# Coasean Bargaining Games with Stochastic Stock Externalities

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**Abstract**

The recent approach 'subgame consistency' in cooperative stochastic differential games by Yeung and Petrosjan (2006) and Yeung and Petrosjan (2004) is applied to the classical Coase theorem in the presence of stochastic stock externalities. The dynamic Coasean bargaining solution is identified involving a negotiated plan of externality trade over time as well as subgame consistent Coasean liability payments flow under different assignments of property rights. The agent with the right to determine the externality has the advantage to choose his own private equilibrium as the initial condition in the dynamic system of the Coasean bargaining solution. The dynamic Coasean bargaining solution is formulated followed by an illustration showing an analytical tractable solution.

**Keywords**: dynamic cooperative games, cooperative stochastic differential games, dynamic stability, Coase theorem

**JEL classification**: C71, C73, Q53, Q56

## 1 Introduction

The paper by Coase (1960) is one of the most well known papers in the economic literature. It contains an illustrative argument that a difference between private and social costs will disappear, resulting in a Pareto efficient outcome, if two agents are allowed to bargain about the level of externality. If the Coase theorem works, all that is necessary to cure the inefficient allocation is a common law that clearly assigns well-defined rights over the level of externality to one of the agents rather than a social planner that enforces a Pareto optimal externality level. There is a vast literature on the Coase theorem covering a wide range of aspects, however, so far analyzes of the Coase theorem have not yet been performed in dynamic game theory with stock externalities, and nevertheless stochastic stock externalities. One possible reason for this is the technical difficulties within cooperative (stochastic) differential game theory.

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An early paper on extending the Nash bargaining solution to cooperative differential games was Haurie (1976) pointing out the difficulties of 'dynamic instability'. However, during the last decade there has been a growing literature on time consistent and dynamic stability in cooperative differential games e.g. Petrosjan and Zenkevich (1996), Petrosjan (1997) and Filar and Petrosjan (2000). There is also a literature on time consistent side payments in upstream-downstream problems e.g. (Jorgensen and Zaccour, 2001) and Haurie and Zaccour (1995). Recently, a literature has emerged on the issue of finding time consistent payoff distributions in cooperative stochastic differential games, e.g. Yeung and Petrosjan (2004) (TU-games), Yeung (2004) and Yeung and Petrosjan (2005) (NTU-games). In this paper, the classical set up of the Coase theorem in the upstream-downstream case, adding stochastic stock externalities, is analyzed using the recent discoveries in cooperative stochastic differential game theory.

Consider first the static case of Coase theorem that is well known in the literature. The upstream agent has the legal right to determine the externality (e.g. emit pollution) and the starting point is upstream agent's private equilibrium activity level. The downstream agent, who suffers from the upstream externality (e.g. pollution), then has an incentive to offer a compensation to the upstream agent provided that he reduces the externality level. Since downstream agent knows upstream agent's objective function, the former can offer an allocation flow, which increases upstream agent's utility and gives him an incentive to accept the offer. Continuing this process of Pareto improvements the agents would eventually end up in a Pareto optimal activity levels. By symmetry, the same result holds if the downstream agent has the right the determine the level of externality (be free from pollution). Pareto improvement is possible because the reductions in externality levels and the private gains can be traded between the agents. Since one agent has the right to determine the externality and the gains to the other agent from reduction in externality level are private, the market mechanism will lead the bargaining agents to a Pareto efficient outcome just as for ordinary goods and services where rights are defined.

Moreover, the Coase theorem states that initial assignment of rights (e.g. whether a polluter is given the right to pollute or a sufferer is given the right to be free from pollution) does not affect the Pareto optimal outcome (efficiency proposition). Some arguments also include that the final allocation of resources will be the same regardless assignments of rights (invariance proposition). The first argument can be considered as a weak version of Coase theorem and both arguments together as a strong version of the Coase theorem. Several papers have discussed the limits of the theorem by setting up assumptions under which the theorem holds or does not hold. The theorem has been stated in different ways, e.g. Regan (1972), Calabresi (1968) and Fresh (1979), but in general the common assumptions usually involve an economy with perfect competition, two actors to each externality bargain, perfect information, no wealth effects and zero transactions costs. For a discussion on assumptions see e.g. Hoffman and Spitzer (1982). The majority of the papers in literature concerns imperfections within the Coase bargaining process such as transaction costs and other imperfections that may disturb the bargaining process, such as insufficient rent to be able to pay liabilities (Tybout, 1972) and (Wellisz, 1964), non-convexity (Starret, 1972), imperfect information in noncooperative game theory (Saraydar, 1983), moral hazard (Kamien et al., 1966) and (Tybout, 1972), private information (Schweizer, 1988) and different kind of transaction costs, e.g. (Allen, 1991) and (Barzel, 1989). This paper does not deal with aspects of imperfections within the Coase bargaining process but will assume that the bargaining process takes place at time $t = 0$ with no transaction costs and results in a Pareto optimal bargaining outcome. The contribution of this paper is rather of another dimension. It illustrates the Coase theorem in a dynamic model with stock dynamics. The essential difference, compared to the static analysis that has been performed hitherto, is that the initial bargaining not only concerns an externality level with a corresponding lump sum reallocation, but rather a 'plan' of Pareto optimal controls and a liability payment flow to the agent that has the legal right to determine the externality level.

## 1.1 The Coase Theorem and Stock Externalities

We suppose that Coasean bargaining between two agents takes place at $t = 0$ and that the agreement contains a future 'plan' of agreed (Pareto optimal) reductions in externality level, as well as a corresponding flow of liability (compensation) payment to the agent that has the legal right to determine the level of externality. What conditions should a Coasean bargaining solution satisfy in a model with stock dynamics? Firstly, the initial conditions of the dynamic system are connected to the assignment of rights. For example, if the upstream agent has the right to determine the externality (pollute), then the initial condition should be the private equilibrium of upstream agent. Conversely, if the downstream agent has the right determine the externality (be free from pollution), the initial condition is the private equilibrium of downstream agent. The Coasean bargaining solution (CBS) policy then involves a move from any of these initial states toward the Coasean steady state. Secondly, the emission flow and the necessary liability payment flow should be such that both agents are at least as well of with the bargain as without the bargain (individual rationality). Thirdly, we expect that the bargaining process at time $t = 0$ goes on until there are no further Pareto improvements, i.e. the bargaining outcome should belong to the Pareto optimal set (group rationality). The fourth issue is time consistency due to the extension to stock externalities. While the process goes on over time, the optimal conditions agreed at time $t = 0$ should remain. As a result, the plan in an agreement signed prior to $t_0$ may no longer be optimal anymore at, say $t \geq t_0$, and rational agents abandon the agreement at $t$ and the planned motions in initial agreement are therefore said to be 'dynamically instable' (Haurie, 1976). The initial plan should be time consistent, viz. include rules how the agents jointly should react to changes in interests in order to maintain the original optimal agreement at each instant of time. There are several definitions of time consistency and in this paper we follow Petrosjan (1997) subgame consistent concept implying that that the individual payoff at each instant of time over the remaining process should be no less than what the agents could get by abandon the agreement. Time consistency then implies that the initial bargaining outcome plan is maintained (i.e. remains optimal) along the process as $t$ goes to infinity and the solution approaches the expected steady state.

The disposition of the paper is as follows: In section 2, threat strategies are identified and tied to the assignment of rights and a time consistent Coasean bargaining solution is formulated introducing time consistent Coasean liability payment flows. Section 3 presents an analytically tractable solution which is followrd by a summary.

