Concept Formation in Mathematics

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Jörgen Sjögren

Abstract

This thesis consists of three overlapping parts, where the first one centers around the possibility of defining a measure of the power of arithmetical theories. In this part a partial measure of the power of arithmetical theories is constructed, where "power" is understood as capability to prove theorems. It is also shown that other suggestions in the literature for such a measure do not satisfy natural conditions on a measure. In the second part a theory of concept formation in mathematics is developed. This is inspired by Aristotle's conception of mathematical objects as abstractions, and it uses Carnap's method of *explication* as a means to formulate these abstractions in an ontologically neutral way. Finally, in the third part some problems of philosophy of mathematics are discussed. In the light of this idea of concept formation it is discussed how the relation between formal and informal proof can be understood, how mathematical theories are tested, how to characterize mathematics, and some questions about realism and indispensability.

The thesis is based on the following texts.

Measuring the Power of Arithmetical Theories. Thesis for the Licentiate degree, Department of Philosophy, University of Göteborg, (2004) Philosophical Communications, Red Series number 39, ISSN: 0347 - 5794. Also available at: http://www.phil.gu.se/posters/jslic.pdf

On Explicating the Concept *The Power of an Arithmetical Theory*. *Journal of Philosophical Logic*, (2008) 37: 183-202. DOI: 10.1007/s10992-007-9077-8

A Note on the Relation Between Formal and Informal Proof. *Acta Analytica*, (2010) 25: 447-458. DOI 10.1007/s12136-009-0084-y

Indispensability, The Testing of Mathematical Theories, and Provisional Realism. Re-submitted paper.

Mathematical Concepts as Unique Explications (with Christian Bennet). Submitted paper.

Preface

This project started as a logic project, and then gradually evolved into a project in philosophy of mathematics. While logic, at least in its modern technical form, is a fairly young discipline, although it all started with Aristotle, philosophy of mathematics traces its origin well back to Pythagoras. It is not without hesitation that I have entered into these disciplines; logic with modern giants as Hilbert, Gödel, etc. has become an advanced part of mathematics, and philosophy of mathematics with contributors as Plato, Kant, etc. has had a prominent share throughout all of western philosophy from ancient times till now. To think I would be able to contribute anything in these connections may seem both presumptuous and in vain. In case I have accomplished something, and not nothing, this is due among other things to people surrounding me, and above all to my supervisors Christian Bennet and Dag Westerståhl. The moral support and the intellectual guidance by Christian Bennet has been of utmost importance. He has always encouraged me to go on with my ideas, and been a source of inspiration in discussions. The experience and expertise of Dag Westerståhl has of course been unvaluable. Another source of inspiration has been the discussions at the seminars with the logic group at (the former) dept. of philosophy. The same goes for the people at the department of philosophy where I once began my studies in philosophy when it was located at Korsvägen. The persons I have met at the department ever since have always been extremely helpful.

Of course there are many friends and colleagues that in one way or another have helped me make this possible. To name but a few, my son Martin Sjögren introduced me to LaTeX, and my colleagues Stefan Karlsson and Yosief Wondmagegne often helped me when things went wrong. Kerstin Pettersson gave me moral support, and helped me get some economic support from University of Skövde that made it easier to finish the work. My old friend Leif Segelström helped me make my English less unreadable in my thesis for the licentiate degree. The late Torkel Franzén, my opponent at the licentiate seminar, came up with many suggestions that improved that thesis. I, furthermore, admire my wife Barbro who, among other things, has had patience with me when I have been working on the problems treated

in the thesis almost every evening during a decade or so. Last but not least my discussions about mysteries of the universe with Lennart Jönsson when we were young roving the woods surrounding Vaggeryd made an ineffacable impression on me. It was these walks and talks that aroused my intellectual curiosity!

Contents

1. Introduction

The main aim of this thesis is to contribute to understanding concept formation in mathematics. When the project started this was however not the goal. The original problem concerned Chaitin's incompleteness theorem, originally announced in the early 1970's, and a suggestion formulated by him later on of how to use this result to construct a measure of the power of an arithmetical theory. Michiel van Lambalgen and Panu Raatikainen had argued, convincingly in my opinion, that the suggestion was untenable.[1] A natural question is then, if there are any other ideas that can be used to construct such a measure. A partial solution to this problem is provided in *Measuring the Power of Arithmetical Theories*.[2] The introduction to that essay also contains a discussion of the applicability of logic, and in a wider sense, of mathematics, motivated by Chaitin's suggestion of an application of a theorem in logic. A preliminary discussion of the problem of applicability was presented in (Sjögren, 2006).[3] These ideas are developed and elaborated in this thesis.

The suggestion in the thesis is that concept formation in mathematics takes place via abstractions, and that the process of refining abstractions can be described as sequences of explications. While talk about abstract objects naturally involves ontological standpoints, formulating explications need not have any ontological implications. To regard mathematical objects as abstract ones is of course not new. For my purposes, Aristotle's view on mathematical objects is more useful than Plato's in relation to the problem of the applicability of mathematics. He is of the opinion that mathematical objects are abstract, but they do not exist as separated forms, like Plato's ideas. They are embodied in matter and can be separated in thought only. In this process of separation we can decide under which point of view we want to regard a substance. The process of separation can be analysed as a sequence of explications, a technique described and used by Carnap from 1945 onwards.[4]

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[1] See (Chaitin, 1971), (Chaitin, 1974), (van Lambalgen, 1989), (Raatikainen, 1998), and (Raatikainen, 2000).

[2] (Sjögren, 2004), thesis for the licentiate degree.

[3] This paper, in Swedish, is not included in the thesis.

[4] (Carnap, 1945), (Carnap, 1947), (Carnap, 1950), (Carnap, 1963a), and (Carnap, 1963b).

One consequence of this analysis of concept formation is that mathematical concepts are partly empirical; they have an empirical origin, and partly logical; they have a position in a deductive system. Since mathematical concepts have an empirical origin, the applicability of mathematics can be explained. But mathematics is not an empirical science. Mathematical propositions relate concepts to each other, and are parts of more or less well developed deductive systems; this is its logical, or analytic, component. Mathematical propositions are tested for consistency, fruitfulness, simplicity, elegance, etc., not against an empirical 'reality'. If propositions, containing new concepts, are considered to be consistent with a relevant part of mathematics, they can in principle be incorporated into this body. If fruitful, the concepts may survive. Compare this with the situation in physics, where the main judge is empirical reality.

Another consequence is that mathematical concepts, when mature, seem to have unique explications; they are robust.[5] The paradigm example is Church-Turing's thesis; the explication of *effectively computable function* as *Turing computable function*. There are several alternative explications, using different ideas, but in the most general case they all determine, extensionally, the same set of functions. This points to a difference between e.g. physics and mathematics; a difference hinted at by Aristotle when he states that the objects of mathematics are more separable than those of physics.[6]

Included in this thesis is my thesis for the licentiate degree. It consists of three parts; an introduction, a pre-study of Kolmogorov complexity resulting in a slight generalization of Chaitin's incompleteness theorem, and the construction of a partial measure of the *power of arithmetical theories*. The remainder of the thesis consists of four papers dealing with different aspects of concept formation in above all mathematics and logic, but also in empirical sciences. The papers are separately written, and some overlap is inevitable. One purpose of this introductory chapter is to present a more detailed background to concepts and ideas used than was possible in the articles. Another purpose is to provide the reader with a brief outline of the ideas in the thesis.

This chapter thus contains a more elaborate exposition of the ideas of

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[5]This idea was suggested by Christian Bennet.

[6]Physics B. 2. 193b-194a.

Carnap and Aristotle. It also makes some comments on *abstract objects*, and the related notion of *idealization*. Finally, there are some remarks on *structuralism*, and how structuralist ideas may be related to results in the papers. There is also a summary of the papers, and some suggestions for further work.[7]

2. On Explications

2.1 Carnap and Explications

The papers in the thesis center around *explication* as a means to generate more exact concepts in science and mathematics. In this section I give a rather detailed survey of Carnap's way of using explications as an instrument to generate exact concepts, as well as some critical comments on some of Carnap's positions. There is also an overview of the explications discussed in the papers.

It is customary to regard the thinking of Carnap as taking place in four, partly overlapping, phases. The first one culminates with *Aufbau*, and the second with *Logische Syntax*.[8] Influenced by among others Tarski, Carnap's semantic period starts after the publication of *Logische Syntax*, and lasts well into the 1950's, when he had already been working on problems concerning the concept *probability*.[9] Although these phases in Carnap's thinking obviously exist, some philosophers emphasise the continuity in his development with regard to both problems and method.[10] One problem that occupied Carnap throughout the years was e.g. how to distinguish the factual (synthetic, empirical) from the logical (analytic, tautological); a distinction subjected to, as it seemed, devastating criticism by Quine in "Two Dogmas of

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[7]For reference the following abbreviations will be used; MPAT, Measuring the Power of an Arithmetical Theory, and MPAT1, MPAT2, and MPAT3 refers to the different sections in MPAT; EPAT: On Explicating the Concept *The Power of an Arithmetical Theory*; FIP: A Note on the Relation between Formal and Informal Proof; ITR: Indispensability, the Testing of Mathematical Theories, and Provisional Realism; CUE: Mathematical Concepts as Unique Explications (jointly written with Christian Bennet).

[8](Carnap, 1928), (Carnap, 1934).

[9]Important books in the semantic phase are (Carnap, 1942), (Carnap, 1943), and (Carnap, 1947), and his main work on probability is (Carnap, 1950). Carnap also published several papers, and for a more complete bibliography the reader could consult e.g. (Carus, 2007) or (Schilpp, 1963).

[10]See e.g. (Creath, 1991), and (Carus, 2007).

Empiricism".[11] There is, however, a current revaluation of Carnap's philosophical ideas, as when, for example, A. W. Carus sees in Carnap a defender of Enlightenment, and the tools Carnap developed, above all explication, as a means in this defence.[12]

> The conceptual framework he created is still the most promising instrument, I will argue, for the very purpose he invented it to serve, in the somewhat utopian Vienna Circle context of the 1920s and the early 1930s: it is still the best basis for a comprehensive and internally consistent Enlightenment world view.[13]

Carnap's interest was, furthermore, not only in technical details, but in an overall view.

> It has come to be realized that there was a good deal more to Carnap than his particular contributions to various specialized fields. There was also a vision that held all these parts together and motivated them, a vision whose importance transcends and outlasts the parts.

> ... Carnap is a much more subtler and sophisticated philosopher [...] than was generally suspected a few years ago.[14]

This thesis is also a contribution to this revaluation, and this renewed interest.

