathscr{T}_{\Bbbk}]$ in $K^1(C(S^{2n-1}))$.

Corollary A.3.4. *The class $[\mathscr{T}_{\Bbbk}] \in Ext(C(S^{2n-1}))$ is independent of $\Bbbk$.*

Proof. The extension $\mathscr{T}_{\Bbbk}$ is equivalent to $\mathscr{T}_{\Bbbk'}$ since it follows from equation (A.6) that the following diagram with exact rows commute

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{K} \arrow[r] \arrow[d, "Ad(Q_{\Bbbk,\Bbbk'})"] & \mathscr{T}_{\Bbbk'} \arrow[r] \arrow[d, "Ad(Q_{\Bbbk,\Bbbk'})"] & C(S^{2n-1}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{K} \arrow[r] & \mathscr{T}_{\Bbbk} \arrow[r] & C(S^{2n-1}) \arrow[r] & 0
\end{tikzcd}.$$

□

So we know that $[\mathscr{T}_{\Bbbk}]$ is independent of $\Bbbk$, this implies that the index of $T_{\Bbbk}(u)$ for $u \in M_n \otimes A$ is independent of $\Bbbk$. But how do we calculate it? The index theorem that allows the calculation involves studying how the coordinate functions on S^{2n-1} act on the monomial base of $\mathscr{L}_0$. We will first review some theory of Toeplitz operators on the Bergman space and then study what happens in complex dimension 1 and 2.

The Bergman space on the unit ball $B_{2n} \subseteq \mathbb{C}^n$ is defined as $A^2(B_{2n}) := L^2(B_{2n}) \cap \mathscr{O}(B_{2n})$, that is; holomorphic functions on B_{2n} which are square integrable. The Bergman space is a closed subspace of $L^2(B_{2n})$ and we will denote the orthogonal projection $L^2(B_{2n}) \to A^2(B_{2n})$ by P_B.

The Bergman projection defines a K-homology class $[P_B] \in K^1(C(S^{2n-1}))$ in the same fashion as for the Landau projections. That is, for $a \in C(\overline{B_{2n}})$ the operator $[P_B, a] \in \mathscr{B}(L^2(B_{2n}))$ is compact. The reason that we can use P_B to define a K-homology class for S^{2n-1} instead of $\overline{B_{2n}}$ is analogously to above that $P_B a|_{A^2(B_{2n})}$ is compact if and only if $a \in C_0(B_{2n})$, see more in [89]. Thus $P_B a|_{A^2(B_{2n})}$ is Fredholm if and only if $a|_{S^{2n-1}}$ is invertible.

Furthermore, $P_B a P_B$ is compact if and only if $a \in C_0(B_{2n})$. So $[P_B]$ is a well defined K-homology class in $K^1(C(S^{2n-1}))$. By [23] the following index formula holds for the Toeplitz operator $P_B a|_{A^2(B_{2n})}$ if the symbol $a_\partial := a|_{S^{2n-1}}$ is smooth:

$$\operatorname{ind}(P_B a|_{A^2(B_{2n})}) = \frac{-(n-1)!}{(2n-1)!(2\pi i)^n} \int_{S^{2n-1}} \operatorname{tr}((a_\partial^{-1} \mathrm{d} a_\partial)^{2n-1}). \tag{A.8}$$

This formula was also proven in [46] by an elegant use of the Atiyah-Singer index theorem.

We will by $\mathscr{T}^n$ denote the C^*-algebra generated by $P_B C(\overline{B_{2n}}) P_B$ in $\mathscr{B}(A^2(B_{2n}))$. The K-homology class $[P_B] \in K^1(C(S^{2n-1}))$ can be represented by the extension class $[\mathscr{T}^n] \in Ext(C(S^{2n-1}))$ defined by means of the short exact sequence

$$0 \to \mathscr{K} \to \mathscr{T}^n \xrightarrow{\sigma^n} C(S^{2n-1}) \to 0. \tag{A.9}$$

A.4 The special cases $\mathbb{C}$ and $\mathbb{C}^2$

In this section we will study the special cases of complex dimension 1 and 2. Dimension 1 has been studied previously in [12] and provides a simpler picture than in higher dimensions. In the 1-dimensional case we have that $K_1(C(S^1)) \cong \mathbb{Z}$ and we can take the coordinate function $z : S^1 \to \mathbb{C}$ to be a generator. So when we want to determine the class $[\mathscr{T}_k]$ we only need to calculate the index of $P_k \lambda(z) P_k$ where λ is as in equation (A.4). We recall the following Proposition from [12]:

Proposition A.4.1 (Proposition 7.3 from [12])**.** *For any $k \in \mathbb{N}$ we have that*

$$\operatorname{ind}(P_k \lambda(z) P_k) = -1.$$

The method used in [12] to prove this Proposition was to show that in a suitable basis $P_k \lambda(z) P_k$ was up to some coefficients a unilateral shift. In higher dimension the proof is based on similar ideas.

Theorem A.4.2. *For $n = 1$ there is an isomorphism $\mathscr{T}_k \cong \mathscr{T}^1$ making $[\mathscr{T}_k] = [\mathscr{T}^1] \in K^1(C(S^1))$.*

Proof. By Proposition 7.3 of [12]

$$[u] \circ [\mathscr{T}_k] = \operatorname{ind}(P_k \lambda(u) P_k) = -\operatorname{wind}(u) = [u] \circ [\mathscr{T}^1] \tag{A.10}$$

for an invertible function $u \in C(S^1)$. Here $\mathsf{wind}(u)$ denotes the winding number of u which is defined for smooth u as

$$\mathrm{wind}(u) := \frac{1}{2\pi i}\int_{S^1} u^{-1}\mathrm{d}u$$

and defines an isomorphism $K_1(C(S^1)) \to \mathbb{Z}$. By the Universal Coefficient Theorem for KK-theory (see Theorem 4.2 of [75]) the mapping

$$K^1(C(S^1)) \to Hom(K_1(C(S^1)), \mathbb{Z})$$

is an isomorphism so equation (A.10) implies that $[\mathscr{T}_k] = [\mathscr{T}^1]$.

By Theorem 13 of [38], the short exact sequence $0 \to \mathscr{K} \to \mathscr{T}_k \to C(S^1) \to 0$ is characterized by an isometry or coisometry v such that $vv^* - 1$ and $v^*v - 1$ are compact and $\mathscr{T}_k$ is generated by v. Then $z \mapsto v$ defines a splitting and the symbol mapping $\mathscr{T}_k \to C(S^1)$ is just $v \mapsto z$. By equation (A.10), $1 - vv^*$ is a rank one projection, so the theorem follows. □

Also in dimension 2 we can find a generator for the odd K-theory. As generator for $K_1(C(S^3)) \cong \mathbb{Z}$ we can take the diffeomorphism $u : S^3 \to SU(2)$ defined as

$$u(z_1, z_2) := \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}.$$

Proposition A.4.3. *The extension class $[\mathscr{T}^2]$ generates $K^1(C(S^3))$ and $[u]$ generates $K_1(C(S^3))$.*

Proof. Recalling that P_B denotes the Bergman projection we will start by calculating the index of the Toeplitz operator $P_B u P_B : A^2(B_4) \otimes \mathbb{C}^2 \to A^2(B_4) \otimes \mathbb{C}^2$. Using the index theorem by Boutet de Monvel ([23]) reviewed above in equation (A.8), the following index formula holds for smooth u:

$$\mathrm{ind}(P_B u P_B) = -\frac{1}{3!(2\pi i)^2}\int_{S^3} \mathrm{tr}((u^*\mathrm{d}u)^3). \tag{A.11}$$

A straight-forward calculation gives that

$$\mathrm{tr}((u^*\mathrm{d}u)^3) = 3(z_1\mathrm{d}\bar{z}_1 - \bar{z}_1\mathrm{d}z_1)\wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 + 3(z_2\mathrm{d}\bar{z}_2 - \bar{z}_2\mathrm{d}z_2)\wedge \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1.$$

Invoking Stokes Theorem on equation (A.11) gives that

$$-\frac{1}{3!(2\pi i)^2}\int_{S^3} \mathrm{tr}((u^*\mathrm{d}u)^3) = \frac{1}{48\cdot vol(B_4)}\int_{B_4} \mathrm{dtr}((u^*\mathrm{d}u)^3) =$$
$$= \frac{1}{4\cdot vol(B_4)}\int_{B_4} \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 = -\frac{1}{vol(B_4)}\int_{B_4} \mathrm{d}V = -1$$

This equation shows that

$$[u] \circ [\mathscr{T}^2] = \operatorname{ind}(P_B u P_B) = -1. \tag{A.12}$$

Consider the split-exact sequence $0 \to C_0(\mathbb{R}^3) \to C(S^3) \to \mathbb{C} \to 0$ where the mapping $C(S^3) \to \mathbb{C}$ is point evaluation. Since the sequence splits, and $K_1(\mathbb{C}) = K^1(\mathbb{C}) = 0$ the embedding $C_0(\mathbb{R}^3) \to C(S^3)$ induces isomorphisms $K_1(C(S^3)) \cong K_1(C_0(\mathbb{R}^3)) = \mathbb{Z}$ and $K^1(C(S^3)) \cong K^1(C_0(\mathbb{R}^3)) = \mathbb{Z}$. So the Kasparov product $K_1(C(S^3)) \times K^1(C(S^3)) \to \mathbb{Z}$ is just a pairing $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, and since $[u] \circ [\mathscr{T}^2] = -1$ it follows that $[\mathscr{T}^2]$ generates $K^1(C(S^3))$ and $[u]$ generates $K_1(C(S^3))$. □

Theorem A.4.4. *For any* $\Bbbk \in \mathbb{N}^2$ *we have*

$$\operatorname{ind}(P_\Bbbk \lambda(u) P_\Bbbk) = -1. \tag{A.13}$$

Therefore $[\mathscr{T}^2] = [\mathscr{T}_\Bbbk]$.

Proof. If equation (A.13) holds, $[\mathscr{T}^2] = [\mathscr{T}_\Bbbk]$ follows directly from equation (A.12) using the Universal Coefficient Theorem for KK-theory (see Theorem 4.2 of [75]). This is a consequence of the fact that the natural mapping

$$K^1(C(S^3)) \to Hom(K_1(C(S^3)), \mathbb{Z})$$

is an isomorphism. The injectivity of this mapping implies that if $[u] \circ [\mathscr{T}^2] = [u] \circ [\mathscr{T}_\Bbbk]$ for a generator $[u]$ then $[\mathscr{T}^2] = [\mathscr{T}_\Bbbk]$.

To prove equation (A.13) we take $\Bbbk = 0$, since Corollary A.3.4 implies that the integer $\operatorname{ind}(P_\Bbbk \lambda(u) P_\Bbbk)$ is independent of $\Bbbk$. We claim that $P_0 \lambda(u) P_0$ is an injective operator and the cokernel of $P_0 \lambda(u) P_0$ is spanned by the $\mathbb{C}^2$-valued function $z \mapsto \mathrm{e}^{-|z|^2/4} \oplus 0$. This statement will prove the theorem.

To prove that $P_0 \lambda(u) P_0$ is injective, assume $f \in \ker(P_0 \lambda(u) P_0)$. Define the functions

$$\xi^{\mathbb{m}}(z) := z^{\mathbb{m}} \mathrm{e}^{-|z|^2/4}$$

for $\mathbb{m} \in \mathbb{N}^2$. The functions $\xi^{\mathbb{m}}$ form an orthogonal basis for $\mathscr{L}_0$ by Theorem 1.63 of [40]. Expand the function f in an L^2-convergent series

$$f = \sum_{\mathbb{m} \in \mathbb{N}^2} c_{\mathbb{m}} \xi^{\mathbb{m}},$$

where $c_{\mathbb{m}} = c_{\mathbb{m}}^{(1)} \oplus c_{\mathbb{m}}^{(2)} \in \mathbb{C}^2$. Since $f \in \ker(P_0 \lambda(u) P_0)$ we have the following orthogonality condition

$$0 = \langle \xi^{\mathbb{m}'} \oplus 0, \lambda(u) f \rangle = \sum_{\mathbb{m}} \int_{\mathbb{C}^2} \left(c_{\mathbb{m}}^{(1)} \frac{\bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_1}}{|z|} + c_{\mathbb{m}}^{(2)} \frac{\bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_2}}{|z|} \right) \mathrm{e}^{|z|^2/2} \mathrm{d}V =$$

$$= \sum_{\mathbb{m}} t_{\mathbb{m},\mathbb{m}'} \int_{S^3} \left(c^{(1)}_{\mathbb{m}} \bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_1} + c^{(2)}_{\mathbb{m}} \bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_2} \right) \mathrm{d}S,$$

for some coefficients $t_{\mathbb{m},\mathbb{m}'}$, for a detailed calculation of $t_{\mathbb{m},\mathbb{m}'}$ see below in Proposition A.5.1. Using that the functions $\xi^{\mathbb{m}}$ are orthogonal we obtain that there exists a $C_{\mathbb{m}} > 0$ such that

$$c^{(1)}_{\mathbb{m}-e_1} = -C_{\mathbb{m}} c^{(2)}_{\mathbb{m}-e_2}. \tag{A.14}$$

On the other hand, we have

$$0 = \langle 0 \oplus \xi^{\mathbb{m}'}, \lambda(u) f \rangle = \sum_{\mathbb{m}} \int_{\mathbb{C}^2} \left(-c^{(1)}_{\mathbb{m}} \frac{\bar{z}^{\mathbb{m}'+e_2} z^{\mathbb{m}}}{|z|} + c^{(2)}_{\mathbb{m}} \frac{\bar{z}^{\mathbb{m}'+e_1} z^{\mathbb{m}}}{|z|} \right) \mathrm{e}^{|z|^2/2} \mathrm{d}V =$$

$$= \sum_{\mathbb{m}} t_{\mathbb{m},\mathbb{m}'} \int_{S^3} \left(-c^{(1)}_{\mathbb{m}} \bar{z}^{\mathbb{m}'+e_2} z^{\mathbb{m}} + c^{(2)}_{\mathbb{m}} \bar{z}^{\mathbb{m}'+e_1} z^{\mathbb{m}} \right) \mathrm{d}S.$$

Again using orthogonality of the functions $\xi^{\mathbb{m}}$ we obtain that there is a $C'_{\mathbb{m}} > 0$ such that

$$c^{(1)}_{\mathbb{m}+e_2} = C'_{\mathbb{m}} c^{(2)}_{\mathbb{m}+e_1}. \tag{A.15}$$

Equation (A.14) implies $c^{(1)}_{\mathbb{m}} = 0$ for $m_2 = 0$. For $m_2 > 0$ equation (A.14) implies

$$c^{(1)}_{\mathbb{m}} = -C_{\mathbb{m}+e_1} c^{(2)}_{\mathbb{m}-e_2+e_1}.$$

Then equation (A.15) for $\mathbb{m} - e_2$ gives

$$c^{(1)}_{\mathbb{m}} \left(1 + \frac{C_{\mathbb{m}+e_1}}{C'_{\mathbb{m}-e_2}} \right) = 0.$$

So $c^{(1)}_{\mathbb{m}} = 0$ for all $\mathbb{m}$. Equation (A.14) implies $c^{(2)}_{\mathbb{m}} = 0$ for all $\mathbb{m}$. Thus $f = 0$ and $\ker(P_0 \lambda(u) P_0) = 0$.

The second statement, that the cokernel of $P_0 \lambda(u) P_0$ is spanned by the $\mathbb{C}^2$-valued function

$$z \mapsto \mathrm{e}^{-|z|^2/4} \oplus 0,$$

is proven analogously. There is a natural isomorphism

$$\operatorname{coker} P_0 \lambda(u) P_0 \cong (\operatorname{im} P_0 \lambda(u) P_0)^{\perp} = \ker P_0 \lambda(u^*) P_0.$$

Analogously to the reasoning above, for $g \in \ker P_0 \lambda(u^*) P_0$ we expand the function g in an L^2-convergent series

$$g = \sum_{\mathbb{m} \in \mathbb{N}^2} d_{\mathbb{m}} \xi^{\mathbb{m}},$$

where $d_{\mathbb{m}} = d^{(1)}_{\mathbb{m}} \oplus d^{(2)}_{\mathbb{m}} \in \mathbb{C}^2$. After taking scalar product by $\xi_{\mathbb{m}'}$, for some $D_{\mathbb{m}}, D'_{\mathbb{m}} > 0$ we obtain the following conditions on the coefficients:

$$d^{(1)}_{\mathbb{m}+e_1} = D_{\mathbb{m}} d^{(2)}_{\mathbb{m}-e_2} \quad \text{and} \tag{A.16}$$

$$d^{(1)}_{\mathbb{m}+e_2} = -D'_{\mathbb{m}} d^{(2)}_{\mathbb{m}-e_1}. \tag{A.17}$$

The second of these equations implies $d^{(1)}_{\mathbb{m}} = 0$ for $m_1 = 0$ and $m_2 > 0$. Also, the first of these equations implies $d^{(1)}_{\mathbb{m}} = 0$ for $m_2 = 0$ and $m_1 > 0$. For $m_1, m_2 > 0$, putting in $\mathbb{m} - e_1$ in the first equation, gives

$$d^{(1)}_{\mathbb{m}} = D_{\mathbb{m}-e_1} d^{(2)}_{\mathbb{m}-e_1-e_2}.$$

Finally, combining this relation with the second equation for $\mathbb{m} - e_2$ we obtain

$$d^{(1)}_{\mathbb{m}} \left(1 + \frac{D_{\mathbb{m}-e_1}}{D'_{\mathbb{m}-e_2}} \right) = 0 \quad \text{for} \quad m_1, m_2 > 0.$$

Therefore $d^{(1)}_{\mathbb{m}} = 0$ for all $\mathbb{m} \neq 0$. The equations in (A.16) imply $d^{(2)}_{\mathbb{m}} = 0$ for all $\mathbb{m}$. However, the function $z \mapsto \mathrm{e}^{-|z|^2/4} \oplus 0$, corresponding to $d^{(1)}_0 = 1$, is in the space $\ker(P_0 \lambda(u^*) P_0)$ which completes the proof. $\square$

A.5 Index formula on the particular Landau levels

In this section we will prove an index formula for the particular Landau levels. On S^{2n-1} we have the complex coordinates $z_1, \dots, z_n$ and we denote by $Z_1, \dots, Z_n$ the image of these coordinate functions under the representation λ which was defined in equation (A.4). So Z_i is the operator on $L^2(\mathbb{C}^n)$ given by multiplication by the almost everywhere defined function $z \mapsto \frac{z_i}{|z|}$. Consider the polar decompositions

$$P_0 Z_i P_0 = V_{i,0} S_{i,0},$$

where $V_{i,0}$ are partial isometries and $S_{i,0} > 0$. An orthonormal basis for $\mathscr{L}_0$ is given by

$$\eta_{\mathbb{m}}(z) := \frac{z^{\mathbb{m}} \mathrm{e}^{-|z|^2/4}}{\sqrt{\pi^n 2^{|\mathbb{m}|+n} \mathbb{m}!}},$$

see more in [40].

Proposition A.5.1. *The operator $V_{i,0}$ is an isometry described by the equation*

$$V_{i,0} \eta_{\mathbb{m}} = \eta_{\mathbb{m}+e_i}$$

and the operator $S_{i,0}$ is diagonal in the basis $\eta_{\mathbb{m}}$ with eigenvalues given by

$$\lambda^{\eta}_{i,\mathbb{m}} = \Gamma\left(|\mathbb{m}| + n + \frac{1}{2}\right) \frac{\sqrt{m_i + 1}}{(|\mathbb{m}| + n)!}. \tag{A.18}$$

Proof. For $\mathbb{m}, \mathbb{m}' \in \mathbb{N}$ we have

$$\begin{aligned}
\langle \eta_{\mathbb{m}'}, Z_i \eta_{\mathbb{m}} \rangle &= \int_{\mathbb{C}^n} \frac{1}{\pi^n \sqrt{2^{|\mathbb{m}+\mathbb{m}'|+2n} \mathbb{m}! \mathbb{m}'!}} \frac{\bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_i}}{|z|} \mathrm{e}^{-|z|^2/2} \mathrm{d}V = \\
&= \frac{1}{\pi^n \sqrt{2^{|\mathbb{m}+\mathbb{m}'|+2n} \mathbb{m}! \mathbb{m}'!}} \int_0^\infty r^{|\mathbb{m}|+|\mathbb{m}'|+n-1} \mathrm{e}^{-r^2/2} \mathrm{d}r \int_{S^{2n-1}} \bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_i} \mathrm{d}S = \\
&= \delta_{\mathbb{m}',\mathbb{m}+e_i} \frac{\Gamma\left(|\mathbb{m}| + n + \frac{1}{2}\right)}{2\pi^n \mathbb{m}! \sqrt{(\mathbb{m}_j + 1)}} \int_{S^{2n-1}} \bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_i} \mathrm{d}S = \\
&= \delta_{\mathbb{m}',\mathbb{m}+e_i} \Gamma(|\mathbb{m}| + n + \frac{1}{2}) \frac{\sqrt{m_i + 1}}{(|\mathbb{m}| + n)!}.
\end{aligned}$$

It follows that $V_{i,0}\eta_{\mathbb{m}} = \eta_{\mathbb{m}+e_i}$ and $S_{i,0}\eta_{\mathbb{m}} = \lambda^{\eta}_{i,\mathbb{m}}\eta_{\mathbb{m}}$, where $\lambda^{\eta}_{i,\mathbb{m}}$ is as in equation (A.18). □

On the other hand, we can, just as on $\mathscr{L}_0$, let $\tilde{Z}_1, \ldots, \tilde{Z}_n \in \mathscr{B}(L^2(B_{2n}))$ be the operators on $L^2(B_{2n})$ defined by the multiplication by the almost everywhere defined function $z \mapsto \frac{z_i}{|z|}$. Consider the polar decompositions

$$P_B \tilde{Z}_i P_B = V_{i,B} S_{i,B},$$

where again $V_{i,B}$ are partial isometries and $S_{i,B} > 0$. An orthonormal basis for $A^2(B_{2n})$ is given by

$$\mu_{\mathbb{m}}(z) := \pi^{-n/2} \sqrt{\frac{(n + |\mathbb{m}|)!}{\mathbb{m}!}} z^{\mathbb{m}}.$$

Similar to the lowest Landau level, the partial isometries $V_{i,B}$ are just shifts in this basis:

Proposition A.5.2. *The operator $V_{i,B}$ is an isometry described by the equation*

$$V_{i,B}\mu_{\mathbb{m}} = \mu_{\mathbb{m}+e_i}$$

and the operator $S_{i,B}$ is diagonal in the basis $\mu_{\mathbb{m}}$ with eigenvalues given by

$$\lambda^{\mu}_{i,\mathbb{m}} = \frac{\sqrt{m_i + 1}}{\sqrt{n + |\mathbb{m}| + 1}}. \tag{A.19}$$

