OMITTING TYPES AND MODEL THEORY

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1. Introduction

This essay is an attempt to create some model theory specific for the context of a type being omitted. Inspiration is sought mainly from three classical model theoretic theorems and from the infinitary logic $\mathcal{L}_{\omega_1\omega}$. The classical theorems are the compactness theorem, the joint consistency theorem and the theorem of interpolation. Each theorem is transformed, mainly through the substitution of consistency with $p\uparrow$-consistency (pronounciated p-consistency), into one or several properties applicable on types. The scope of these properties is then being examined to some extent. The project is mainly inspired by the work of Fredrik Engström in [1] and especially [2].

The theorems 2, 3, 6, 11 and 13 can all be considered classical results, that have only been more or less redrafted to fit the present setting and notation. Theorems 5, 8, 10 and 12 are, to the authors knowledge, new.

None of the definitions are to be understood in any way as conventional notation. With the exeption of $p\uparrow$-consistency (definition 1), which is borrowed from Engström, all explicit, enumerated definitions are made up specifically for this essay.

2. Preliminaries

2.1. **Theories.** A theory, here generally denoted T, is a set of sentences, called axioms. The theory is consistent iff no contradictions can be derived from these axioms. A theorem of T is a sentences derivable from the axioms of T. The set of all theorems of T is written $\mathrm{Th}(T)$. A theory T is complete iff for every sentence φ expressible in the language, either φ or $\neg\varphi$ is a theorem of T.

2.2. **Types.** A type is a set of open formulas with a certain arity. In many contexts only interesting, i.e. infinite, non-contradictory sets of formulas are considered to be types. Others are only concerned with complete types. We will make no such inscrimination, even though our theorems will only concern countable types. We will generally use the letter p to denote types.

A type p is *realized* in a model if there is a tuple of elements satisfying every formula of p. If it is not realized it is *omitted.* We let $\mathcal{M} \models p\downarrow$ mean that p is realized in $\mathcal{M}$ and $\mathcal{M} \models p\uparrow$ that it is omitted.

A set of formulas p is a type *over* a theory T if T has a model that realizes p. This is equivalent with $T \vdash \forall\bar{x}\neg\Delta(\bar{x})$ for no finite conjunction $\Delta(\bar{x})$ of formulas of p. A type p over T is *isolated* in T if there exists a formula $\varphi(\bar{x})$ such that $\exists\bar{x}\varphi(\bar{x})$ is consistent with T and $T \vdash \forall\bar{x}(\varphi(\bar{x}) \to \delta(\bar{x}))$ for every $\delta(\bar{x})$ of p. If $\varphi(\bar{x})$ isolates p in T and $T \vdash \exists\bar{x}\varphi(\bar{x})$, then p is *strongly isolated* in T. If p is not isolated in T, then it is a *limit* in T.

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3. Expressibility

Using types, we can express things that are inexpressible using only finite first order sentences. Whether or not any element satisfies a certain infinite set of formulas can make a big difference. For example, in first order arithmetic, PA, the property that every element is finite is inexpressible. If $p(x) = \{(x > i) : i \in \omega\}$, then for every model $\mathcal{M} \models \text{PA}$, $\mathcal{M} \models p\upharpoonright$ expresses this very property.

Types are however not the only way of doing this. Allowing countably infinite conjunctions or disjunctions but only finite quantification gives at least the same expressibility. Such limited infinitary logic is called $\mathcal{L}_{\omega_1\omega}$. If $p = \{\delta_i : i \in \omega\}$, it is easy to see how $p\upharpoonright$ is expressed in $\mathcal{L}_{\omega_1\omega}$, simply by

$$\neg\exists\bar{x}\big(\bigwedge_{i\in\omega}\delta_i(\bar{x})\big)$$

Expressing arbitrary $\mathcal{L}_{\omega_1\omega}$-sentences using types is more difficult and generally involves infinitely many types of an expanded language being omitted. We will see more about how this is done in section 5.

4. Consistency

Definition 1. A theory T is $p\upharpoonright$-*consistent* iff it has a model that omits p.

