Morley's number of countable models

Rasmus Blanck

Department of Philosophy
University of Göteborg
2004

Morley's number of countable models*

Rasmus Blanck

Abstract

A theory formulated in a countable predicate calculus can have at most $2^{\aleph_0}$ nonisomorphic countable models. In 1961 R. L. Vaught [9] conjected that if such a theory has uncountably many countable models, then it has exactly $2^{\aleph_0}$ countable models. This would of course follow immediately if one assumed the continuum hypothesis to be true. Almost ten years later, M. Morley [5] proved that if a countable theory has strictly more than $\aleph_1$ countable models, then it has $2^{\aleph_0}$ countable models.

This leaves us with the possibility that a theory has exactly $\aleph_1$, but not $2^{\aleph_0}$ countable models — and even today, Vaught's question remains unanswered.

This paper is an attempt to shed a little light on Morley's proof.

1 Preliminaries

In this section we will establish some notions used in the paper. First, a *relation structure* $\underline{A} = \langle A, R_i \rangle$ for $i \in I$, is a set A together with finitary relations R_i indexed over a countable set I. The *similarity type* of $\underline{A}$ is the function $\tau : I \to \omega$ such that $\tau(i) = n$ if R_i is an n-ary relation. Corresponding to each similarity type τ is an applied predicate language $L_0(\tau)$ containing an n-ary relation symbol $\mathrm{R_i}$ for each $i \in I$ with $\tau(i) = n$.

An *atomic formula* is a sequence of the form $v_i = v_j$ or $R_i v_{j1} \dots v_{jn}$ where $\tau(i) = n$. We take $\wedge, \neg$ and $\exists$ to be primitive, and the formulas of $L_0(\tau)$ are defined inductively by

F1 an atomic formula is a formula,

F2 if φ and ψ are formulas, then $(\varphi \wedge \psi)$ is a formula,

F3 if φ is a formula, then $\neg\varphi$ is a formula,

F4 if φ is a formula and v_i is a variable, then $\exists v_i \varphi$ is a formula.

We will feel free to use the derived connectives $\vee, \to$, and $\leftrightarrow$ when convenient.

The infinitary language $L_{\omega_1,\omega}(\tau)$ is defined by adding a new formation rule of infinite conjunctions:

*This thesis was submitted for the degree of Bachelor of Arts at the Department of Philosophy, University of Göteborg.

F5 if Ξ is a countable set of formulas whose only free variables are among $v_0, \ldots, v_n$ for some n, then $\bigwedge \Xi$ is a formula.

Thus one allows infinite conjunctions, but only if the number of free variables remains finite. The notion of satisfaction, etc., can be defined in the obvious way. In particular, every element of Ξ is a subformula of $\bigwedge \Xi$.

It will be convenient to speak of "substituting v_i for v_j in φ, renaming variables to prevent clashes of bound variables." The renaming process is somewhat awkward in $L_{\omega_1,\omega}$. We could define it explicitly but instead we simply assume there is a function $Subst(i,j,\varphi)$ defined for all $i, j \in \omega$ and all formulas φ in $L_{\omega_1,\omega}$ such that

SB1 $Subst(i,i,\varphi) = \varphi$,

SB2 $i \neq j \Rightarrow v_i$ does not occur free in $Subst(i,j,\varphi)$,

SB3 $\models v_i = v_j \rightarrow (\varphi \leftrightarrow Subst(i,j,\varphi))$.

The set of formulas in $L_{\omega_1,\omega}$ is uncountable. We will be interested in certain countable subsets of this set, and we say a set L of formulas is *regular* if

R1 L satisfies F1-F5,

R2 $\varphi \in L \Rightarrow$ every subformula of $\varphi \in L$,

R3 $\varphi \in L \Rightarrow Subst(i,j,\varphi) \in L$,

R4 L is countable.

It is clear that every countable set of formulas is contained in a smallest regular subset.

Next, we want to define a certain *topology* on a given set X. Generally, a set $T \subseteq \mathscr{P}(X)$ is a topology on X if

T1 T is closed under arbitrary unions

T2 T is closed under finite intersections

T3 $T \neq \varnothing$

The elements in T are by definition *open*, and the complement of an open set is *closed*. A *topological space* is a pair of a non-empty set X and a topology T on X.

A class B of open sets is called a *base* for a topological space if each open set in X can be represented as a union of elements of B. These basis sets are then said to generate the topology T.