Proof. The proof is the analogous to that of Proposition A.5.1. For $\mathbb{m}, \mathbb{m}' \in \mathbb{N}$ we have

$$\langle \mu_{\mathbb{m}'}, \tilde{Z}_i \mu_{\mathbb{m}} \rangle = \int_{B_{2n}} \pi^{-n} \sqrt{\frac{(n+|\mathbb{m}|)!(n+|\mathbb{m}'|)!}{\mathbb{m}!\mathbb{m}'!}} \frac{\bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_i}}{|z|} \mathrm{d}V =$$

$$= \pi^{-n} \sqrt{\frac{(n+|\mathbb{m}|)!(n+|\mathbb{m}'|)!}{\mathbb{m}!\mathbb{m}'!}} \int_0^1 r^{|\mathbb{m}|+|\mathbb{m}'|+2n-1} \mathrm{d}r \int_{S^{2n-1}} \bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_i} \mathrm{d}S =$$

$$= \delta_{\mathbb{m}', \mathbb{m}+e_i} \frac{(n+|\mathbb{m}|)! \sqrt{n+|\mathbb{m}|+1}}{(2|\mathbb{m}|+2n)\mathbb{m}! \sqrt{m_i+1}} \int_{S^{2n-1}} \bar{z}^{\mathbb{m}'} z^{\mathbb{m}+e_i} \mathrm{d}S =$$

$$= \delta_{\mathbb{m}', \mathbb{m}+e_i} \frac{\sqrt{m_i+1}}{\sqrt{n+|\mathbb{m}|+1}}.$$

It follows that $V_{i,B}\mu_{\mathbb{m}} = \mu_{\mathbb{m}+e_i}$ and $S_{i,B}\mu_{\mathbb{m}} = \lambda^{\mu}_{i,\mathbb{m}}\mu_{\mathbb{m}}$ where the eigenvalues $\lambda^{\mu}_{i,\mathbb{m}}$ are given in equation (A.19). □

Lemma A.5.3. *If a is a real number then*

$$\frac{\Gamma(x+a)}{\Gamma(x)} = x^a + \mathcal{O}(x^{-1+a}) \quad \textit{as} \quad x \to +\infty.$$

Proof. By Stirling's formula

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \frac{\ln 2\pi}{2} + \mathcal{O}(x^{-1}).$$

After Taylor expanding $\ln \Gamma(x+a)$ around $a = 0$ we obtain that

$$\ln \Gamma(x+a) - \ln \Gamma(x) = a \ln x + \mathcal{O}(x^{-1}).$$

□

Lemma A.5.4. *With the unitary $U : A^2(B_{2n}) \to \mathscr{L}_0$ defined by $\mu_{\mathbb{m}} \mapsto \eta_{\mathbb{m}}$, the operators $S_{i,0}$ and $S_{i,B}$ satisfy*

$$U^* S_{i,0} U - S_{i,B} \in \mathscr{K}.$$

Proof. The operators $U^* S_{i,0} U$ and $S_{i,B}$ are both diagonal in the basis $\mu_{\mathbb{m}}$. So it is sufficient to prove that $|\lambda^{\eta}_{\mathbb{m}} - \lambda^{\mu}_{\mathbb{m}}| \to 0$. The proof of this statement is based on the estimate from Lemma A.5.3. When $|\mathbb{m}| \to \infty$, Lemma A.5.3 implies

$$|\lambda^{\eta}_{\mathbb{m}} - \lambda^{\mu}_{\mathbb{m}}| = \left| \frac{\Gamma\left(|\mathbb{m}| + n + \frac{1}{2}\right) \sqrt{m_i+1}}{(|\mathbb{m}|+n)!} - \frac{\sqrt{m_i+1}}{\sqrt{|\mathbb{m}|+n-1}} \right| =$$

$$= \sqrt{m_i+1} \left| \frac{\Gamma\left((|\mathbb{m}|+n+1) - \frac{1}{2}\right)}{\Gamma(|\mathbb{m}|+n+1)} - (|\mathbb{m}|+n-1)^{-1/2} \right| = \mathcal{O}(|\mathbb{m}|^{-1}).$$

Therefore we have that $U^*S_{i,0}U - S_{i,B} \in \mathscr{L}^{n+}(A^2(B_{2n}))$, the n:th Dixmier ideal. In particular $U^*S_{i,0}U - S_{i,B}$ is compact. □

Theorem A.5.5. *The unitary U induces an isomorphism $Ad(U) : \mathscr{T}_0 \xrightarrow{\sim} \mathscr{T}^n$ such that*

$$\sigma^n \circ Ad(U) = \sigma_0.$$

where σ^n and σ_0 are the symbol mappings.

Proof. Lemma A.5.4 and the Propositions A.5.1 and A.5.2 imply

$$U^*(P_0 Z_i P_0)U = P_B \tilde{Z}_i P_B + K_i, \tag{A.20}$$

for some compact operators K_i. Since $\mathscr{T}^n$ contains the compact operators, $U^*(P_0 Z_i P_0)U \in \mathscr{T}^n$. Corollary A.3.2 therefore implies $U^*\mathscr{T}_0 U \subseteq \mathscr{T}^n$. Theorem A.3.1 states that $\mathscr{T}_0$ acts irreducibly on $\mathscr{L}_0$, so $U^*\mathscr{T}_0 U$ acts irreducibly on $A^2(B_{2n})$. Therefore $\mathscr{K} \subseteq U^*\mathscr{T}_0 U$ and $P_B \tilde{Z}_i P_B \in U^*\mathscr{T}_0 U$. The operators $P_B \tilde{Z}_i P_B$ together with $\mathscr{K}$ generate $\mathscr{T}^n$ so $U^*\mathscr{T}_0 U \supseteq \mathscr{T}^n$. The relation $\sigma^n \circ Ad(U) = \sigma_0$ holds since by equation (A.20) it holds on the generators of $C(S^{2n-1})$. □

Corollary A.5.6. *Let $[\mathscr{T}^n] \in Ext(C(S^{2n-1}))$ denote the Toeplitz quantization of the Bergman space defined in equation* (A.9) *and $[\mathscr{T}_{\Bbbk}] \in Ext(C(S^{2n-1}))$ the Toeplitz quantization of the particular Landau level of height $\Bbbk$ defined in equation* (A.7)*. Then*

$$[\mathscr{T}^n] = [\mathscr{T}_{\Bbbk}].$$

So for $u \in A \otimes M_N(\mathbb{C})$ such that $u_\partial := \pi_\partial(u)$ is invertible and smooth

$$\operatorname{ind}(P_{\Bbbk} u|_{\mathscr{L}_{\Bbbk} \otimes \mathbb{C}^N}) = \frac{-(n-1)!}{(2n-1)!(2\pi i)^n} \int_{S^{2n-1}} \operatorname{tr}((u_\partial^{-1} \mathrm{d}u_\partial)^{2n-1}). \tag{A.21}$$

Proof. By Corollary A.3.4 the class $[\mathscr{T}_{\Bbbk}]$ is independent of $\Bbbk$, so take $\Bbbk = 0$. In this case Theorem A.5.5 implies that the unitary U makes the following diagram commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{T}_0 & \xrightarrow{\sigma_0} & C(S^{2n-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Ad(U)} & & \downarrow{\scriptstyle Ad(U)} & & \| & & \\
0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{T}^n & \xrightarrow{\sigma^n} & C(S^{2n-1}) & \longrightarrow & 0
\end{array}.$$

Therefore $[\mathscr{T}^n] = [\mathscr{T}_0] = [\mathscr{T}_{\Bbbk}]$ and the index formula (A.21) follows from [46]. □

Paper B

Analytic formulas for degree of non-smooth mappings: the odd-dimensional case

Abstract

The notion of topological degree is studied for mappings from the boundary of a relatively compact strictly pseudo-convex domain in a Stein manifold into a manifold in terms of index theory of Toeplitz operators on the Hardy space. The index formalism of non-commutative geometry is used to derive analytic integral formulas for the index of a Toeplitz operator with Hölder continuous symbol. The index formula gives an analytic formula for the degree of a Hölder continuous mapping from the boundary of a strictly pseudo-convex domain.

Introduction

This paper is a study of analytic formulas for the degree of a mapping from the boundary of a relatively compact strictly pseudo-convex domain in a Stein manifold. The degree of a continuous mapping between two compact, connected, oriented manifolds of the same dimension is abstractly defined in terms of homology. If the function f is differentiable, an analytic formula can be derived using Brouwer degree, see [69], or the more global picture of de Rham-cohomology. For any form ω of top degree the form $f^*\omega$ satisfies

$$\int_X f^*\omega = \deg f \int_Y \omega.$$

Without differentiability conditions on f, there are no known analytic formulas beyond the special case of a Hölder continuous mapping $S^1 \to S^1$ which can be found in Chapter $2.\alpha$ of [34]. The degree of a Hölder continuous function $f : S^1 \to S^1$ of exponent α is expressed by an analytic formula by replacing de Rham cohomology with the cyclic homology of the algebra of Hölder continuous functions as

$$\deg(f) = \frac{1}{(2\pi i)^{2k}} \int \overline{f(z_0)} \frac{f(z_1) - f(z_0)}{z_1 - z_0} \cdots \frac{f(z_0) - f(z_{2k})}{z_0 - z_{2k}} \mathrm{d}z_0 \ldots \mathrm{d}z_{2k}, \tag{B.1}$$

whenever $\alpha(2k+1) > 1$. Later, the same technique was used in [76] and [77] in constructing index formulas for pseudo-differential operators with operator-valued symbols. Our aim is to find new formulas for the degree in the multidimensional setting by expressing the degree of a Hölder continuous function as the index of a Toeplitz operator and using the approach of [34].

The motivation to calculate the degree of a non-smooth mapping comes from non-linear σ-models in physics. For instance, the Skyrme model, describing self-interacting mesons in terms of a field $f : X \to Y$, see [10], only have a constant solution if one does not pose a topological restriction and since the solutions are rarely smooth, but rather in the Sobolev space $W^{1,d}(X,Y)$, one needs a degree defined on non-continuous functions. In the paper [27], the notion of a degree was extended as far as to VMO-mappings in terms of approximation by continuous mappings. See also [25] for a study of the homotopy structure of $W^{1,d}(X,Y)$.

The main idea that will be used in this paper is that the cohomological information of a continuous mapping $f : X \to Y$ between odd dimensional manifolds can be found in the induced mapping $f^* : K^1(X) \to K^1(Y)$ using the Chern character. The analytic formula will be obtained by using index theory of Toeplitz operators. The index theory of Toeplitz operators is a well studied subject for many classes of symbols, see for instance [14], [23], [34] and [46]. If $X = \partial\Omega$, where Ω is a strictly pseudo-convex domain in a complex manifold, and

$f : \partial\Omega \to Y$ is a smooth mapping the idea can be expressed by the commutative diagram:

$$\begin{array}{ccccc} K^1(Y) & \xrightarrow{f^*} & K^1(\partial\Omega) & \xrightarrow{\text{ind}} & \mathbb{Z} \\ \big\downarrow \text{ch}_Y & & \big\downarrow \text{ch}_{\partial\Omega} & & \big\downarrow \\ H^{odd}_{dR}(Y) & \xrightarrow{f^*} & H^{odd}_{dR}(\partial\Omega) & \xrightarrow{\chi_{\partial\Omega}} & \mathbb{C} \end{array} \tag{B.2}$$

where the mapping $\mathsf{ind} : K^1(\partial\Omega) \to \mathbb{Z}$ denotes the index mapping defined in terms of suitable Toeplitz operators on $\partial\Omega$ and

$$\chi_{\partial\Omega}(x) := -\int_{\partial\Omega} x \wedge Td(\Omega).$$

The left part of the diagram (B.2) is commutative by naturality of the Chern character and the right part of the diagram is commutative by the Boutet de Monvel index formula.

The K-theory is a topological invariant and the picture of the index mapping as a mapping from a local homology theory via Chern characters can be applied to more general classes of functions than the smooth functions. The homology theory present through out all the index theory is cyclic homology. For a Hölder continuous mapping $f : \partial\Omega \to Y$ of exponent α and Ω being a relatively compact strictly pseudo-convex domain in a Stein manifold the analogy of the diagram (B.2) is

$$\begin{array}{ccccc} K_1(C^\infty(Y)) & \xrightarrow{f^*} & K_1(C^\alpha(\partial\Omega)) & \xrightarrow{\text{ind}} & \mathbb{Z} \\ \big\downarrow \text{ch}_Y & & \big\downarrow \text{ch}_{\partial\Omega} & & \big\downarrow \\ HC_{odd}(C^\infty(Y)) & \xrightarrow{f^*} & HC_{odd}(C^\alpha(\partial\Omega)) & \xrightarrow{\tilde{\chi}_{\partial\Omega}} & \mathbb{C} \end{array} \tag{B.3}$$

where the mapping $\tilde{\chi}_{\partial\Omega} : HC_{odd}(C^\alpha(\partial\Omega)) \to \mathbb{C}$ is a cyclic cocycle on $C^\alpha(\partial\Omega)$ defined as the Connes-Chern character of the Toeplitz operators on $\partial\Omega$, see more in [32] and [34]. The condition on Ω to lie in a Stein manifold ensures that the cyclic cocycle $\tilde{\chi}_{\partial\Omega}$ can be defined on Hölder continuous functions, see below in Theorem B.4.2. The right-hand side of the diagram (B.3) is commutative by Connes' index formula, see Proposition 4 of Chapter IV.1 of [34]. The dimension in which the Chern character will take values depends on the Hölder exponent α. More explicitly, the cocycle $\tilde{\chi}_{\partial\Omega}$ can be chosen as a cyclic $2k+1$-cocycle for any $2k+1 > 2n/\alpha$.

The index of a Toeplitz operator T_u on the vector valued Hardy space $H^2(\partial\Omega)\otimes\mathbb{C}^N$ with smooth symbol $u : \partial\Omega \to \mathrm{GL}_N(\mathbb{C})$ can be calculated using the Boutet de Monvel index formula as $\mathsf{ind}\, T_u = -\int_{\partial\Omega} \mathsf{ch}_{\partial\Omega}[u]$ if the Chern character $\mathsf{ch}_{\partial\Omega}[u]$ only contains a top degree term. In particular, if $g : Y \to \mathrm{GL}_N(\mathbb{C})$ satisfies that all terms, except for the top-degree term, in $\mathsf{ch}_{\partial\Omega}[g]$ are exact and $f : \partial\Omega \to Y$ is smooth we can consider the matrix symbol $g \circ f$ on $\partial\Omega$. Naturality of the Chern character implies the identity

$$\deg f \int_Y \mathrm{ch}_Y[g] = -\mathrm{ind}\, T_{g\circ f}$$

where $T_{g\circ f}$ is a Toeplitz operator on $H^2(\partial\Omega)\otimes\mathbb{C}^N$ with symbol $g\circ f$. This result extends to Hölder continuous functions in the sense that if we choose g which also satisfies the condition $\int_Y \mathrm{ch}_Y[g] = 1$ we obtain the analytic degree formula:

$$\deg f = \tilde{\chi}_{\partial\Omega}(\mathrm{ch}_{\partial\Omega}[g \circ f]).$$

A drawback of our approach is that it only applies to boundaries of strictly pseudo-convex domains in Stein manifolds. We discuss this drawback at the end of the fourth, and final, section of this paper. The author intends to return to this question in a future paper and address the problem for even-dimensional manifolds.

The paper is organized as follows; in the first section we reformulate the degree as an index calculation using the Chern character from odd K-theory to de Rham cohomology. This result is not remarkable in itself, since the Chern character is an isomorphism after tensoring with the complex numbers. However, the constructions are explicit and allows us to obtain explicit expressions for a generator of the de Rham cohomology. We will use the complex spin representation of $\mathbb{R}^{2n}$ to construct a smooth function $u : S^{2n-1} \to SU(2^{n-1})$ such that the Chern character of u is a multiple of the volume element on S^{2n-1}. The function u will then be used to construct a smooth mapping $\tilde{g} : Y \to GL_{2^{n-1}}(\mathbb{C})$ for arbitrary odd-dimensional manifold Y whose Chern character coincide with $(-1)^n \mathrm{d}V_Y$ where $\mathrm{d}V_Y$ is a normalized volume form on Y, see Theorem B.1.6. Thus we obtain for any continuous function $f : \partial\Omega \to Y$ the formula $\deg f = (-1)^{n+1}\mathrm{ind}\, T_{g\circ f}$, as is proved in Theorem B.2.1.

In the second section we will review the theory of Toeplitz operators on the boundary of a strictly pseudo-convex domain. The material in this section is based on [23], [34], [41], [46], [50] and [71]. We will recall the basics from [41], [50] and [71] of integral representations of holomorphic functions on Stein manifolds and the non-orthogonal Henkin-Ramirez projection. We will continue the section by recalling some known results about index formulas and how a certain Schatten class condition can be used to obtain index formulas. The focus will be on the index formula of Connes, see Proposition 4 in Chapter IV.1 of [34], involving cyclic cohomology and how the periodicity operator S in cyclic cohomology can be used to extend cyclic cocycles to larger algebras. In our case the periodicity operator is used to extend a cyclic cocycle on the algebra $C^\infty(\partial\Omega)$ to a cyclic cocycle on $C^\alpha(\partial\Omega)$. We will also review a theorem of Russo, see [80], which gives a sufficient condition for an integral operator to be of Schatten class.

The third section is devoted to proving that the Szegö projection $P_{\partial\Omega} : L^2(\partial\Omega) \to H^2(\partial\Omega)$ satisfies the property that for any $p > 2n/\alpha$ the commutator $[P_{\partial\Omega}, a]$ is a Schatten class operator of order p for any Hölder continuous functions a on $\partial\Omega$ of exponent α. The statement about the commutator $[P_{\partial\Omega}, a]$ can be reformulated as the corresponding big Hankel operator with symbol a being of Schatten class. We will in fact not look at the Szegö projection, but rather at the non-orthogonal Henkin-Ramirez projection P_{HR} mentioned above. The projection P_{HR} has a particular behavior making the estimates easier and an application of Russo's Theorem implies that $P_{HR} - P_{\partial\Omega}$ is Schatten class of order $p > 2n$, see Lemma B.3.6.

In the fourth section we will present the index formula and the degree formula for Hölder continuous functions. Thus if we let $C_{\partial\Omega}$ denote the Szegö kernel and $\mathrm{d}V$ the volume form on $\partial\Omega$ we obtain the following index formula for u invertible and Hölder continuous on $\partial\Omega$:

$$\mathrm{ind}\, T_u = -\int_{\partial\Omega^{2k+1}} \mathrm{tr}\left(\prod_{i=0}^{2k}(1 - u(z_i)^{-1}u(z_{i+1}))C_{\partial\Omega}(z_i, z_{i+1})\right)\mathrm{d}V$$

for any $2k+1 > 2n/\alpha$. Here we identify z_{2k+1} with z_0. Using the index formula for mapping degree we finally obtain the following analytic formula for the degree of a Hölder continuous mapping from $\partial\Omega$ to a connected, compact, orientable, Riemannian manifold Y. If $f : \partial\Omega \to Y$ is a Hölder continuous function of exponent α, the degree of f can be calculated for $2k+1 > 2n/\alpha$ from the formula:

$$\deg(f) = (-1)^n \int_{\partial\Omega^{2k+1}} \tilde{f}(z_0, z_1, \ldots, z_{2k}) \prod_{j=0}^{2k} C_{\partial\Omega}(z_{j-1}, z_j)\mathrm{d}V$$

where $\tilde{f} : \partial\Omega^{2k+1} \to \mathbb{C}$ is a function explicitly expressed from f, see more in equation (B.25).

B.1 The volume form as a Chern character

In order to represent the mapping degree as an index we look for a matrix symbol whose Chern character is cohomologous to the volume form $\mathrm{d}V_Y$ on Y. We will start by considering the case of a $2n-1$-dimensional sphere and construct a mapping into the Lie group $SU(2^{n-1})$ using the complex spin representation of $Spin(\mathbb{R}^{2n})$. In the complex spin representation a vector in S^{2n-1} defines a unitary matrix, this construction produces a matrix symbol on odd-dimensional spheres such that its Chern character spans $H^{2n-1}_{dR}(S^{2n-1})$. The matrix symbol on S^{2n-1} generalizes to an arbitrary connected, compact, oriented manifold Y of dimension $2n-1$ such that its Chern character coincides with $(-1)^n \mathrm{d}V_Y$.

Let V denote a real vector space of dimension $2n$ with a non-degenerate inner product g. We take a complex structure J on V which is compatible with the metric and extend the mapping J to a complex linear mapping on $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. Since $J^2 = -1$ we can decompose $V_{\mathbb{C}} := V^{1,0} \oplus V^{0,1}$ into two eigenspaces of J corresponding to the eigenvalues $\pm i$. If we extend g to a complex bilinear form $g_{\mathbb{C}}$ on $V_{\mathbb{C}}$ and using the isomorphism $\mathbb{C}l(V,g) \cong Cl(V_{\mathbb{C}}, g_{\mathbb{C}})$, we can identify the complexified Clifford algebra of V with the complex algebra generated by $2n$ symbols $e_{1,+}, \dots, e_{n,+}, e_{1,-}, \dots, e_{n,-}$ satisfying the relations

$$\{e_{j,+}, e_{k,+}\} = \{e_{j,-}, e_{k,-}\} = 0 \quad \text{and} \quad \{e_{j,+}, e_{k,-}\} = -2\delta_{jk},$$

where $\{\cdot,\cdot\}$ denotes anti-commutator. The complex algebra $\mathbb{C}l(V,g)$ becomes a $*$-algebra in the $*$-operation $e_{j,+}^* := -e_{j,-}$.