## 2 Coasean Bargaining with Stock Externalities

Consider two agents, agent 1 is the upstream agent that owns and invests in an upstream stock $k_1(t)$. Agent 2 is the downstream agent that owns and invests in the downstream stock $k_2(t)$. The state space of the game is $K \in R^2$ and the state dynamics is described by the stochastic differential equations

$$\begin{aligned} dk_1(t) &= f_1[k_1(t), c_1(t)] + \sigma_1[k_1(t), t]dz_1(t) & (1) \\ dk_2(t) &= f_2[k_1(t), k_2(t), c_2(t)] + \sigma_2[k_2(t), t]dz_2(t) & (2) \end{aligned}$$

The upstream stock $k_1(t)$ also enters (2), generating an upstream-downstream stock externality. The growth contain a stochastic growth process of each capital stock where $dz_i$ is the increment of a Wiener process $z_i(t)$ with variance $\sigma_i^2 \geq 0$ and $cov(z_i z_j) = 0$ for $i = 1, 2$ and $i \neq j$. The instantaneous payoff at time $s \in [0, \infty)$ to each agent $i = 1, 2$ is $g_i[c_i(t)]$ which may be referred to as consumption activity at time $s$. The control space contains the feasible activities $(c_2, c_1) \subseteq R_+^2$. The state space is the two-dimensional space $(k_2, k_1) \subseteq R_+^2$. We assume that the instantaneous payoffs can be transfered across agents and time. The payoff function to agent $i = 1, 2$ is then

$$\int_0^\infty g_i[c_i(t)]e^{-\rho_i t}dt \quad i = 1, 2 \tag{3}$$

where $\rho_i > 0$ is an agent-specific discount rate.

### 2.1 Coasean Bargaining Solution (CBS)

In static analysis it is well-known that the Coase theorem states that two agents in the upstream-downstream problem could gain if there are well-defined property rights and a negotiated compensation is paid by one agent to the other agent for a corresponding decrease in externality level. Recall the following facts about Coasean bargaining situation:

(1) Let agent $i$ have the right to determine the level of externality (e.g. the right to pollute if upstream agent or the right to be free from pollution if downstream agent).

(2) Agent $j$, where $i \neq j$, that lacks the right to determine the externality level collects private gains from a negotiated reduction in externality level (a reduction in damage rate as downstream agent or an increase in permitted pollution level as a upstream agent)

Pareto improvement is possible because the reductions in externality levels in (1) and the gains in (2) can be traded between the agents. Consider the case when two agents can negotiate at $t = 0$. Since the dynamic model involves changes in stock levels over time, a Coasean bargaining solution (CBS) policy is a *plan of trade* that spans over time and contains (1) agreed reduction in externality flow and (2) corresponding liability payment flows paid by $j$ to $i$ as compensation for the successive decreases in externality level over time. Moreover, when societies come together and negotiate at $t = 0$, they may a mistake and therefore the CBS should involve closed loop solutions so that the initial plan remains optimal. Define a time consistent CBS policy as

$$\begin{aligned}\Pi(k_1, k_2, t) &= \left\{\phi_1^o(k_1, k_2, t), \phi_2^o(k_1, k_2, t), L_i^o(k_1, k_2, t)\right\} \quad i = 1, 2 \\ k_1^o(0) &= k_1^i, \quad k_2^o(0) = k_2^i\end{aligned} \tag{4}$$

as a bargaining outcome that is agreed before the process starts at $t_0$ and is valid for $t \in [t_0, \infty)$ and where $L_i^o(k_1, k_2, t)$ is the instantaneous flow of Coasean liability payment to agent $i$ that has the right to determine the externality level.

We require that the bargaining solution $\Pi(k_1, k_2, t)$ satisfies individual rationality

$$W_i(k_1, k_2, t) \geq V_i(k_1, k_2, t) \quad i = 1, 2 \tag{5}$$

and that the bargaining outcome belongs to the Pareto optimal set, i.e. group rationality holds

$$\sum_{m=1}^{2} W_i(k_1, k_2, t) = W(k_1, k_2, t) \tag{6}$$

We follow (Petrosjan, 1997) and Yeung and Petrosjan (2004) and require that $W_i(k_1, k_2, t)$ for $i = 1, 2$ are continuously twice differentiable in $t$ and $(k_1, k_2)$ and that

$$W_i^t(k_1, k_2, t) = W_i^\tau(k_1, k_2, t)e^{\rho_i(\tau - t)} \quad t_0 \leq t \leq \tau \quad \text{for} \quad i = 1, 2 \tag{7}$$

The bargained outcome before time $t = 0$ remains optimal over time in any possible future state that results from current optimal behavior of the actors as well as stochastic changes in the process.

#### 2.1.1 Closed-Loop Threat Strategies

Before the bargaining begins, the agents $i = 1, 2$ make clear their threat strategies $c_i^*(t)$ given the assignment of rights. Agent $i = 1, 2$ maximizes utility by choosing optimal activity level $c_i(t)$ in the optimal control problems given that the other agent $j \neq i$ maximizes utility in a non-cooperative solution.

$$\max_{c_i(t)} \int_0^\infty g_i[c_i(t)]e^{-\rho_i t}dt \quad i = 1, 2 \tag{8}$$

s.t.

$$f_1[k_1(t), c_1(t)] + \sigma_1[k_1(t), t]dz_1(t) \tag{9}$$

$$f_2[k_1(t), k_2(t), c_2(t)] + \sigma_2[k_2(t), t]dz_2(t) \tag{10}$$

$$k_1(0) = k_{1,0} \quad k_2(0) = k_{2,0} \tag{11}$$

Clearly, in the Coasean bargaining the upstream agent has an advantage of being unaffected by the downstream agent, however, the assignment of rights will be conclusive for this advantage. Maximizing the corresponding dynamic programming equations corresponding to the problems above give the closed-loop threat strategies

$$c_i^*(k_i, t) = \phi_i[k_i(t), t] \quad i = 1, 2 \tag{12}$$

given the initial conditions $(k_1(0), k_2(0))$ determined by the agent that has the right to determine the externality level. If the dynamic programming equation is time autonomous with infinite time horizon, the controls in (12) will be Markov stationary and subgame perfect as it also holds off the equilibrium path.

#### 2.1.2 Assignment of Rights and Initial Conditions

The initial conditions $(k_1(0), k_2(0))$ could be any point in the feasible state space, though, it is reasonable that the initial states be connected to the assignment of rights. If upstream agent has the right to determine the of externality level, then the initial conditions should be the private equilibrium $PE_1$ steady state and downstream agent has to take this as given. Conversely, if downstream agent has the right to determine the externality level, the initial condition is the $PE_2$ steady state and the upstream agent has to take this choice of initial conditions as given.

Suppose upstream agent 1 has the right to determine the externality. The initial condition is the private equilibrium $PE_1(0)$ of upstream agent 1, denoted as

$$k_1^o(0) = \bar{k_1^u} \geq 0 \tag{13}$$
$$k_2^o(0) = \bar{k_2^u} \geq 0$$

Suppose instead that downstream agent 2 has the right to determine the externality generated by upstream activity. The initial condition is the steady state private equilibrium $PE_2(0)$ of downstream agent 2

$$k_1^o(0) = \bar{k_1^d} = 0 \tag{14}$$
$$k_2^o(0) = \bar{k_2^d} \geq 0$$

Since agent 2 receives no benefit but damage costs from the externality, he would not allow any externality in a non-cooperative equilibrium, and hence, the initial condition in a bargaining solution is $k_1^o(0) = 0$. This actually implies that agent 2 imposes the restriction $k_1(t) = 0$ on agent 1 in the non-cooperative equilibrium, implying that t' agent 1's disagreement value function is $V_1 \equiv 0$.

