On abstract model theory and defining well-orderings.

Magisteruppsats i Logik
Tommi Salo, Institutionen för filosofi, lingvistik och vetenskapsteori

Abstract

In this paper we will study the expressive power, measured by the ability to define certain classes, of some extensions of first order logic. The central concepts will be definability of classes of ordinals and the well-ordering number w of a logic.

First we discuss the partial orders $\leq$, $\leq_{PC}$ and $\leq_{RPC}$ on logics and how these relate to each other and to our definability concept. Then we study the division between bounded and unbounded logics. An interresting result in this direction is the theorem due to Lopez-Escobar stating that $\mathscr{L}_{\infty\omega}$ is weak in the sense that it does not define the entire class of well-orderings, even though it has no well-ordering number, whereas $\mathscr{L}_{\omega_1\omega_1}$ is strong in the same sense.

1 Introduction

This thesis began as a question about how to compare the expressive power of logics stronger than first order logic and indeed how the term "expressive power" should be understood in this context. This question in term arose from the observation that some concepts which were not first order were readily expressed in second order logic and in logics with additional quantifiers. Choosing the definability of well-orders as a central theme, for reasons which will become apparent, this text is an attempt to bring clarity to these questions.

Since many basic mathematical concepts like "finitely many", "countably many", "uncountably many" and notions stemming from these are not definable in first order logic, it is natural to ask how to deal with these formally. In 1957 Mostowski presented a kind of quantifiers being able to express these concepts (Mostowski [1957]). He also showed that the methods of model theory could be extended to apply in these logics as well. Mostowski's was one of the earliest explicit proposals to expand the study of logic in this new direction (Barwise [1985]). The new logics containing Mostowski's quantifiers were shown to behave very differently from first order logic (Mostowski [1957]). This in turn led to questions about which properties of first order logic could be retained when expanding it to encompass non first order notions. Mostowski's idea was later also extended to concepts other than cardinality, yielding quantifiers based on measure theory, probability theory and more (Barwise [1985]). Furthermore, in 1966, Lindström proposed a definition of "generalized quantifiers", expanding Mostowski's idea further (Lindström [1966]).

Prior to Mostowski's work there had been several studies made on what we will be calling *infinitary* logics by Zermelo, Novikoff and Bochvar and others (Barwise et al.). These new logics arose from different needs in several different fields. For example it was shown by Mostowski (Mostowski [1968]) that, in a logic with *finite* syntax in which the notion of finite is definable, the Craig interpolation theorem fails. To amend this it would suffice, as was shown by Lopez-Escobar (Lopez-Escobar [1965]), to express the same notion in an *infinite* syntax. Further motivations for studying infinitary logics came from within logic as a way to make explicit inductive definitions (Barwise [1985]).

A third general category of logics, as described in (Väänänen [2008]), which emerged as an extension of first order logic was *higher order* logics, and of these, most notably *second order* logic. The main theme behind these logics, quantification over arbitrary sets, had been implicit in mathematics since at least the 19th century in the works on the foundations of calculus. In mathematics it was (and still is) often presupposed that quantifying over sets of numbers or indeed relations on numbers and sets of functions is non-problematic. Second order logic gave a framework for a straightforward formalization of these procedures. At first *full* second order logic ($\mathscr{L}^2$) was deemed unmanageable when compared to first order logic; it lacked completeness, compactness and many other features which made working with first order logic smoother. $\mathscr{L}^2$ recieved less attention than it perhaps deserved at first, quoting Barwise: "In fact, in the early days of extended model theory, many of us saw ourselves as chipping away manageable fragments of second-order logic." (Barwise [1985], p. 11). In later years, though, work in the field of full second order model theory has been very fruitful (Barwise [1985]).

The definition of a model theoretic logic, together with the regularity properties in Section 2.1 are all from (Ebbinghaus [1985]). The notion of "definable in a logic" is new, while the definitions of elementary, projective and relativized projective classes are all from (Ebbinghaus [1985]). Theorem 2.2.8 is due to (Ebbinghaus [1985]). In Section 2.3, the orderings $\leq$, $\leq_{PC}$, $\leq_{RPC}$ and $\leq_{\equiv}$ are all from (Ebbinghaus [1985]).

In sections 3.1 and 3.2, the definition of *well ordering number* is due to (Ebbinghaus [1985]). The distinction between *strong* and *unbounded* is new here, although both words are used for similar properties throughout several of the texts cited. The proof of Proposition 3.1.2 is new, but the result is stated in (Ebbinghaus [1985]). Theorem 3.2.5 is from (Ebbinghaus [1985]) and Theorem 3.1.4 is due to (Dickmann [1975]).

Propositions 3.3.7 and 3.3.8 are both due to (Dickmann [1975]). Theorem 3.3.9 as it is presented and proven here is also from (Dickmann [1975]).

Example 2.3.5 is essentially from (Mundici [1985]) and Example 3.1.3 is from (Ebbinghaus [1985])

2 Model theoretic logics

As this paper is aimed at those with little to no prior knowledge about abstract model theory and model theoretic logics this chapter will be dedicated to introducing the central concepts. The chapter is divided into three parts; in the first we discuss the notion of "a logic" and try to extract the kernel of this concept. Further we will name a few basic results of abstract model theory. In the second part we look at definability in a more general sense and clarify what we mean by "definable in a logic" via elementary and projective classes. Finally in the third part we will compare some of the principal regular logics regarding their expressive power.

2.1 Extending first order logic

In this text we will be using the expression $\mathscr{L}_{\omega\omega}$ interchangeably with the phrase "first order logic" for reasons which will become clear as we progress in our definition of generalized logics. Also, $\mathscr{L}^2$ will be used as a symbol for second order logic.

Although historically the many extensions of first order logic preceded the notion of a generalized logic, we will begin this section at the other end. By first abstracting some central properties of first order logic which make it "a logic" we will then be able to study a few special cases of this abstraction, all the while retaining the link to what is well known to us.

A vocabulary (or signature) is defined just like in the first order case, containing predicate, function and constant symbols. We will be using lower case greek letters τ, σ, υ to denote vocabularies. Given a vocabulary τ, the τ-*terms* are built up just like terms of first order logic. For example, if $\tau = \{f, c, d\}$, where c and d are constant symbols and f is a two-place function symbol, then c, d and $f(c, d)$ are all τ-terms. Central to the logics we will be studying are *models* (or *structures*). For a vocabulary τ the definition of a τ-structure is exactly the same as in the first order case, including the interpretation of the symbols in the vocabulary. In this text we will be using fracture style upper case roman letters (like this: $\mathfrak{A}$) to denote structures. When referencing the set underlying the structure we will use the same letter, but in straight style (thus: A) and to denote the interpretation (or extension) of a non-logical symbol $\S \in \tau$ in a τ-structure $\mathfrak{A}$ we will write $\S^{\mathfrak{A}}$.

Given a sentence ϕ in the signature τ, the class $\{\mathfrak{A} | \mathfrak{A} \vDash \phi\}$ of models of ϕ is denoted $mod(\phi)$. If we wish only to have the class of τ-structures which are models of ϕ, we write $mod_\tau(\phi)$. This terminology generalizes naturally to model classes of sets of sentences.

We are now ready to define what Feferman calls a "model theoretic language" (Feferman [1974]). The use of this term is motivated by the fact that the general form of this definition, taken from (Ebbinghaus [1985]), does not mention logical constants, but rather is all about models and model theory. Throughout this text we will be using the term "a logic":

2.1.1 Definition.

A *logic* is a pair $(\mathscr{L}, \vDash_{\mathscr{L}})$, where $\mathscr{L}$ is a mapping taking a vocabulary τ and yielding a class $\mathscr{L}[\tau]$, *the class of* $\mathscr{L}$*-sentences of vocabulary* τ*,* and $\vDash_{\mathscr{L}}$ is a relation between structures and $\mathscr{L}$-sentence. Further, $\mathscr{L}$ and $\models_{\mathscr{L}}$ satisfy the following:

(i) *If* $\tau \subseteq \sigma$*, then* $\mathscr{L}[\tau] \subseteq \mathscr{L}[\sigma]$*,*

(ii) *If* $\mathfrak{A} \vDash_{\mathscr{L}} \phi$*, then* $\phi \in \mathscr{L}[\tau_{\mathfrak{A}}]$*, where* $\tau_{\mathfrak{A}}$ *is the vocabulary of* $\mathfrak{A}$*,*

(iii) *If* $\mathfrak{A} \vDash_{\mathscr{L}} \phi$ *and* $\mathfrak{A} \cong \mathfrak{B}$*, then* $\mathfrak{B} \vDash_{\mathscr{L}} \phi$ *(the Isomorphism Property),*

(iv) *If* $\phi \in \mathscr{L}[\tau]$ *and* $\tau \subseteq \tau_{\mathfrak{A}}$*, then* $\mathfrak{A} \vDash_{\mathscr{L}} \phi$ *iff* $\mathfrak{A} \upharpoonright \tau \vDash_{\mathscr{L}} \phi$ *(the Reduct Property) and*

(v) *If* $\rho : \tau \to \sigma$ *is a renaming (a bijection from* τ *to* σ *such that* $\S \in \tau$ *and* $\rho(\S) \in \sigma$ *are of the same kind and the same arity), then for every* $\phi \in \mathscr{L}[\tau]$ *there is a* $\psi \in \mathscr{L}[\sigma]$ *such that, for all* τ*-structures* $\underline{A}$*,* $\underline{A} \vDash_{\mathscr{L}} \phi$ *iff* $\underline{A}^{\rho} \vDash_{\mathscr{L}} \psi$ *(the Renaming Property).*

In the last clause of the definition, $\mathfrak{A}^{\rho}$ is *the renaming of* $\mathfrak{A}$ *through* ρ. This is the unique structure $\mathfrak{B}$ having the same universe as $\mathfrak{A}$ in which $\S^{\mathfrak{A}} = \rho(\S)^{\mathfrak{B}}$ for every symbol $\S \in \tau_{\mathfrak{A}}$. is interpreted in the same way as the he same way.

The notation in this definition is too unwieldy to be practical. For that reason, whenever it is clear from context which the underlying logic is, we will omit the index of $\vDash$. Further we will use the abbreviation $\mathscr{L}$ to denote the logic $(\mathscr{L}, \vDash_{\mathscr{L}})$.

Now, this definition is far too general for our purposes since it allows for all kinds of construction to be called logics. Since we wish to study only logics which are in some way comparable to first order logic we will be needing the notion of a *regular logic.*

2.1.2 Definition.

A logic $\mathscr{L}$ is said to have the *basic closure properties* if it satisfies the following conditions:

(i) For all τ and all atomic sentences $\phi \in \mathscr{L}_{\omega\omega}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau]$ such that $\mathrm{Mod}^{\tau}_{\mathscr{L}}(\psi) = \mathrm{Mod}^{\tau}_{\mathscr{L}_{\omega\omega}}[\phi]$. *(Atom property)*

(ii) For all τ and all $\phi \in \mathscr{L}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau]$ such that $\mathrm{Mod}^{\tau}_{\mathscr{L}}(\psi) = \mathrm{Str}[\tau] \setminus \mathrm{Mod}^{\tau}_{\mathscr{L}}(\phi)$. *(Negation property)*

(iii) For all τ and all $\phi_0, \phi_1 \in \mathscr{L}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau]$ such that $\mathrm{Mod}^{\tau}_{\mathscr{L}}(\psi) = \mathrm{Mod}^{\tau}_{\mathscr{L}}(\phi_0) \cap \mathrm{Mod}^{\tau}_{\mathscr{L}}(\phi_1)$. *(Conjunction property)*

(iv) For all τ, if $c \in \tau$ is a constant, then for any $\phi \in \mathscr{L}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau \setminus \{c\}]$ such that for all $(\tau \setminus \{c\}) -$ structures $\mathfrak{A}$,

$\mathfrak{A} \vDash \psi$ iff $(\mathfrak{A}, a) \vDash \phi$ for some $a \in A$, where $c^{(\mathfrak{A},a)} = a$. *(Particularization property)*

If a logic $\mathscr{L}$ has all of these properties it is guaranteed to have at least the same expressive power as first order logic, in which case we will use the same symbol ϕ for both the first order and the $\mathscr{L}$-sentence, as long as this does not cause confusion. In case a logic has both the *Negation* and the *Conjunction properties* we say that it has the *Boolean property* and write the sentences posited in the definition $\neg\phi$ and $\phi_0 \wedge \phi_1$. Likewise the sentence posited for the *Particularization property* is often written $\exists c\phi$ or $\exists x\phi[x\backslash c]$.

Apart from the basic closure properties we will also require that the logics studied have the *Substitution* and *Relativization properties.* Since *terms* are syntactically the same in a generalized logic $\mathscr{L}$ as in $\mathscr{L}_{\omega\omega}$ it makes sense to talk about "substituting" a formula for a predicate in $\mathscr{L}$.

