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Okounkov bodies and geodesic rays in Kähler geometry

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ABSTRACT

This thesis presents three papers dealing with questions in Kähler geometry.

In the first paper we construct a transform, called the Chebyshev transform, which maps continuous hermitian metrics on a big line bundle to convex functions on the associated Okounkov body. We show that this generalizes the classical Legendre transform in convex and toric geometry, and also Chebyshev constants in pluripotential theory. Our main result is that the integral of the difference of two transforms over the Okounkov body is equal to the Monge-Ampère energy of the two metrics. The Monge-Ampère energy, sometimes also called the Aubin-Mabuchi energy or the Aubin-Yau functional, is a well-known functional in Kähler geometry; it is the primitive function to the Monge-Ampère operator. As a special case we get that the weighted transfinite diameter is equal to the mean over the unit simplex of the weighted directional Chebyshev constants. As an application we prove the differentiability of the Monge-Ampère on the ample cone, extending previous work by Berman-Boucksom.

In the second paper we associate to a test configuration for a polarized variety a filtration of the section ring of the line bundle. Using the recent work of Boucksom-Chen we get a concave function on the Okounkov body whose law with respect to Lebesgue measure determines the asymptotic distribution of the weights of the test configuration. We show that this is a generalization of a well-known result in toric geometry.

In the third paper, starting with the data of a curve of singularity types, we use the Legendre transform to construct weak geodesic rays in the space of positive singular metrics on an ample line bundle L. Using this we associate weak geodesics to suitable filtrations of the algebra of sections of L. In particular this works for the natural filtration coming from an algebraic test configuration, and we show how this in the non-trivial case recovers the weak geodesic ray of Phong-Sturm.

Keywords: ample line bundles, Okounkov bodies, Monge-Ampère operator, Legendre transform, Chebyshev constants, test configurations, weak geodesic rays.

Preface

This thesis consists of an introduction and the following papers.

- David Witt Nyström,
  Transforming metrics on a line bundle to the Okounkov body,
  submitted.
- David Witt Nyström,
  Test configurations and Okounkov bodies,
  accepted for publication in Compositio Mathematica.
- Julius Ross and David Witt Nyström,
  Analytic test configurations and geodesic rays,
  submitted.

In order not to loose focus, the following paper is not included in this thesis.

- Robert Berman, Sebastien Boucksom and David Witt Nyström,
  Fekete points and convergence towards equilibrium measure on complex manifolds,
  Acta Mathematica 207 (2011), no. 1, 1-27.

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My third big thank goes to my co-author Julius Ross in Cambridge. I know that we will continue writing papers together, in fact we are right now working on a new exciting thing, but that is a different story...

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David Witt Nyström
Göteborg, March 2012

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Abandon all hope, ye who enter here.

Dante

Part I

INTRODUCTION

0
INTRODUCTION

At the International Congress of Mathematicians (abbreviated ICM) in Madrid, 2006, four mathematicians were awarded the Fields medal, generally regarded as the highest accolade in mathematics. One of them stole the show, without even showing up. The russian Grigori Perelman had in 2002 posted on the internet a solution to one of the longstanding problems in mathematics, the Poincaré conjecture. By the time of the ICM in 2006 a consensus had been reached among the experts that Perelman's solution was correct. Perelman however had gotten disillusioned with the mathematical community, and refused to come to Madrid to pick up his medal. The three other recipients of the Fields medal that year were Terence Tao, Wendelin Werner and Andrei Okounkov. A large part of this thesis revolves around a mathematical invention due to the other russian in the bunch, Andrei Okounkov.

If one divides mathematics into its major subfields, say algebra, number theory, analysis, geometry, probability theory and discrete mathematics, then this thesis belongs to the land of geometry. If one wants to be more specific, we are doing algebraic geometry, which means that one uses algebraic equations to define the geometries of interest. The geometric shapes one studies are in general very complicated and either hard or impossible to visualize. For one thing we allow the dimension of our spaces to be arbitrarily large.

In 1996 Okounkov published a paper titled "Brunn-Minkowski inequality for multiplicities." In it he took one of the complicated geometric objects that we are interested in and showed how to produce a simplified image of it. These images sort of look like blobs, and are called Okounkov bodies, after their inventor. Since they are so simple they do not tell us everything about the complicated object we started with, but they still give us some clues.

It took more than ten years though until other mathematicians started to realize the usefulness of these images, and by then Okounkov himself was doing different things. In 2008 two research teams working independently developed the ideas of Okounkov much further (see [18, 22]), and found new applications. Other researchers (including myself) hopped on the train.

In the modern take on geometry one starts with a very flexible object called a manifold. Up close it should look just like flat space of some chosen dimension, but its global behaviour can be complicated. One can twist and stretch the manifold however one likes, as long as one does not tear or break it. The manifold is the white canvas for the geometer. One should note that already the white canvas has a lot of structure in itself, which is studied in the field of topology. One then proceeds to give the manifold some additional structure, making it more rigid. What kind of structure depends on what kind of geometry one works with. In algebraic geometry the additional structure tells you what functions on the manifold should be thought of as polynomials or rational functions, i.e. quotients of polynomials. In Riemannian geometry the structure one imposes on the manifold is that of a metric. A metric enables you to measure the lenght of curves along the manifold, and also areas and volumes. This is what gives a manifold a precise geometric shape, so the object is now very rigid. The most important concept in this area is that of curvature. The curvature of a manifold with metric measures the local non-flatness of it in a precise mathematical way. In two dimensions the curvature is just a function on the manifold, often called the Gaussian curvature. The ordinary plane has zero curvature since it is flat, the sphere has constant positive curvature and there are also spaces with constant negative curvature. These negatively curved (hyperbolic) spaces locally look like the surface of a saddle. However, for most spaces the curvature will vary from point to point. In higher dimensions a function is not enough to capture the non-flatness of a space, so the curvature is a much more complicated kind of object (a $(1, 3)$-tensor for those in the know). Even though the curvature in dimensions higher than two is a complicated object one can still form a function from it called the scalar curvature. The scalar curvature at a point measures how the size of a ball with radius r centered at that point compares to the size of a ball in flat space with the same radius as the radius shrinks to zero.

The combination of algebraic geometry with Riemannian geometry is called Kähler geometry. In Kähler geometry one equips the manifolds with metrics, giving rise to curvature. The metrics one uses are not arbitrary though, they are supposed to be adapted to the algebraic structure of the manifold.

Most researchers studying Okounkov bodies have focused on algebraic geometric aspects. The first and second article in this thesis uses Okounkov bodies rather in the setting of Kähler geometry. Recall that the Okounkov body is a simplified image of a manifold. In Kähler geometry we add a metric to the manifold, and the point of the first paper is to show that this extra information can be incorporated as a graph over the original image.

The second and third paper are motivated by one of the big open problems in Kähler geometry, the Yau-Tian-Donaldson conjecture. The metrics one looks at in Kähler geometry come in classes. The Yau-Tian-Donaldson conjecture says something about when a class contains a metric such that the scalar curvature of the space becomes constant.

In the formulation of the conjecture objects called test configurations come in. To each test configuration there is an associated sequence of numbers, and the asymptotics of these numbers conjecturally decides if one can find this special kind of metric or not. In the second paper, using the work of Boucksom-Chen in [7] we show how to draw a graph over the Okounkov body which encodes some of this number asymptotics.

The third paper is a collaboration with Julius Ross from the University of Cambridge.

A geodesic is a curve whose length between any two nearby points is minimal among all curves between those points. A geodesic ray is a geodesic which continues indefinately in some direction.

The space of metrics in a Kähler class is infinite dimensional, nevertheless the work of Mabuchi, Semmes and Donaldson (see [24], [33], [12]) has showed that this space has a beautiful geometry, and one can talk about its geodesics. Given a geodesic ray in there one can calculate a number, and there is a conjecture due to Donaldson which says that if all these numbers are positive there will be a metric in the class with constant scalar curvature.

Phong-Sturm showed in [28] how to use the data of a test configuration to construct weak versions of these geodesic rays. Inspired by this, we present in the third paper a general construction of weak geodesic rays. We define objects called analytic test configurations, and show how to construct weak geodesic rays using these. We also prove that ordinary test configurations give rise to analytic ones, and in the case of a non-trivial analytic test configuration we show that our geodesic rays coincide with those constructed by Phong-Sturm.

Before going into the details of the different papers we will start by recalling some basic material on algebraic and Kähler geometry.

0.1 Kähler geometry

The exposition given here is by necessity extremely sketchy. For a proper treatment of this material see e.g. [17] and [21].

In algebraic geometry one studies the geometry of the set of solutions to polynomial equations. The simplest case is the circle which is given by the equation $x^2 + y^2 = 1$. It gets more interesting when looking at the solutions of an equation like $y^2 = x^3 - x$. The geometric object one thus gets is an example of an elliptic curve. The theory of elliptic curves is extremely rich, e.g. a result on elliptic curves was the key in Andrew Wiles proof of Fermat's Last Theorem in 1994.

0.1.1 Projective manifolds

Often one lets the variables take complex values, not only real ones. When doing this the circle transforms into a sphere, and the elliptic curve becomes a torus. Actually, this is only after adding points at infinity, making the shapes compact, i.e. of finite extent.

Let us be more precise. Complex n-dimensional space $\mathbb{C}^n$ consists of all n-tuples $(z_1, ..., z_n)$ of complex numbers z_i. If we want to compactify $\mathbb{C}^n$ adding all the points at infinity, we construct the n-dimensional complex projective space $\mathbb{P}^n$. Points in $\mathbb{P}^n$ correspond to complex lines in $\mathbb{C}^{n+1}$ going through the origin. If we pick $n+1$ complex numbers $Z_0, ..., Z_n$ not all zero, then the set of points $\lambda(Z_0, ..., Z_n)$ with $\lambda \in \mathbb{C}$ gives a complex line in $\mathbb{C}^{n+1}$ through the origin, so we get a point in $\mathbb{P}^n$. This point is denoted by $[Z_0 : ... : Z_n]$ and the Z_i:s are called homogeneous coordinates.

If p is a homogeneous polynomial in the variables $Z_0, ..., Z_n$ then p is zero at a point $(Z_0, ..., Z_n)$ if and only if p is zero on the whole line generated by $(Z_0, ..., Z_n)$. Thus the equation $p = 0$ on $\mathbb{C}^{n+1}$ descends to an equation on $\mathbb{P}^n$. If we have a polynomial equation on $\mathbb{C}^n$ we can homogenize it and thus by the above procedure get an equation on $\mathbb{P}^n$ which has the effect of adding points at infinity to the solution, making it compact. This is what we did to get the sphere and the torus in our previous example.

A subset X of $\mathbb{P}^n$ is called a projective algebraic set if it is the common zero set of some collection of homogeneous polynomials. If X does not happen to be the union of two proper algebraic subsets, then X is called a projective variety. If X is smooth as well, i.e. locally looks like $\mathbb{C}^m$ for some m, then X is called a projective manifold.

Note here that the elliptic curve given by $y^2 = x^3 - x$ consists of two disconnected pieces when viewed as a subset of the (x, y)-plane. But when we move to the complex projective picture as above, what we get is one connected piece, a torus. This showcases some of the advantages one has in using complex numbers in geometry.

0.1.2 Holomorphic functions

When doing complex analysis in $\mathbb{C}$ the main object of study is usually the set of holomorphic functions, i.e. complex valued functions f satisfying the Cauchy-Riemann equations

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Instead of thinking of a function f as depending on the real variables x and y one can just as well think of it as depending on the complex parameters z and $\bar{z}$. Then the Cauchy-Riemann equation becomes equivalent to the d-bar equation

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Intuitively it says that a function is holomorphic if it only depends on z and not $\bar{z}$. One thus sees that any polynomial in z (and not $\bar{z}$) is holomorphic. In fact any holomorphic function f can locally araound a point a be written as a convergent power series

$$f(z) = \sum_i a_i(z-a)^i,$$

i.e. holomorphic functions are analytic.

One can generalize this to higher dimensions, thus a complex valued function on $\mathbb{C}^n$ is holomorphic if for all $1 \leq i \leq n$,

$$\frac{\partial f}{\partial \bar{z}_i} = 0.$$

Consider the complex projective space $\mathbb{P}^n$. A point p in $\mathbb{P}^n$ has homogeneous coordinates $[Z_0 : ... : Z_n]$. At least one of these coordinates must by definition be non-zero, so let us say that $Z_0 = 1$. The set of points in $\mathbb{P}^{\ltimes}$ with $Z_0 = 1$ is naturally identified with $\mathbb{C}^n$, and we say that a function on that part of $\mathbb{P}^{\ltimes}$ is holomorphic if it is holomorphic on $\mathbb{C}^n$. If we happen to be at a point where $Z_0 = 0$ then for some other index i we have that $Z_i = 1$, and then we get another identification with $\mathbb{C}^n$. We have thus defined what it means for a function to be locally holomorphic on $\mathbb{P}^n$.

Let X be a projective manifold as defined above, sitting inside some $\mathbb{P}^n$. We say that a function f on some part of X is holomorphic if it locally is the restriction of a holomorphic function on some piece of $\mathbb{P}^n$. We know that X looks like $\mathbb{C}^m$ for some m ($m \leq n$). In fact locally around each point in X we can find m holomorphic functions giving us holomorphic coordinates z_i. A manifold with this property is called a complex manifold, so a projective manifold is also a complex manifold.

A map between two complex manifolds is called holomorphic if the composition with any holomorphic coordinte on the target manifold is holomorphic. If there exists a holomorphic bijection between two complex manifolds, then we think of them as just two incarnations of the same complex manifold. In this sense, the embedding of a projective manifold into projective space is not unique, each manifold will have infinately many different (biholomorphic) embeddings.

0.1.3 Line bundles and sections

Because a projective manifold X is compact, the only functions that are holomorphic on the whole of X are the constants. This is one reason for introducing holomorphic line bundles on X. A holomorphic line bundle L on X is a family of complex lines $L_x \cong \mathbb{C}$ holomorphically parametrized by the points x in X, and such that the parametrization is locally trivial. The last statement means that around a point x there is a neighbourhood U such that the collection of lines L_y, $y \in U$ looks like $U \times \mathbb{C}$. A section s of L is a function which maps each point $x \in X$ to some point on its associated line L_x. Since the line bundle is locally trivial, locally around a point x a section just looks like an ordinary complex valued function. However, since the line bundle can "twist," globally the section does not in general correspond to an ordinary function. A section is called holomorphic if it locally looks holomorphic. Thanks to the twisting of a line bundle, we can have non-trivial holomorphic sections, even though there are no non-trivial holomorphic functions.

We can look at the example $\mathbb{P}^n$. There is a natural line bundle on $\mathbb{P}^n$ denoted by $\mathcal{O}(1)$. The line of $\mathcal{O}(1)$ corresponding to a point $[Z_0 : ... : Z_n]$ in $\mathbb{P}^n$ is defined as the dual of the line generated by $(Z_0, ..., Z_n)$. A homogeneous coordinate Z_i is not a well-defined function on $\mathbb{P}^n$ but to each point $[Z_0 : ... : Z_n]$ it gives an element in the dual space by mapping $\lambda(Z_0, ..., Z_n)$ to λZ_i. Thus each homogeneous coordinate Z_i correspond to a holomorphic section of $\mathcal{O}(1)$.

The set of holomorphic sections of a line bundle L is denoted by $H^0(X, L)$. It is a vector space, and a fundamental fact is that it is always finite dimensional. This is in stark contrast to the local picture, where the vector space of holomorphic functions on an open subset of $\mathbb{C}^n$ has infinite dimensions.

0.1.4 Chern classes, self-intersection and volume

Any manifold M has an associated collection of algrebraic objects (groups) called the homology groups $H_k(M, \mathbb{Z})$, where k ranges from zero to the real dimension of M, say m. They are real vector spaces, and heuristically the dimension of the homology groups measure the number of holes in M of different dimensions. A submanifold of dimension k gives you an element in $H_k(M, \mathbb{Z})$, but two different submanifolds does not necessarily give you two different elements. There are also cohomology groups $H^k(M, \mathbb{Z})$, whose elements can be reperesented by differential forms of degree k on M (when M is smooth). For oriented compact manifolds (such as projective manifolds) the Poincaré duality states that for any k there is a canonical isomorphism between the homology group $H_k(M, F)$ and the cohomology group $H^{m-k}(M, \mathbb{Z})$, where m was the real dimension of M.

Recall that a function is called meromorphic if it locally can be written as the quotient of two holomorphic functions where the denominator is not identically zero. Similarly one can talk about meromorphic sections of a line bundle. Even if a line bundle has no non-trivial holomorphic section one can always find a meromorphic one. If f is a meromorphic section, let $Z(f)$ denote the zero set counted with multiplicities, and let $P(f)$ denote the polar set, again counted with multiplicities. One can show that the homology class of $Z(f) - P(f)$ in $H_2(X, \mathbb{Z})$ is independent of the particular choice of f, thus we get an invariant of the line bundle L. By taking the Poincare dual we end up with a cohomology class in $H^2(X, \mathbb{Z})$ which is called the first Chern class of L, denoted by $c_1(L)$. The element $c_1(L)$ can be represented by a differential form ω of degree 2, and $c_1(L)^n$ denotes the element represented by ω^n, i.e. ω wedged with itself n times. Since X has real dimension $2n$ this is a form of full degree, so we can integrate it over X to get an integer (L^n) which is called the self-intersection of L. If L has n holomorphic sections whose common zero set is a finite collection of points then (L^n) is the number of these points counted with multiplicity. This explains why $(L)^n$ is called the self-intersection.

If we have two holomorphic line bundles L_1 and L_2 we can take their pointwise tensor product and this will again be a holomorphic line bundle, denoted by $L_1 \otimes L_2$. Sometimes, instead of this multiplicative notation one uses additive notation, i.e. $L_1 + L_2$. This is because of the association between line bundles and divisors, and divisors are thought of as being added, not multiplied. This is the convention we will use in this thesis. A line bundle L tensored with itself k times will thus be denoted by kL. The k:th power of $\mathcal{O}(1)$ is usually written as $\mathcal{O}(k)$. If we have a homogeneous polynomial of degree k then by the same kind of argument as above this yields a holomorphic section of $\mathcal{O}(k)$. In fact, the space of holomorphic sections of $\mathcal{O}(k)$ is isomorphic to the set of homogeneous polynomial of degree k. An easy calculation thus gives that

$$\dim H^0(\mathbb{P}^n, \mathcal{O}(k)) = \binom{n+k}{n} = \frac{k^n}{n!} + o(k^n).$$

One can prove that for any line bundle L there exists a constant C such that

$$\dim H^0(X, kL) = C\frac{k^n}{n!} + o(k^n).$$

The constant C for a particular line bundle L is called the volume of L, denoted $\mathrm{vol}(L)$, and it is an important invariant of the line bundle. A line bundle with positive volume is called big.

If X is a projective manifold which is embedded in $\mathbb{P}^N$ for some N, then one can restrict the line bundle $\mathcal{O}(1)$ to X and get a holomorphic line bundle on X. A line bundle L on X which is the restriction of $\mathcal{O}(1)$ under some embedding of X into projective space is called very ample. If some positive power of L is very ample then L is called ample.

From the definition one sees that the volume of a line bundle is always non-negative, but it does not have to be an integer, in fact it can even be irrational. The self-intersection on the other hand is always an integer, but it can be negative. However, from the asymptotic Riemann-Roch theorem it follows that for ample line bundles the self-intersection and the volume coincide.

An interesting property of the self-intersection of ample line bundles is that it is $1/n$-concave. In other words, if L_1 and L_2 are two ample line bundles then

$$((L_1 + L_2)^n)^{1/n} \geq (L_1^n)^{1/n} + (L_2^n)^{1/n}. \tag{1}$$

Since for ample line bundles the self-intersection and the volume coincides, the volume is $1/n$-concave in the ample case. Using a result due to Fujita one can in fact prove that the inequality

$$\mathrm{vol}(L_1 + L_2)^{1/n} \geq \mathrm{vol}(L_1)^{1/n} + \mathrm{vol}(L_2)^{1/n} \tag{2}$$

extends to the whole class of big line bundles.

Using Jensen's inequality it follows that $1/n$-concavity implies log-concavity (see e.g. [16]), thus the volume functional is also log-concave.

0.1.5 The Brunn-Minkowski inequality

It was in order to explain $1/n$-concavity inequalities such as (1) and (2) that Okounkov introduced Okounkov bodies. To understand his motivation we need to recall a classic result in convex geometry, the Brunn-Minkowski inequality.

A convex body in $\mathbb{R}^n$ is a compact convex set with non-empty interior. That it is convex means that the line segment between any two points in the body lies in the body. Examples include the ball and the hypercube. If A and B are any subsets of $\mathbb{R}^n$ their Minkowski sum $A + B$ is defined as

$$A + B := \{x + y : x \in A, y \in B\}.$$

If A and B are convex bodies, one easily sees that their Minkowski sum $A + B$ also will be a convex body. The Brunn-Minkowski inequality relates the Lebesgue volume of the sum $A + B$ with the volumes of A and B.

THEOREM 1. *Let A and B be two convex bodies in $\mathbb{R}^n$. Then we have that*

$$\mathit{vol}(A + B)^{1/n} \geq \mathit{vol}(A)^{1/n} + \mathit{vol}(B)^{1/n}. \tag{3}$$

For an exposition on the Brunn-Minkowski inequality see [16].

Note the similarity between (2) and (3). Okounkov's idea in [26] and [27] was to, given a line bundle L, construct a convex body $\Delta(L)$, with the property that its volume equals (L^n) or vol(L). If the construction works so that

$$\Delta(L_1 + L_2) \supseteq \Delta(L_1) + \Delta(L_2),$$

then the inequalities (1) and (2) would follow from the Brunn-Minkowski inequality.

0.1.6 Okounkov bodies

Okounkov found a way to associate to any ample line bundle L a convex body $\Delta(L)$, now called the Okounkov body of L. This had the right kind of properties, making inequality (2) a consequence of the Brunn-Minkowski inequality. Later, Lazarsfeld-Mustaţă in [22] and Kaveh-Khovanskii in [18] independently showed that Okounkov's construction worked in a much more general setting, e.g. L could be big and it would still have the same properties, thus showing that inequality (2) also follows from Brunn-Minkowski.