Theorem 2 (Omitting types theorem, Henkin-Orey)**.** *If p is a limit in T then T is $p\upharpoonright$-consistent.*

Proof. We will prove this theorem under the condition that everything is countable. To keep everything simple we will also assume that p is unary, even though this is not actually nessessary for the proof to go through. Let L be the language of the type and the theory. We will create an expansion $T' \supset T$ in $L' = L \cup \{c_1, c_2, \ldots\}$, where $\{c_1, c_2, \ldots\}$ is an enumeration of new individual constants. We will then construct a so called Henkin model $\mathcal{M} \models T'$ whose L-reduct will be a model of T that omits p. To do this, we need T' to be complete, consistent and such that every existential sentence is witnessed by an individual constant.

We will let T' be the union of an enumeration of theories $T_0, T_1, \ldots$ created in the following manner. $T_0 = T$. Let $\{\phi_1, \phi_2, \ldots\}$ be an enumeration of all sentences expressible in L' and suppose we have already defined T_n. As will soon be seen, we only add a finite number of sentences to every new theory, so T_n will be the union of T and a finite set of sentences $\{\gamma_1, \ldots, \gamma_k\}$. We can assume that no x_is occur in the axioms of T_0. If they do occur, we start by replacing them with y_is. We now let $\gamma_1'(x_1, \ldots, x_m), \ldots, \gamma_k'(x_1, \ldots, x_m)$ be formulas identical with the sentences that have been added to T_n, but with all new constants replaced by individual variables x with the same index, so that c_1 is replaced with x_1, c_2 with x_2 and so forth.

Let $\gamma(x_n) = \exists x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_m(\gamma_1'(x_1, \ldots, x_m) \wedge \ldots \wedge \gamma_k'(x_1, \ldots, x_m))$.

$\exists x_n\gamma(x_n)$ is consistent with T and since p is a limit in T, there will be some $\delta(x) \in p$ such that $\exists x(\gamma(x) \wedge \neg\delta(x))$ is consistent with T. Create T_{n+1} by adding to T_n

i) $\neg\delta(c_n)$

ii) ϕ_n, if $T_n + \neg\delta(c_n) + \phi_n$ is consistent, $\neg\phi_n$ otherwise.

iii) $\psi(c_i)$, if $\phi_n = \exists x\psi(x)$ and ϕ_n was added to T_{n+1}, where c_i is the individual constant with the least index not ocurring in the sentences of T_n or in ϕ_n.

T' is complete, consistent and every existential sentence is witnessed by some c. We can therefore create a model $\mathcal{M} \models T' = \cup_{n\in\omega} T_n$ by letting equivalence classes under identity of the individual constants themselves form a domain. No constant a of L can realize p, since the formula $(x = a)$ would then isolate p in T. No new constant c can realize p, because of the way T' was constructed, so $\mathcal{M}$, as well as its L-reduct, will omit p. □

4.1. **Compactness.**

Theorem 3 (Compactness theorem)**.** *A theory is consistent iff every finite subset of the theory is consistent.*

We will not prove this theorem here, but it is actually rather intuitive. If a theory is contradictory, then the contradiction is derivable from a finite number of theorems.

Definition 4. We say that a type p is *compact* if it is true for every theory T that T is $p\uparrow$-consistent iff every finite subset of T is $p\uparrow$-consistent.

Clearly not all types are compact. Let the formulas $\delta_i(x)$ say that there are at least i different elements non-identical with x. Consider the type $p^{\text{inf}}(x) = \{\delta_i(x) : i \in \omega\}$ and the theory $T = \{\exists x \delta_i(x) : i \in \omega\}$. Every finite subset of T is consistent with omitting p^{inf}, but T is not.

Theorem 5. *A type p is compact iff there is a finite formula $\varphi(\bar{x})$ such that for every model $\mathcal{M}$, p is realized in $\mathcal{M}$ iff $\mathcal{M} \models \exists\bar{x}\varphi(\bar{x})$*

Proof. If p is realized in $\mathcal{M}$ iff $\mathcal{M} \models \exists\bar{x}\varphi(\bar{x})$, then any theory T is $p\uparrow$-consistent iff $\exists\bar{x}\varphi(\bar{x})$ is not a theorem of T. It is a theorem of T iff it is a theorem of a finite subset of T, so p is compact.