### 2.1.3 Pareto Optimal Trajectories

We suppose that negotiation, which takes place at $t = 0$, proceeds until no further Pareto improvements are possible. Let $W_i$ be the payoff (before distribution of joint payoff) to agent $i = 1, 2$, that results from a negotiation that goes on until the total payoff $W = W_1 + W_2$ is maximized. Thus, the agents agree to solve the joint stochastic optimal control problem.

**Definition 1** *Let $E$ be the expectation operator, then if there exists a value function $W(k_1, k_2, t)$ that satisfies*

$$W(k_1, k_2, t) = \tag{15}$$
$$E\left\{ \int_0^\infty \sum_{i=1}^{2} g_i[k_i^o(t), c_i^o(t)] e^{-\rho_i t} dt \right\}$$
$$\geq E\left\{ \int_0^\infty \sum_{i=1}^{2} g_i[k_i(t), c_i(t)] e^{-\rho_i t} dt \right\}$$

*for all $c_i(k_1, k_2, t)$ in the feasible set $c_i \subseteq R_+^2$ for $i = 1, 2$ which satisfy the stochastic growth processes in the $(k_2, k_1)$ state space*

$$dk_1(t) = [f_1[k_1^o(t), c_1^o(t)]\, dt + \sigma_1 k_1^o dz_1 \tag{16}$$

$$dk_2(t) = [f_2[k_1^o(t), k_2^o(t), c_2^o(t)]\, dt + \sigma_2 k_2^o dz_2 \tag{17}$$

$$k_1^o(0) = \bar{k^x}_1 \quad k_2^o(0) = \bar{k^x}_2 \quad x = u, d \tag{18}$$

*then the closed loop Pareto optimal control paths are*

$$c_i^o(t) = \phi_i\big(k_1^o, k_2^o, t\big) \tag{19}$$

*for* $i = 1, 2$

The value function $W(k_1, k_2, t)$ in (15) and the dynamic system (16) - (18) should to satisfy the Isaacs-Bellman-Fleming partial differential equation system (Basar and Olsder, 1999).

$$\begin{aligned} -\frac{\partial W(k_1, k_2, t)}{\partial t} &= \\ \max_{c_1, c_2}\left[\sum_{i=1}^{2} g_i[k_i(t), c_i(t)]e^{-\rho_i t}\right]e^{-\rho_i t} \\ +\frac{\partial W}{\partial k_1(t)} f_1[k_1(t), c_1(t)] + \frac{1}{2}\frac{\partial^2 W}{\partial k_1^2}\sigma_1^2 k_1(t)^2 \\ +\frac{\partial W}{\partial k_2(t)} f_2[k_1(t), k_2(t), c_2(t)] + \frac{1}{2}\frac{\partial^2 W}{\partial k_2^2}\sigma_2^2 k_2(t)^2 \end{aligned} \tag{20}$$

Maximizing (20) and identifying the value function $W(k_1, k_2, t)$ yield the optimal controls

$$c_i^o(t, k_1(t)) \quad i = 1, 2 \tag{21}$$

In general, the Pareto optimal solution involves a shift of net growth from the stock of the agent that has the right to determine the level of externality to the stock of the agent that does not have the right to determine the externality. If the dynamic programming equation in (20) is time autonomous with infinite time horizon, the controls will be Markov stationary (i.e. functions of only state variables).

#### 2.1.4 Coasean Liability Payment Flows

Since the Isaacs-Bellman-Fleming differential equation is satisfied, the Pareto optimal trajectories are also time consistent, however, this does *not* imply that the payoff distribution after the instantaneous liability payment is time consistent. Recall that a Pareto improvement bargaining is possible because agent $i$ has the right to determine the level of externality (the right to pollute or the right to be free from pollution) while agent $j$ receives gains from negotiation (e.g. a reduction in damage rate as downstream agent or an increase in permitted pollution level as an upstream agent) and that these reductions in externality and private gains can be traded over time. We suppose that the egalitarian principle holds. The expected present value of agent $i$'s share in a Pareto optimal allocation resulting from a negotiation taking place at $t = 0$ is

$$W_i \equiv V_i + \frac{1}{2}\left[W(k_1,k_2,t) - \sum_{m=1}^{2} V_m(k_1,k_2,t)\right] \geq 0 \tag{22}$$

where $W_i(k_1,k_2,t)$ is the payoff to agent $i = 1,2$ after liability payment and $V_i$ is identified as the payoff to agent $i$ in agent $i$'s private equilibrium solution $PE_i$ given the assignment of property rights. The formulation suggests that individual payoffs are transferable across agents and time. The expression within brackets in (22) is by nature of joint maximization always nonnegative. The total payoff is

$$\begin{aligned} W(k_1,k_2,t) &= W_1(k_1,k_2,t) + W_2(k_1,k_2,t) \\ &= E\int_{t_0}^{\infty} \sum_{m=1}^{2} g_i[k_i^o(t)]e^{-\rho_i t}dt \end{aligned} \tag{23}$$

Let agent $i$ be the agent with the right to determine the externality, then (22) implies that the total liability payment over the whole planning period $t \in [0,\infty)$ by agent $j$ to $i$ is

$$\Lambda_i^j(k_1,k_2,t) = \frac{1}{2}[(W_j - V_j) - (W_i - V_i)] \leq 0 \quad i \neq j \tag{24}$$

using (23) the total liability payment by $j$ to $i$ in exchange for a reduction in externality level is

$$\Lambda_i^j(k_1,k_2,t) = W_j - \int_{t_0}^{\infty} g_j[k_j^o(t), \phi_j[k_j^o(t)]]e^{-\rho_j t}dt \leq 0 \quad i \neq j \tag{25}$$

or in other terms, using (22)

$$\begin{aligned} \Lambda_i^j(k_1,k_2,t) = V_j + \frac{1}{2}\left[W(k_1,k_2,t) - \sum_{m=1}^{2} V_m(k_1,k_2,t)\right] \\ - \int_{t_0}^{\infty} g_j[k_j^o(t), \phi_j[k_j^o(t)]]e^{-\rho_j t}dt \leq 0 \quad i \neq j \end{aligned} \tag{26}$$

**Proposition 1** *If agent $i$ is the agent with the right to determine the externality, then the instantaneous liability payment flow $L_i^j(k_1,k_2,t)$ from agent $j$ to agent $i$ for a Pareto optimal reduction in externality flow, should satisfy*

$$L_i^j(k_1,k_2,t) \equiv \frac{\partial \Lambda_i^j}{\partial t}(k_1,k_2,t) = \tag{27}$$

$$-\frac{1}{2}\left[\frac{\partial W}{\partial t} + \sum_{m=1}^{2} \frac{\partial W}{\partial k_m}\frac{\partial k_m}{\partial t}(c_1^o,c_2^o,k_1,k_2,t) + \sum_{m=1}^{2} \frac{\partial^2 W}{\partial k_m^2}\sigma_m(k_m,t)\right]$$

$$+\frac{1}{2}\left[\frac{\partial V_i}{\partial t}+\sum_{m=1}^{2}\frac{\partial V_i}{\partial k_m}\frac{\partial k_m}{\partial t}(c_1^o,c_2^o,k_1,k_2,t)+\sum_{m=1}^{2}\frac{\partial^2 V_i}{\partial k_m^2}\sigma_m(k_m,t)\right]$$
$$-\frac{1}{2}\left[\frac{\partial V_j}{\partial t}+\sum_{m=1}^{2}\frac{\partial V_m}{\partial k_m}\frac{\partial k_m}{\partial t}(c_1^o,c_2^o,k_1,k_2,t)+\sum_{m=1}^{2}\frac{\partial^2 V_j}{\partial k_m^2}\sigma_m(k_m,t)\right]$$
$$-g_j[k_j^o(t),\phi_i[k_j^o(t)]]e^{-\rho_j t}\leq 0$$
$$i\neq j$$

**Proof:** Using (26) and applying theorem 3.1 in Yeung and Petrosjan (2004).

Then we are ready to define a subgame consistent Coasean bargaining solution as follows.