Let us now describe the construction of the Okounkov body of a big line bundle L.

When defining the Okounkov body, one can either use a flag of irreducible subvarieties, or work with local coordinates. For simplicity we choose here to work with local coordinates.

Suppose we have chosen a point p in X, and local holomorphic coordinates $z_1, ..., z_n$ centered at p, and let $e \in H^0(U, L)$ be a local trivialization of L around p. If we divide a section $s \in H^0(X, L)$ by e we get a local holomorphic function. It has an unique represention as a convergent power series in the variables z_i,

$$\frac{s}{e} = \sum a_\alpha z^\alpha,$$

which for convenience we will simply write as

$$s = \sum a_\alpha z^\alpha.$$

We consider the lexicographic order on the multiindices α, and let $v(s)$ denote the smallest index α such that $a_\alpha \neq 0$. Recall that the lexicographic order is defined so that $\alpha < \beta$ if for some index j, $\alpha_i = \beta_i$ when $i < j$ and $\alpha_j < \beta_j$. Let

$$\Delta_1(L) := \left\{v(s) : s \in H^0(X, L) \setminus \{0\}\right\}.$$

If k is a positive integer, e^k is a local trivialization of kL. By looking at the power series of s/e^k for sections $s \in H^0(X, kL)$ we get sets

$$\Delta_k(L) := \left\{\frac{v(s)}{k} : s \in H^0(X, kL) \setminus \{0\}\right\}.$$

If $s \in H^0(X, L)$, then $s^k \in H^0(X, kL)$, and one easily sees that $v(s^k) = kv(s)$. This is the why in the definition of $\Delta_k(L)$ the points $v(s)$ are scaled by $1/k$.

DEFINITION 1. *The Okounkov body $\Delta(L)$ of a big line bundle L is defined as*

$$\Delta(L) := \overline{\bigcup_{k=1}^{\infty} \Delta_k(L)}.$$

Remark. Note that the Okounkov body $\Delta(L)$ of a line bundle L in fact depends on the choice of point p in X and local coordinates z_i. We will however supress this in the notation, writing $\Delta(L)$ instead of the perhaps more proper but cumbersome $\Delta(L, p, (z_i))$.

From the article [22] by Lazarsfeld-Mustaţă we recall some results on Okounkov bodies of line bundles.

LEMMA 2. *The number of points in $\Delta_k(L)$ is equal to the dimension of the vector space $H^0(kL)$.*

LEMMA 3. *The Okounkov body $\Delta(L)$ of a big line bundle is a convex body.*

The most important property of the Okounkov body is its relation to the volume of the line bundle, described in the following theorem.

THEOREM 4. *For any big line bundle it holds that*

$$vol(L) = n! vol_{\mathbb{R}^n}(\Delta(L)),$$

where the volume of the Okounkov body is measured with respect to the standard Lesbesgue measure on $\mathbb{R}^n$.

Using a result of Khovanskii on semigroups one can show that the points in $\Delta_k(L)$ almost fill the intersection of $\Delta(L)$ with the scaled integer lattice $(1/k)\mathbb{Z}^n$. Since the number of points in this intersection is easily computed to be

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(L))k^n + o(k^n),$$

Theorem 8 then follows using Lemma 5 and the definition of $\mathrm{vol}(L)$. For a detailed proof see [22].

0.1.7 Toric geometry and moment polytopes

In general the Okounkov body of a line bundle is difficult to compute. There are certain interesting cases though where we know exactly what they look like.

Toric manifolds are manifolds that are extremely symmetric. By definition a toric manifold is a manifold which has an action of the algebraic torus $(\mathbb{C}^*)^n$ with an open dense orbit. The easiest example is given by the sphere. Yet they do not have to be that simple, so the class of toric manifolds is sufficiently rich geometrically to attract a lot of interest. If the toric manifold X is projective as well, it means that X is the compactification of an embedded copy of $(\mathbb{C}^*)^n$ in some projective space $\mathbb{P}^N$. If we restrict $\mathcal{O}(1)$ to X, we get that the algebraic torus action lifts to an action on the restricted line bundle. Such a line bundle is called a toric line bundle.

Let us think about how one can embed $(\mathbb{C}^*)^n$ into projective space to get a projective toric manifold. A lattice polytope P in $\mathbb{R}^n$ is by definition the convex hull of a finite collection of points in the integer lattice $\mathbb{Z}^n$. Let N_P denote the number of lattice points in P, and let α_i, $1 \leq i \leq N_P$ be an enumeration of these lattice points. We get a map f_P from $(\mathbb{C}^*)^n$ into $\mathbb{P}^{N_P-1}$ by letting

$$f_P(z) := [z^{\alpha_1}, ..., z^{\alpha_{N_P}}].$$

This might not be an embedding. But we can do the same thing for the lattice polytopes kP, for $k \in \mathbb{N}$, and we thus a get sequence of maps f_{kP}. It turns out that f_{kP} will be an embedding for k sufficiently large. By taking the closure of the image of f_{kP} we get what is called a toric variety, i.e. it has the right kind of algebraic torus action, but it might not be smooth manifold. We will denote the corresponding toric variety by X_P. The polytope P is called the moment polytope of the toric variety X_P.

The unit n-simplex Σ_n is the lattice polytope in $\mathbb{R}^n$ spanned by the origin and the unit vectors e_i. One can show that $X_{\Sigma_n} = \mathbb{P}^n$. Polytopes which give rise to toric manifolds, i.e. smooth toric varieties, are called Delzant polytopes, so we see that Σ_n is Delzant. In fact, a lattice polytope P is Delzant if and only if a neighbourhood of any vertex in P can be transformed to a neighbourhood of the origin in Σ_n by an element in $GL(n, \mathbb{Z})$ and a translation.

Given a Delzant polytope P we thus get a projective toric manifold X_P. The map f_P extends to a map from X_P to $\mathbb{P}^{N_P-1}$, and we denote the pullback of $\mathcal{O}(1)$ to X_P by L_P. One can see that for any k, $kL_P = L_{kP}$. Since for large k f_{kP} is an embedding it follows that L_P is an ample toric line bundle

One can understand the spaces of holomorphic sections to the powers of L_p by looking at the polytope P. Indeed we have that

$$H^0(X_P, kL_P) \cong \bigoplus_{\alpha \in kP \cap \mathbb{Z}^n} < z^\alpha > . \tag{4}$$

More specifically there is a basis $\{s_\alpha\}$ for the space of sections $H^0(X_P, kL_P)$ such that for any $\alpha, \beta \in kP \cap \mathbb{Z}^n$,

$$s_\alpha / s_\beta = z^{\alpha - \beta}$$

on the copy of $(\mathbb{C}^*)^n$. The coordinates z_i are given by map f_{kP}.

As in the case of Okounkov bodies, from (4) one sees that

$$\text{vol}(L_P) = n!\text{vol}(P).$$

So what is the relation between P and the Okounkov body $\Delta(L)$? Note that the Okounkov body depended upon us choosing local coordinates around some point. We know that P is Delzant, so we can transform it so that one of its vertices lies at the origin, and locally P looks like the unit simplex around that point. This shape of P easily implies that the compactification of $(\mathbb{C}^*)^n$ includes $\mathbb{C}^n$. Thus we can use the origin in $\mathbb{C}^n$ as our point and z_i as our local coordinats. Using s_0 as the local trivialization, we get from (4) that

$$\Delta_k(L) = P \cap (1/k)\mathbb{Z}^n,$$

and thus

$$\Delta(L) = P.$$

This means that one can think of Okounkov bodies as generalizing the correspondence between toric line bundles and polytopes in toric geometry.

For a proper exposition of toric geometry we refer the reader to the book [15] by Fulton.

0.1.8 Symplectic geometry and moment maps

We have not yet explained why we call the polytope P corresponding to a toric manifolds X_P the moment polytope of X_P. This leads us into the field of symplectic geometry.

DEFINITION 2. *A 2-form ω on a manifold M is symplectic if it is non-degenerate and closed. The pair (M, ω) is then called a symplectic manifold.*

The non-degeneracy means that for any point $p \in M$ and any element v in the tangent space at p there is a tangent vector $w \in T_pM$ such that $\omega_p(v, w) \neq 0$. It turns out that symplectic manifolds must be of even real dimension $2n$ for some n, and then one can formulate the non-degeneracy as $\omega^n \neq 0$.

Say that S^1 acts on (M, ω), i.e. ω is left invariant under the S^1-action on M. Then the action is generated by a vector field X. We say that the action is Hamiltonian if there is a function H solving the equation

$$dH = \omega(X, \cdot).$$

Indeed one can show that the one form $\omega(X, \cdot)$ always is closed, so the action is Hamiltonian if $\omega(X, \cdot)$ is exact.

Let now the real n-torus $T^n := (S^1)^n$ act on (M, ω). We can decompose the torus action into n commutative circle actions. We say that the torus action is Hamiltonian if each of these circle actions are Hamiltonian. If we let H_i denote the Hamiltonian of the i:th S^1-action we can put these together to get a map μ from M to $\mathbb{R}^n$, $\mu := (H_1, ..., H_n)$. The map μ is called the moment map of the torus action.

Remark. Instead of just looking at T^n one can look at any Lie group G acting on (M, ω). If the action is Hamiltonian one can define a moment map μ in a more invariant way than we did above, namely as a map from M to the dual of the Lie algebra of G.

0.1.9 Plurisubharmonic functions

A function u on some open subset U of $\mathbb{C}$ is called harmonic if $\Delta u = 0$. Here Δ denotes the Laplacian, i.e.

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

If u is an upper semicontinuous function from U to $[-\infty, \infty)$ such that the Laplacian Δu is positive in the sence of distributions then u is called subharmonic. An upper semicontinuous function u from an open subset of C^n to $[-\infty, \infty)$ is called plurisubharmonic if the restriction of u to any complex line is locally subharmonic. If u happens to be C^2 then this is equivalent to the complex Hessian

$$\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right)$$

being positive semidefinite. If the Hessian is positive definite u is said to be strictly plurisubharmonic. The notion of plurisubharmonicity is preserved by biholomorphisms, hence it makes sense to talk about functions being plurisubharmonic locally on a complex manifold, and in particular on a projective manifold.

0.1.10 Hermitian metrics on line bundles

Given a projective manifold X with an ample line bundle L, there is a natural class of symplectic structures ω on X, namely those symplectic forms that belong to the first Chern class of L, $c_1(L)$. There is a more geometric way to think about these symplectic forms ω.

A 2-form ω is said to be $(1,1)$ if

$$\omega(\cdot,\cdot) = \omega(J\cdot, J\cdot).$$

Here J denotes the almost complex structure on the tangent space defined by

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \qquad J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}.$$

That a $(1,1)$-form ω is strictly positive means that for any $v \in T_pX$

$$\omega_p(v, Jv) > 0.$$

Such an ω is clearly non-degenerate, thus is a symplectic form. A strictly positive closed $(1,1)$-form is called a Kähler form. A Kähler form ω gives rise to a Riemannian metric g_ω on X by saying that if v and w lie in T_pX then $g_\omega(v,w) := \omega(v, Jw)$.

A hermitian metric $h = e^{-\phi}$ on L is a smooth choice of scalar product on the complex line L_p at each point p on the manifold. If f is a local holomorphic frame for L in some neighbourhood U_f, then one writes

$$|f|_h^2 = h_f = e^{-\phi_f},$$

where ϕ_f is a smooth function on U_f. We will use the convention to let ϕ denote the metric $h = e^{-\phi}$, thus if ϕ is a metric on L, $k\phi$ is a metric on kL. Sometimes

ϕ is intead referred to as the weight of the metric $h = e^{-\phi}$. This convention is used in paper I, but not in the rest of the thesis.

The curvature of a smooth metric is given by $dd^c\phi$ which is the $(1,1)$-form locally defined as $dd^c\phi_f$, where f is any local holomorphic frame. Here d^c is short-hand for the differential operator

$$\frac{i}{4\pi}(\bar{\partial} - \partial),$$

so $dd^c = i/2\pi\partial\bar{\partial}$. A classic fact is that the curvature form of a smooth metric ϕ lies in the first Chern class of L.

The metric ϕ is said to be positive if the curvature $dd^c\phi$ is strictly positive as a $(1,1)$-form, which is equivalent to the function ϕ_f being strictly plurisubharmonic for any local frame f. We let $\mathcal{H}(L)$ denote the space of positive metrics on L. A famous theorem, the Kodaira embedding theorem, states that a line bundle has a positive metric iff it is ample. That an ample line bundle has a positive metric is easy to show, it is the converse which is hard to prove.

The curvature form $dd^c\phi$ of a positive metric ϕ is thus a Kähler form in $c_1(L)$. On the other hand, if L is ample, by the dd^c-lemma (see e.g. [17]), any form ω in $c_1(L)$ can be written as the curvature form of a smooth metric $\phi + u$, where ϕ is a positive metric and u is a smooth function. At the point where u attains its minimum we have that $\omega \geq dd^c\phi$, and thus ω is strictly positive at that point. If ω is symplectic it follows that ω will be strictly positive on the whole of X, thus $\phi + u$ is a positive metric. This means that any symplectic form in $c_1(L)$ is the curvature form of some positive metric. If two positive metrics ϕ and ψ have the same curvature, then $dd^c(\phi - \psi) = 0$. This implies that $\phi - \psi$ is harmonic on X, which by the maximum principle gives that $\phi - \psi$ is a constant.

Therefore any symplectic (Kähler) form in $c_1(L)$ correspond to a positive metric on L, which is unique up to a constant.

0.1.11 The moment polytope revisited

Since $T^n \subset (\mathbb{C}^*)^n$ any toric manifold X_P has a natural T^n-action on it. And as we have seen in the previous section, given an ample toric line bundle L_P

we have natural symplectic structures on our manifold, coming from positive metrics on L_P. By averaging a symplectic form ω over the action we get a symplectic form ω_{av} which is invariant under the action.

We can trivialize of L over $(\mathbb{C}^*)^n$ so that $s_\alpha = z^\alpha$. With respect to this trivialization a positive metric ϕ corresponds to a plurisubharmonic function $\tilde{\phi}$ on $(\mathbb{C}^*)^n$. That ϕ extends to the whole manifold forces a growth condition on the function $\tilde{\phi}$, namely that

$$\tilde{\phi} - \ln\big(\sum_{\alpha \in P\cap\mathbb{Z}^n} |z^\alpha|^2\big) \tag{5}$$

remains bounded.

If $dd^c\phi$ is T^n-invariant then ϕ must be T^n-invariant, and thus $\tilde{\phi}(z_1, ..., z_n) = \tilde{\phi}(|z_1|, ..., |z_n|)$. Let f denote the function $f(w_1, ..., w_n) := (e^{w_1}, ..., e^{w_n})$. It follows that $u := \tilde{\phi} \circ f$ is plurisubharmonic and independent of the imaginary parts y_i of w_i. It is a well-known fact that any such function is a convex function of the real parts x_i of w_i. An easy calculation yields that

$$dd^c u = \frac{1}{4\pi}\sum \frac{\partial^2 u}{\partial x_i \partial x_j} dy_i \wedge dx_j. \tag{6}$$

We observe that

$$(1/2)d\frac{\partial u}{\partial x_i} = dd^c u(2\pi\frac{\partial}{\partial y_i}, \cdot).$$

Now $dd^c\tilde{\phi} = f^{-1*}dd^c u$ and the pushforward of $2\pi\frac{\partial}{\partial\theta_i}$ under f^{-1} is $2\pi\frac{\partial}{\partial y_i}$. Since $2\pi\frac{\partial}{\partial\theta_i}$ generates the i:th circle action on $(\mathbb{C}^*)^n$ it follows that

$$H_i = (1/2)\frac{\partial u}{\partial x_i} \circ f^{-1}$$

solves the Hamiltonian equation. Thus the T^n-action is Hamiltonian and a moment map is given by $(1/2)\nabla u \circ f^{-1}$.

What is the image of the moment map μ? Looking at the growth condition (5) we get that

$$u - \ln\big(\sum_{\alpha \in P\cap\mathbb{Z}^n} e^{2\alpha\cdot x}\big)$$

is bounded. Clearly

$$\nabla \ln(e^{2\alpha\cdot x}) = 2\alpha.$$

It is not hard to see that the image of

$$(1/2)\nabla \ln \big(\sum_{\alpha \in P \cap \mathbb{Z}^n} e^{\alpha \cdot x} \big)$$

is the interior of P. It follows that the image of μ is also the interior of P.

This shows why we call P the moment polytope, since it is the image of a moment map μ.

0.1.12 The Monge-Ampère energy

We have seen above that a postive metric ϕ gives rise to a symplectic form in $c_1(L)$ by taking the curvature $dd^c\phi$. Taking the top power of this form gives us a volume form $(dd^c\phi)^n$. This form is called the Monge-Ampère measure of ϕ and is also denoted by $MA(\phi)$. Since $dd^c\phi \in c_1(L)$, integrating $MA(\phi)$ over X we get

$$\int_X MA(\phi) = \int_X c_1(L)^n = \text{vol}(L),$$

the volume of L.

An important bifunctional on the space of positive metrics is the Monge-Ampère energy $\mathcal{E}$. It first appeared in the works of Mabuchi and Aubin (see [3] for references). Because of this it sometimes goes under the name of Aubin-Mabuchi energy, e.g. in paper III. The Aubin-Yau functional is yet another name given to it.

Given two positive metrics ϕ and ψ the Monge-Ampère energy of the pair is defined as

$$\mathcal{E}(\phi, \psi) := \frac{1}{n+1} \sum_{j=0}^{n} \int_X (dd^c\phi)^j \wedge (dd^c\psi)^{n-j}.$$

The Monge-Ampère energy has the cocycle property, i.e. for all positive metrics ψ, φ and ψ'

$$\mathcal{E}(\psi, \varphi) + \mathcal{E}(\varphi, \psi') + \mathcal{E}(\psi', \psi) = 0.$$

It is very strongly connected with the Monge-Ampère operator, which maps a metric to its Monge-Ampère measure. For any positive metric ϕ and smooth function u,

$$\frac{d}{dt}_{|t=0}\mathcal{E}(\phi + tu, \psi) = \int_X uMA(\phi). \tag{7}$$

One way of formulating (7) is to say that the Monge-Ampère operator is the differential of the Monge-Ampère energy.

0.1.13 The real Monge-Ampère operator

Let u be a smooth strictly convex function on $\mathbb{R}^n$. There is a real version of the Monge-Ampère operator, mapping convex functions to Borel measures on $\mathbb{R}^n$. When u is smooth we can write this as

$$MA_{\mathbb{R}}(u) := \frac{n!}{2^n}\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) dx,$$

where dx denotes the Lebesgue measure on $\mathbb{R}^n$. Since u is strictly convex the Hessian matrix

$$\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$$

is positive definite, thus $MA_{\mathbb{R}}(u)$ is a positive measure.

Depending on the growth of u at infinity the measure $MA(u)$ can have either finite or infinite mass.

Let P be a lattice polytope in $\mathbb{R}^n$. Assume that there is a constant C such that

$$\ln\big(\sum_{\alpha \in P\cap\mathbb{Z}^n} e^{2\alpha\cdot x}\big) - C \leq u \leq \ln\big(\sum_{\alpha \in P\cap\mathbb{Z}^n} e^{2\alpha\cdot x}\big) + C. \tag{8}$$

As we saw above this implies that the image of ∇u is the interior of $2P$. What is the mass of $MA(u)$? We have that $det(\nabla^2 u)dx = (\nabla u)^* dx$, which means that the Monge-Ampère measure is the pullback of the Lebesgue measure under the gradient map times $n!/2^n$. This implies that the mass of the Monge-Ampère measure is the volume of $2P$ times $n!/2^n$, i.e.

$$\int_{\mathbb{R}^n} MA_{\mathbb{R}}(u) = n!\text{vol}(P).$$

One can also define the real Monge-Ampère energy of a pair of smooth strictly convex functions u and v, both being bounded as in (8). Let $u_t :=$

$tu + (1-t)v$. Then we define

$$\mathcal{E}_{\mathbb{R}}(u,v) := \int_{t=0}^{1} \int_{\mathbb{R}^n} (u-v) MA(u_t).$$

If we differentiate $\mathcal{E}_{\mathbb{R}}(u_t, v)$ with respect to t we clearly get

$$\int_{\mathbb{R}^n} \dot{u_t} MA(u_t),$$

and this makes it the natural real analog of the Monge-Ampère energy.

One can show that the real Monge-Ampère energy also has the cocycle property.

0.1.14 The Legendre transform

Let u be a smooth strictly convex function on $\mathbb{R}^n$, and let $\Delta(u)$ denote the image of the gradient ∇u. The Legendre transform of u, denoted by $\mathcal{L}u$ is the function on $\Delta(u)$ defined by

$$\mathcal{L}u(y) := \sup\{x \cdot y - u(x) : x \in \mathbb{R}^n\}.$$

We have that $\mathcal{L}u$ is convex since it is the supremum over the linear and hence convex functions $x \cdot y - u(x)$. One can show that it is smooth and strictly convex as long as u is smooth and strictly convex.

Since u is smooth and strictly convex, we see that $u(x) - x \cdot y$ is minimized exactly where $\nabla u(x) = y$, and thus $\mathcal{L}u(y) = x \cdot y - u(x)$ where x solves the equation $\nabla u(x) = y$. Using this one can show that $\nabla \mathcal{L}u(y) = x$, where again x solves the equation $\nabla u(x) = y$.

We can take the Legendre transform of $\mathcal{L}u$ and we similarly get that

$$\mathcal{L}(\mathcal{L}u)(x) = y \cdot x - \mathcal{L}u(y),$$

where y solves the equation $\nabla \mathcal{L}u(y) = x$. From the above we see that it means that $\nabla u(x) = y$ and thus

$$\mathcal{L}(\mathcal{L}u)(x) = y \cdot x - (x \cdot y - u(x)) = u(x).$$

This shows that the Legendre transform is an involution.

Let u and v be smooth strictly convex functions both satisfying the bound (8) for some constant C. Thus the Legendre transforms $\mathcal{L}u$ and $\mathcal{L}v$ are convex functions on $2P^\circ$. We claim that

$$\mathcal{E}(u,v) = \frac{n!}{2^n}\int_{2P^\circ}(\mathcal{L}v - \mathcal{L}u)dx, \tag{9}$$

where dx denotes the Lebesgue measure.