The space $S_V := \wedge^* V^{1,0}$ becomes a complex Hilbert space equipped with the sesquilinear form induced from g and J. The vector space S_V will be given the orientation from the lexicographic order on the basis $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$ for $i_1 < i_2 < \ldots < i_k$. Define $c : V_{\mathbb{C}} \to End(S_V)$ by

$$c(v).w := \sqrt{2} v \wedge w, \quad \text{for} \quad v \in V^{1,0} \quad \text{and}$$

$$c(v').w := -\sqrt{2} v' \neg w \quad \text{for} \quad v' \in V^{0,1}.$$

The linear mapping c satisfies

$$c(v^*) = c(v)^* \quad \text{and} \quad c(w)c(v) + c(v)c(w) = -2g(w,v)$$

so by the universal property of the Clifford algebra $\mathbb{C}l(V,g)$ we can extend c to a $*$-representation $\varphi : \mathbb{C}l(V) \to End_{\mathbb{C}}(S_V)$. The space S_V is a 2^n-dimensional Hilbert space which we equip with a grading as follows

$$S_V = S_V^+ \oplus S_V^- := \wedge^{even} V^{1,0} \oplus \wedge^{odd} V^{1,0}.$$

Consider the subalgebra $\mathbb{C}l(V)_+$ consisting of an even number of generators. The representation φ restricts to a representation $\mathbb{C}l(V)_+ \to End_{\mathbb{C}}(S_V^+)$ and $\mathbb{C}l(V)_+ \to End_{\mathbb{C}}(S_V^-)$. We define the 2^{n-1}-dimensional oriented Hilbert space $E_n := S_{\mathbb{C}^n}^+$ when n is even and $E_n := S_{\mathbb{C}^n}^-$ when n is odd. The representation $\mathbb{C}l(\mathbb{C}^n)_+ \to End_{\mathbb{C}}(E_n)$ will be denoted by φ_+. For a vector $v \in \mathbb{C}^n$ we can use the fact that $\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n \oplus \mathbb{C}^n$ and define

$$v_+ := \varphi_+(v \oplus 0) \in End_{\mathbb{C}}(E_n) \quad \text{and} \quad v_- := \varphi_+(0 \oplus v) \in End_{\mathbb{C}}(E_n).$$

We will now define a symbol calculus for S^{2n-1}. We choose the standard embedding $S^{2n-1} \subseteq \mathbb{C}^n$ by taking coordinates $z_i : S^{2n-1} \to \mathbb{C}$ satisfying $|z_1|^2 + |z_2|^2 \cdots + |z_n|^2 = 1$. Define the smooth mapping $u : S^{2n-1} \to \mathbb{C}l(\mathbb{R}^{2n})_+$ by

$$u(z) := \frac{1}{2}(e_{1,+} + e_{1,-})(z_+ + \bar{z}_-). \tag{B.4}$$

Proposition B.1.1. *The mapping u satisfies*

$$u(z)^*u(z) = u(z)u(z)^* = 1$$

so $u : S^{2n-1} \to SU(2^{n-1}) \subseteq End_{\mathbb{C}}(E_n)$ is well defined.

The proof of this proposition is a straight-forward calculation using the relations in the Clifford algebra $\mathbb{C}l(V,g)$. Observe that if $n=2$ the mapping u is a diffeomorphism since we can choose 1 and $e_1 \wedge e_2$ as a basis for S_V^+ and in this basis

$$u(z_1,z_2) = \begin{pmatrix} -z_1 & -\bar{z}_2 \\ z_2 & -\bar{z}_1 \end{pmatrix}$$

For any N we can consider the subgroup $SU(N-1) \subseteq SU(N)$ of elements of the form $1 \oplus x$. Denoting by e_1 the first basis vector in $\mathbb{C}^N$, we can define a mapping $q : SU(N) \to S^{2N-1}$ by $q(v) := ve_1$. A straight-forward calculation shows that q factors over the quotient $SU(N)/SU(N-1)$ and induces a diffeomorphism $SU(N)/SU(N-1) \cong S^{2N-1}$. The function u is in a sense a splitting to q:

Proposition B.1.2. *If $\iota : S^{2n-1} \to S^{2^n-1}$ is defined by*

$$\iota(z_1,z_2,\dots z_n) := \begin{cases} (-z_1,z_2,\dots z_n,0,\dots,0) & \textit{for } n \textit{ even} \\ (-\bar{z}_1,z_2,\dots z_n,0,\dots,0) & \textit{for } n \textit{ odd} \end{cases}$$

and $q : SU(2^{n-1}) \to S^{2^n-1}$ is the mapping constructed above, the following identity is satisfied

$$q \circ u = \iota.$$

Proof. We will start with the case when n is even. The first n basis vectors of S_V^+ are $1, e_1 \wedge e_2, e_1 \wedge e_3, \dots, e_1 \wedge e_n$ and

$$q(u(z)) = u(z)1 = -z_1 + z_2 e_1 \wedge e_2 + z_3 e_1 \wedge e_3 + \dots + z_n e_1 \wedge e_n.$$

If n is odd, the first basis vectors of S_V^- are $e_1, e_2, \dots, e_n$. Therefore we have the equality

$$q(u(z)) = u(z)e_1 = -\bar{z}_1 e_1 + z_2 e_2 + \cdots z_n e_n.$$

□

Consider $\alpha_+ := \varphi_+(\mathrm{d}z \oplus 0)$ and $\alpha_- := \varphi_+(0 \oplus \mathrm{d}\bar{z})$ as elements in $T^*S^{2n-1} \otimes End_{\mathbb{C}}(E_n)$. For an element $\mathbb{k} = (k_1,\dots,k_{2l-1}) \in \{+,-\}^{2l-1}$ we define $\alpha_{\mathbb{k}} := \alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_{2l-1}} \in \wedge^{2l+1}T^*S^{2n-1} \otimes End_{\mathbb{C}}(E_n)$. Define the set Γ_l^+ as the set of $\mathbb{k} \in \{+,-\}^{2l-1}$ such that the number of $+$ in $\mathbb{k}$ is l. Similarly Γ_l^- is defined as the set of $\mathbb{k} \in \{+,-\}^{2l-1}$ such that the number of $-$ in $\mathbb{k}$ is l. The number of elements in $\Gamma_l^\pm$ can be calculated as

$$|\Gamma_l^+| = |\Gamma_l^-| = \binom{2l-1}{l-1} = \frac{(2l-1)!}{l!(l-1)!}.$$

Lemma B.1.3. *For any* $\Bbbk \in \{+,-\}^{2l-1}$ *we have the equalities*

$$\mathrm{tr}(z_+\alpha_{\Bbbk}) = \begin{cases} 0 \quad \text{if} \quad \Bbbk \notin \Gamma_l^- \\ (-1)^n 2^{n-1} l! \sum_{m_1,m_2,\dots m_l} z_{m_1} \mathrm{d}\bar{z}_{m_1} \bigwedge_{j=2}^l \mathrm{d}z_{m_j} \wedge \mathrm{d}\bar{z}_{m_j} \quad \text{if} \quad \Bbbk \in \Gamma_l^- \end{cases}$$

$$\mathrm{tr}(\bar{z}_-\alpha_{\Bbbk}) = \begin{cases} 0 \quad \text{if} \quad \Bbbk \notin \Gamma_n^+ \\ (-1)^{n+1} 2^{n-1} l! \sum_{m_1,m_2\dots,m_l} \bar{z}_{m_1} \mathrm{d}z_{m_1} \bigwedge_{j=2}^l \mathrm{d}z_{m_j} \wedge \mathrm{d}\bar{z}_{m_j} \quad \text{if} \quad \Bbbk \in \Gamma_l^+ \end{cases}$$

Here tr *denotes the matrix trace in* $End_{\mathbb{C}}(E_n)$.

The proof is a straight-forward, but rather lengthy, calculation using the relations in the Clifford algebra, so we omit the proof. We will use the notation $\mathrm{d}V$ for the normalized volume measure on S^{2n-1}:

$$\mathrm{d}V = \frac{(n-1)!}{2\pi^n} \sum_{k=1}^{2n} (-1)^{k-1} x_k \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{k-1} \wedge \mathrm{d}x_{k+1} \wedge \cdots \wedge \mathrm{d}x_{2n} = \tag{B.5}$$

$$= \frac{(n-1)!}{2(2\pi i)^n} \sum_{k=1}^{n} \bar{z}_k \mathrm{d}z_k \wedge_{j\neq k} (\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j) - z_k \mathrm{d}\bar{z}_k \wedge_{j\neq k} (\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j). \tag{B.6}$$

That $\mathrm{d}V$ is normalized follows from that the $2n-1$-form ω on S^{2n-1}, defined by

$$\omega = \sum_{k=1}^{2n} (-1)^{k-1} x_k \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{k-1} \wedge \mathrm{d}x_{k+1} \wedge \cdots \wedge \mathrm{d}x_{2n},$$

satisfies that, if we change to spherical coordinates, the form $r^{2n-1}\mathrm{d}r \wedge \omega$ coincide with the volume form on $\mathbb{C}^n$. Since $\int_{\mathbb{C}} \mathrm{e}^{-|z|^2} \mathrm{d}m = \pi$, where m denotes Lebesgue measure, Fubini's Theorem implies that $\int_{\mathbb{C}^n} \mathrm{e}^{-|z|^2} \mathrm{d}m = \pi^n$ and

$$\pi^n = \int_{\mathbb{C}^n} \mathrm{e}^{-|z|^2} \mathrm{d}m = \int_0^\infty \mathrm{e}^{-r^2} r^{2n-1} \mathrm{d}r \int_{S^{2n-1}} \omega = \frac{(n-1)!}{2} \int_{S^{2n-1}} \omega.$$

Recall that if $g : Y \to \mathrm{GL}_N(\mathbb{C})$ is a smooth mapping, the Chern character of g is an element of the odd de Rham cohomology $H^{odd}_{dR}(Y)$ defined as

$$\mathrm{ch}[g] = \sum_{k=0}^{\infty} \frac{(k-1)!}{(2\pi i)^k (2k-1)!} \mathrm{tr}(g^{-1}\mathrm{d}g)^{2k-1}.$$

See more in Chapter 1.8 in [91]. We will denote the $2k-1$-degree term by $\mathrm{ch}_{2k-1}[g]$. The cohomology class of $\mathrm{ch}[g]$ only depends on the homotopy class of g so the Chern character induces a group homomorphism $\mathrm{ch} : K_1(C^\infty(Y)) \to H^{odd}_{dR}(Y)$.

Lemma B.1.4. *The mapping* u *defined in* (B.4) *satisfies*

$$\mathrm{ch}[u] = (-1)^n \mathrm{d}V.$$

Proof. Since the odd de Rham cohomology of S^{2n-1} is spanned by the volume form it will be sufficient to show that $\mathrm{ch}_{2n-1}[u] = (-1)^n \mathrm{d}V$. First we observe the identity $u^*\mathrm{d}u = -\mathrm{d}u^* u$, which follows from Proposition B.1.1. This fact implies

$$(u^*\mathrm{d}u)^{2n-1} = (-1)^{n-1} u^* \underbrace{\mathrm{d}u\,\mathrm{d}u^* \cdots \mathrm{d}u^*\,\mathrm{d}u}_{2n-1 \quad \text{factors}}.$$

Our second observation is

$$u^*\mathrm{d}u = -\frac{1}{2}(z+\bar{z})(\mathrm{d}z + \mathrm{d}\bar{z}) \quad \text{and} \quad \mathrm{d}u^*\,\mathrm{d}u = -\frac{1}{2}(\mathrm{d}z+\mathrm{d}\bar{z})(\mathrm{d}z+\mathrm{d}\bar{z}).$$

Therefore

$$(u^*\mathrm{d}u)^{2n-1} = -\frac{1}{2^n}(z+\bar{z})(\mathrm{d}z+\mathrm{d}\bar{z})^{2n-1}.$$

Because of Lemma B.1.3 we have the equalities

$$\begin{aligned}
\mathrm{tr}((z+\bar{z})(\mathrm{d}z+\mathrm{d}\bar{z})^{2n-1}) &= \sum_{\Bbbk\in\Gamma_n^+} \mathrm{tr}(\bar{z}\alpha_{\Bbbk}) + \sum_{\Bbbk\in\Gamma_n^-} \mathrm{tr}(z\alpha_{\Bbbk}) = \\
&= \sum_{\Bbbk\in\Gamma_n^+} (-1)^{n+1} 2^{n-1}(n-1)!n! \sum_{k=1}^{n} \bar{z}_k \mathrm{d}z_k \wedge_{j\neq k} (\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j) + \\
&+ \sum_{\Bbbk\in\Gamma_n^-} (-1)^{n} 2^{n-1}(n-1)!n! \sum_{k=1}^{n} z_k \mathrm{d}\bar{z}_k \wedge_{j\neq k} (\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j) = \\
= (-1)^{n+1}2^{n-1}(2n-1)! &\sum_{k=1}^{n} \Big(\bar{z}_k \mathrm{d}z_k \wedge_{j\neq k} (\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j) - z_k \mathrm{d}\bar{z}_k \wedge_{j\neq k} (\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j)\Big) = \\
&= \frac{(-1)^{n+1}2^n(2\pi i)^n(2n-1)!}{(n-1)!}\mathrm{d}V.
\end{aligned}$$

Finally, adding all results together we come to the conclusion of the Lemma:

$$\mathrm{tr}(u^*\mathrm{d}u)^{2n-1} = -\frac{1}{2^n}\mathrm{tr}((z+\bar{z})(\mathrm{d}z+\mathrm{d}\bar{z})^{2n-1}) = (-1)^n \frac{(2\pi i)^n(2n-1)!}{(n-1)!}\mathrm{d}V.$$

□

To generalize the construction of u to an arbitrary manifold we need to cut down u at "infinity". We define the smooth function $\xi_0 : [0,\infty) \to \mathbb{R}$ as

$$\xi_0(x) := \begin{cases} \mathrm{e}^{-\frac{4}{x^2}}, & x > 0 \\ 0, & x = 0 \end{cases}$$

and the smooth function $\xi : S^{2n-1} \to \mathbb{C}^n$ by

$$\xi(z) := \xi_0(|1 - \Re(z_1)|)z + \big(\xi_0(|1 - \Re(z_1)|) - 1, 0, 0, \dots, 0\big).$$

By standard methods it can be proved that for any natural number k and any vector fields $X_1, X_2, \dots X_l$ on S^{2n-1} the function ξ satisfies

$$|\xi(z) - (-1, 0, \dots, 0)| = \mathcal{O}(|1 - \Re(z_1)|^k) \quad \text{and} \tag{B.7}$$

$$|X_1 X_2 \cdots X_l \xi(z)| = \mathcal{O}(|1 - \Re(z_1)|^k) \quad \text{as} \quad z \to (1, 0, \dots, 0). \tag{B.8}$$

Furthermore, the length of $\xi(z)$ is given by

$$|\xi(z)|^2 = 2(\Re(z_1) + 1)(\xi_0(|1 - \Re(z_1)|)^2 - \xi_0(|1 - \Re(z_1)|)) + 1$$

so $|\xi(z)| > 0$ for all $z \in S^{2n-1}$.

Using the function ξ we define the smooth function $\tilde{u} : S^{2n-1} \to GL_{2^{n-1}}(\mathbb{C})$ by

$$\tilde{u}(z) := \frac{1}{2}(e_{1,+} + e_{1,-})(\xi(z)_+ + \overline{\xi(z)}_-).$$

The function $\tilde{u}$ is well defined since

$$\tilde{u}(z)^* \tilde{u}(z) = |\xi(z)|^2 > 0.$$

Observe that we may express $\tilde{u}$ in terms of u as

$$\tilde{u}(z) = \xi_0(|1 - \Re(z_1)|)(u(z) - 1) + 1.$$

If we choose a diffeomorphism $\tau : B_{2n-1} \cong S^{2n-1} \setminus \{(1, 0, \dots, 0)\}$ the equation (B.7) and (B.8) implies that the function $\tau^* \tilde{u}$ can be considered as a smooth function $B_{2n-1} \to GL_{2^{n-1}}(\mathbb{C})$ such that $\tau^* \tilde{u} - 1$ vanishes to infinite order at the boundary of B_{2n-1}. The particular choice of τ as the inverse of the stereographic projection

$$\tau(y) := \left(2|y|^2 - 1, 2\sqrt{1 - |y|^2}\, y\right)$$

will give a function $\tau^* \tilde{u}$ of the form

$$\begin{aligned} \tau^* \tilde{u}(y) &= \mathrm{e}^{-\frac{1}{(1-|y|^2)^2}}(u(\tau(y)) - 1) + 1 = \\ &= \frac{\mathrm{e}^{-\frac{1}{(1-|y|^2)^2}}}{2}(e_{1,+} + e_{1,-})(\tau(y)_+ + \overline{\tau(y)}_-) + 1 - \mathrm{e}^{-\frac{1}{(1-|y|^2)^2}}. \end{aligned}$$

Lemma B.1.5. *There is a homotopy of smooth functions $S^{2n-1} \to GL_{2^{n-1}}(\mathbb{C})$ between $\tilde{u}$ and u. Therefore $\mathrm{ch}[\tilde{u}] - \mathrm{ch}[u]$ is an exact form.*

Proof. We can take the homotopy $w : S^{2n-1} \times [0,1] \to GL_{2^{n-1}}(\mathbb{C})$ as

$$w(z,t) = \xi_t(|1 - \Re(z_1)|)(u(z) - 1) + 1,$$

where

$$\xi_t(x) := \mathrm{e}^{-\frac{4(1-t)}{x^2}}.$$

Clearly, $w : S^{2n-1} \times [0,1] \to GL_{2^{n-1}}(\mathbb{C})$ is a smooth function and $w(z,0) = \tilde{u}(z)$ and $w(z,1) = u(z)$. □

In the general case, let Y be a compact, connected, orientable manifold of odd dimension $2n-1$. If we take an open subset U of Y with coordinates $(x_i)_{i=1}^{2n-1}$ such that

$$U = \{x : \sum_{i=1}^{2n-1} |x_i(x)|^2 < 1\},$$

the coordinates define a diffeomorphism $\nu : U \cong B_{2n-1}$. We can define the functions $g, \tilde{g} : Y \to GL_{2^{n-1}}(\mathbb{C})$ by

$$g(x) := \begin{cases} u(\tau\nu(x)) & \text{for} \quad x \in U \\ 1 \quad \text{for} \quad x \notin U \end{cases} \tag{B.9}$$

$$\tilde{g}(x) := \begin{cases} \tilde{u}(\tau\nu(x)) & \text{for} \quad x \in U \\ 1 \quad \text{for} \quad x \notin U \end{cases} \tag{B.10}$$

If we let $\tilde{\nu} : Y \to S^{2n-1}$ be the Lipschitz continuous function defined by

$$\tilde{\nu}(x) = \begin{cases} \tau(\nu(x)) & \text{for} \quad x \in U \\ (1,0,\dots,0) & \text{for} \quad x \notin U \end{cases} \tag{B.11}$$

the functions $\tilde{g}$ and g can be expressed as $g = \tilde{\nu}^* u$ and $\tilde{g} = \tilde{\nu}^* \tilde{u}$. The function $\tilde{g}$ is smooth and the function g is Lipschitz continuous.

Theorem B.1.6. *Denoting the normalized volume form on Y by $\mathrm{d}V_Y$, the function $\tilde{g}$ satisfies*

$$\mathrm{ch}[\tilde{g}] = (-1)^n \mathrm{d}V_Y,$$

in $H^{odd}_{dR}(Y)$. Thus, if $f : X \to Y$ is a smooth mapping

$$\deg(f) = (-1)^n \int_X f^* \mathrm{ch}[\tilde{g}]$$

Proof. By Lemma B.1.5 and Lemma B.1.4 we have the identities

$$\int_Y \mathrm{ch}[\tilde{g}] = \int_U \mathrm{ch}_{2n-1}[\tilde{g}] = \int_U \tilde{v}^* \mathrm{ch}_{2n-1}[\tilde{u}] =$$
$$= \int_{S^{2n-1}} \mathrm{ch}_{2n-1}[\tilde{u}] = \int_{S^{2n-1}} \mathrm{ch}_{2n-1}[u] = (-1)^n.$$

Therefore we have the identity $\mathrm{ch}_{2n-1}[\tilde{g}] = (-1)^n \mathrm{d}V_Y$. Since $\mathrm{ch}[\tilde{g}] - \mathrm{ch}_{2n-1}[\tilde{g}]$ is an exact form on U and vanishes to infinite order at ∂U the Theorem follows. □

B.2 Toeplitz operators and their index theory

In this section we will give the basics of integral representations of holomorphic functions and the Henkin-Ramirez integral representation, we will more or less pick out the facts of [41], [50] and [71] relevant for our purposes. After that we will review the theory of Toeplitz operators on the Hardy space on the boundary of a strictly pseudo-convex domain. We will let M denote a Stein manifold and we will assume that $\Omega \subseteq M$ is a relatively compact, strictly pseudo-convex domain with smooth boundary.

Consider the Hilbert space $L^2(\partial\Omega)$, in some Riemannian metric on $\partial\Omega$. We will use the notation $H^2(\partial\Omega)$ for the Hardy space, that is defined as the space of functions in $L^2(\partial\Omega)$ with holomorphic extensions to Ω. The subspace $H^2(\partial\Omega) \subseteq L^2(\partial\Omega)$ is a closed subspace so there exists a unique orthogonal projection $P_{\partial\Omega} : L^2(\partial\Omega) \to H^2(\partial\Omega)$ called the Szegö projection. We will consider the Henkin-Ramirez projection, see [49], [70] and the generalization in [50] to Stein manifolds, which we will denote by $P_{HR} : L^2(\partial\Omega) \to H^2(\partial\Omega)$ and call the HR-projection. The HR-projection is not necessarily orthogonal but is often possible to calculate explicitly, see [71], and easier to estimate. We will briefly review its construction in the case $M = \mathbb{C}^n$ following Chapter VII of [71]. The construction of the HR-projection on a general Stein manifold is somewhat more complicated, but the same estimates hold so we refer the reader to the construction in [50].

The kernel of the HR-projection should be thought of as the first terms in a Taylor expansion of the Szegö kernel. This idea is explained in [56]. The HR-kernel contains the most singular part of the Szegö kernel and the HR-kernel can be very explicitly estimated at its singularities. This is our reason to use the HR-projection instead of the Szegö projection. If Ω is defined by the strictly pluri-subharmonic function ρ a function $\Phi = \Phi(w,z)$ is defined as the smooth global extension of the Levi polynomial

$$F(w,z) := \sum_{j=1}^n \frac{\partial\rho}{\partial w_j}(w)(w_j - z_j) - \frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2\rho}{\partial w_j \partial w_k}(w)(w_j - z_j)(w_k - z_k)$$

from the diagonal in $\Omega\times\Omega$ to the whole of $\overline{\Omega}\times\overline{\Omega}$, see more in Chapter V.1.1 and Chapter VII.5.1 of [71]. If we take $c>0$ such that $\partial\bar{\partial}\rho\geq c$ there is an $\varepsilon>0$ such that

$$2\Re\Phi(w,z)\geq\rho(w)-\rho(z)+c|z-w|^2,\quad\text{for}\quad|z-w|<\varepsilon,\tag{B.12}$$

see more in equation 1.6, Chapter V.1.1 of [71]. By Lemma 1.5 of Chapter VII of [71] the function Φ satisfies the following estimate

$$\int_{\partial\Omega}\frac{\mathrm{d}V(w)}{|\Phi(w,z)|^{n+\beta}}\lesssim 1\tag{B.13}$$

where $\mathrm{d}V$ denotes the volume measure on $\partial\Omega$ if $\beta<0$ and a similar estimate with the roles of z and w interchanged. Here we used the standard notation $a\lesssim b$ for the statement that there exists a constant $C>0$ such that $a\leq Cb$.