2.1.3 Definition.

A logic $\mathscr{L}$ is said to have the *Substitution property* iff, for any τ and τ', where $\tau \subseteq \tau'$, and every $\phi \in \mathscr{L}[\tau']$, *if,* for every symbol $\S\in \tau'\backslash\tau$, there is a formula $\psi_{\S}(c_{\S,1}, ..., c_{\S,n}) \in \mathscr{L}[\tau \cup \{c_{\S,1}, ..., c_{\S,n}\}]$, with new $c_{\S,i}$, where n is the arity of §, *then* there exists an $\mathscr{L}[\tau]$-sentence ϕ^* such that, in any $\tau \cup \{c_{\S,1}, ..., c_{\S,n}\}$-model $\mathfrak{A}$ and expansion of $\mathfrak{A}$ by § to $\mathfrak{A}^*$, if

$$\S^{\mathfrak{A}^*} = \left\{ \langle a_i \rangle_{i<n} \,|\, \text{if } c_{\S,i}^{\mathfrak{A}} = a_i \text{ for all } i, \text{ then } \mathfrak{A} \vDash \psi(c_{\S,1}, ..., c_{\S,n}) \right\}$$

then $\mathfrak{A}^* \vDash \phi$ iff $\mathfrak{A} \vDash \phi^*$ for every $\S\in\tau' \setminus \tau$.

Note. Intuitively substitution means that a sentence containing symbols we do not want can be reformulated without these symbols, using another formula as a proxy. This is most often used to replace function and constant symbols by predicate symbols in circumstances where the former cause complications. This may be the case for example when using *Relativization* defined below and the original sentence contains constant symbols.

In the next definition we will be using the concept of *closedness* under a signature. Recall that, for a structure $\mathfrak{A}$, we say that a subset B of A is τ*-closed* iff the interpretation of any constant symbol in $\mathfrak{A}$ belongs to B and, for any $f \in \tau$, applying $f^{\mathfrak{A}}$ to any element in B does not bring us outside of B.

2.1.4 Definition.

A logic $\mathscr{L}$ is said to have the *Relativization property* iff for any $\chi \in \mathscr{L}[\sigma]$ and $\phi \in \mathscr{L}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau \cup \sigma]$ such that for any $\tau \cup \sigma$-structure $\mathfrak{A}$, if $\chi^{\mathfrak{A}} = \{a \in A | \mathfrak{A} \vDash \chi[a]\}$ is τ-closed in $\mathfrak{A}$, then $\mathfrak{A} \models \psi$ iff $(\mathfrak{A} \restriction \tau) | \chi^{\mathfrak{A}} \vDash \phi$.

Note. Relativization here means that, starting with a sentence ϕ and a structure $\mathfrak{A}$, we can extend that structure so that, as long as we can define the domain of $\mathfrak{A}$ within the extension with a formula χ, we can also find a formula ψ which is true of the extension precisely when ϕ is true of $\mathfrak{A}$. In fact, by the definition, ψ must also be independent of $\mathfrak{A}$.

Finally, a logic which has all of the above properties is called *regular*:

2.1.5 Definition.

A logic $\mathscr{L}$ is said to be *regular* if it has

(i) the *basic closure properties,*

(ii) the *substitution property* and

(iii) the *relativization property.*

Being *regular* ensures that a logic has *at least* the expressive power of $\mathscr{L}_{\omega\omega}$ in a sense we will define later. Moreover it has many of the intuitive properties of first order logic, such as the sentences forming new ones with the help of boolean operations and quantification and the "meaning" of a sentence being independent of the names of the symbols used. Being regular is in no way a "mandatory" property of a logic, although the ones we will be studying are.

One other point is also important to stress: While we have argued for the exclusion of non-regular logics from this study, there are other classes of logics we omited already in the beginning. All logics we consider here are "model theoretic" in the sense that they have a Tarskian semantics. We will not, for example, study modal or intuitionistic logics or dependence logic here. In these logics Tarskian semantics are generally inappropriate (cf. Ebbinghaus [1985]).

With this in mind we are ready to study some specific abstract logics. These are divided into three primary categories: infinitary logics, quantifier extensions and higher order logics. First we turn to

Infinitary Logics

2.1.6 Definition.

Given two infinite cardinals κ and λ, κ regular, the logic $\mathscr{L}_{\kappa\lambda}$ is first order logic together with the new formation rules

(CF) If $\{\phi_\alpha\}_{\alpha<\mu}$, $\mu < \kappa$, is a set of formulae with free variables among $\{x_\beta\}_{\beta<\nu}$, $\nu < \lambda$, then $\bigwedge\{\phi_\alpha\}_{\alpha<\mu}$ is a formula.

(EQF) If ϕ is a formula and $\mu < \lambda$, then $\underset{\alpha<\mu}{\exists} x_\alpha\phi$ is a formula.

and the new interpretation rules

(CI) A formula $\bigwedge\{\phi_\alpha\}_{\alpha<\mu}$ is true in a structure $\mathfrak{A}$ if, for all $\alpha < \mu$, $\mathfrak{A} \models \phi_\alpha$.

(EQI) A formula $\underset{\alpha<\mu}{\exists} x_\alpha \phi$ is true in a structure $\mathfrak{A}$ if there exists a sequence $\langle a_\alpha \rangle_{\alpha \in \mu}$ such that ϕ is satisfied in $\mathfrak{A}$ by $\langle a_\alpha \rangle_{\alpha \in \mu}$.

Here we omit the details of how the satisfaction relation behaves in cases with free variables, i.e. when the formulae are not sentences. This is done as a generalization of Tarski's ideas in the first order case, for example through interpretation mappings. Furthermore, showing that $\mathscr{L}_{\kappa\lambda}$ is regular is done via induction over formulae and this is left as an exercise for the reader.

There are two important notes regarding this definition.

Note 1 For any given sentence ϕ of a logic $\mathscr{L}_{\kappa\lambda}$, ϕ is built up through a finite iteration of conjunctions and disjunctions, all of which involve less than κ sentences. Therefore, if κ is regular, there must be less than κ different variables in ϕ, so quantifying over less than κ variables always suffices to bind all variables of ϕ. For this reason we will be regarding only logics in which $\kappa \geq \lambda$. (cf. Dickmann [1975])

Note 2 The demand that κ is regular is there to ensure that quantifications are always sufficient to bind all variables in the logics $\mathscr{L}_{\theta\theta}$. For example, in $\mathscr{L}_{\aleph_\omega \aleph_\omega}[\{P\}]$ (where P is a unary predicate symbol) , let $\chi = \bigwedge_{n<\omega} \{\phi_n\}$, where the ϕ_n is the formula $\bigwedge_{\alpha<\omega_n} \{P(x_\alpha)\}$ respectively. Then, the formula χ contains $\aleph_\omega$ of the variables x_α. Since quantifications in $\mathscr{L}_{\aleph_\omega \aleph_\omega}[\{P\}]$ are restricted to sets of variables of cardinality *less than* $\aleph_\alpha$, we cannot bind all variables in χ.

By now the symbol $\mathscr{L}_{\omega\omega}$ introduced in the beginning of this section will also begin making sense. Namely, if we allow quantification to work over only finite sequences of variables and conjuntion to take only finite sets of formulae we end up with first order logic.

In speaking of infinitary logics, we presuppose that all constructions are *small* in the sense that the sentences of a logic form a *set*. There are, however, two special cases of logics we will encounter in the following text which are not *small* in this sense:

2.1.7 Examples

$\mathscr{L}_{\infty\omega}$ is thought of as being the *union* of the logics $\mathscr{L}_{\kappa\omega}$ for *all* cardinals κ. For this reason the sentences do not form a set, but a *proper class*. Syntax and semantics for this logic are straightforward generalizations of the "smaller" infinitary case. Note however that every sentence formed in $\mathscr{L}_{\infty\omega}$ is an $\mathscr{L}_{\kappa\omega}$-sentence for *some* κ. This kind of construction can be generalized to yield the logics $\mathscr{L}_{\infty\lambda}$ and $\mathscr{L}_{\infty\infty}$. For more about these logics and their properties, (Dickmann [1975]) is a rich source.

Next among the three primary categories of logics we have

Quantifier extensions

A question which arises naturally in the context of quantifier extensions is "what is *a quantifier*?". Syntactically we might say that a quantifier is a logical constant which binds variables to formulae much like the other logical constants bind formulae together. The semantics of quantifiers is somewhat more complicated in the general case. Since a deeper analysis of this field would bring us far beyond the scope of this paper, we will make do with a few examples and finally a short description of *generalized quantifiers.*

2.1.8 Example.

The cardinality quantifiers Q_α, meaning "there are at least $\aleph_\alpha$ many", have already been mentioned. When extending first order logic these yield the logics $\mathscr{L}(Q_\alpha)$. These are true extensions of $\mathscr{L}_{\omega\omega}$ in the sense that it is possible in these logics to express concepts which are undefinable in $\mathscr{L}_{\omega\omega}$ such as "finitely many", "countably many", "uncountably many" and so on.

2.1.9 Example (Equicardinality Quantifiers).

One other kind of quantifier we will be discussing is the *equicardinality quantifier* (or *Härtig quantifier*) I and its relatives. Unlike all quantifiers we have seen this far the *Härtig* quantifier does not bind *one* variable to *one* formula. Instead it operates on two of each. The new formula $Ixy\{\phi(x), \psi(y)\}$ is then interpreted as "the set of elements satisfying ϕ is equipotent to the set of elements satisfying ψ". Two common relatives of I which we will also look at are the *Rescher quantifier* Q^R and the *Chang quantifier* Q^C.

Q^R is similar to I. It binds two variables and two formulae into one in the same way as I. On the other hand, while the quantifier I expressed *equal* cardinality, the interpretation of the sentence $Q^R xy\{\phi, \psi\}$ is that "ϕ is satisfied by *fewer* elements than ψ". Since $Ixy\{\phi(x), \psi(y)\}$ in $\mathscr{L}(I)$ is equivalent to $\neg Q^R xy\{\phi(x), \psi(y)\} \wedge \neg Q^R xy\{\phi(x), \psi(y)\}$ in $\mathscr{L}(Q^R)$ (due to the *Schröder-Bernstein* theorem) for any ϕ and ψ, we might ask whether $\mathscr{L}(Q^R)$ in a sense *can define* $\mathscr{L}(I)$. We will see that this is the case, making this idea more precise in Example 2.3.5.

The Chang quantifier is more like the quantifiers we encountered earlier in that it binds one variable to one formula. Given a formula ϕ, the interpretation of the new formula $Q^C x\phi$, again somewhat informally, is that the set $\{a \in A | \mathfrak{A} \vDash \phi[a]\}$ of individuals satisfying ϕ in a model $\mathfrak{A}$ is equipotent to A. Now, given a formula ϕ, the formula $Ixy\{\phi, y = y\}$ is also true in a model $\mathfrak{A}$ iff ϕ is satisfied by a set equipotent to A. Just like above, this will be seen to mean that Q^C is *definable* in terms of I.

2.1.10 Example (Branching Quantifiers).

In a first order formula ϕ in *prenex normal form*, all of the quantifiers are collected at the beginning. In such a formula, if x is an existentially quantified variable, then the choise of x depends on all of the universally quantified variables preceeding it. For example, in the sentence $\forall x_1 \exists x_2 \forall x_3 \phi(x_1, x_2, x_3)$, it is stated that *no matter the choice of* x_1, there can always be found a matching x_2 to satisfy the formula $\phi(x_1, x_2, x_3)$ (and also, the choice of $x_3$is independent of these). The choice of x_2 may therefore depend on which x_1, but is independent of x_3. We say that the quantifiers are *linearly ordered.*

It is also possible to order quantifiers *partially* in a sentence. The simplest possible example of such a sentence is $\left[\begin{array}{cc} \forall x & \exists y \\ \forall z & \exists w \end{array}\right] \phi$, with the interpretation that *two independent choices are possible*: For every x you can find a y and at the same time for every z you can find a w (independent of x and y) such that they together satisfy $\phi(x, y, z, w)$. This construction is the *Henkin quantifier* Q^H (we will be using the expression $\left[\begin{array}{cc} \forall x & \exists y \\ \forall z & \exists w \end{array}\right] \phi$ interchangeably with the alternative form $Q^H \left[\begin{array}{cc} x & y \\ z & w \end{array}\right] \phi$). Quantifiers such as this are called *branching quantifiers.*

To see that $\left[\begin{array}{cc} \forall x & \exists y \\ \forall z & \exists w \end{array}\right] \phi$ is not the same thing as the formula

$$\forall x \exists y \forall z \exists w \phi(x, y, z, w)$$

of first order logic, note that in the latter formula the choice of w is allowed to depend on x, y and z whereas in the branching-quantified formula, given a z, the w is already fixed and cannot vary with our choice of x or y. To see how this seemingly small detail makes a difference we note the following:

Claim. All of the quantifiers Q^R, I and Q^C are definable in terms of Q^H.

Proof. Consider the sentence χ

$$Q^H \left[\begin{array}{cc} x & y \\ z & w \end{array}\right] [(x = z \leftrightarrow y = w) \wedge (\phi(x) \rightarrow \psi(y))] \wedge$$
$$\neg Q^H \left[\begin{array}{cc} x & y \\ z & w \end{array}\right] [(x = z \leftrightarrow y = w) \wedge (\psi(x) \rightarrow \phi(y))]$$

Note, first, that, for any formula ϕ, $\mathfrak{A} \models \forall x \exists y \phi(x, y)$ iff we can define a function $f : A \rightarrow A$ such that, for every $a \in A$, $\mathfrak{A} \vDash \phi(x, y)[a, f(a)]$. This function (called a *Skolem function*) picks, for every element of A, one of the elements posited by the existential quantification.