The class of *Borel* sets is the smallest collection of sets that contains the open sets and is closed under complementation and countable unions. This class is also closed under countable intersections, since these are countable unions of complements.

Let X be a countable set and let 2^X denote the set of all functions from X to $\{0,1\}$. We may identify this with the set of all subsets of X. If $Y \supset X$ we define $\pi_{YX} : 2^Y \to 2^X$ — the projection on X — by restricting the domain of $g \in 2^Y$ to X, in symbols $\pi_{YX}(g) = g \mid X$. A set $A \subset 2^X$ is *analytic* if it is a projection of a Borel set.

To generate our topology, we let U be a basis set if there is a finite $X_0 \subset X$ and an $f_0 \in 2^{X_0}$ such that $U = \{f : \pi_{XX_0}(f) = f_0\}$. Thus, a basis set in this topology is the set of extensions of some finite set.

Theorem 1.1. *An uncountable analytic set has power* $2^{\aleph_0}$.

We will just give the outlines of the proof, as a detailed proof would lead beyond the scope of this paper.[1]

Sketch of proof. The *Baire space* $\mathcal{N}$ is the set of infinite sequences of integers, with a certain topology on it. It is clear that $\mathcal{N}$ is of power $2^{\aleph_0}$. $\mathcal{N}$ is what is known as a *Polish space*, and a classical result of descriptive set theory shows that every uncountable projection of a Polish space is of power $2^{\aleph_0}$. By another theorem, each Borel set is a projection of $\mathcal{N}$, and by definition an analytic set is a projection of a Borel set. Thus, every uncountable analytic set is of power $2^{\aleph_0}$. □

2 Enumerated models

An *enumerated structure* of similarity type τ is a countable structure $\underline{A}$ of similarity type τ together with an enumeration $\langle a_0, a_1, \dots \rangle$ of A. Thus a given countable structure $\underline{A}$ corresponds to continuum many enumerated structures. Let L be a regular subset of $L_{\omega_1,\omega}(\tau)$. With each enumerated structure $\underline{A}$ we can associate the subset of L consisting of the formulas of L satisfied by the sequence $\langle a_0, \dots, a_n, \dots \rangle$. This subset corresponds to a point t of 2^L.

Theorem 2.1. *The set* $\{t : t$ *corresponds to an enumerated model*$\}$ *is a Borel subset of* 2^L.

Proof. Consider the following conditions on $t \in 2^L$:

C1 for each $\varphi \in L$, exactly one of $\varphi, \neg\varphi \in t$,

C2 $\varphi_1 \wedge \varphi_2 \in t$ iff $\varphi_1 \in t$ and $\varphi_2 \in t$,

C3 $\bigwedge \psi \in t$ iff $\psi \subseteq t$,

C4 $\exists v_i \varphi \in t$ iff for some j $Subst(i,j,\varphi) \in t$,

C5 $(v_i = v_j) \in t$ and $\varphi \in t$ implies $Subst(i,j,\varphi) \in t$ and $(v_j = v_i) \in t$.

C6 $(v_i = v_i) \in t$ for all i.

[1]The interested reader could turn for example to [2] or [6] for a thorough discussion of this matter.

The idea is to prove that a necessary and sufficient condition for a t to correspond to an enumerated model is that the set of those t's satisfies each one of the conditions C1-C6. Then we note that each such set is a Borel set and thus the intersection of those sets is also Borel, since the class of Borel sets is closed under intersections.

Suppose t corresponds to an enumerated structure. We prove that t satisfies each one of C1-C6 by a straightforward induction on the length of formulas. If φ contains no free variables exactly one of φ and $\neg\varphi$ is in L, since otherwise both φ and $\neg\varphi$ would be satisfied by all sequences. Suppose t satisfies C1 for formulas containing n free variables. t then satisfies C1 for formulas containing $n+1$ free variables, since otherwise both φ and $\neg\varphi$ would be satisfied by the objects $\langle a_0, \dots, a_n, \dots \rangle$.[2]

The proof that t satisfies C2-C6 is carried out similarily.

Now, suppose t satisfies C1-C6. We wish to construct an enumerated structure that corresponds to t. First, we note that C5 and C6 imply that $\{\langle i, j\rangle : v_i = v_j \in t\}$ is an equivalence relation on ω. Denote the equivalence class of i by $[i]$. We form the model whose universe is the set of equivalence classes $\{[i]\}$ under this relation and for each n-ary relation define $R([i_1], \dots, [i_n])$ to hold if $R(v_{i1}, \dots, v_{in}) \in t$. Let the enumeration of this model be the map which sends i into $[i]$. Again, induction on the length of formulas proves that t corresponds to the enumerated structure defined above.