Proof. Let us consider $f(t) := \mathcal{E}(u_t, v)$ as a function of t, where as above $u_t = tu + (1-t)v$. Let also

$$g(t) := \frac{n!}{2^n}\int_{2P^\circ}(\mathcal{L}v - \mathcal{L}u_t)dx.$$

We wish to prove that $f(1) = g(1)$. The derivative of f with respect to t was given by

$$\dot{f}(t) = \int \dot{u}_t MA(u_t).$$

The derivative of g is given by

$$\dot{g}(t) = -\frac{n!}{2^n}\int_{2P^\circ}(\frac{d}{dt}\mathcal{L}u_t)dx.$$

Since $\mathcal{L}u_t(y) = x\cdot y - u_t(x)$ where $\nabla u_t(x) = y$, one easily gets that

$$\frac{d}{dt}\mathcal{L}u_t(y) = -\dot{u}_t(x) = -\dot{u}_t \circ (\nabla u)^{-1}(y).$$

Therefore

$$\dot{g}(t) = -\frac{n!}{2^n}\int_{2P^\circ}(\frac{d}{dt}\mathcal{L}u_t)dx = \frac{n!}{2^n}\int_{2P^\circ}\dot{u}_t \circ (\nabla u_t)^{-1}dx =$$
$$= \frac{n!}{2^n}\int_{\mathbb{R}^n}\dot{u}_t(\nabla u_t)^* dx = \int_{\mathbb{R}^n}\dot{u}_t MA(u_t).$$

In the last step we used the fact that

$$\frac{n!}{2^n}(\nabla u)^* dx = MA(u_t).$$

Since $f(0) = g(0) = 0$ and $\dot{f}(t) = \dot{g}(t)$ for all t we get that $f(1) = g(1)$, which is what we wanted. □

One can interpret this result as saying that the Legendre transform linearizes the real Monge-Ampère operator.

0.1.15 Symplectic potentials

Let u be a plurisubharmonic function on $\mathbb{C}^n$ which does not depend on the y-variables. Then as we noted above u is a convex function of the x-variables. We have chosen the definition of the real Monge-Ampère measure of u so that it equals the pushforward of $MA_{\mathbb{C}}(u)$ on the strip $\mathbb{R}^n \times [0, 2\pi)^n$ to $\mathbb{R}^n$.

Let ϕ and ψ be two T^n-invariant positive metrics of an ample toric line bundle L_P. Let as before $f(w_1, ..., w_n) := (e^{w_1}, ..., e^{w_n})$, but we now think of f as mapping the cylinder $\mathbb{R}^n \times T^n$ biholomorphically to $(\mathbb{C}^*)^n$. If we equate the metrics with their standard trivialization over $(\mathbb{C}^*)^n$ then $u_\phi := \phi \circ f^{-1}$ and $u_\psi := \psi \circ f^{-1}$ are plurisubharmonic functions independent of the y-variables on T^n, and thus we can think of them as convex functions on $\mathbb{R}^n$. We get that

$$\int_X (\phi - \psi) MA(\phi_t) = \int_{\mathbb{R}^n \times T^n} (u_\phi - u_\psi) MA_{\mathbb{C}}(u_{\phi_t}) = \\ = \int_{\mathbb{R}^n} (u_\phi - u_\psi) MA_{\mathbb{R}}(u_{\phi_t}).$$

Since the left-hand-side is the t-derivative of $\mathcal{E}(\phi_t, \psi)$ and the right-hand-side is the derivative of $\mathcal{E}(u_{\phi_t}, u_\psi)$, integrating over t gives us that

$$\mathcal{E}(\phi, \psi) = \mathcal{E}(u_\phi, u_\psi). \tag{10}$$

We know that the image of the gradient of u_ϕ and u_ψ is the interior of $2P$. From section (0.1.14) we know that the Legendre transforms $\mathcal{L}u_\phi$ and $\mathcal{L}u_\psi$ are convex functions on $2P^\circ$, so the Legendre transforms of $u_\phi/2$ and $u_\psi/2$ are convex functions on P°. It is easy to see that for any $y \in P^0$ we have that

$$\mathcal{L}u_\phi(2y) = 2\mathcal{L}(u_\phi/2)(y). \tag{11}$$

The Legendre transform $\mathcal{L}(u_\phi/2)$ times 2 is called the symplectic potential of ϕ, and we will denote it by $\mathcal{L}\phi$ (hopefully not causing to much confusion). Putting (9) and (10) together and using equation (11) we get that

$$\mathcal{E}(\phi, \psi) = n! \int_{P^\circ} (\mathcal{L}\psi - \mathcal{L}\phi) dx. \tag{12}$$

0.2 Paper I

As we have seen, Okounkov bodies can be seen as a generalization of the moment polytope associated to a toric line bundle. Inspired by that, one can wonder if there is a generalized Legendre transform which associates to a positive metric of a line bundle L a convex function on the interior of the Okounkov body L. Of course, in order to be interesting it would have to have some nice properties. For instance, one would like to be able to calculate the Monge-Ampère energy by integrating the difference of the transforms over the Okounkov body, as in (12).

The first paper in this thesis proposes a transform of this kind, the Chebyshev transform. The Chebyshev transform of a positive metric ϕ, denoted by $c[\phi]$, is a convex function on $\Delta(L)^{\circ}$, and we prove the corresponding formula

$$\mathcal{E}(\phi,\psi)=n!\int_{\Delta(L)^{\circ}}(c[\psi]-c[\phi])dx.$$

In fact, the setting of the paper is more general. Instead of only looking at ample line bundles we widen our scope to include all big line bundles, i.e. those having positive volume. A big line bundle that is not ample has no positive metrics though. So we are forced to look at more general types of metrics.

DEFINITION 3. *A continuous hermitian metric $h=e^{-\psi}$ on a line bundle L is a continuous choice of scalar product on the complex line L_p at each point p on the manifold. If f is a local frame for L on U_f, then one writes*

$$|f|^2=h_f=e^{-\psi_f},$$

where ψ_f is a continuous function on U_f.

Note that a continuous metric does not have to be locally plurisubharmonic. Thus any line bundle will have (lots of) continuous metrics. We also have to consider metric that are locally plurisubharmonic but not necessarily continuous.

DEFINITION 4. *A positive singular metric ψ is a metric that can be written as $\psi:=\phi+u$, where ϕ is a continuous metric and u is a $dd^c\phi$-psh function, i.e. u is upper semicontinuous and $dd^c\psi:=dd^c\phi+dd^cu$ is a positive $(1,1)$-current.*

The last statement in the definition simply means that $\phi + u$ is locally plurisubharmonic.

We let $PSH(L)$ denote the space of positive singular metrics on L.

As an important example, if $\{s_i\}$ is a finite collection of holomorphic sections of kL, we get a positive metric $\psi := \frac{1}{k}\ln(\sum |s_i|^2)$ which is defined by letting for any local frame f,

$$e^{-\psi_f} := \frac{|f|^2}{(\sum |s_i|^2)^{1/k}}.$$

A line bundle which has a singular positive metric is called pseudoeffective. When L is big, then we know that for k large, kL has lots of holomorphic sections, so we get positive singular metrics on L by the above procedure, which means that L is pseudoeffective.

A positive singular metric ϕ is said to have minimal singularities if for any other positive singular metric ψ we have that $\psi \leq \phi + C$ for some constant C. If a positive singular metric is continuous, then clearly it has minimial singularities, but in general there are no continuous positive singular metrics. A standard way to find positive singular metrics with minimal singularities is by envelopes.

Let ϕ be a continuous metric (not necessarily positive) on a big line bundle L. Then we can form the envelope $P(\phi)$ by letting

$$P(\phi) := \sup\{\psi \leq \phi : \psi \in PSH(L)\}.$$

It is easy to show that $P(\phi)$ is a positive singular metric bounded by ϕ, and that it has minimal singularities. The way to think of this is that we get $P(\phi)$ by projecting ϕ down to the space of positive singular metrics.

There is a theory for defining the (non-pluripolar) Monge-Ampère operator and the Monge-Ampère energy for positive singular metrics with minimal singularities. This is based upon the pioneering work of Bedford and Taylor in the 80's on extending the Monge-Ampère operator to the class of locally bounded plurisubharmonic functions. Later e.g. Boucksom, Eyssidieux, Guedj and Zeriahi have extended this to the geometric setting of big line bundles on manifolds.

It means that one can define the Monge-Ampère energy $\mathcal{E}(\phi,\psi)$ when ϕ and ψ are two positive singular metrics with minimal singularities. When ϕ and ψ are smooth positive metrics on an ample line bundle then this coincides with the classical definition. If ϕ and ψ are two continuous metrics, not positive, then we can define their Monge-Ampère energy $\mathcal{E}(\phi,\psi)$ as the Monge-Ampère energy of the envelopes $P(\phi)$ and $P(\psi)$.

0.2.1 Capacity, transfinite diameter and Chebyshev constants

The question is how to transform a given metric to a convex function on the Okounkov body? The method we found was inspired by classical constructions in potential theory of the complex plane and generalizations of these introduced by Leja, Zaharjuta, Bloom-Levenberg et. al. in the context of pluripotential theory in $\mathbb{C}^n$ (see [5, 32, 38]).

Let K be a compact subset of $\mathbb{C}$. One way of measuring its size would be to measure its area. However, in potential theory one is interested in different notions of size, called capacities. One of these is called logarithmic capacity (see e.g. [32]). Intuitively, the logarithmic capacity of a set K, denoted $\text{cap}(K)$, measures how much electric charge one can fit in the set while maintaining a unit amount of potential energy.

The Lelong class $\mathcal{L}$ is the class of subharmonic functions u on $\mathbb{C}$ that are bounded from above by $\ln(1+|z|^2)$ plus some constant. If we think of $\mathbb{C}$ as embedded in $\mathbb{P}^1$, then the Lelong class exactly corresponds to the positive singular metrics on $\mathcal{O}(1)$.

Given a compact K, we let ϕ_K denote the envelope

$$\phi_K := (\sup\{\phi : \phi \leq 0 \text{ on } K, \phi \in \mathcal{L}\})^* .$$

The star * means that we take the uppersemicontinuous regularization of the supremum. The Monge-Ampère measure of ϕ_K (i.e. $dd^c\phi_K$) describes the equilibrium distribution of charge on K. Using this, one can show that if K and K' are two compacts, then

$$\mathcal{E}(\phi_K,\phi_{K'}) = \ln \text{cap}(K') - \ln \text{cap}(K). \tag{13}$$

A collection $\{x_i\}$ of k points in K are called Fekete points if they maximize the quantity

$$\Big(\prod_{1\leq i<j\leq k}|x_i - x_j|\Big)^{1/\binom{k}{2}}$$

among all sets of k points in K. We denote this maximum by δ_k. As k tends to infinity δ_k converges to a quantity $\delta(K)$ which is called the transfinite diameter of K. Fekete proved (see e.g. [32] for references) that this transfinite diameter coincides with the logarithic capacity of K.

Yet another quantity is the Chebyshev constant. We let $||.||_K$ denote the norm which takes the supremum of the absolute value on K. Let P_k denote the space of polynomials in z with z^k as highest degree term. Let for any k

$$Y_k(K) := \inf\left\{||p||_K : p \in P_k\right\}.$$

One defines the Chebyshev constant $C(K)$ of K as the following limit

$$C(K) := \lim_{k\to\infty}\left(Y_k(K)\right)^{1/k}.$$

It is well known that this number also coincides with the aforementioned capacity and transfinite diameter.

In $\mathbb{C}^n$ things get more complicated, but there are still analogs of the one-variable quantities. Given a compact K there is a similar notion of logarithmic capacity. The Lelong class $\mathcal{L}$ is now the class of plurisubharmonic functions on $\mathbb{C}^n$ that are bounded from above by $\ln(1+\sum|z_i|^2)$. Again if we think of $\mathbb{C}^n$ as embedded in $\mathbb{P}^n$, then the Leleong class corresponds to the positive singular metrics on $\mathcal{O}(1)$. Given K we get an envelope ϕ_K exactly as in $\mathbb{C}$, and for two compacts K and K' equation (13) holds.

Leja (see e.g. [38] for references) defined the notion of a transfinite diameter $\delta(K)$ for compacts K in $\mathbb{C}^n$, using Vandermonde determinants. Leja got a sequence of numbers δ_k and he defined the transfinite diameter $\delta(K)$ as the $\limsup$ of this sequence.

By extending the concept of a Chebyshev constant Zaharjuta in [38] was able to prove that that Leja's sequence actually converged to the transfinite diamater. Zaharjuta got his new Chebyshev constant by defining a certain convex function on a simplex, and then taking the exponent of the mean value of this.

For any $\alpha \in \mathbb{N}^n$ let P_α denote the space of polynomials of the form

$$z^\alpha + \sum_{\beta \in I} a_\beta z^\beta$$

where $\beta \in I$ if $|\beta| \leq |\alpha|$ and $\beta \leq \alpha$ with respect to the lexicographic order (see section 0.1.6). Let

$$Y_\alpha(K) := \inf\{||p||_K : p \in P_\alpha\}.$$

Let Σ^{n-1} denote the set of $\alpha = (\alpha_1, ..., \alpha_n)$ where $\alpha_i \geq 0$ for all i and $\sum \alpha_i = 1$. Let α_k be a sequence in $\mathbb{N}^n$ such that $|\alpha_k| = k$ and α_k/k converges to some point α in Σ^{n-1}. Then Zaharjuta defined the directional Chebyshev constant $C(K, \alpha)$ as

$$C(K, \alpha) := \lim_{k \to \infty} (Y_{\alpha_k}(K))^{1/k}.$$

He proved that this number did not depend on the specific choice of sequence α_k, and that the function $\ln C(K, \alpha)$ was convex on Σ^{n-1}. For this he used the submultiplicativity of Y, namely that

$$Y_{\alpha+\beta}(K) \leq Y_\alpha(K) Y_\beta(K).$$

Then Zaharjuta proved that

$$\ln \delta(K) = \frac{1}{\text{vol}(\Sigma^{n-1})} \int_{\Sigma^{n-1}} \ln C(K, \alpha) d\alpha.$$

Let us go back to the complex plane and introduce the weighted setting. Here we are considering weighted compact sets (K, ϕ) where ϕ is a continuous function on K. The weight ϕ is to be understood as an external field. We can define weighted analogs of the capacity, transfinite diameter and Chebyshev constant. Let $P(K, \phi)$ be the envelope defined as

$$P(K, \phi) := \left(\sup\{\psi : \phi \leq \phi \text{ on } K, \psi \in \mathcal{L}\}\right)^* .$$

Then given two weighted compacts (K, ϕ) and (K', ψ) we have that

$$\mathcal{E}(P(K, \phi), P(K', \psi)) = \ln \text{cap}(K', \psi) - \ln \text{cap}(K, \phi).$$

The weighted transfinite diameter is defined as the unweighted one, only that we look at the weighted quantity

$$\Big(\prod_{1\leq i<j\leq k}|x_i-x_j|e^{-\phi(x_i)/2-\phi(x_j)/2}\Big)^{1/\binom{k}{2}}.$$

We thus get a weighted transfinite diameter $\delta(K,\phi)$ by taking the limit as k tends to infinity.

The weighted Chebyshev constant is computed as the unweighted one only using the supremum norm

$$||p||_{K,k\phi}:=\sup\left\{|p(x)|e^{-k\phi(x)/2}:x\in K\right\}$$

instead of $||.||_K$on the spaces P_k.

As before the weighted capacity and transfinite diameter will coincide, but in general the weighted Chebyshev constant will not yield the same number. There is a formula relating the different quantities, but we will not go into it here.

Also in the weighted setting in $\mathbb{C}^n$ we have this phenomena. The weighted logarithmic capacity was proven to coincide with the weighted transfinite diameter by Berman-Boucksom in [3]. Bloom-Levenberg studied the weighted directional Chebyshev constants in [5] and they found a formula relating the integral of the convex function $\ln C(K,\phi,\alpha)$ with the logaritm of the transfinite diameter, showing that they do not coincide in general.

The space of polynomials in z of degree k or less is naturally isomorphic to the space of holomorphic sections to $\mathcal{O}(k)$ on $\mathbb{P}^1$, $H^0(\mathbb{P}^1,\mathcal{O}(k))$ The supremum norms $||.||_K$ and $||.||_{K,k\phi}$ are also norms on this space. Since not only P_k but P_m for all $0\leq m\leq k$ are affine subspaces of $H^0(\mathbb{P}^1,\mathcal{O}(k))$, inspired by the idea of directional Chebyshev constants we get new directional quantities by looking at

$$Y_{k,m}(K,\phi):=\inf\{||p||^2_{K,k\phi}:p\in P_m\}.$$

Of course, if $\phi=0$, corresponding to the unweighted setting $Y_{k,m}=Y_m$, so we get nothing new. But in general we get something different. If we let m_k be a sequence of natural numbers so that $m_k\leq k$ and m_k/k converges to some

$\alpha \in [0,1]$ then we define $C(K,\phi,\alpha)$ as the limit of $(Y_{k,m_k}(K,\phi))^{1/k}$. In the same way as Zaharjuta did, because of submultiplicativity one can prove that $C(K,\phi,\alpha)$ is well-defined and that $\ln C(K,\phi,\alpha)$ is a convex function on the unit interval. In the paper we call this convex function the Chebyshev transform of (K,ϕ), and we denote it by $c[K,\phi]$ It will follow as a consequence of our main theorem that if the set is regular, i.e. $P(K,\phi)$ is continuous, then the integral of the Chebyshev transform is equal to the logaritm of the transfinite diameter and hence the logaritm of the logarithmic capacity. We will postpone the discussion of our proof till later. Note that the weighted Chebyshev constant is the exponential of the value of the Chebyshev transform at the point one. In the unweighted case, the Chebyshev transform is linear, and goes from zero at zero to twice the logarithm of the classical Chebyshev constant, so integrating gives us back the classical constant.

One can do the same thing in $\mathbb{C}^n$. Given a weighted compact (K,ϕ) we thus get a convex function on the full unit simplex Σ_n in $\mathbb{R}^n$ which we again call the Chebyshev transform of (K,ϕ). When $\phi = 0$ then the function is linear, zero at the origin and equal to a constant times the logarithm of Zaharjuta's function on the boundary Σ^{n-1}, giving back Zaharjuta's integral when integrating over the whole unit simplex. We get, again as a consequence of our main theorem, that when (K,ϕ) is regular, then $n!$ times the integral of the Chebyshev transform equals the logarithm of the transfinite diameter. This means that by the work of Berman-Boucksom in [3] we get that

$$\mathcal{E}(P(K,\phi),P(K',\psi)) = n!\int_{\Sigma_n}(c[K',\psi]-c[K,\phi])d\alpha. \qquad (14)$$

Note here that the unit simplex is the Okounkov body of $\mathcal{O}(1)$. If we let K be the whole $\mathbb{P}^n$, and let ϕ be a continuous metric on $\mathcal{O}(1)$, then this gives us a transform from metrics to convex functions on the Okounkov body with the desired property that equation (14) should hold.

Since $\mathbb{P}^n$ is toric, there is already the Legendre transform mapping metrics that are T^n-invariant to convex functions on Σ_n. It is not hard to show that for T^n-invariant metrics the Chebyshev transform is nothing else than the Legendre transform. So one can say that we have extended the Legendre transform to also include non-invariant metrics.

0.2.2 The Chebyshev transform

We will now describe how to extend this construction to the case of an arbitrary big line bundle L on a projective manifold X. Given a continuous metric ϕ we get supremum norms $||.||_{k\phi}$ on the spaces $H^0(X,kL)$ by letting

$$||s||^2_{k\phi} := \sup\left\{|s(x)|^2 e^{-k\phi(x)} : x \in X\right\}.$$

Say that we have chosen a point $p \in X$ and local coordinates z_i around that point. We want affine subspaces in $H^0(X,kL)$ that can play the role that the P_m:s played in the case of $\mathcal{O}(1)$. Let α be a point in $\Delta_k(L)$. It means that there is a section $s \in H^0(X,kL)$ that locally can be written as $z^{k\alpha}$ plus higher order terms. We let $A_{\alpha,k}$ denote the affine space of sections in $H^0(X,kL)$ locally of the form

$$z^{k\alpha} + \text{higher order terms}.$$

Let $F[\phi](k\alpha,k)$ be defined as

$$F[\phi](k\alpha,k) := \inf\{\ln ||s||^2_{k\phi} : s \in A_{\alpha,k}\}.$$

This mimicks the definition of the directional Chebyshev constants, only that here we take the logarithm from the start.

Let s be a section in $H^0(X,kL)$ and s' a section in $H^0(X,mL)$, such that locally $s = z^{k\alpha} +$ higher order terms and $s' = z^{m\beta} +$ higher order terms. Then ss' lies in $H^0(X,(k+m)L)$ and $ss' = z^{k\alpha+m\beta} +$ higher order terms. We also have that $||ss'||^2_{(k+m)\phi} \leq ||s||^2_{k\phi}||s'||^2_{m\phi}$. This implies that $F[\phi]$ is subadditive, meaning that

$$F[\phi](k\alpha + m\beta, k+m) \leq F[\phi](k\alpha,k) + F[\phi](m\beta,m).$$

Using this we show that if α_k is a sequence in Δ_k, where k tends to infinity and α_k tends to some point $\alpha \in \Delta(L)^\circ$, then the sequence $1/kF[\phi](k\alpha,k)$ converges to a number $c[\phi](\alpha)$. This number does not depend of the particular choice of sequence α_k, and the function $c[\phi](\alpha)$ is convex in α. We call $c[\phi]$ the Chebyshev transform of the metric ϕ.

Our main theorem is then as follows.

THEOREM 5. *For any two continuous metrics ϕ and ψ on a big line bundle L, we have that*

$$\mathcal{E}(\phi, \psi) = n! \int_{\Delta(L)^\circ} (c[\psi] - c[\phi]) dx. \tag{15}$$

It is easy to see that if $\phi = \psi + 1$, then $c[\psi] = c[\phi] + 1$. We also have that $\mathcal{E}(\psi + 1, \psi) = \text{vol}(L)$. So in this special case equation (15) reduces to the fact that $n!$ times the volume of the Okounkov body is equal to the volume of the line bundle.