Now, in the first conjunct in the sentence χ, the Henkin quantification can be interpreted to mean that there exist functions f and g such that

$$\forall x \forall z \, [(x = z \leftrightarrow f(x) = g(z)) \wedge (\phi(x) \rightarrow \psi(f(x)))]$$

. Since $x = z \rightarrow f(x) = g(z)$ for all x and z, the functions f and g are identical. So, substituting f for g, we have $f(x) = f(z) \rightarrow x = z$ universally. This means that f is injective, which, taken together with $\phi(x) \rightarrow \psi(f(x))$, means that ϕ is satisfied by at most as many elements as ψ. By analogous reasoning we see that the second conjunct of χ is true of a model $\mathfrak{A}$ iff the cardinality of $\{a \in A|\mathfrak{A} \vDash \phi[a]\}$ is *strictly* less than the cardinality of $\{a \in A|\mathfrak{A} \vDash \psi[a]\}$. This shows that χ is equivalent to $Q^R xy\phi(x)\psi(y)$ in the sense that it is true of precisely the same models. By Example 2.1.9 above this suffices to show the claim. □

Finally we end this section with a discussion about *generalized quantifiers*. Somewhat informally, defining a generalized quantifier Q amounts to presenting the class of interpretations (models) in which Q is satisfied. For the *Rescher* quantifier, Q^R, above, for example, we say that $Q^R xy\phi(x)\psi(y)$ is satisfied in a structure $\mathfrak{A}$ iff the triplet $(A, \{a \in A \,|\, \mathfrak{A} \vDash \phi[a]\}, \{b \in B \,|\, \mathfrak{A} \vDash \psi[b]\})$ belongs to the class $K_{Q^R} = \{(A, B, C) \mid B \subseteq A,\ C \subseteq A \text{ and } |B| < |C|\}$. For the sake of clarity we restrict ourselves to quantifiers binding one variable to one formula when formulating the definition.

2.1.11 Definition.

The extension of a logic $\mathscr{L}$ by a quantifier Q, denoted $\mathscr{L}(Q)$, is $\mathscr{L}$ extended with a rule for constructing new formulae with Q (QF) and a new rule for interpreting formulae containing Q (QI) such that:

(QF) ϕ is an $\mathscr{L}(Q)$-formula iff either ϕ is an $\mathscr{L}$-formula or ϕ is $Qx\psi$, where ψ is an $\mathscr{L}(Q)$-formula.

(QI) If $Qx\phi$ is an $\mathscr{L}(Q)$-formula and $\mathfrak{A}$ a structure, then $\mathfrak{A} \vDash Qx\phi$ iff $(A, \{x \in A|\mathfrak{A} \vDash \phi\,[x]\})$ is in K_Q.

Note here, that in order to extend a logic with the quantifier Q, we must also provide the class K_Q, in which we specify the interpretation of Q as above. For simplicity we have restricted the definition to single variable, non-branching quantifiers. For a more thorough discussion of this field, cf. (Mundici [1985]).

As the last part of the trinity of extensions of first order logic we have:

Second order logic

Second order logic, $\mathscr{L}^2$, is an extension of first order logic in which we introduce two new kind of variables, called *predicate* and *function variables*, also called *second order variables*, as well as quantification over these. For example, predicate variables function syntactically exactly like predicate symbols in $\mathscr{L}_{\omega\omega}$. Given a signature τ, let $\phi \in \mathscr{L}_{\omega\omega}[\tau]$ be a first order sentence which contains a n-ary predicate symbol P. Then, exchanging all occurences of P in ϕ for an n-ary predicate variable X, calling the resulting formula ψ, and quantifying over X yields an $\mathscr{L}^2[\tau]$-sentence $\forall X\psi$. The semantics for $\mathscr{L}^2$ is an extensions of that for first order logic. Pure first order sentences are interpreted like in $\mathscr{L}_{\omega\omega}$. As an exampel of how second order variables are handled, let $\forall X\psi$ be as above. This sentence is true in a structure $\mathfrak{A}$ iff ψ is true in $\mathfrak{A}$ *regardless* of which extension $X^{\mathfrak{A}} \subseteq A^n$ we choose for X.

Second order logic is a strong logic in regard to what it can express. In it we can formalize most mathematical concepts, many of which are are well beyond the grasp of first order logic. For this reason it is reasonable to ask why first order logic seems to be such a staple logic in many presentations, including the one present here, what Barwise calls "the first order thesis" (Barwise [1985]). One of the main reasons for this is the fact that while first order logic has many nice properties such as a sound and complete proof system, compactness and the Löwenheim-Skolem-property (down to $\aleph_0$), most of the logics we study in this paper fail on some of these points (cf. "Lindströms Theorem" in (Ebbinghaus [1985])) . In the case of $\mathscr{L}^2$ we loose all of them.

Note.

Väänänen mentions three main groups (or traces) of extensions to first order logic in (Väänänen [2008]). Since we have been using the same terminology in this exposé it is worth noting that the usage differs between these two texts. While our grouping has been done mainly on a syntactical basis, Väänänen groups the logics semantically, meaning for example that that $\mathscr{L}(Q_1)$ belongs a category of countably compact quantifier extensions of first order logic, while $\mathscr{L}(Q^H)$ for example belong to the category of "higher order logics" together with the larger infinitary logics $\mathscr{L}_{\kappa\kappa}$. This separation is motivated by the fact that, in the latter logics, quantification is possible at least over predicates of cardinality at most κ (Väänänen [2008]).

2.2 Definability and projective classes

In this part we will be looking at the concept of "definable in the logic $\mathscr{L}$". Identifying a property with the class of structures which have that property we can pin down what this means. In this setting defining a property P would be the same as pinning down the class K_P of structures which have that property.

As a first idea for defining classes of structures and through them study the expressibility of a logic elementary classes go far. They are exactly the classes which correspond to sentences in the logic and are therefore a kind of "basis" for the other definability concepts we will encounter. The elementary approach however means that a definition of a property would only be allowed to use the non-logical constants of the property itself in its formulation. The definition of an *infinite set*, for example, could not contain any function symbols and thus could not take the route via *Dedekind infinity* and *bijections*. This restriction is somewhat arbitrary and also does not tie in with the expressibility of a logic, but rather the choice of vocabulary. For this reason we introduce alternative allowing a definition to involve any non-logical constants. This is the idea behind *projective classes*, studying not directly the model classes of a logic with a specific vocabulary, but rather the reducts of model classes with larger vocabularies, formally:

2.2.1 Definition.

Given a logic $\mathscr{L}$ and a vocabulary τ a class of $\mathscr{L}(\tau)$-structures K is an $\mathscr{L}(\tau)$-*projective class* iff there is a vocabulary $\sigma \supseteq \tau$ and a class K' such that

(i) K' is an $\mathscr{L}(\sigma)$-elementary class and

(ii) $K = \{\mathfrak{A} | \mathfrak{A} = \mathfrak{B} \upharpoonright \tau \text{ and } \mathfrak{B} \in K'\}$, the class of τ-reducts of structures in K'.

Using projective classes it is possible to express properties which are not definable by elementary classes:

2.2.2 Example.

The property "is infinite", represented by the class $\{\mathfrak{A} : |A| \geq \aleph_0\}$ has $\emptyset$ as its vocabulary. It is not elementary in $\mathscr{L}_{\omega\omega}$ since its complement (the class of finite structures) is not, due to compactness. The same class is however projective in $\mathscr{L}_{\omega\omega}$ since a formula which says that "the function f is injective but not surjective" is satisfied in every (Dedekind) infinite τ-structure where $\tau = \{f\}$. Now the reducts of these structures to the empty vocabulary is just the class of infinite $\emptyset$-structures.

This alternative way of defining "definability" is obviously not the only one. We have chosen it for the above reasons, and because it is relatively simple to graps. Occasionally, however, we will have reason to look at other ways of defining properties of classes in a logic. Thus, whenever we say that some property or class is "definable in the logic $\mathscr{L}$", this should be interpreted as the class being *projective* in $\mathscr{L}$, but to lessen the confusion a little we will try to always state explicitly which "kind" of definable the property is. Continuing the investigation of model classes, we first state this fact about the relation between projective and elementary classes:

2.2.3 Fact

If a class K is elementary in $\mathscr{L}$ then it is projective in $\mathscr{L}$.

The reverse on the other hand is not generally true, as we saw in example 2.2.2. This means that, going from elementary definitions to projective, we have *broadened* the class of definable classes, thereby increasing the expressional power of the logic. Of course there might be, and indeed there are, logics in which all projective classes are elementary ($\mathscr{L}^2$ is one such example). In such a logic we would have no reason to choose PC over EC as the "definable" classes.

As mentioned above there are other "kinds" of definable among which we could choose. Apart from elementary and projective classes we will have reason to study one more kind, namely *relativized* projective classes, RPC for short. As the name implies RPC is a further refinement of projective classes, as seen in the following definition:

2.2.4 Definition.

Given a logic $\mathscr{L}$ and a vocabulary τ, a class of structures K is RPC in $\mathscr{L}(\tau)$, written $K \in \mathrm{RPC}_{\mathscr{L}(\tau)}$, if there is a vocabulary $\sigma \supseteq \tau$, a unary predicate U in σ but not in τ and a class K' elementary in $\mathscr{L}(\sigma)$ such that $\mathfrak{A} \in K$ iff there is a $\mathfrak{B} \in K'$ such that $\mathfrak{A} = (\mathfrak{B} \restriction \tau)\,|U^{\mathfrak{B}}$ and $U^{\mathfrak{B}}$ is τ-closed in $\mathfrak{B}$.

While PC was an extension of the concept of elementary classes by allowing definitions to contain auxiliary symbols, RPC allows definitions to be "interior" in a structure. The predicate U marks the boundary of the inner model and the definition can use the elements not in U in an external way. We will later see an example of this type of definability.

As we saw in Example 2.2.2 the property of being finite is not elementary in first order logic, but the property of being *infinite* is projective. It turns out that finiteness is not definable in first order logic.

2.2.5 Proposition.

The class of all finite structures is not a projective class in $\mathscr{L}_{\omega\omega}$.

Proof. Let K be the class of finite $\emptyset$-structures. We assume for a contradiction that K is PC in $\mathscr{L}_{\omega\omega}$ and that the class K' is the elementary class associated with K. Further we let $K' = mod(\phi)$ and ϕ have vocabulary τ. Note first that since K is the class of $\emptyset$-reducts of K', K' contains structures of arbitrarily large *finite* cardinality. Now, let $\sigma = \tau \cup \{c_i | i \in \mathbb{N}\}$ and let $\Psi = \{\psi_i\}_{i\in\omega}$, where each ψ_i states that there exist $x_0, ..., x_i$ such that all are different. Now since every finite subset of $\{\phi\} \cup \Psi$ has a model (they can be chosen as expansions of models in K') $\{\phi\} \cup \Psi$ must have a model $\mathfrak{A}$ (by compactness). Since $\mathfrak{A}$ is a model of ϕ, it is in K', but $\mathfrak{A}$ cannot be finite, since for any finite cardinality n, any model of that size is too small to satisfy ψ_n. Therefore K contains the infinite model $\mathfrak{A}|\emptyset$ and we have a contradiction. □

Now, we turn to stronger logics and try to define finiteness in those. The two candidates will be the small infinitary logic $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}(Q_0)$, where Q_0 is Mostowskis infinite quantifier:

2.2.6 Proposition

The class of all finite $\emptyset$-structures is elementary in $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}(Q_0)$.

Proof. In $\mathscr{L}(Q_0)$ the sentence $Q_0x(x = x)$ is satisfiable in exactly all *infinite* structures, so its negation $\neg Q_0x(x = x)$ is true of the class of finite structures. For the case of $\mathscr{L}_{\omega_1\omega}$, let ψ_i be as in Proposition 2.2.5. If we let ϕ be the conjunction $\bigwedge_{i\in\omega} \{\psi_i\}$, then $mod(\phi)$ is the class of infinite models and $\neg\phi$ defines the class of finite models in $\mathscr{L}_{\omega_1\omega}$. □

As a last part of this investigation we show a result of abstract model theory, relating the definability of ω in a logic to compactness. To this end we first introduce a generalized type of compactness, called *κ-compactness.*

2.2.7 Definition.

Given a cardinal κ, a logic $\mathscr{L}$ is called *κ-compact* iff, for all τ and all $\Phi \subseteq \mathscr{L}[\tau]$ such that $|\Phi| \leq \kappa$, if every finite subset of Φ has a model, then all of Φ has a model.

In terms of this definition, $\mathscr{L}_{\omega\omega}$, for example, is κ-compact for all infinite cardinals κ.

The notion of *defining an ordinal* presented here will be more rigorously studied in the next chapter. In short, a set Φ of τ-sentences defines ω iff τ contains a binary predicate symbol $<$ which is interpreted as a well-ordering of its field in any model of Φ and that at in least one of the models of Φ the field of $<$ has order type ω.

2.2.8 Theorem.

$\mathscr{L}$ is $\aleph_0$-compact iff no countable set of sentences of $\mathscr{L}$ can define ω.

Proof. The implication from left to right makes use of a compactness argument. Assume for a contradiction that $\mathscr{L}$ is $\aleph_0$-compact and that the countable set Φ of $\mathscr{L}$-sentences defines ω, with $\mathfrak{A}$ a model order isomorphic to ω. Then, expanding the signature of Φ with countably many new constant symbols c_i, let, for every n, ψ_n be the sentence

$$c_n < c_{n-1} \wedge \ldots \wedge c_1 < c_0.$$

Let $\Psi = \{\psi_n \,|\, n \in \omega\}$. Now we can expand the signature of $\mathfrak{A}$ with n new constant symbols so that $\mathfrak{A} \vDash \psi_n$. This is possible since $\mathfrak{A}$ is infinite. Since $\mathfrak{A}$ is also a model of Φ,this means that every finite subset of $\Phi \cup \Psi$ has a model. By the assumption, then, there exists a model $\mathfrak{B} \models \Phi \cup \Psi$, but the sequence $\left\langle c_i^{\mathfrak{B}} \right\rangle_{i \in \omega}$ is strictly descending, so $\mathfrak{B}$ is a non-well ordered model of Φ, contradicting that Φ pins down ω.