Each set A of t's satisfying for example C1 is obviously Borel — A is open and the complement of A is easily found by letting A^c be the set of t's not satisfying C1. As for the intersection of two sets A, B satisfying C1, C2 respectively, it is the set of t's satisfying both C1 and C2.

Therefore, the set of t's satisfying all six conditions C1-C6 is Borel. □

Corollary 2.2. *Let T be a countable set of sentences of L. The set $\{t : t$ represents an enumerated model which satisfies all the sentences of $T\}$ is a Borel set.*

Proof. This is the intersection of the Borel set defined above with the Borel set $\{t : T \subset t\}$. □

For each regular $L \subset L_{\omega_1,\omega}$ the set $L^n = \{\varphi \in L :$ the free variables of φ are a subset of $\{v_0, \dots, v_{n-1}\}\}$. For example, L^0 is the set of sentences of L. If $\underline{A}$ is a structure of similarity type τ and $\langle a_0, \dots, a_{n-1}\rangle$ a sequence of elements of A, the L-type of this sequence is the subset of L^n satisfied by $\langle a_0, \dots, a_{n-1}\rangle$ in $\underline{A}$. In particular, the L-type of $\underline{A}$ is the type of the empty sequence. A class K of models is called an *axiomatic class* in $L_{\omega_1,\omega}(\tau)$ if there is a countable set T of sentences in $L_{\omega_1,\omega}(\tau)$ such that $K = \{\underline{A} : \underline{A} \models T\}$. The set of L-types of n-tuples which occur in the members of K is denoted by $S_n^L(K)$.

[2] Note that a_n is the n+1:th object in the sequence.

Theorem 2.3. *If K is an axiomatic class, the set $S_n^L(K)$ is an analytic subset of L^n.*

Proof. Let L' be a regular set such that $L \cup T \subset L'$. By Theorem 2.1 the set $\{t : t \in 2^{L'}$ and t corresponds to an enumerated model of $T\}$ is a Borel set $B \subset 2^{L'}$. But $S_n^L(K) = \pi_{L'L^n}(B)$ and is therefore analytic. □

Corollary 2.4. *If K is an axiomatic class, $S_n^L(K)$ is either countable or of power $2^{\aleph_0}$.*

Proof. This is immediate from Theorem 1.1 and Theorem 2.3. □

A theory T is *scattered* if $S_n^L(K)$ is countable for every regular L, where K is the class of models of T.

Theorem 2.5. *If T has fewer than $2^{\aleph_0}$ isomorphism types of countable models then T is scattered.*

Proof. Suppose that T where not scattered and that K is the class of models of T. Then there is a regular subset L of $L_{\omega_1,\omega}$ and an integer n such that $S_n^L(K)$ has power $2^{\aleph_0}$. Only a countable number of n-types can be realized in each countable model. Further, types realized in isomorphic models must be the same. This implies that there are $2^{\aleph_0}$ nonisomorphic models. □

3 Scattered theories

In the previous section we established the result that a theory with fewer than $2^{\aleph_0}$ isomorphism types of countable models is scattered. Our objective in this section will be to prove that a scattered theory has at most $\aleph_1$ such models. This would be trivial if we assumed the continuum hypothesis to be true. Scott [7] has shown that the isomorphism type of every countable structure is determined by a sentence of $L_{\omega_1,\omega}$, and Morley's proof consists of an elaboration of this result. For the rest of this section we assume that T is a fixed scattered theory and that K is the class of its countable models.

Define an increasing sequence $\{L_\alpha : \alpha < \omega_1\}$ of languages inductively by:

L_α is the smallest regular language such that for each $\beta < \alpha$:

(1) $L_\beta \subset L_\alpha$

(2) for each $n \in \omega$ and each $\psi \in S_n^{L_\beta}(K)$, $\bigwedge \psi \in L_\alpha$

In other words, in each step we make sure that the (possibly infinite) conjunction of all L_β-types is also in L_α. Note that L_0 is the usual finitary language, and that for limit ordinals δ, L_δ in the usual transfinite fashion is the union of all L_β for $\beta < \delta$. The assumption that T is scattered is needed to ensure that L_α is countable.