Carnap introduces the concept *explication* in a paper on *probability* in 1945.[15] In explicating a concept the question is not, as is often the case in science and mathematics,

> one of defining a new concept but rather of redefining an old one. Thus we have here an instance of that kind of problem [...] where a concept already in use is to be made more exact or, rather, is to be replaced by a more exact new concept.[16]

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[11] Originally published in *Philosophical Review* 1951; reprinted in (Quine, 1953).

[12] For the revaluation of the philosophy of Carnap, see e.g. (Carus, 2007), and (Awodey and Klein, 2004). See also (Stein, 1992), and (Gregory, 2003) on Carnap and Quine, and the relation between the analytic and the synthetic.

[13] (Carus, 2007), p. 8.

[14] Gottfried Gabriel; Both quotations are from the Introduction to (Awodey and Klein, 2004), p. 3.

[15] (Carnap, 1945).

[16] *Ibid.*

In an explication the *explicandum* is the more or less vague concept, and the new, more exact one, is the *explicatum*.[17] As an example Carnap mentions Frege's and Russell's explication of the cardinal number *three* as the class of all triplets. His concern in this paper is to clarify explicanda concerning two concepts of probability; probability as *degree of confirmation*, and as *relative frequency in the long run*.

In *Meaning and Necessity* Carnap describes the concept *explication* in the following manner.

> The task of making more exact a vague or not quite exact concept used in every day life or in an earlier stage of scientific or logical development, or rather of replacing it by a newly constructed, more exact concept, belongs among the most important tasks of logical analysis and logical construction. We call this the task of explicating, or of giving an *explication* for, the earlier concept; this earlier concept, or sometimes the term used for it, is called the *explicandum*; and the new concept, or its term, is called an *explicatum* of the old one.[18]

As before Carnap exemplifies with *cardinal number*, but he now adds *truth*, his own efforts to handle concepts like *L-truth* (logical truth, analytic), and phrases of the form *the so-and-so*, etc. He also briefly mentions how the meaning relation between explicandum and explicatum ought to be understood.

> Generally speaking, it is not required that an explicatum have, as nearly as possible, the same meaning as the explicandum; it should, however, correspond to the explicandum in such a way that it can be used instead of the latter.[19]

Concerning the possible correctness of an explication, Carnap states that

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[17] For the terminology Carnap refers to Kant and Husserl, although the use they make of the term "explication" differs to a great extent from Carnap's (Beaney, 2004).

[18] (Carnap, 1947), pp. 7f.

[19] *Ibid.*, p. 8.

there is no theoretical issue of right or wrong between the various conceptions, but only the practical question of the comparative convenience of different methods.[20]

Finally, Carnap devotes chapter one of *Logical Foundations of Probability* to the concept *explication*.[21] The main problem in this book is to construct explications of concepts like *degree of confirmation*, *induction*, and *probability*. The process of making explications is described as above, and now Carnap emphasises the need to clarify explicanda in order to make clear which sense of a vague and unclear explicandum it is that needs to be explicated. It was a clarification of this type the 1945 paper discussed. Now he exemplifies with the concept *salt* where one meaning is the way it is used in chemistry, and another as it is used in household language. The latter can be explicated as *NaCl*, and the former as a substance formed by the union of an anion of an acid and a cation of a base. Carnap did not present, except in some vague phrases, any criteria the explicatum must fulfil in the 1945 paper or in *Meaning and Necessity*, but in *Logical Foundations of Probability* this is taken care of.

> 1. The explicatum is to be *similar to the explicandum* in such a way that, in most cases in which the explicandum has so far been used, the explicatum can be used; however close similarity is not required, and considerable differences are permitted.
> 2. The characterization of the explicatum, that is the rules of its use [...], is to be given in an *exact* form, so as to introduce the explication into a well-connected system of scientific concepts.
> 3. The explicatum is to be a *fruitful* concept, that is, useful for the formulation of many universal statements (empirical laws in the case of a nonlogical concept, logical theorems in the case of a logical concept).
> 4. The explicatum should be as *simple* as possible; this means as simple as the more important requirements (1), (2), and (3) permit.[22]

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[20] *Ibid.*, p. 33.

[21] (Carnap, 1950).

[22] *Ibid*, p. 7.

Concerning the possible correctness, or truth, of an explication, Carnap reinforces the statement from *Meaning and Necessity* that there is no question of right or wrong. Since the explicandum is not an exact concept, the problem of explication is not stated in exact terms, so

> the question whether the solution is right or wrong makes no good sense because there is no clear-cut answer. The question should rather be whether the proposed solution is satisfactory, whether it is more satisfactory than another one, and the like.[23]

One example Carnap discusses in some detail in *Logical Foundations of Probability* is the explication of the pre-scientific concept *fish* (animal living in water) as *pisces* (in biological taxonomy). It can be, and has been, maintained that *pisces* is not more precise than *fish*. It is more narrow, but still vague, and considering the frequent changing of borders between different taxa in taxonomical systems, this seems correct. Still, by creating a taxonomical system, the concept *pisces* gets a position in a well-connected system of concepts, even though the system is provisional, and will change. Concepts in biology are difficult to handle because of the diversity of biological phenomena. A concept such as *species* is an extremely fundamental concept, but there is no unequivocal explication of it.[24]

This process of making vague or pre-scientific concepts more exact so that they may be used in science or mathematics fits well into the totality of Carnap's thinking. Carus traces the idea back to the *principle of tolerance*, formulated in *Logische Syntax*, a principle Carnap describes in the following way in his intellectual autobiography.

> I wished to show that everyone is free to choose the rules of his language and thereby his logic in any way he wishes. This I called the "principle of tolerance"; it might perhaps be called more exactly the "principle of the conventionality of language forms".[25]

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[23] *Ibid*, p. 4.

[24] See (Kuipers, 2007) on explications in empirical sciences, CUE for some comments on the concept *pisces*, and FIP on *species*.

[25] (Carnap, 1963a), p. 55.

In *Logische Syntax* Carnap states that

> *[i]t is not our business to set up prohibitions, but to arrive at conventions.*[26]

And a little bit further down in the text the principle is formulated:

> *In logic, there are no morals.* Everyone is at liberty to build up his own logic, i.e. his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments.[27]

Carnap himself takes this principle to be implicit already in *Aufbau*, where he allows different languages in the project of rational reconstruction.[28] Seen in this way, the process of making explications permeates all of Carnap's philosophy.[29]

In papers subsequent to *Logical Foundations of Probability* Carnap uses the concept of explication as a well-known idea, and does not bother to explain it.[30] The concept was also almost immediately introduced into the secondary literature.[31]

In *Word & Object* Quine sees the method of explication as paradigmatic, and illustrates its use with the concept *ordered pair*.[32] The noun "ordered pair" is, according to Quine, a defective one like e.g. the geometrical noun "line". It is, however, much more easy to come to grips with how to treat two objects as one via explications in the case of "ordered pair", than it is with "line" as denoting an ideal or abstract object. Quine starts with an analysis of the explicandum resulting in the usual criterion of identity between two ordered pairs, i.e.

$$\langle x, y \rangle = \langle z, w \rangle \text{ iff } x = z \text{ and } y = w.$$

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[26] (Carnap, 1937), p. 51.

[27] *Ibid.*, p. 52.

[28] (Carnap, 1963a), pp. 17 f., and p. 44.

[29] As mentioned above this is in line with Carus's view on Carnap (Carus, 2007). This is also Michael Beaney's position in (Beaney, 2004).

[30] See e.g. papers added in the supplement to the second edition (1957) of (Carnap, 1947).

[31] See e.g. (Hempel, 1952), pp. 11-13.

[32] (Quine, 1960), §§53, 54.

He then formulates several explicata like Kuratowski's well-known suggestion

$$\{\{x\}, \{x, y\}\}.$$

Using the explicatum instead of the explicandum is called "elimination" by Quine.

Quine points to an important aspect of explications which is a consequence of Carnap's ideas, but not so much discussed by Carnap himself. It is that this process of making explications is not affected by the so-called *paradox of analysis*, since there is no demand for synonymy between explicandum and explicatum. The paradox of analysis concerns how an analysis could be both correct and informative. If correct, the two concepts, or terms, have the same meaning, and no information is conveyed. If the two terms differ in meaning, the analysis is incorrect.[33]

The criteria that an explicatum ought to satisfy are (deliberately?) vague. This has the advantage that there are no (or few) formal obstacles to the process of explication. The scientist or philosopher can concentrate on the content and not on whether he is formally doing the right thing. One possible disadvantage might be that disagreements concerning a proposed explication will focus on whether it really is an explication, and not on whether it is fruitful, etc.[34]

2.2   Some Problems with Carnap's Position

Not much was written on Carnap and explications after the 1960's, perhaps due to the decline of logical positivism, until the renewed interest in recent times. This subsection presents some critical views centering on problems of provability, exactness, and vagueness concerning explication. I begin with provability. In his contribution to the International Colloquium in the Philosophy of Science in London 1965, Kreisel discusses informal rigour and how intuitive notions are made precise as follows.

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[33] (Quine, 1960), p. 258. See also (Beaney, 2004) for a fuller account for the paradox of analysis in relation to Carnap.

[34] Compare the view on the relation between *formal* and *informal proof* in FIP with Leitgeb's in (Leitgeb, 2009). In FIP it is suggested that the former is explicatum of the latter, while Leitgeb denies this on the ground that the two concepts are too different in meaning (p. 272). There is no disagreement on how to regard the two concepts. In FIP the focus is on fruitfulness, and in (Leitgeb, 2009) on difference in meaning.

> The 'old fashioned' idea is that one obtains rules and definitions by analyzing intuitive notions and putting down their properties. This is certainly what mathematicians thought they were doing when defining length or area, or for that matter, logicians when finding rules of inference or axioms (properties) of mathematical structures such as the continuum.[35]

Kreisel tries to show how intuitive notions can figure in exact proofs. He exemplifies with the intuitive concept *logical validity*, and argues that it can be strictly related to *formal derivability* and *truth in all set-theoretic structures* (via the completeness theorem for first-order logic).[36] On a direct question from Bar-Hillel in the following discussion on the relation between informal rigour *vs.* formal rigour, and Carnap's notions of clarification of the explicandum *vs.* formulation of the explicatum, Kreisel elaborates his point.[37] He opposes Carnap's idea of the impossibility of correctness of informal concepts, and argues that he has proved that Carnap is wrong in the above-mentioned example. Kreisel sees a danger with Carnap's position in that people will not bother to look for proofs since they believe there are none. Carnap would presumably deny the possibility of informal rigour as well as the possibility of finding *the* correct explication together with a proof of its correctnes.