**Definition 2** *Let agent i be the agent with the right to determine the externality level at each* $t\in(0,\infty)$ *and agent j where* $i\neq j$ *the agent that gains from a reduction in externality level with defined optimal control problems (8) - (11). Let agent i and j negotiate a Coasean bargaining solution* $\Pi(k_1,k_2,t)$ *at time* $t=0$ *that satisfies*

*i)* *Individual rationality in (5)*
*ii)* *Group rationality in (6)*
*iii)* *Time consistency in (7)*
*iii)* *Subgame consistent Coasean liability payment flow in proposition 1*
*iv)* *Initial conditions* ($k_1^i$, $k_2^i$) *identified as agent i's* $PE_i$ *steady state*

*then* $\Pi(k_1,k_2,t)$ *is a subgame consistent Coasean bargaining solution (CBS).*

# 3 Analytical Illustration

In this section we illustrate the suggested CBS in definition 2 by an analytically tractable upstream-downstream problem. Consider two agents, 1 and 2, that may be considered as an upstream region and downstream region. Upstream agent 1 owns and uses capital stock $k_1(t)$ while downstream agent 2 owns and uses capital stock $k_2(t)$ as input in the production functions $y_i(t)=\phi_i k_i(t)^{1/2}$, where $\phi_i>0$ is the technology level and $i=1,2$. Upstream activity generates pollution flow $P(t)=\varphi y_1(t)$ (where $\varphi>0$ is a pollution parameter), which damages downstream capital stock $k_2$ of agent 2 as described by the dynamics (29) - (31). To simplify calculation, it assumed that agent 1's capital stock is not damaged by own emissions without loss of generality. The growth process of each capital stock follows a stochastic process implying that externality is stochastic as well. Both agents $i=1,2$ maximize utility by choosing optimal consumption level $c_i(t)$ in the stochastic optimal control problems (28) and (31), which are described hereinafter

$$\max_{c_i(t)}\int_0^\infty c_i(t)^{1/2}e^{-\rho_i t}dt\quad i=1,2 \tag{28}$$

s.t.

$$dk_1(t) = \left[\phi_1 k_1(t)^{1/2} - \delta_1 k_1(t) - c_1(t)\right] dt + \sigma_1 k_1 dz_1 \tag{29}$$

$$dk_2(t) = \left[\phi_2 k_2(t)^{1/2} - \delta_2 k_2(t) - c_2(t) - \varphi \frac{y_1(t)}{y_2(t)} k_2(t)\right] dt + \sigma_2 k_2 dz_2 \tag{30}$$

$$k_1(0) = k_{1,0} \quad k_2(0) = k_{2,0} \quad i = 1,2 \tag{31}$$

$\rho_i > 0$ are agent-specific discount rates and $\delta_i > 0$ in (29) and (30) are depreciation rates for $i = 1, 2$. The fourth term on the RHS in (30) is the endogenous damage rate to downstream capital. The greater $P(t) = \varphi y_1(t)$ is relative to $y_2(t)$, the greater is the decay rate. Finally, (29) and (30) contain the stochastic growth process of each capital stock where $dz_i$ is the increment of a Wiener process $z_i(t)$ with variance $\sigma_i^2 \geq 0$ and $cov(z_i z_j) = 0$ for $i = 1, 2$ and $i \neq j$.

Since agent 1 is unaffected by agent 2's activities, the non-cooperative equilibrium of problems (28) - (31) coincides with agent 1's private equilibrium that is solved in appendix A.1. The closed-loop strategies are

$$c_i^*(k_i, t) = \left(2\rho_i + \delta_i + \frac{\sigma_i^2}{4}\right)^2 k_i(t) \quad i = 1,2 \tag{32}$$

Equation (32) shows that activity $c_i$ is proportional to the size of own capital stock $k_i$. A myopic (high $\rho_i$) agent would prefer greater activity today (decreasing saving rate). If the variance $\sigma_i^2$ of the stochastic process of capital increases, current activity increases for given levels of stock (saving rate falls). This is connected to a decrease in the shadow prices for given levels of own stock in (33) where $u_1(k_1, t)$ and $d_1(k_2, t)$ are agent 1's closed loop shadow prices and $u_2(k_1, t)$ and $d_2(k_2, t)$ agent 2's shadow prices. The greater volatility of capital accumulation, the smaller is the value of a unit of capital and the smaller is the saving rate with the corresponding increase in current consumption and emissions for given stock level.

$$\begin{aligned}
u_1(k_1,t) &\equiv \frac{\partial V_1}{\partial k_1}(k_1,t) = \frac{1}{2}\left(\frac{1}{2\rho_1 + \delta_1 + \frac{\sigma_1^2}{2}}\right)^{\frac{1}{2}} \frac{1}{k_1(t)^{\frac{1}{2}}} \\
d_1(k_2,t) &\equiv \frac{\partial V_1}{\partial k_1}(k_2,t) = 0 \\
u_2(k_1,t) &\equiv \frac{\partial V_2}{\partial k_1}(k_1,t) = -\frac{1}{2}\left(\frac{1}{2\rho_1 + \delta_1 + \frac{\sigma_1^2}{4}}\right)\left(\frac{1}{2\rho_2 + \delta_2 + \frac{\sigma_2^2}{4}}\right)^{\frac{1}{2}} \frac{\varphi}{k_1(t)^{\frac{1}{2}}} \\
d_2(k_2,t) &\equiv \frac{\partial V_2}{\partial k_2}(k_2,t) = \frac{1}{2}\left(\frac{1}{2\rho_2 + \delta_2 + \frac{\sigma_2^2}{2}}\right)^{\frac{1}{2}} \frac{1}{k_2(t)^{\frac{1}{2}}}
\end{aligned} \tag{33}$$

An increase in the depreciation rates give the same effect. Not surprisingly, upstream agent 1 has a zero shadow price $d_1(k_2, t)$ of downstream stock, while downstream agent 2 has a shadow cost $u_2(k_1, t)$ of upstream stock, which is proportional to the pollution parameter $\varphi$.

Agent 2's downstream optimal control problem is tied to agent 1's optimal control problem. An increase in activity by upstream agent 1, reduces downstream stock levels via the damage rate $\varphi y_1(t)/y_2(t)$, which in turn increases downstream agent's shadow price $d_2(k_2, t)$ of own stock. Downstream optimal activity decreases. The lower stock level of downstream agent, the greater is the damage rate and the greater is the reduction in activity, *ceteris paribus*. Without rights there is nothing else that downstream agent can do than to choose optimal activity given the upstream agent's optimal control activity.

### 3.1 Upstream Agent Determines Externality - Initial Conditions

Suppose upstream agent 1 has the right to determine the externality. The initial condition is the private equilibrium $PE_1$ steady state of upstream agent 1, denoted as $(k_1^u, k_2^u)$ and derived in appendix A.1.

$$\begin{aligned} k_1^o(0) = \bar{k_1^u} &= \left[\frac{\phi_1}{2\rho_1 + \delta_1 + \sigma_1^2/4)}\right]^2 \\ k_2^o(0) = \bar{k_2^u} &= \left[\frac{\phi_2 - \frac{\varphi\phi_1/\phi_2}{2\rho_1+\delta_1+\sigma_1^2/4}}{2\rho_2 + \delta_2 + \sigma_2^2/4}\right]^2 \end{aligned} \tag{34}$$

Note that $\varphi > 0$ holds back the steady state of downstream capital stock compared to agent 1's private equilibrium steady state. The level $\bar{k_2^u}$ is the steady state when agent 2 is maximizing utility, given the choice $\bar{k_1^u}$ of upstream agent 1.