For the converse we suppose there is no finite formula $\varphi(\bar{x})$ such that $\mathcal{M} \models p\downarrow$ iff $\mathcal{M} \models \exists\bar{x}\varphi(\bar{x})$ and show that p is not compact by presenting a theory T such that T is not $p\uparrow$-consistent even though every finite subset of T is.

Suppose $p = \{\delta_i : i \in \omega\}$ is an L-type. We then consider the $L \cup \{\bar{a}\}$-theory $T = \{\delta_i(\bar{a}) : i \in \omega\}$. T is clearly not $p\uparrow$-consistent. We now suppose that T has a finite subtheory Γ that is not $p\uparrow$-consistent and derive a contradiction.

As every axiom of T is a formula $\delta_i(\bar{a})$ with $\delta_i \in p$, so is every axiom of Γ. We let $\varphi(\bar{x})$ be the conjunction of the δ_is from p that occur in Γ. Since a does not occur in p and Γ is finite, $\exists\bar{x}\varphi(\bar{x})$ is logically equivalent with Γ. If $\Gamma \models p\downarrow$, then $\exists\bar{x}\varphi(\bar{x}) \models p\downarrow$ and since φ is a conjunction of formulas from p, $p\downarrow \models \exists\bar{x}\varphi(\bar{x})$. This contradicts our initial supposition. □

This theorem effectively makes compact types uninteresting. If a property is expressible with a single formula, then there is really no reason to involve the unneccessarily complex notion of a type.

4.2. **Joint Consistency.**

Theorem 6 (Joint Consistency Theorem, Robinson)**.** *Let T be a complete theory in the language L. Suppose T_1 and T_2 are consistent extensions of T, in the languages L_1 and L_2 respectively, such that $L_1 \cap L_2 = L$. Then $T_1 \cup T_2$ is consistent.*

This theorem can be used to prove consistency for complex theories by proving consistency for their subtheories in the obvious way. Also $\mathcal{L}_{\omega_1\omega}$ has a joint consistency theorem, but with the premiss that the joint theory (T) is complete for $\mathcal{L}_{\omega_1\omega}$[3, p.281s].

A similair theorem for $p{\uparrow}$-consistency could be useful. We shall therefore define an analogue property for types and see if we can find some criteria determining whether a certain type has it or not. The property would be most useful if it would prove to be universal, that is if every type would have it.

Definition 7. We say that an L-type p has the *Joint Consistency Property (JCP)*, if the following is fulfilled. If T is a complete $p{\uparrow}$-consistent L-theory and T_1 and T_2 are $p{\uparrow}$-consistent extensions of T, in the languages L_1 and L_2, such that $L_1 \cap L_2 = L$, then $T_1 \cup T_2$ is $p{\uparrow}$-consistent.

The type p^{inf} from above has the JCP, simply because it can never fulfill the premisses. p^{inf} is not a type over any complete $p^{\text{inf}}{\uparrow}$-consistent theories. For any complete theory T, either

$$T \vdash \exists x \delta_i(x)$$

for all $\delta_i \in p^{\text{inf}}$, in which case every element of every model of T realises p^{inf}, or

$$T \vdash \neg\exists x \delta_i(x)$$

for some $i \in \omega$, in which case p^{inf} is not a type over T, according to our definition.

This is of course not the way we intend for types to have the JCP. We would want them to have the property in a non-trivial way, the way that is described in the definition. It remains an open question throughout this essay, whether or not any type actually has the JCP in the intended way. What is certain, however, is that the JCP is not a universal property.

Theorem 8. *Not every type has the JCP.*

Proof. We first define a set of formulas δ_i and γ_i in the language $L = \{D, G\}$, where D and G are binary predicate symbols, such that

$$\delta_i(x) = \exists y_0 \dots y_i \big(\bigwedge_{j \neq k} y_j \neq y_k \bigwedge_{k \leq i} D(x, y_k) \big)$$

and

$$\gamma_i(x) = \exists y_0 \dots y_i \big(\bigwedge_{j \neq k} y_j \neq y_k \bigwedge_{k \leq i} G(x, y_k) \big)$$

We then let

$$T_0 = \{\exists x \exists y (\delta_i(x) \wedge \gamma_i(y)) : i \in \omega\}$$

and

$$p(x, y) = \{\delta_i(x) \wedge \gamma_i(y) : i \in \omega\}$$

The axioms of T_0 say that every δ_i and γ_i is satisfied by some object, whereas p states that there are two specific objects satisfying every δ_i and γ_i, thereby being D– and G–related to infinitely many objects.