Building on this Yuan show in [37] how to construct a Chebyshev transform in the setting of Arakelov geometry. He proves a statement like theorem 5 but with the Monge-Ampère energy replaced by the adelic volume.

0.2.3 Proof of main theorem

The proof of Theorem 5 relies on the use of certain L^2-norms instead of supremum norms. If we pick a smooth volume form dV on X, then we get a L^2-norm $||.||_{k\phi,dV}$ on the space $H^0(X, kL)$, by letting

$$||s||_{k\phi,dV} := \int_X |s|^2 e^{-k\phi} dV.$$

If $\alpha \in \Delta_k$, we can consider the number $F_2[\phi](k\alpha, k)$, defined as

$$F_2[\phi](k\alpha, k) := \inf\{\ln ||s||^2_{k\phi,dV} : s \in A_{\alpha,k}\}.$$

The supremum norm and the L^2 norm are asymptotically equivalent (one says that dV has the Bernstein-Markov property) which means that one can just as well use F_2 to compute the Chebyshev transform. F_2 is however not subadditive, which is why we chose to use the supremum norm $||.||_{k\phi}$ in the definition. The point of using a L^2 norm is that we can construct an orthonormal basis for $H^0(X, kL)$. Let s_α be the section in $A_{k\alpha,k}$ which minimized the L^2-norm.

Then one notes that

$$\left\{e^{-F_2[\phi](k\alpha,k)} s_\alpha : \alpha \in \Delta_k(L)\right\} \tag{16}$$

is an orthonormal basis for $H^0(X, kL)$.

Using this observation, it is easy to show that the Chebyshev transform coinsides with the Legendre transform on T^n-invariant metrics on toric line bundles.

In [3] Boucksom-Berman introduce bifunctionals $\mathcal{L}_k$ of Donaldson type. If we have a pair of continuous metrics ϕ and ψ, $\mathcal{L}_k(\phi, \psi)$ is defined as

$$\mathcal{L}_k(\phi, \psi) := \frac{n!}{2k^{n+1}} \ln\left(\frac{\mathrm{vol}\mathcal{B}(k\phi, dV)}{\mathrm{vol}\mathcal{B}(k\psi, dV)}\right).$$

Here $\mathcal{B}(k\phi, dV)$ denotes the unit ball in $H^0(X, kL)$ with respect to the L^2-norm $||.||_{k\phi,dV}$. The volume of this unit ball is not well-defined without picking a measure, so we choose a linear isomorphism with $\mathbb{C}^m$, m being the dimension of the vector space, and take the Lebesgue measure there. When measuring the quotient of the volume of two unit balls, then this quotient is independent of the choice of isomorphism we made, thus it is well-defined.

Let s_i be an orthonormal basis with respect to $||.||_{k\phi,dV}$ and t_i an orthonormal basis with respect to $||.||_{k\psi,dV}$. If A_k is the change of basis matrix between them, then one observes that

$$|\det A_k|^2 = \frac{\mathrm{vol}\mathcal{B}(k\psi, dV)}{\mathrm{vol}\mathcal{B}(k\phi, dV)}. \tag{17}$$

Consider the orthonormal basis (16) and the corresponding one for the norm $||.||_{k\psi,dV}$. Since we know that s_α and t_α lies in $A_{\alpha,k}$, and the difference of two elements in $A_{\alpha,k}$ can be written as a linear combination of elements in $A_{\beta,k}$ for $\beta > \alpha$, it follows that the change of basis matrix A_k is triangular. Thus

$$|\det A_k|^2 = \Big(\prod_{\alpha\in\Delta_k(L)} e^{F_2[\phi](k\alpha,k) - F_2[\psi](k\alpha,k)}\Big)^2.$$

Combining this with equation (17) and the definition of $\mathcal{L}_k$ we get that

$$\mathcal{L}_k(\phi, \psi) = \frac{n!}{k^n} \sum_{\alpha\in\Delta_k(L)} (1/kF_2[\psi](k\alpha, k) - 1/kF_2[\phi](k\alpha, k)).$$

Since $1/kF_2[\phi]$ and $1/kF_2[\psi]$ converge to $c[\phi]$ and $c[\psi]$ respectively, it is not hard to show that

$$\lim_{k\to\infty} \mathcal{L}_k(\phi,\psi) = n! \int_{\Delta(L)^\circ} (c[\psi] - c[\phi])dx.$$

The main theorem in [3] states that the $\mathcal{L}_k$ functionals converge to the Monge-Ampère energy, so this gives us our main theorem.

0.2.4 Differentiability of Monge-Ampère energy

The paper also contains an application of the Main Theorem to the study of the Monge-Ampère energy.

In [22] Lazarsfeld-Mustaţă showed how one can use Okouknov bodies to prove the differentiability of the volume functional on the space of big line bundles. This was already known, but the new proof using Okounkov bodies was in a sense more intuitive.

In [3] Berman-Boucksom proved that the Monge-Ampère energy was differentiable as a functional on the space of continuous metrics on a fixed big line bundle.

In our paper we combine the result of Berman-Boucksom and the strategy of Lazarsfeld and Mustaţă, using our formula for the Monge-Ampère energy, and prove that the Monge-Ampère energy is differentiable as a functional on the space of continuous metrics on line bundles in the ample cone. That is, in contrast to Berman-Boucksom, we let the underlying line bundle vary as well.

0.3 Paper II

The second paper relates Okounkov bodies with test configurations. The main motivation for introducing and studying test configurations comes from the Yau-Tian-Donaldson conjecture.

0.3.1 The Yau-Tian-Donaldson conjecture

We have seen that given an ample line bundle L on a projective manifold X there is a natural class of symplectic structures on X, namely the Kähler forms in $c_1(L)$. We could interpret these as the curvature forms of positive metrics on L (see Section 0.1.10). We also saw that a Kähler form ω gives rise to a Riemannian metric g_ω. Such a metric is called a Kähler metric. The curvature tensor R of a Riemannian metric at a point takes as input two tangent vectors and produces an endomorphism of the tangent space, so it has type $(1,3)$. By contraction of the full curvature tensor R one gets the Ricci curvature Ric which has type $(0,2)$, i.e. it is a 2-form, and then the scalar curvature which is a function. Of course, in the process one loses more and more information.

For a Riemann surface the scalar curvature is twice the Gaussian curvature. The uniformization theorem tells us that any Riemann surface (which we think of as a complex manifold) has a J-invariant Riemannian metric with constant scalar curvature.

The Yau-Tian-Donaldson conjecture concerns the question when there exists a Kähler form in $c_1(L)$ whose Kähler metric has constant scalar curvature. We will use the abbreviation $cscK$ for constant scalar curvature. When L is a multiple of the canonical bundle K_X then the Kähler metric is cscK iff it is a Kähler-Einstein metric. The Ricci curvature form $Ric(g_\omega)$ can be seen to lie in $c_1(K_X^{-1})$. A metric is Kähler-Einstein if the Ricci curvature $Ric(g_\omega)$ is a multiple μ of the Kähler form $\omega \in c_1(L)$,

$$Ric(g_\omega) = \mu\omega.$$

From the uniformazation theorem follows that any compact Riemann surface has a Kähler-Einstein metric.

By scaling the constant μ can be chosen to be zero or plus or minus one. Yau proved the existence of a Kähler-Einstein metric when $\mu = 0$, i.e. when K_X is trivial. It was in fact a special case of the Calabi conjecture which was proved by Yau (see e.g. [29] for references). Existence of such metrics when $\mu = -1$, i.e. K_X being ample was proved independently by Aubin and Yau (again see [29]). A manifold with K_X^{-1} (called the anticanonical bundle) ample is called Fano. There are known obstructions to the existence of Kähler-Einsten metrics in the Fano case. The obstuctions due to Matsushima and Futaki both uses non-trivial automorphisms of X. E.g. the Futaki-invariant is a function on the space of holomorphic vector fields on X which if non-zero rules out the existence of a Kähler-Einstein metric. Tian later found an example of a Fano manifold without holomorphic vector fields but still no Kähler-Einstein metric, so this was not the end of the story.

Yau conjectured that the existence of a cscK metric with Kähler form in $c_1(L)$ should be equivalent to the pair (X, L) being stable in the sense of GIT (geometric invariant theory) (see e.g. [29] and [35]).

When Yau stated his conjecture, it was not clear what it should mean for a pair (X, L) to be stable in this context. In fact several new notions of stability has been put forward since.

The Yau-Tian-Donaldson conjecture is Yau's conjecture with the stability condition interpreted as K-stability. K-stability is a stability notion due to Donaldson building on the work of Tian. The definition of K-stability relies on the concept of a test configuration, which we will describe next.

0.3.2 Test configurations

Heuristically a test configuration can be thought of as a degenerated symmetry. Even though a manifold has no holomorphic symmetries it has lots of these degenerated symmetries.

Let us be more precise.

DEFINITION 5. *A test configuration of a pair (X, L) (X projective and L ample) consists of:*

(i) a scheme $\mathcal{X}$ with a $\mathbb{C}^*$-action ρ,

(ii) an $\mathbb{C}^*$-equivariant line bundle $\mathcal{L}$ over $\mathcal{X}$,

(iii) and a flat $\mathbb{C}^*$-equivariant projection $\pi : \mathcal{X} \to \mathbb{C}$ such that $\mathcal{L}$ restricted to the fiber over 1 is isomorphic to rL for some $r > 0$.

The zero fiber $X_0 := \pi^{-1}(0)$ of the test configuration is the degeneration of X which has a C^*-symmetry. In general the degeneration X_0 can be very singular, it might not even be a scheme.

As we remarked, given a holomorphic vector field one can compute its Futaki invariant, and this being nonzero excludes the possibility of finding a Kähler-Einstein metric. But in general there might not be any non-zero holomorphic vector fields, so this stability notion (Futaki invariant being zero) turns out to be too weak.

Donaldson shows how to define a Futaki invariant of a test configuration, generalizing the one of vector fields, and defines K-stability using this. A pair (X, L) is K-stable if the Futaki invariant of any non-trivial test configuration of (X, L) is positive.

An example of a test configuration is the deformation to the normal bundle of a submanifold (or more generally to the normal cone of a subscheme) $Y \subset X$. Consider $X \times \mathbb{C}$ and let $\mathcal{X}$ be the blow-up of $X \times \mathbb{C}$ along Y at zero. Let $\bar{L}$ be the pullback of the line bundle $L \boxtimes \mathbb{C}$ on $X \times \mathbb{C}$ using π. Let E denote the exceptional divisor of the blow-up. E is a $\mathbb{P}^m$-bundle over Y, $m =$ codim(Y), and is naturally identified with the compactified normal bundle of Y in X.If $c > 0$ is sufficiently small the line bundle $\mathcal{L} := \bar{L} - cE$ will be ample. The trivial C^*-action on $X \times \mathbb{C}$ lifts to $\mathcal{X}$ since $Y \times \{0\}$ is invariant, and it is shown in [31] that this defines a test configuration. The central fiber is the union of X with E, where the C^*-action on X is trivial while for smooth Y the action on E is the natural one on the normal bundle seen as a holomorphic vector bundle.

0.3.3 Toric test configurations

When X is toric and L is a toric line bundle, then one can speak of toric test configurations. In [13] Donaldson describes the relationship between toric test configurations and the geometry of polytopes. Let g be a positive concave rational piecewise affine function defined on P, where P is the moment polytope of L. One may define a polytope Q in $\mathbb{R}^{n+1}$ with P as its base and the graph of

g as its roof, i.e.

$$Q := \{(x, y) : x \in P, y \in [0, g(x)]\}.$$

That g is a concave rational piecewise affine function means precisely that Q is a rational polytope, i.e. it is the convex hull of a finite set of rational points in $\mathbb{R}^n$. In fact, by scaling we can without loss of generality assume that Q is a lattice polytope. Then Q corresponds to a toric line bundle L_Q over a toric variety X_Q of dimension $n + 1$. We may write the correspondence between integer lattice points of kQ and basis elements for $H^0(kL_Q)$ as

$$(\alpha, \eta) \in kQ \cap \mathbb{Z}^{n+1} \leftrightarrow t^{-\eta} z^{\alpha} \in H^0(kL_Q).$$

There is a natural $\mathbb{C}^*$-action ρ given by multiplication in the t-coordinate. We also get a projection π of X_Q down to $\mathbb{P}^1$, by letting

$$\pi(x) := \frac{t^{-\eta+1} z^{\alpha}(x)}{t^{-\eta} z^{\alpha}(x)}$$

for any η, α such that this is well defined. Donaldson shows in [13] that if one excludes $\pi^{-1}(\infty)$, then the triple L_Q, ρ and π is a test configuration, so π is flat and the fiber over 1 of (X_Q, L_Q) is isomorphic to (X_P, L_P).

Donaldson also has a formula for the Futaki invariant F of a toric test configurations associated to a function g. Consider the affine subspace spanned by a facet of P. Pick an integer basis for the $n-1$-dimensional lattice inside. This yields an isomorphism with $\mathbb{R}^{n-1}$, and we let $d\sigma$ be the pullback of Lebesgue measure. This measure will not depend on a particular choice of basis. Doing this for all facets gives us a measure $d\sigma$ on ∂P. The formula for the Futaki invariant F now reads:

$$F = \frac{1}{2\text{vol}(P)} \left(a \int_P f dx - \int_{\partial P} f d\sigma \right),$$

where $a = \text{vol}(\partial P)/\text{vol}(P)$.

Using this formula (and some hard analysis) Donaldson proved in [14] the Yau-Tian-Donaldson conjecture for toric surfaces.

As we have seen, heuristically, the relationship between a general line bundle L and its Okounkov body is supposed to mimic the relationship between a toric line bundle and its associated polytope. Therefore, one would hope that one could translate a general test configuration into some geometric data on the Okounkov body, as in the toric case.

0.3.4 Filtrations of the section ring

The collection of vector spaces of sections $H^0(X, kL)$ can be given the structure of a graded algebra by using tensor multiplications ($s \in H^0(X, kL)$ and $t \in H^0(X, mL)$ implies that $st := s \otimes t \in H^0(X, (k+m)L)$). The total algebra is called the section ring of L and denoted by $R(L)$.

In [7] Boucksom-Chen showed how certain filtrations of the section ring give rise to concave functions on the Okounkov body, and how this concave function captures information about the asymptotic behaviour of the filtration.

Let us first define what we mean by a filtration.

DEFINITION 6. *By a filtration $\mathcal{F}$ of a graded algebra $\oplus_k V_k$ we mean a vector space-valued map from $\mathbb{R} \times \mathbb{N}$,*

$$\mathcal{F} : (\lambda, k) \longmapsto \mathcal{F}_\lambda V_k,$$

such that for any k, $\mathcal{F}_\lambda V_k$ is a family of subspaces of V_k that is decreasing and left-continuous in λ.

In [7] Boucksom-Chen consider filtrations which behaves well with respect to the multiplicative structure of the algebra.

They give the following definition.

DEFINITION 7. *Let $\mathcal{F}$ be a filtration of a graded algebra $\oplus_k V_k$. We shall say that*

(i) $\mathcal{F}$ is multiplicative if

$$(\mathcal{F}_\lambda V_k)(\mathcal{F}_\eta V_m) \subseteq \mathcal{F}_{\lambda+\eta} V_{k+m}$$

for all $k, m \in \mathbb{N}$ and $\lambda, \eta \in \mathbb{R}$.

(ii) *$\mathcal{F}$ is pointwise left-bounded if for each k $\mathcal{F}_\lambda V_k = V_k$ for some λ.*

(iii) *$\mathcal{F}$ is linearly right-bounded if there exist a constant C such that for all k, $\mathcal{F}_{kC} V_k = \{0\}$.*

A filtration $\mathcal{F}$ is said to be admissible (in lack of a better word) if it is multiplicative, pointwise left-bounded and linearly right-bounded.

Let $\mathcal{F}$ be an admissible filtration of the section ring. For any real number λ we consider the subring $R(L,\lambda) := \oplus_k \mathcal{F}_{k\lambda} H^0(X, kL)$.

Following [22] one can define the Okounkov body of $R(L,\lambda)$, simply using the sections in the subring in the construction. We thus get a family of convex sets $\Delta(L,\lambda)$. We get a function $G[\mathcal{F}]$ on $\Delta(L)$ by letting

$$G[\mathcal{F}](x) := \sup\{\lambda : x \in \Delta(L,\lambda)\}.$$

The fact that $\mathcal{F}$ is multiplicative makes $G[\mathcal{F}]$ concave, thus $G[\mathcal{F}]$ is called the concave transform of the filtration $\mathcal{F}$.

Given a filtration $\mathcal{F}$ one can encode the way the dimension of $\mathcal{F}_\lambda H^0(X, kL)$ jumps as a function of λ by considering the positive measures

$$\mu_k := \frac{d}{d\lambda}(-\dim \mathcal{F}_{k\lambda} H^0(X, kL)).$$

The next theorem is the main result in [7].

THEOREM 6. *The sequence μ_k/k^n converges weakly as measures to the push-forward measure $G[\mathcal{F}]_*(dx_{|\Delta(L)})$, where $dx_{|\Delta(L)}$ denotes the restriction of Lebesgue measure to the Okounkov body.*

0.3.5 The concave transform of a test configuration

The main observation in paper 2 is that a test configuration gives rise to an admissible filtration of the section ring. Building on this, it was subsequently noted by Szekylyhidi in [34] that specifying a test configurations is equivalent to specifying a finitely generated filtration of the section ring, which means that one can recreate the test configuration from the filtration.

We will briefly describe how one gets the filtration starting from a test configuration.

For simplicity assume $\mathcal{L}_1 \cong L$, i.e. $r = 1$. Let $s \in H^0(X, kL)$ be a holomorphic section. Then using the $\mathbb{C}^*$-action ρ on $\mathcal{L}^{\otimes k}$ we get a canonical extension $\bar{s} \in H^0(\mathcal{X} \setminus X_0, \mathcal{L}^{\otimes k})$ which is invariant under the action ρ, simply by letting

$$\bar{s}(\rho(\tau)x) := \rho(\tau)s(x) \tag{18}$$

for any $\tau \in \mathbb{C}^*$ and $x \in X$.

We prove that this section extends over X_0 to a global meromorphic section of $\mathcal{L}^{\otimes k}$. We then get a filtration by letting $\mathcal{F}_\lambda H^0(X, kL)$ consist of those sections whose extension to $\mathcal{X}$ vanish along X_0 to order λ or more (when $\lambda < 0$ the section is allowed a pole of at most order $-\lambda$). Another way to put it is to say that a section lies in $\mathcal{F}_\lambda H^0(X, kL)$ if its extension multiplied by $z^{-\lambda}$ extends to a global holomorphic section. Here z is the holomorphic variable on the base $\mathbb{C}$.

That the filtration we get is multiplicative is immediate, and the boundedness follows from a result due to Phong-Sturm (see [28]). Thus from the work of Boucksom-Chen we see that get a concave function on the Okounkov body. We will call this function the concave transform of the test configuration, denoted by $G[\mathcal{T}]$ if $\mathcal{T}$ denotes the test configuration.

We furthermore show that in the case of a toric test configuration, our concave transform coincides with the piecewise linear function g considered by Donaldson.

Using Theorem 6 we know that the function $G[\mathcal{T}]$ captures the leading term in the asymptotics of the measures μ_k associated to the filtration.

In order to calculate the Futaki invariant one usually look at the weights of the $\mathbb{C}^*$ action on $H^0(X_0, L_0^{\otimes k})$, where $L_0 := \mathcal{L}_{|X_0}$. These weights are also encoded in the filtration, and summing the weights is the same as taking the first moment of the measure μ_k.

Let w_k be the first moment of μ_k (i.e. $w_k := \int x\mu_k$) and

$$d_k := \dim H^0(X, kL).$$

Then from the equivariant Riemann-Roch theorem one gets that there is an asymptotic expansion in powers of k of the expression w_k/kd_k (see e.g. [13]),

$$\frac{w_k}{kd_k} = F_0 - k^{-1}F_1 + O(k^{-2}).$$

F_1 is the Futaki invariant of $\mathcal{T}$ that we have denoted by F or $F(\mathcal{T})$.

Since $d_k = \text{vol}(L)k^n/n! + o(k^n)$ and μ_k/k^n converges weakly as measures to $G[\mathcal{T}]_* dx_{|\Delta(L)}$ we get that

$$F_0 = \frac{n!}{\text{vol}(\Delta(L))}\int_{\Delta(L)} G[\mathcal{T}]dx.$$

The Futaki invariant also depends on the second order asymptotics of w_k and d_k which is therefore in general not captured by $G[\mathcal{T}]$ and $\Delta(L)$ alone. The toric case is very special in this sense. E.g. since the lattice points in kP enumerate a basis for $H^0(X_P, kL_P)$ the shape of P determines the Hilbert polynomial of L_P, i.e. the full asymptotics of d_k and not only the leading term. However, in [34] Szekylyhidi shows how one can use these ideas to study the Futaki invariant.

0.3.6 Product test configurations and geodesic rays

There is a natural Riemannian metric we can put on $\mathcal{H}(L)$, the space of positive metrics on L. The tangent space of $\mathcal{H}(L)$ is naturally identified with $C^\infty(X)$. Let $dV_\phi := MA(\phi)/n!$. The norm of a tangent vector u at a point ϕ in $\mathcal{H}(L)$ is then given by

$$||u||^2_\phi := \int_X |u|^2 MA(\phi)/n!.$$

This metric is called the Mabuchi metric (see [24], [33], [12]).

If we have a $\mathbb{C}^*$-action ρ on X which lifts to L we can think of it as a simple kind of test configuration where $\mathcal{X} \cong X \times \mathbb{C}$ and $\mathcal{L} = L \boxtimes \mathbb{C}$. The weight measures μ_k then correspond to the weight measures of the $\mathbb{C}^*$-action on $H^0(X, kL)$.