### 3.2 Downstream Agent Determines Externality - Initial Conditions

Suppose instead that downstream agent 2 has the right to determine the externality. The initial condition is the steady state private equilibrium $PE_2$ of downstream agent 2 $(k_1^d, k_2^d)$ which is derived in appendix A.2.

$$\begin{aligned} k_1^o(0) &= 0 \\ k_2^o(0) &= \bar{k_2^d} = \left[\frac{\phi_2}{2\rho_2 + \delta_2 + \sigma_2^2/4}\right]^2 \end{aligned} \tag{35}$$

Since agent 2 receives no benefit but damage costs from $P$ and $y_1$ it would not allow any upstream activity, and hence, the initial condition in a bargaining solution is $k_1^o(0) = 0$ and $P = 0$. The level $\bar{k_2^d}$ is the steady state when agent 2 is maximizing utility given $k_1^d = 0$. This actually implies that agent 2 imposes the restriction $dk_1/dt = 0$ on agent 1 implying that $V_1 \equiv 0$.

### 3.3 Pareto Optimal Trajectories

We suppose that negotiation, which takes place prior to $t = 0$, proceeds until no further Pareto improvements are possible. Let $W_i$ be the payoff to agent $i = 1, 2$, then negotiation goes on until the total payoff $W = W_1 + W_2$ is maximized. Thus, the agents agree to solve the joint stochastic optimal control problem

**Definition 3** *Let $E$ be the expectation operator, then if there exists a value function $W(k_1, k_2, t)$ that satisfies*

$$W(k_1, k_2, t) = \tag{36}$$

$$E\left\{ \int_0^\infty \left[c_1^o(t)^{\frac{1}{2}} + c_2^o(t)^{\frac{1}{2}}\right] e^{-\rho_i t} dt \right\}$$

$$\geq E\left\{ \int_0^\infty \left[c_1(t)^{\frac{1}{2}} + c_2(t)^{\frac{1}{2}}\right] e^{-\rho_i t} dt \right\}$$

*for all $c_i(k_1, k_2, t)$ in the feasible set $c_i \subseteq R_+^2$ for $i = 1, 2$ which satisfy the stochastic growth processes in the $(k_2, k_1)$ state space*

$$dk_1(t) = \left[\phi_1 k_1^o(t)^{1/2} - \delta_1 k_1^o(t) - c_1\right] dt + \sigma_1 k_1^o dz_1 \tag{37}$$

$$dk_2(t) = \left[\phi_2 k_2^o(t)^{1/2} - \delta_2 k_2^o(t) - c_2 - \varphi \frac{y_1^o(t)}{y_2^o(t)} k_2^o(t)\right] dt + \sigma_2 k_2^o dz_2 \tag{38}$$

$$k_1^o(0) = \bar{k^x}_1 \quad k_2^o(0) = \bar{k^x}_2 \quad x = u, d \tag{39}$$

*then the closed loop Pareto optimal control paths are*

$$c_i^o\big(k_1, k_2, t\big) \tag{40}$$

*for $i = 1, 2$ of the problems defined in (28) - (31).*

The value function in definition 1 and the dynamic system (37) - (39) should to satisfy the Bellman-Fleming partial differential equation system for stochastic optimal control (Fleming and Richel, 1975).

$$\frac{\partial W(k_1, k_2, t)}{\partial t} = \tag{41}$$

$$\max_{c_1, c_2} \left[c_1(t)^{\frac{1}{2}} + c_2(t)^{\frac{1}{2}}\right] e^{-\rho_i t}$$

$$+\frac{\partial W}{\partial k_1(t)}\Big[\phi_1 k_1(t)^{1/2} - \delta_1 k_1(t) - c_1(t)\Big]\,dt + \frac{1}{2}\frac{\partial^2 W}{\partial k_1^2}\sigma_1^2 k_1(t)^2$$

$$+\frac{\partial W}{\partial k_2(t)}\Big[\phi_2 k_2(t)^{1/2} - \delta_2 k_2(t) - c_2(t) - \varphi\frac{y_1(t)}{y_2(t)}k_2(t)\Big]\,dt + \frac{1}{2}\frac{\partial^2 W}{\partial k_2^2}\sigma_2^2 k_2(t)^2$$

Maximizing (41) and solving for $c_i^o(k_1, k_2, t)$, $i = 1, 2$.

$$c_1^o(k_1(t)) = \left(\frac{1}{a}\right)^2 k_1(t) \tag{42}$$

$$c_2^o(k_2(t)) = \left(\frac{1}{b}\right)^2 k_2(t) \tag{43}$$

The values of $a$ and $b$ are solved explicitly in appendix A.3. Since $a \leq a_1$ the joint maximized upstream consumption is higher than in agent 1's $PE_1$ in (32), i.e. the investment and growth of agent 1 is lower than in $PE_1$. Since the damage rate is reduced, the optimal consumption level of agent 2 is greater than in $PE_1$. Thus the Pareto optimal solution involves a shift of net growth between the agents where upstream net growth falls and downstream net growth increases.

Since (41) is time autonomous with infinite time horizon, the controls are Markov stationary (i.e. functions of only state variables). The value function in definition 1 is next to be defined in order to specify the optimal controls in (42) and (43).

**Proposition 2** *The value function in (44) satisfy definition 1 and the indirect HJB partial differential equation system (20)*

$$W\big(k_1(t), k_2(t)\big) = \Big(ak_1(t)^{\frac{1}{2}} + bk_2(t)^{\frac{1}{2}} + c\Big)e^{-\rho_i t} \tag{44}$$

**Proof**: Appendix A.3 which also defines $(a, b, c)$ explicitly. Q.E.D.

Since the Bellman-Fleming differential equation is satisfied, the Pareto optimal trajectories are also time consistent.

## 3.4 Upstream Agent Determines the Externality Flow

Consider first the case when upstream agent 1 has the right to pollute and downstream agent 2 has to pay a liability payment flow to 1 for reducing the emissions flow. The initial condition is the private equilibrium $PE_1$ steady state (34) where agent 1 is producing at the private equilibrium level. After bargaining, the process starts from (34) and moves along the Pareto optimal controls in (42) and (43) with the stochastic processes

$$dk_1(t) = \phi_1 k_1^o(t)^{1/2} - \delta_1 k_1^o(t) - \left(\frac{1}{a}\right)^2 k_1^o(t) + \sigma_1 k_1 dz_1 \qquad (45)$$

$$dk_2(t) = \phi_2 k_2^o(t)^{1/2} - \delta_2 k_2^o(t) - \left(\frac{1}{b}\right)^2 k_2^o(t) - \varphi\frac{y_1^o(t)}{y_2^o(t)} k_2^o(t) + \sigma_2 k_2 dz_2$$

This is a process of reallocation of net growths where net growth of agent 1 falls and net growth of agent 2 increases along the process between the $PE_1$ steady state at $t = 0$ as initial condition and the CSS. Since 1 has the right to pollute, 2 must pay the instantaneous liability payment flow $L_1^2(k_1(t))$ to 1 to compensate the lower net growth in agent 1. Applying proposition 1 to the problems (28) - (31) and simplifying by using (64), (65) and (75) give the instantaneous Coasean liability payment flow from agent 2 to 1 as compensation for reduction in the externality by 1

$$L_1^2(k_1(t)) = \frac{1}{2}\left\{(a - a_1 + a_2)\left(2\rho_i + \delta_1 + \frac{1}{2a^2}\right) + \frac{b\varphi\phi_1}{\phi_2}\right\} k_1(t)^{\frac{1}{2}} \leq 0 \qquad (46)$$