By Theorem 3.6, Chapter VII of [71] we can associate with Φ a function $H_{\partial\Omega}$ in $\Omega\times\Omega$ holomorphic in its second variable such that if $g\in L^1(\Omega)$ is holomorphic it has the integral representation:

$$g(z)=\int_{\partial\Omega}H_{\partial\Omega}(w,z)g(w)\mathrm{d}V(w).$$

For the function $H_{\partial\Omega}$ the estimate

$$|H_{\partial\Omega}(z,w)|\lesssim|\Phi(w,z)|^{-n},\tag{B.14}$$

holds in $\partial\Omega\times\partial\Omega$, see more in Proposition 3.1, Chapter VII of [71]. Since Φ satisfies the estimate (B.12) where c is the infimum of $\partial\bar{\partial}\rho$ the construction of a HR-projection does give an L^2-bounded projection for strictly pseudo-convex domains. If Ω is weakly pseudo-convex the situation is more problematic and not that well understood partly due to problems estimating solutions to the $\bar{\partial}$-equation in weakly pseudo-convex domains. By Proposition 3.8 of Chapter VII.3.1 in [71] the kernel $H_{\partial\Omega}$ satisfies the estimate

$$|H_{\partial\Omega}(z,w)-\overline{H_{\partial\Omega}(w,z)}|\lesssim|\Phi(z,w)|^{-n+1/2}.\tag{B.15}$$

The estimate (B.15) will be crucial when proving that $P_{\partial\Omega}-P_{HR}$ is in the Schatten class. The kernel $H_{\partial\Omega}$ determines a bounded operator P_{HR} on $L^2(\partial\Omega)$ by Theorem 3.6 of Chapter VII.3 in [71]. Since the range of P_{HR} is contained in $H^2(\partial\Omega)$ and $g=P_{HR}g$ for any $g\in H^2(\partial\Omega)$ it follows that $P_{HR}:L^2(\partial\Omega)\to H^2(\partial\Omega)$ is a projection.

We will now present some facts about Toeplitz operators on the Hardy space of a relatively compact strictly pseudo-convex domain Ω in a complex manifold M. Our operators are associated with the Szegö projection since the theory becomes somewhat more complicated when a non-orthogonal projection is involved. For any dimension N we denote by $C(\partial\Omega, M_N(\mathbb{C}))$ the C^*-algebra of continuous functions $\partial\Omega \to M_N(\mathbb{C})$, the algebra of complex $N \times N$-matrices. The algebra $C(\partial\Omega, M_N(\mathbb{C}))$ has a representation $\pi : C(\partial\Omega, M_N(\mathbb{C})) \to \mathscr{B}(L^2(\partial\Omega) \otimes \mathbb{C}^N)$ which is given by pointwise multiplication. We define the linear mapping

$$T : C(\partial\Omega, M_N(\mathbb{C})) \to \mathscr{B}(H^2(\partial\Omega) \otimes \mathbb{C}^N), \quad a \mapsto P_{\partial\Omega}\pi(a)P_{\partial\Omega}.$$

Here we identify $P_{\partial\Omega}$ with the projection $L^2(\partial\Omega) \otimes \mathbb{C}^N \to H^2(\partial\Omega) \otimes \mathbb{C}^N$. An operator of the form $T(a)$ is called a Toeplitz operator on $\partial\Omega$. Toeplitz operators are well studied, see for instance [23], [34], [46] and [76]. The representation π satisfies $[P_{\partial\Omega}, \pi(a)] \in \mathscr{K}(L^2(\partial\Omega) \otimes \mathbb{C}^N)$ for any $a \in C(\partial\Omega, M_N(\mathbb{C}))$, see for instance [23] or Theorem B.3.1 below. Here we use the symbol $\mathscr{K}$ to denote the algebra of compact operators. The fact that $P_{\partial\Omega}$ commutes with continuous functions up to a compact operator implies the property

$$T(ab) - T(a)T(b) \in \mathscr{K}(H^2(\partial\Omega) \otimes \mathbb{C}^N). \tag{B.16}$$

Furthermore, $T(a)$ is compact if and only if $a = 0$. Let us denote the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ by $\mathscr{C}(\mathscr{H})$ and the quotient mapping $\mathscr{B}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ by $\mathfrak{q}$. Equation (B.16) implies that the mapping

$$\beta := \mathfrak{q} \circ T : C(\partial\Omega, M_N(\mathbb{C})) \to \mathscr{C}(H^2(\partial\Omega) \otimes \mathbb{C}^N)$$

is an injective $*$-homomorphism.

By the Boutet de Monvel index formula, from [23], if the symbol a is invertible and smooth the index of the Toeplitz operator $T(a)$ has the analytic expression:

$$\mathrm{ind}\,(T(a)) = -\int_{\partial\Omega} \mathrm{ch}[a] \wedge Td(\Omega), \tag{B.17}$$

see more in Theorem 1 in [23], and the remarks thereafter. The mapping $a \mapsto \mathrm{ind}\,(T(a))$, defined on functions $a : \partial\Omega \to \mathrm{GL}_N(\mathbb{C})$ is homotopy invariant, so it extends to a mapping $\mathrm{ind} : K_1(C^\infty(\partial\Omega)) \to \mathbb{Z}$. Here $K_1(C^\infty(\partial\Omega))$ denotes the odd K-theory of the Frechet algebra $C^\infty(\partial\Omega)$ which is defined as homotopy classes of invertible matrices with coefficients in $C^\infty(\partial\Omega)$, see more in [22].

Theorem B.2.1. *Suppose that $\Omega \subseteq M$ is a relatively compact strictly pseudoconvex bounded domain with smooth boundary, Y is a compact, orientable manifold of dimension $2n-1$ and $g : Y \to GL_{2^{n-1}}(\mathbb{C})$ is the mapping defined in* (B.10). *If $f : \partial\Omega \to Y$ is a continuous function, then*

$$\deg(f) = (-1)^{n+1}\mathrm{ind}\,(P_{\partial\Omega}\pi(g \circ f)P_{\partial\Omega}). \tag{B.18}$$

Proof. If we assume that f is smooth, the index formula of Boutet de Monvel, see above in equation (B.17), implies that the index of $P_{\partial\Omega}\pi(g\circ f)P_{\partial\Omega}$ satisfies

$$\operatorname{ind}(P_{\partial\Omega}\pi(g\circ f)P_{\partial\Omega}) = -\int_{\partial\Omega} f^*\mathrm{ch}[\tilde{g}]\wedge Td(\Omega) = -\int_{\partial\Omega} f^*\mathrm{ch}[\tilde{g}] = (-1)^{n+1}\deg(f),$$

where the first equality follows from g and $\tilde{g}$ being homotopic, see Lemma B.1.5, and the last two equalities follows from Theorem B.1.6. The general case follows from the fact that both hand sides of (B.18) is homotopy invariant. □

Theorem B.2.1 does in some cases hold with even looser regularity conditions on f. Since both sides of the equation (B.18) are homotopy invariants the Theorem holds for any class of functions which are homotopic to smooth functions in such sense that both sides in (B.18) are well defined and depend continuously on the function. For instance, if Ω is a bounded symmetric domain we may take $f:\partial\Omega\to Y$ to be in the VMO-class. It follows from [17] that if $w:\partial\Omega\to \mathrm{GL}_N$ has vanishing mean oscillation and Ω is a bounded symmetric domain, the operator $P_{\partial\Omega}wP_{\partial\Omega}$ is Fredholm. By [27] the degree of a VMO-function is well defined and depends continuously on f without any restriction on the geometry. To be more precise, there is a one-parameter family $(f_t)_{t\in(0,1)}\subseteq C(\partial\Omega,Y)$ such that $f_t\to f$ in VMO when $t\to 0$ and $\deg(f)$ is defined as $\deg(f_t)$ for t small enough. Since the index of a Fredholm operator is homotopy invariant the degree of a function $f:\partial\Omega\to Y$ in VMO satisfies

$$\deg f = (-1)^{n+1}\operatorname{ind}(P_{\partial\Omega}\pi(g\circ f_t)P_{\partial\Omega}) = (-1)^{n+1}\operatorname{ind}(P_{\partial\Omega}\pi(g\circ f)P_{\partial\Omega}).$$

Our next task will be calculating the index of Toeplitz operators with non-smooth symbol. For $p\geq 1$, let $\mathscr{L}^p(\mathscr{H})\subseteq\mathscr{B}(\mathscr{H})$ denote the ideal of Schatten class operators on a separable Hilbert space $\mathscr{H}$, so $T\in\mathscr{L}^p(\mathscr{H})$ if and only if $\mathrm{tr}((T^*T)^{p/2})<\infty$. An exact description of integral operators belonging to this class exists only for $p=2$. However, for $p>2$ there exists a convenient sufficient condition on the kernel, found in [80]. We will return to this subject a little later. Suppose that $\pi:\mathscr{A}\to\mathscr{B}(\mathscr{H})$ is a representation of a $\mathbb{C}$-algebra $\mathscr{A}$ and P is a projection such that $[P,\pi(a)]\in\mathscr{L}^p(\mathscr{H})$ for all $a\in\mathscr{A}$ and $P-P^*\in\mathscr{L}^p(\mathscr{H})$. Atkinson's Theorem implies that if a is invertible, $P\pi(a)P$ is Fredholm. The operator $F:=2P-1$ has the properties

$$F^2=1 \quad\text{and}\quad F-F^*,\ [F,\pi(a)]\in\mathscr{L}^p(\mathscr{H}). \tag{B.19}$$

If π and F satisfy the conditions in equation (B.19) the pair (π,F) is called a p-summable odd Fredholm module. If the pair (π,F) satisfies the requirement in equation (B.19) but with $\mathscr{L}^p(\mathscr{H})$ replaced by $\mathscr{K}(\mathscr{H})$ the pair (π,F) is a bounded odd Fredholm module. For a more thorough presentation of Fredholm modules, e.g. Chapter VII and VIII of [22]. Since our focus is on Toeplitz operators we will call (π, P) a *Toeplitz pair* if $(\pi, 2P-1)$ is a bounded odd Fredholm module and (π, P) is said to be p-summable if $(\pi, 2P-1)$ is.

The condition that $L := P^* - P \in \mathscr{L}^p(\mathscr{H})$ can be interpreted in terms of the orthogonal projection $\tilde{P}$ to the Hilbert space $P\mathscr{H}$. Using that $\tilde{P}P = P$ and $P\tilde{P} = \tilde{P}$ we obtain the identity

$$\tilde{P}L = \tilde{P}P^* - \tilde{P}P = \tilde{P} - P. \tag{B.20}$$

Thus the condition $P^* - P \in \mathscr{L}^p(\mathscr{H})$ is equivalent to the property $\tilde{P} - P \in \mathscr{L}^p(\mathscr{H})$.

A Toeplitz pair (π, P) over a topological algebra $\mathscr{A}$ defines a mapping $a \mapsto \mathrm{ind}(P\pi(a)P)$ on the invertible elements of $\mathscr{A} \otimes M_N(\mathbb{C})$ for any N. Since the index is homotopy invariant, the association $a \mapsto \mathrm{ind}(P\pi(a)P)$ induces the mapping $\mathrm{ind} : K_1(\mathscr{A}) \to \mathbb{Z}$, where $K_1(\mathscr{A})$ denotes the odd K-theory of $\mathscr{A}$, see [22].

A. Connes placed the index theory for p-summable Toeplitz pairs in a suitable homological picture using cyclic homology in [32]. We will consider Connes' original definition of cyclic cohomology which simplifies the construction of the Chern-Connes character. The notation $\mathscr{A}^{\otimes k}$ will be used for the k-th tensor power of $\mathscr{A}$. The Hochschild differential $b : \mathscr{A}^{\otimes k} \to \mathscr{A}^{\otimes k-1}$ is defined as

$$\begin{aligned} b(x_0 \otimes x_1 \otimes \cdots \otimes x_k \otimes x_{k+1}) &:= (-1)^{k+1} x_{k+1} x_0 \otimes x_1 \otimes \cdots \otimes x_k + \\ &+ \sum_{j=0}^{k} (-1)^j x_0 \otimes \cdots \otimes x_{j-1} \otimes x_j x_{j+1} \otimes x_{j+2} \otimes \cdots \otimes x_{k+1}. \end{aligned}$$

The cyclic permutation operator $\lambda : \mathscr{A}^{\otimes k} \to \mathscr{A}^{\otimes k}$ is defined by

$$\lambda(x_0 \otimes x_1 \otimes \cdot \otimes x_k) = (-1)^k x_k \otimes x_0 \otimes \cdots \otimes x_{k-1}.$$

The complex $C^k_\lambda(\mathscr{A})$ is defined as the space of continuous linear functionals μ on $\mathscr{A}^{\otimes k+1}$ such that $\mu \circ \lambda = \mu$. The Hochschild coboundary operator $\mu \mapsto \mu \circ b$ makes $C^*_\lambda(\mathscr{A})$ into a complex. The cohomology of the complex $C^*_\lambda(\mathscr{A})$ will be denoted by $HC^*(\mathscr{A})$ and is called the cyclic cohomology of $\mathscr{A}$. There is a filtration on cyclic cohomology coming from a linear mapping $S : HC^k(\mathscr{A}) \to HC^{k+2}(\mathscr{A})$ which is called the periodicity operator. For a definition of the periodicity operator, see [34].

The additive pairing between $HC^{2k+1}(\mathscr{A})$ and the odd K-theory $K_1(\mathscr{A})$ is defined by

$$\langle \mu, u \rangle_{2k+1} := d_{2k+1} \, (\mu \otimes \mathrm{tr}) \left(\underbrace{(u^{-1} - 1) \otimes (u - 1) \otimes \cdots \otimes (u^{-1} - 1) \otimes (u - 1)}_{2k+2 \quad \text{factors}} \right)$$

where we choose the same normalization constant d_k as in Proposition 3 of Chapter III.3 of [34]:

$$d_{2k+1} := \frac{2^{-(2k+1)}}{\sqrt{2i}}\Gamma\left(\frac{2k+3}{2}\right)^{-1}.$$

The choice of normalization implies that for a cohomology class in $HC^{2k+1}(\mathscr{A})$ represented by the cyclic cocycle μ, the pairing satisfies

$$\langle S\mu, u\rangle_{2k+3} = \langle \mu, u\rangle_{2k+1},$$

see Proposition 3 in Chapter III.3 of [34]. Following Definition 3 of Chapter IV.1 of [34] we define the Connes-Chern character of a p-summable Toeplitz pair as the cyclic cocycle:

$$\mathrm{cc}_{2k+1}(\pi,P)(a_0,a_1,\ldots,a_{2k+1}) := c_{2k+1}\mathrm{tr}(\pi(a_0)[P,\pi(a_1)]\cdots[P,\pi(a_{2k+1})]),$$

for $2k+1\geq p$ where

$$c_{2k+1} := -\sqrt{2i}2^{2k+1}\Gamma\left(\frac{2k+3}{2}\right).$$

This choice of normalization constant implies that

$$S\mathrm{cc}_{2k+1}(\pi,P) = \mathrm{cc}_{2k+3}(\pi,P),$$

by Proposition 2 in Chapter IV.1 of [34].

Theorem B.2.2 (Proposition 4 of Chapter IV.1 of [34]). *If (π,P) is a p-summable Toeplitz pair over $\mathscr{A}$, $2k+1\geq p$ and a is invertible in $\mathscr{A}\otimes M_N(\mathbb{C})$ the index of $P\pi(a)P : P\mathscr{H}\otimes\mathbb{C}^N\to P\mathscr{H}\otimes\mathbb{C}^N$ may be expressed as*

$$\begin{aligned}\mathrm{ind}\,(P\pi(a)P) &= \langle \mathrm{cc}_{2k+1}(\pi,P), a\rangle_{2k+1} = \\ &= -\mathrm{tr}\left(\pi(a^{-1})[P,\pi(a)][P,\pi(a^{-1})]\cdots[P,\pi(a^{-1})][P,\pi(a)]\right) = \\ &= -\mathrm{tr}(P-\pi(a^{-1})P\pi(a))^{2k+1}.\end{aligned}$$

The role of the periodicity operator S in the context of index theory is to extend index formulas to larger algebras. Suppose that μ is a cyclic k-cocycle on an algebra $\mathscr{A}$ which is a dense $*$-subalgebra of a C^*-algebra A. As is explained in [34] for functions on S^1 and in [77] for operator valued symbols, a representative for the cyclic $k+2m$-cohomology class that $S^m\mu$ defines can be extended to a cyclic cocycle on a larger $*$-subalgebra $\mathscr{A}\subseteq\mathscr{A}'\subseteq A$. When μ is the cyclic cocycle $f_0\otimes f_1\mapsto\int f_0\mathrm{d}f_1$ on $C^\infty(S^1)$, the $2m+1$-cocycle $S^m\mu$ is cohomologous to a cocycle that extends to $C^\alpha(S^1)$ whenever $\alpha(2m+1)>1$ by Proposition 3 in Chapter III2.α of [34] and a formula for that representative is given above in (B.1). Cyclic cocycles of the form $\mu = \mathrm{cc}(\pi, P)$ appear in index theory and the periodicity operator can be used to extend index formulas to larger algebras.

The index formula of Theorem B.2.2 holds for Toeplitz operators under a Schatten class condition and to deal with this condition we will need the following theorem of Russo [80] to give a sufficient condition on an integral operator for it to be Schatten class. Let X denote a σ-finite measure space. As in [18], for numbers $1 \leq p, q < \infty$, the mixed (p,q)-norm of a measurable function $k : X \times X \to \mathbb{C}$ is defined by

$$\|k\|_{p,q} := \left(\int_X \left(\int_X |k(x,y)|^p \mathrm{d}x \right)^{\frac{q}{p}} \mathrm{d}y \right)^{\frac{1}{q}}.$$

We denote the space of measurable functions $k : X \times X \to \mathbb{C}$ with finite mixed (p,q)-norm by $L^{(p,q)}(X \times X)$. By Theorem 4.1 of [18] the space $L^{(p,q)}(X \times X)$ becomes a Banach space in the mixed (p,q)-norm which is reflexive if $1 < p, q < \infty$.

The hermitian conjugate of the function k is defined by $k^*(x,y) := \overline{k(y,x)}$. Clearly, if a bounded operator K has integral kernel k, the hermitian conjugate K^* has integral kernel k^*.

Theorem B.2.3 (Theorem 1 in [80])**.** *Suppose that $K : L^2(X) \to L^2(X)$ is a bounded operator given by an integral kernel k. If $2 < p < \infty$ then*

$$\|K\|_{\mathscr{L}^p(L^2(X))} \leq (\|k\|_{p',p} \|k^*\|_{p',p})^{1/2}, \tag{B.21}$$

where $p' = p/(p-1)$.

In the statement of the Theorem in [80], the assumption $k \in L^2(X \times X)$ is made. This assumption implies that K is Hilbert-Schmidt and $K \in \mathscr{L}^p(L^2(X))$ for all $p > 2$ so for our purposes it is not interesting. But since L^2-kernels are dense in $L^{(p,q)}$, the non-commutative Fatou lemma, see Theorem $2.7d$ of [82], implies (B.21) for any k for which the right-hand side of (B.21) is finite. Using Theorem B.2.3, we obtain the following formula for the trace of the product of integral operators:

Theorem B.2.4. *Suppose that $K_j : L^2(X) \to L^2(X)$ are operators with integral kernels k_j for $j = 1, \ldots, m$ such that $\|k_j\|_{p',p}, \|k_j^*\|_{p',p} < \infty$ for certain $p > 2$. Then for $m \geq p$ the operator $K_1 K_2 \cdots K_m$ is a trace class operator and we have the trace formula*

$$\mathrm{tr}(K_1 K_2 \cdots K_m) = \int_{X^m} \left(\prod_{j=1}^m k_j(x_j, x_{j+1}) \right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_m,$$

where we identify x_{m+1} with x_1.

Proof. The case $p = m = 2$ follows if for any $k_1, k_2 \in L^2(X \times X)$ we have the trace formula

$$\mathrm{tr}(KL^*) = \int_{X\times X} k(x,y)\overline{l(x,y)}\mathrm{d}x\mathrm{d}y.$$

Consider the sesquilinear form on $\mathscr{L}^2(L^2(X))$ defined by

$$(K,L) := \mathrm{tr}(KL^*) - \int k(x,y)\overline{l(x,y)}\mathrm{d}x\mathrm{d}y.$$

Since $\mathrm{tr}(K^*K) = \int_{X\times X} |k(x,y)|^2\mathrm{d}x\mathrm{d}y$ the sesquilinear form satisfies $(K,K) = 0$ and the polarization identity implies $(K,L) = 0$ for any $K, L \in \mathscr{L}^2(L^2(X))$.

If the operators $K_j : L^2(X) \to L^2(X)$ are Hilbert-Schmidt, or equivalently they satisfy $k_j \in L^2(X \times X)$, we may take $K = K_1$ and $L^* = K_2K_3 \cdots K_m$ so the case $p = m = 2$ implies that the operators $K_1, K_2, \ldots, K_m$ satisfy the statement of the Theorem. In the general case, the Theorem follows from the non-commutative Fatou lemma, see Theorem $2.7d$ of [82], since $\mathscr{L}^2$ is dense in $\mathscr{L}^p$ for $p > 2$. □

B.3 The Toeplitz pair on the Hardy space

As explained in section 2, for the representation $\pi : C(\partial\Omega) \to \mathscr{B}(L^2(\partial\Omega))$ and the Szegö projection $P_{\partial\Omega}$ the commutator $[P_{\partial\Omega}, \pi(a)]$ is compact for any continuous a. Thus $(\pi, P_{\partial\Omega})$ is a Toeplitz pair over $C(\partial\Omega)$. To enable the use of the index theory of [34] we will show that the Toeplitz pair $(\pi, P_{\partial\Omega})$ restricted to the subalgebra of Hölder continuous functions $C^\alpha(\partial\Omega) \subseteq C(\partial\Omega)$ becomes p-summable. These results will give us analytic degree formulas for Hölder continuous mappings.

Theorem B.3.1. *If Ω is a relatively compact strictly pseudo-convex domain in a Stein manifold of complex dimension n and P denotes either P_{HR} or $P_{\partial\Omega}$ the operator $[P, \pi(a)]$ belongs to $\mathscr{L}^p(L^2(\partial\Omega))$ for $a \in C^\alpha(\partial\Omega)$ and for all $p > 2n/\alpha$.*

The proof will be based on Theorem B.2.3. We will start our proof of Theorem B.3.1 by some elementary estimates. We define the measurable function $k_\alpha : \partial\Omega \times \partial\Omega \to \mathbb{C}$ by

$$k_\alpha(z,w) := \frac{|z-w|^\alpha}{|\Phi(w,z)|^n}.$$

Lemma B.3.2. *The function k_α satisfies*

$$k_\alpha(z,w) \lesssim |\Phi(w,z)|^{-(n-\frac{\alpha}{2})}$$

for $|z-w| < \varepsilon$.

Proof. By (B.12) we have the estimate

$$|z-w|^{\alpha} \lesssim |\Phi(w,z)|^{\alpha/2}.$$

From this estimate the Lemma follows. □

We will use the notation $\mathrm{d}V$ for the volume measure on $\partial\Omega$.