Lemma 3.1. *Suppose that $\underline{A}$ and $\underline{B}$ are models of T and that $\langle a_0, \ldots, a_{n-1}\rangle$ and $\langle b_0, \ldots, b_{n-1}\rangle$ are sequences having the same type in $L_{\alpha+1}$. Then for every a_n in $\underline{A}$ there is a b_n in $\underline{B}$ such that $\langle a_0, \ldots, a_n\rangle$ and $\langle b_0, \ldots, b_n\rangle$ have the same type in L_α.*

Proof. Let $\psi \in S^{L_\alpha}_{n+1}(K)$ be the type of $\langle a_0, \ldots, a_n\rangle$. Then, by the construction above, $\exists v_n \bigwedge \psi$ is a formula of $L_{\alpha+1}$ satisfied by $\langle a_0, \ldots, a_{n-1}\rangle$ and hence by $\langle b_0, \ldots, b_{n-1}\rangle$. □

Now we will give a characterization of the isomorphism types of the countable models of T in terms of sentences of $L_{\omega_1,\omega}$.

Theorem 3.2 (Scott). *Let $\underline{A}$ be a countable model of T. Then there is an $\alpha_0 < \omega_1$ and a sentence φ_0 in L_{α_0} such that a countable structure satisfies φ_0 iff it is isomorphic to $\underline{A}$.*

Proof. Consider any two finite sequences of elements $\langle a_0, \ldots, a_{n-1}\rangle$ and $\langle b_0, \ldots, b_{n-1}\rangle$ of $\underline{A}$ and $\underline{B}$ respectively. Either they have the same L_α-type for each $\alpha < \omega_1$ or there is a least α such that they have different L_α-types. Since there is only a countable number of pairs of finite sequences, there must be a $\delta < \omega_1$ such that if two sequences have the same L_δ-type, then they have the same type for all $\alpha < \omega_1$. Each L_δ-type corresponds to a single sentence in $L_{\delta+1}$, according to the construction above. The set of L_δ-types realized in $\underline{A}$ can therefore be described by a single sentence in $L_{\delta+2}$. Similarly the set of $L_{\delta+1}$-types realized in $\underline{A}$ can be described by a single sentence in $L_{\delta+3}$. Thus, there is a single sentence φ in $L_{\delta+3}$ which describes which L_δ-types of finite sequences are realized in $\underline{A}$ and also asserts that the $L_{\delta+1}$-type of a finite sequence is determined by its L_δ-type. We will show that any countable model $\underline{B}$ satisfying φ must be isomorphic to $\underline{A}$.

Let $\underline{B}$ satisfy φ and let $\langle a_0, a_1, \ldots\rangle$ and $\langle b_0, b_1, \ldots\rangle$ be enumerations of $\underline{A}$ and $\underline{B}$ respectively. Then it is possible to inductively define sequences $\langle c_0, c_1, \ldots\rangle$ and $\langle d_0, d_1, \ldots\rangle$ in $\underline{A}$ and $\underline{B}$ respectively such that

(1) $\langle c_0, c_1, \ldots\rangle$ has the same L_δ-type as $\langle d_0, d_1, \ldots\rangle$,

(2) $c_{2n} = a_n$

(3) $d_{2n+1} = b_n$

Suppose we have defined such a sequence for all $m < n$. If n is even, we let $c_n = a_{2/n}$. Since the sequences $\langle c_0, \ldots, c_{n-1}\rangle$ and $\langle d_0, \ldots, d_{n-1}\rangle$ have the same L_δ-type, they have the same $L_{\delta+1}$-type. According to Lemma 3.1, there is a d_n such that $\langle c_0, \ldots, c_n\rangle$ and $\langle d_0, \ldots, d_n\rangle$ have the same L_δ-type. For n odd we let $d_n = b_{(n-1)/2}$ and apply the lemma to get c_n. The function that maps c_n on d_n is the required isomorphism. □

Theorem 3.3. *A scattered theory can have at most $\aleph_1$ nonisomorphic countable models.*

Proof. Each isomorphism type is characterised by a sentence in some L_α, $\alpha < \omega_1$, but each L_α has only a countable number of sentences.

□

Combining Theorem 2.5 and Theorem 3.3 we get

Theorem 3.4. *If a theory T in $L_{\omega_1,\omega}$ has more than $\aleph_1$ nonisomorphic countable models, it has $2^{\aleph_0}$ nonisomorphic models.*

4 Relativized reducts

We have established a result on the number of countable models of theories in the infinitary language $L_{\omega_1,\omega}$. In this section we prove that the result is applicable to the ordinary finitary predicate calculus as well. This is done by introducing the concept of relativized reducts — we restrict the theories to be in L_0 and note that they have classes of models for which the argument in the preceding section still goes through.