To reinforce Kreisel's argument against Carnap, consider the possible truth, or even provability, of Church's thesis.[38] Joseph Shoenfield argues that it may be possible to find a proof of the thesis in spite of the vagueness of the explicandum.

> Since the notion of a computable function has not been defined precisely, it may seem that it is impossible to give a proof of Church's thesis. However, this is not necessarily the case. We understand the notion of a computable function well enough to make some statements about it. In other words, we can write down some axioms about computable functions which most

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[35] (Kreisel, 1967a), p. 138.

[36] *Ibid.*, pp. 152-157. See also FIP.

[37] See (Bar-Hillel, 1967), p. 172, and Kreisel's reply in (Kreisel, 1967b), pp. 175-177.

[38] Robert Black reviews possible ways of proving the thesis (Black, 2000).

people would agree are evidently true. It might be possible to prove Church's thesis from such axioms.[39]

Recently a proof of Church's thesis along these lines has been presented by Nacum Dershowitz and Yuri Gurevich via an axiomatization of the explicandum.[40] These arguments are strong ones against Carnap's idea that there is never any question of right or wrong in the process of making explications. In some cases an explication may be provably correct, but note that this possibility also can depend on what is accepted as a proof.

As mentioned above, Carnap identifies two steps in the construction of an explication where the first concerns the clarification of the explicandum, and the second is to make the explicandum more precise, i.e. constructing the explicatum. Looking at Carnap's examples, the clarification of the explicandum often consists in removing ambiguities, and these are in several examples related to paradoxes.[41] Concerning the meaning relation between the explicandum and the explicatum Carnap is of the opinion that the meanings can differ considerably, while others, Tarski e.g., think they must coincide.[42]

Joseph Hanna takes as his goal to make clear this meaning relation between the explicandum and the explicatum.[43] In his analysis of explications he distinguishes two types of explications exemplified by the explication of *effectively computable function* and *ordered pair*, respectively. Concerning the first the "categorial domain" is according to Hanna clear; there is no question about what constitutes a function, and the choice of domain is unproblematic. He thus takes for granted that the functions are partial functions on the natural numbers, and also that the function concept is determined. Given all this, we want to provide a sharp dividing line between the computable and the non-computable functions via an explication. This type of explication, Hanna calls "explication$_1$", and he calls its vagueness "external".[44] In

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[39](Shoenfield, 1993), p. 26.

[40](Dershowitz and Gurevich, 2008).

[41]See (Hanna, 1968).

[42]On Carnap, see above, on Tarski see (Tarski, 1944).

[43](Hanna, 1968). Giovanni Boniolo points out that Hanna's paper is one of few relevant texts on Carnap and explications. He does this in a paper critical to what he conceives of as ideal-language-philosophy in Carnap (Boniolo, 2003). See also below on this issue.

[44]As to terminology he refers to Kaplan. For references see (Hanna, 1968).

the second example the vagueness is of a different kind. Given the ordinary criterion for *ordered pair* there is never any question of ordinary vagueness, i.e. whether an object is an ordered pair or not. It is more a question of which entities we want to regard as ordered pairs, i.e. the categorial domain is not clear. We can, with Kuratowski, explicate *ordered pair* as certain classes of classes, and in this way determine the categorial domain, but other choices are possible.[45] Hanna calls this type of explication "explication$_2$", and the vagueness "internal". These two types of explications are not independent. Determining a categorial domain may produce external vagueness. Consider once again the concept *effectively computable function*. It is of course possible to choose categorial domain in other ways than suggested above, but this will not remove external vagueness. With *ordered pair* no external vagueness is introduced; Kuratowski's explication determines both the categorial domain and the explicatum. Hanna also gives a technical analysis of external vagueness, but these details are not relevant here. The distinction between these two types of explications is informative. It parallels in a way, but is not identical with, the two steps Carnap points out. In the clarification of the explicandum a categorial domain may be determined, and in the construction of the explicatum external vagueness may be removed.

Critical voices against the demand of exactness were raised early on from what is usually called "ordinary-language philosophy". In the Carnap volume of Schilpp's *Library of Living Philosophers*, Strawson compares his own method of *natural linguistics* with Carnap's method of *rational reconstruction*.[46] Strawson emphasises that the introduction of an explication is to take place in an *exact* scientific or logico-mathematical language, and draws a sharp dividing line between scientific and non-scientific discourse. And, according to Strawson, philosophical problems normally arise using non-scientific language, so

> it seems prima facie evident that to offer formal explanations of key terms of scientific theories to one who seeks philosophical illumination of essential concepts of non-scientific discourse, is

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[45]Quine gives examples of explications where the categorial domain is the natural numbers, coding pairs of natural numbers as a natural number; (Quine, 1960), §§53, 54.

[46](Strawson, 1963), pp. 503-518.

> to do something utterly irrelevant ...[47]

He also states that the

> use of scientific language could not replace the use of non-scientific
> language for non-scientific purposes.[48]

Introducing a scientific vocabulary changes the subject, and does not lead to an illumination of the *philosophical* problem.[49] The clarification of problems using explications cannot be achieved

> unless extra-systematic points of contact are made [...] at *every*
> point where the relevant problems and difficulties concerning
> the unconstructed concepts arise.[50]

In Carnap's reply to Strawson he clarifies his view on the exactness demand explicata are to satisfy.[51] His first objection to Strawson is that it is not totally clear what he means by explication as "clarification": whether it concerns the clarification of the explicandum, or the formulation of the explicatum. Carnap then states that he sees no sharp dividing line between scientific and non-scientific language. Scientific languages arise from non-scientific ones, and scientific vocabulary works its way into non-scientific languages. Strawson's interpretation of Carnap's position, that explications are to be formulated in an exact, formal language, may seem straightforward, since explications are "to be given in an exact form".[52] But in the chapter containing the quotation, Carnap also introduces the above-mentioned explication of *fish* as *pisces*; an example that ought to cast doubt on the belief that explications *always* must be formulated exactly, and in an exact context. In his answer to Strawson, Carnap claimes that an

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[47] *Ibid.*, p. 505.

[48] *Ibid.*

[49] *Ibid.*, p. 506.

[50] *Ibid.*, p. 513.

[51] (Carnap, 1963b), pp. 933-940.

[52] See above, and (Carnap, 1950), p. 7.

> explication replaces the imprecise explicandum by a more precise explicatum. Therefore, whenever greater precision in communication is desired, it will be advisable to use the explicatum instead of the explicandum. The explicatum may belong to the ordinary language, although perhaps to a more exact part of it.[53]

And

> [t]he only essential requirement is that the explicatum be more precise than the explicandum; it is unimportant to which part of the language it belongs.[54]

Not surprisingly Carnap feels that the method of rational reconstruction (explication) has greater possibility of casting light on philosophical issues than ordinary-language philosophy, and he objects to Strawson's opinion that his analyses are "utterly irrelevant". Carnap also means that it is, among other things, the solving of philosophical problems he has devoted his career to. Of course, the spirits of these two philosophers are very different.

In conclusion, Carnap's position concerning the possible correctness of an explication may be too defensive. It may sometimes be possible to prove the correctness of an explication. Furthermore, an explication need not take place in an exact setting. The requirement is that greater exactness, or less vagueness, is accomplished by the explication. Finally, there are, over and above Carnap's way of describing the process, two kinds of processes going on in making explications; determining a categorial domain, and removing or diminishing vagueness.

2.3   On the Use of Explications in the Thesis

In the papers included here not so much attention is paid to the second item in Carnap's list of properties that the explicata ought to satisfy. Some of

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[53] (Carnap, 1963b), p. 935.

[54] *Ibid.*, p. 936. Other philosophers who protest against what they take as a strict demand of exactness are Frank A. Tillman and Giovanni Boniolo. Tillman, in a paper comparing the methods of Carnap and Strawson, makes the same mistake as Strawson when he declares that the explicatum must be "constructed with the help of a formalized language" (Tillman, 1965). So does Boniolo when he believes that explicatum must be exact, fit into a formal system, and that *the mathematical method* must not enter into philosophical analysis (Boniolo, 2003).

the explications discussed take place in an exact context, others in a more informal setting. Support for this being in accordance with Carnap's ideas can be found, as we just saw, in his reply to Strawson.

Mathematical concepts, being abstractions of more concrete ones, sometimes have a distant empirical origin, but there are differences between physics, or science in a wider sense, and mathematics in how the respective concepts are used.[55] While mathematical concepts, at least mature ones, are robust, physical concepts change with theory (r)evolution. Mature mathematical concepts have unique explications. Furthermore, mathematical concepts, when introduced into a theory, must fit into that theory in a consistent way, while concepts of physics (science) are tested with ordinary, empirical means. Even though it may seem that e.g. the concept *continuity*, in its topological sense, is far removed from an empirical origin, it can be traced back to the idea that movements do not take place in jumps. In this way many mathematical concepts have a more or less distant origin in empirical reality, and this can be more or less obvious. This means that empirical science is important for the development of mathematics. I will argue that the relationship between mathematics and science can be partly understood via ideas presented in this thesis, in which several examples of explications are presented and discussed. There are also situations where it seems to be impossible to make a vague or unclear concept precise. In these cases it seems impossible to produce an explication, and thus to mathematize the discipline in question.

In the papers included in this thesis, one (EPAT) contains a negative claim. In this paper it is argued that the concept *the power of an arithmetical theory* is impossible to explicate. The other three papers provide several positive instances of explications. FIP focuses on the long discussed relation between *formal* and *informal proof* in mathematics, while ITR and CUE treat problems of a more philosophical kind. In ITR a strengthened version of the indispensability argument is used to argue for realism in mathematics, a strengthening that is possible due to the ideas of concept formation presented. In CUE a characterization of mathematics is formulated, founded on the special character of mathematical concepts .

The setting in this thesis is that of classical mathematics, although there

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[55] This idea is developed in CUE.

are some occasional comments on constructive proofs in FIP. A study of the development and robustness within constructivist traditions of e.g. the function concept would certainly be worthwhile, but that must be left for another occasion.

2.4   An Overview of Treated Explications

Here is a brief survey of some of the explications discussed in the papers.

- The concept *set* has an informal origin in the concept *collection of objects*. The first systematic effort to explicate the concept is due to Cantor. Via Frege's use of the concept, involving the principle of abstraction, and the discovery of Russell's paradox, a new explication was called for. The dominant explication is the axiomatization of set theory by Zermelo, Fraenkel, and Skolem (ZF, or ZFC with the axiom of choice included). These axioms express one view what is to be regarded as a set. Discussions of alternative axiomatizations are still going on (ITR, CUE).