It follows that the payment flow can also be derived as

$$L_2^1(k_1(t)) = \frac{1}{2}\left\{(a + a_1 - a_2)\left(2\rho_i + \delta_1 + \frac{1}{2a^2}\right) - \frac{1}{a}\right\} k_1(t)^{\frac{1}{2}} \geq 0 \qquad (47)$$

By using (64), (65) and (75), it is straightforward to double check that

$$L_1^2(k_1(t)) + L_2^1(k_1(t)) \equiv 0 \qquad (48)$$

The first term on the RHS in (46) is nonpositive since $a_2 \leq 0$ and $a \leq a_1$. The last term on the RHS is a share of the instantaneous gain (reduction in instantaneous cost) that downstream agent gets from the reduction in upstream emissions. The greater is $\varphi$ the greater is the instantaneous gain.[1] The liability flow can alternatively, $L_1^2(k_1, t)$ be expressed as a fixed share of 1's production

$$L_1^2(k_1, t) = \frac{1}{2\phi_1}\left\{(a + a_1 - a_2)\left(2\rho_i + \delta_1 + \frac{1}{2a^2}\right) - \frac{1}{a}\right\} y_1(t) \qquad (50)$$

[1] It can be shown that a sufficient condition for a bargaining solution that satisfies individual rationality and also time consistency *at each instant of time*, i.e. it also satisfies Petrosjan (1997), is

$$\varphi \geq 8\frac{\phi_2}{\phi_1}\sqrt{\rho_i + \frac{\delta_2}{2}} \qquad (49)$$

The pollution parameter must not be too low relative to the ratio between downstream and upstream technology level. Moreover, volatility and depreciation rate of downstream stock must not be too high as this may violate the inequality.

Agent 2 has to compensate 1's loss by a fixed share of 1's production. The greater 1's production, the greater is the instantaneous flow that 2 must pay to 1 at each instant of time.

The corresponding steady state liability payment flow from 2 to 1 is found by substituting the expected steady state $E[k_{1,\infty}]$ into (46)

$$E[L_{1,\infty}^2] = \left\{ \frac{1}{2\phi_1}(a + a_1 - a_2) \left( 2\rho_i + \delta_1 + \frac{1}{2a^2} \right) - \frac{1}{a} \right\} E[k_{1,\infty}^{1/2}] \tag{51}$$

This illustrates the redistribution effects over time from different assignments of rights. Agent 2 must continue paying the steady state Coasean liability flow $\bar{L}_1^2$ for ever to agent 1 in order to keep 1 at down at the CSS level, $\bar{k}_1^o < \bar{k}_u = k_1^o(0)$, otherwise 1 would use its right to pollute at the $PE_1$ level $\bar{k}_1^u$.

### 3.5 Downstream Agent determines the Externality

Suppose instead that downstream agent 2 has the right to be free from upstream pollution $P_1$. The initial conditions (35) is instead defined by agent 2's private equilibrium $PE_2$ $(k_1^d, k_2^d)$ derived in appendix A.2. Since 2 has no benefit but costs from upstream emissions generating production, it would set $y_1 = P = 0$ imposing the constraint $dk_1/dt = 0$. As a result, agent 1's disagreement value function is $V_1 \equiv 0$. Using this in the rule (28) gives the liability payment from agent 1 to agent 2.

$$L_2^1(k_1(t)) = \left\{ \frac{1}{2}(a + a_1 - a_2) \left( 2\rho_i + \delta_1 + \frac{1}{2a^2} \right) - \frac{1}{a} \right\} k_1(t)^{\frac{1}{2}} \leq 0 \tag{52}$$

we also have

$$\begin{aligned} L_1^2(k_1(t)) = \left\{ \frac{1}{2}(a - a_1 + a_2) \left( 2\rho_i + \delta_1 + \frac{1}{2a^2} \right) \right\} k_1(t)^{\frac{1}{2}} \\ - \frac{b\varphi\phi_1}{2\phi_2} k_1(t)^{\frac{1}{2}} \geq 0 \end{aligned} \tag{53}$$

Again one can check that $L_2^1(k_1(t)) = -L_1^2(k_1(t))$ by using (64), (65) and (75). The expected steady state payment by 1 to 2 can be derived by similar manner as in the previous section.

### 3.6 Sensitivy Analysis of CBS - the Deterministic Case

Suppose that $\sigma_i = 0$ for $i = 1, 2$ then the qualitative analysis of the CBS vector field for the deterministic case in the production space $(y_1, y_2)$ is found around the Coasean steady state (CSS). Agent 2's private equilibrium $PE_2$ is located at the $dk_2/dt = 0$ isocline where it intersects the $y_2$-axis, while $PE_1$ is located below the same isocline. The bargaining solution when agent 1 has the right to determine the externality level consists of the unique trajectory between

$PE_1$ as the initial condition and the CSS. On the other hand, when agent 2 has the right to determine the externality level, the bargaining solution would have been the unique trajectory between $PE_2$ as the initial condition and the CSS provided that 1's individuality condition had been satisfied. If the pollution parameter $\varphi$ increases, $PE_1$ moves downwards and the $dk_2/dt = 0$ isocline pivots clockwise around its intersection with the $y_2$-axis, implying that downstream capital grows slower in the beginning towards the CSS when the polluter has the right to pollute. The growth in downstream agent is further delayed by a low downstream productivity level $\phi_i$ and/or high downstream depreciation rate. If $\varphi$ is great, the initial condition in $PE_1$ may have a zero downstream production. The growth of the downstream stock when the upstream agent has the right to determine the externality ' is further delayed by a low downstream productivity level $\phi_i$ and/or high downstream depreciation rate of downstream stock and nevertheless a high pollution parameter. Even though both agents will gain in the long run, the downstream agent has to wait longer than the upstream agent before it can collect the gains in terms of higher net growth.

# 4 Summary

For the first time, a subgame consistent cooperative solution ha been used to analyze the classical Coasean bargaining solution (CBS) with stochastic stock externalities in a differential bargaining game. A time consistent Coasean bargaining solution has been formulated where a Coasean bargaining takes place at $t = 0$ and results in a Pareto optimal 'plan of trade' containing agreed future successive exchanges of reductions in externality and instantaneous flow of liability payment between two agents. It is essential that this initial plan be time consistent. For this purpose, a rule was derived that identifies time consistent Coasean liability (compensation) payment flows. A CBS concept with the following properties had been introduced. (1) The assignment of property rights will not affect the expected Coasean steady state and the Pareto optimal paths toward it. The agent that has the right to determine the externality level has the advantage to choose initial conditions in the Coasean dynamic system. i.e. whether the process starts in agent 1's or agent 2's private equilibrium steady state. An analytical illustration was presented with a time consistent Coasean liability payment flow that was a monotonous function of upstream stock level. Growth in the downstream stock is delayed when upstream agent has the right to pollute. When the bargaining game approaches the expected Coasean steady state as $t \to \infty$, the agent that lacks the right to determine the externality level has to continue paying the expected steady state liability flow to the agent that has the right to determine the externality level.