For the implication from right to left, assume that $\mathscr{L}$ is not $\aleph_0$-compact and that $\Theta = \{\theta_n \,|\, n \in \omega\}$ is a set of $\mathscr{L}[\tau]$-sentences which has no model, but every finite subset of which has. We can assume that τ is relational since $\mathscr{L}$ allows for elimination of function and constant symbols. In this context, let Θ' be the set of sentences

(i) $<$ *is a linear ordering,*

(ii) $\forall x \in field(<) \exists y R(x, y)$,

(iii) $\forall x \in field(<) \left[\exists^{\geq n} z (z \leq x) \rightarrow \theta_n^{\{y \,|\, R(x,y)\}}\right]$ for $n \in \omega$.

Now we can construct a model for Θ'. First, let A be a copy of ω and the interpretation of $<$ be $\in$ restricted to ω. Further, for every $n \in A$, we extend A with a relativization of a model $\mathfrak{A}_n \models \theta_0 \wedge \ldots \wedge \theta_n$ and let $R \supset \{\langle x, y\rangle \,|\, x = n \text{ and } y \in \mathfrak{A}_n\}$. This is possible since every finite subset of Θ has a model. Through this construction we get a model $\mathfrak{B}$ such that for any element b in the field of $<^{\mathfrak{B}}$, if b has at least n predecessors, then b is related through $R^{\mathfrak{B}}$ to a set which constitutes a model for (the relativization of) θ_n, so Θ' is consistent.

Also, in any model $\mathfrak{C} \models \Theta'$, every element in the field of $<^{\mathfrak{C}}$ has finitely many predecessors. To see this, if $c \in \mathfrak{C}$ has arbitrarily many predecessors, the relativizations $\theta_n^{\{y \mid R(c,y)\}}$ are true for all n by clause (iii). That is, the substructure with the domain $\{y \mid R(c,y)\}$ is a model for all of Θ, contradicting the assumption about Θ being inconsistent. Since $<^{\mathfrak{C}}$ is also a linear ordering by (i), $\langle field(<^{\mathfrak{C}}), <^{\mathfrak{C}} \rangle$ must be isomorphic to $\langle \omega, \in \restriction \omega \rangle$. □

2.3 An ordering of logics

In the preceding parts of this chapter we have mentioned logics being "strictly stronger than" or "equivalent". In this part we will formalize the notion of "stronger than" as an ordering of logics. From our choice of formalism equivalence will follow naturally. Note first that there are different possible approaches to comparing and separating logics. One idea stems from the intuitive notion that if $\mathscr{L}$ can tell two structures $\mathfrak{A}$ and $\mathfrak{B}$ apart, then any logic $\mathscr{L}^*$ nominally stronger than $\mathscr{L}$ should also be able to. If we substitute "tell apart" for "satisfy different sentences" we get an ordering based on $\mathscr{L}$-equivalence:

2.3.1 Definition

A logic $\mathscr{L}^*$ is said to be *equivalently* at least as strong as $\mathscr{L}$, written $\mathscr{L} \leq_{\equiv} \mathscr{L}^*$, if, for every $\mathfrak{A}$, $\mathfrak{B}$, if $\mathfrak{A} \equiv_{\mathscr{L}^*} \mathfrak{B}$ then $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$, where $\equiv_{\mathscr{L}}$ is the relation "satisfy the same $\mathscr{L}$-sentences".

Whilst this is an intuitive approach we will not focus much on this type of comparison. Instead we will look at a relation based on elementary classes. What we wish to formalize is the idea:

> If for any sentence ϕ in $\mathscr{L}$ there is a sentence ψ in $\mathscr{L}^*$ which has the same meaning as ϕ, then $\mathscr{L}^*$ is at least as strong as $\mathscr{L}$.

That is, the stronger logic must contain equivalents of any sentence in the weaker one. For an "equivalence" relation between sentences in these two logics we choose the one generated by taking the model classes of sentences as equivalence classes.

2.3.2 Definition

A logic $\mathscr{L}^*$ is said to be at least as strong as $\mathscr{L}$, written $\mathscr{L} \leq \mathscr{L}^*$, iff for every sentence ϕ of $\mathscr{L}$ there is a sentence ψ of $\mathscr{L}^*$ such that $\{\mathfrak{A} | \mathfrak{A} \vDash \phi\} = \{\mathfrak{A} | \mathfrak{A} \vDash \psi\}$.

Equivalently one could say that $\mathscr{L}^*$ must have at least the same elementary classes as $\mathscr{L}$. From here on we will be using the usual terminology of orderings. If $\mathscr{L} \leq \mathscr{L}^*$ and $\mathscr{L}^* \nleq \mathscr{L}$ then $\mathscr{L}^*$ is strictly stronger than $\mathscr{L}$, in symbols, $\mathscr{L} < \mathscr{L}^*$. Conversely, if $\mathscr{L} \leq \mathscr{L}^*$ and $\mathscr{L}^* \leq \mathscr{L}$ then $\mathscr{L} \equiv \mathscr{L}^*$, that is, "$\mathscr{L}$ is equivalent to $\mathscr{L}^*$". Moreover, it should be obvious what we mean by $\mathscr{L} \geq \mathscr{L}^*$.

Before continuing, we will be needing the concept of *Löwenheim number*, which extends the downward part of the Löwenheim-Skolem theorem to abstract logics.

2.3.3 Definition.

The *Löwenheim number* of a logic $\mathscr{L}$ is the least cardinal λ such that every satisfiable sentence of $\mathscr{L}$ has a model of cardinality at most λ.

Now that we have an order relation among logics it is of interest to see how some of the principal logics are related. Note first that the ordering is not total. For example the *concept of finiteness* is expressible in $\mathscr{L}(Q_0)$, for example through the sentence $\neg Q_0 x(x = x)$, but since $\mathscr{L}(Q_1)$ is $\aleph_0$-compact (Fuhrken [1964]) it cannot express that same idea. For this reason we have $\mathscr{L}(Q_0) \not\leq \mathscr{L}(Q_1)$. On the other hand $\mathscr{L}(Q_0)$ has the property that any satisfiable sentence in the logic has models of cardinality at most $\aleph_0$, or equivalently, the *Löwenheim number* of $\mathscr{L}(Q_0)$ is $\aleph_0$ (Ebbinghaus [1985]), which means that *being uncountable* is not expressible in that logic as opposed to to in $\mathscr{L}(Q_1)$. By this reasoning we see that $\mathscr{L}(Q_0) \not\geq \mathscr{L}(Q_1)$ so these two logics are incomparable in our ordering.

Since we are considering only regular logics, all of them are at least as strong as $\mathscr{L}_{\omega\omega}$. In fact the question "in which cases is a logic *equivalent* to first order logic?" was one of the motivating factors behind the study of abstract model theory and generalized logics in the beginning. For more on this topic, the reader is refered to Lindströms original article (Lindström [1969]) and (Ebbinghaus [1985]).

Continuing the example with cardinality quantifiers from above, we get:

2.3.4 Example.

There are sentences in $\mathscr{L}(Q_0)$ which have no first-order counterparts, for example $Q_0 x(x = x)$. Thus $\mathscr{L}_{\omega\omega} < \mathscr{L}(Q_0)$. Furthermore, since the Löwenheim number of any $\mathscr{L}(Q_\alpha)$ is $\aleph_\alpha$ for $\alpha > 0$ (Ebbinghaus [1985]), $\mathscr{L}_{\omega\omega}$ is strictly weaker than any $\mathscr{L}(Q_\alpha)$.

2.3.5 Example

As we mentioned earlier the *Härtig* quantifier I and the *Rescher* quantifier Q^R are related. For any two formulae ϕ and ψ the formula $Ixy\phi\psi$ has the same model class as the formula $\neg Q^R xy\phi\psi \wedge \neg Q^R yx\psi\phi$. In the logic of this section this means that $\mathscr{L}(Q^R)$ is at least as strong as $\mathscr{L}(I)$. On the other hand $\mathscr{L}(I)$ is not equivalent to $\mathscr{L}(Q^R)$ (Mundici [1985]). Finally, $\mathscr{L}(I)$ pins down ω through the formula $\forall x \forall y\, (x = y \leftrightarrow Iuw\, [u < x, w < y])$ together with the axioms of a linear ordering without endpoint, which means that it must be stronger than first order logic. To summarize we have $\mathscr{L}_{\omega\omega} < \mathscr{L}(I) < \mathscr{L}(Q^R)$.

2.3.6 Example

A logic between first and second order logic in terms of expressibility is *weak* second order logic, $\mathscr{L}^{w2}$, where the predicate variables range only over *finite* sets. To see how the weak version is actually weaker than second order logic we first note that $\forall X\phi$ means "for all *finite* extensions of the predicate X..." in $\mathscr{L}^{w2}$ as opposed to the unrestricted interpretation of similar formulae in $\mathscr{L}^2$. It is easy to see however that finiteness is expressible in full second order logic via Dedekind infinity and because of this any $\mathscr{L}^{w2}$-formula can be converted to an $\mathscr{L}^2$-formula. Thus $\mathscr{L}^{w2} \leq \mathscr{L}^2$. This ordering is in fact strict. To see this, first let $\phi(f, X, Y)$ be the conjunction

$$\forall x \forall y \left[(X(x) \wedge X(y)) \rightarrow (Y(f(x)) \wedge f(x) = f(y) \rightarrow x = y)\right] \wedge \\ \forall z \left[Y(z) \rightarrow \exists w \left(X(w) \wedge f(w) = z\right)\right]$$

where f is a function symbol and X and Y are unary predicate symbols. This formula states that "f restricted to X is a bijection $X \rightarrow Y$". Further, let $\psi(X)$ be the formula

$$\exists f_X \exists Y \left[\forall x \left(Y(x) \rightarrow X(x)\right) \wedge \exists y \left(X(y) \wedge \neg Y(y)\right) \wedge \phi(f_X, X, Y)\right].$$

This is a formula stating that X is Dedekind-infinite. Now, the formula

$$\exists X \exists Y \left[\psi(X) \wedge \psi(Y) \wedge \neg \exists f \phi(f, X, Y) \wedge \forall Z \left(\psi(Z) \rightarrow \exists g \left[\phi(g, X, Z) \vee \phi(g, Y, Z)\right]\right)\right]$$

is true in exactly those structures which have cardinality $\aleph_1$. Now, any elementary class of $\mathscr{L}^{w2}$ which has only infinite models is a projective class of $\mathscr{L}(Q_0)$ (Shapiro [2001]). Also, $\aleph_1$ is not definable in $\mathscr{L}(Q_0)$ (Chang [1990]), so we may conclude that $\mathscr{L}^{w2} < \mathscr{L}^2$.
On the other hand, the formula $\forall X(X = X)$ in weak second order logic is true in all finite structures and by compactness this has no counterpart in first order logic. Summing up, thus, we have: $\mathscr{L}_{\omega\omega} < \mathscr{L}^{w2} < \mathscr{L}^2$.

Next, we will look at what happens if we change the kind of classes used in defining the above ordering. By inserting 'projective' for 'elementary' in Definition 2.3.2 we get the relation $\leq_{PC}$:

2.3.7 Definition

A logic $\mathscr{L}^*$ is said to be *projectively* at least as strong as a logic $\mathscr{L}$, in symbols $\mathscr{L} \leq_{PC} \mathscr{L}^*$, iff for every class of models K, if K is projective in $\mathscr{L}$, then K is projective in $\mathscr{L}^*$.

As a partial result about this relation we prove

2.3.8 Proposition

If $\mathscr{L}_1 \leq \mathscr{L}_2$ then $\mathscr{L}_1 \leq_{PC} \mathscr{L}_2$.

Proof. Let $\mathscr{L}_1 \leq \mathscr{L}_2$. Then for any vocabulary τ and any class K projective in $\mathscr{L}_1[\tau]$ there is a class K' elementary in $\mathscr{L}_1[\sigma]$, where $\sigma \supseteq \tau$, which is a class of expansions of the models in K to the vocabulary σ. Now K' is elementary also in $\mathscr{L}_2[\sigma]$ so for every $\mathfrak{A} \in K$ there is a $\mathfrak{B} \in K'$ such that $\mathfrak{A} = \mathfrak{B} \mid \tau$ and vice versa, but this is exactly the definition of projective class, proving the proposition. □

As a last variant, exchanging *relativized projective* for *projective* in the definition of the ordering of logics, we get yet another ordering of logics:

2.3.9 Definition

A logic $\mathscr{L}^*$ is said to be *relatively projectively at least as strong as* $\mathscr{L}$ iff for every class K of models, if K is RPC in $\mathscr{L}$ then K is RPC in $\mathscr{L}^*$.

Now a natural question to ask about all these orderings of logics is how they are interrelated. As we saw above, $\leq$ is a subordering of $\leq_{PC}$ in the following sense:

2.3.10 Definition.

Given two orderings $\leq$ and $\preceq$ with the same field, we say that $\leq$ is a *subordering* of $\preceq$ iff, for every x and y, $x \leq y$ implies $x \preceq y$.