- As to the concept *function*, it has an origin in a vague idea of a *(causal?) dependency* between two entities. The development of this concept took a long time, and Euler's idea of a function as an analytic expression was an early attempt. With Dirichlet the modern logical (or set-theoretical) concept is almost arrived at, and this concept has in turn been generalized to other function-like concepts (FIP, ITR, CUE).

- Closely related to the function concept is that of *continuity*. The origin of the idea is, that *changes do not take place in jumps*. Euler and his contemporaries regarded continuity as a property of functions. Functions were in the seventeenth and the early eighteenth century associated with geometrical curves. It gradually became evident that this conception of continuity was untenable, and with Cauchy a new approach to continuity, via the concept *limit*, appeared. This idea was later made precise with the $\epsilon - \delta$ definition of Weierstraß. The concept of continuity then split into the concepts *pointwise* and *uniform continuity*.

Still later the *topological* concept of continuity was formulated (FIP, CUE).

- As to the concept *speed*, or *rate of change*, and the geometrical counterpart *inclination*, Newton and Leibniz independently found fruitful explications via the concepts *fluxion* and *infinitesimal*. The inconsistencies in Newton's use of fluxions were pointed out by Berkeley, but mathematicians and scientists continued to use these new, fruitful concepts. The precise notion of limit enabled mathematicians to eliminate these concepts, so their use, from a mathematical point of view, became more a way of speaking. With the development of nonstandard analysis, we know how to handle infinitesimals in a consistent way (ITR).

- The concept *effectively computable function* received several explications from the 1930's onward by the work of Church, Turing, Post, *et al*. This case is illuminating, since all these explicata are provably equivalent, and the example is paradigmatic in the proposal that (many) fruitful mathematical concepts are uniquely explicable, or, as in CUE, that a concept is mathematical if it is uniquely explicable.

- The historical origin of the concept *(natural) number* seems impossible to trace, but it may very well be that it is still most natural to regard numbers as a qualitative concept, assigning a property to collections of objects as *pair* or *many*.[56] The route to a comparative concept like *more than* or *larger than* is then rather direct. Seeing numbers as quantitative concepts involves introducing operations on numbers, and possibly speaking of the numbers themselves, thus raising problems of existence, etc. In ancient Greece philosophers like Plato 'defined' numbers via a generating unit, and the concept was exact enough to satisfy mathematicians well into the nineteenth century, when Frege defined numbers as the extension (Wertverlauf) of certain properties. With

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[56]See the contribution of Cappelletti and Giardino (Cappelletti and Giardino, 2007), and Maddy ((Maddy, 2007), pp. 318-328) on a competence of numerosity, a competence to see groups of objects as having a certain size, in children and animals. See also (Devlin, 2000) for a more full account on these matters.

the development of set theory natural numbers are, in foundational studies, identified with e.g. von Neumann ordinals (CUE).

- The concept *formal proof* can be regarded as an explication of *informal proof*, where "informal proof" is understood as proofs used in all their diversity in mathematics. Its distant origin is the ancient observation that *reasoning* could be more or less precise. The first known and worked out codification (or explication) of *correct reasoning* is due to Aristotle in his logic. Euclid, in his compilation of the mathematics of his time in the *Elements*, sets the standard of mathematical reasoning for a long time, and thus indirectly defines *correct reasoning*. Not much happened, related to this type of problems, until the nineteenth century with the development of non-Euclidean geometry (changing the view of the role of the axioms), the axiomatization of arithmetic (Dedekind, Peano), and the new logic of Frege. In the twentieth century, the effort to provably avoid inconsistency required an even deeper understanding of the concept *mathematical proof*. It will be argued that *formal proof* as it is defined in a first-order context is an adequate explication in this project (FIP, CUE).

To point to the difference between the use of concepts in mathematics and science, and to highlight the process of mathematization in science, some fundamental concepts of empirical science are mentioned in the papers.

- The classificatory concept *species* as it is used in biology has many different explications, both historically, and at present. This may be due to the diversity of biological phenomena, but also to an unclear conception of what it is that constitutes a species (ITR, CUE).

- The concept *element* in chemistry, on the other hand, is also a classificatory concept, but it is quite precise, having a history developing from the atomism and theory of the elements of ancient Greek philosophers. With the periodic table, originally based on valency and atomic weight, the elements were systematized. The discovery of Germanium by Winkler 1886 after Mendelejeff's prediction of its existence and properties in 1871 was a scientific triumph in the same

category as the discovery of Neptune. With the work on Brownian motion of Einstein and Perrin, atomism was fully accepted by (almost) all scientists (ITR).

- *Gravitation* found its first fruitful explication with Newton using the problematic concept *force*. This concept was 'replaced' by the geometrical conception of Einstein's general theory of relativity. This theory is also problematic, since it is classical, i.e. not quantized, and scientists are searching for a new concept of gravitation unifying quantum mechanics and gravitation. This new theory, if developed, may be something completely different from both Newton's and Einstein's conceptions.

Examples like these point to a difference between mathematics and science. Concepts of mathematics, when mature, have robust, unique explicata. Their development is normally towards more precise and more general concepts. They have a central place in mathematical theories. These theories may be fruitful in developing mathematics, or in scientific applications; alternatively they are interesting in there own right. They can remain a central theme in mathematics for a long time, or they can lose their force of attraction, and be left without further notice by the mathematical society. Nevertheless, the theories are, as it seems, consistent, and part of the mathematical architecture.

Explicated concepts in empirical science, on the other hand, tend to be of a more temporary kind. They change when theories change, just as the concepts *species*, *element*, and *gravitation* mentioned above. Furthermore, a fruitful mathematization of a theory is almost always necessary for the rapid development of the theory, and here the ideal is to reach quantitative, measurable explicata. Carnap illustrates with the classificatory concept *warmth*, specified as the comparative concept *warmer*, and finally explicated as the quantitative concept *temperature*.

3. Philosophy of Mathematics in Aristotle

As indicated in section 1, Aristotle's philosophy of mathematics can be a starting-point for understanding both the applicability of mathematics, and how mathematical concepts are separable in thought in abstracting processes. In this section I will make some comments on Aristotle's views on mathematical objects as abstractions, on the existence of mathematical objects, and on mathematical truth.

3.1 Mathematical Objects as Abstractions

Aristotle's philosophy of mathematics is part of his general philosophy, and consequently he has to relate concepts of mathematics to his distinction between *form* and *matter*, *genus* and *differentia specifica*, *essential* and *non-essential attributes*, etc., and this may make it difficult to extract just what is relevant in the context of philosophy of mathematics. There is, furthermore, no (known) treatise on the philosophy of mathematics by Aristotle. His remarks on mathematics and philosophy of mathematics are scattered throughout all of his texts. Concerning the first distinction, mathematical objects are not pure forms, and they are not sensible objects, but they are separable from sensible objects in thought. This process of separation is described as a process of abstraction. In this activity the mathematician, or metaphysician, eliminates non-essential attributes, or attributes not to be taken into consideration.[57] The mathematician

> investigates abstractions (for in his investigation he eliminates all the sensible qualities, e.g. weight and lightness, hardness and its contrary, and also heat and cold and the other sensible contraries, and leaves only the quantitative and continuous [...] and the attributes of things *qua* quantitative and continuous, and does not consider them in any other respect ...)[58]

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[57](Heath, 1998), pp. 42, 220, 224. In this book Heath has collected and commented on most of the writings of Aristotle on mathematics and philosophy of mathematics.

[58]Metaph. K. 3. $1061^a 28 - 35$. I use (Barnes, 1984) when quoting Aristotle.

Thomas Heath illustrates the process of abstraction *vs.* the process of adding elements or conditions by the contrast between a *unit*, a substance without position, and a *point*, a substance having position.[59] The process of abstracting in Aristotle is not a process of finding common properties among individuals, but rather a process of subtracting.[60] According to John J. Cleary it is not an epistemological theory, but a logical theory with ontological consequences. He furthermore maintains that clarifying the 'qua' locution in the quotation above is *the* crucial point in understanding Aristotle's mathematical ontology.[61] This is exactly the strategy of Jonathan Lear, who introduces a "*qua*-operator" to analyse the abstraction process in Aristotle's philosophy of mathematics. My focus will be on Lear's analysis.[62]

To consider b as an F, b *qua* F, is to consider a substance, in Aristotle's sense, in a certain aspect. For an object b *qua* F to be true of a predicate G, it is required that $F(b)$ is true, and that an object's having the property G follows of necessity from its being an F; in symbols

$$G(b \text{ qua } F) \longleftrightarrow F(b) \wedge (F(x) \vdash G(x)),$$

where the turnstile signifies the relation *follows of necessity*. This *qua*-operator generates a kind of filtering process. Consider a bronze, isosceles triangle b;

$$B(b) \wedge I(b) \wedge T(b).$$

The operator b *qua* T filters out as inessential the other properties, and we are allowed to conclude whatever is possible for the substance b considered as a triangle. This filtering process determines in what aspect, or under what description, a substance or object is being considered. And it is in this aspect a property may be essential or not. The substance b may have other properties, but it is the *qua*-operator that determines under what description the object is to be considered; which properties are to be regarded as essential. This is the reason for writing "$F(x) \vdash G(x)$" and not "$F(b) \vdash G(b)$" in the above

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[59](Heath, 1998), p. 66.

[60](Halper, 1989).

[61](Cleary, 1989).

[62](Lear, 1982). Lear holds that the analysis via the *qua* operator is relevant not only with regard to mathematical concepts; it concerns the whole of Aristotle's metaphysics.

definition of *G(b qua F)*; the result must not depend on any other properties of *b* than its being an *F*.

Since mathematical objects are not pure forms they must inhere in some kind of matter, called intelligible matter, as distinct from sensible matter.

> Even the straight line ... may be analysed into its matter, continuity (more precisely continuity in space, extension, or length), and its form. 'Though the geometer's line is length without breadth or thickness, and therefore abstract, yet extension is a sort of geometrical matter which enables the conception of mathematics to be after all concrete'.[63]

If mathematical objects are not separable from sensible objects, and if they, in some way, are inherent in sensible objects, how are they related to the objects of physics and metaphysics? Physical objects have attributes in addition to mathematical ones. They can be moving, for example, but mathematics abstracts from movement. Physical objects, like mathematical, contain planes, etc., but the mathematician does not treat planes and points *qua* attributes of physical bodies, and he does not study them *qua* limits or boundaries of physical bodies, as the physicist does.[64] The relation between the objects of mathematics, physics, and metaphysics are described in the following way by Aristotle.