Lemma B.3.3. *The function k_α satisfies*

$$\int_{\partial\Omega} |k_\alpha(z,w)|^{p'}\mathrm{d}V(z) \lesssim 1$$

$$\int_{\partial\Omega} |k_\alpha(z,w)|^{p'}\mathrm{d}V(w) \lesssim 1$$

whenever

$$(2n-\alpha)p' < 2n.$$

Proof. We will only prove the first of the estimates in the Lemma. The proof of the second estimate goes analogously. Using (B.12) for Φ, we obtain

$$\int_{\partial\Omega} |k_\alpha(z,w)|^{p'}\mathrm{d}V(z) \lesssim \int_{B(r,w)} |k_\alpha(z,w)|^{p'}\mathrm{d}V(z),$$

since the function Φ satisfies $|\Phi(w,z)| > r^2$ outside the ball $B(r,w)$ of radius r around w. By Lemma B.3.2 we can estimate the kernel pointwise by Φ so (B.13) implies

$$\int_{B(r,w)} |k_\alpha(z,w)|^{p'}\mathrm{d}V(z) \lesssim \int_{B(r,w)} |\Phi(w,z)|^{-p'(n-\frac{\alpha}{2})}\mathrm{d}V(z) \lesssim 1$$

if $(n-\frac{\alpha}{2})p' < n$. □

Lemma B.3.4. *The function k_α satisfies $\|k_\alpha\|_{p',p} < \infty$ and $\|k_\alpha^*\|_{p',p} < \infty$ for $p > 2n/\alpha$.*

Proof. By the first estimate in Lemma B.3.3 we can estimate the mixed norms of k_α as

$$\|k_\alpha\|_{p',p}^{p} \lesssim 1,$$

whenever $(2n-\alpha)p' < 2n$. The statement $(2n-\alpha)p' < 2n$ is equivalent to

$$\frac{1}{p} = 1 - \frac{1}{p'} < \frac{\alpha}{2n}$$

which is equivalent to $p > 2n/\alpha$. Similarly, the second estimate in Lemma B.3.3 implies $\|k_\alpha^*\|_{p',p} < \infty$ under the same condition on p. □

Lemma B.3.5. *Suppose that $a \in C^\alpha(\partial\Omega)$ and let κ_a denote the integral kernel of $[P_{HR}, \pi(a)]$. The kernel κ_a satisfies*

$$|\kappa_a(z,w)| \leq |a|_\alpha |k_\alpha(z,w)|. \tag{B.22}$$

Proof. The integral kernel of $[P_{HR}, \pi(a)]$ is given by

$$\kappa_a(z,w) = (a(z) - a(w))H_{\partial\Omega}(w,z).$$

Since a is Hölder continuous and $H_{\partial\Omega}$ satisfies equation (B.14) the estimate (B.22) follows. □

Lemma B.3.6. *The HR-projection P_{HR} satisfies $P_{HR} - P_{HR}^* \in \mathscr{L}^q(L^2(\partial\Omega))$ for any $q > 2n$. Therefore $P_{HR} - P_{\partial\Omega} \in \mathscr{L}^q(L^2(\partial\Omega))$ for any $q > 2n$.*

Proof. Let us denote the kernel of the operator $P_{HR} - P_{HR}^*$ by b. By (B.15) we have the pointwise estimate $|b(z,w)| \lesssim |\Phi(w,z)|^{-n+1/2}$. Applying Lemma B.3.4 with $\alpha = 0$ and p' such that $(n-1/2)q' = np'$ we obtain the inequality $\|b\|_{q',q} < \infty$ for any $q > 2n$. The fact that $P_{HR} - P_{\partial\Omega} \in \mathscr{L}^q(L^2(\partial\Omega))$ follows now from (B.20). □

Proof of Theorem B.3.1. By Lemma B.3.5 the integral kernel κ_a of $[P_{HR}, \pi(a)]$ satisfies $|\kappa_a| \leq |a|_\alpha k_\alpha$. Theorem B.2.3 implies the estimate

$$\|[P_{HR}, \pi(a)]\|_{\mathscr{L}^p(L^2(\partial\Omega))} \leq |a|_\alpha (\|k_\alpha\|_{p',p} \|k_\alpha^*\|_{p',p})^{1/2}.$$

By Lemma B.3.4, $\|k_\alpha\|_{p',p}, \|k_\alpha^*\|_{p',p} < \infty$ for $p > 2n/\alpha$ so $[P_{HR}, \pi(a)] \in \mathscr{L}^p(L^2(\partial\Omega))$ for $p > 2n/\alpha$. By Lemma B.3.6, $P_{HR} - P_{\partial\Omega} \in \mathscr{L}^p(L^2(\partial\Omega))$, so

$$[P_\Omega, \pi(a)] = [P_{HR}, \pi(a)] + [P_\Omega - P_{HR}, \pi(a)] \in \mathscr{L}^p(L^2(\partial\Omega))$$

for $p > 2n/\alpha$ and the proof of the Theorem is complete. □

B.4 The index- and degree formula

We may now combine our results on summability of the Toeplitz pairs (P_{HR}, π) and $(P_{\partial\Omega}, \pi)$ into index theorems and degree formulas. The index formula will be proved using the index formula of Connes, see Theorem B.2.2.

Theorem B.4.1. *Suppose that Ω is a relatively compact strictly pseudo-convex domain with smooth boundary in a Stein manifold of complex dimension n and denote the corresponding HR-kernel by $H_{\partial\Omega}$ and the Szegö kernel by $C_{\partial\Omega}$. If*

$a : \partial\Omega \to \mathrm{GL}_N$ is Hölder continuous with exponent α, then for $2k+1 > 2n/\alpha$ the index formulas hold

$$\begin{aligned}
&\operatorname{ind}(P_{\partial\Omega}\pi(a)P_{\partial\Omega}) = \operatorname{ind}(P_{HR}\pi(a)P_{HR}) = \\
&\quad = -\int_{\partial\Omega^{2k+1}} \operatorname{tr}\left(\prod_{j=0}^{2k}(1-a(z_{j-1})^{-1}a(z_j))H_{\partial\Omega}(z_{j-1},z_j)\right)\mathrm{d}V = \qquad \text{(B.23)}\\
&\quad = -\int_{\partial\Omega^{2k+1}} \operatorname{tr}\left(\prod_{j=0}^{2k}(1-a(z_{j-1})^{-1}a(z_j))C_{\partial\Omega}(z_{j-1},z_j)\right)\mathrm{d}V, \qquad \text{(B.24)}
\end{aligned}$$

where the integrals in (B.23) *and* (B.24) *converge.*

Proof. By Theorem B.2.2 we have

$$\operatorname{ind}(P_{\partial\Omega}\pi(a)P_{\partial\Omega}) = -\operatorname{tr}(P_{\partial\Omega} - \pi(a^{-1})P_{\partial\Omega}\pi(a))^{2k+1}$$

and by Theorem B.2.4 the trace has the form

$$\begin{aligned}
&-\operatorname{tr}(P_{\partial\Omega} - \pi(a^{-1})P_{\partial\Omega}\pi(a))^{2k+1} = \\
&\qquad = -\int_{\partial\Omega^{2k+1}} \operatorname{tr}\left(\prod_{j=0}^{2k}(1-a(z_{j-1})^{-1}a(z_j))C_{\partial\Omega}(z_{j-1},z_j)\right)\mathrm{d}V.
\end{aligned}$$

Similarly, the index for $P_{HR}\pi(a)P_{HR}$ is calculated. The Theorem follows from the identity $\operatorname{ind}(P_{\partial\Omega}\pi(a)P_{\partial\Omega}) = \operatorname{ind}(P_{HR}\pi(a)P_{HR})$ since Lemma B.3.6 implies that $P_{\partial\Omega}\pi(a)P_{\partial\Omega} - P_{HR}\pi(a)P_{HR}$ is compact. □

Theorem B.4.1 has an interpretation in terms of cyclic cohomology. Define the cyclic $2n-1$-cocycle $\chi_{\partial\Omega}$ on $C^\infty(\partial\Omega)$ by

$$\chi_{\partial\Omega} := \sum_{k=0}^{n} S^k\omega_k,$$

where ω_k denotes the cyclic $2n-2k-1$-cocycle given by the Todd class $Td_k(\Omega)$ in degree $2k$ as

$$\omega_k(a_0,a_1,\ldots,a_{2n-2k-1}) := \int_{\partial\Omega} a_0\mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_{2n-2k-1} \wedge Td_k(\Omega).$$

Similarly to Proposition 13, Chapter III.3 of [34], we have the following:

Theorem B.4.2. *The cyclic cocycle $S^m\chi_{\partial\Omega}$ defines the same cyclic cohomology class on $C^\infty(\partial\Omega)$ as*

$$\begin{aligned}
&\tilde{\chi}_{\partial\Omega}(a_0,a_1,\ldots,a_{2n+2m-1}) := \\
&\quad := \int_{\partial\Omega^{2n+2m-1}} \operatorname{tr}\left(a_0(z_0)\prod_{j=1}^{2n+2m-1}(a_j(z_j)-a_j(z_{j-1}))C_{\partial\Omega}(z_{j-1},z_j)\right)\mathrm{d}V,
\end{aligned}$$

where we identify $z_{2n+2m-1} = z_0$. *Furthermore, the cyclic cocycle* $\tilde{\chi}_{\partial\Omega}$ *extends to a cyclic* $2n+2m-1$*-cocycle on* $C^\alpha(\partial\Omega)$ *if* $m > (2n(1-\alpha)+\alpha)/2\alpha$.

Returning to the degree calculations, to express the degree of a Hölder continuous function we will use Theorem B.2.1 and Theorem B.4.1. In order to express the formulas in Theorem B.4.1 directly in terms of f we will need some notations. Let $\langle\cdot,\cdot\rangle$ denote the scalar product on $\mathbb{C}^n$. The symmetric group on m elements will be denoted by S_m. We will consider S_m as the group of bijections on the set $\{1,2,\ldots,m\}$ and identify the element $m+1$ with 1 in the set $\{1,2,\ldots,m\}$.

For $2l \leq m$ we will define a function $\varepsilon_l : S_m \to \{0,1,-1\}$ which we will refer to the order parity. If $\sigma \in S_m$ satisfies that there is an $i \in \{\sigma(1),\sigma(2),\ldots\sigma(2l-1),\sigma(2l)\}$ such that $i+1, i-1 \notin \{\sigma(1),\sigma(2),\ldots\sigma(2l-1),\sigma(2l)\}$ we set $\varepsilon_l(\sigma) = 0$. If σ does not satisfy this condition the order parity of σ is set as $(-1)^k$, where k is the smallest number of transpositions needed to mapping the set $\{\sigma(1),\sigma(2),\ldots\sigma(2l-1),\sigma(2l)\}$, with j identified with $j+m$, to a set of the form $\{j_1, j_1+1, j_2, j_2+1, \ldots, j_l, j_l+1\}$ where $1 \leq j_1 < j_2 < \cdots < j_l \leq m$.

Proposition B.4.3. *The function* u *satisfies*

$$\mathrm{tr}\left(\prod_{i=0}^{2k}(1-u(z_{i-1})^*u(z_i))\right) =$$

$$= \sum_{l=0}^{2k+1}\sum_{\sigma\in S_{2(2k+1)}}(-1)^l 2^{n-l-1}\varepsilon_l(\sigma)\langle z_{\sigma(1)}, z_{\sigma(2)}\rangle\langle z_{\sigma(3)}, z_{\sigma(4)}\rangle\cdots\langle z_{\sigma(2l-1)}, z_{\sigma(2l)}\rangle,$$

where we identify z_m *with* z_{m+2k+1} *for* $m = 0,1,\ldots,2k$.

Proof. The product in the lemma satisfies the equalities

$$\prod_{i=1}^{2k-1}(1-u(z_{i-1})^*u(z_i)) = \prod_{i=1}^{2k-1}\left(1+\frac{1}{2}(z_{i-1,+}+\bar{z}_{i-1,-})(z_{i,+}+\bar{z}_{i,-})\right) =$$

$$= \sum_{l=0}^{2k-1}\sum_{i_1<i_2<\ldots<i_l}2^{-l}\prod_{j=1}^{l}\left((z_{i_j-1,+}+\bar{z}_{i_j-1,-})(z_{i_j,+}+\bar{z}_{i_j,-})\right).$$

The Lemma follows from these equalities and degree reasons. □

Let us choose an open subset $U \subseteq Y$ such that there is a diffeomorphism $\nu : U \to B_{2n-1}$. Let $\tilde{\nu}$ be as in equation (B.11) and define the function $\tilde{f} : \partial\Omega^{2k+1} \to \mathbb{C}$ by

$$\tilde{f}(z_0,z_1,\ldots,z_{2k}) := \sum_{\sigma\in S_{2(2k-1)}}\sum_{l=0}^{2k-1}(-1)^l 2^{n-l-1}\varepsilon_l(\sigma)\prod_{i=1}^{l}\langle\tilde{\nu}(f(z_{\sigma(2j-1)})), \tilde{\nu}(f(z_{\sigma(2j)}))\rangle \tag{B.25}$$

where we identify z_m with z_{m+2k+1}.

Theorem B.4.4. *Suppose that Ω is a relatively compact strictly pseudo-convex domain with smooth boundary in a Stein manifold of complex dimension n and that Y is a connected, compact, orientable, Riemannian manifold of dimension $2n-1$. If $f:\partial\Omega\to Y$ is a Hölder continuous function of exponent α the degree of f can be calculated by*

$$\begin{aligned}\deg(f) =&(-1)^n\langle\tilde{\chi}_{\partial\Omega}, g\circ f\rangle_{2k+1} =\\ &=(-1)^n\int_{\partial\Omega^{2k+1}}\tilde{f}(z_0,z_1,\ldots,z_{2k})\prod_{j=0}^{2k}H_{\partial\Omega}(z_{j-1},z_j)\mathrm{d}V =\\ &=(-1)^n\int_{\partial\Omega^{2k+1}}\tilde{f}(z_0,z_1,\ldots,z_{2k})\prod_{j=0}^{2k}C_{\partial\Omega}(z_{j-1},z_j)\mathrm{d}V\end{aligned}$$

whenever $2k+1>2n/\alpha$.

Proof. By Theorem B.2.1 and Theorem B.4.1 we have the equality

$$\deg(f)=(-1)^n\int_{\partial\Omega^{2k+1}}\mathrm{tr}\left(\prod_{j=0}^{2k}(1-g(f)(z_j)^*g(f)(z_{j+1}))H_{\partial\Omega}(z_{j-1},z_j)\right)\mathrm{d}V.$$

Proposition B.4.3 implies

$$\mathrm{tr}\left(\prod_{j=0}^{2k}(1-g(f)(z_j)^*g(f)(z_{j+1}))\right)=\tilde{f}(z_0,z_1,\ldots,z_{2k}),$$

from which the Theorem follows. □

Let us end this paper by a remark on the restriction in Theorem B.4.4 that the domain of f must be the boundary of a strictly pseudo-convex domain in a Stein manifold. The condition on a manifold M to be a a Stein manifold of complex dimension n implies that M has the same homotopy type as an n-dimensional CW-complex since the embedding theorem for Stein manifolds, see for instance [45], implies that a Stein manifold of complex dimension n can be embedded in $\mathbb{C}^{2n+1}$ and by Theorem 7.2 of [65] an n-dimensional complex submanifold of complex Euclidean space has the same homotopy type as a CW-complex of dimension n.

Conversely, if X is a real analytic manifold, then for any choice of metric on X, the co-sphere bundle S^*X is diffeomorphic to the boundary of a strictly pseudo-convex domain in a Stein manifold, see for instance Proposition 4.3 of [47] or Chapter V.5 of [45]. So the degree of f coincides with the mapping $H^{2n-1}_{dR}(S^*Y)\to H^{2n-1}_{dR}(S^*X)$ that f induces under the Thom isomorphism $H^n_{dR}(X)\cong H^{2n-1}(S^*X)$. Thus the degree of a function $f:X\to Y$ can be expressed using our methods for any real analytic X.

Paper C

Analytic formulas for degree of non-smooth mappings: the even-dimensional case

Abstract

Topological degrees of continuous mappings between manifolds of even dimension are studied in terms of index theory of pseudo-differential operators. The index formalism of non-commutative geometry is used to derive analytic integral formulas for the index of a 0:th order pseudo-differential operator twisted by a Hölder continuous vector bundle. The index formula gives an analytic formula for the degree of a Hölder continuous mapping between even-dimensional manifolds. The paper is an independent continuation of Paper B.

Introduction

This paper is an independent continuation of Paper B where the degree of a mapping from the boundary of a strictly pseudo-convex domain was given in terms of an explicit integral formula involving the Szegö kernel. In this paper analytic formulas are given for general even-dimensional manifolds in terms of the signature operator. The classical approach to mapping degree is to define in an abstract way the degree of a continuous mapping between two compact connected oriented manifolds of the same dimension in terms of homology. If the function f is differentiable, an analytic formula for the degree can be derived using Brouwer degree, see [69], or the more global picture of de Rham cohomology. Without differentiability conditions on f, the only known analytic degree formula beyond Paper B is a formula of Connes which only holds in one dimension, see more in Chapter $2.\alpha$ of [34]. Our aim is to find another formula for the degree, that is valid for a Hölder continuous function, by expressing the degree as the index of a pseudo-differential operator and using the approach of [34] and Paper B.

Throughout the paper we will use the idea that the Chern character extracts cohomological information of a continuous mapping $f : X \to Y$ between even dimensional manifolds from the induced mapping $f^* : K^0(Y) \to K^0(X)$. The K-theory is a topological invariant and the picture of the index mapping as a pairing in a local homology theory via Chern characters in the Atiyah-Singer index theorem can be applied to more general classes of functions than the smooth functions. The homology theory present throughout all the index theory is the cyclic homology. For a Hölder continuous mapping $f : X \to Y$ of exponent α and an elliptic pseudo-differential operator A of order at least α, this idea can be read out from the commutativity of the diagram:

$$\begin{array}{ccccc} K_0(C^\infty(Y)) & \xrightarrow{f^*} & K_0(C^\alpha(X)) & \xrightarrow{\mathrm{ind}_A} & \mathbb{Z} \\ \downarrow \mathrm{ch}_Y & & \downarrow \mathrm{ch}_X & & \downarrow \\ HC_{even}(C^\infty(Y)) & \xrightarrow{f^*} & HC_{even}(C^\alpha(X)) & \xrightarrow{\tilde{\mu}_A} & \mathbb{C} \end{array} \tag{C.1}$$

where the mapping $\tilde{\mu}_A : HC_{even}(C^\alpha(X)) \to \mathbb{C}$ is a cyclic cocycle on $C^\alpha(X)$ defined as the Connes-Chern character of the bounded K-homology class that A defines, see more in [32] and [34]. The right-hand side of the diagram (C.1) is commutative by Connes' index formula, see Proposition 4 of Chapter IV.1 of [34]. The dimension in which the Chern character will take values depends on the Hölder exponent α. More explicitly, for $2n$-dimensional manifolds, the cocycle $\tilde{\mu}_A$ can be chosen as a cyclic $2k$-cocycle for any $k > n/\alpha$.

To describe this idea more explicitely, when $E \to X$ is a smooth vector bundle defined by the smooth projection-valued function $p : X \to \mathscr{K}$, the index of the twisted pseudo-differential operator $A_E := p(A \otimes \mathrm{id}_{\mathscr{K}})p$ can be calculated in terms of the de Rham cohomology using the Atiyah-Singer index formula as

$$\mathrm{ind} A_E = \int_{T^*X} \pi^* \mathrm{ch}[E] \wedge \mathrm{ch}[A] \wedge \pi^* Td(X),$$

where $Td(X)$ denotes the Todd class of the complexified tangent bundle. In particular, if $E \to Y$ is a line bundle on an even-dimensional manifold Y such that $\mathrm{ch}[E]$ only contains a constant term and a top-degree term and $f : X \to Y$ is smooth we can consider the line bundle $f^*E \to X$. Naturality of the Chern character implies the identity

$$\deg(f)\, \mathrm{ch}_0[A] \int_Y \mathrm{ch}_Y[E] = \mathrm{ind} A_{f^*E} - \mathrm{ind}\,(A).$$

In Theorem C.2.2, we construct an explicit line bundle $E_Y \to Y$ satisfying the above conditions together with the condition $\int_Y \mathrm{ch}_Y[E_Y] = 1$. In the correct analytic setting the above degree formula extends to Hölder continuous functions. The analytic setting we choose in Theorem C.4.2 is to associate a Fredholm module $(\tilde{\pi}, \tilde{F}_A)$ with an elliptic pseudo-differential operator A of order at least α. The Fredholm module $(\tilde{\pi}, \tilde{F}_A)$ is q-summable over the algebra of Hölder continuous functions $C^\alpha(X)$, for any $q > \dim(X)/\alpha$. Thus the Connes-Chern character $\tilde{\mu}_A := \mathrm{cc}_k(\tilde{\pi}, \tilde{F}_A)$ is well defined for dimensions $2k > \dim(X)/\alpha$. In Theorem B.4.4 we take A to be the signature operator and show that if $f : X \to Y$ is Hölder continuous the following analytic degree formula holds:

$$\deg(f) = \tilde{\mu}_A(\mathrm{ch}_X[f^*E_Y]) - \mathrm{sign}(X).$$

The drawback exhibited in Paper B, where the results were restricted to boundaries of strictly pseudo-convex domains in Stein manifolds, is not present in this paper. The restriction that X and Y must be even-dimensional does not really pose a problem since when X and Y are odd-dimensional we can consider the mapping $f \times \mathrm{id} : X \times S^1 \to Y \times S^1$ instead which is a mapping between even-dimensional manifolds and $\deg(f) = \deg(f \times \mathrm{id})$. The drawback of the degree formula in Theorem B.4.4 is that it is in general quite hard to calculate explicit integral kernels for pseudo-differential operators.

C.1 K-theory and Connes' index formula

To formulate the calculation of mapping degrees in a setting fitting with non-smooth mappings, we need a framework for "differential geometry" where there are no classical differentials. The framework we will use is Alain Connes' non-commutative geometry, see [32] and [34]. We will recall some basic concepts of non-commutative geometry in this section.

The K-theory of a compact topological space Y is defined as the Grothendieck group of the abelian semigroup of isomorphism classes of vector bundles under direct sum. The Serre-Swan theorem states a one-to-one correspondence between the isomorphism classes of vector bundles over a compact space Y and projection valued functions $p : Y \to \mathscr{K}$, see [85]. Here $\mathscr{K}$ denotes the C^*-algebra of compact operators on some separable, infinite dimensional Hilbert space. Following the Serre-Swan theorem, an equivalent approach to K-theory is to use equivalence classes of projections $p \in C(Y) \otimes \mathscr{K}$. The K-theory is denoted by $K_0(C(Y))$. To read more about K-theory, see [22]. The formulation of K-theory in terms of projections can be defined for any algebra $\mathscr{A}$ as equivalence classes of projections $p \in \mathscr{A} \otimes \mathscr{K}$.