# Appendix

## A.1 Agent 1's Private Equilibrium ($\mathrm{PE}_1$) Solution

**Definition 4** *If there exist value functions $V_i(k_1, k_2, t)$ for $i = 1, 2$ that satisfy*

$$V_i(k_1, k_2, t) = \int_0^\infty c_i^u(t)^{\frac{1}{2}} e^{-\rho_i t} dt \geq \int_0^\infty c_i(t)^{\frac{1}{2}} e^{-\rho_i t} dt \tag{54}$$

*for all strategies $c_i(k_1(t), k_2(t), t)$ in the feasible set $c_i(t) \in [0, 1] \subseteq R^2$ which satisfy the growth processes in the $(k_2, k_1)$ state space*

$$dk_1(t) = \left[\phi_1 k_1(t)^{1/2} - \delta_1 k_1(t) - c_1(t)\right] dt \tag{55}$$

$$dk_2(t) = \left[\phi_2 k_2(t)^{1/2} - \delta_2 k_2(t) - c_2(t) - \varphi \frac{y_1(t)}{y_2(t)} k_2(t)\right] dt \tag{56}$$

$$k_1^u(0) = k_{1,0} \quad k_2^u(0) = k_{2,0} \tag{57}$$

*then the closed loop Pareto optimal control paths are*

$$c_i^u\left(k_1, k_2, t\right) \tag{58}$$

*of the problems defined in (28) to (31).*

The value functions in definition 1 and the dynamic system formed by (55) and (56) have to satisfy the HJB partial differential equation system

$$\begin{aligned} -\frac{\partial V_i(k_1, k_2, t)}{\partial t} = \\ \max_{c_i(t)} c_i(t)^{\frac{1}{2}} e^{-\rho_i t} + \frac{\partial V_i}{\partial k_1}\left[\phi_1 k_1(t)^{1/2} - \delta_1 k_1(t) - c_1(t)\right] + \\ + \frac{\partial V_i}{\partial k_2}\left[\phi_2 k_2(t)^{1/2} - \delta_2 k_2(t) - c_2(t) - \varphi \frac{y_1(t)}{y_2(t)} k_2(t)\right] \\ i = 1, 2 \end{aligned} \tag{59}$$

The closed loop controls $c_i^u(k_1, k_2, t)$ for $i = 1, 2$ are given by maximizing the HJB differential equations (59) and solving for the control variable.

$$c_1^u(k_1(t)) = \left(\frac{1}{2} \Big/ \frac{\partial V_1}{\partial k_1}\right)^2 k_1(t) \tag{60}$$

$$c_2^u(k_2(t)) = \left(\frac{1}{2}\Big/\frac{\partial V_2}{\partial k_2}\right)^2 k_2(t) \tag{61}$$

The value function in definition 3 must be identified in order to specify the optimal controls in (60) and (61).

**Proposition 3** *The value functions in (62) satisfy definition 3 and the indirect HJB differential equation system formed by (59)*

$$V_i\big(k_1(t), k_2(t)\big) = \left(a_i k_1(t)^{\frac{1}{2}} + b_i k_2(t)^{\frac{1}{2}} + c_i\right) e^{-\rho_i t} \quad i = 1, 2 \tag{62}$$

**Proof**: Substituting (60) into the differential equations (59) forms the indirect HJB differential equation system for $i = 1, 2$ as follows

$$-\frac{\partial V_i(k_1, k_2, t)}{\partial t} = \tag{63}$$

$$\left[\frac{1}{2}\Big/\frac{\partial V_i}{\partial k_i} k_i^u(t)\right] + \frac{\partial V_i}{\partial k_1}\left[\phi_1 k_1^u(t)^{1/2} - \delta_1 k_1^u(t) - \left(\frac{1}{2}\Big/\frac{\partial V_i}{\partial k_1}\right)^2 k_1^u(t)\right]$$

$$+\frac{\partial V_i}{\partial k_2}\left[\phi_2 k_2^u(t)^{1/2} - \delta_2 k_2^u(t) - \left(\frac{1}{2}\Big/\frac{\partial V_i}{\partial k_2}\right)^2 k_2^u(t) - \varphi\frac{y_1^u(t)}{y_2^u(t)} k_2^u(t)\right] k_2^u(t)$$

The coefficients of the values functions in (62) are determined by an equation system formed by (63) resulting in six equations in (64) and (65) and the six unknowns $(a_1, b_1, c_1, a_2, b_2, c_2)$.

$$\begin{aligned} a_1 &= \left(\frac{1}{2\rho_i + \delta_1}\right)^{\frac{1}{2}} \\ b_1 &= 0 \\ c_1 &= \frac{a_1\phi_1}{2\rho_i} \end{aligned} \tag{64}$$

$$\begin{aligned} a_2 &= -\frac{b_2}{2} \cdot \frac{\varphi\phi_1/\phi_2}{2\rho_i + \delta_1} \\ b_2 &= \left(\frac{1}{2\rho_i + \delta_2}\right)^{\frac{1}{2}} \\ c_2 &= \frac{b_2\phi_2}{2\rho_i} \end{aligned} \tag{65}$$

By differentiating (62), the closed loop optimal controls in (60) and (61) can be expressed in terms of parameters

$$c_1^u(k_1(t)) = \left(\frac{1}{a_1}\right)^2 k_1(t) \tag{66}$$

$$c_2^u(k_2(t)) = \left(\frac{1}{b_2}\right)^2 k_2(t) \tag{67}$$

Since the HJB equations in (59) are time autonomous with infinite time horizon, the controls are Markov stationary (i.e. functions of only state variables).

Substituting the optimal controls in the state equations (55) and (56) yields the stock dynamics that constitutes the private equilibrium $PE_1$ vector field in the $(k_2, k_1)$ space.

$$dk_1(t) = \left[\phi_1 k_1^u(t)^{1/2} - \delta_1 k_1^u(t) - \left(\frac{1}{a_1}\right)^2 k_1^u(t)\right] dt \tag{68}$$

$$dk_2(t) = \left[\phi_2 k_2^u(t)^{1/2} - \delta_2 k_2^u(t) - \left(\frac{1}{b_2}\right)^2 k_2^u(t) - \varphi \frac{y_1^u(t)}{y_2^u(t)} k_2^u(t)\right] dt \tag{69}$$

$$k_1^u(0) = k_{1,0} \quad k_2^u(0) = k_{2,0} \tag{70}$$

Using (68) and (69) with (64) and (65) agent 1's private equilibrium steady state levels $(k_1^u, k_2^u)$ can be expressed in terms of parameter

$$\begin{aligned} \bar{k_1^u} &= \left[\frac{\phi_1}{2\rho_i + \delta_1}\right]^2 \\ \bar{k_2^u} &= \left[\frac{\phi_2 - \frac{\varphi\phi_1/\phi_2}{2\rho_i+\delta_1}}{2\rho_i + \delta_2}\right]^2 \end{aligned} \tag{71}$$

## A.2 Agent 2's Private Equilibrium (PE$_2$) Solution

Suppose that agent 2 has the right to be free from pollution $P$. Clearly, 2 would set emissions generating production $y_1$ to zero as 2 has no benefit but damage from $P$. Given that $P = 0$, agent 2 maximizes (28) - (31) subject to $\bar{k_1^d} = 0$ for all $t$ resulting in the optimal control

$$c_2^d(k_1(t)) = \left(\frac{1}{b_2}\right)^2 k_2(t) \tag{72}$$

and with the corresponding dynamics of $k_2$ in (69) the steady state $(k_1^d, k_2^d)$ is

$$\begin{aligned} \bar{k}_1^d &= 0 \\ \bar{k}_2^d &= \left[\frac{\phi_2}{2\rho_i + \delta_2}\right]^2 \end{aligned} \tag{73}$$

agent 1 has to take the restriction $dk_1/dt = 0$ as given in $PE_2$ implying that the disagreement payoff $V_1 \equiv 0$.