> The physicist is he who concerns himself with all the properties active and passive of bodies or materials thus or thus defined; attributes not considered as being of this character he leaves to others, in certain cases it may be to a specialist, e.g. a carpenter or a physician, in others (*a*) where they are inseparable in fact, but are separable from any particular kind of body by an effort of abstraction, to the mathematician, (*b*) where they are separate, to the First Philosopher.[65]

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[63] (Heath, 1998), p. 67. Heath refers to Hicks and De Anima, but the references in Heath's book are incomplete.

[64] *Ibid.*, pp. 10f, 98.

[65] De An. A. 1. 403^b 12 − 16.

Note that Aristotle's process of abstraction does not give rise to *abstract ideas*, and it is in that way not affected by e.g. Berkeley's attack on abstract ideas, or Frege's attack on psychologism.[66]

3.2 On the Existence of Mathematical Objects

The above analysis is relevant for the question of the existence of mathematical objects, and Lear's conclusion, in the light of his analysis, is as follows.

> Thus, for Aristotle, one can say truly that separable objects and mathematical objects exist, but all this statement amounts to - when properly analyzed - is that mathematical properties are truly instantiated in physical objects and, by applying a predicate filter, we can consider these objects as solely instantiating the appropriate properties.[67]

Other commentators on Aristotle's view on the existence of mathematical objects give similar accounts. Edward Halper means that mathematical objects exist as attributes of sensible things; they exist potentially in bodies. This existence is real, and mathematicians treat the objects as separated.[68] Taking this for granted, Halper's main concern is how mathematical objects, being attributes, can have attributes. This is close to the position of H. G. Apostle, who maintains that mathematical objects exist as potentialities in a secondary way.[69] According to Edward Hussey, Aristotle takes it for granted that there are mathematical objects, and that mathematical objects

> (A) do not exist 'apart from' sensible objects; (B) are prior to sensible objects in definition, but (C) posterior to them in being/substance.[70]

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[66](Lear, 1982). Berkeley's criticism is in the introduction to (Berkeley, 1710), and Frege's of abstractions, or psychologism, appears e.g. in his review of Husserl's *Philosophie der Arithmetik*, 1891; see (Frege, 1894).

[67](Lear, 1982).

[68](Halper, 1989).

[69](Apostle, 1952), ch. 1, sec. 5

[70](Hussey, 1991).

Alfred E. Taylor states that mathematical objects are inherent in matter, and Heath that they subsist in matter.[71] Finally, according to Aristotle himself,

> ... some parts of mathematics deal with things which are immovable, but probably not separable, but embodied in matter; ...[72]

Clearly, commentators agree that mathematical objects exist. What they possibly disagree about is the manner of existence, and it is also worth mentioning that, since mathematical objects are separated in thought, some take it that mathematical objects exist in thought. That they *only* exist in thought is a neo-Platonist idea; an idea that modern commentators usually do not accept.[73]

Aristotle's strategy does not say much about arithmetic. The only result reached is that substances can be singled out as units in which to count.[74] But note that Halper, for example, focuses on *number* when he discusses how the attribute number in turn can have an attribute such as even.[75]

3.3 Questions of Truth

Concerning questions of truth and falsity, Aristotle remarks in a couple of places that no falsehoods enter into the argument in the process of abstraction.

> Now, the mathematician, though he too treats of these things, nevertheless does not treat of them as the limits of a natural body; nor does he consider the attributes indicated as the attributes of such bodies. That is why he separates them; for in thought they are separable from motion, and it makes no difference, nor does any falsity result, if they are separated.[76]

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[71] (Taylor, 1912), p. 17, and (Heath, 1998), p. 66.

[72] Metaph. E. 1. $1026^a 14 - 16$.

[73] See e.g. (Mendell, 2004), (Halper, 1989), and (Cleary, 1989).

[74] (Lear, 1982).

[75] (Halper, 1989).

[76] Phys. B. 2. $193^b 31 - 35$.

> Thus if we suppose things separated from their attributes and make any inquiry concerning them as such, we shall not for this reason be in error, any more than when one draws a line on the ground and calls it a foot long when it is not; for the error is not included in the proposition.[77]

First, since a separated triangle does not exist, it is to be regarded as a fiction, but this will not result in falsities. Furthermore, drawing a line and saying it is one foot long is only for heuristic purposes. The figure is not part of the argument. Though the drawn line is not really one foot, we never use this. According to Lear, it does not matter if we use a separated triangle c, or use c considered as a triangle in an argument. His argument is as follows. Let c be a separated triangle that have properties only because it is a triangle; i.e. $G(c) \leftrightarrow G(c \text{ qua } T)$. Suppose we prove, as in the *Elements* I:32, that c has the sum of its interior angles equal to two right angles, $2R(c)$. Since we have concluded that $2R(c)$ only because c is a triangle, it follows that $\forall x(T(x) \rightarrow 2R(x))$, and so for *any* triangle b, that $2R(b)$. No falsity thus results in considering c as a separated triangle, if we only use what can be proved of it as a triangle.

Lear raises two issues related to Hartry Field's efforts to show that mathematics is not necessary to physics.[78] These issues are related to topics discussed in ITR, and I will make some brief comments on them here. Concerning the first issue, Aristotle argues for the truth of mathematics, while Field is of the opinion that only the consistency of mathematics is needed for it to be a conservative extension of physics. In this case Lear does not refer explicitly to Aristotle, but takes his ideas to be Aristotelian in spirit. The key to the truth of mathematics, he says, is not a referential question since separated mathematical objects do not exist, but lies in the usefulness of mathematics. To understand this usefulness, bridges are needed between the physical world and the world of mathematical objects, and one way to understand these bridges is via the *qua*-operator that reveals structural features.

That there must exist bridges between the physical world and

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[77] Metaph. M. 3. 1078^a 16 − 21.

[78] (Field, 1980), (Lear, 1982).

those portions of mathematics which are applicable to it implies that the mathematics must reproduce (to a certain degree of accuracy) certain structural features of the physical world. It is in virtue of this accurate structural representation of the physical world that applicable mathematics can fairly be said to be true.[79]

3.4 On the Relation between Sciences

The other issue, also in relation to Field's ideas, is that mathematics, according to Aristotle, is a conservative extension of physics. This means that if M is a mathematical theory, P a physical theory, and S a sentence that does not contain any terms from the language of M, then S can be proved from P alone, if it can be proved from $M + P$. Lear uses his *qua*-operator to argue that, in Aristotle, geometry is a conservative extension of physics. When we prove that a physical triangle c has the $2R$ property, we can either make use of an abstract triangle, or consider, with the help of the *qua*-operator, c as a triangle.

In my opinion this argument is not a convincing account of Aristotle's views on the relation between science and applied mathematics. Consider the following example from Aristotle. It is

> for the empirical scientists to know the fact and for the mathematical to know the reason why; for the latter have the demonstrations of the explanations, and often they do not know the fact ...[80]

> ... it is for the doctor to know the fact that circular wounds heal more slowly, and for the geometer to know the reason why.[81]

In the context where these quotations occur Aristotle discusses relations and subordinations between sciences. A result in one science can be used in another if it subordinates the other, as when e.g. facts about the rainbow can

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[79](Lear, 1982). See below on structuralism, and the relation between mathematics and the physical world. See also ITR, and Donald Gillies (Gillies, 2000) for an example of an empiricist philosophy of mathematics inspired by Aristotle.

[80]An. Post. B. 13. $79^a 3 - 6$.

[81]An. Post. B. 13. $79^a 13 - 16$.

be explained in (mathematical) optics, and still more generally in geometry. Concerning the example above some of the properties are geometrical, and even though medicine is not subordinated to geometry, it is still possible to use mathematics in this case.[82] To know "the reason why" is to know the formal cause, and this may be an example where mathematics enter into an explanation of a physical fact in an essential way.[83] In this and similar cases mathematics may enter essentially into explanations. To know both the hows and the whys both insights are needed.[84] Lear might think that it should be possible to prove the relevant theorem with the "*qua* strategy", but it is also possible that Aristotle's philosophy of mathematics contains more aspects than can be seen wearing Lear's glasses. Thus, when Lear states that

> Aristotle treated geometry as though it were a conservative extension of physical science[85]

I believe he exaggerates the force of his own explanation of parts of Aristotle's philosophy of mathematics via the *qua*-operator. Aristotle is not only interested in *how*, but also in *why*; that is, he is interested in formal causes. Also, at that time geometry was considered as intimately connected with, indeed a description of, physical space.

3.5   Concluding Remarks

Aristotle is of the opinion that mathematical objects do not exist as separated; they can be separated in thought via a process of abstraction or subtraction. Lear uses the *qua* operator to analyse this process. No falsehoods enter into an argument, because we do not use the special properties e.g. a figure may have. It is just a heuristic device. Mathematics may enter into explanations in

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[82](Mendell, 2004).

[83]It is a theorem, probably proved by Zenodorus some time between 200 B.C. and A.D. 100, that the circle is the geometrical object, bounded by a curve of a fixed length, that has maximum area, and Aristotle ought to have been aquainted with this intuitively very plausible result. See (Kline, 1972), p. 126 on Zenodorus and isoperimetric problems, and chapter 24 for details on the development of the calculus of variations. See also ITR on the applicability of mathematics.

[84]Apostle means that the definitions of mathematics are formal causes and the starting point of demonstrations ((Apostle, 1952), p. 50). See also Hussey, who finds it puzzling when Aristotle says that mathematics is concerned with forms; i.e. formal causes (Hussey, 1991).

[85](Lear, 1982).

an essential way, since mathematics may provide the whys, the formal cause, of a phenomenon. Finally, mathematical propositions are true or false, and their truth is related to the applicability of mathematics; it is not a referential issue since mathematical objects do not exist as separated.

In Aristotle definitions are made via *genus* and *differentia specifica* using essential attributes or attributes we want to pay attention to. An explication can be seen as a device to accomplish something analogous. A mathematician may try to isolate aspects of objects or problems to arrive at the essential ones, and "essential" must not be taken in any metaphysical meaning, but just referring to aspects that may make it possible to analyse the problem, aspects to pay attention to. This can be seen as a process of abstraction, or as a process of idealization. But before entering into this discussion some comments on the difference between abstract and concrete objects are in place.