Let ϕ be a positive S^1-invariant metric on L. Using the action ρ, we get a geodesic ray ϕ_t in $\mathcal{H}(L)$ such that $\phi_1 = \phi$. Let us denote the t derivative at the point one by $\dot{\phi}$. It is a real-valued function on X. By the function $\dot{\phi}/2$ we can push forward the measure dV_ϕ to a measure on $\mathbb{R}$, which we denote by μ_ϕ. This measure does not depend on the particular choice of positive S^1-invariant metric ϕ. In fact, we have the following.

THEOREM 7. *If we denote the product test configuration by $\mathcal{T}$, and the corresponding concave transform by $G[\mathcal{T}]$, then for any positive S^1-invariant metric ϕ it holds that*

$$\mu_\phi = G[\mathcal{T}]_* dx_{|\Delta(L)}. \tag{19}$$

The rescaled weight measures μ_k/k^n converges weakly to the right-hand-side of (6). Using the approach of Berndtsson in [4], one can also show that the sequence μ_k/k^n converges weakly to the left-hand-side of (6), thus proving the theorem.

What this result tells us is that the level sets of the moment map $\dot{\phi}/2$ are related to to the level sets of $G[\mathcal{T}]$. This relationship is investigated further by Julius Ross and the author in a paper still in preparation.

0.4 Paper III

The third paper is joint with Julius Ross from the University of Cambridge. It presents a general construction of weak geodesic rays in the space of positive singular metrics of an ample line bundle L.

But let us first review some background material.

In the last section we touched upon the subject of geodesics in the space of positive metrics on a line bundle. An important motivation for studying such geodesics is again the Yau-Tian-Donaldson conjecture.

0.4.1 The Mabuchi K-energy

Let $S = S(\phi)$ denote the scalar curvature of the Kähler metric associated to $dd^c\phi$. We want to understand when we can find a metric ϕ in $\mathcal{H}(L)$ such that the scalar curvature $S(\phi)$ is constant (i.e. a cscK metric). Let $\bar{S}$ be the average of S,

$$\bar{S} := \frac{1}{\text{vol}(X)} \int_X S(\phi) MA(\phi).$$

In fact

$$S(\phi) MA(\phi) = nRic(\phi) \wedge (dd^c\phi)^{n-1},$$

so it follows that $\bar{S}$ is a topological constant depending only on $c_1(L)$ and $c_1(K_X)$.

Similar to the variational defininition of the Monge-Ampère energy

$$\frac{d}{dt}\mathcal{E}(\phi_t, \psi) := \int_X \dot{\phi}_t MA(\phi_t)$$

one can define a functional K that involves the scalar curvature by letting

$$\frac{d}{dt}K(\phi_t, \psi) := \frac{1}{\text{vol}(X)} \int_X (\bar{S} - S(\phi_t)) MA(\phi_t).$$

The bifunctional K thus defined is called the Mabuchi K-energy. It has the cocycle property just as the Monge-Ampère energy, and therefore we can fix ψ and identify the K-energy with the functional $K(\cdot, \psi)$ on $\mathcal{H}(L)$. We see that if $S(\phi)$ is constant then ϕ is a critical point of the K-energy.

One can show that the K-energy is convex along any geodesic, and even strictly convex if the automorphism group is discrete. So in that case, if any two points in $\mathcal{H}(L)$ could be joined by a geodesic it would imply that a metric with constant scalar curvature would be unique. A recent result of Lempert-Vivas in [23] however shows that there are situations where $\mathcal{H}(L)$ is not geodesically connected.

The question of uniqueness of cscK metrics is thus related to geodesic segments in $\mathcal{H}(L)$. The question of existence is conjecturally related to geodesic rays rather than segments.

Heuristically, if the K-energy is bounded from below and proper in $\mathcal{H}(L)$, one should be able to find a critical point ϕ, which would give us a cscK metric. So the non-existence of such a metric should imply something about the boundary behaviour of the K-energy. A conjecture due to Donaldson in [11] says that there is no cscK metric in $\mathcal{H}(L)$ iff there is a geodesic ray ϕ_t, $t \in [0, \infty)$ in $\mathcal{H}(L)$ with

$$\frac{d}{dt} K(\phi_t, \psi) < 0$$

for all t.

0.4.2 Geodesic rays

Thus understanding geodesics and in particular geodesic rays has a possible bearing on the existence problem for cscK metric. A number of authors have studied this (e.g. Chen-Tian, Donaldson, Phong-Sturm, Mabuchi and Semmes among others).

It was realized in [11, 24, 33] that a curve ϕ_t is a geodesic iff it solves a homogeneous Monge-Ampère equation on $X \times A$ where A is an annulus or a puntured disc, the case of a punctured disc corresponds to geodesic rays.

Specifically, let $A := \{e^a < |z| < e^b\}$ be an annulus and let π be the projection from $X \times A$ to X. Given a curve ϕ_t, $a < t < b$, of positive metrics, consider the metric $\Phi(x, w) := \phi_{\ln|w|}(x)$ on $\pi^*(L)$. Then the geodesic equation for ϕ_t is equivalent to the degenerate homogeneous Monge-Ampère equation

$$\Omega^{n+1} = 0 \quad \text{on } X \times A, \tag{20}$$

where $\Omega = p_1^*\omega_0 + dd^c\Phi$ and $p_1^*\omega_0$ is the pullback of the curvature of the initial metric.

Arrezo-Tian in [1] have shown how to construct geodesic rays given some special data using the Cauchy-Kowaleski theorem. See also the work of Chen in [10].

It is generally hard to construct geodesic rays, since the assumptions are so strong. Because of the difficulties many authors (e.g. Chen, Phong-Sturm, Song-Zelditch and Sun) have studied a more general class of curves in the space of positive singular metrics called weak geodesics.

Let ϕ_t be a curve in $PSH(L)$. This curve is called a weak geodesic if the metric $\Phi(x, w) := \phi_{\ln|w|}(x)$ is a locally bounded positive singular metric on $\pi^*(L)$ which solves the homogeneous Monge-Ampère equation $MA(\phi_t) = 0$

on $X \times A$. When A is a punctured disc ϕ_t is called a weak geodesic ray. In this case we usually think of the parameter t as going from zero to plus infinity.

These weak geodesic rays are easier to construct than true ones. E.g. one can use envelope techniques.

Let ϕ be a smooth metric on an ample line bundle L. Then we can consider the envelope $P(\phi)$ which is defined as

$$P(\phi) := \sup\{\psi : \psi \leq \phi, \psi \in PSH(L)\}.$$

Then $P(\phi)$ is a positive singular metric (i.e. not smooth) and by the work of Bedford-Taylor the Monge-Ampère of $P(\phi)$ is zero on the set where the envelope $P(\phi)$ is strictly less than ϕ (see [2]). By looking at the product $X \times \mathbb{P}^1$ say and a suitable smooth metric on $L \boxtimes \mathcal{O}(1)$ one can use a variant of this envelope construction to produce weak geodesic rays. In order to get something non-trivial though one has to impose some condition on the allowed behaviour at the origin for the candidate metrics in the envelope.

0.4.3 Phong-Sturm rays

Because of the connection to the Yau-Tian-Donaldson conjecture one is particularly interested in (weak) geodesic rays associated to test configurations. Pick a positive metric ϕ and a test configuration $\mathcal{T}$. In [28] Phong-Sturm show how to construct a weak geodesic ray ϕ_t emanating from ϕ using the test configuration. The behaviour of ϕ_t as t tends to infinity is then governed by the test configuration.

We will briefly describe their construction here.

Using ϕ one gets scalar products $(\cdot,\cdot)_{k\phi}$ on the vector spaces $H^0(X, kL)$, by letting

$$(s_1, s_2)_{k\phi} := \int_X s_1 \overline{s_2} e^{-k\phi} MA(\phi).$$

Recall that a test configuration give rise to a filtration $\mathcal{F}$ of each vector space $H^0(X, kL)$ (see section 0.3.5). Let λ_i be an enumeration in increasing order of the numbers where $\mathcal{F}_\lambda H^0(X, kL)$ drops in dimension. There is a unique decomposition of $H^0(X, kL)$ into a direct sum of mutually orthogonal subspaces

V_i such that $\mathcal{F}_\lambda V$ is the sum of V_i over the indices i such that $\lambda_i \geq \lambda$. If we allow for λ_i to be equal to λ_j even when $i \neq j$, we can assume that all the subspaces V_i are one dimensional. This additional decomposition is not unique, but that will not matter in what follows. Let s_i be a section in V_i of unit length, then $\{s_i\}$ is an orthonormal basis. Consider the curve of positive singular metrics

$$\Phi_k(t) := \frac{1}{k} \ln \big(\sum_i e^{t\lambda_i} |s_i|^2 \big).$$

The *Phong-Sturm* ray is the limit

$$\phi_t := \lim_{k \to \infty} (\sup_{l \geq k} \Phi_l(t))^*. \tag{21}$$

Phong-Sturm prove that the curve ϕ_t is a weak geodesic ray. That $\phi_0 = \phi$ follows from a celebrated result by Bouche-Catlin-Tian-Zelditch [6, 9, 36, 39] on Bergman kernel asymptotics.

Note that in linking geodesics with the Mabuchi K-energy the regularity was essential. Therefore it is important to know what kind of regularity the weak geodesic rays possess. In [30] Phong-Sturm prove that the weak geodesic rays ϕ_t are of class $C^{1,\alpha}$ for any $0 < \alpha < 1$.

0.4.4 The Legendre transform

A test configuration is an algebraic object, while positive singular metrics and weak geodesic rays belong to the more analytical realm of pluripotential theory. The object of the third paper is to show that one can use a larger class than the (algebraic) test configurations to construct weak geodesic rays. We call the objects in this larger class analytical test configurations, because in this context they play a role similar to that of an ordinary test configurations.

Before giving the definition of an analytic test configuration we will discuss the Kiselman minimum principle and the Legendre transform of curves of positive metrics.

A very important property of the class of plurisubharmonic functions is that it is closed under taking maximum, and the supremum of a family of plurisubharmonic functions is plurisubharmonic after regularization. Onthe other hand, the minimum of two plurisubharmonic functions is very seldom plurisubharmonic. There is one situation though where one gets plurisubharmonic functions after taking infimum, this is described by Kiselman's minimum principle (see [20]).

THEOREM 8. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n \times \mathbb{C}^m$ such that if $(x, y) \in \Omega$ then $(x, y') \in \Omega$ for all y' such that* $\operatorname{Re} y = \operatorname{Re} y'$. *Let π be the projection $\pi(x, y) = x$, and assume for simplicity that the fibers are connected. Let u be a plurisubharmonic function on Ω which is independent of* $\operatorname{Im} y$. *Then the function v on $\pi(\Omega)$ defined by*

$$v(x) := \inf\{u(x, y) : (x, y) \in \Omega\}$$

is plurisubharmonic.

Let ϕ_t be a weak geodesic ray in $PSH(L)$. Thinking of t as a complex coordinate where ϕ_t is independent of $\operatorname{Im} t$ means that ϕ_t is a positive singular metric on $X \times \{\text{t} \geq 0\}$. By choosing a suitible open cover of X by pseudoconvex domains the Kiselman minimum principle implies that the metric

$$\psi_0 := \inf\{\phi_t : t \geq 0\}$$

lies in $PSH(L)$, unless it is identically $-\infty$. For any real number λ the curve $\phi_t - t\lambda$ is still a weak geodesic ray. Thus we get a family of positive singular metrics

$$\psi_\lambda := \inf\{\phi_t - t\lambda : t \geq 0\}$$

parametrized by λ. That ϕ_t is locally plurisubharmonic in all variables and independent of $\operatorname{Im} t$ implies that ϕ_t is convex in t. Since ψ_λ is the infimum of metrics that are linear and thus concave in λ we get that the curve ψ_λ is concave in λ.

A curve ϕ_t in $PSH(L)$ independent of $\operatorname{Im} t$ which is a locally bounded positive singular metric on $X \times \{\text{t} \geq 0\}$ is called a weak subgeodesic ray. From the procedure above we see that any weak subgeodesic ray produces a concave curve of positive singular metrics. Note that even if the original curve ϕ_t is smooth, the concave curve ψ_λ can be highly singular (e.g. it will be identically $-\infty$ for $\lambda >> 0$).

We call ψ_λ the Legendre transform of the subgeodesic ϕ_t. Kiselman in [20] and others have used the local version of this Legendre transform to study singularities of plurisubharmonic functions. Here it gives us information of the limit behaviour of ϕ_t as t tends to infinity.

There is also a Legendre transform going in the opposite direction. Let ψ_λ, $\lambda \in \mathbb{R}$ be a concave curve of positive singular metrics such that ψ_λ is equal to some locally bounded metric $\psi_{-\infty}$ for $\lambda << 0$, and $-\infty$ for $\lambda >> 0$. Then we can consider the curve ϕ_t defined by

$$\phi_t := (\sup\{\psi_\lambda + t\lambda : \lambda \in \mathbb{R}\})^*,$$

where the star means we take the upper semicontinuous regularization. Then ϕ_t is a weak subgeodesic ray.

If we start with a weak subgeodesic ray ϕ_t, since the Legendre transform is an involution it follows that the Legendre transform of the concave curve ψ_λ is our original ϕ_t. So if ϕ_t happened to be a weak geodesic the Legendre transform of ψ_λ would subsequently be a weak geodesic.

0.4.5 Analytic test configurations

The objective of paper 3 was to find a more general analytical procedure to construct weak geodesics than the one proposed by Phong-Sturm. Recall that in the heuristics behind using test configurations to construct weak geodesic rays the test configuration encodes the limiting behaviour of the ray as t tends to infinity. We have seen that a way to understand this limiting behaviour is to apply the Legendre transform and thus get a concave curve of positive singular metrics. In our paper we consider certain equivalence classes of concave curves of metrics that we call analytic test configurations. Given this data and a starting metric ϕ, we show how to construct a weak geodesic ray by an envelope procedure followed by a Legendre transform.

Let us be more precise.

First some definitions from pluripotential theory. A set is called complete pluripolar if it can be described locally as the set where a plurisubharmonic function is minus infinity. We say that a positive singular metric ψ has small unbounded locus if the set where ψ fails to be locally bounded is contained in a closed complete pluripolar subset of X (see [8]).

We can now define what we mean by a test curve.

DEFINITION 8. *A map $\lambda \mapsto \psi_\lambda$ from $\mathbb{R}$ to $PSH(L)$ is called a* test curve *if*

(i) ψ_λ is concave in λ,

(ii) and ψ_λ has small unbounded locus whenever $\psi_\lambda \not\equiv -\infty$.

There should also exist a constant C such that

(iii) ψ_λ is equal to some locally bounded positive metric $\psi_{-\infty}$ for $\lambda < -C$,

(iiii) and $\psi_\lambda \equiv -\infty$ for $\lambda > C$.

We say that two metrics ψ_1 and ψ_2 in $PSH(L)$ have the same singularity type if for some constant C we have that

$$\psi_1 - C \leq \psi_2 \leq \psi_1 + C.$$

We let $\mathrm{Sing}(L)$ denote the set of singularity types in $PSH(L)$.

If we have a test curve ψ_λ we get a curve $[\psi_\lambda]$ in $\mathrm{Sing}(L)$, where of course $[\psi]$ denotes the equivalence class of ψ.

DEFINITION 9. *A concave curve $[\psi_\lambda]$ in Sing(L) which is the image of a test curve ψ_λ under the natural projection $\psi \mapsto [\psi]$ is called an analytic test configuration.*

We say that the analytic test configuration $[\psi_\lambda]$ is trivial if there exists a number λ_c such that $[\psi_\lambda] = [\phi]$ for $\lambda < \lambda_c$ and $[\psi_\lambda] = [-\infty]$ for $\lambda > \lambda_c$.

0.4.6 Maximal envelopes

Our process of constructing weak geodesics has two steps. The first step consists in taking certain envelopes, called maximal envelopes.

Let ϕ be a continuous metric and ψ a positive singular metric. Then the envelope $P_\psi \phi$ is defined as

$$P_\psi \phi := \sup\{\psi' \leq \min\{\phi, \psi\} : \psi' \in PSH(L)\}.$$

This is a new positive singular metric. We can also form an envelope with respect to the singularity type $[\psi]$ rather than ψ, by letting

$$P_{[\psi]}\phi := \lim_{C \to \infty} P_{\psi+C}\phi.$$

After taking the usc regularization we get a positive singular metric $\phi_{[\psi]}$ which we call the maximal envelope of ϕ with respect to the singularity type $[\psi]$.

DEFINITION 10. *If $\psi \in PSH(L)$, then ψ is said to be* maximal *with respect to a metric ϕ if $\psi \leq \phi$ and furthermore $\psi = \phi$ a.e. with respect to $MA(\psi)$.*

Since ψ might not be locally bounded by $MA(\psi)$ we mean the nonpluripolar Monge-Ampère measure of ψ (see [8]).

A key technical result in our paper is the proof that the maximal envelope $\phi_{[\psi]}$ of any continuous metric ϕ with respect to any singularity type $[\psi]$ is maximal with respect of ϕ.

Given a positive metric ϕ and an analytic test configuration $[\psi_\lambda]$ we get a new concave curve by considering the curve of maximal envelopes $\phi_{[\psi_\lambda]}$. It is not hard to show that this is a test curve. So we can let ϕ_t be the Legendre transform

$$\phi_t := (\sup\{\phi_{[\psi_\lambda]} + t\lambda : \lambda \in \mathbb{R}\})^*.$$

Our main theorem is then the following.

THEOREM 9. *For any positive metric ϕ and any analytic test configuration $[\psi_\lambda]$ it holds that the Legendre transform*

$$\phi_t := (\sup\{\phi_{[\psi_\lambda]} + t\lambda : \lambda \in \mathbb{R}\})^*$$

is a weak geodesic ray emanating from ϕ.

0.4.7 Proof of main theorem

We have already remarked that the Legendre transform ϕ_t is a weak subgeodesic. It is well-known that in order to prove that a weak subgeodesic ϕ_t is a weak geodesic it suffices to show that the Monge-Ampère energy $\mathcal{E}(\phi_t, \phi)$ is linear in t. E.g. this is what Phong-Sturm did to prove that their ray was a weak geodesic.

To calculate the energy $\mathcal{E}(\phi_t, \phi)$ we use the following lemma.

LEMMA 10. *Suppose that ψ is maximal with respect to a positive metric ϕ with small unbounded locus, and let $t > 0$. Then we have that*

$$t \int_X MA(\psi) \leq \mathcal{E}(\max\{\psi + t, \phi\}, \phi) \leq t \int_X MA(\phi). \tag{22}$$

Remark. This is one place where the somewhat annoying assumption of the metrics in the test curve having small unbounded locus comes into the argument.

Let N be a big natural number and let $\varphi_i := \phi_{[\psi_{i/N}]}$, $i \in \mathbb{Z}$. Then ϕ_t can be approximated by

$$\phi_t^N := \max_i \{\varphi_i + ti/N\}.$$

Let

$$\phi_t^{N,M} := max_{i \leq M}\{\varphi_i + ti/N\}.$$

$$= \sum_j \mathcal{E}(\varphi_{\leq j}, \varphi_{\leq j-1}).$$

Using the concavity of $\phi_t^{N,M}$ we show that

$$\mathcal{E}(\phi_t^{N,M+1}, \phi_t^{N,M}) = \mathcal{E}(\max\{\varphi_{M+1} + t/N, \varphi_M\}, \varphi_M).$$

Since φ_{M+1} is maximal with respect to φ_M we can use Lemma 10 to conclude that

$$\frac{t}{N} \int_X MA(\varphi_{M+1}) \leq \mathcal{E}(\phi_t^{N,M+1}, \phi_t^{N,M}) \leq \frac{t}{N} \int_X MA(\varphi_M).$$

Let i be such that $\varphi_i = \phi$. Then $\phi_t^{N,i} = \phi + ti/N$. By the cocycle property of the Monge-Ampère energy we get that

$$\mathcal{E}(\phi_t^N, \phi) = \mathcal{E}(\phi_t^N, \phi_t^{N,i}) + \mathcal{E}(\phi_t^{N,i}, \phi) =$$
$$= \sum_{M \geq i} \mathcal{E}(\phi_t^{N,M+1}, \phi_t^{N,M}) + ti/N \int_X MA(\phi).$$

By letting N tend to infinity and using continuity properties of the Monge-Ampère energy one arrives at the following equation:

$$\mathcal{E}(\phi_t, \phi) = -t \int_{\lambda=-\infty}^{\infty} \lambda dF(\lambda),$$

where

$$F(\lambda) := \int_X MA(\phi_{[\psi_\lambda]}).$$

In particular it shows that the Monge-Ampère energy is linear in t, thus theorem 9 follows.

0.4.8 Connection to the work of Phong-Sturm

One natural question is how this relates to the construction of Phong-Sturm.

Let $\mathcal{F}$ be an admissible filtration of the section ring $R(L)$ (see section 0.3.4 for the definition). Let us also choose a positive metric ϕ. We saw above that we have an orthonormal basis $\{s_i\}$ of $H^0(X, kL)$ with respect to the scalar product $(\cdot, \cdot)_{k\phi}$, where the section s_i is associated to the weight λ_i (see section 0.4.3). Let $\psi_{k,\lambda}$ denote the metric

$$\psi_{k,\lambda} := \frac{1}{k} \ln \big(\sum_{i \in I_{k,\lambda}} |s_i|^2 \big),$$

where $I_{k,\lambda} := \{i : \lambda_i \geq k\lambda\}$. Thus $\psi_{k,\lambda}$ is the Bergman metric associated to the projection down to the subspace $\mathcal{F}_{k\lambda} H^0(X, kL)$.

In our paper we prove that for any λ the sequence of metrics $\psi_{k,\lambda}$ converges and the usc regularization of the limit is a positive singular metric denoted $\phi_\lambda^{\mathcal{F}}$ (with the possibility that $\phi_\lambda^{\mathcal{F}} \equiv -\infty$). This is in fact a test curve, so it defines an analytic test configuration. Using a a Skoda-type division theorem we are also able to prove that the metrics $\phi_\lambda^{\mathcal{F}}$ are maximal with respect to ϕ, thus it follows that the Legendre transform of this test curve is a weak geodesic ray emanating from ϕ.

In the case of an (algebraic) test configuration we can use the associated filtration to thus get weak geodesic rays emanating from any positive metric ϕ we choose.