None of the axioms of T_0 contradict the formulas of p, so p is a type over T_0. To see that T_0 is $p{\uparrow}$-consistent, just consider the model $\mathcal{M}_0$ with $dom(\mathcal{M}_0) = \omega$ and let $\mathcal{M}_0 \models D(c, d)$ iff $d < c$ and $\mathcal{M}_0 \models G(c, d)$ iff $d < c$.

We do however need a complete theory and will therefore create $\mathcal{M} \models T_0 + p{\uparrow}$ in order to let $T = \text{Th}(\mathcal{M})$. The model $\mathcal{M}_0$ is not very well suited for this purpose, because $\mathcal{M}_0 \models D(x, y) \leftrightarrow G(x, y)$ and because it makes D and G discrete orders with maximal elements. This means that the place an element has in the order D can be used to determine which place it has in G, making it impossible for us to reorder one predicate without affecting the other. We will therefore separate the domains of the predicates, expanding T_0 to

$$T_0' = T_0 \cup \{\forall x\big((\exists y(D(x,y) \vee D(y,x))) \leftrightarrow \neg(\exists y(G(x,y) \vee G(y,x)))\big)\}$$

To ensure preserved $p\!\uparrow$-consistency, consider the model $\mathcal{M}$ with $\mathrm{dom}(\mathcal{M}) = \{c_i : i \in \omega\} \cup \{d_i : i \in \omega\}$ such that $D(x,y)$ iff $x = d_i$, $y = d_j$ and $i > j$ and $G(x,y)$ iff $x = c_i$, $y = c_j$ and $i > j$.

Since $\mathcal{M} \models T_0' \cup p\!\uparrow$, it suits our purposes and we let $T = \mathrm{Th}(\mathcal{M})$. We then expand T into two new theories T_1 and T_2. Let $L_1 = L \cup \{a\}$, $T_1 = T \cup \{\delta_i(a) : i \in \omega\}$, $L_2 = L \cup \{b\}$ and $T_2 = T \cup \{\gamma_i(b) : i \in \omega\}$.

We let $\mathcal{N} \models \mathrm{PA}$ be non standard and construct models of $T_1 \cup p\!\uparrow$ and $T_2 \cup p\!\uparrow$.

Let $\mathrm{dom}(\mathcal{A}) = \{\mathrm{dom}(\mathcal{N}) \cup \{c_i : i \in \omega\}\}$, interprete a as any non standard element of $\mathcal{N}$ and let $D(x,y)$ iff $\mathcal{N} \models x > y$ and $G(x,y)$ iff $x = c_i$, $y = c_j$ and $i > j$. $\mathcal{A} \models T_1 \cup p\!\uparrow$.

We create $\mathcal{B}$ in basically the same way, but let the domain of G be non-standard and let the domain of D be as in $\mathcal{M}$.

In $T_1 \cup T_2$, p is strongly isolated by the formula $x = a \wedge y = b$, so this theory is not $p\!\uparrow$-consistent. □

If JCP would have been a universal property among types, we could have used it to build $p\!\uparrow$-consistent extensions of theories for all types p. As this was not the case, we could try to define a weaker property, either by strengthening the premisses in the definition or by weakening the conclusion.

Demanding that p should be non-isolated in T_1 and in T_2 is not enough, since it is non-isolated already in this counterexample. To see this we can regard p as a composition of two types, $q = \{\delta_i(x) : i \in \omega\}$ and $r = \{\gamma_i(x) : i \in \omega\}$, so that (c_1, c_2) realizes p iff c_1 realizes q and c_2 realizes r. $T_1 = T \cup \{\delta_i(a) : i \in \omega\}$ implies $T_1 \vdash \neg\gamma_i(a)$ for every $\gamma_i \in r$. There are no further additions to T_1 so r will not be isolated in T_1 unless it is isolated in T. The same is true for q and T_2. In a complete theory, a type is either strongly isolated or a limit. Both q and r can therefore not be isolated in T, as both types, and thereby p, would be strongly isolated in T.