4. Abstract Objects and Idealizations

4.1 Abstract Objects *versus* Concrete Objects

In ITR it is remarked *in passim* that there is no sharp dividing line between abstract and concrete objects. Also, since the proposal is that mathematical entities are abstractions, the distinction needs to be defused. In our ontologies we, implicitly or explicitly, presuppose entities that are more or less abstract, more or less concrete. It is fairly easy to display examples of both kinds but every effort to provide a dividing line has failed, and the main argument that there is no sharp dividing line is just the failure to produce one.

One of the most authoritative discussion on abstract and concrete objects later on is David Lewis's. His discussion is followed up by John P. Burgess and Gideon Rosen.[86] What will follow here is an account of their analyses of the problem. Lewis recognizes four ways to explain the distinction between the abstract and the concrete, where the first one is the *way of example*. Concrete entities are things like donkeys, protons, and stars, whereas abstract entities are things like numbers. The idea is that everybody knows how to distinguish between abstract and concrete objects, so all that is needed is to hint at it.

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[86](Lewis, 1986), (Burgess and Rosen, 1997).

The second way is the *way of conflation*; the distinction between abstract and concrete is thought to be the same one as that between sets and individuals, or between universals and particulars, etc. The third way is the *negative way*; abstract entities have no spatiotemporal location, and they do not enter into causal interaction. Finally, the fourth way is the *way of abstraction*; abstract entities are abstractions from concrete ones.[87]

Lewis's interest in the distinction is related to the existence of possible worlds and entities in them, and whether these worlds, and the objects therein, are abstract or concrete, but this is not the issue at stake here. However, he notes that these four ways do not necessarily produce a clearcut demarcation between abstract and concrete entities, and that the demarcations they generate are not necessarily the same. The first way, the way of example, is not specific enough.

> ... there are just too many ways that numbers differ from donkeys *et al.* and we still are none the wiser about where to put the border between donkey-like and number-like.[88]

But Lewis sees no opposition between this way and the second way, the way of conflation. There is e.g. no conception of number agreed upon, but if numbers are (abstract) sets, they are abstract according to both ways. He sees, however, a conflict between the way of conflation and the way of negation. It seems, according to Lewis, that at least some sets and universals might be located, and according to the way of negation they should not be classified as abstract. Sets of concrete objects, e.g. the singleton set containing David Lewis, is located where David Lewis is.[89] Similarly, universals are located where the corresponding particulars are located. Furthermore, Lewis thinks that abstract entities can enter into causal interactions. Something can, for example, cause a set of effects, and a set of causes can cause something. Finally, concerning the fourth way, the way of abstraction, we cannot just identify abstractions with universals. In making abstractions we focus on some suitable aspect or aspects, and all these aspects need not be suitable candidates for

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[87](Lewis, 1986), pp. 81-86.

[88]*Ibid.*, p. 82.

[89]Penelope Maddy has a similar argument for the reality of some sets in (Maddy, 1990).

genuine universals.[90]

Burgess and Rosen develop Lewis's analysis further.[91] They take departure in Lewis's four ways, and see the way of example as the most common one, used by e.g. Goodman and Quine, as the introducers of modern nominalism, and Hartry Field, as one of the most prominent defender of nominalism later on.[92] The distinction between abstract and concrete is to be a distinction of kinds, not of degrees.[93] Within the category of abstract entities Burgess and Rosen distinguish several levels, where mathematical objects, *mathematicalia*, like sets and numbers, are the most paradigmatic abstract ones.[94] At the next level are *metaphysicalia*, objects postulated in metaphysical speculations like universals and possibilia. If Platonic forms are identified with universals, Burgess and Rosen reverse the order of ideas in Plato's world of ideas. Further down the list come what they call *characters*, entities that are equivalent in some way, sharing some common trait. Burgess and Rosen mention biological species, geometric shapes, meanings, and expression types as examples from this level. At the lowest level are e.g. institutions of different kinds.

Concrete objects can be divided into *physicalia*, observable physical objects, and *events*. Further down the list are theoretical objects like quarks and black holes, and even further down things like *mentalia* and physical objects postulated by metaphysicians such as arbitrary conglomerates.

The way of example has, according to Burgess and Rosen, lead to sufficient concensus among nominalist philosophers for them to be able to pursue their projects. But if one is to get a better understanding of the distinction between *abstracta* and *concreta* one has to rely on something else, e.g. the three other ways described by Lewis. One might say that we have to use the way af abstraction to see the common traits of the entities on the list of *abstracta*. Burgess and Rosen see the same problems with the additional ways as Lewis does, and they add some more. However, they think the understanding of

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[90](Lewis, 1986), pp. 83-85. This is also in line with the discussion of mathematical objects as abstractions, as it is presented in the section on Aristotle above.

[91]See (Burgess and Rosen, 1997), especially pp. 13-25.

[92](Goodman and Quine, 1947) and (Field, 1980), pp. 1f.

[93](Lewis, 1986), p. 81, (Burgess and Rosen, 1997), p. 14.

[94](Burgess and Rosen, 1997), pp. 14f.

the dividing line between the abstract and the concrete is clear enough for themselves to go on with their own project, i.e. analysing nominalist strategies.

Concerning the degrees of abstraction discussed by Burgess and Rosen, we may note that the most paradigmatically abstract objects, according to them, are *mathematicalia*, whereas e.g. geometrical forms are on a lower level together with characters, as they call objects at that level. But it is not clear, it seems to me, why e.g. mathematical circles should be on another level of abstraction than functions and relations. After all, the relation $x^2 + y^2 = 1$ 'is' a circle. Furthermore, if some mathematical entities are abstractions specified via explications, then these entities ought to be on the level of characters. Thus, there is room for regarding at least some mathematical objects as less abstract than they are according to Burgess and Rosen.

4.2   Abstractions and Idealizations

Closely related to the process of abstraction is the process of idealization. Speaking in the language of Aristotle, the process of abstraction can be seen as an elimination of non-essential properties; properties that we do not want to pay attention to. Triangles can be e.g. isosceles, right-angled, etc., and in a process of abstraction we may disregard features such as these. When studying composition of functions we may leave out of account traits such as whether the functions are odd or even and arrive at e.g. the group structure. In a process of idealization the mathematician or empirical scientist may disregard properties such as friction when studying mechanical systems. In cases such as these the aim is rather to arrive at a problem description that can be analysed using mathematics at some suitable level.

One way to understand the difference between abstraction and idealization is implicit in Lewis's ideas of possible worlds.

> Idealizations are unactualized things to which it is useful to compare actual things. An idealized theory is a theory known to be false at our world, but true at worlds thought to be close to ours. The frictionless planes, the ideal gases, the ideally rational belief systems - one and all, these are things that exists as parts

> of other worlds than our own. The scientific utility of talking of idealizations is among the theoretical benefits to be found in the paradise of *possibilia*.[95]

These ideal objects thus exist in possible worlds close to ours, but not in our world. Abstract objects on the other hand, at least *mathematicalia*, are to be applicable in every possible world. Someone taking possible worlds seriously may find that this casts some light on the distinction.

When a mathematician or scientist makes an idealization, he is aware of the introduction of falsehoods, so there is no risk of jumping to ontological conclusions in this case.[96] In making abstractions, on the other hand, the focus is on arriving at essential traits of the objects studied. The realistically inclined mathematician might think he has analysed some existing object, as when numbers are explicated as properties of classes. The non-realist mathematician can make the same abstraction, without taking the abstract objects to have independent existence.

Still another way of looking at the difference between idealization and abstraction is that, when already developed mathematical theories are used in applications, we try to idealize (simplify) the physical situation so that it fits the mathematics. In a first approximation we think of frictionless planes, non-elastic collisions, etc. These appoximations are often taken as limits of existing states. We can imagine surfaces of less and less degree of friction, and ideally we can imagine frictionless surfaces. But, as was noted above, Aristotle does not regard mathematical planes as limits of physical bodies. The case with abstraction, at least when trying to find new concepts, is different. Here we try to see what is essential in the given problem situation, or to see patterns in structures that can be developed into fruitful mathematics, as when Newton discovered (defined, explicated) that velocity is a time derivative, or that force is proportional to the second derivative of position.

ITR argues for a provisional realism with the help of an indispensability argument. The fruitfully explicated object is thought to correspond to some

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[95] (Lewis, 1986), p. 27.

[96] One of Maddy's arguments against the indispensability argument is that the use of idealizations in applications of mathematics involves no ontological commitments. But this misses the point, since everybody knows, to mention one exemple Maddy discusses, that there are no infinitely deep oceans, so one would not use this type of examples in an indispensability argument to argue for the existence of infinities anyway. See (Maddy, 1992), and ITR.

kind of reality. But, as Wittgenstein puts it, denying as well as affirming that mathematical propositions are about a mathematical reality leads to peculiarities.[97]

5. Structuralism

Lear ends the paper mentioned earlier (section 3.1) with a discussion of how the truth of mathematics can be understood when mathematics in a way reflects the structure of the physical world. To oversimplify, the forms of Aristotle exist in the substances as non-separated, while the forms of Plato exist in a separated world of ideas. In both versions the forms give the world its structure, and both versions are versions of realism. ITR and CUE contain brief remarks on structuralism related to this issue. Here I will give an account of various structuralist ideas, ending with some remarks on how these ideas relate to conceptions formulated in the included papers.

Structuralism has developed in several directions and is an influential philosophy of mathematics in the twentieth century. In an authoritative survey Erick H. Reck and Michael P. Price analyse different versions of structuralism, without taking a definitive standpoint for or against specific versions.[98] I will use this paper as an appropriate starting-point.

5.1 General Remarks

Structuralism in recent philosophy of mathematics is thus not one single school of thought. Reck and Price identify three intuitive ideas behind structuralism.

> (1) that mathematics is primarily concerned with "the investigation of structures"; (2) that this involves an "abstraction from the nature of individual objects"; or even (3) that mathematical objects "have no more to them than can be expressed in terms

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[97](Wittgenstein, 1989), p. 141.

[98](Reck and Price, 2000)

of the basic relations of the structure".[99]

They then distinguish between at least four versions of structuralism, classified as follows:

- Formalist structuralism

- Relativist structuralism

- Absolutist structuralism, including modal variants

- Pattern structuralism

Before the essence of three of these versions of structuralism is sketched a word on the evolution of modern structuralism is in place. Reck and Price see one of the beginnings in the study of what has become known as abstract algebra.[100] Another source is the interest in the development of axiomatic systems such as Dedekind's and Peano's axiomatizations of arithmetic, and Hilbert's of geometry. Important steps in the development of structuralist ideas are also the development of axiomatic set theory, and Bourbaki's idea that set theory can form the basis of all of mathematics via "mother-stuctures", such as algebraic and topological structures.