Let us compare the k:th step approximation of this ray with the corresponding approximation of the Phong-Sturm ray. Note that

$$\psi_{k,\lambda_i/k} + t\lambda_i/k = \frac{1}{k}\ln\big(\sum_{j\in I_{k,\lambda_i}} e^{t\lambda_i}|s_j|^2\big).$$

We thus get that our k:th step approximation is

$$\Psi_k(t) := \sup_\lambda\{\psi_{k,\lambda} + t\lambda\} = \sup_{\lambda_i}\{\psi_{k,\lambda_i/k} + t\lambda_i/k\} =$$
$$= \sup_{\lambda_i}\big\{\frac{1}{k}\ln\big(\sum_{j\in I_{k,\lambda_i}} e^{t\lambda_i}|s_j|^2\big)\big\}.$$

Compare this to the approximation of the Phong-Sturm ray:

$$\Phi_k(t) := \frac{1}{k}\ln\big(\sum_i e^{t\lambda_i}|s_i|^2\big).$$

Here is an elementary lemma.

LEMMA 11. *If $\{a_i : i \in I\}$ is a finite set of real numbers and k a positive integer then*

$$\max_{i\in I} a_i \leq \frac{1}{k}\ln\big(\sum_{i\in I} e^{ka_i}\big) \leq \max_{i\in I} a_i + \frac{1}{k}\ln|I|. \tag{23}$$

It shows that for k large, $1/k$ times the logarithm of a sum of exponetials e^{ka_i} is closely approximated by the maximum of the numbers a_i. If we let

$$a_i = \frac{1}{k}\ln|s_i(x)|^2 + t\lambda_i/k,$$

then we see that $\Phi_k(t)(x)$ is approximated by $\max\{a_i\}$. On the other hand, by the same lemma we get that $\max\{a_i\}$ also approximates

$$\frac{1}{k}\ln\big(\sum_{i,j:\lambda_j\geq\lambda_j} e^{t\lambda_i}|s_j(x)|^2\big).$$

But another application of the lemma tells us that this is approximated by $\max\{b_i\}$, where

$$b_i = \frac{1}{k}\ln\big(\sum_{\lambda_j\geq\lambda_j}|s_j(x)|^2\big) + t\lambda_i/k = \psi_{k,\lambda_i/k}(x) + t\lambda_i/k.$$

This shows that $\Psi_k(t) - \Phi_k(t)$ tends to zero as k tends to infinity.

This makes it very probable that the two weak geodesic rays coincide, but it is not a proof since one has to take into account the different limiting procedures in the definitions. We do prove that when the analytic test configuration we get is non-trivial, then the two rays do coincide.

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Part II

PAPERS

PAPER I

David Witt Nyström
Transforming metrics on a line bundle to the Okounkov body

1

Transforming metrics on a line bundle to the Okounkov body

ABSTRACT

Let L be a big holomorphic line bundle on a compact complex manifold X. We show how to associate a convex function on the Okounkov body of L to any continuous metric $e^{-\psi}$ on L. We will call this the Chebyshev transform of ψ, denoted by $c[\psi]$. Our main theorem states that $n!$ times the integral of the difference of the Chebyshev transforms of two weights is equal to the Monge-Ampère energy of the weights, which is a well-known functional in Kähler-Einstein geometry and Arakelov geometry. We show that this can be seen as a generalization of classical results on Chebyshev constants and the Legendre

transform of invariant metrics on toric manifolds. As an application we prove the differentiability of the Monge-Ampère energy in the ample cone.

1.1 Introduction

In [8] and [9] Kaveh-Khovanskii and Lazarsfeld-Mustaţă initiated a systematic study of Okounkov bodies of divisors and more generally of linear series. Our goal is to contribute with an analytic viewpoint.

It was Okounkov who in his papers [10] and [11] introduced a way of associating a convex body in $\mathbb{R}^n$ to any ample divisor on a n-dimensional projective variety. This convex body, called the Okounkov body of the divisor and denoted by $\Delta(L)$, can then be studied using convex geometry. It was recognized in [9] that the construction works for arbitrary big divisors.

We will restrict ourselves to a complex projective manifold X, and instead of divisors we will for the most part use the language of holomorphic line bundles. Because of this, in the construction of the Okounkov body, we prefer choosing local holomorphic coordinates instead of the equivalent use of a flag of subvarieties (see [9]). We use additive notation for line bundles, i.e. we will write kL instead of $L^{\otimes k}$ for the k:th tensor power of L. We will also use the additive notation for metrics. If h is a hermitian metric on a line bundle, we may write it as $h = e^{-\psi}$, and call ψ a weight. Thus if ψ is a weight on L, $k\psi$ is a weight on kL.

The main motivation for studying Okounkov bodies has been their connection to the volume function on divisors. Recall that the volume of a line bundle L is defined as

$$\mathrm{vol}(L) := \limsup_{k\to\infty} \frac{n!}{k^n}\dim(H^0(kL)).$$

A line bundle is said to be big if it has positive volume. From here on, all line bundles L we consider will be assumed to be big. By Theorem A in [9], for any big line bundle L it holds that

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)) = \frac{1}{n!}\mathrm{vol}(L).$$

We are interested in studying certain functionals on the space of weights on L that refine $\mathrm{vol}(L)$ (see below).

A weight ψ is said to be *psh* if

$$dd^c\psi \geq 0$$

as a current. Given two locally bounded psh weights ψ and φ we define $\mathcal{E}(\psi, \varphi)$ as

$$\frac{1}{n+1}\sum_{j=0}^{n}\int_X (\psi - \varphi)(dd^c\psi)^j \wedge (dd^c\varphi)^{n-j},$$

which we will refer to as the Monge-Ampère energy of ψ and φ. This bifunctional first appeared in the works of Mabuchi and Aubin in Kähler-Einstein geometry (see [1] and references therein).

If ψ and φ are continuous but not necessarily psh, we may still define a Monge-Ampère energy, by first projecting them down to the space of psh weights,

$$P(\psi) := \sup\{\psi' : \psi' \leq \psi, \psi' \text{ psh}\}.$$

We are therefore led to consider the functional

$$\mathcal{E}(\psi, \varphi) := \frac{1}{n+1}\sum_{j=0}^{n}\int_\Omega (P(\psi) - P(\varphi))(dd^cP(\psi))^j \wedge (dd^cP(\varphi))^{n-j}, \quad (1.1)$$

where Ω denotes the dense Zariski-open set where both $P(\psi)$ and $P(\varphi)$ are locally bounded. For psh weights ψ, trivially $P(\psi) = \psi$, therefore there is no ambiguity in the notation. The Monge-Ampère energy can be seen as a generalization of the volume since if we let ψ be equal to $\varphi + 1$, from e.g. [1] we have that

$$\mathcal{E}(\psi, \varphi) = \int_\Omega (dd^cP(\varphi))^n = \mathrm{vol}(L).$$

Given a continuous weight ψ, we will show how to construct an associated convex function on the interior of the Okounkov body of L which we will call the Chebyshev transform of ψ, denoted by $c[\psi]$. The construction can be seen to generalize both the Chebyshev constants in classical potential theory and the Legendre transform of convex functions (see subsections 9.2 and 9.3 respectively).

First we construct $\Delta(L)$. Choose a point $p \in X$ and local holomorphic coordinates $z_1, ..., z_n$ centered at p. Choose also a trivialization of L around p. With respect to this trivialization any holomorphic section $s \in H^0(L)$ can be written as a convergent power series in the coordinates z_i,

$$s = \sum_{\alpha} a_\alpha z^\alpha.$$

Consider the lexicographic order on $\mathbb{N}^n$, and let $v(s)$ denote the smallest index α (i.e. with respect to the lexicographic order) such that

$$a_\alpha \neq 0.$$

We let $v(H^0(L))$ denote the set $\{v(s) : s \in H^0(L)\}$, and finally let the Okounkov body of L, denoted by $\Delta(L)$, be defined as closed convex hull in $\mathbb{R}^n$ of the union

$$\bigcup_{k \geq 1} \frac{1}{k} v(H^0(kL)).$$

Observe that the construction depends on the choice of p and the holomorphic coordinates. For other choices, the Okounkov bodies will in general differ.

Now let ψ be a continuous weight on L. There are associated supremum norms on the spaces of sections $H^0(kL)$,

$$||s||^2_{k\psi} := \sup_{x \in X} \{|s(x)|^2 e^{-k\psi(x)}\}.$$

If $v(s) = k\alpha$ for some section $s \in H^0(kL)$, we let $A_{\alpha,k}$ denote the affine space of sections in $H^0(kL)$ of the form

$$z^{k\alpha} + \text{ higher order terms.}$$

We define the discrete Chebyshev transform $F[\psi]$ on $\bigcup_{k \geq 1} v(H^0(kL)) \times \{k\}$ as

$$F[\psi](k\alpha, k) := \inf\{\ln ||s||^2_{k\psi} : s \in A_{\alpha,k}\}.$$

THEOREM 1. *For any point $p \in \Delta(L)^\circ$ and any sequence $\alpha(k) \in \frac{1}{k} v(H^0(kL))$ converging to p, the limit*

$$\lim_{k \to \infty} \frac{1}{k} F[\psi](k\alpha(k), k)$$

exists and only depends on p. We may therefore define the Chebyshev transform of ψ by letting

$$c[\psi](p) := \lim_{k\to\infty} \frac{1}{k} F[\psi](k\alpha(k), k),$$

for any sequence $\alpha(k)$ converging to p.

The main observation underlying the proof is the fact that the discrete Chebyshev transforms are subadditive. Our proof is thus very much inspired by the work of Zaharjuta, who in [14] used subadditive functions on $\mathbb{N}^n$ when studying directional Chebyshev constants, and also by the article [3] where Bloom-Levenberg extend Zaharjutas results to a more general weighted setting, but still in $\mathbb{C}^n$ (we show in section 7 how to recover the formula of Bloom-Levenberg from Theorem 1).

We prove a general statement concerning subadditive functions on subsemigroups of $\mathbb{N}^d$ that generalizes a result of Zaharjuta.

THEOREM 2. *Let $\Gamma \subseteq \mathbb{N}^d$ be a semigroup which generates $\mathbb{Z}^d$ as a group, and let F be a subadditive function on Γ which is locally bounded from below by some linear function. Then for any sequence $\alpha(k) \in \Gamma$ such that $|\alpha(k)| \to \infty$ and $\frac{\alpha(k)}{|\alpha(k)|} \to p \in \Sigma(\Gamma)^\circ$ ($\Sigma(\Gamma)$ denotes the convex cone generated by Γ) for some point p in the interior of $\Sigma(\Gamma)$, the limit*

$$\lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

exists and only depends on F and p. Furthermore the function

$$c[F](p) := \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

thus defined on $\Sigma(\Gamma)^\circ \cap \Sigma^\circ$ is convex.

Theorem 1 will follow from Theorem 2.

It should be pointed out that related Chebyshev transforms play an important role in [12] in the context of Arakelov geometry.

Our main result on the Chebyshev transform is the following.

THEOREM 3. *Let ψ and φ be two continuous weights on L. Then it holds that*

$$\mathcal{E}(\psi,\varphi) = n! \int_{\Delta(L)^\circ} (c[\varphi] - c[\psi]) d\lambda, \tag{1.2}$$

where $d\lambda$ denotes the Lebesgue measure on $\Delta(L)$.

The proof of Theorem 3 relies on the fact that one can use certain L^2-norms related to the weight, called Bernstein-Markov norms, to compute the Chebyshev transform. With the help of these one can interpret the right-hand side in equation (1.2) as a limit of Donaldson bifunctionals $\mathcal{L}_k(\psi,\varphi)$. On the other hand, the main theorem in [1] says that the bifunctionals $\mathcal{L}_k(\psi,\varphi)$ converges to the Monge-Ampère energy when k tends to infinity, which gives us our theorem.

Because of the homogeneity of the Okounkov body, i.e.

$$\Delta(kL) = k\Delta(L),$$

one may define the Okounkov body of an arbitrary $\mathbb{Q}$-divisor D by letting

$$\Delta(D) := \frac{1}{p}\Delta(pD),$$

for any integer p clearing all denominators in D. Theorem B in [9] states that one may in fact associate an Okounkov body to an arbitrary big $\mathbb{R}$-divisor, such that the Okounkov bodies are fibers of a closed convex cone in $\mathbb{R}^n \times N^1(X)_{\mathbb{R}}$, where $N^1(X)_{\mathbb{R}}$ denotes the Neron-Severi space of $\mathbb{R}$-divisors. We show that this can be done also on the level of Chebyshev transforms, i.e. there is a continuous and indeed convex extension of the Chebyshev transforms to the space of continuous weights on big $\mathbb{R}$-divisors. We prove Theorem 3 for weights on ample $\mathbb{R}$-divisors.

As an application we prove that the Monge-Ampère energy is differentiable in the ample cone. In [1] Berman-Boucksom consider as a function of t the Monge-Ampère of weights ψ_t and φ, where ψ_t vary smoothly with t. Theorem B in [1] states that the function

$$F(t) := \mathcal{E}_L(\psi_t,\varphi)$$

then is differentiable in t, and that the derivative is given by

$$\dot{F}(t) = \int_{\Omega} \dot{\psi}_t (dd^c P(\psi_t))^n,$$

where $\dot{\psi}_t$ denotes the derivative of ψ_t at t. In section 9 we prove a generalization of this in the ample setting where the underlying $\mathbb{R}$-divisor L_t varies with t within the ample cone.

THEOREM 4. *Let A_i, $i = 1, ..., m$ be a finite collection of ample line bundles, and for each i let ψ_i and φ_i be two continuous weights on A_i. Let O denote the open cone in $\mathbb{R}^m$ such that $a \in O$ iff $\sum a_i A_i$ is an ample $\mathbb{R}$-divisor. Then the function*

$$F(a) := \mathcal{E}_{\sum a_i A_i}(\sum a_i \psi_i, \sum a_i \varphi_i)$$

is $\mathcal{C}^1$ on O.

We also calculate the differential. If we consider the special case where A is ample and φ is some positive continuous weight on A, and let

$$F(t) := \mathcal{E}_{L+tA}(\psi_1 + t\varphi, \psi_2 + t\varphi)$$

for some continuous weights ψ_1 and ψ_2 on an ample divisor L. Then our calculations show that

$$\dot{F}(0) = \sum_{j=0}^{n-1} \int_X (P(\psi_1) - P(\psi_2)) dd^c\varphi \wedge (dd^c P(\psi_1))^j \wedge (dd^c P(\psi_2))^{n-j-1}. \tag{1.3}$$

Another special case is the following. If A is an ample divisor and s_A is a defining section for A, by multiplying with $s_A^{\otimes tk}$ we get embeddings of the spaces $H^0(k(L - tA))$ into $H^0(kL)$. There is also an associated map between the spaces of weights, where ψ_L maps to

$$\psi_{L-tA} := \psi_L - t \ln |s_A|^2.$$

It follows from the proof of Theorem 4 that

$$\frac{d}{dt}_{|0} \mathcal{E}_X(\psi_{L-tA}, \varphi_{L-tA}) = -\mathcal{E}_A(\psi_L, \varphi_L).$$

Our proof uses the same approach as the proof of the differentiability of the volume in [9]. Since the Monge-Ampère energy is given by the integral of Chebyshev transforms over Okounkov bodies, when we differentiate we get one term coming from the variation of the Okounkov body, as studied in [9], and one term coming from the variation of the Chebyshev transforms. We observe that if one in formula (1.3) as Ψ chooses the positive weight $\ln|s|^2$, and let $\psi_0 = \varphi_0 + 1$, using the Lelong-Poincare formula one recovers the formula for the derivative of the volume in the ample cone, i.e.

$$\frac{d}{dt}_{|0}\mathrm{vol}_X(L + tA) = n\mathrm{vol}_{[A]}(L_{|[A]}),$$

where $[A]$ denotes the divisor $\{s = 0\}$.

1.1.1 Organization

In section 2 we start by defining the Okounkov body of a semigroup, and we recall a result on semigroups by Khovanskii that will be of great use later on.

Section 3 deals with subadditive functions on subsemigroups of $\mathbb{N}^{n+1}$ and contains the proof of Theorem 2.

The definition of the Okounkov body of a line bundle follows in section 4.

In section 5 we define the discrete Chebyshev transform of a weight, and prove that this function has the properties needed for Thereom 2 to be applicable. We thus prove Theorem 1. We also show that the difference between two Chebyshev transforms is bounded on the interior of the Okounkov body.

The Monge-Ampère energy of weights is introduced in section 6. Here we also state our main theorem, Theorem 3.

In section 7 we show how one can use Bernstein-Markov norms instead of supremum norms in the construction of the Chebyshev transform.

The proof of Theorem 3 follows in section 8.

Section 9 discusses previuos results.

In subsection 9.1 we observe that if we in (1.2) let φ be equal to $\psi + 1$, then we recover Theorem A in [9], i.e. that

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)) = \frac{1}{n!}\mathrm{vol}(L).$$

In subsection 9.2 we move on to clarify the connection to the classical Chebyshev constants. We see that if we embed $\mathbb{C}$ into $\mathbb{P}^1$ and choose our weights wisely then formula (1.2) gives us the classical result in potential theory that the Chebyshev constant and transfinite diamter of a regular compact set in $\mathbb{C}$ coincides. See subsection 9.2 for definitions.

Subsection 9.3 studies the case of a toric manifold, with a torus invariant line bundle and invariant weights. We calculate the Chebyshev transforms, and observe that for invariant weights, the Chebyshev transform equals the Legendre transform of the weight seen as a function on $\mathbb{R}^n$.

We show in section 10 that if the line bundle is ample, the Chebyshev transform is defined on the zero-fiber of the Okounkov body, not only in the interior. Using the Ohsawa-Takegoshi extension theorem we prove that

$$\mathcal{E}_Y(P(\varphi)_{|Y}, P(\psi)_{|Y}) = (n-1)! \int_{\Delta(L)_0} (c[\psi] - c[\varphi])(0,\alpha) d\alpha, \tag{1.4}$$

where $\Delta(L)_0$ denotes the zero-fiber of $\Delta(L)$, and Y is a submanifold locally given by the equation $z_1 = 0$.

In section 11 we show how to translate the results of Bloom-Levenberg to our language of Chebyshev transforms. We reprove Theorem 2.9 in [3] using our Theorem 3, equation (1.4) and a recursion formula from [1].

We show in section 12 how to construct a convex and therefore continuous extension of the Chebyshev transform to arbitrary big $\mathbb{R}$-divisors.

In section 13 we move on to prove Theorem 4 concerning the differentiability of the Monge-Ampère energy in the ample cone.

1.1.2 Acknowledgement

First of all I would like to thank Robert Berman for proposing the problem to me. In addition to Robert Berman I would also like to thank Bo Berndtsson and Sebastien Boucksom for their numerous valuable comments and suggestions concerning this article.

1.2 The Okounkov body of a semigroup

Let $\Gamma \subseteq \mathbb{N}^{n+1}$ be a semigroup. We denote by $\Sigma(\Gamma) \subseteq \mathbb{R}^{n+1}$ the closed convex cone spanned by Γ. By $\Delta_k(\Gamma)$ we will denote the set

$$\Delta_k(\Gamma) := \{\alpha : (k\alpha, k) \in \Gamma\} \subseteq \mathbb{R}^n.$$

DEFINITION 1. *The Okounkov body* $\Delta(\Gamma)$ *of the semigroup* Γ *is defined as*

$$\Delta(\Gamma) := \{\alpha : (\alpha, 1) \in \Sigma(\Gamma)\} \subseteq \mathbb{R}^n.$$

It is clear that for all non-negative k,

$$\Delta_k(\Gamma) \subseteq \Delta(\Gamma).$$

The next theorem is a result of Khovanskii from [7].

THEOREM 5. *Assume that* $\Gamma \subseteq \mathbb{N}^{n+1}$ *is a finitely generated semigroup which generates* $\mathbb{Z}^{n+1}$ *as a group. Then there exists an element* $z \in \Sigma(\Gamma)$, *such that*

$$(z + \Sigma(\Gamma)) \cap \mathbb{Z}^{n+1} \subseteq \Gamma.$$

When working with Okounkov bodies of semigroups it is sometimes useful to reformulate Theorem 5 into the following lemma.

LEMMA 6. *Suppose that* Γ *is finitely generated, generates* $\mathbb{Z}^{n+1}$ *as a group, and also that* $\Delta(\Gamma)$ *is bounded. Then there exists a constant* C *such that for all* k, *if*

$$\alpha \in \Delta(\Gamma) \cap \left(\frac{1}{k}\mathbb{Z}\right)^n$$

and if the distance between α *and the boundary of* $\Delta(\Gamma)$ *is greater than* C/k, *then in fact we have that*

$$\alpha \in \Delta_k(\Gamma).$$

Proof. By definition we that

$$\alpha \in \Delta(\Gamma) \cap \left(\frac{1}{k}\mathbb{Z}\right)^n \qquad \text{iff} \qquad (k\alpha, k) \in \Sigma(\Gamma) \cap \mathbb{Z}^{n+1}.$$

Also by definition

$$\alpha \in \Delta_k(\Gamma) \qquad \text{iff} \qquad (k\alpha, k) \in \Gamma.$$

By Theorem 5 we have that

$$(k\alpha, k) \in \Gamma \qquad \text{if} \qquad (k\alpha, k) - z \in \Sigma(\Gamma),$$

and since $\Sigma(\Gamma)$ is a cone, $(k\alpha, k) - z \in \Sigma(\Gamma)$ iff $(\alpha, 1) - z/k \in \Sigma(\Gamma)$. If $(\alpha, 1)$ lies further than $|z|/k$ from the boundary of $\Sigma(\Gamma)$, then trivially $(\alpha, 1) - z/k \in \Sigma(\Gamma)$. Since by assumtion the Okounkov body is bounded, the distance between $(\alpha, 1)$ and the boundary of $\Sigma(\Gamma)$ is greater than some constant times the distance between α and the boundary of $\Delta(\Gamma)$. The lemma follows. □

COROLLARY 7. *Suppose that Γ generates $\mathbb{Z}^{n+1}$ as a group, and also that $\Delta(\Gamma)$ is bounded. Then $\Delta(\Gamma)$ is equal to the closure of the union $\cup_{k\geq 0}\Delta_k(\Gamma)$.*

Proof. That

$$\overline{\cup_{k\geq 0}\Delta_k(\Gamma)} \subseteq \Delta(\Gamma)$$

is clear. For the opposite direction, we exhaust $\Delta(\Gamma)$ by Okounkov bodies of finitely generated subsemigroups of Γ. Therefore, without loss of generality we may assume that Γ is finitely generated. We apply Lemma 6 which says that all the $(\frac{1}{k}\mathbb{Z})^n$ lattice points in $\Delta(\Gamma)$ whose distance to the boundary of $\Delta(\Gamma)$ is greater that some constant depending on the element z in (5), divided by k, actually lie in $\Delta_k(\Gamma)$. The corollary follows. □

1.3 Subadditive functions on semigroups

Let Γ be a semigroup. A real-valued function F on Γ is said to be *subadditive* if for all $\alpha, \beta \in \Gamma$ it holds that

$$F(\alpha + \beta) \leq F(\alpha) + F(\beta).$$

If $\alpha \in \mathbb{R}^{n+1}$, we denote the sum of its coordinates $\sum \alpha_i$ by $|\alpha|$. We also let $\Sigma^0 \subseteq \mathbb{R}^{n+1}$ denote the set

$$\Sigma^0 := \{(\alpha_1, ..., \alpha_{n+1}) : |\alpha| = 1, \theta_i > 0\}.$$

In [3] Bloom-Levenberg observe that one can extract from [14] the following theorem on subadditive functions on $\mathbb{N}^{n+1}$.