Clearly, the abelian group $K_0(\mathscr{A})$ depends covariantly on the algebra $\mathscr{A}$ so K_0 defines a functor. In particular, the functor K_0 has many properties making the K-theory of a C^*-algebra manageable to calculate, for instance; homotopy invariance, half exactness and stability under tensoring by a matrix algebra. Furthermore, a dense embedding of topological algebras $\mathscr{A}' \hookrightarrow \mathscr{A}$ which is isoradial induces an isomorphism on K-theory, see more in [37]. For instance, if Y is a compact manifold all of the embeddings $C^\infty(Y) \subseteq C^\alpha(Y) \subseteq C(Y)$ induce isomorphisms on K-theory. Here $C^\alpha(Y)$ denotes the algebra of Hölder continuous functions of exponent $\alpha \in]0,1]$. The isomorphism $K_0(C(Y)) \cong K_0(C^\infty(Y))$ enables us to define the Chern character $\mathrm{ch} : K_0(C(Y)) \to H^{even}_{dR}(Y)$ by representing a class $[p] \in K_0(C(Y))$ by a smooth $p : Y \to \mathscr{K}$ and define

$$\mathrm{ch}[p] := \sum_{j=0}^{\infty} \frac{1}{(2\pi)^j j!} \mathrm{tr}(p\mathrm{d}p\mathrm{d}p)^j.$$

We choose the trace as the fiberwise operator trace in $\wedge^* T^* Y \otimes \mathscr{K}$ which is well defined since a compact projection is of finite rank. The term in the sum of degree $2j$ is denoted by $\mathrm{ch}_j[p]$.

However, we will need a Chern character defined on Hölder continuous projections. The homology theory fitting with index theory of more complicated geometries than smooth functions on smooth manifolds is cyclic homology. We will consider Connes' original definition of cyclic homology which simplifies the construction of the Chern character and the Chern-Connes character. We will let $\mathscr{A}$ denote a topological algebra and we will use the notation $\mathscr{A}^{\otimes k}$ for the k-th tensor power of $\mathscr{A}$. The Hochschild differential $b : \mathscr{A}^{\otimes k} \to \mathscr{A}^{\otimes k-1}$ is defined by

$$b(x_0 \otimes x_1 \otimes \cdots \otimes x_k \otimes x_{k+1}) := (-1)^{k+1} x_{k+1} x_0 \otimes x_1 \otimes \cdots \otimes x_k + \\ + \sum_{j=0}^{k} (-1)^j x_0 \otimes \cdots \otimes x_{j-1} \otimes x_j x_{j+1} \otimes x_{j+2} \otimes \cdots \otimes x_{k+1}.$$

The cyclic permutation operator $\lambda : \mathscr{A}^{\otimes k} \to \mathscr{A}^{\otimes k}$ is defined as

$$\lambda(x_0 \otimes x_1 \otimes \cdot \otimes x_k) = (-1)^k x_k \otimes x_0 \otimes \cdots \otimes x_{k-1}.$$

We define a complex of $\mathbb{C}$-vector spaces $C_*^\lambda(\mathscr{A})$ by

$$C_k^\lambda(\mathscr{A}) := \mathscr{A}^{\otimes k+1}/(1-\lambda)\mathscr{A}^{\otimes k+1},$$

with differential given by b. The homology of the complex $C_*^\lambda(\mathscr{A})$ is called the cyclic homology of $\mathscr{A}$ and will be denoted by $HC_*(\mathscr{A})$. A cycle in $C_\lambda^k(\mathscr{A})$ will be called a cyclic k-cycle.

The complex $C_\lambda^k(\mathscr{A})$ is defined as the space of continuous linear functionals μ on $\mathscr{A}^{\otimes k+1}$ such that $\mu \circ \lambda = \mu$. The Hochschild coboundary operator $\mu \mapsto \mu \circ b$ makes $C_\lambda^*(\mathscr{A})$ into a complex. The cohomology of the complex $C_\lambda^*(\mathscr{A})$ will be denoted by $HC^*(\mathscr{A})$ and is called the cyclic cohomology of $\mathscr{A}$. Cyclic cohomology is an algebraic generalization of de Rham homology. The difference lies in that the dimension defines a grading on the de Rham theories, while the dimension defines a filtration on the cyclic theories. This difference can be explained by a Theorem of Connes [34] stating that if X is a compact manifold, there is an isomorphism

$$HC^k(C^\infty(X)) \cong Z_k(X) \oplus \bigoplus_{j>0} H_{k-2j}^{dR}(X), \tag{C.2}$$

where $Z_k(X)$ denotes the space of closed k-currents on X. The filtration on cyclic cohomology can be described by the linear mapping $S : HC^k(\mathscr{A}) \to HC^{k+2}(\mathscr{A})$ called the periodicity operator. For a definition of the periodicity operator, see [34].

The Chern character $\mathrm{ch}_{2k} : K_0(\mathscr{A}) \to HC_{2k}(\mathscr{A})$ in degree $2k$ is defined as in Proposition 3 of Chapter III.3 of [34] by

$$\mathrm{ch}_{2k}[p] := (k!)^{-1} \mathrm{tr} \left(\underbrace{p \otimes p \otimes \cdots \otimes p \otimes p}_{2k+1 \quad \text{factors}} \right). \tag{C.3}$$

The additive pairing between $HC^{2k}(\mathscr{A})$ and the K-theory $K_0(\mathscr{A})$ is defined by

$$\langle \mu, x \rangle_{2k} := \mu.\mathrm{ch}_{2k}[x].$$

The choice of normalization implies that for a cohomology class in $HC^{2k}(\mathscr{A})$ represented by the cyclic cocycle μ, the pairing satisfies

$$\langle S\mu, x\rangle_{2k+2} = \langle \mu, x\rangle_{2k},$$

see Proposition 3 in Chapter III.3 of [34].

The homology theory dual to K-theory is K-homology. Analytic K-homology is described by Fredholm modules, a theory fitting well with index theory and cyclic cohomology. For $q \geq 1$, let $\mathscr{L}^q(\mathscr{H}) \subseteq \mathscr{B}(\mathscr{H})$ denote the ideal of Schatten class operators on a separable Hilbert space $\mathscr{H}$, so $T \in \mathscr{L}^q(\mathscr{H})$ if and only if $\mathrm{tr}((T^*T)^{q/2}) < \infty$. For $q \neq 2$ there is no exact description for an integral operator to belong to the Schatten class of order q. However, for $q > 2$ there exists a convenient sufficient condition on the kernel, found in [80]. We will return to this subject a little later.

A graded Hilbert space is a Hilbert space $\mathscr{H}$ equipped with an involutive mapping γ, that is, $\gamma^2 = 1$. While γ is an involution, we can decompose $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$, where $\mathscr{H}_\pm = \ker(\gamma \mp 1)$. An operator T on $\mathscr{H}$ is called even if $T\mathscr{H}_\pm \subseteq \mathscr{H}_\pm$ and odd if $T\mathscr{H}_\pm \subseteq \mathscr{H}_\mp$. Suppose that $\mathscr{H}$ is a graded Hilbert space and $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ is an even representation of a trivially graded $\mathbb{C}$-algebra $\mathscr{A}$. If $F \in \mathscr{B}(\mathscr{H})$ is an odd operator such that

$$F^2 = 1,\ F = F^* \quad \text{and} \quad [F, \pi(a)] \in \mathscr{L}^q(\mathscr{H})\ \forall a \in \mathscr{A}, \tag{C.4}$$

the pair (π, F) is called a q-summable even Fredholm module. The conditions $F^2 = 1$ and $F = F^*$ simplifies many calculations, but in practice it is sufficient if they hold modulo q-summable operators. If we decompose the graded Hilbert space $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ into its even and odd part, the odd operator F decomposes as:

$$F = \begin{pmatrix} 0 & F_+ \\ F_- & 0 \end{pmatrix}$$

where $F_+ : \mathscr{H}_- \to \mathscr{H}_+$ and $F_- : \mathscr{H}_+ \to \mathscr{H}_-$. Similarly we can decompose $\pi = \pi_+ \oplus \pi_-$ where $\pi_\pm : \mathscr{A} \to \mathscr{B}(\mathscr{H}_\pm)$ are representations. The first and second condition in (C.4) are equivalent to the conditions $F_+ = F_-^{-1} = F_-^*$ and the commutator condition is equivalent to

$$F_-\pi_+(a) - \pi_-(a)F_- \in \mathscr{L}^q(\mathscr{H}_+, \mathscr{H}_-) \quad \text{and} \quad F_+\pi_-(a) - \pi_+(a)F_+ \in \mathscr{L}^q(\mathscr{H}_-, \mathscr{H}_+).$$

If the pair (π, F) satisfies the requirement in equation (C.4) but with $\mathscr{L}^q(\mathscr{H})$ replaced by $\mathscr{K}(\mathscr{H})$ the pair (π, F) is a bounded even Fredholm module. The set of homotopy classes of bounded even Fredholm modules forms an abelian group under direct sum called the even analytic K-homology of $\mathscr{A}$ and is denoted by $K^0(\mathscr{A})$. For a more thorough presentation of Fredholm modules, e.g. Chapter VII and VIII of [22].

Following Definition 3 of Chapter IV.1 of [34] we define the Connes-Chern character $\mathrm{cc}_{2k}(\pi,F)$ of a q-summable even Fredholm module (π,F) for $2k\geq q$ as the cyclic $2k$-cocycle:

$$\mathrm{cc}_{2k}(\pi,F)(a_0,a_1,\ldots,a_{2k}):=(-1)^k k!\,\mathrm{str}(\pi(a_0)[F,\pi(a_1)]\cdots[F,\pi(a_{2k})]),$$

where $\mathrm{str}(T):=\mathrm{tr}(\gamma T)$ for $T\in\mathscr{L}^1(\mathscr{H})$ and γ denotes the grading on $\mathscr{H}$. This choice of normalization leads to $S\mathrm{cc}_{2k}(\pi,F)=cc_{2k+2}(\pi,F)$, see Proposition 2 of Chapter IV.1 of [34].

If (π,F) is a bounded Fredholm module over $\mathscr{A}$ we can define the index mapping $\mathrm{ind}_F:K_0(\mathscr{A})\to\mathbb{Z}$ as

$$\mathrm{ind}_F[p]:=\mathrm{ind}((\pi_+\otimes\mathrm{id})(p)F_+(\pi_-\otimes\mathrm{id})(p)),$$

where we represent $[p]$ by a finite-dimensional projection $p\in\mathscr{A}\otimes\mathscr{K}(\mathbb{C}^N)$ and we consider $(\pi_+\otimes\mathrm{id})(p)F_+(\pi_-\otimes\mathrm{id})(p)$ as an operator

$$(\pi_+\otimes\mathrm{id})(p)F_+(\pi_-\otimes\mathrm{id})(p):(\pi_-\otimes\mathrm{id})(p)\left(\mathscr{H}_-\otimes\mathbb{C}^N\right)\to(\pi_+\otimes\mathrm{id})(p)\left(\mathscr{H}_+\otimes\mathbb{C}^N\right).$$

The association $[p]\times(\pi,F)\mapsto\mathrm{ind}_F(p)$ is homotopy invariant and defines a bilinear pairing $K_0(\mathscr{A})\times K^0(\mathscr{A})\to\mathbb{Z}$ which is non-degenerate after tensoring with $\mathbb{Q}$, see more in [22]. To simplify the notation, we suppress the dimension N and identify (π,F) with the Fredholm module $(\pi\otimes\mathrm{id}_{\mathscr{K}(\mathbb{C}^N)},F\otimes\mathrm{id}_{\mathbb{C}^N})$.

Theorem C.1.1 (Proposition 4 of Chapter IV.1 of [34]). *If (π,F) is a q-summable even Fredholm module and $2k\geq q$ the index mapping ind_F can be calculated as*

$$\mathrm{ind}_F[p]=\langle\mathrm{cc}_k(\pi,F),p\rangle_k.$$

In Theorem C.1.1, the conditions (C.4) on the Fredholm module (π,F) requires some caution. If for instance, we remove the condition $F^2\neq 1$ one can choose F such that $\mathrm{ind}(F_+)\neq 0$. On the other hand if we require π to be unital, we have that $\mathrm{ind}(F_+)=\mathrm{ind}_F(1)$ but $\langle\mathrm{cc}_k(\pi,F),1\rangle_k=0$, therefore $\mathrm{ind}(F_+)\neq\langle\mathrm{cc}_k(\pi,F),1\rangle_k$.

In the context of index theory, the periodicity operator S plays the role of extending index formulas such as that in Theorem C.1.1 to larger algebras. Suppose that μ is a cyclic k-cocycle on an algebra $\mathscr{A}_0$ which is a dense $*$-subalgebra of a C^*-algebra $\mathscr{A}$. The cyclic $k+2m$-cocycle $S^m\mu$ can sometimes be extended by continuity to a cyclic cocycle on a larger $*$-subalgebra $\mathscr{A}_0\subseteq\mathscr{A}'\subseteq\mathscr{A}$. In Paper B the properties of $S^m\mu$ were studied for Ω being a strictly pseudo-convex domain in a Stein manifold of complex dimension n and μ being the cyclic $2n-1$-cocycle on $\mathscr{A}_0=C^\infty(\partial\Omega)$ defined by

$$\mu:=\sum_{k=0}^{n}S^k\omega_k,$$

where ω_k denotes the cyclic $2n-2k-1$-cocycle given by the Todd class $Td_k(\Omega)$ in degree $2k$ as

$$\omega_k(a_0, a_1, \dots, a_{2n-2k-1}) := \int_{\partial\Omega} a_0 \mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_{2n-2k-1} \wedge Td_k(\Omega).$$

It was proved in Paper B that the cyclic cocycle $S^m\mu$ defines the same cyclic cohomology class on $C^\infty(\partial\Omega)$ as

$$\begin{aligned} &\tilde{\mu}(a_0, a_1, \dots, a_{2n+2m-1}) := \\ &:= \int_{\partial\Omega^{2n+2m-1}} \mathrm{tr}\left(a_0(z_0) \prod_{j=1}^{2n+2m-1} (a_j(z_j) - a_j(z_{j-1})) K_{\partial\Omega}(z_{j-1}, z_j) \right) \mathrm{d}V, \end{aligned} \tag{C.5}$$

where $K_{\partial\Omega}$ denotes the Szegö kernel or the Henkin-Ramirez kernel. The cyclic cocycle $\tilde{\mu}$ is the *odd* Connes-Chern character of the Toeplitz operators on the Hardy space and $\tilde{\mu}$ extends to a cyclic cocycle on the algebra of Hölder continuous functions on $\partial\Omega$. We will later on use a cyclic cocycle of the form $\mu = \mathrm{cc}_{2k}(\pi, F)$ and the periodicity operator to extend a formulation of the Atiyah-Singer index theorem to pseudo-differential operators twisted by a Hölder continuous vector bundle.

The index formula of Theorem C.1.1 holds for q-summable Fredholm modules and to deal with the q-summability of pseudo-differential operators we will need the following theorem of Russo [80] to give a sufficient condition for an integral operator to be Schatten class. Following [18], when X is a σ-finite measure space and $1 \leq p, q < \infty$, the mixed (p,q)-norm of a function $k : X \times X \to \mathbb{C}$ is defined by

$$\|k\|_{p,q} := \left(\int_X \left(\int_X |k(x,y)|^p \mathrm{d}x \right)^{\frac{q}{p}} \mathrm{d}y \right)^{\frac{1}{q}}.$$

The space of measurable functions with finite mixed (p,q)-norm is denoted by $L^{(p,q)}(X \times X)$. By Theorem 4.1 of [18] the space $L^{(p,q)}(X \times X)$ becomes a Banach space in the mixed (p,q)-norm which is reflexive if $1 < p, q < \infty$. If a bounded operator K has integral kernel k, the hermitian conjugate K^* has integral kernel $k^*(x,y) := \overline{k(y,x)}$.

Theorem C.1.2 (Theorem 1 in [80]). *Suppose that $K : L^2(X) \to L^2(X)$ is a bounded operator given by an integral kernel k. If $2 < q < \infty$*

$$\|K\|_{\mathcal{L}^q(L^2(X))} \leq (\|k\|_{q',q} \|k^*\|_{q',q})^{1/2}, \tag{C.6}$$

where $q' = q/(q-1)$.

In the statement of the Theorem in [80], the completely unnecessary assumption $k \in L^2(X \times X)$ is made. For the discussion on how to remove the condition $k \in L^2(X \times X)$ and the proof of the next Theorem, we refer to Paper B.

Theorem C.1.3. *Suppose that $K_j : L^2(X) \to L^2(X)$ are operators with integral kernels k_j for $j = 1, \dots, m$ such that $\|k_j\|_{q',q}, \|k_j^*\|_{q',q} < \infty$ for certain $q > 2$. Whenever $m \geq q$ the operator $K_1 K_2 \cdots K_m$ is a trace class operator and we have the trace formula*

$$\mathrm{tr}(K_1 K_2 \cdots K_m) = \int_{X^m} \left(\prod_{j=1}^{m} k_j(x_j, x_{j+1}) \right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_m,$$

where we identify x_{m+1} with x_1.

C.2 The projection with Chern character being the volume form

In order to obtain a formulation of the degree as an index, we start by constructing a line bundle E_Y over an arbitrary even dimensional manifold Y such that the only non-constant term in $\mathrm{ch}[E_Y]$ is of top degree. The idea is to use the tautological line bundle over S^2 and the fact that S^{2n} is the smashed products of n copies of S^2 to define a line bundle over S^{2n} for arbitrary n. Under the diffeomorphism $S^2 \cong P^1_{\mathbb{C}}$, the projective complex line, we take the complex coordinate z on S^2 corresponding to one of the affine charts. In the z-chart the tautological line bundle is defined by the projection valued function $p_0 : \mathbb{C} \to M_2(\mathbb{C})$ which is given as

$$p_0(z) := \frac{1}{1 + |z|^2} \begin{pmatrix} |z|^2 & z \\ \bar{z} & 1 \end{pmatrix}.$$

If we define $v : \mathbb{C} \to \mathbb{C}^2$ by $v(z) := \frac{1}{\sqrt{1+|z|^2}}(z, 1)$ then for $w \in \mathbb{C}^2$ we have that

$$p_0(z)w = \langle w, v(z) \rangle v(z). \tag{C.7}$$

It follows that the function p_0 satisfies $p_0^2 = p_0^* = p_0$, so p_0 is a hermitian projection. Let us denote by ω the Fubini-Study metric on S^2, so

$$\omega := \frac{i \mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2}.$$

A straight-forward calulation shows that

$$\mathrm{ch}[p_0] = 1 + \frac{\omega}{2\pi}$$

In higher dimensions we define $p_T : \mathbb{C}^n \to M_{2^n}(\mathbb{C})$ by

$$p_T(z_1, z_2, \dots, z_n) := p_0(z_1) \otimes p_0(z_2) \otimes \dots \otimes p_0(z_n). \tag{C.8}$$

Let us use the notation

$$p_j(z_1, z_2, \dots, z_n) := 1 \otimes \cdots \otimes 1 \otimes p_0(z_j) \otimes \dots \otimes 1 \dots \otimes 1,$$

with $p_0(z_j)$ in the j:th position. It follows that

$$\mathrm{ch}[p_T] = \mathrm{ch}[p_1 p_2 \cdots p_n] = \sum_{I \subseteq \{1,2,\dots,n\}} (2\pi)^{-|I|} \bigwedge_{j \in I} \omega_j, \tag{C.9}$$

where ω_j denotes the Fubini-Study metric depending on the j:th variable z_j.

Lemma C.2.1. *The projection p_T extends to a projection valued C^1-function on S^{2n} such that*

$$\mathrm{ch}[p_T] = 1 + \mathrm{d}V_{S^{2n}}$$

in $H^{even}_{dR}(S^{2n})$.

Proof. Since $H^{even}_{dR}(S^{2n})$ is spanned by the constant function and the volume form, it is sufficient to show that $\mathrm{ch}_0[p_T] = 1$ and $\mathrm{ch}_n[p_T] = \mathrm{d}V_{S^{2n}}$. The function p_T takes values as a rank-one projection, therefore $\mathrm{ch}_0[p_T] = 1$. That $\mathrm{ch}_n[p_T] = \mathrm{d}V_{S^{2n}}$ follows from the following calculation

$$\int_{S^{2n}} \mathrm{ch}_k[p_T] = (2\pi)^{-n} \int_{\mathbb{C}^n} \bigwedge_{j=1}^{n} \omega_j =$$
$$= 2^n (2\pi)^{-n} \int_{\mathbb{C}^n} \prod_{j=1}^{n} \frac{\mathrm{d}V_{\mathbb{C}}(z_j)}{(1+|z_j|^2)^2} = \left(\int_{\mathbb{C}} \frac{2r\mathrm{d}r}{(1+r^2)^2} \right)^n = 1.$$

□

In the general case, let Y be a compact connected orientable manifold of dimension $2n$. If we take an open subset U of Y with coordinates $(x_i)_{i=1}^{2n}$ such that

$$U = \{x : \sum_{i=1}^{2n} |x_i(x)|^2 < 1\},$$

the coordinates define a diffeomorphism $\nu : U \cong B_{2n}$. Let us also choose a diffeomorphism $\tau : B_{2n} \cong \mathbb{C}^n$. We can define the projection valued functions $p_Y : Y \to M_{2^n}(\mathbb{C})$ by

$$p_Y(x) := \begin{cases} p_T(\tau\nu(x)) & \text{for} \quad x \in U \\ p_T(\infty) & \text{for} \quad x \notin U \end{cases} \tag{C.10}$$

If we let $\tilde{\nu} : Y \to S^{2n}$ be the Lipschitz continuous function defined by

$$\tilde{\nu}(x) = \begin{cases} \tau(\nu(x)) & \text{for} \quad x \in U \\ \infty & \text{for} \quad x \notin U \end{cases} \tag{C.11}$$

the C^1-function p_Y can be expressed as $p_Y = \tilde{\nu}^* p_T$.

Theorem C.2.2. *If Y is a compact connected orientable manifold of even dimension and $\mathrm{d}V_Y$ denotes the normalized volume form on Y, the projection p_Y satisfies*

$$\mathrm{ch}[p_Y] = 1 + \mathrm{d}V_Y,$$

in $H^{even}_{dR}(Y)$. Thus, if $f : X \to Y$ is a smooth mapping

$$\deg(f) = \int_X f^* \mathrm{ch}[p_Y]$$

Proof. By Lemma C.2.1 we have the identities

$$\int_Y \mathrm{ch}[p_Y] = \int_U \mathrm{ch}_n[p_Y] = \int_U \tilde{\nu}^* \mathrm{ch}_n[p_T] = \int_{S^{2n}} \mathrm{ch}_k[p_T] = 1.$$

Therefore we have the identity $\mathrm{ch}_n[p] = \mathrm{d}V_Y$. Since $\mathrm{ch}[p_Y] = 1 + \mathrm{ch}_n[p_Y]$ up to an exact form on U and vanishes to first order at ∂U the Theorem follows. □

Later on we will also need the Chern character of p_Y in cyclic homology as is defined in (C.3).