## A.3 Proof of Proposition 1 in CBS

Substituting (42) and (43) into the partial differential equation (41) forms the stochastic dynamic programming equation as follows

$$\begin{aligned} -\frac{\partial W(k_1,k_2,t)}{\partial t} = \\ \left[\frac{1}{2}\Big/\frac{\partial W}{\partial k_1}k_1(t) + \frac{1}{2}\Big/\frac{\partial W}{\partial k_2}k_2(t)\right] \\ +\frac{\partial W}{\partial k_1}\left[\phi_1 k_1(t)^{1/2} - \delta_1 k_1(t) - \left(\frac{1}{2}\Big/\frac{\partial W}{\partial k_1}\right)^2 k_1(t)\right] \\ +\frac{\partial W}{\partial k_2}\left[\phi_2 k_2(t)^{1/2} - \delta_2 k_2(t) - \left(\frac{1}{2}\Big/\frac{\partial W}{\partial k_2}\right)^2 k_2(t) - \varphi\frac{y_1(t)}{y_2(t)}k_2(t)\right]k_2(t) \end{aligned} \tag{74}$$

The coefficients of the values function in (44) is determined by the partial differential equation system (74). Solving for the three unknowns $(a, b, c)$ yields

$$\begin{aligned} a &= -\frac{b\varphi\phi_1/\phi_2}{2\rho_i + \delta_2} + \left[\left(\frac{b\varphi\phi_1/\phi_2}{2\rho_i + \delta_2}\right)^2 + \frac{1}{2\rho_i + \delta_2}\right]^{\frac{1}{2}} \\ b &= \left(\frac{1}{2\rho_i + \delta_2}\right)^{\frac{1}{2}} = b_1 \\ c &= \frac{a\phi_1 + b\phi_2}{2} \end{aligned} \tag{75}$$

where $(a, b, c)$ are uniquely determined.

## A.4 Stability around CSS - Deterministic case

Variable transformations of the system, (37) and (38), defining production as $y_1(t) = \phi_1 k_1(t)^{\frac{1}{2}}$ and $y_2(t) = \phi_2 k_2(t)^{\frac{1}{2}}$, yield the linear differential equation system when $\sigma_i = 0$.

$$\begin{bmatrix} dy_1(t) \\ dy_2(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\delta_1 - 1/a^2 & 0 \\ -\varphi & -\delta_2 - 1/b^2 \end{bmatrix}\begin{bmatrix} y_1(t)dt \\ y_2(t)dt \end{bmatrix} + \begin{bmatrix} \frac{\phi_1^2}{2} \\ \frac{\phi_2^2}{2} \end{bmatrix}$$

The last column vector on the RHS is the isoclines' intersections with the $y_1$ and $y_2$-axis. The isoclines in the production space $(y_1, y_2)$ are

$$y_1|_{\dot{y_1}=0} = \frac{\phi^1}{\delta_1 + 1/a^2} \tag{76}$$

$$y_2|_{\dot{y_2}=0} = \frac{\phi^2 - \varphi y_1}{\delta_2 + 1/a^2} \tag{77}$$

In the state space $(k_2, k_1)$ the isoclines are

$$k_1|_{\dot{k_1}=0} = \left[\frac{\phi^1}{\delta_1 + 1/a^2}\right]^2 \tag{78}$$

$$k_2|_{\dot{k_2}=0} = \left[\frac{\phi^2 - \varphi y_1}{\delta_2 + 1/b^2}\right]^2 \tag{79}$$

The isoclines (76) and (78) have zero slopes in the, while (77) and (79) have negative slopes $(y_1, y_2)$ and $(k_1, k_2)$ spaces respectively. Computing vectors in each of the areas bordered by isoclines in (76) and (77) and the $y_1$-axis and the $y_2$-axis in the $(y_1, y_2) \subseteq R_+^2$ space reveals the vector field. If $\varphi = 0$ the vector field reduces to a star node. The characteristic roots of the Jacobian matrix are

$$\lambda_{1,2} = \frac{-(\delta_1 + 1/a^2) - (\delta_1 + 1/b^2)}{2} \tag{80}$$
$$\pm \frac{\sqrt{[-(\delta_1 + 1/a^2) - (\delta_1 + 1/b^2)]^2 - 4[(\delta_1 + 1/a^2)(\delta_1 + 1/b^2)]}}{2}$$

Both eigenvalues have negative real parts. The characteristic roots are real for all nonnegative parameter values since

$$[(\delta_1 + 1/a^2) - (\delta_1 + 1/b^2)]^2 \geq 0 \tag{81}$$

## References


Allen, D. (1991), 'What are transaction costs?', *Research in Law and Economics* **14**, 1–18.

Barzel, Y. (1989), *Economic Analysis of Property Rights*, Cambridge University Press.

Basar, T. and Olsder, G. (1999), *Dynamic Noncooperative Game Theory*, second edition edn, Philadelpiha SIAM.

Calabresi, G. (1968), 'Transaction costs, resource allocations and liability rules: A comment', *Journal of Law and Economics* **11**, 67–73.

Coase, R. (1960), 'The problem of social cost', *Journal of Law and Economics* **3**, 1–44.

Filar, J. and Petrosjan, L. (2000), 'Dynamic cooperative games', *International Game Theory Review* **2**, 47–65.

Fleming, W. and Richel, R. (1975), *Determinstic and Stochastic Optimal Control*, Springer Verlag.

Fresh, H. (1979), 'The extended coase theorem and long run equilbrium: The non-equivalence of liability rules and property rights', *Economic Inquiry* **27**, 254–268.

Haurie, A. (1976), 'A note on nonzero-sum differential games with bargaining solution', *Journal of Optimization Theory and Applications* **18**, 31–39.

Haurie, A. and Zaccour, G. (1995), 'Differential game models of global environment management', *Annals of the International Society of Dynamic Games* **2**, 3–24.

Hoffman, E. and Spitzer, M. (1982), 'The coase theorem: Some experimental tests', *Journal of LAw and Economics* **25**, 73–98.

Jorgensen, S. and Zaccour, G. (2001), 'Time consistent side payments in a dynamic game of downstream pollution', *Journal of Economic Dynamic and Control* **25**, 1973–1987.

Kamien, M., Schwartz, N. and Dolbear, F. (1966), 'Asymmetry between bribes and charges', *Water Resources Research* **2**, 147–157.

Petrosjan, L. (1997), 'Agreeable solutions in differential games', *International Journal of Mathematics, Game Theory and Algebra* **7**, 165–177.

Petrosjan, L. and Zenkevich, N. (1996), *Game Theory*, World Scientific Singapore.

Regan, D. (1972), 'The problem of social cost revisited', *Journal of Law and Economics* **15**, 427–437.

Saraydar, E. (1983), ‘Bargaining power, dissimulation anf the coase theorem’, *Journal oif Institutional and Theorethical Economics* **139**, 599–611.

Schweizer, U. (1988), ‘Externalities and the coase theorem: Hypothesis or result?’, *Journal of Institutional and Theorethical Economics* **144**, 245–266.

Starret, D. (1972), ‘Fundamental nonconvexities in the theory of externalites’, *Journal of Economic Theory* **4**, 180–199.

Tybout, R. (1972), ‘Pricing pollution and other negative externalities’, *Bell Journal of Economics* **3**, 252–266.

Wellisz, S. (1964), ‘On external diseconomies and the government - assisted invisible hand’, *Economica* **31**, 345–362.

Yeung, D. (2004), ‘Technical note: Nontransferable individal payoffs in cooperative stochastic differential games’, *International Game Theory Review* **6**, 281–289.

Yeung, D. and Petrosjan, L. (2004), ‘Subgame consistent cooperative solutions in stochastic differential games’, *Journal of Optimization Theory and Applications* **120**, 651–666.

Yeung, D. and Petrosjan, L. (2005), ‘Subgame consistent solutions of a cooperative stochastic differential game with nontransferable payoffs’, *Journal of Optimization Theory and Applications* **124**, 701–724.

Yeung, D. and Petrosjan, L. (2006), *Cooperative Stochastic Differential Games*, Springer Science Business Media, Inc.