THEOREM 8. *Let F be a subadditive function on $\mathbb{N}^{n+1}$ which is bounded from below by some linear function. Then for any sequence $\alpha(k) \in \mathbb{N}^{n+1}$ such that $|\alpha(k)| \to \infty$ when k tends to infinity and such that*

$$\alpha(k)/|\alpha(k)| \to \theta \in \Sigma^0,$$

it holds that the limit

$$c[F](\theta) := \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

exists and does only depend on θ. Furthermore, the function $c[F]$ thus defined is convex on Σ^0.

We will give a proof of this theorem which also shows that it holds locally, i.e. that F does not need to be subadditive on the whole of $\mathbb{N}^{n+1}$ but only on some open convex cone and only for large $|\alpha|$. Then Zaharjuta's theorem still holds for the part of Σ^0 lying in the open cone. We will divide the proof into a couple of lemmas.

LEMMA 9. *Let O be an open convex cone in $\mathbb{R}_+^{n+1}$ and let F be a subadditive function on $(O \setminus B(0, M)) \cap \mathbb{N}^{n+1}$, where $B(0, M)$ denotes the ball of radius M centered at the origin, and M is any positive number. Then for any closed convex cone $K \subseteq O$ there exists a constant C_K such that*

$$F(\alpha) \leq C_K|\alpha|$$

on $(K \setminus B(0, M)) \cap \mathbb{N}^{n+1}$.

Proof. Pick points in $(O \setminus B(0, M)) \cap \mathbb{N}^{n+1}$ such that if we denote by Γ the semigroup generated by the points, the convex cone $\Sigma(\Gamma)$ should contain $(K \setminus B(0, M))$ and the distance between the boundaries should be positive. The points should also generate $\mathbb{Z}^{n+1}$ as a group. Then from Theorem 5 it follows that there exists an M' such that

$$(K \setminus B(0, M')) \cap \mathbb{N}^{n+1} \subseteq \Gamma. \tag{1.5}$$

Let α_i denote the generators of Γ we picked. The inclusion (1.5) means that for all $\alpha \in (K \setminus B(0, M')) \cap \mathbb{N}^{n+1}$ there exist non-negative integers a_i such that

$$\alpha = \sum a_i \alpha_i.$$

By the subadditivity we therefore get that

$$F(\alpha) \leq \sum a_i F(\alpha_i) \leq C \sum a_i \leq C|\alpha|.$$

Since only finitely many points in $(K \setminus B(0, M)) \cap \mathbb{N}^{n+1}$ do not lie in $(K \setminus B(0, M')) \cap \mathbb{N}^{n+1}$ the lemma follows. □

LEMMA 10. *Let O, K and F be as in the statement of Lemma 9. Let α be a point in $(K^\circ \setminus B(0, M)) \cap \mathbb{N}^{n+1}$, and let $\gamma(k)$ be a sequence in $(K \setminus B(0, M)) \cap \mathbb{N}^{n+1}$ such that*

$$|\gamma(k)| \to \infty$$

when k tends to infinity and that

$$\frac{\gamma(k)}{|\gamma(k)|} \to p \in K^\circ$$

for some point p in the interior of K. Let l be the ray starting in $\alpha/|\alpha|$, going through p, and let q denote the first intersection of l with the boundary of K. Denote by t the number such that

$$p = t\frac{\alpha}{|\alpha|} + (1-t)q.$$

Then there exists a constant C_K depending only of F and K such that

$$\limsup_{k\to\infty} \frac{F(\gamma(k))}{|\gamma(k)|} \leq t\frac{F(\alpha)}{|\alpha|} + (1-t)C_K.$$

Proof. We can pick points β_i in $(K \setminus B(0, M)) \cap \mathbb{N}^{n+1}$ with $\beta_i/|\beta_i|$ lying arbitrarily close to q, such that if Γ denotes the semigroup generated by the points β_i and α, Γ generates $\mathbb{Z}^{n+1}$ as a group and

$$p \in \Sigma(\Gamma)^\circ.$$

Therefore from Theorem 5 it follows that for large k $\gamma(k)$ can be written

$$\gamma(k) = a\alpha + \sum a_i\beta_i$$

for non-negative integers a_i and a. The subadditivity of F gives us that

$$F(\gamma(k)) \leq aF(\alpha) + \sum a_i F(\beta_i) \leq aF(\alpha) + C_K \sum a_i|\beta_i|,$$

where we in the last inequality used Lemma 9. Dividing by $|\gamma(k)|$ we get

$$\frac{F(\gamma(k))}{|\gamma(k)|} \leq \frac{a|\alpha|}{|\gamma(k)|}\frac{F(\alpha)}{|\alpha|} + C_K \sum \frac{a_i|\beta_i|}{|\gamma(k)|}.$$

Our claim is that $\frac{a|\alpha|}{|\gamma(k)|}$ will tend to t and that $\sum \frac{a_i|\beta_i|}{|\gamma(k)|}$ will tend to $(1-t)$. Consider the equations

$$\frac{\gamma(k)}{|\gamma(k)|} = \frac{a|\alpha|}{|\gamma(k)|}\frac{\alpha}{|\alpha|} + \sum \frac{a_i|\beta_i|}{|\gamma(k)|}\frac{\beta_i}{|\beta_i|}$$

and

$$p = t\frac{\alpha}{|\alpha|} + (1-t)q.$$

Observe that

$$t = \frac{|p - \frac{\alpha}{|\alpha|}|}{|q-\alpha|}.$$

If $|\frac{\gamma(k)}{|\gamma(k)|} - p| < \delta$ and $|\frac{\beta_i}{|\beta_i|} - q| < \delta$ for all i, then we see that

$$\frac{a|\alpha|}{|\gamma(k)|} \leq \frac{|p - \frac{\alpha}{|\alpha|}| + \delta}{|q - \frac{\alpha}{|\alpha|}| - \delta} \leq t + \varepsilon(\delta),$$

where $\varepsilon(\delta)$ goes to zero as δ goes to zero. Similarly we have that

$$\frac{a|\alpha|}{|\gamma(k)|} \geq \frac{|p - \frac{\alpha}{|\alpha|}| - \delta}{|q - \frac{\alpha}{|\alpha|}| + \delta} \geq t - \varepsilon'(\delta), \tag{1.6}$$

where $\varepsilon'(\delta)$ goes to zero as δ goes to zero. Since

$$\frac{a|\alpha|}{|\gamma(k)|} + \sum \frac{a_i|\beta_i|}{|\gamma(k)|} = 1,$$

inequality (1.6) implies that

$$\sum \frac{a_i|\beta_i|}{|\gamma(k)|} \leq 1 - t + \varepsilon'(\delta).$$

The lemma follows. □

COROLLARY 11. *Let O and F be as in the statement of Lemma 9. Then for any sequence $\alpha(k)$ in $O \cap \mathbb{Z}^{n+1}$ such that $|\alpha(k)| \to \infty$ when k tends to infinity and such that $\alpha(k)/|\alpha(k)|$ converges to some point p in O the limit*

$$\lim_{k\to\infty} \frac{F(\alpha)}{|\alpha(k)|}$$

exists and only depends on F and p.

Proof. Let $\alpha(k)$ and $\beta(k)$ be two such sequences converging to p. Let $K \subseteq O$ be some closed cone such that $p \in K^\circ$. Let us as in Lemma 10 write

$$p = t_k \frac{\beta(k)}{|\beta(k)|} + (1 - t_k)q_k.$$

For any $\varepsilon > 0$, t_k is greater than $1 - \varepsilon$ when k is large enough. By Lemma 10 we have that for such k

$$\limsup_{m\to\infty} \frac{F(\alpha(m))}{|\alpha(m)|} \leq (1 - t_k)\frac{F(\beta(k))}{|\beta(k)|} + \varepsilon C_K \leq \frac{F(\beta(k))}{|\beta(k)|} + \varepsilon C_K + \varepsilon C,$$

where C comes from the lower bound

$$\frac{F(\beta)}{|\beta|} \geq C$$

which holds for all β by assumption. Since ε tends to zero when k gets large we have that

$$\limsup_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|} \leq \liminf_{k\to\infty} \frac{F(\beta(k))}{|\beta(k)|}.$$

By letting $\alpha(k) = \beta(k)$ we get existence of the limit, and by symmetry the limit is unique. □

PROPOSITION 12. *The function $c[F]$ on $O \cap \Sigma^\circ$ defined by*

$$c[F](p) := \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

for any sequence $\alpha(k)$ such that $|\alpha(k)| \to \infty$ and $\frac{\alpha(k)}{|\alpha(k)|} \to p$, which is well-defined according to Corollary 11, is convex, and therefore continuous.

Proof. First we wish to show that $c[F]$ is lower semicontinuous. Let p be a point in $O \cap \Sigma^\circ$ and q_n a sequence converging to p. From Lemma 10 it follows that

$$c[F](p) \leq \liminf_{q_n \to p} c[F](q_n),$$

which is equivalent to lower semicontinuity.

Using this the lemma will follow if we show that for any two points p and q in $O \cap \Sigma^\circ$ it holds that

$$2c[F](\frac{p+q}{2}) \leq c[F](p) + c[F](q). \tag{1.7}$$

Choose sequences $\alpha(k), \beta(k) \in O \cap \mathbb{N}^{n+1}$ such that

$$\frac{\alpha(k)}{|\alpha(k)|} \to p, \qquad \frac{\beta(k)}{|\beta(k)|} \to q,$$

and for simplicity assume that $|\alpha(k)| = |\beta(k)|$. Then

$$\frac{\alpha(k)+\beta(k)}{|\alpha(k)+\beta(k)|} \to \frac{p+q}{2}.$$

Hence

$$2c[F](\frac{p+q}{2}) = \lim_{k\to\infty} \frac{F(\alpha(k)+\beta(k))}{|\alpha(k)|} \leq \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|} + \lim_{k\to\infty} \frac{F(\beta(k))}{|\beta(k)|} =$$
$$= c[F](p) + c[F](q).$$

□

Together with Theorem 5 these lemmas yield a general result for subadditive functions on subsemigroups of $\mathbb{N}^{n+1}$.

A function F defined on a cone O is said to be *locally linearly bounded from below* if for each point $p \in O$ there exists an open subcone $O' \subseteq O$ containing p and a linear function λ on O' such that $F \geq \lambda$ on O'.

THEOREM 13. *Let $\Gamma \subseteq \mathbb{N}^{n+1}$ be a semigroup which generates $\mathbb{Z}^{n+1}$ as a group, and let F be a subadditive function on Γ which is locally linearly bounded from below. Then for any sequence $\alpha(k) \in \Gamma$ such that $|\alpha(k)| \to \infty$ and $\frac{\alpha(k)}{|\alpha(k)|} \to p \in \Sigma(\Gamma)^\circ$ for some point p in the interior of $\Sigma(\Gamma)$, the limit*

$$\lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

exists and only depends on F and p. Furthermore the function

$$c[F](p) := \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

thus defined on $\Sigma(\Gamma)^\circ \cap \Sigma^\circ$ is convex.

Proof. By Theorem 5 it follows that for any point $p \in \Sigma(\Gamma)^\circ$ there exists an open convex cone O and a number M such that

$$(O \setminus B(0,M)) \cap \mathbb{N}^{n+1} \subseteq \Gamma.$$

We can also choose O such that F is bounded from below by a linear function on O. Therefore the theorem follows immediately from Corollary 11 and Proposition 12. □

We will show how this theorem can be seen as the counterpart to Theorem 5 for subadditive functions.

DEFINITION 2. *Let Γ be a subsemigroup of $\mathbb{N}^{n+1}$ and let F be a subadditive function of Γ which is locally linearly bounded from below. One defines the convex envelope of F, denoted by $P(F)$, as the supremum of all linear functions on $\Sigma(\Gamma)^\circ$ dominated by F, or which ammounts to the same thing, the supremum of all convex one-homogeneous functions on $\Sigma(\Gamma)^\circ$ dominated by F.*

THEOREM 14. *If Γ generates $\mathbb{Z}^{n+1}$ as a group, then for any subadditive function F on Γ which is locally linearly bounded from below it holds that*

$$F(\alpha) = P(F)(\alpha) + o(|\alpha|)$$

for $\alpha \in \Gamma \cap \Sigma(\Gamma)^\circ$.

Proof. That

$$F(\alpha) \geq P(F)(\alpha)$$

follows from the definition. If we let $c[F]$ be defined on the whole of $\Sigma(\Gamma)^\circ$ by letting

$$c[F](\alpha) := |\alpha| c[F](\frac{\alpha}{|\alpha|}),$$

it follows from Theorem 13 that $c[F]$ will be convex and one-homogeneous. It will also be dominated by F since by the subadditivity

$$\frac{F(\alpha)}{|\alpha|} \geq \frac{F(k\alpha)}{|k\alpha|}$$

for all positive integers and therefore

$$\frac{F(\alpha)}{|\alpha|} \geq \lim_{k\to\infty} \frac{F(k\alpha)}{|k\alpha|} = c[F](\frac{\alpha}{|\alpha|}).$$

It follows that

$$P(F) \geq c[F].$$

For $\alpha \in \Gamma$ by definition we have that

$$P(F)(\alpha) \leq \frac{F(k\alpha)}{k}$$

for all positive integers k. At the same time

$$c[F](\alpha) = \lim_{k\to\infty} \frac{F(k\alpha)}{k},$$

hence we get that

$$P(F)(\alpha) \leq c[F](\alpha)$$

for $\alpha \in \Gamma$ Since both $P(F)$ and $c[F]$ are convex they are continuous, so by the homogeneity we get that

$$P(F) \leq c[F]$$

on $\Sigma(\Gamma)^\circ$, and therefore $P(F) = c[F]$. The theorem now follows from Theorem 13. □

1.4 The Okounkov body of a line bundle

In this section, following Okounkov, we will show how to associate a semigroup to a line bundle.

DEFINITION 3. *An order $<$ on $\mathbb{N}^n$ is additive if $\alpha < \beta$ and $\alpha' < \beta'$ implies that*

$$\alpha + \alpha' < \beta + \beta'.$$

One example of an additive order is the lexicographic order where

$$(\alpha_1, ..., \alpha_n) <_{\text{lex}} (\beta_1, ..., \beta_n)$$

iff there exists an index j such that $\alpha_j < \beta_j$ and $\alpha_i = \beta_i$ for $i < j$.

Let X be a compact projective complex manifold of dimension n, and L a holomorphic line bundle, which we will assume to be big. Suppose we have chosen a point p in X, and local holomorphic coordinates $z_1, ..., z_n$ around that point, and let $e_p \in H^0(U, L)$ be a local trivialization of L around p. Any holomorphic section $s \in H^0(X, kL)$ has an unique represention as a convergent power series in the variables z_i,

$$\frac{s}{e_p^k} = \sum a_\alpha z^\alpha,$$

which for convenience we will simply write as

$$s = \sum a_\alpha z^\alpha.$$

We consider the lexicographic order on the multiindices α, and let $v(s)$ denote the smallest index α such that $a_\alpha \neq 0$.

DEFINITION 4. *Let $\Gamma(L)$ denote the set*

$$\bigcup_{k \geq 0} \big(v(H^0(kL)) \times \{k\}\big) \subseteq \mathbb{N}^{n+1}.$$

It is a semigroup, since for $s \in H^0(kL)$ and $t \in H^0(mL)$

$$v(st) = v(s) + v(t). \tag{1.8}$$

The Okounkov body of L, denoted by $\Delta(L)$, is defined as the Okounkov body of the associated semigroup $\Gamma(L)$.

We write $\Delta_k(\Gamma(L))$ simply as $\Delta_k(L)$.

Let us recall some basic facts on Okounkov bodies (see e.g. [9] for proofs).

LEMMA 15. *The number of points in $\Delta_k(L)$ is equal to the dimension of the vector space $H^0(kL)$.*

LEMMA 16. *The Okounkov body of a big line bundle is bounded, hence compact.*

LEMMA 17. *If L is a big line bundle, $\Gamma(L)$ generates $\mathbb{Z}^{n+1}$ as a group. In fact $\Gamma(L)$ contains a translated unit simplex.*

Remark. Note that the additivity of v as seen in equation (1.8) only depends on the fact that the lexicographic order is additive. Therefore we could have used any total additive order on $\mathbb{N}^n$ to define a semigroup $\tilde{\Gamma}(L)$, and the associated Okounkov body $\tilde{\Delta}(L)$. We will only consider the case where the Okounkov body $\tilde{\Delta}(L)$ is bounded, and the semigroup $\tilde{\Gamma}(L)$ generates $\mathbb{N}^n$ as a group.

LEMMA 18. *For any closed set K contained in the convex hull of $\Delta_M(L)$ for some M, there exists a constant C_K such that if*

$$\alpha \in K \cap (\frac{1}{k}\mathbb{Z})^n$$

and the distance between α and the boundary of K is greater than $\frac{C_K}{k}$, then $\alpha \in \Delta_k(L)$.

Proof. Let Γ be the semigroup generated by the elements $(M\beta, M)$ where $\beta \in \Delta_M(L)$, and some unit simplex in $\Gamma(L)$. Applying Lemma 6 gives the lemma. □

LEMMA 19. *If K is relatively compact in the interior of $\Delta(L)$, there exists a number M such that for $k > M$,*

$$\alpha \in K \cap (\frac{1}{k}\mathbb{Z})^n$$

implies that $\alpha \in \Delta_k(L)$.

Proof. This is a consequence of Lemma 18 by choosing M such that the distance between K and the convex hull of $\Delta_M(L)$ is strictly positive, therefore greater than $\frac{C_K}{k}$ for large k. □

1.5 The Chebyshev transform

DEFINITION 5. *A continuous hermitian metric $h = e^{-\psi}$ on a line bundle L is a continuous choice of scalar product on the complex line L_p at each point p on the manifold. If f is a local frame for L on U_f, then one writes*

$$|f|^2 = h_f = e^{-\psi_f},$$

where ψ_f is a continuous function on U_f. If $h = e^{-\psi}$ is a metric, ψ is called a weight.

We will show how one to a given continuous weight associates a subadditive function on the semigroup $\Gamma(L)$.

For all $(k\alpha, k) \in \Gamma(L)$, let us denote by $A_{\alpha,k}$ the affine space of sections in $H^0(kL)$ of the form

$$z^{k\alpha} + \text{higher order terms}.$$

Consider the supremum norm $||.||_{k\psi}$ on $H^0(kL)$ given by

$$||s||^2_{k\psi} := \sup_{x\in X}\{|s(x)|^2 e^{-k\psi(x)}\}.$$

DEFINITION 6. *We define the discrete Chebyshev transform $F[\psi]$ on $\Gamma(L)$ by*

$$F[\psi](k\alpha, k) := \inf\{\ln ||s||^2_{k\psi} : s \in A_{\alpha,k}\}.$$

A section s in $A_{\alpha,k}\}$ which minimizes the supremum norm is called a Chebyshev section.

LEMMA 20. *The function $F[\psi]$ is subadditive.*

Proof. Let $(k\alpha, k)$ and $(l\beta, l)$ be two points in $\Gamma(L)$, and denote by γ

$$\gamma := \frac{k\alpha + l\beta}{k + l}.$$

Thus we have that

$$(k\alpha, k) + (l\beta, l) = ((k+l)\gamma, k+l).$$

Let s be some section in $A_{\alpha,k}$ and s' some section in $A_{\beta,l}$. Since

$$ss' = (z^{k\alpha} + \text{higher order terms})(z^{l\beta} + \text{higher order terms}) = \\ = z^{(k+l)\gamma} + \text{higher order terms},$$

we see that $ss' \in A_{\gamma,k+l}$. We also note that the supremum of the product of two functions is less or equal to the product of the supremums, i.e.

$$||ss'||^2_{(k+l)\psi} \le ||s||^2_{k\psi}||s'||^2_{l\psi}.$$

It follows that

$$\inf\{||s||^2_{k\psi} : s \in A_{\alpha,k}\} \inf\{||s'||^2_{l\psi} : s' \in A_{\beta,l}\} \le \inf\{||t||^2_{\gamma,k+l} : t \in A_{\gamma,k+l}\},$$

which gives the lemma by taking the logarithm. □

LEMMA 21. *There exists a constant C such that for all $(k\alpha, k) \in \Gamma(L)$,*

$$F[\psi](k\alpha, k) \ge C|(k\alpha, k)|$$

.