This definition is then expanded in the natural way: $\leq$ and $\preceq$ are *equal* iff both orderings are suborderings of eachother and $\leq$ is a *strict subordering* of $\preceq$ iff $\leq$ is a subordering of $\preceq$ and they are *not* equal.

For this discussion we will be introducing two new logics, only briefly. First, we have *monadic* second order logic $\mathscr{L}^{m2}$, which is second order logic, with the second order variables ranging only over unary predicates (cf. Mundici [1985]). The other is the *game theoretical* logic $\mathscr{L}_{\infty G}$. The sentences of $\mathscr{L}_{\infty G}$ are sentences of $\mathscr{L}_{\infty\omega}$, possibly extended with an an *infinite string of quantifications* $\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots$. The semantics for this quantification is beyond the scope of this paper. For a full presentation of $\mathscr{L}_{\infty G}$ and other game theoretical logics, cf. (Kolaitis [1985]).

By an argument similar to the one in Proposition 2.3.8, we can show that $\leq_{PC}$ is a subordering of $\leq_{RPC}$, which also implies that $\leq$ is a subordering of $\leq_{RPC}$. Next we will be needing the following result about monadic second order logic (cf. (Väänänen [1977]):

2.3.11 Theorem.

Given a monadic signature τ, $\mathscr{L}^{m2}[\tau] \leq \mathscr{L}_{\omega\omega}[\tau]$.

This means that the class of infinite structures cannot be elementary in $\mathscr{L}^{m2}[\emptyset]$. Since $\mathscr{L}(Q^H)[\emptyset]$ defines infinity (Mundici [1985]), we have $\mathscr{L}(Q^H) \not\leq \mathscr{L}^{m2}$. It is also shown in (Mundici [1985]) that $\mathscr{L}(Q^H) \leq_{RPC} \mathscr{L}^{m2}$. It then follows that that $\leq_{RPC}$ cannot be a subordering of $\leq$, and thus that $\leq$ is a strict subordering of $\leq_{RPC}$.

Whether the subordering relations between $\leq$ and $\leq_{PC}$ and between $\leq_{PC}$ and $\leq_{RPC}$ are strict is unknown to the author.

It is shown in (Kolaitis [1985]) that $\mathscr{L}_{\omega G}$ and $\mathscr{L}_{\infty\omega}$are equivalent under $\leq_{\equiv}$. Also, $\mathscr{L}_{\infty G}$ defines well-orderings as an elementary class (Kolaitis [1985]). As we will show, however, the finite quantifier logic $\mathscr{L}_{\infty\omega}$ does *not* define well-orderings, and is therefore strictly weaker than $\mathscr{L}_{\infty G}$ under $\leq$. This means that $\leq_{\equiv}$ is not a subordering of $\leq$. On the other hand, for any two logics $\mathscr{L}_1$, $\mathscr{L}_2$ such that $\mathscr{L}_1 \leq \mathscr{L}_2$, let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures which satisfy different sentences in $\mathscr{L}_1$. Further, let $\phi \in \mathscr{L}_1[\tau]$ be such that $\mathfrak{A} \vDash \phi$ and $\mathfrak{B} \nvDash \phi$. Now, since $\mathscr{L}_1 \leq \mathscr{L}_2$, $mod(\phi)$ is an elementary class of $\mathscr{L}_2$ also. This means that there is an $\mathscr{L}_2$-sentence ψ such that $\mathfrak{A} \in mod(\phi) = mod(\psi)$, so $\mathfrak{A} \vDash \psi$. Similarly, $\mathfrak{B} \nvDash \psi$, so $\mathscr{L}_1 \leq_{\equiv} \mathscr{L}_2$. In conclusion, then, $\leq$ is a strict subordering of $\leq_{\equiv}$.

3 Well-orderings

In this chapter we will look at the special case of defining *well orderings* in model theoretic logics. Recall that, what we mean by a property being definable in a logic $\mathscr{L}$, is that the class of models with that property is projective in $\mathscr{L}$. In some instances we will also allow "definable" to mean "relativized projective". By the class of well orderings we mean the class $\{\mathfrak{A} \,|\, \mathfrak{A}$ is a $\{<\}$-model and $<^{\mathfrak{A}}$ well orders A. Note also, that if $\mathfrak{A}$ is in this class, then $\langle A, <^{\mathfrak{A}} \rangle$ is order isomorphic to some ordinal. First we will look at two examples.

3.0.1 Example

WO is not a projective class in any compact logic. To see this, for a compact logic $\mathscr{L}$ and a signature τ containing $\{<\}$, take any sentence $\phi \in \mathscr{L}[\tau]$ which has arbitrarily large linear orders. Add to τ a set $\{c_i : i \in \omega\}$ of constants not already in τ. By compactness the set of sentences $\{c_i < c_j | j < i,\, i \in \omega,\, j \in \omega\} \cup \{\phi\}$ has a model $\underline{A}$ and, being a model of this set, it is not well-ordered by $<^A$. By a similar argument, the class of well-orderings is not RPC in any compact logic $\mathscr{L}$.

3.0.2 Example

The class WO is elementary in second order logic, $\mathscr{L}^2$, for example through the sentence

$$\forall X \left(\exists x X(x) \rightarrow [\exists y (X(y) \wedge \forall z (X(z) \rightarrow z \leq y)]\right).$$

3.1 Well-ordering numbers

Next we turn to *well-ordering numbers.* As we saw above, the class of well-orderings was not definable in $\mathscr{L}_{\omega\omega}$ and definable in $\mathscr{L}^2$. These are the two extremes of a spectrum of definability when it comes to well orderings. For every logic there is a boundary for *how large a subset* of WO is definable. In the case of first order logic we have already mentioned the problem of finding this boundary. By Theorem 2.2.8 ω is not definable in $\mathscr{L}_{\omega\omega}$. As we will soon see, since regular logics pin down all finite well-orderings, this means that ω must be the upper bound for the well-orderings definable in $\mathscr{L}_{\omega\omega}$. In $\mathscr{L}^2$, on the other hand, there can be no upper bound for the definable well orderings. This notion is made more precise in the following definition:

3.1.1 Definition.

A set of sentences Φ in a logic $\mathscr{L}[\tau]$, where $<\in\tau$, *pins down* an ordinal α iff

(i) For all $\mathfrak{A} \models \Phi$, $<^{\mathfrak{A}}$ is a well-ordering *of its field* and

(ii) There is a $\mathfrak{B} \models \Phi$ such that $<^{\mathfrak{B}}$ is of order type α.

Furthermore we let $w_\kappa = sup\{\alpha | \Phi \text{ pins down } \alpha \text{ and } |\Phi| \leq \kappa\}$ and call $w_1(\mathscr{L})$ the *well-ordering number* of $\mathscr{L}$, denoted $w(\mathscr{L})$. Note, that the bounds are defined to be over *all* sets of sentences of *any* signature. Also, this definition allows for relativized projective classes to be used as the interpretation of the order predicate needs only be partial in any structure.

Now, if an infinite ordinal α is pinned down by a sentence ϕ in $\mathscr{L}$, then $\alpha+1$ is pinned down (in the signature $\{\prec\}$) through the sentence

$$\phi \wedge \forall x \forall y \left[\exists z (z < x \wedge z < y) \rightarrow (x < y \leftrightarrow x \prec y)\right] \wedge \\ \forall z (x < z \vee x = z) \rightarrow \forall w (w \prec x \vee w = x)$$

Also, if an ordinal β is pinned down by a sentence ψ in $\mathscr{L}$ then so is any initial segment of β (again in the signature $\{\prec\}$) through the sentence

$$\psi \wedge \exists z \forall x \forall y \left[(x < y \wedge y < z) \leftrightarrow (x \prec y)\right]$$

By similar arguments, w_κ is closed under ordinal addition, multiplication and exponentiation. (Ebbinghaus [1985]). Thus we may conclude that $w(\mathscr{L})$ must be a limit ordinal greater than all ordinals pinned down by single sentences in $\mathscr{L}$.

3.1.2 Proposition.

The well-ordering number of $\mathscr{L}_{\omega_1\omega}$ is at least ω_1.

Proof. Let the formulae

$$\mu_\beta(x) = \forall y \left[y < x \leftrightarrow \bigvee_{\gamma < \beta} \mu_\gamma(y) \right]$$

be defined by transfinite recursion for all β. For countable β, μ_β is a $\mathscr{L}_{\omega_1\omega}$-formula, so for countable α, the sentence

$$\forall x \bigvee_{\beta<\alpha} \mu_\beta(x)$$

is an $\mathscr{L}_{\omega_1\omega}$-sentence. Further, let ϕ_α be the conjunction of this sentence with the first order formula stating that $<$ is a linear ordering. This sentence pins down the ordinal α. We will show this through induction on α. Note, that any model of one of the sentences ϕ_α is linearly ordered by$<$. Since $\mu_0(x)$ is not defined by the schema above, define it to be the formula $\forall y\,[\neg y < x]$. Then, $\mu_0(x)$ is satisfied in $\mathfrak{A} \vDash \phi_\alpha$, for any α, only by a *least element*. Now, we will begin with the sentence ϕ_1. This is true in a linearly ordered model $\mathfrak{A}$ iff $\mathfrak{A} \vDash \forall x \forall y\,[\neg y < x]$, which means that $\mathfrak{A}$ can have only one element, which means that $\mathfrak{A}$ is order isomorphic to

For the induction step, the assumption will be that, for every $\beta < \alpha$, there is a model$\mathfrak{A} \vDash \phi_\beta$ such that $\langle A, <^{\mathfrak{A}} \rangle$ and every model of ϕ_β is order isomorphic to some ordinal $< \beta$.

Here we will have to start by examining the formulae $\mu_\beta(x)$. These have the property that, for any linearly ordered model $\mathfrak{B}$ and element $b \in B$, $\mathfrak{B} \vDash \mu_\beta(x)[b]$ iff the ordered set $S_b = \left\langle \{x|x < b\}, <^{\mathfrak{A}} \restriction \{x|x < b\} \right\rangle$ is order isomorphic to β. The proof of this is also done by induction, in this case over β.

First, note that $\mu_0(x)$ is satisfied in any linearly ordered model only by a *least element*, giving us a base case for the induction. For the induction step, assume that, for every $\gamma < \beta$, $\mu_\gamma(x)$ is satisfied in a linearly ordered model $\mathfrak{B}$ by at most one element b and that the ordered set $S_b = \left\langle \{x|x < b\}, <^{\mathfrak{A}} \restriction \{x|x < b\} \right\rangle$ is order isomorphic to γ. Also, note that, by the definition of the sets S_a, a set S_b is an initial segment of S_a iff $b < a$. Now, assume that $b' \in B$ such that $\mathfrak{B} \vDash \mu_\beta(x)[b']$. We cannot have $b' \leq b$ for any b that satisfies one of the formulas $\mu_\gamma(x)$, where $\gamma < \beta$, for by the definition of $\mu_\beta(x)$, $\mathfrak{B} \vDash \mu_\gamma(x)[b]$ implies $b < b'$ in $\mathfrak{B}$. On the other hand, if we assume that $c \in B$ is less than b' in $\mathfrak{B}$, again, by the definition of $\mu_\beta(x)$, there is an ordinal $\gamma < \delta$ such that $\mathfrak{A} \vDash \mu_\gamma(x)[c]$. Together, this means that b' is the least (and only) element of B such that $c < b'$ in $\mathfrak{B}$ iff c satisfies some $\mu_\gamma(x)$, where $\gamma < \beta$, again in $\mathfrak{B}$. Now, by the induction hypothesis, this means that we have a one-one correspondence between the initial segments of $S_{b'}$ and the ordinals $\gamma < \beta$, so the order type of $S_{b'}$ must be β.

Returning to the first induction, this means that, if $\mathfrak{A}$ is a model of ϕ_α, then for every $a \in A$, the initial segment S_a as defined above is order isomorphic to some ordinal $\beta < \alpha$, so $\mathfrak{A}$ must be well-ordered.. Also, if $\mathfrak{A}$ is order isomorphic to α, then all initial segments of $\mathfrak{A}$ are order isomorphic to some $\beta < \alpha$, so $\mathfrak{B} \vDash \phi_\alpha$.

These ϕ_α are thus sentences of $\mathscr{L}_{\omega_1\omega}$ which pin down all countable ordinals, which implies that $w(\mathscr{L}_{\omega_1\omega}) \geq \omega_1$. □

3.1.3 Example.

$w_\kappa(\mathscr{L}_{\omega\omega}) = \omega$ for any κ, by compactness.
Above, we saw that $w(\mathscr{L}_{\omega_1\omega}) \geq \omega_1$. The opposite inequality is also true (Ebbinghaus [1985]), so the well-ordering number of $\mathscr{L}_{\omega_1\omega}$ is ω_1.

We can show, using the same method as in Theorem 3.1.2, that $w(\mathscr{L}_{\kappa\omega})$ is at least κ, proving the following result:

3.1.4 Theorem.

$w(\mathscr{L}_{\infty\omega}) = \infty$.

Now we may ask how far this brings us toward defining well-orderings. The disjunction of the *proper class* of formulae $\{\mu_\alpha | \alpha \in \mathbf{ON}\}$ pins down every ordinal and so defines the class WO, but this is not a sentence even in $\mathscr{L}_{\infty\omega}$. In section 3.3 we will be looking at a result by Lopez-Escobar stating that no single formula of $\mathscr{L}_{\infty\omega}$ suffices for defining WO.