Clearly, some mathematical theories, like abstract algebra and topology, study structures, but it does not follow that *all* mathematics do. Mathematicians also study rather specific and concrete problems. Saunders Mac Lane exemplifies with problems related to analytic functions (holomorphic functions), questions of numbers (whether e is transcendental e.g.), studies of special partial differential equations and their solutions, and argues that issues such as these can hardly be regarded as structural.[101]

Another observation is that while Bourbaki tries to define what a structure is via set theory, and states that mathematics is the study of structures, this is just a declaration. Bourbaki does not bother to treat mathematics structurally to any greater extent. According to Leo Corry, the informal

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[99](Reck and Price, 2000). The qoutations in the quotation are from Geoffrey Hellman (Hellman, 1989), and Charles Parsons (Parsons, 1990).

[100]See (Corry, 2004) on the development of abstract algebra, and (Maddy, 2008) on the development of pure mathematics.

[101](Mac Lane, 1996).

structuralist thesis has an ideological function in the writings of Bourbaki, but is not really used at all.[102]

Structuralist methodologies have grown along the historical lines sketched above. Philosophically, but not necessarily mathematically, these methodologies have certain semantical and metaphysical consequences that need to be analysed. Note also that structuralist positions are compatible with several different ontological standpoints. Structuralists often start from a second-order axiomatizations of arithmetic, a categorical theory, and so do Reck and Price, as well as Shapiro. But different kinds of axiom systems play different roles in mathematics. By axiom systems for arithmetic, mathematicians try to describe a definite structure, while e.g. the group axioms are definitional in character.[103] This choice of example as a starting-point in the analysis is of course important, but one should be aware of that this is not the only type of mathematical theory.

5.2 Relativist Structuralism

Initially Reck and Price make a distinction between formalist and more substantive versions of structuralism. *Formalist structuralism* denies that mathematical terms refer, and that mathematical propositions have truth-values. Metaphysical questions are dispensed with. From the perspective of this thesis these ideas are not so relevant. The more substantive versions, the thicker ones to speak with Reck and Price, answer, however, questions about reference and truth in different ways.

According to *relativist structuralism*, the axioms of e.g. PA_2 are true in different models. The non-logical constants in the language of PA_2 refer to elements in these models. We can pick any model we want as long as we are consistent, i.e. stick to the chosen one. We could pick any other model, and the non-logical constants would have other referents. Since all models are isomorphic, any proposition has the same truth-value in each model. This idea works for categorical theories like PA_2, but not for, say, group theory. It is true of every group that the inverse element is unique, but we cannot

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[102]Boubaki's structuralist program is presented in the last chapter of (Bourbaki, 1968). See also (Corry, 1992) on Bourbaki's program.

[103]See also (Oliveri, 2007), pp. 122f.

study just one group to come to this conclusion. This must be proved from the group axioms. Since the group axioms are definitional in character, their roles is rather to exhibit similarities in *different* structures, and these axioms are best regarded as a unique explication of the required properties.[104]

5.3 Universalist Structuralism

So, along relativist structuralist lines referents of terms are relative to the chosen model; the reference is not fixed. If φ is a sentence of arithmetic, formulated in the language of PA_2 with non-logical symbols 0 for zero and s for successor, then in φ, 0 refers to different elements in the different models. The idea with *universalist structuralism* is to let 0 and s refer to their interpretations in every model at the same time. The non-logical constant N, with the intended interpretation of $N(x)$ as "x is a natural number", is introduced, and the axioms of PA_2 are relativized to $N(x)$. The conjunction of these axioms is written $PA_2(0, s, N)$. An arithmetical sentence φ, translated so that it only contains the non-logical constants 0, s, and N, is denoted $\varphi(0, s, N)$. Starting from

$$PA_2(0, s, N) \rightarrow \varphi(0, s, N),$$

the non-logical constants are quantified out to get

$$\forall x \forall f \forall X (PA_2(x, f, X) \rightarrow \varphi(x, f, X)).$$

This sentence, call it ψ, thus abstracts away from what is peculiar about special models, and is about *all* relevant objects. According to this analysis a mathematician really means ψ, when he states φ, and Reck and Price suggest that ψ is to be regarded, not as an analysis, but as an explication of φ.

One common problem for universalist and relativist structuralism, is what the range of the variables is supposed to be. An axiom of infinity seems to be needed not to get into the situation where sentences are vacuously true since $PA_2(0, s, N)$ would be false. Other ways to try solve this problem is to modalize and to quantify over possible objects, or to add a necessity operator

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[104]Compare this with axiomatizations of set theory. In this case it is not at all clear which structure we ought to aim at, and different axiom systems can be seen as *different ways* to explicate the set concept. See ITR and CUE.

to ψ. This last version is Geoffrey Hellman's suggestion, but it leads to the same vacuity problems according to Reck and Price.[105]

Both relativist and universalist structuralism are consistent with versions of eliminativism. This means that it is possible to appeal to space-time points or quasi-abstract objects like strokes, as well as to use set theory and an ontology associated with sets. What then about semantics and ontology so far? Formalist structuralism is deflationary concerning both semantics and ontology. According to relativist structuralism mathematical terms refer to their interpretations in chosen models, while, according to universalist structuralism terms refer to their interpretation in all models at once. Both are compatible with ontologies ranging from set-theoretic Platonism to physical nominalism.

Reck and Price distinguish *particular* and *universal structures*. A particular structure may be e.g. the cyclical group of order four, while a universal structure is what particular structures share, e.g. the group properties as described in the group axioms. In relativist structuralism the structure is more of a particular, and in absolute structuralism more of a universal; hence the terminology.

5.4 Pattern Structuralism

To get a grasp of *pattern structuralism*, imagine different models of PA_2 laying, as it were, alongside each other. Since all models are isomorphic there will be a total correspondence between objects. The pattern that all these models share is fundamental in this form of structuralism.[106] The patterns are fundamental, not the 'points', 'positions', or 'roles' that make up the patterns. An element of a model of PA_2 can be said to occupy a 'position' or to play a 'role'. Numbers, on this view, are not objects at all and Shapiro talks of the positions or places in either of two ways; "places-as-offices", meaning that the slots could be filled by objects, and "places-as-objects", where we treat the positions themselves as objects. For arithmetic, there is a unique entity, the natural number pattern, and then reference can be explained; the symbol 0

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[105] (Hellman, 1989).

[106] Essential contributions are made by e.g. Stewart Shapiro and Michael D. Resnik in (Shapiro, 1997) and (Resnik, 1997).

refers to the first position in the pattern, etc. Truth can be understood along the same lines in the usual Tarskian way. According to Reck and Price there are three basic aspects to these patterns. A pattern has many instantiations or exemplifications, it has an internal composition, and there is nothing more to the positions than that they are just positions in a pattern.

Pattern structuralism is not one single idea. Shapiro distinguishes between *ante rem* structuralism, where the objects in the places-are-objects are independent of physical structures and which is not eliminative, and *in re* structuralism, which he considers as an eliminative, realist variant. According to Reck and Price, pattern structuralism is, however, not eliminativist at all since it is not trying to get rid of abstract objects, because the patterns are in themselves abstract. Shapiro's terminology has its origin in discussions of the ontological status of universals, where the universals are *ante rem* in Platonistic variants, and *in re* in Aristotelian ones.[107] He writes:

> A structure is like a universal, a one over many of sorts, and we conceive of mathematical structures as freestanding and *ante rem*.[108]

One reason for Shapiro to prefer *ante rem* structuralism is that eliminative, *in re* structuralism demands a robust background ontology to fill the places in the places-as-offices positions.[109] The *in re* structuralist is in need of an infinite number of objects when dealing with e.g. arithmetic, to avoid the vacuity problem discussed above. Another solution, also presented above, is to go modal and presuppose infinitely many *possible* objects. In contrast to these options, Shapiro

> avoids the eliminative program altogether and adopts an *ante rem* realism towards structures. Structures exist whether they are exemplified in a nonstructural realm or not. On this option, statements in the places-are-object mode are taken literally, at face value.[110]

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[107] See also (Oliveri, 2007), p. 119.

[108] (Shapiro, 1997), p. 132.

[109] *Ibid.*, p. 86.

[110] *Ibid.*, p. 89.

Nevertheless, the options discussed by Shapiro are "definitionally equivalent, and neither is to be preferred to any other on ontological/ideological grounds".[111] He maintains that his *ante rem* structuralism fits mathematical practice better, but there is thus no fact of the matter in choosing between *in re* and *ante rem* structuralism.

There are of course problems with these ideas too. What does it mean that two patterns are identical? Resnick and Shapiro come up with different strategies to meet this problem. The idea of placing models of PA_2 alongside each other might work, but the picture is hard to use discussing groups, i.e. models of the group axioms. The idea might work for particular structures, but not for universal ones using the above terminology. It fits second-order arithmetic well, but the proposals made by Reck, Price, Shapiro, *et al* are not at all obvious, perhaps not even reasonable, in other mathematical contexts; a point they also recognize.

According to Reck and Price, spokesmen for formalist or relativist structuralism are seldom seen in debates on the philosohy of mathematics, although relativist structuralism seems to be rather common among mathematicians. As mentioned above modalized structuralism is proposed by Hellman, and pattern structuralism by Resnik and Shapiro among others.

5.5 Structuralism in the Thesis

ITR uses the idea of concept formation in mathematics sketched above. It contains a rather detailed discussion of in what way mathematical theories are tested, and in what way it can reasonably be said that a mathematical theory has an empirical support. There is in addition a case for realism via an indispensability argument. Since several mathematical concepts have an origin in a physical reality, revealing structures or patterns via abstractions mediated by explications, the position in the paper is a version of pattern structuralism. In Shapiro's terminology, it is best regarded as a version of *in re* structuralism, since it, in an Aristotelian way, argues that mathematical patterns inhere in physical reality.[112] The place-holders in the offices are

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[111] *Ibid.*, p. 242.

[112] *Cf.* the last quotation in section 3.3.

not physical objects, but explicated entities, sets in model-theoretic versions, or objects isolated in thought. To avoid vacuity problems one needs an infinite number of objects, at least potentially, but this is not a very strong or controversial assumption.

An advantage of speaking of *in re* structuralism in this way, is that it is then possible to have an embryo of an epistemology. We cannot perceive abstract objects, but we can recognize patterns and properties among physical objects.[113],[114]

6. Summaries of the papers

6.1 Measuring the Power of an Arithmetical Theory

This thesis for the licentiate degree centers around how to measure the power of an arithmetical theory. It consists of three parts; an introduction, and two separately written sections where the first contains generalizations of Chaitin's incompleteness theorem, and the second the construction of a partial measure of the power of arithmetical theories.