Another possible addition to the premisses could be that both extensions should be complete, but we can easily replace T_1 with $\mathrm{Th}(\mathcal{A})$ and T_2 with $\mathrm{Th}(\mathcal{B})$ in the counterexample, making it an exception from such a property as well.

We could weaken the conclusion by stating that T' might not be $p\!\uparrow$-consistent, but at least p is not strongly isolated in T', but again, the counterexample is already at hand.

4.3. **Joint isolation.** There is however a certain dividedness over our counterexample, which we can describe schematically as $T_1 \rightarrow A$ and $T_2 \rightarrow (A \rightarrow p\!\downarrow)$. We shall formally define this phenomenon as *joint isolation* and prove that, in the setting above, it is equivalent with p being isolated in $T_1 \cup T_2$ by a conjunction, with one conjunct from T_1 and one from T_2. There might not be any obvious way of using the resulting theorem in the construction of $p\!\uparrow$-consistent theories, but it still gives some characterisation within the field.

Definition 9. Let T_1 and T_2 be theories in L_1 and L_2, such that $L_1 \cap L_2 = L$ and let $p = \{\delta_i(\bar{x}) : i \in \omega\}$ be an n-ary L-type over both theories. If there are L-formulas $\theta_i(\bar{x})$, an L_1-formula $\varphi(\bar{x})$ and an L_2-formula $\psi(\bar{x})$ such that
$T_1 \vdash \forall \bar{x}(\varphi(\bar{x}) \to \theta_i(\bar{x}))$
and
$T_2 \vdash \forall \bar{x}(\psi(\bar{x}) \to (\theta_i(\bar{x}) \to \delta_i(\bar{x})))$
for all $i \in \omega$, then we say that p is *jointly isolated* in T_1 and T_2 by $\varphi(\bar{x})$ and $\psi(\bar{x})$.

Theorem 10. *Let T_1 and T_2 be consistent theories in L_1 and L_2 respectively, such that $L_1 \cap L_2 = L$. Let p be an n-ary L-type over both theories and let $\varphi(\bar{x})$ and $\psi(\bar{x})$ be n-ary formulas in L_1 and L_2 respectively.*

p is isolated in $T' = T_1 \cup T_2$ by $\varphi(\bar{x}) \wedge \psi(\bar{x})$ iff p is jointly isolated in T_1 and T_2 by $\varphi(\bar{x})$ and $\psi(\bar{x})$.

Proof. $\Rightarrow$

Suppose p is isolated in T' by the formula $\varphi(\bar{x}) \wedge \psi(\bar{x})$. This means that

$$T' \vdash \forall \bar{x}(\varphi(\bar{x}) \wedge \psi(\bar{x}) \to \delta_i(\bar{x}))$$

for every $\delta_i \in p$.

Since all proofs have finite sets of premisses and every theorem of T' is implied by a conjunction of T_1- and T_2-theorems, there must exist sentences $\mu_i \in T_1$ and $\nu_i \in T_2$ such that

$$\vdash \mu_i \wedge \nu_i \to \forall \bar{x}(\varphi(\bar{x}) \wedge \psi(\bar{x}) \to \delta_i(\bar{x}))$$

for every $\delta_i \in p$.