3.2 Boundedness and its neighborhood

This far we have found that some logics can characterize all well-orderings, while others fail somewhere on the way. A logic which contains a sentence characterizing the entire class WO is called *strong*. On the other hand, a logic which is not *strong* is called *weak*. A logic which has a well-ordering number is called *bounded*. When there exists no such upper bound for the definable well-orderings we call the logic *unbounded*.

Note. A strong logic is by necessity unbounded, but the opposite is not true, as we will see later.

We will begin by looking at a few examples of both unbounded and bounded logics.

Given an $\mathscr{L}_{\omega\omega}$-formula ψ, if we define

$$\phi_n = \exists x_0 ... \exists x_{n-1} \left[\bigwedge \{x_i \neq x_j \text{ for } i \neq j\} \wedge \bigwedge \{\psi(x_i) \text{ for } i < n\} \right]$$

for every natural number n, then $\phi = \bigwedge\limits_{n<\omega} \{\phi_n\}$ is a sentence of $\mathscr{L}_{\omega_1\omega}$. Now, for any $\mathfrak{A} \models \phi$, $\mathfrak{A}$ must contain arbitrarily large (and therefore infinite) sets satisfying ϕ. This means that ϕ is equivalent to $Q_0 x \psi(x)$. Since ψ is arbitrary, $\mathscr{L}(Q_0) \leq \mathscr{L}_{\omega_1\omega}$. Using this we can give some examples of boundedness.

3.2.1 Example.

As we saw in Example 3.1.3, $\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_{\omega_1\omega}$ are bounded by ω and ω_1 respectively.
By the argument above, $\mathscr{L}(Q_0)$ must also be bounded since any ordinal which is pinned down by $\mathscr{L}(Q_0)$ logic is also pinned down by the stronger $\mathscr{L}_{\omega_1\omega}$.

3.2.2 Examples.

As we have already seen, $\mathscr{L}^2$ characterizes well-orderings through a straightforward formula. For other examples of unbounded langaugages, recall the alternative formulation of well-ordering: "$\langle A, < \rangle$ is well-ordered" is equivalent to "there exists no infinite, strictly descending chain in $\langle A, < \rangle$".
This statement about infinite descending chains is formalizable in $\mathscr{L}_{\omega_1\omega_1}$ through the sentence

$$\neg \left[\underset{n<\omega}{\exists} x_n \bigwedge \{x_{i+1} < x_i | i \in \mathbf{N}\} \right].$$

This shows that $\mathscr{L}_{\omega_1\omega_1}$ is strong, with WO an elementary class, and thus also unbounded.
A third example of an unbounded logic is $\mathscr{L}(I)$, first order logic expanded with the *Härtig* quantifier. Elementarily we can characterize ω through the sentence $\phi = \forall x \forall y \, (x = y \leftrightarrow Iuv \, \{u < x, v < y\})$, which reads out "x and y are the same iff they have the same number of predecessors", together with the usual first order axiomatization of a linear order without a last element. [Barwise/Flum].
Through a variant of the ϕ above we can now show that WO is a relativized projective class of $\mathscr{L}(I)$. The sentence ψ, which consists of

$$\forall x \forall y \, [(Iuv \, \{u < x, v < y\} \wedge U(x) \wedge U(y)) \rightarrow x = y]$$

in conjunction with the first order axioms of a discrete linear order with first, but without last, element, does the job here. In any model $\mathfrak{A}$ of ψ, $\left\langle U^{\mathfrak{A}}, <^{\mathfrak{A}} \restriction U^{\mathfrak{A}} \right\rangle$ must be order isomorphic to an ordinal number. To see this, let $\mathfrak{A} = \left\langle A, U^{\mathfrak{A}}, <^{\mathfrak{A}} \right\rangle$, where $|A| = \aleph_\alpha$. By the sentence above, the ordering $\left\langle A, <^{\mathfrak{A}} \right\rangle$ has ω as an initial segment. Now, we define the function $f : \alpha \rightarrow U^{\mathfrak{A}}$ in the following way: For every cardinal $\aleph_\beta < \aleph_\alpha$, there exists exactly one $b \in U^{\mathfrak{A}}$ such that $|\{x | x < b\}| = \aleph_\beta$, by ψ. Let $\langle \beta, b \rangle \in f$ for every $\beta < \alpha$. By construction, $\beta < \gamma$ iff $\aleph_\beta < \aleph_\gamma$ iff $|\{x | x < f(\beta)\}| < |\{x | x < f(\gamma)\}|$, and both being initial segments of $\mathfrak{A}$, this is true iff $f(\beta) < f(\gamma)$. This means that f is order preserving, so $\left\langle U^{\mathfrak{A}}, <^{\mathfrak{A}} \restriction U^{\mathfrak{A}} \right\rangle$ is order isomorphic to $\omega + \alpha$. Whether WO is also an elementary or projective class of $\mathscr{L}(I)$ is unknown to the author.

Before turning to the main result of this paper we will need one more idea from abstract model theory. The *Hanf number* of a logic $\mathscr{L}$ is an analogy to the well-ordering number, giving a measure of how well cardinal numbers are distinguished in $\mathscr{L}$. The following definition is the cardinality analogue of pinning down ordinals as in Definition 3.1.1.

3.2.3 Definition.

We say that a set of sentences $\Phi \subseteq \mathscr{L}$ *pins down the cardinal* κ if Φ has a model of cardinality κ but does not have models of arbitrarily high cardinalities. Furthermore, if we let $h_\kappa(\mathscr{L})$ be $sup \, \{\lambda | \Phi \text{ pins down } \lambda, \Phi \in \mathscr{L} and \ |\Phi| \leq \kappa\}$, we say that $h(\mathscr{L}) = h_1(\mathscr{L})$ is the *Hanf number* of $\mathscr{L}$.

We will use the term "Φ *pins down cardinals*" if the set of sentences Φ does not have arbitrarily large models. Recall the definition of *Beth-numbers* from set theory:

3.2.4 Definition (The $\beth$-sequence).

For any cardinal κ, any ordinal α and any limit ordinal β:

(1) $\beth_0(\kappa) = \kappa$,

(2) $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$ and

(3) $\beth_\beta(\kappa) = sup\{\beth_\gamma(\kappa)|\gamma < \beta\}$.

Using a construction which partly mirrors that of the set-theoretic universes V_α we can show that the well-ordering number of a logic forces the hanf number upward. The idea here is that we use the ordinals pinned down by $\mathscr{L}$ as a "back bone" for constructing ever larger structures, at the same time making sure that the constructed formulae pin down cardinals. This construction is adopted from [Barwise] with only slight variations.

Let $\mathscr{L}$ be a regular. Given a set of sentences $\Phi \subset \mathscr{L}[\tau]$ of size $\leq \kappa$ such that Φ pins down the cardinal $\lambda < h_\kappa(\mathscr{L})$, expand the vocabulary with two new unary predicate symbols P_0 and P_1 (for distinguishing the inner models) and a binary predicate symbol E (for the set membership-relation) . Then, let Ψ be the set of formulae

(i) $\exists x P_0(x)$

(ii) $\forall x(P_0(x) \vee P_1(x))$

(iii) $\phi^{\{x|P_0(x)\}}$, the relativization of ϕ to the "set" P_0, for every $\phi \in \Phi$. This means that $\{x|P_0(x)\}$, seen as a substructure, is a model for Φ.

(iv) $\forall x \forall y[\forall z(E(z,x) \leftrightarrow E(z,y)) \rightarrow x = y]$, stating that the interpretation of E is an extensional relation and

(v) $\forall x[P_1(x) \rightarrow \forall y(E(y,x) \rightarrow P_0(y))]$, which, together with (iv) above, bounds $\{x|P_1(x)\}$ to have at most one element for every E-subset of $\{x|P_0(x)\}$.

Now, for any structure $\mathfrak{A} \models \Psi$ let $A_0 = \{a|P_0(a)\}$ and $A_1 = \{a|P_1(a)\}$. Then, by (i), $|\mathfrak{A}| = |A_0| + |A_1|$. By (v) there is a function $f : A_1 \rightarrow P(A_0)$ defined by

$$f = \{\langle a,b\rangle | a \in A_1 \text{ and } b = \{c|E(c,a)\}\}.$$

From (iv) follows that f is injective, so $|A_1| \leq 2^{|A_0|}$. Since Φ pins down cardinals, $|A_0| < h_\kappa(\mathscr{L})$, so $A < 2^{h_\kappa(\mathscr{L})}$ and Ψ pins down cardinals. On the other hand we can construct a structure $\mathfrak{B} \models \Psi$ such that $|\mathfrak{B}| = |\alpha| + 2^{|\alpha|}$, so $h_\kappa(\mathscr{L}) > 2^{|\alpha|}$, or equivalently, $h_\kappa(\mathscr{L}) > \beth_1(|\alpha|)$. Constructing a set of sentences Φ_α, where the power set operation is iterated α times, for each ordinal α which is "pinnable" in $\mathscr{L}$ it is possible to prove (cf. (Ebbinghaus [1985])):

3.2.5 Theorem.

Given a cardinal κ, assume that, for every ordinal $\alpha < w_\kappa(\mathscr{L})$, there is a set of sentences Φ_α and a structure $\mathfrak{A}_\alpha$ such that

(i) α is pinned down by Φ_α,

(ii) $|\Phi_\alpha| < \kappa$,

(ii) $\mathfrak{A}_\alpha \models \Phi_\alpha$,

(iv) $|\mathfrak{A}_\alpha| < h_\kappa(\mathscr{L})$ and

(iii) $<^{\mathfrak{A}_\alpha}$ is of order type α.

Then, for every $\lambda < h_\kappa(\mathscr{L})$, $h_\kappa(\mathscr{L}) \geq \beth_{w_\kappa(\mathscr{L})}(\lambda)$.

3.3 $\mathscr{L}_{\infty\omega}$ is weak.

The result that $\mathscr{L}_{\infty\omega}$ is weak was first proven by Lopez-Escobar (Lopez-Escobar [1966a]). First, by generalizing results by Morley (Morley [1965]) and Helling (Helling [1964]) on an upper bound for $h_\kappa(\mathscr{L}_{\omega_1\omega})$, Lopez-Escobar showed that $h(\mathscr{L}_{\kappa\omega}) < \beth_{(2^\kappa)^+}$. Then, using methods introduced by Hanf (Hanf [1962]), he could show that *if* $\mathscr{L}_{\kappa\omega}$ is strong, *then* $\mathscr{L}_{\kappa\omega}$ pins down the cardinal $\beth_\lambda$, where $\lambda = 2^{2^{2^\kappa}}$, yielding a lower bound for $h(\mathscr{L}_{\kappa\omega})$. In the proof we will be investigating, due to (Dickmann [1975]), this is done in a way similar to the construction in Theorem 3.2.5. From this we then derive a contradiction.

In this section we hope to shed some light on the constructions of both the upper and lower bounds for $h(\mathscr{L}_{\kappa\omega})$ mentioned above. We start with a central theorem:

3.3.1 Theorem.

$h(\mathscr{L}_{\kappa^+\omega}) < \beth_{(2^\kappa)^+}$

The proof of this theorem depends on some preliminary results. Before we can look at these, however, we will be needing the concept of *types* from first order logic:

3.3.2 Definition.

Given a signature τ, an *n-type over* τ is a set Σ of first order formulae such that

(1) All $\sigma \in \Sigma$ are $\mathscr{L}_{\omega\omega}[\tau]$-formulae,

(2) The free variables of all $\sigma \in \Sigma$ are found among $x_0, ..., x_{n-1}$ and

(3) Σ is consistent, which is to say that there exists a model $\mathfrak{A}$ and a sequence $a_i \in A$, $0 \leq i < n$ such that $\mathfrak{A} \models \sigma[a_0, ..., a_{n-1}]$ for all $\sigma \in \Sigma$ and

(4) Σ is maximal in the sense that Σmay not be extended to contain new formulae without breaking this consistency.

Often the signature and the number of free variables of a type is implicit from the context, in which case we may ommit them, referring to a type simply as "a type over τ" or even just "a type". Also, since any consistent set of formulae which share a number of free variables may be expanded to be maximal, we will at some points be using the term "type" somewhat loosely to refer to such sets.

For completenes we also repeat what it means for a type to be omitted:

3.3.3 Definition.

Let Σ be an n-type over τ and $\mathfrak{A}$ a τ-structure. If there exist $a_0, ..., a_{n-1}$ in A such that $\mathfrak{A} \models \sigma[a_0, ..., a_{n-1}]$ for all $\sigma \in \Sigma$ we say that Σ is *realized* in $\mathfrak{A}$ (or that $\mathfrak{A}$ *realizes* Σ). Otherwise Σ is *omitted* in $\mathfrak{A}$ ($\mathfrak{A}$ *omits* Σ) . Further, if $\mathbf{S}$ is a set of types, we say that $\mathbf{S}$ is *omitted* in $\mathfrak{A}$ (or that $\mathfrak{A}$ *omits* $\mathbf{S}$) iff all Σ in $\mathbf{S}$ are omitted in $\mathfrak{A}$. Otherwise we say that $\mathbf{S}$ is *realized* in $\mathfrak{A}$ ($\mathfrak{A}$ *realizes* $\mathbf{S}$).