Lemma C.2.3. *The Chern character of p_Y is given by $\tilde{\nu}^* \mathrm{ch}[p_T]$ and the Chern character of p_T in cyclic homology is given by the formula*

$$\begin{aligned} \mathrm{ch}[p_T](z_0, z_1, \dots, z_{2k}) &= (k!)^{-1} \mathrm{tr}_{\mathbb{C}^{2^n}} \left(\prod_{l=0}^{2k} p_T(z_l) \right) = \\ &= (k!)^{-1} \prod_{j=1}^{n} \prod_{l=1}^{2k} \frac{1 + \bar{z}_{j,l} z_{j,l+1}}{1 + |z_{j,l}|^2}, \end{aligned}$$

where we identify $z_{j,2k+1} = z_{j,0}$.

Proof. We may write the Chern character of p_T as the product of traces depending only on the j:th coordinate using (C.8). However, (C.7) implies that

$$\mathrm{tr}_{\mathbb{C}^2} \left(\prod_{l=0}^{2k} p_0(z_{j,l}) \right) = \prod_{l=0}^{2k} \langle \nu(z_{j,l}), \nu(z_{j,l+1}) \rangle = \prod_{l=0}^{2k} \frac{1 + \bar{z}_{j,l} z_{j,l+1}}{1 + |z_{j,l}|^2}.$$

□

C.3 Index theory for pseudo-differential operators

The index of an elliptic pseudo-differential operator can be expressed in terms of local formulas depending on the symbol via the Atiyah-Singer index theorem. In this section we will use elliptic pseudo-differential operators and the Atiyah-Singer index theorem to give an index formula for the degree of a continuous mapping. The theory of pseudo-differential operators can be found in [53] and for an introduction to the Atiyah-Singer index theorem we refer the reader to the survey article [4].

The Atiyah-Singer index theorem, see Theorem 1 of [4], states that the index of an elliptic pseudo-differential operator A on the compact manifold X without boundary can be calculated using de Rham cohomology as:

$$\mathrm{ind}\,(A) = \int_{T^*X} \mathrm{ch}[A] \wedge \pi^* Td(X),$$

where $\mathrm{ch}[A]$ is the Chern character of A and $Td(X)$ is the Todd class of the complexified tangent bundle of X. Since the Chern character is a ring homomorphism we have the following lemma:

Lemma C.3.1. *For a smooth projection valued function $p : X \to \mathcal{K}$ and a pseudo-differential operator A, the Chern character of the pseudo-differential operator $A_p := p(A \otimes id)p$ is given by $\mathrm{ch}[A_p] = \mathrm{ch}[A] \wedge \pi^*\mathrm{ch}[p]$.*

Later on, in Theorem C.4.5, the pseudo-differential operator will play a different role compared to the role in the Atiyah-Singer theorem. In the Atiyah-Singer theorem the elliptic pseudo-differential operator defines an element in K-theory which pairs with the K-homology class whose Connes-Chern character is the Todd class under the isomorphism (C.2) and gives an index. We will use the elliptic pseudo-differential operator in the dual way as a K-homology class that we pair with projections over $C^\infty(X)$ in terms of an index. The heuristic explanation of this method is that $K^0(X)$ is a ring and $K_c^*(T^*X)$ is a $K^*(X)$-module. If A is an elliptic pseudo-differential operator, A defines both a K-homology class on X and the symbol of A defines an element $[A] \in K_c^0(T^*X)$. Furthermore, $\mathrm{ind}_A(x) = \mathrm{ind}\,(x \cdot [A])$ for $x \in K^0(X)$, where $\mathrm{ind} : K_c^0(T^*X) \to \mathbb{Z}$ is the index mapping of Atiyah-Singer. On the level of de Rham cohomology this is exactly the content of Lemma C.3.1.

Theorem C.3.2. *If $f : X \to Y$ is a smooth mapping between even-dimensional manifolds, p_Y is as in* (C.10) *and A is an elliptic pseudo-differential operator on X we have the following degree formula:*

$$\mathrm{ch}_0[A]\deg(f) = \mathrm{ind}\,(A_{p_Y \circ f}) - \mathrm{ind}\,(A),$$

where $\mathrm{ch}_0[A]$ *denotes the constant term in* $\pi_*\mathrm{ch}[A]$*. If* A *is of order* 0*, the same statement holds for continuous* f*.*

Proof. By Lemma C.3.1, Theorem C.2.2 and the Atiyah-Singer index theorem we have that

$$\begin{aligned}\mathrm{ind}\,(A_{p_Y\circ f}) = \int_X \mathrm{ch}[f^*p_Y]\wedge \pi_*\mathrm{ch}[A]\wedge Td(X) = \\ = \int_X (1+f^*\mathrm{d}V_Y)\wedge \pi_*\mathrm{ch}[A]\wedge Td(X) = \\ = \mathrm{ch}_0[A]\deg(f) + \int_X \pi_*\mathrm{ch}[A]\wedge Td(X) = \\ = \mathrm{ch}_0[A]\deg(f)+\mathrm{ind}\,(A).\end{aligned}$$

Since both $\mathrm{ind}\,(A_{p_Y\circ f})$ and $\deg(f)$ are well defined homotopy invariants for continuous f when A is of order 0 the final statement of the Theorem follows. □

To deal with analytic formulas for the mapping degree when f is not smooth will require some more concrete information about Schatten class properties of pseudo-differential operators.

Lemma C.3.3. *A pseudo-differential operator* b *of order* -1 *satisfies* $b\in\mathscr{L}^q(L^2(X))$ *for any* $q>\dim(X)$*.*

Lemma C.3.3 is proved by using a rather standard technique for pseudo-differential operators. In Theorem C.3.5, when we prove a similar result for Hölder continuous functions we will need some heavier machinery. We include a sketch of the proof of Lemma C.3.3 just to highlight the difference in methods. Letting Δ_X denote the second order Laplace-Beltrami operator, the operator $(1-\Delta_X)^{1/2}b$ is of order 0 whenever b is of order -1. Thus $b=(1-\Delta_X)^{-1/2}(1-\Delta_X)^{1/2}b$ and since $(1-\Delta_X)^{1/2}b$ is a bounded operator the Lemma follows if $(1-\Delta_X)^{-1/2}$ is in the Schatten class for any $q>\dim(X)$. This fact follows from the fact that the k:th eigenvalue of the Laplacian behaves like $-k^{2/\dim(X)}$ as $k\to\infty$, a statement that goes back to [90].

Lemma C.3.4. *The pseudo-differential operator* b *of order* 0 *has an integral kernel* $T\in C^\infty(X\times X\setminus D)$ *satisfying the estimate* $|T(x,y)|\lesssim|x-y|^{-\dim(X)-\varepsilon}$ *almost everywhere for any* $\varepsilon>0$*, here* D *denotes the diagonal in* $X\times X$*.*

Here we use the notation $a\lesssim b$ if there is a constant $C>0$ such that $a\leq Cb$. Observe that the estimate on T only holds for $x\neq y$ so the integral operator defined by T must be realized as a principal value. We will not prove Lemma C.3.4, but refer to Theorem 2.53 of [40].

Theorem C.3.5. *If F is a pseudo-differential operator of order 0 on a compact manifold X without boundary and $a \in C^\alpha(X)$, then the operator $[F, \pi(a)]$ is Schatten class of order q for any $q > \max(\dim(X)/\alpha, 2)$.*

We will prove Theorem C.3.5 by two lemmas describing Schatten class properties of the commutator $[F, \pi(a)]$ in terms of its local nature as in Theorem C.1.2. The first of the two lemmas is a direct consequence of Theorem C.1.2:

Lemma C.3.6. *Let $n/2 < \beta < n$ and suppose that T is an integral operator on $\mathbb{R}^n$ whose integral kernel is bounded by $|x-y|^{-\beta}$, for any $\chi, \chi' \in C_c^\infty(\mathbb{R}^n)$ and $q > \max(n/(n-\beta), 2)$ the operator $\chi T \chi'$ is q-summable.*

Proof. First we observe that the condition $\beta < n$ implies that the kernel is locally integrable and the integral operator defined by T can be defined without taking any principal values, so we may apply Russo's theorem directly to $\chi T \chi'$. Let us use the notation $K_\beta(x,y) := |x-y|^{-\beta}$. If T has an integral kernel bounded by K_β Theorem C.1.2 implies that for some large $R > 0$ and for $q \geq 2$

$$\begin{aligned} \|\chi T \chi'\|_{\mathscr{L}^q(L^2(X))} &\leq \left(\|\chi K_\beta \chi'\|_{q',q} \|\chi' K_\beta \chi\|_{q',q}\right)^{1/2} \lesssim \\ &\lesssim \left(\int_{B(0,R)} \left(\int_{B(0,R)} \frac{\mathrm{d}V(x)}{|x-y|^{\beta q'}}\right)^{q/q'} \mathrm{d}V(y)\right)^{1/q} < \infty \end{aligned}$$

if $\beta q' < n$ which is equivalent to $q > n/(n-\beta)$. □

Lemma C.3.7. *Suppose that F is a properly supported pseudo-differential operator of order 0 in $\mathbb{R}^n$ and $a \in C^\alpha(\mathbb{R}^n)$ has compact support, then the operator $[F, \pi(a)]$ is Schatten class of order q for any $q > \max(n/\alpha, 2)$.*

Proof. Since F is a pseudo-differential operator of order 0, the operator F can by Lemma C.3.4 be represented by an integral kernel which is pointwise bounded by $|x-y|^{-n-\varepsilon}$ for some $\alpha > \varepsilon > 0$. Thus the integral kernel of $[F, \pi(a)]$ is bounded by $|a(x) - a(y)||x-y|^{-n-\varepsilon}$. It follows from the Hölder continuity of a that the integral kernel of $[F, \pi(a)]$ is pointwise bounded by the kernel $|x-y|^{-(n+\varepsilon-\alpha)}$. While F is properly supported and a has compact support, we can take $\chi, \chi' \in C_c^\infty(\mathbb{R}^n)$ such that $\chi[F, \pi(a)]\chi' = [F, \pi(a)]$ and Lemma C.3.6 implies that the operator $\chi[F, \pi(a)]\chi'$ is Schatten class of order q for any $q > \max(n/(\alpha - \varepsilon), 2)$. Since ε is arbitrary the Lemma follows. □

Theorem C.3.5 follows from Lemma C.3.7 since Theorem C.3.5 can be reduced to a local claim where Lemma C.3.7 applies.

C.4 Index of a Hölder continuous twist

In this section we will combine Theorem C.1.1 with Theorem C.3.5 into index formulas for certain elliptic pseudo-differential operators twisted by Hölder continuous vector bundles. If $p : X \to \mathscr{K}(\mathbb{C}^N)$ is a continuous projection-valued function, we will use the notation E_p for the vector bundle over X corresponding to p via Serre-Swan's theorem.

When A is an elliptic differential operator from the vector bundle E to the vector bundle E', we want to consider the, possibly unbounded, operator $A_p := p(A \otimes \mathrm{id}_{\mathbb{C}^N})p$ which is called the twist of A by p and acts between the vector bundle $E \otimes E_p$ and $E' \otimes E_p$. However, unless A is of order 0, we must assume that p is smooth to ensure that A_p is a densely defined Fredholm operator. In the case p is smooth, Lemma C.3.1 and the Atiyah-Singer index theorem implies that

$$\mathrm{ind}\,(A_p) = \int_{T^*X} \pi^*\mathrm{ch}[p] \wedge \mathrm{ch}[A] \wedge \pi^* Td(X),$$

and we also have the identity

$$\mathrm{ind}\,(A_p) = \mathrm{ind}\,(p(A(1+A^*A)^{-1/2} \otimes \mathrm{id}_{\mathbb{C}^N})p), \tag{C.12}$$

because $1+A^*A$ is strictly positive. The right-hand side of (C.12) is well defined for continuous p and, as we will see, it can be calculated for Hölder continuous projections by means of Theorem C.1.1 using a certain Fredholm module we associate with A. To construct this Fredholm module, we start by defining the odd, self-adjoint operator

$$\tilde{A} := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

on $L^2(X, E \oplus E')$ which is graded by letting $L^2(X,E)$ be the even part and let $L^2(X,E)$ be the odd part. We define the mapping

$$\varphi : \mathbb{R} \to \mathbb{R}, \quad u \mapsto u(1+u^2)^{-1/2} \quad \text{and the operator} \quad F_A := \varphi(\tilde{A}).$$

The operator F_A is an odd, self-adjoint operator of order 0. However, the square of the operator F_A can be calculated as

$$F_A^2 = \tilde{A}^2(1+\tilde{A}^2)^{-1} = 1 - (1+\tilde{A}^2)^{-1} \neq 1.$$

To mend the problem $F_A^2 \neq 1$, we replace φ by the function $\tilde{\varphi} : \mathbb{R} \to M_4(\mathbb{C})$ defined as

$$\tilde{\varphi}(u) := \begin{pmatrix} \varphi(u) & 0 & 0 & i(1+u^2)^{-1/2} \\ 0 & \varphi(u) & -i(1+u^2)^{-1/2} & 0 \\ 0 & i(1+u^2)^{-1/2} & -\varphi(u) & 0 \\ -i(1+u^2)^{-1/2} & 0 & 0 & -\varphi(u) \end{pmatrix}.$$

The function $\tilde{\varphi}$ satisfies $\tilde{\varphi}(u)^2 = 1$. If we equip $\mathbb{C}^4$ with the grading from the involution

$$\gamma_{\mathbb{C}^4} := 1 \oplus (-1) \oplus 1 \oplus (-1),$$

the operator $\tilde{F}_A := \tilde{\varphi}(\tilde{A})$ is an odd, self adjoint operator on the graded tensor-product $\mathbb{C}^4 \otimes L^2(X, E \oplus E')$. To simplify notations we set $\mathscr{E} := \mathbb{C}^4 \otimes (E \oplus E')$. The operator $\tilde{F}_A$ does satisfy that $\tilde{F}_A^2 = 1$. The operator $\tilde{F}_A$ can be written as a matrix of operators as

$$\tilde{F}_A := \begin{pmatrix} F_A & 0 & 0 & i(1+\tilde{A}^2)^{-1/2} \\ 0 & F_A & -i(1+\tilde{A}^2)^{-1/2} & 0 \\ 0 & i(1+\tilde{A}^2)^{-1/2} & -F_A & 0 \\ -i(1+\tilde{A}^2)^{-1/2} & 0 & 0 & -F_A \end{pmatrix}. \tag{C.13}$$

Since pseudo-differential operators are only pseudo-local, we will use a parameter t to extract the singular part of $\tilde{F}_A$. Let $A(t)$ denote the elliptic differential operator defined from A dilated by the action of $t > 0$ on T^*X. Set $\tilde{F}(t) := \tilde{F}_{A(t)}$. Define W_0 as the smooth pseudo-differential operator given by the orthogonal finite-rank projection onto $\ker(\tilde{A})$ and set $W_0^\perp := 1 - W_0$. Observe that, since $\tilde{A}$ is elliptic, we have that $W_0 \in C^\infty(X, E \oplus E') \otimes_{alg} C^\infty(X, (E \oplus E')^*)$. We define

$$W := \begin{pmatrix} 0 & 0 & 0 & iW_0 \\ 0 & 0 & -iW_0 & 0 \\ 0 & iW_0 & 0 & 0 \\ -iW_0 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma C.4.1. *If A is an elliptic differential operator of order at least n/p, the operator valued function $t \mapsto \tilde{F}(t)$ satisfies*

$$\|\tilde{F}(t) - \tilde{F} - W\|_{\mathscr{L}^p(L^2(X))} = \mathscr{O}(t^{-1}) \quad \text{as} \quad t \to \infty,$$

where $\tilde{F}$ is the 0-homogeneous part of $\tilde{F}_A$.

Proof. The operator $\tilde{F}(t) - \tilde{F} - W$ is a matrix consisting of terms of the form

$$\begin{aligned} a_1 &= i(1+t^2\tilde{A}^2)^{-1/2} - iW_0 = iW_0^\perp(1+t^2\tilde{A}^2)^{-1/2}W_0^\perp = \\ &= it^{-1}W_0^\perp\tilde{A}^{-1}W_0^\perp T_1 \qquad \text{and} \\ a_2 &= \tilde{A}(t(1+t^2\tilde{A}^2)^{-1/2} - W_0^\perp|\tilde{A}|^{-1}W_0^\perp) = \\ &= \tilde{A}(t(1+t^2\tilde{A}^2)^{-1/2} - |\tilde{A}|^{-1}) = t^{-1}W_0^\perp|\tilde{A}|^{-1}W_0^\perp T_2, \end{aligned}$$

for some $T_1, T_2 \in \mathscr{B}(L^2(X))$. Since both a_1 and a_2 are pseudo-differential operators of order lower than $-n/p$, it follows that their $\mathscr{L}^p$-norm behaves like t^{-1} as $t \to \infty$ and the Lemma follows. ☐

We let $\pi : C^\alpha(X) \to \mathscr{B}(L^2(X, E \oplus E'))$ denote the even representation given by pointwise multiplication. Define $\tilde{\pi} : C^\alpha(X) \to \mathscr{B}(L^2(X, \mathscr{E}))$ by letting $\tilde{\pi} := \pi \oplus 0$ under the isomorphism $L^2(X, \mathscr{E}) \cong \mathbb{C}^4 \otimes L^2(X, E \oplus E')$.

Theorem C.4.2. *If A is an elliptic differential operator of strictly positive order between two vector bundles E and E' over X and $q > \max(\dim(X)/\alpha, 2)$, then the pair $(\tilde{\pi}, \tilde{F}_A)$ is a q-summable even Fredholm module over $C^\alpha(X)$.*

Proof. The operator $\tilde{F}_A$ is an elliptic, self-adjoint pseudo-differential operator of order 0 and $\tilde{F}_A^2 = 1$. Under the isomorphism $L^2(X, \mathscr{E}) \cong \mathbb{C}^4 \otimes L^2(X, E \oplus E')$ the decomposition (C.13) implies that

$$[\tilde{F}_A, \tilde{\pi}(a)] = \begin{pmatrix} [F_A, \pi(a)] & 0 & 0 & -i\pi(a)(1+\tilde{A}^2)^{-1/2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i(1+\tilde{A}^2)^{-1/2}\pi(a) & 0 & 0 & 0 \end{pmatrix}. \tag{C.14}$$

Since the order of A is strictly positive, these facts together with Theorem C.3.5 and Lemma C.3.3 imply that $(\tilde{\pi}, \tilde{F}_A)$ is a q-summable even Fredholm module for any $q > \max(\dim(X)/\alpha, 2)$. $\square$

Let us represent the pseudo-differential operator $\tilde{F}$ by the integral kernel $\tilde{K}_A$ which is a conormal distribution section of the big Hom-bundle $\mathrm{Hom}(\mathscr{E}, \mathscr{E})$. By (C.13), we can write $\tilde{K}_A = K_A \oplus K_A \oplus (-K_A) \oplus (-K_A)$ where K_A is a conormal distribution section of the big Hom-bundle $\mathrm{Hom}(E \oplus E', E \oplus E')$. Observe that K_A is defined by a smooth section $C^\infty(X \times X \setminus D, \mathrm{Hom}(E \oplus E', E \oplus E'))$.

We will use the notation Γ_k for the subset of $\{1, 2, 3, 4\}^{2k}$ consisting of all sequences $(s_l)_{l=1}^{2k}$ satisfying the conditions that $s_1 \neq 4$, $s_{2k} \neq 3$ and $s_l = 3$ for some l if and only if $s_{l+1} = 4$. These conditions are motivated by the form of the commutator (C.14). Let $w(I)$ denote the numeral of occurrences of 3 in I. Take $\Gamma_k^w \subseteq \Gamma_k$ as the subset of sequences with $w(I) = w$. To a sequence $I \in \Gamma_k$ we will associate the sequence $\Lambda(I) = (i_l)_{l=1}^{2k} \in \{1, 2, \dots 2k\}^{2k}$ defined by

$$i_l = \begin{cases} l, & \text{if} \quad s_l = 1,3 \\ l+1, & \text{if} \quad s_l = 2,4 \end{cases},$$

where we identify $2k+1$ with 1. For a sequence $I \in \Gamma_k$ we let $\iota(I) := (-1)^{\sum_{l=1}^{2k} i_l - l} \cdot i^{w(I)}$. With a projection-valued function $p : X \to \mathscr{K}$ and a sequence $I \in \Gamma_k$ we associate the function

$$Q_I^p(x_1, \dots, x_{2k}) := \mathrm{tr}_{\mathbb{C}^{2^n}} \left(p(x_1) \prod_{l=1}^{2k} p(x_{i_l}) \right). \tag{C.15}$$

We also define

$$H_A(x_1,\dots,x_{2k}) := \mathrm{str}_{E\oplus E'}\left(\prod_{l=1}^{2k} K_A(x_i,x_{i+1})\right).$$

A straight-forward calculation implies the following lemma:

Lemma C.4.3. *For any projection-valued function $p : X \to \mathcal{K}(\mathbb{C}^N)$, the following identity holds:*

$$\mathrm{str}_{(E\oplus E')\otimes\mathbb{C}^N}\left(p(x_1)\prod_{j=1}^{2k}(p(x_{j+1})-p(x_j))K_A(x_j,x_{j+1})\right) =$$
$$= \sum_{(s_l)_{l=1}^{2k}\in\Gamma_k^0} \iota(I)Q_I^p(x_1,\dots,x_{2k})H_A(x_1,\dots,x_{2k}).$$

This Lemma describes the diagonal terms in the decomposition (C.14). To describe the products with the off-diagonal terms, we need some notation for the integral kernels. Set $K_1 = K_2 = K_A$ and $K_3 = K_4 = W_0$. For $I \in \Gamma_k$ we define the integral kernel

$$H_{A,I}(x_1,\dots,x_{2k}) := \mathrm{str}_{E\oplus E'}\left(\prod_{l=1}^{2k} K_{s_l}(x_i,x_{i+1})\right).$$

Lemma C.4.4. *The function*

$$(x_1,\dots,x_{2k}) \mapsto \sum_{(s_l)_{l=1}^{2k}\in\Gamma_k^w} \iota(I)Q_I^p(x_1,\dots,x_{2k})H_{A,I}(x_1,\dots,x_{2k}) \tag{C.16}$$

is absolutely integrable over X^{2k} for all $w = 0,1,\dots,k$.

This Lemma is a direct consequence of Theorem C.3.5 and Theorem C.1.3 since the sum over Γ_k^w corresponds to the sum of the supertraces of the products between p and $2(k-w)$ commutators between p and K_A and $2w$ commutators between p and W_0. In fact, when $w > 0$ the integral of (C.16) will be the trace of a finite rank operator. One can decompose the function (C.16) even further by decomposing Γ_k^w into equivalence classes under the equivalence relation $1 \sim 2$ and $3 \sim 4$, which again will be a decomposition into absolutely integrable functions.