This can be rewritten as

$$\vdash \forall \bar{x}(\mu_i \wedge \varphi(\bar{x}) \to (\nu_i \wedge \psi(\bar{x}) \to \delta_i(\bar{x})))$$

We now add new constants $\bar{c}$ to get closed formulas

$$\vdash \mu_i \wedge \varphi(\bar{c}) \to (\nu_i \wedge \psi(\bar{c}) \to \delta_i(\bar{c}))$$

According to Craigs theorem of interpolation (Theorem 11), this implication will have an interpolant θ_i in $L \cup \{\bar{c}\}$ such that

$$\vdash \varphi(\bar{c}) \wedge \mu_i \to \theta_i$$

and

$$\vdash \theta_i \to (\psi(\bar{c}) \wedge \nu_i \to \delta_i(\bar{c})).$$

The new constants do not occur anywhere else, so we can replace them with universal quantification and get

$$\vdash \forall \bar{x}(\varphi(\bar{x}) \wedge \mu_i \to \theta_i(\bar{x}))$$

and

$$\vdash \forall \bar{x}(\theta_i(\bar{x}) \to (\psi(\bar{x}) \wedge \nu_i \to \delta_i(\bar{x}))).$$

Every μ_i is a theorem of T_1 and every ν_i is a theorem of T_2, so for every $i \in \omega$

$$T_1 \vdash \forall \bar{x}(\varphi(\bar{x}) \to \theta_i(\bar{x}))$$

and

$$T_2 \vdash \forall \bar{x}(\theta_i(\bar{x}) \wedge \psi(\bar{x}) \to \delta_i(\bar{x}))$$

$\Leftarrow$ Very straightforward from the definition of joint isolation

$T_1 \vdash \forall\bar{x}(\varphi(\bar{x}) \to \theta_i(\bar{x}))$ and $T_2 \vdash \forall\bar{x}(\theta_i(\bar{x}) \wedge \psi(\bar{x}) \to \delta_i(\bar{x}))$ imply $T_1 \cup T_2 \vdash \forall\bar{x}(\varphi(\bar{x}) \wedge \psi(\bar{x}) \to \delta_i(\bar{x}))$ □

5. Interpolation

Theorem 11 (Theorem of interpolation, Craig)**.** *This theorem states for formulas φ and ψ of the languages L_1 and L_2 respectively, that if $\vdash \varphi \to \psi$ then there is a formula θ in $L_1 \cap L_2$ such that $\vdash \varphi \to \theta$ and $\vdash \theta \to \psi$*

We have already seen the use one can have of this theorem.

Several analogues can be formulated within the realm of omitting types. We will prove some rather trivial theorems and discuss the difficulties involved in strengthening them.

Theorem 10 can actually be regarded as an interpolation theorem for isolation. The likeness with Craigs theorem becomes even bigger if we let both theories be empty but let φ and ψ be from different languages and p from the intersecting language. If

$$\vdash \forall\bar{x}(\varphi(\bar{x}) \wedge \psi(\bar{x}) \to \delta_i(\bar{x})) \text{ for every } \delta_i \in p$$

then for every $\delta_i \in p$ there is a θ_i from the intersecting language such that

$$\vdash \forall\bar{x}(\varphi(\bar{x}) \to \theta_i(\bar{x})$$

and

$$\vdash \forall\bar{x}(\theta_i(\bar{x}) \to \psi(\bar{x}) \to \delta_i(\bar{x})$$

$q = \{\theta_i : i \in \omega\}$ is thereby an interpolating type.

We could also investigate interpolation between finite sentences of different languages under the premiss that a certain type is omitted, the case when

$$\mathrm{Th}(p\uparrow) \vdash \varphi \to \psi.$$

If φ is an L_1-sentence and ψ an L_2-sentence, then $\mathrm{Th}(p\uparrow)$ will be a $L_1 \cup L_2$-theory and there is no obvious reason why an interpolant should exist. If however the L_1- or L_2-reduct of $\mathrm{Th}(p\uparrow)$ is enough to derive the implication from, then there will be an interpolating $L_1 \cap L_2$ sentence.

If $\mathrm{Th}(p\uparrow)|L_1 \vdash \varphi \to \psi$ then an L_1-sentence $\Gamma \in \mathrm{Th}(p\uparrow)$ exists such that

$$\vdash \Gamma \wedge \varphi \to \psi.$$

According to Craigs theorem of interpolation there is an $L_1 \cap L_2$-formula θ such that $\vdash \varphi \wedge \Gamma \to \theta$ and $\vdash \theta \to \psi$.