In the introduction three natural conditions such a measure ought to fulfil are formulated. It is shown that these cannot be simultaneously satisfied, and discussed how they can be modified. There is also an account of Michiel van Lambalgen's and Panu Raatikainen's criticism of Chaitin's extra-logical interpretation of his own incompleteness theorem. Finally, there are some remarks on the applicability of mathematics and logic, and a brief discussion on the use and misuse of results in logic, especially Gödel's incompleteness theorems.

In the second section two versions of Chaitin's incompleteness theorem are described and generalized. Both versions use the Kolmogorov complexity of a string defined as e.g.

$$K(s) = \min\{l(p) : U(p){\downarrow} = s\},$$

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[113] See (Shapiro, 1997), part II:4, and (Jenkins, 2005) on this issue.

[114] In CUE it is stated without arguments that mathematics is about structures. The same idea of concept formation is used, and the same version of structuralism is implicit in CUE.

i.e. the length of the shortest 'program' p which causes the universal Turing machine U to stop with output s. One version of Chaitin's incompleteness theorem then reads:

Theorem *There is a constant c such that for every formal program p, if $U(p)$ generates a sentence of the form $K(s) > n$ only if it is true, and $U(p)$ outputs $K(s) > n$, then $n < l(p) + c$.*

In the theorem $U(p)$ can be regarded as a formal system, and the content of the theorem can then be stated: If $n \geq l(p) + c$, then the formal system cannot prove that $K(s) > n$.

The generalization consists in using a more general function than l in the above definition of Kolmogorov complexity, and a more general pairing function than Chaitin uses in his proof of the theorem. It then follows that the constant c also depends on the choice of these functions, and this is a strengthening of the criticism by van Lambalgen and Raatikainen of Chaitin's purported interpretation of the constant c as depending on the associated theory only.

The other version of Chaitin's incompleteness theorem is a consequence of the set $B = \{x : K(x) \leq f(x)\}$ being simple for suitable choices of f. Also in this case the generalization consists in using a more general function than l in the definition of $K(s)$ thus giving rise to different simple sets B.

In the third section, the main part of the thesis, a partial measure of the power of extensions of Peano Arithmetic is defined. In order to do this, a fragment of the modal logic GL, the letterless modal sentences, and its Lindenbaum algebra are used. A probability-like measure on certain finite parts of this fragment is defined, and numbers are thus assigned to the equivalence classes of this part of the Lindenbaum algebra of GL, or rather to representatives of the equivalence classes. Via a translation this structure is uniquely embedded into the Lindenbaum algebra of Peano Arithmetic. The translation of a letterless sentence is called a constant sentence. Using the same measure a measure on the equivalence classes of the corresponding finite fragment of the Lindenbaum algebra of Peano Arithmetic is generated. This measure can then be extended to certain non-constant sentences. Finally, the measure is extended to some extensions of Peano Arithmetic of the type $PA + \Phi$, where Φ is a set of constant sentences.

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There is also a discussion of problems that must be dealt with if one tries to extend our measure to a larger class of extensions.

One conclusion of this part, then, is that it is possible to design a partial measure m on certain extensions of PA. The constructed measure takes rational numbers as values, and the measure is such that $m(S) \leq m(T)$, if $S \dashv T$.

6.2 On Explicating the Concept *The Power of an Arithmetical Theory*

This paper can be seen as a continuation of MPAT, but the perspective is different. In focus here is the possibility of formulating a quantitative explication of the concept *The Power of an Arithmetical Theory*. The paper begins with a clarification of the explicandum similar to the one in MPAT1. The best we can arrive at, if m is a measure on extensions T_i of Peano Arithmetic, is that $m(T_1) \leq m(T_2)$, if $T_1 \dashv T_2$, m is a partial recursive function, and the range of m is some dense subset of $\mathbf{R}$.

Chaitin has suggested four ways to construct a measure: using Kolmogorov complexity, the Halting probability, simple sets, and ordinal numbers, respectively. Concerning the first two suggestions, van Lambalgen and Raatikainen showed the impossibility of using these. As to the other two suggestions, it is shown that these too are impossible to use in the construction of a measure.

There is, finally, a review of the measure constructed in MPAT3. One difference between this measure and the others is that it takes a 'bird's eye view' on extensions of Peano Arithmetic, and considers the theories from the 'outside' via the Lindenbaum algebra, while Chaitin looks at them from the 'inside', and suggests measures that depend on what is provable in the respective theories. It is indicated how the measure could be extended to a vector-valued one in order to cope with a larger fragment of the Lindenbaum algebra of extensions of Peano Arithmetic.

The conclusion of the paper is that it seems impossible to construct an explication of *The Power of an Arithmetical Theory*. The argument is pragmatical; none of the suggested measures, with the possible exception of the

one constructed in MPAT3, a measure that also has some shortcomings, fulfil the natural conditions laid down in the introduction of the paper.

6.3 A Note on the Relation Between Formal and Informal Proof

While the result in EPAT is essentially negative, rejecting the possibility of explicating *The Power of an Arithmetical Theory*, this paper presents a positive result formulating an explication of *informal (mathematical) proof*, proof as used in mathematics.

The paper starts with presenting Carnap's idea of *explication* as a means to construct more exact concepts. There follow several examples of explications from mathematics as well as from empirical sciences. These examples illustrate different ways explications are used in mathematics and science to arrive at fruitful concepts, thus suggesting that this is *the* way to form concepts in these contexts. Most of the examples are briefly described in Section 2.3 of this introduction.

The main thesis of the paper is that *formal proof in first order logic* is an explicatum for *informal proof*. To be able to prove results concerning informal, mathematical proofs an exact counterpart is needed and this explication is the most fruitful one. The relation between formal and informal proof is a widely discussed one, and the suggestion in the paper is meant as part of an answer to this question. There are also some comments on whether weaker versions such as constructive proof, or stronger ones such as proof in second-order logic, are to be preferred as explicata for informal proof, with the conclusion that proof as analysed in first-order logic is preferred, since this makes metamathematics a fruitful part of logic or really of mathematics.

6.4 Indispensability, The Testing of Mathematical Theories, and Provisional Realism

In this paper the idea of concept formation in mathematics as abstractions from vague, intuitive empirical concepts formulated via explications is put on trial in a more philosophical setting. I argue for a provisional realism

concerning the existence of mathematical objects based on a strengthened version of the indispensability argument.

By way of introduction an account of the indispensability argument is given, and its weak point is identified as the use of the thesis of confirmational holism. This problem is in turn due to an erroneous conception of how mathematical theories are tested, how they are related to an empirical 'reality'. To analyse this, three different kinds of examples are discussed. It is argued that when the mathematical theories or concepts are representational in character, they are intimately connected with empirical theories. In this case the mathematical theories are tested together with the empirical ones.

So-called quasi-empirical views of mathematics are also discussed, and it is argued that they do not provide an empirical ground for mathematics. In this connection the possible empirical character of computer-assisted proofs and numerical examples is discussed, but in neither of these cases anything empirical is involved.

Then, the above-mentioned model of concept formation in mathematics is described and illustrated with some examples. An empirical and a deductive component are identified and I illustrate how the deductive component is logically tested by the development of set theory, and different explications of the set concept. The empirical component is connected to fruitfulness, and tested together with empirical theories in applications as when e.g. velocity is explicated as a time derivative

So far no ontological commitments are made. One feature in this process of concept formation is that it seems that mature, mathematical concepts are robust; they are uniquely explicable, as illustrated with the concept effectively computable function. These robust concepts are regarded as 'mathematical kinds'. If these concepts, or rather the theories involving them, are confirmed in test situations together with empirical theories, claims of existence concerning these concepts are confirmed.

The strengthening of the indispensability argument is thus a weakening of the use of confirmational holism. It is just the empirical parts of mathematics that are tested together with empirical theories, but precisely these parts are of interest for making ontological claims.

6.5 Mathematical Concepts as Unique Explications

In this note, jointly written with Christian Bennet, a characterization of mathematics in terms of the tools discussed in the previous papers is made. It is suggested that a concept is mathematical if and only if it is uniquely explicable, or has emerged within the study of such concepts. Mathematical concepts are compared with concepts from empirical sciences. The latter change with the development of the theories in which they are used, while mature mathematical concepts normally have received a stable, unique explication. To support the claim a number of examples are discussed.

There are also some comments on educational issues related to this characterization of mathematics.

The suggested explication of mathematics is that it is the study of uniquely explicable concepts. In this way mathematics is analytic, but it is also synthetic since these explicata have an empirical origin.

There are also some comments on educational issues related to this characterization of mathematics.

7. Future Work - Some Ideas

The ideas in this thesis can be developed and extended in several ways, and below are some suggestions for further work.

- According to the indispensability argument, metaphysical assumptions in mathematics get empirical support when physical theories, using the mathematics, are verified.[115] In the sharpening of the indispensability argument, an empirical and a logical component in the concept formation process in mathematics were identified. This study of the empirical component in mathematics could be continued, and possibly cast some light on the distinction between the synthetic and the analytic.

- Closely related to the suggestion above is the problem of the applicability of mathematics.[116] The ideas in the thesis provide means to analyse this problem further.

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[115] See e.g. Colyvan's contribution (Colyvan, 2007) in (Leng et al., 2007).

[116] This is a problem that e.g. Steiner has studied in great detail. See (Steiner, 1998), and (Steiner, 2005).

- Concept formation in mathematics is related to concept aquisition in mathematics education. It could be fruitful to investigate if and how the ideas of concept formation in this thesis can help to understand the process of concept aquisition by students of mathematics. This also involves investigating the relevance of the history of mathematics in mathematics education.

- To further support the idea of concept formation in mathematics in the thesis, a thoroughgoing historical study of a central, concept formation process in mathematics, e.g. the concept *area*, would be both desirable and valuable.

- Another possible field of study is the mathematization of science; e.g. biology. What is needed for a fruitful mathematization? Are concepts, well suited to describe physical phenomena, well suited for the study of biological phenomena, or is it necessary to try to develop new mathematical concepts? Are biological phenomena too complex to make the relevant abstractions and necessary idealizations?

- In the thesis an epistemology of mathematics is missing. The ideas in the thesis could be relevant in understanding the context of discovery, but hardly in the context of justification (to speak in terms of philosophy of science).[117]

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[117]One idea to follow up may be Jenkin's idea of *concept grounding* as presented in (Jenkins, 2005), and see also e.g. (Leng et al., 2007).

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