Let $\underline{A} = \langle A, R_i \rangle$ for $i \in I$ be a relation structure of type τ and let $J \subset I$. If $\underline{B} = \langle A, R_i \rangle$ for $i \in J$, then $\underline{B}$ is said to be $\underline{A}$ *reduced* to J and we write $\underline{B} = \underline{A} \mid J$. The similarity type of $\underline{B}$ is denoted by $\tau \mid J$.

If $\varphi_0(v_0)$ is a formula of $L_{\omega_1,\omega}$ with one free variable, then A^φ is defined to be

$$A^\varphi = \{a \in A : \underline{A} \models \varphi[a]\}$$

and R_i^φ denotes the relation R_i restricted to A^φ. The relation system $\underline{A}^\varphi = \langle A^\varphi, R_i^\varphi \rangle$ is $\underline{A}$ *relativized* to φ. If the preceding operations, reduction and relativization, are combined, then the result $\underline{A}^\varphi \mid J$ is called a *relativized reduct* of $\underline{A}$.

A class K of models is called a *pseudo-axiomatic* class in $L_{\omega_1,\omega}(\tau \mid J)$ if there is a set T of sentences and a formula φ in $L_{\omega_1,\omega}$ such that

$$K = \{\underline{B} : \text{there exists } \underline{A} \text{ of type } \tau, \underline{A} \models T \text{ and } \underline{B} = \underline{A}^\varphi \mid J\}$$

If both T and φ are restricted to be in L_0, then K is a pseudo-axiomatic class of $L_0(\tau \mid J)$.

Theorem 4.1. *If L is a regular subset of $L_{\omega_1,\omega}(\tau \mid J)$ and K is a pseudo-axiomatic class, then $S_n^L(K)$ is analytic.*

Proof. Let K' be an axiomatic class in $L_{\omega_1,\omega}(\tau)$ satisfying $K = \{\underline{B} :$ there exists $\underline{A} \in K'$ and $\underline{B} = \underline{A}^\varphi \mid J\}$. Let L' be a regular subset of $L_{\omega_1,\omega}(\tau)$ such that $L \subset L'$, $\varphi \in L$ and the axioms T for K' are contained in L'.

Let B be the set of all $t \in 2^{L'}$ satisfying

(1) $T \subset t$,

(2) t satisfies C1-C6,

(3) $\varphi(v_i) \in t$, for all $i = 0, 1, \ldots, n-1$.

Then B is Borel by an argument analogous to the argument of the proof of Theorem 2.1, and $S_n^L(K)$ — the projection $\pi_{L'L^n}$ of B — is analytic by definition. □

Repeating the argument of the preceding section we get

Theorem 4.2. *If a pseudo-axiomatic class has more than $\aleph_1$ nonisomorphic countable models, it has $2^{\aleph_0}$ nonisomorphic models.*

Theorem 4.3. *There is a pseudo-axiomatic class in $L_{\omega_1,\omega}$ with exactly $\aleph_1$ isomorphism types of countable models.*

Whether Theorem 4.3 can be extended to axiomatic classes or pseudo-axiomatic classes in L_0 is not known.

References

[1] H. Friedman, *Countable models of set theory* in **Lecture notes in mathematics, vol. 337**, Springer-Verlag, New York, 1973, pp. 539-573.

[2] A. S. Kechris, **Classical descriptive set theory**, Springer-Verlag, New York, 1995.

[3] K. Kuratowski, **Topology**, Academic Press, New York and London, 1966.

[4] D. Lascar, *Why some people are excited about Vaught's conjecture* in **The Journal of Symbolic logic**, Volume 50, Number 4, December 1985, pp. 973-982.

[5] M. Morley, *The number of countable models* in **The Journal of Symbolic Logic**, Volume 35, Number 1, March 1970, pp. 14-18.

[6] Y. N. Moschovakis, **Descriptive set theory**, North-Holland, Amsterdam, 1980.

[7] D. Scott, *Logic with denumerably long formulas and finite strings of quantifiers* in **Theory of models**, North-Holland, Amsterdam, 1965, pp. 329-341.

[8] G. F. Simmons, **Introduction to topology and modern analysis** (reprint), Krieger Publishing Company, Malabar, 2003.

[9] R. L. Vaught, *Denumerable models of complete theories* in **Proceedings of the symposium in foundations of mathematics, infinitistic methods**, Pergamon Press, New York, 1961, pp. 303-321.