If $\mathrm{Th}(p\uparrow)|L_2 \vdash \varphi \to \psi$ then an L_2-sentence $\Gamma \in \mathrm{Th}(p\uparrow)$ exists such that

$$\vdash \varphi \to (\Gamma \to \psi).$$

There is then an $L_1 \cap L_2$-formula θ such that $\vdash \varphi \to \theta$ and $\vdash \theta \to (\Gamma \to \psi)$.

Either way, a θ exists in $L_1 \cap L_2$, such that $\mathrm{Th}(p\uparrow) \vdash \varphi \to \theta$ and $\mathrm{Th}(p\uparrow) \vdash \theta \to \psi$.

Finally, we could try to investigate interpolation between types of different languages. There is an interpolation theorem for $L_{\omega_1\omega}$, where the interpolant may be an infinite sentence (Lopez-Escobar). An analogue might be defined for types of different languages. If the types imply one another formula by formula, then defining the interpoling type is trivial using the same method that we used in the proof of theorem 10.

Theorem 12. *If* $p = \{\delta_i(\bar{x}) : i \in \omega\}$ *is a type in* L_1*,* $q = \{\gamma_i(\bar{x}) : i \in \omega\}$ *is a type in* L_2 *and for every* $\gamma_i \in q$ *there is a finite subset* Δ_i *of* p *such that* $\vdash \forall\bar{x}(\Delta_i(\bar{x}) \to \gamma_i(\bar{x}))$*, then there is a type* $r = \{\theta_i(\bar{x}) : i \in \omega\}$ *in* $L_1 \cap L_2$ *such that* $\vdash \forall\bar{x}(\Delta_i(\bar{x}) \to \theta_i(\bar{x}))$ *and* $\vdash \forall\bar{x}(\theta_i(\bar{x}) \to \gamma_i(\bar{x}))$ *for every* $i \in \omega$*.*

Note that the premiss here is that if a tuple realizes p, then it also realizes q. This implies $\models p\downarrow \to q\downarrow$, but the two are not equivalent.

Proof. We simply add new individual constants $\bar{c}$ to relize the types q and $p' = \{\Delta_i : i \in \omega\}$. We then apply the ordinary interpolation theorem for every $i \in \omega$. This gives us a set of interpolants in $L_1 \cap L_2 \cup \{\bar{c}\}$. We then replace the new constants $\bar{c}$ with $\bar{x}$ in the formulas to get the desired interpoling type r in $L_1 \cap L_2$. □

If omitting a type of one language merely implies omitting one of another language, then the task is more interesting. This is the case when, if $p = \{\delta_i(\bar{x}) : i \in \omega\}$ is a type in L_1 and $q = \{\gamma_i(\bar{x}) : i \in \omega\}$ is a type in L_2, p is omitted in every model that omits q. This is easily translated to $L_{\omega_1\omega}$, the resulting formula will be

$$\vdash \neg\exists\bar{x} \bigwedge_{i\in\omega} \gamma_i(\bar{x}) \to \neg\exists\bar{x} \bigwedge_{i\in\omega} \delta_i(\bar{x}).$$

As mentioned above, there will be an $L_{\omega_1\omega}$-interpolant θ in $L_1 \cap L_2$. The problem occurs when translating θ back to first order types. New symbols as well as an infinite amount of types might be neccessary to do this.

Theorem 13 (Chang)**.** *Given a sentence* φ *of* $\mathcal{L}_{\omega_1\omega}$ *in the countable language* L*, there is a countable* $L' \supseteq L$ *and a set* S *of first order types such that for every model* $\mathcal{M}$*,*

$$\mathcal{M} \models \varphi \text{ iff there is an expansion } \mathcal{M}' \text{ of } \mathcal{M} \text{ such that } \mathcal{M}' \text{ omits } S$$

Proof. This theorem is proven through the very construction of L' and S. L' will entail a new n-ary predicate symbol R_σ for every subformula σ of φ with n free variables. An L'-formula φ' is then defined as the conjunction of the following axioms.