Now we turn back to proving Theorem 3.3.1. For this we need the concept of *Morley number*:

3.3.4 Definition.

For any cardinal λ and any signature τ, we say that a set $\mathbf{S}$ of types over τ *omits the cardinal* λ iff $\mathbf{S}$ is omitted in a structure of cardinality λ but is not omitted in arbitrarily large structures.
Given a cardinal κ, let $\mathbf{U}_\kappa$ be the set of all sets of cardinality $\leq \kappa$ of types over signatures of cardinality at most κ. Then, the *Morley number* $\mathfrak{m}_\kappa$ is defined as $sup\{\lambda|\mathbf{S}$ omits λ, $\mathbf{S} \in \mathbf{U}_\kappa\}$. We also define a *second Morley number* $\mathfrak{n}_\kappa$ as $sup\{\lambda|\mathbf{S}$ omits λ, $\mathbf{S} \in \mathbf{U}\}$, where $\mathbf{U}$ is the union of $\mathbf{U}_\kappa$ over all κ.

From this definition it is obvious that $\mathfrak{m}_\kappa \leq \mathfrak{n}_\kappa$. Given a signature τ of cardinality κ, there are at most κ $\mathscr{L}_{\omega\omega}[\tau]$-formulae. Since each τ-type is a set of formulae chosen from among the κ τ-formulae, there are at most 2^κ τ-types. Denote this set $Type[\tau]$. Now, any $\mathbf{S} \in \mathbf{U}_\kappa$ is a subset of $Type[\tau]$ of cardinality at most κ. There are $(2^\kappa)^\kappa$ such subsets, so $|\mathbf{U}_\kappa| \leq (2^\kappa)^\kappa = 2^\kappa$. By a similar argument, $|\mathbf{U}| \leq 2^{2^\kappa}$. Taking the signature τ to be a κ-large set of unary predicates, we can construct exactly 2^{2^κ}. As can be seen in the proof for Corollary 3.3.7, this means that $\mathfrak{m}_\kappa < \mathfrak{n}_\kappa$.

The following result describes the size of the second Morley number $\mathfrak{n}_\kappa$. This is given without proof.

3.3.5 Theorem

$\mathfrak{n}_\kappa = \beth_{(2^\kappa)^+}$

This result can be found in (Chang [1968]), building on Morleys proof of $\mathfrak{m}_\omega = \beth_{\omega_1}$ in (Morley [1965]). The proof makes use of a construction with indiscernibles and the Erdős-Rado theorem. For another proof, see (Dickmann [1975]).

We will also be needing the following consequence of the Löwenheim-Skolem theorem (downward):

3.3.6 Fact

Given a type Σ of signature τ and a structure $\mathfrak{A}$ omitting Σ there are structures omitting Σ in every cardinality λ such that $max\{|\tau|, \aleph_0\} \leq \lambda \leq |\mathfrak{A}|$.

From theorem 3.3.3 and this fact we may now derive an upper bound for the *Morley number* $\mathfrak{m}_\kappa$:

3.3.7 Corollary

$\mathfrak{m}_\kappa < \beth_{(2^\kappa)^+}$

Proof. Given a signature μ of cardinality at most κ, there are at most 2^κ sets of types over μ of cardinality $\leq \kappa$ (compare with the number of subsets of κ). Let $\langle \mathbf{S}_\xi | \xi < 2^\kappa \rangle$ be a listing of these. For every ξ, let λ_ξ be the least cardinal such that *if* $\mathbf{S}_\xi$ *is omitted in a structure of cardinality at least* λ_ξ *then it is omitted in arbitrarily large structures.* The morley number $\mathfrak{m}_\kappa$ is the supremum of these λ_ξ by definition. Now, any $\lambda \geq \beth_{(2^\kappa)^+}$ must have the mentioned property by Theorem 3.3.5. Also, if $\mathbf{S}_\xi$ is omitted in the cardinal $\lambda \geq \beth_{(2^\kappa)^+}$, it is omitted in cardinalities less than $\beth_{(2^\kappa)^+}$ by Fact 3.3.6. By this reasoning, $\lambda_\xi < \beth_{(2^\kappa)^+}$ for all $\xi < 2^\kappa$. Now, since $cf(\beth_{(2^\kappa)^+}) = (2^\kappa)^+$ and $(2^\kappa)^+$ is regular, $sup\{\lambda_\xi | \xi < 2^\kappa\} < \beth_{(2^\kappa)^+}$. □

This result gives us an upper bound for Morley numbers, concerning sets of $\mathscr{L}_{\omega\omega}$-types. We will also be needing a way to relate sets of first order types to sentences in $\mathscr{L}_{\kappa^+\omega}$. Intuitively such a sentence corresponds to a set of cardinality κ of types in the following way: If $\mathbf{S}$ is a set of types over μ, $\mathbf{S}$ is ommitted in a structure $\mathfrak{A}$ exactly when *no* type in $\mathbf{S}$ is realised in $\mathfrak{A}$, corresponding to the $\mathscr{L}_{\kappa^+\omega}$-sentence $\bigwedge_{\Sigma \in \mathbf{S}} \neg \exists x \bigwedge_{\phi \in \Sigma} \phi(x)$ being true in $\mathfrak{A}$. In this way we can codify $\mathscr{L}_{\kappa^+\omega}$-sentences as κ-large sets of types and vice versa. More formally:

3.3.8 Proposition

$h(\mathscr{L}_{\kappa^+\omega}) = \mathfrak{m}_\kappa$

(Sketch of proof). The direction that $\mathfrak{m}_\kappa \leq h(\mathscr{L}_{\kappa^+\omega})$, i.e that, if a set $\mathbf{S}$ of at most κ types is ommitted by a structure of cardinality at least $h(\mathscr{L}_{\kappa^+\omega})$, then it is omitted by arbitrarily large structures, is proven by constructing a sentence ϕ which is the conjunction over all $\Sigma \in \mathbf{S}$ of the sentences$\neg \exists x \bigwedge_{\phi \in \sigma} \phi(x)$. This has the property

$$\mathfrak{A} \models \phi \iff \mathfrak{A} \text{ omits every type of } \mathbf{S}$$

and is a sentence in $\mathscr{L}_{\kappa^+\omega}$.
For the other direction, i.e $h(\mathscr{L}_{\kappa^+\omega}) \leq \mathfrak{m}_\kappa$, given any sentence $\phi \in \mathscr{L}_{\kappa^+\omega}$ (these sentences correspond to *sets of sentences* of cardinality at most κ^+) one can expand the signature of ϕ with one predicate symbol R_ψ for each subformula ψ of ϕ and construct a set $\mathbf{S}$ of types, codifying the satisfiability of subformulae of ϕ. Now one can show that for any structure $\mathfrak{A}$, $\mathfrak{A} \models \phi$ iff there is an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ (expanded with all the R_ψ) omitting $\mathbf{S}$. Now if there is a model $\mathfrak{B} \models \phi$ of cardinality at least $\mathfrak{m}_\kappa$ then there is an expansion $\mathfrak{B}'$ omitting $\mathbf{S}$, which is also of cardinality at least $\mathfrak{m}_\kappa$, but this in turn means there are arbitrarily large structures omitting $\mathbf{S}$ and therefore arbitrarily large models of ϕ. □

This was first proven for the case $\kappa = \omega$ by López-Escobar (Dickmann [1975]). The result cited here, in turn, is due to Chang (Chang [1968]).

Now, Theorem 3.3.1, stating that $h(\mathscr{L}_{\kappa^+\omega}) < \beth_{(2^\kappa)^+}$, is a straightforward consequence of Proposition 3.3.8 and Corollary 3.3.7. Using this upper bound for the hanf number of a finite quantifier logic we can now prove the non-definability of well-orderings in those logics. More generally:

3.3.9 Theorem (López-Escobar)

WO is not PC in $\mathscr{L}_{\infty\omega}$.

In the proof of this theorem we start by assuming that well orderings are PC in $\mathscr{L}_{\kappa\omega}$ and contruct a theory which has large, but not arbitrarily large, models. This then results in a lower bound on the hanf number of $\mathscr{L}_{\kappa\omega}$ which is larger than the upper bound fixed in Theorem 3.3.1, leading to a contradiction. The construction in this proof is much like that in the discussion leading up to Theorem 3.2.5. We will also be using the formulae μ_α from Proposition 3.1.2.

Proof. Assume for a contradiction that WO is a projective class in $\mathscr{L}_{\infty\omega}$. Then there are λ σ, ϕ such that $\phi \in \mathscr{L}_{\lambda\omega}[\sigma]$ and $WO = mod(\phi) \restriction \{<\}$. We may assume that λ is a successor cardinal such that $\lambda = \kappa^+$ and $|\sigma| < \kappa$. The construction we are aiming at now is a set of sentences Θ in a signature $\tau \supseteq \sigma$ such that

(I) there is $\mathfrak{A} \vDash \phi \wedge \Theta$ such that $|\mathfrak{A}| = \beth_\mu$, where $\mu = 2^{2^{2^\kappa}}$, and

(II) for all $\mathfrak{A}$, $\mathfrak{A} \vDash \phi \wedge \Theta \implies |\mathfrak{A}| \leq \beth_\mu$.

This would then mean that $h(\mathscr{L}_{\lambda\omega}) \geq \beth_{2^{2^\kappa}}$, but as we already know, $h(\mathscr{L}_{\lambda\omega}) = h(\mathscr{L}_{\kappa^+\omega}) < \beth_{(2^\kappa)^+} < \beth_{2^{2^\kappa}}$.

For this end we expand the signature σ to a signature τ containing the additional symbols

(i) $P_1, ..., P_5$ (binary predicates)
(ii) P_6 (trinary predicate)
(iii) P_7 (unary predicate)
(iv) $c_0, ..., c_6$ (constants).

Now, let $\Phi_\alpha(x)$ be the formula $\forall y[y < x \leftrightarrow \bigvee_{\beta<\alpha} \mu_\beta]$. This has the property that, in a well-ordered structure $\mathfrak{A}$, $\mathfrak{A} \vDash \Phi_\alpha(x)[a]$ iff the initial segment $\{b|b < a\}$ is order isomorphic to α. Finally, we let Θ be the following set of sentences:

(1) $\forall x\forall y[P_1(x, y) \leftrightarrow x < y \wedge \neg\exists z(x < z \wedge z < y)]$
(2) $\forall x[P_7(x) \leftrightarrow c_0 < x \wedge \forall y(y < x \to \exists z(y < z \wedge z < x))]$
(3) $\Phi_0(c_0)$
(4) $\Phi_\omega(c_1)$
(5) $\Phi_\kappa(c_2)$
(6) $\forall x\forall y\,[x < c_3 \wedge y < c_3 \to (\forall z\,(P_2(x, z) \leftrightarrow P_2(y, z)) \to x = y)]$
(7) $\forall x\forall y\,[x < c_4 \wedge y < c_4 \to (\forall z\,(P_3(x, z) \leftrightarrow P_3(y, z)) \to x = y)]$
(8) $\forall x\forall y\,[x < c_5 \wedge y < c_5 \to (\forall z\,(P_4(x, z) \leftrightarrow P_4(y, z)) \to x = y)]$
(9) $\forall x\forall y\,[P_2(x, y) \to y < c_2]$
(10) $\forall x\forall y\,[P_3(x, y) \to y < c_3]$
(11) $\forall x\forall y\,[P_4(x, y) \to y < c_4]$
(12) $\forall x\forall y\forall z\,[y < x \wedge z < x\,(\forall w(P_6(x, y, w) \leftrightarrow P_6(x, z, w)) \to y = z)]$
(13) $\forall x\forall y\forall z\,[P_5(x, y) \wedge P_5(x, z) \to y = z]$
(14) $\forall x[\exists y P_5(x, y) \leftrightarrow x \leq c_5]$
(15) $P_5(c_0, c_1) \wedge P_5(c_5, c_6)$
(16) $\forall x\forall y\forall z\forall w\,[P_1(x, y) \wedge P_5(x, z) \wedge P_5(y, w) \to \forall u\forall v\,(P_6(w, u, v) \to v < z)]$
(17) $\forall x\forall y\,[P_7(x) \wedge P_5(x, y) \to \forall z\,(z < y \to \exists u\exists v\,(u < x \wedge P_5(u, v) \wedge z < v))]$
(18) $\forall x(x \leq c_6)$

To show (II) above, let $\mathfrak{B} \vDash \phi \wedge \Theta$. Since $\mathfrak{B}$ is a model of ϕ, the reduct $\mathfrak{B} \restriction \{<\}$ is order isomorphic to some ordinal δ. We may assume WLOG that $\mathfrak{B} \restriction \{<\}$ is $\langle\delta, \in\restriction\delta\rangle$. This means that the interpretations of the constants c_i in $\mathfrak{B}$ are ordinals; call these ξ_i. By (3)-(5) above, $\xi_0 = 0$, $\xi_1 = \omega$ and $\xi_2 = \kappa$. Now, by (6) the function $f = \{\langle\alpha, A\rangle|\alpha < \xi_3$ and $A = \{\beta|\langle\alpha, \beta\rangle \in P_2^{\mathfrak{B}}\}$ is injective and by (9) $f_2(\alpha) \subseteq \xi_2$ for every $\alpha < \xi_3$, so $|\xi_3| \leq 2^\kappa$. By similar arguments, $|\xi_4| \leq 2^{2^\kappa}$ and $|\xi_5| \leq 2^{2^{2^\kappa}} < \mu^+$. For $\alpha \in \delta$, we define the functions

$$g_\alpha = \{\langle\beta, B_\alpha\rangle|\beta < \alpha \text{ and } B_\alpha = \{\gamma|\langle\alpha, \beta, \gamma\rangle \in P_6^{\mathfrak{B}}\}\}.$$

By (12), then,

(#) $\qquad g_\alpha$ is an injection on the set α.