Theorem C.4.5. *Suppose that A is an elliptic differential operator of strictly positive order on the compact manifold X without boundary, acting from E to E'. If $p : X \to \mathcal{K}(\mathbb{C}^N)$ is a projection valued, Hölder continuous function of* *exponent α and $2k-1 > \max(\dim(X)/\alpha, 2)$, the following index formula holds:*

$$\begin{aligned}
&\operatorname{ind}(p(F_{A,+}\otimes 1)p) = \langle \mathrm{cc}_k(\tilde{\pi}, \tilde{F}_A), p\rangle_{2k} = \\
&\quad = (-1)^k \int_{X^{2k}} \mathrm{str}\left(p(x_1)\prod_{j=1}^{2k} (p(x_{j+1}) - p(x_j))K_A(x_j, x_{j+1})\right) \mathrm{d}V_{X^{2k}} + \\
&\quad + (-1)^k \sum_{w=1}^{k} \int_{X^{2k}} \sum_{(s_l)_{l=1}^{2k}\in\Gamma_k^w} \iota(I) Q_I^p(x_1,\ldots,x_{2k}) H_{A,I}(x_1,\ldots,x_{2k}) \mathrm{d}V_{X^{2k}}
\end{aligned}$$

where the trace is taken over $(E\oplus E')\otimes\mathbb{C}^N$ and we identify $x_1 = x_{2k+1}$. All of the integrals are absolutely convergent.

Observe that if the dimension of X is odd, $\operatorname{ind}(p(F_{A,+}\otimes 1)p) = 0$ for continuous p since the Atiyah-Singer index theorem implies that it holds for smooth projection-valued functions.

Proof. The first equality follows from Theorem C.1.1 and Theorem C.4.2 since

$$\tilde{\pi}(p)(\tilde{F}_{A,+}\otimes 1)\tilde{\pi}(p) = \Big(\pi(p)(F_{A,+}\otimes 1)\pi(p)\Big) \oplus 0.$$

Recall that, for any $t > 0$, the operator $A(t)$ satisfies the same conditions as A so by homotopy invariance of the Chern-Connes character and Lemma C.4.1,

$$\begin{aligned}
&\langle \mathrm{cc}_k(\tilde{\pi}, \tilde{F}_A), p\rangle_{2k} = \langle \mathrm{cc}_k(\tilde{\pi}, \tilde{F}_{A(t)}), p\rangle_{2k} = \\
&\quad = (-1)^k \mathrm{str}_{L^2(X,\mathscr{E}\otimes\mathbb{C}^N)}\Big(\tilde{\pi}(p)[\tilde{F}_{A(t)}, \tilde{\pi}(p)]\cdots[\tilde{F}_{A(t)}, \tilde{\pi}(p)]\Big) = \\
&\quad = (-1)^k \mathrm{str}_{L^2(X,\mathscr{E}\otimes\mathbb{C}^N)}\Big(\tilde{\pi}(p)[\tilde{F}+W, \tilde{\pi}(p)]\cdots[\tilde{F}+W, \tilde{\pi}(p)]\Big) + \mathscr{O}(t^{-1}).
\end{aligned}$$

Since p is Hölder continuous, $(p(x_{j+1}) - p(x_j))K_A(x_j, x_{j+1})$ is locally integrable and of finite mixed $L^{(q',q)}$-norm by Theorem C.3.5. Let us set $\tilde{p} := \tilde{\pi}(p)$ and $\tilde{K}_A^W := \tilde{K}_A + W$. Theorem C.1.2 and the calculations above imply the equality

$$\begin{aligned}
&\mathrm{str}_{L^2(X,\mathscr{E}\otimes\mathbb{C}^N)}\Big(\tilde{\pi}(p)[\tilde{F}+W, \tilde{\pi}(p)]\cdots[\tilde{F}+W, \tilde{\pi}(p)]\Big) = \\
&\quad = \int_{X^{2k}} \mathrm{str}\left(\tilde{p}(x_1)\prod_{j=1}^{2k}\Big(\tilde{p}(x_{j+1})\tilde{K}_A^W(x_j, x_{j+1}) - \tilde{K}_A^W(x_j, x_{j+1})\tilde{p}(x_j)\Big)\right) \mathrm{d}V_{X^{2k}} \\
&\quad = \int_{X^{2k}} \sum_{(s_l)_{l=1}^{2k}\in\Gamma_k} \iota(I) Q_I^p(x_1,\ldots,x_{2k}) H_{A,I}(x_1,\ldots,x_{2k}) \mathrm{d}V_{X^{2k}}.
\end{aligned}$$

While the term $\langle \mathrm{cc}_k(\tilde{\pi}, \tilde{F}_A), p\rangle_k$ is constant, the second identity stated in the theorem follows from Lemma C.4.3 by letting $t\to\infty$. $\square$

C.5 Degrees of Hölder continuous mappings

Returning now to the degree calculations, we will use Theorem C.4.5 for a particular choice of pseudo-differential operator. Assume that X is a compact Riemannian manifold of dimension $2n$ without boundary. The Hodge grading on $\bigwedge^* T^*X \otimes \mathbb{C}$ is defined by the involution τ defined on a p-form ω by

$$\tau\omega = i^{p(p-1)+n} * \omega,$$

where $*$ denotes the Hodge duality. Observe that if $(e_j)_{j=1}^{2n}$ is an oriented, orthonormal basis of the cotangent space, the operator τ can be written as

$$\tau = \left(\frac{i}{2}\right)^n \prod_{j=1}^{2n}(e_j \wedge - e_j \neg). \tag{C.17}$$

We let E_+ denote the sub-bundle of $\bigwedge^* T^*X \otimes \mathbb{C}$ consisting of even vectors with respect to the Hodge grading and E_- the sub-bundle of odd vectors with respect to the Hodge grading. The operator τ anti-commutes with $\mathrm{d}+\mathrm{d}^*$ so $A = \mathrm{d}+\mathrm{d}^*$ is a well defined operator from E_+ to E_-. The operator A is called the signature operator. Observe that $\tilde{A} = \mathrm{d}+\mathrm{d}^*$ as an operator on $\bigwedge^* T^*X \otimes \mathbb{C}$ and $\tilde{A}^2$ is the Laplace-Beltrami operator on X. By Theorem C.4.2 the pair $(\tilde{\pi}, \tilde{F}_{\mathrm{d}+\mathrm{d}^*})$ is a q-summable Fredholm module over $C^\alpha(X)$.

Assume that $f : X \to Y$ is a Hölder continuous function where Y is a $2n$-dimensional manifold. We can choose an open subset $U \subseteq Y$ such that there is a diffeomorphism $\nu : U \to B_{2n}$. In fact, by the closed mapping lemma, since X is compact and Y Hausdorff, the mapping f is open. Therefore, it is possible to choose U such that there is an open set $U_0 \subseteq X$ satisfying $U \subseteq f(U_0)$ and E_+ and E_- are trivial over U_0.

Taking $\tilde{\nu} : Y \to \mathbb{C}^n$ as the Lipschitz continuous function defined in (C.11) we can for $k > n/\alpha$ define the integrable function $\tilde{f}_k \in L^1(X^{2k})$ as:

$$\tilde{f}_k(x_1, \ldots, x_{2k}) := \tag{C.18}$$

$$= \sum_{I \in \Gamma_k} \iota(I) Q_I^{p_T}(\tilde{\nu} f(x_1), \ldots, \tilde{\nu} f(x_{2k})) H_{\mathrm{d}+\mathrm{d}^*, I}(x_1, \ldots, x_{2k}) = \tag{C.19}$$

$$= \sum_{w=0}^{k} \sum_{I \in \Gamma_k^w} \iota(I) H_{\mathrm{d}+\mathrm{d}^*, I}(x_1, \ldots, x_{2k}) \prod_{j=1}^{n} \prod_{l=0}^{2k} \frac{1 + \overline{f(x_{i_l})}_j f(x_{i_{l+1}})_j}{1 + |f(x_{i_l})_j|^2}, \tag{C.20}$$

where the second formula follows from Lemma C.2.3. The kernel $H_{\mathrm{d}+\mathrm{d}^*, I}$ is in general quite hard to find. In local coordinates, the operator $K_{\mathrm{d}+\mathrm{d}^*}$ will be similar to a Riesz transform. The operator W_0 is the projection onto the finite-dimensional space of harmonic forms on X. We will demonstrate this by calculating the kernels $H_{\mathrm{d}+\mathrm{d}^*, I}$ explicitly on S^{2n} in the next section.

Theorem C.5.1. *Suppose that* X *and* Y *are smooth compact connected manifolds without boundary of dimension* $2n$ *and* $f : X \to Y$ *is Hölder continuous of exponent* α*. When* $k > n/\alpha$ *the following integral formula holds:*

$$\deg(f) = -2^{-n}\mathrm{sign}(X) + 2^{-n}(-1)^k \int_{X^{2k}} \tilde{f}_k(x_1, \dots, x_{2k}) \mathrm{d}V_{X^{2k}}$$

where $\tilde{f}_k$ *is as in* (C.18)–(C.20)*.*

Proof. While $F_{\mathrm{d}+\mathrm{d}^*} = (\mathrm{d}+\mathrm{d}^*)(1+\Delta)^{-1/2}$, we have that

$$\mathrm{ind}\,(F_{\mathrm{d}+\mathrm{d}^*,+}) = \mathrm{ind}\,(\mathrm{d}+\mathrm{d}^*) = \mathrm{sign}(X).$$

The operator $\mathrm{d}+\mathrm{d}^*$ satisfies $\mathrm{ch}_0[\mathrm{d}+\mathrm{d}^*] = 2^n L_0(T^*X) = 2^n$, since the constant term in the L-genus is 1. Therefore Theorem C.3.2 and Theorem C.4.5 implies that

$$\begin{aligned}\deg(f) &= 2^{-n}\mathrm{ind}\,((p_Y \circ f \otimes \mathrm{id})F_{\mathrm{d}+\mathrm{d}^*,+}(p_Y \circ f \otimes \mathrm{id})) - 2^{-n}\mathrm{ind}\,(F_{\mathrm{d}+\mathrm{d}^*,+}) = \\ &= 2^{-n}\langle \mathrm{cc}_k(\tilde{\pi}, \tilde{F}_A), f^* p_Y\rangle_{2k} - 2^{-n}\mathrm{sign}(X) = \\ &= 2^{-n}(-1)^k \int_{X^{2k}} \tilde{f}_k(x_1, \dots, x_{2k})\mathrm{d}V_{X^{2k}} - 2^{-n}\mathrm{sign}(X).\end{aligned}$$

□

A couple of remarks on the choice of A as the signature operator are in order. This choice is rather superfluous since any pseudo-differential operator A of order 1 with $\mathrm{ch}_0[A] \neq 0$ will give a degree formula similar to that in Theorem C.5.1. If one can find an invertible A on X such that $\mathrm{ch}_0[A] \neq 0$, the formula of Theorem C.5.1 would be much simpler since $\tilde{f}_k$ will not contain any contributions from $\Gamma_k \setminus \Gamma_k^0$.

The signature operator has been studied on Lipschitz manifolds, see [86], which gives an analytic degree formula for Hölder continuous mappings between even-dimensional Lipschitz manifolds. On Lipschitz manifolds, the Atiyah-Singer theorem is replaced by Teleman's index theorem from [87]. Of course, there are some analytic difficulties in the proof of Theorem C.3.5 for Lipschitz manifolds, that more or less manifests themselves on a notational level.

C.6 Example on S^{2n}

Let us end this paper by writing down the integral kernels in the case of a Hölder continuous function $f : S^{2n} \to Y$ where Y is a compact, connected $2n$-dimensional manifold without boundary. We will compare the method of using pseudo-differential operators from Theorem C.5.1 with that of using Henkin-Ramirez kernels from Paper B. The degree formula of Paper B is based on the usage of the cocycle (C.5).

To apply Theorem C.5.1 the kernels $H_{\mathrm{d}+\mathrm{d}^*,I}$ need to be calculated. Define $U_0 : L^2(S^{2n}, \bigwedge T^*S^{2n}) \to L^2(\mathbb{R}^{2n}, \bigwedge T^*\mathbb{R}^{2n})$ by pulling back along the mapping $\lambda : \mathbb{R}^{2n} \to S^{2n}$ defined by

$$\mathbb{R}^{2n} \ni x \mapsto \left(\frac{|x|^2 - 1}{|x|^2 + 1}, \frac{2x}{|x|^2 + 1} \right) \in S^{2n} \subseteq \mathbb{R}^{2n+1}$$

and equipping $\mathbb{R}^{2n}$ with the pull-back metric. Since the metric is positive definite we can define the unitary mapping $U : L^2(S^{2n}, \bigwedge T^*S^{2n}) \to L^2(\mathbb{R}^{2n}) \otimes \mathbb{C}^{2^{2n}}$ by composing U_0 with the unitary mapping $L^2(\mathbb{R}^{2n}, \bigwedge T^*\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n}) \otimes \mathbb{C}^{2^{2n}}$ defined by the metric. Clearly, we have that $UK_{\mathrm{d}+\mathrm{d}^*}U^*$ is a pseudo-differential operator on $\mathbb{R}^{2n}$ with symbol $\xi \mapsto i(\xi \wedge - \xi \neg)/\sqrt{2}|\xi|$, if we identify $\mathbb{C}^{2^{2n}}$ with $\bigwedge \mathbb{C}^{2n}$. For any $a \in C^\alpha(S^{2n})$ we have that $UaU^* = a \circ \lambda$.

Let us find the integral kernel K_1 of $UK_{\mathrm{d}+\mathrm{d}^*}U^*$. Since the symbol of K_1 is a homogeneous function that commutes with the $SU(2n)$-action on $\bigwedge \mathbb{C}^{2n}$, the integral kernel K_1 is given by

$$K_1(x,y) = c_n \frac{(x-y) \wedge -(x-y)\neg}{\sqrt{2}|x-y|^{2n+1}}$$

and the constant $c_n = (n-1)!/\pi^n$ is calculated in Chapter III in [83]. So the operator defined by K_1 is a matrix of Riesz transforms. While the harmonic forms on S^{2n} are spanned by the constant function and the volume form, the kernel W_0 is the constant projection $\bigwedge T^*S^{2n} \to \mathbb{C} \oplus \bigwedge^{2n} T^*S^{2n}$. So we can write the kernel K_3 of UW_0U^* as

$$K_3(x,y) = g(x)g(y)(1 + \prod_{j=1}^{2n} e_j \wedge e_j \neg),$$

where $g(x) := c'_n(1+|x|^2)^{-n}$ and $c'_n = 2n\pi^n n!(n-2)!/(2n+1)!$. Using these expressions for K_1 and K_3 it follows from (C.17) that

$$H_{\mathrm{d}+\mathrm{d}^*,I}(x_1, \dots, x_{2k}) = \mathrm{str}\left(\prod_{l=1}^{2k} K_{s_l}(x_l, x_{l+1}) \right) = \mathrm{tr}\left(\tau \prod_{l=1}^{2k} K_{s_l}(x_l, x_{l+1}) \right) =$$

$$= \sum_{\sigma \in S_{2(k-w)}} \sum_{m=0}^{n} i^n \mathrm{sign}(\sigma) H_{\sigma,m,I}(x_1, \dots, x_{2k}),$$

when $w(I)=w$ and where

$$H_{\sigma,m,I}(x_1,\ldots,x_{2k}) := c_n^{2k-m}\prod_{s_l\neq 1,2} g(x_l)g(x_{l+1})\prod_{l=1,s_l=1,2}^{m}\frac{\langle e_j, x_{\sigma(l)}-x_{\sigma(l)+1}\rangle}{|x_{\sigma(l)}-x_{\sigma(l)+1}|^{2n+1}}\cdot$$
$$\cdot\prod_{l=m+1,s_l=1,2}^{k}\frac{\langle x_{\sigma(l)}-x_{\sigma(l)+1}, x_{\sigma(l+1)}-x_{\sigma(l+1)+1}\rangle}{|x_{\sigma(l)}-x_{\sigma(l)+1}|^{2n+1}|x_{\sigma(l+1)}-x_{\sigma(l+1)+1}|^{2n+1}}.$$

By this notation we mean that an element σ in the symmetric group $S_{2(k-w)}$, on $2(k-w)$ elements, acts on the indices l such that $s_l = 1,2$. Here we use the notation $\langle\cdot,\cdot\rangle$ for the scalar product. For a Hölder continuous function $f : S^{2n} \to Y$ we take an open set $U \subseteq Y$ such that there is a diffeomorphism $\nu : U \to B_{2n}$ and consider the Hölder continuous function $f_0 := \tilde{\nu} f \lambda : \mathbb{R}^{2n} \to S^{2n}$, where $\tilde{\nu}$ is as in equation (C.11). The signature of a sphere is 0 so Theorem C.5.1 and (C.20) implies the following degree formula:

$$\deg(f) =$$

$$= (-1)^k \sum_{w=0}^{k}\int_{\mathbb{R}^{4nk}}\sum_{I\in\Gamma_k^w}\sum_{\sigma\in S_{2(k-w)}}\sum_{m=0}^{n} i^n \operatorname{sign}(\sigma) c_n^{2k-m}\iota(I)\cdot$$
$$\cdot\quad\prod_{j=1}^{n}\prod_{l=1}^{2k}\frac{1+\overline{f_0(x_j)}_{i_l} f_0(x_j)_{i_{l+1}}}{1+|f_0(x_j)_{i_l}|^2}\quad\cdot$$
$$\cdot\quad\prod_{s_l\neq 1,2} g(x_l)g(x_{l+1})\prod_{l=1,s_l=1,2}^{m}\frac{\langle e_j, x_{\sigma(l)}-x_{\sigma(l)+1}\rangle}{|x_{\sigma(l)}-x_{\sigma(l)+1}|^{2n+1}}\quad\cdot$$
$$\cdot\prod_{l=m+1,s_l=1,2}^{k}\frac{\langle x_{\sigma(l)}-x_{\sigma(l)+1}, x_{\sigma(l+1)}-x_{\sigma(l+1)+1}\rangle}{|x_{\sigma(l)}-x_{\sigma(l)+1}|^{2n+1}|x_{\sigma(l+1)}-x_{\sigma(l+1)+1}|^{2n+1}}\mathrm{d}V_{\mathbb{R}^{2nk}},$$

where $f_0(x)_i$ denotes the i:th complex coordinate of $f_0(x)$ if $f_0(x)\in U$ and ∞ if $f_0(x)\notin U$.

If we attempt using the degree formula of Paper B to a function $f : S^{2n} \to Y$ we must in some way change the dimension. We will do so by finding a strictly pseudo-convex domain Ω in $\mathbb{C}^{n+1}$ such that $\partial\Omega = S^{2n}\times S^1$ and use the degree formula of Paper B to calculate the degree of $f\times \mathrm{id} : S^{2n}\times S^1 \to Y\times S^1$. To find such domain Ω we define the function $\rho \in C^\infty(\mathbb{C}^{n+1})$ as

$$\rho(z_1,\ldots,z_{n+1}) := 4|1-z_1 z_2|^2 + |z|^2 - 3.$$

We let $\Omega := \{z \in \mathbb{C}^{n+1} : \rho(z) < 0\}$. Clearly, ρ is strictly pluri-subharmonic in Ω and $\mathrm{d}\rho \neq 0$ on $\partial\Omega$. Furthermore

$$\rho(z) = 4(1 - 2\Re(z_1 z_2) + |z_1 z_2|^2) + |z|^2 - 3.$$

So Ω is a relatively compact strictly pseudo-convex domain in $\mathbb{C}^{n+1}$ with smooth boundary. Writing down the function ρ in its real argument verifies that $\Omega \cong B_{2n+1} \times S^1$ and $\partial\Omega \cong S^{2n} \times S^1$.

Let us find the Henkin-Ramirez kernel H_R of the boundary $\partial\Omega$. In order to do this we will use the Fornaess embedding theorem, see [41]. This approach to construct the Henkin-Ramirez kernel goes as follows; if we find a proper, holomorphic mapping from Ω into a convex domain, we obtain the holomorphic support function of $\partial\Omega$ by pulling back the support function in the convex domain. Define the holomorphic function $\Psi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ by $(z_1, z_2, \ldots, z_{n+1}) \mapsto (2(1 - z_1 z_2), z_1, z_2, \ldots, z_{n+1})$. Clearly Ω is mapped in to the ball of radius 3, $3B_{2n+4}$, and $\Omega = \Psi^{-1}(3B_{2n+4})$. Therefore Ω has the support function

$$a(z,\zeta) := (\Psi(\zeta) - \Psi(z)) \cdot \overline{\Psi(\zeta)} = \bar{\zeta} \cdot (\zeta - z) + 4(\zeta_1\zeta_2 - z_1 z_2)\overline{\zeta_1\zeta_2}. \tag{C.21}$$

Thus the integral kernel of the Henkin-Ramirez kernel for $S^{2n} \times S^1$ is given by

$$H_R(z,\zeta)\mathrm{d}V_{S^{2n}\times S^1} = \frac{1}{(2\pi i)^n} \frac{s \wedge (\bar{\partial}_\zeta s)^{2(n-1)}}{a(z,\zeta)^n},$$

where

$$s(z,\zeta) = \partial|\zeta|^2 + 2(1 - \overline{\zeta_1\zeta_2})(z_2 \mathrm{d}\zeta_1 + \zeta_1 \mathrm{d}\zeta_2).$$

Because of (C.21) the function $a(z,\zeta)$ is symmetric in the sense that for $z, \zeta \in \partial\Omega$ we have $a(z,\zeta) = \overline{a(\zeta,z)}$. However, the kernel H_R is not symmetric as is seen from the expression for s.

Concluding the case of a Hölder continuous function $f : S^{2n} \to Y$, we take an open subset $U \subseteq Y$ diffeomorphic to a ball and choose a diffemorphism $\nu : U \times (S^1 \setminus \{pt\}) \to B_{2n+1}$. If we let $f_1 : S^n \times S^1 \to S^{n+1}$ be the Hölder continuous function constructed by extending $\nu(f \times \mathrm{id})$ as in (C.11), Theorem B.4.4 implies the degree formula

$$\deg(f) = \frac{(-1)^n}{(2\pi i)^{2kn}} \int_{(S^{2n}\times S^1)^{2k+1}} \sum_{\sigma \in S_{2(2k-1)}} \sum_{l=0}^{2k-1} c_{l,\sigma} \prod_{i=1}^{l} \langle f_1(z_{\sigma(2i-1)}), f_1(z_{\sigma(2i)}) \rangle \cdot \\ \cdot \prod_{j=0}^{2k} \frac{s(z_j, z_{j+1}) \wedge (\bar{\partial}_{z_{j+1}} s(z_j, z_{j+1}))^{2(n-1)}}{a(z_j, z_{j+1})^n},$$

where $c_{l,\sigma} = (-1)^l 2^{n-l-1} \varepsilon_l(\sigma)$ and $\varepsilon_l(\sigma)$ is the order parity of σ.

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