(i) $\forall\bar{x}(R_\sigma(\bar{x}) \leftrightarrow \sigma(\bar{x}))$, if σ is atomic;
(ii) $\forall\bar{x}(R_\sigma(\bar{x}) \leftrightarrow \neg R_\psi(\bar{x}))$, if σ is $\neg\psi$;
(iii) $\forall\bar{x}(R_\sigma(\bar{x}) \leftrightarrow \bigwedge_{\xi<\kappa} R_{\psi\kappa}(\bar{x}))$, if σ is $\bigwedge_{\xi<\kappa} \psi_\xi$;
(iv) $\forall\bar{x}(R_\sigma(\bar{x}) \leftrightarrow \exists y R_\psi(\bar{x}, y))$, if σ is $\exists y\psi$;

These axioms presuppose that σ has free variables. If this is not the case, we still don't want the new predicates to be nullary, as we are going to use them to define a set of types. For closed formulas σ we therefore add to φ' the axiom $\forall x R_\sigma(x) \leftrightarrow \exists x R(x)$ and

(i') $\forall x(R_\sigma(x) \leftrightarrow \sigma)$, if σ is atomic;
(iv') $\forall x(R_\sigma(x) \leftrightarrow \exists y R_\psi(y))$, if σ is $\exists y\psi$.

We finally add to φ'

(v) $\forall x R_\psi(x)$.

This means that $\mathcal{M} \models \varphi$ iff there is an expansion $\mathcal{M}'$ of $\mathcal{M}$ to L' such that $\mathcal{M}' \models \varphi'$. We let φ' serve as a pedagogical bridge between φ and S. We shall construct S so that $\mathcal{N} \models \varphi'$ iff $\mathcal{N} \models S\uparrow$ for every model $\mathcal{N}$ of L'. We do this by transforming the axioms of φ' into types that will together compose S. For axioms of the form (i),(ii),(iv) and (v), the following one-formula types are added to S:

(i) $\{\neg(R_\sigma(\bar{x}) \leftrightarrow \sigma(\bar{x}))\}$;

(ii) $\{\neg(R_\sigma(\bar{x}) \leftrightarrow \neg R_\psi(\bar{x}))\}$;
(iv) $\{\neg(R_\sigma(\bar{x}) \leftrightarrow \exists y R_\psi(\bar{x}, y))\}$;
(v) $\{\neg R_\psi(x)\}$.
For every axiom of the form (iii) we add the one-formula types
$\{\neg(R_\sigma(\bar{x}) \to R_{\psi\kappa}(\bar{x}))\}$ for every $\xi < \kappa$ and the proper type
$\{R_{\psi\xi}(\bar{x}) \wedge \neg R_\sigma(\bar{x}) | \xi < \kappa\}$.
Thereby S is omitted whenever φ' is true. □

This means that if we apply the theorem on the $L_{\omega_1\omega}$-interpolant θ, it does not give us an interpolating type in $L_1 \cap L_2$, but a countable set of types that entails many new predicates but only the symbols of $L_1 \cup L_2$ that are also in $L_1 \cap L_2$.

This raises the question if such a countable set of types can be comprised into a single type. The method of simply conjoining S to a single type will not work. If $S = \{p_i : i \in \omega\}$ and $p_i = \{\delta_{ij} : j \in \omega\}$, then $p' = \{\delta_{ij} : i, j \in \omega\}$ is also a type, but as different tupels of elements may satisfy the formulas of different types, every p_i can be realized in a model that omits p'. In the special case where all but a finite amount of the p_is are nullary, we can construct p' so that it gets the same arity as the sum of the arities of the p_is.

6. Discussion and conclusion

The compact types have successfully been categorized. They do however not include any interesting types.

At least one type has been found that had the JCP, but only by not fulfilling the premisses. No type has been found that had the JCP in the way intended.

In theorem 10, a connection has been established between how a type is isolated in a compositional theory and its subtheories. The very interesting question of whether a type can be isolated in the compositional theory without being jointly isolated is left unanswered. A natural next step, if this project were to continue, would be to define a weaker JCP, where the type is demanded to be non jointly isolated in the extensions and to examine the scope of that property.

Several specific interpolation results have been derived from Craigs theorem of interpolation, but no new general theorem. Also the $\mathcal{L}_{\omega_1\omega}$-theorem of interpolation has been examined, but it turned out not to be easily manipulated into a general interpolation theorem for types. Still nothing contradicts that such a theorem may eventually be proven.

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