Furthermore, by (1) the interpretation of P_1 must be the successor-relation in any well-ordered structure, and by (2) P_7 must be the set of limit ordinals. Also, from (13)-(15) we see that $P_5^{\mathfrak{B}}$ is a functional relation with domain $\xi_5 + 1$ such that $P_5^{\mathfrak{B}}(0) = \omega$ and $P_5^{\mathfrak{B}}(\xi_5) = \xi_6$.

Now, using (16) and (#) we can show that

(*) $\qquad \left|P_5^{\mathfrak{B}}(\alpha + 1)\right| \leq 2^{\left|P_5^{\mathfrak{B}}(\alpha)\right|}$ for all $\alpha \leq \xi_5$,

that is, $P_5^{\mathfrak{B}}$ grows at most as fast as the $\beth$-sequence for successor ordinals. Using (17) an (*), then, we can show through induction over α that

$$\left|P_5^{\mathfrak{B}}(\alpha)\right| \leq \beth_\alpha \text{ for all } \alpha \leq \xi_5.$$

Finally, since (18) gives us that $\delta = \xi_6 + 1$, we can conclude that $|\mathfrak{B}| = |\delta| = |\xi_6| = \left|P_5^{\mathfrak{B}}(\xi_5)\right| \leq \beth_{\xi_5} < \beth_{\mu^+}$.

For (I), let $\mathfrak{C} \models \phi$, $\mathfrak{C}$ order equivalent to $\nu = \beth_\mu + 1$. This structure exists, since $mod(\phi) = WO$. Take, for exampel, $\mathfrak{C} = \langle \nu, \in \restriction \nu \rangle$. Now, by describing how $\mathfrak{C}$ is expanded into a τ-structure $\mathfrak{D} \models \Theta \wedge \phi$ we will present a structure satisfying (I). Note first that (18) must be true in any structure extending $\mathfrak{C}$. In $\mathfrak{D}$ we intrerpret the extra symbols of τ as follows: $c_0^{\mathfrak{D}} = 0$, $c_1^{\mathfrak{D}} = \omega$, $c_2^{\mathfrak{D}} = \kappa$, $c_3^{\mathfrak{D}} = 2^\kappa$, $c_4^{\mathfrak{D}} = 2^{2^\kappa}$, $c_5^{\mathfrak{D}} = 2^{2^{2^\kappa}}$, $c_6^{\mathfrak{D}} = \beth_\mu$,

$$\begin{aligned} P_1^{\mathfrak{D}} &= \{\langle \alpha, \alpha+1 \rangle \,| \alpha \in \aleph_\mu\}, \\ P_5^{\mathfrak{D}} &= \{\langle \alpha, \beth_\alpha \rangle | \alpha \leq \mu\}, \\ P_7^{\mathfrak{D}} &= \{\alpha | 0 < \alpha \leq \beth_\mu,\ \alpha \text{ limit ordinal}\}. \end{aligned}$$

By the relative sizes of the ordinals $c_2^{\mathfrak{D}}$, $c_3^{\mathfrak{D}}$ and $c_4^{\mathfrak{D}}$ it is now possible to find bijections $f_i : c_{i+1}^{\mathfrak{D}} \to P(c_i^{\mathfrak{D}})$ for $i = 2, 3, 4$. Using these we interpret

$$P_i^{\mathfrak{D}} = \{\langle \alpha, \beta \rangle | \alpha < c_i \text{ and } \beta \in f_i(\alpha)\}.$$

Similarly, for each $\alpha < \mu$, let $g_\alpha : \beth_{\alpha+1} \to P(\beth_\alpha)$ be a bijection. These exist by the definition of the $\beth$-sequence. This makes the interpretation

$$P_6^{\mathfrak{D}} = \{\langle \beth_{\alpha+1}, \beta, \gamma \rangle | \alpha < \mu,\ \beta \in \beth_{\alpha+1} \text{ and } \gamma \in g_\alpha(\beta)\}$$

well-defined. By this construction it is clear that $\mathfrak{D} \models \Theta$, concluding the proof. □

In his original article, Lopez-Escobar showed that "WO is not RPC in $\mathscr{L}_{\infty\omega}$" (Lopez-Escobar [1966b]). Since every PC class is RPC, this result by Lopez-Escobar might be stronger that the one we have proven here, though this is not clear (cf. (Dickmann [1975])).

In conclusion we give an existence result as a corollary.

3.3.10 Corollary.

There exists a regular logic $\mathscr{L}$ which is both unbounded and weak.

Proof. $\mathscr{L}_{\infty\omega}$ is a regular logic (Dickmann [1975]). It is unbounded by Theorem 3.1.4, but weak by Theorem 3.3.9. □

4 Conclusion

We have seen that there are many ways of extending first order logic to logics with greater expressive power. It also became apparent that extending the logic destroys some of the "nice" properties of first order logic, such as completeness and compactness. We have, however, only briefly mentioned the question of how strong a logic we can have without loosing the "niceness". For more on this topic, see (Flum [1985]), as well as (Nadel [1985]).

The idea of this paper was in a sense to compare the expressiveness of logics. To make this idea more precise we chose to define the expressiveness of a logic $\mathscr{L}$ as the class of elementary or *projective* classes of $\mathscr{L}$. This made it possible to not only pinpoint what was expressible in a logic, but also to compare two logics. Here we could have chosen to study *relativized projective* classes and define expressiveness in terms of these. This choise was informed by two facts. As one moves from elementary to projective and relativized projective classes, the expressibility increases, since $EC_{\mathscr{L}} \subseteq PC_{\mathscr{L}} \subseteq RPC_{\mathscr{L}}$. This increase in expressiveness, on the other hand, is due to the fact that the definition of a model class uses an increasingly larger machinery with extra non-logical symbols and inner models as we traverse the hierarchy. So, elementary classes might not capture all there is to a logic because the definition of a model class is restricted to not using any symbols beyond the definiendum. The increase in expressiveness found in relativized projective classes, on the other hand, might be more due to the use of an intricate construction than it is to the expressiveness of the underlying logic.

That being said, we have mostly not entered into these discussion, but rather tried to give an overview of what the different kinds of expressiveness amount to.

In the last part of this paper we looked specifically at what well-orderings we could define in a logic. We saw there that $\mathscr{L}_{\omega_1\omega_1}$ was expressive enough to define all of WO, the class of well-orderings. We saw also that it was impossible to do the same in $\mathscr{L}_{\kappa\omega}$, for any κ, implying the undefinability of WO in $\mathscr{L}_{\infty\omega}$.

This result was proven through an argument about Hanf numbers. The same result follows from other, related, properties of abstract logics, however. For this we are going to be needing the *Craig* property of a logic:

Definition.

Given a logic $\mathscr{L}$, if, for every signature τ and every pair of model classes K_1 and K_2, projective in $\mathscr{L}[\tau]$ there exists an elementary $\mathscr{L}[\tau]$-class K such that $K_1 \subseteq K$ and $K_2 \cap K = \emptyset$, then we say that $\mathscr{L}$ has the *interpolation property.*

If, K_1 and K_2 are as in the definition and K can always be found in another (stronger) logic $\mathscr{L}'$, then we say that $\mathscr{L}'$ *interpolates* $\mathscr{L}$, symbolically, $\text{Craig}(\mathscr{L}, \mathscr{L}')$.

Due to a result by Gostanian and Hrbáček, if $\mathscr{L}_{\kappa^+\omega}$ pins down a regular ordinal α, then $\neg\text{Craig}(\mathscr{L}_{\kappa^+\omega}, \mathscr{L}_{\alpha\alpha})$. On the other hand, Malitz has shown that, for regular κ, $\text{Craig}(\mathscr{L}_{\kappa\omega}, \mathscr{L}_{(2^{<\kappa})^+\omega})$. This together with the assumption that $\mathscr{L}_{\infty\omega}$ is strong yields a contradiction. (Väänänen [2008]).

A few loose ends have been left in the preceeding text for various reasons. These are new, interresting questions arising from the study of model theoretic logics or the definability of well-orderings.

The discussion leading up to Theorem 3.1.4 left open the question of an upper bound for the well-ordering numbers of the logics $\mathscr{L}_{\kappa\omega}$. The interrested reader should find more about this in (Dickmann [1975]) and (Dickmann [1985]).

The section about suborderings gives rise to some questions. There are examples of logics showing that $\leq_{RPC}$ is not a subordering of (and thus not equivalent to) $\leq$, but the author is still to find examples showing that $\leq_{PC}$ is not equivalent to $\leq_{RPC}$. It is also known that $\mathscr{L}^{w2} \nleq \mathscr{L}(Q_0)$ (Shapiro [2001]). On the other hand, the proposition $\mathscr{L}^{w2} \leq_{PC} \mathscr{L}(Q_0)$ is dependent on the open question in computational complexity theory whether NP is equal to PH.

The question about comparing the expressive power of logics through the definability of well-orderings, which introduced the subject of abstract model theory to the author, seems to have been answered partly. While there is some information to be gathered in the well-ordering numbers of logics and the weak/strong-dichotomy, these pieces of information alone do not settle the question about relative expressive power of logics. For example they say little about the finite model theory of a logic. Also, the ordering on logics we introduced were all seen to be partial, whereas an ordering based on well-ordering numbers would be linear (even a well-ordering).

Related to this there is another direction in which further study would be of interrest. Given *any* mathematical structure or property of structures P, which is not definable in first order logic, the question about which logics define P yields more information about the expressiveness of model theoretic logics. Examples of interresting classes are *Complete Partial Orderings* (Dickmann [1985]) and $\aleph_1$-*free abelian groups* vs. *free abelian groups* (Nadel [1985]).

References

J. Barwise. *Model-Theoretic Logics*, volume 8 of *Perspectives in Mathematical Logic*, chapter 1, pages 3–23. Springer Verlag, New York, 1985.

C. C. Chang. Some remarks on the model theory of infinitary languages. In J (ed.) Barwise, editor, *The syntax and semantics of infinitary languages*, Lecture Notes in Mathematics. Springer Verlag, Berlin, 1968.

H. J. Chang, C. C.; Keisler. *Model Theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North Holland, Amsterdam, 1990.

M. A. Dickmann. *Large infinitary languages*. North-Holland, 1975.

M. A. Dickmann. *Model theoretic logics*, volume 8 of *Perspectives in Mathematical Logic*, chapter 9, pages 317–363. Springer Verlag, New York, 1985.

H.-D. Ebbinghaus. *Model theoretic logics*, volume 8 of *Perspectives in Mathematical Logic*, chapter 2, pages 25–73. Springer Verlag, New York, 1985.

S. Feferman. Two notes on abstract model theory, i: Properties invariant on the range of definable relations between structures. *Fundamenta Mathematicae*, 82:153–165, 1974.

J. Flum. *Model theoretic logics*, volume 8 of *Perspectives in Mathematical Logic*, chapter 3, pages 77–120. Springer Verlag, New York, 1985.

G. Fuhrken. Skolem-type normal forms for first-order languages with a generalized quantifier. *Fundamenta Mathematicae*, 54:291–302., 1964.

W. P. Hanf. *Some fundamental problems concerning languages with infinitely long expressions*. PhD thesis, University of California, Berkley, 1962.

M. Helling. Hanf numbers for some generalizations of first-order languages. *Notices of the American Mathematical Society*, 11:679., 1964.

Ph. G. Kolaitis. *Model theoretic logics*, volume 8 of *Perspectives in Mathematical Logic*, chapter 10, pages 365–421. Springer Verlag, New York, 1985.

P. Lindström. First order predicate logic with generalized quantifiers. *Theoria*, 32:186–195, 1966.

P. Lindström. On extensions of elementary logic. *Theoria*, 35:1–11, 1969.

E. G. K. Lopez-Escobar. An interpolation theorem for denumerably long formulas. *Fundamenta Mathematicae*, 57:253–272, 1965.

E. G. K. Lopez-Escobar. On defining well-orderings. *Fundamenta Mathematicae*, 59:13–21, 1966a.

E. G. K. Lopez-Escobar. An addition to "on defining well-orderings". *Fundamenta Mathematicae*, 59:299–300, 1966b.

M. Morley. Omitting classes of elements. In *The Theory of Models*. North-Holland, 1965.

A. Mostowski. On a generalization of quantifiers. *Fundamenta Mathematicae*, 44:12–36, 1957.

A. Mostowski. Craigs interpolation theorem in some extended systems of logic. In *Logic, methodology and philosophical science III (Proceedings of Third International Congress)*. North-Holland, 1968.

D. Mundici. *Model theoretic logics*, volume 8 of *Perspectives in Mathematical Logic*, chapter 6, pages 211–233. Springer Verlag, New York, 1985.

M. Nadel. *Model theoretic logics*, volume 8 of *Perspectives in Mathematical Logic*, chapter 8, pages 271–287. Springer Verlag, New York, 1985.

S. Shapiro. Systems between first-order and second-order logic. In F Gabbay, D M; Guenthner, editor, *Handbook of Philosophical Logic*, volume 1. Kluwver academic publishers, Dordrecht, 2nd edition, 2001.

J. Väänänen. Remarks on generalized quantifiers and second-order logics. *Set theory and hierarchy theory*, 14:117–123, 1977.

J. Väänänen. The craig interpolation theorem in abstract model theory. *Synthese*, 164:401–420, 2008.