Proof. Let $r > 0$ be such that the polydisc D of radius r centered at p is fully contained in the coordinate chart of $z_1, ..., z_n$. We can also assume that our trivialization $e_p \in H^0(U, L)$ of L is defined on D, i.e. $D \subseteq U$. Let s be a section in $A_{\alpha,k}$ and let

$$\tilde{s} := \frac{s}{e_p^k}.$$

Denote by ψ_p the trivialization of ψ. Hence

$$|s|^2 e^{-k\psi} = |\tilde{s}|^2 e^{-k\psi_p}.$$

Since ψ_p is continuous,

$$e^{-\psi_p} > A$$

on D for some constant A. This yields that

$$||s||^2 \geq \sup_{x\in D}\{|\tilde{s}(x)|^2 e^{-k\psi_p(x)}\} \geq A^k \sup_{x\in D}\{|\tilde{s}(x)|^2\}.$$

We claim that

$$\sup_{x\in D}\{|\tilde{s}(x)|^2\} \geq r^{k|\alpha|}.$$

Observe that

$$\sup_{z\in D}\{|z^{k\alpha}|^2\} = r^{k|\alpha|}.$$

One now shows that

$$\sup_{z\in D}\{|z^{k\alpha}|^2\} \leq \sup_{z\in D}\{|z^{k\alpha} + \text{higher order terms}|^2\}$$

by simply reducing it to the case of one variable where it is immediate. We get that

$$||s||^2 \geq A^k r^{k|\alpha|}$$

and hence

$$F[\psi](k\alpha, k) \geq k\ln A + k|\alpha|\ln r \geq C(k + k|\alpha|),$$

if we choose C to be less than both $\ln A$ and $\ln r$. □

DEFINITION 7. *We define the Chebyshev transform of* ψ*, denoted by* $c[\psi]$ *as the convex envelope of* $F[\psi]$ *on* $\Sigma(\Gamma)^\circ$*. It is convex and one-homogeneous. We will also identify it with its restriction to* $\Delta(L)^\circ$*, the interior of the Okounkov body of L. Recall that by definition*

$$\Delta(L) := \Sigma(L) \cap (\mathbb{R}^n \times \{1\}).$$

PROPOSITION 22. *For any sequence* $(k\alpha(k), k)$ *in* $\Gamma(L)$*,* $k \to \infty$*, such that*

$$\lim_{k\to\infty} \alpha(k) = p \in \Delta(L)^\circ,$$

it holds that

$$c[\psi](p) = \lim_{k\to\infty} \frac{1}{k} \ln ||t_{\alpha(k),k}||^2,$$

where $t_{\alpha(k),k}$ *is a Chebyshev section in* $A_{\alpha((k),k}$*.*

Proof. By Lemma 20 and Lemma 21 we can apply Theorem 14 to the function $F[\psi]$ and get that

$$c[\psi](p) = |(p,1)|c[\psi](\frac{(p,1)}{|(p,1)|}) = |(p,1)| \lim_{k\to\infty} \frac{F[\psi](k\alpha,k)}{k|(\alpha(k),1)|} =$$
$$= \lim_{k\to\infty} \frac{F[\psi](k\alpha,k)}{k} = \lim_{k\to\infty} \frac{1}{k} \ln ||t_{\alpha(k),k}||^2.$$

□

LEMMA 23. *Let ψ be a continuous weight on L and consider the continuous weight on L given by $\psi + C$ for some constant C. Then it holds that*

$$F[\psi + C](k\alpha, k) = F[\psi](k\alpha, k) - kC, \tag{1.9}$$

and that

$$c[\psi + C] = c[\psi] - C$$

on $\Delta(L)^\circ$.

Proof. For any section $s \in H^0(kL)$ we have that

$$||s||^2_{k(\psi+C)} = e^{-kC}||s||^2_{k\psi},$$

therefore

$$\ln ||s||^2_{k(\psi+C)} = \ln ||s||^2_{k\psi} - kC.$$

The lemma thus follows from the definitions. □

LEMMA 24. *If ψ and φ are two continuous weights such that*

$$\psi \leq \varphi,$$

then

$$F[\psi] \geq F[\varphi],$$

and also

$$c[\psi] \geq c[\varphi].$$

Proof. Follows immediately from the definitions. □

PROPOSITION 25. *For any two continuous weights on L, ψ and φ, the difference of the Chebyshev transforms, $c[\psi] - c[\varphi]$, is continuous and bounded on $\Delta(L)^\circ$.*

Proof. It is the difference of two convex hence continuous functions, and is therefore continuous. Since $\psi - \varphi$ is a continuous function on the compact space X, there exists a constant C such that

$$\psi \leq \varphi + C.$$

Thus by Lemma 24 and Lemma 23 we have that

$$c[\psi] \leq c[\varphi + C] = c[\varphi] - C.$$

By symmetry we see that $c[\psi] - c[\varphi]$ is bounded on $\Delta(L)^\circ$. □

For Okounkov bodies we have that

$$\Delta(mL) = m\Delta(L),$$

see e.g. [9]. The Chebyshev transforms also exhibit a homogeneity property.

PROPOSITION 26. *Let ψ be a continuous weight on L. Consider the weight $m\psi$ on mL. For any $p \in \Delta(L)^\circ$ it holds that*

$$c[m\psi](mp) = mc[\psi](p).$$

Proof. We observe that trivially $A_{m\alpha,k} = A_{\alpha,km}$, as affine subspaces of $H^0(kmL)$, and hence

$$F[m\psi](km\alpha, k) = F[\psi](km\alpha, km).$$

Let $\alpha(k) \to p \in \Delta(L)^\circ$.We get that

$$\begin{aligned} c[m\psi](mp) = |(mp,1)|c[m\psi](\frac{(mp,1)}{|(mp,1)|}) = \\ = |(mp,1)| \lim_{k\to\infty} \frac{F[m\psi](km\alpha(k),k)}{k|(m\alpha(k),1)|} = \\ = \lim_{k\to\infty} \frac{F[\psi](km\alpha(k),km)}{k} = mc[\psi](p). \end{aligned}$$

□

1.6 The Monge-Ampère energy of weights

One may define a partial order on the space of weights to a given line bundle. Let $\psi <_w \varphi$ if

$$\psi \leq \varphi + O(1)$$

on X. If a weight is maximal with respect to the order $<_w$, it is said to have minimal singularities. It is a fact that a weight with minimal singularities on a big line bundle is locally bounded on a dense Zariski-open subset of X (see e.g. [1]). On an ample line bundle, the weights with minimal singularities are exactly those who are locally bounded.

Let ψ and φ be two locally bounded psh-weights. By $\mathrm{MA}_m(\psi, \varphi)$ we will denote the positive current

$$\sum_{j=0}^{m} (dd^c\psi)^j \wedge (dd^c\varphi)^{m-j},$$

and by $\mathrm{MA}(\psi)$ we will mean the positive measure $(dd^c\psi)^n$.

DEFINITION 8. *If ψ and φ are two psh weights with minimal singularities, then we define the Monge-Amp*
'ere energy of ψ with respect to φ as

$$\mathcal{E}(\psi, \varphi) := \frac{1}{n+1} \int_\Omega (\psi - \varphi) MA_n(\psi, \varphi),$$

where Ω is a dense Zariski-open subset of X on which ψ and φ are locally bounded.

Remark. In [1] Berman-Boucksom use the notation $\mathcal{E}(\psi) - \mathcal{E}(\varphi)$ for what we denote by $\mathcal{E}(\psi, \varphi)$. Thus they consider $\mathcal{E}(\psi)$ as a functional defined only up to a constant.

An important aspect of the Monge-Ampère energy (and a motivation for calling it an energy) is its cocycle property, i.e. that

$$\mathcal{E}(\psi, \varphi) + \mathcal{E}(\varphi, \psi') + \mathcal{E}(\psi', \psi) = 0$$

for all weights ψ, φ and ψ' (see e.g. [1]).

DEFINITION 9. *If ψ is a continuous weight and K a compact subset of X, the psh envelope of ψ with respect to K, $P_K(\psi)$, is given by*

$$P_K(\psi) := \sup\{\varphi : \varphi \text{ psh weight on } L, \varphi \leq \psi \text{ on } K\}.$$

For any ψ and K, as one may check, $P_K(\psi)$ will be psh and have minimal singularities. When $K = X$, we will simply write $P(\psi)$ for $P_X(\psi)$.

If ψ and φ are continuous weights, we will call

$$\mathcal{E}(P(\psi), P(\varphi))$$

the Monge-Ampère energy of ψ with respect to φ, and we will denote it by $\mathcal{E}(\psi, \varphi)$. Since for psh weights ψ, trivially $P(\psi) = \psi$, therefore the notation is unambiguous.

We refer the reader to [5] for a more thorough exposition on Monge-Ampère measures and psh envelopes.

We now state our main result.

THEOREM 27. *Let ψ and φ be continuous weights on L. Then it holds that*

$$\mathcal{E}(\varphi, \psi) = n! \int_{\Delta(L)^\circ} (c[\psi] - c[\varphi]) d\lambda, \tag{1.10}$$

where $d\lambda$ denotes the Lebesgue measure on $\Delta(L)^\circ$.

The proof of Theorem 27 will depend on the fact that one can also use L^2-norms to compute the Chebyshev transform of a continuous weight. This will be explained in the next section.

1.7 Bernstein-Markov norms

DEFINITION 10. *Let μ be a positive measure on X, and ψ a continuous weight on a line bundle L. One says that μ satisfies the Bernstein-Markov property with respect to ψ if for each $\varepsilon > 0$ there exists $C = C(\varepsilon)$ such that for all non-negative k and all holomorphic sections $s \in H^0(kL)$ we have that*

$$\sup_{x \in X}\{|s(x)|^2 e^{-k\psi(x)}\} \leq C e^{\varepsilon k} \int_X |s|^2 e^{-k\psi} d\mu. \tag{1.11}$$

If ψ is a continuous weight on L and μ a Bernstein-Markov measure on X with respect to ψ, we will call the L^2-norm on $H^0(kL)$ defined by

$$||s||^2_{k\psi,\mu} := \int_X |s|^2 e^{-k\psi} d\mu$$

a Bernstein-Markov norm. We will also call the pair (ψ, μ) a Bernstein-Markov pair on (X, L).

For any continuous weight ψ on L there exist measures μ such that (ψ, μ) is a Bernstein-Markov pair. In fact it is easy to show that any smooth volume form dV on X satisfies the Bernstein-Markov property with respect to any continuous weight, see e.g. [1].

A pair (E, ψ) where E is a subset of X and ψ is a continuous weight on L is called a weighted subset. The equilibrium weight ψ_E of (E, ψ) is defined as

$$\psi_E := \sup\{\varphi : \varphi \text{ is psh}, \varphi \leq \psi \text{ on } E\}.$$

A weighted set (E, ψ) is said to be regular if the equilibrium weight ψ_E is upper semicontinuous.

DEFINITION 11. *If a compact $K \subseteq X$ is the support of a positive measure μ, one says that μ satisfies the Bernstein-Markov property with respect to the weighted set (K, ψ) if for all k and $s \in H^0(kL)$ inequality (1.11) holds when X is replaced with K.*

LEMMA 28. *If μ is a smooth volume form and (K, ψ) is a compact regular weighted subset, then the restriction of μ to K satisfies the Bernstein-Markov property with respect to (K, ψ).*

For a proof we refer to [1].

We want to be able to use a Bernstein-Markov norm instead of the supremum norm to calculate the Chebyshev transform of a continuous weight ψ.

We pick a positive measure μ with the Bernstein-Markov property with respect to ψ. For all $(k\alpha, k) \in \Gamma(L)$, let $t_{\alpha,k}$ be the section in $H^0(kL)$ of the form

$$z^{k\alpha} + \text{higher order terms}$$

that minimizes the L^2-norm

$$||t_{\alpha,k}||^2_{k\psi,\mu} := \int_X |t_{\alpha,k}|^2 e^{-k\psi} d\mu.$$

It follows that

$$< t_{\alpha,k}, t_{\beta,k} >_{k\psi} = 0$$

for $\alpha \neq \beta$, since otherwise the sections $t_{\alpha,k}$ would not be minimizing. Hence

$$\{t_{\alpha,k} : \alpha \in \Delta_k(L)\}$$

is an orthogonal basis for $H^0(kL)$ with respect to $||.||_{k\psi,\mu}$. Indeed they are orthogonal, and by Lemma 5 we have that

$$\#\{t_{\alpha,k} : \alpha \in \Delta_k(L)\} = \#\Delta_k(L) = \dim(H^0(kL)),$$

therefore it must be a basis.

DEFINITION 12. *We define the discrete Chebyshev transform $F[\psi,\mu]$ of (ψ,μ) on Γ by*

$$F[\psi,\mu](k\alpha,k) := \ln ||t_{\alpha,k}||^2_{k\psi,\mu}.$$

We also denote $\frac{1}{k}F[\psi,\mu](k\alpha,k)$ by $c_k[\psi,\mu](\alpha)$.

We will sometimes write $c_k[\psi]$ when we mean $c_k[\psi,\mu]$, considering μ as fixed.

PROPOSITION 29. *For any sequence $(k\alpha(k),k)$ in $\Gamma(L)$, $k\to\infty$, such that*

$$\lim_{k\to\infty} \alpha(k) = p \in \Delta(L)^\circ,$$

it holds that

$$c[\psi](p) = \lim_{k\to\infty} c_k[\psi,\mu](\alpha(k)).$$

Proof. For a point $(k\alpha,k) \in \Gamma$, let $t_{\alpha,k}$ be the minimizer with respect to the Bernstein-Markov norm. By the Bernstein-Markov property we get that

$$||t_{\alpha,k}||^2_{\sup} \leq Ce^{\varepsilon k}||t^{\mu}_{\alpha,k}||^2_{\mu},$$

and hence

$$F[\psi](k\alpha, k) \le F[\psi, \mu](k\alpha, k) + \ln C + \varepsilon k. \tag{1.12}$$

Let s be any section in $A_{\alpha,k}$. We have that by definition

$$||t_{\alpha,k}||_\mu^2 \le ||s||_\mu^2 \le \mu(X)||s||_{\sup}^2,$$

so

$$F[\psi, \mu](k\alpha, k) \le F[\psi](k\alpha, k) + \ln \mu(X). \tag{1.13}$$

Equations (1.12) and (1.13) put together gives that

$$F[\psi](k\alpha, k) - \ln C - \varepsilon k \le F[\psi, \mu](k\alpha, k) \le F[\psi](k\alpha, k) + \ln \mu(X). \tag{1.14}$$

It follows that

$$\lim_{k\to\infty} \frac{F[\psi, \mu](k\alpha(k), k)}{k} = \lim_{k\to\infty} \frac{F[\psi](k\alpha(k), k)}{k} = c[\psi](p),$$

which gives the proposition. □

LEMMA 30. *Let ψ be a continuous weight on L and consider the continuous weight on L given by $\psi + C$ for some constant C. Then it holds that*

$$F[\psi + C, \mu](k\alpha, k) = F[\psi, \mu](k\alpha, k) - kC.$$

Proof. This follows exactly as in the case of the suprumum norm, see proof of Lemma 23. □

PROPOSITION 31. *Let (ψ, μ) and (φ, ν) be two Bernstein-Markov pairs, and assume that*

$$\psi \le \varphi$$

Then for every $\varepsilon > 0$ there exists a constant C' such that

$$F[\psi, \mu](k\alpha, k) \ge F[\varphi, \nu](k\alpha, k) - C' - \varepsilon k.$$

Proof. Let $t^{\psi}_{\alpha,k}$ and $t^{\varphi}_{\alpha,k}$ be the minimizing sections with respect to the Bernstein-Markov norms $||.||_{k\psi,\mu}$ and $||.||_{k\varphi}$ respectively. From equation (1.14) and Proposition 31 we get that

$$F[\psi,\mu](k\alpha,k) \geq F[\psi](k\alpha,k) - \ln C - \varepsilon k \geq F[\varphi](k\alpha,k) - \ln C - \varepsilon k \geq$$
$$\geq F[\varphi,\nu] - \ln \nu(X) - \ln C - \varepsilon k.$$

□

PROPOSITION 32. *For any two Bernstein-Markov pairs on (X,L), (ψ,μ) and (φ,ν) the difference of the discrete Chebyshev transforms*

$$c_k[\psi,\mu] - c_k[\varphi,\nu]$$

is uniformly bounded on $\Delta(L)^{\circ}$.

Proof. By symmetry it suffices to find an upper bound. Let $\tilde{C}$ be a constant such that $\psi \leq \varphi + \tilde{C}$. By Lemma 30 and Proposition 31 we get that

$$c_k[\psi,\mu](\alpha) = \frac{1}{k}F[\psi,\mu](k\alpha,k) \geq \frac{1}{k}F[\varphi + C,\nu](k\alpha,k) - \frac{C'}{k} - \varepsilon =$$
$$= \frac{1}{k}F[\varphi,\nu](k\alpha,k) - C - \frac{C'}{k} - \varepsilon = c_k[\varphi,\nu](\alpha) - C - \frac{C'}{k} - \varepsilon.$$

The proposition follows. □

1.8 Proof of main theorem

1.8.1 Preliminary results

Let $\mathcal{B}^2(\mu,k\varphi)$ denote the unit ball in $H^0(kL)$ with respect to the norm

$$|| \cdot ||_{k\phi,\mu} := \int_X |.|^2 e^{-k\varphi} d\mu,$$

i.e.

$$\mathcal{B}^2(\mu,k\varphi) := \{s \in H^0(kL) : \int_X |s|^2 e^{-k\varphi} d\mu \leq 1\}.$$

Consider the quotient of the volume of two unit balls

$$\frac{\mathrm{vol}\mathcal{B}^2(\mu, k\varphi)}{\mathrm{vol}\mathcal{B}^2(\nu, k\psi)}$$

with respect to the Lebesgue measure on $H^0(kL)$, where we by some linear isomorphism identify $H^0(kL)$ with $\mathbb{C}^N$, $N = h^0(kL)$. In fact the quotient of the volumes does not depend on how we choose to represent $H^0(kL)$.

LEMMA 33.

$$\frac{vol\mathcal{B}^2(\mu, k\varphi)}{vol\mathcal{B}^2(\nu, k\psi)} = \frac{det(\int s_i \bar{s}_j e^{-k\psi} d\nu)_{ij}}{det(\int s_i \bar{s}_j e^{-k\varphi} d\mu)_{ij}}, \tag{1.15}$$

where $\{s_i\}$ *is any basis for* $H^0(kL)$.

Proof. First we show that the right hand side does not depend on the basis. Let $\{t_i\}$ be some orthonormal basis with respect to $\int |.|^2 e^{-k\psi} d\nu$, and let $A = (a_{ij})$ be the matrix such that

$$s_i = \sum a_{ij} t_j.$$

Then we see that

$$\int s_i \bar{s}_j e^{-k\psi} d\nu = \int (\sum a_{ik} t_k)(\overline{\sum a_{jl} t_l}) e^{-k\psi} d\nu = \sum a_{ik} \bar{a}_{jk}. \tag{1.16}$$

Therefore by linear algebra we get that

$$\det\left(\int s_i \bar{s}_j e^{-k\psi} d\nu\right)_{ij} = \det(AA^*) = |\det A|^2. \tag{1.17}$$

If we let $\{s'_i\}$ be a new basis,

$$s'_i = \sum b_{ij} s_j, \qquad B = (b_{ij}),$$

then

$$\det\left(\int s'_i \bar{s}'_j e^{-k\psi} d\nu\right)_{ij} = |\det B|^2 \det\left(\int s_i \bar{s}_j e^{-k\psi} d\nu\right)_{ij}.$$

Since $|\det B|^2$ also will show up in the denominator, we see that the quotient does not depend on the choice of basis.

Let as above $\{t_i\}$ be an orthonormal basis with respect to $\int |.|^2 e^{-k\psi} d\nu$ and let $\{s_i\}$ be an orthonormal basis with respect to $\int |.|^2 e^{-k\varphi} d\mu$ and let

$$s_i = \sum a_{ij} t_j, \qquad A = (a_{ij}).$$

It is clear that

$$\frac{\mathrm{vol}\mathcal{B}^2(\mu, k\varphi)}{\mathrm{vol}\mathcal{B}^2(\nu, k\psi)} = |\mathrm{det}A|^2.$$

Note that the square in the right-hand side comes from the fact that we take the determinant of A as a complex matrix. By equations (1.16) and (1.17) we also have that

$$\det \left(\int s_i \bar{s}_j e^{-k\psi} d\nu \right)_{ij} = |\mathrm{det}A|^2,$$

and since $\{s_i\}$ were chosen to be orthonormal

$$\det \left(\int s_i \bar{s}_j e^{-k\varphi} d\mu \right)_{ij} = 1.$$

The lemma follows. □

DEFINITION 13. *Let (φ, μ) and (ψ, ν) be two Bernstein-Markov pairs on (X, L). The Donaldson $\mathcal{L}_k$ bifunctional on (φ, ψ) is defined as*

$$\mathcal{L}_k(\varphi, \psi) := \frac{n!}{2k^{n+1}} \ln \left(\frac{vol\mathcal{B}^2(\mu, k\varphi)}{vol\mathcal{B}^2(\nu, k\psi)} \right).$$

Theorem A in [1] states that for Bernstein-Markov pairs the Donaldson $\mathcal{L}_k$ bifunctional converges to the Monge-Ampère energy.

THEOREM 34. *Let (φ, μ) and (ψ, ν) be two Bernstein-Markov pairs on (X, L). Then it holds that*

$$\lim_{k \to \infty} \mathcal{L}_k(\varphi, \psi) = \mathcal{E}(\varphi, \psi).$$

We will use this result to prove our main result, Theorem 27, stating that the Monge-Ampère energy of two continuous weights is equal to the integral of the difference of the respective Chebyshev transforms over the Okounkov body.

1.8.2 Proof of Theorem 27

Proof. We let $\{s_i\}$ be a basis for $H^0(kL)$ such that

$$s_i = z^{k\alpha_i} + \text{higher order terms},$$

where $\alpha_i \in \Delta_k(L)$ is some ordering of $\Delta_k(L)$. Let

$$s_i = \sum a_{ij} t^{\psi}_{\alpha_j,k}, \qquad A = (a_{ij}).$$

From the proof of Lemma 33 we see that

$$\det\left(\int_X s_i \bar{s}_j e^{-k\psi} d\nu\right)_{ij} = |\det A|^2 \det\left(\int_X t^{\psi}_{\alpha_i,k} \bar{t}^{\psi}_{\alpha_j,k} e^{-k\psi} d\nu\right)_{ij} = \\ = |\det A|^2 \prod_{\alpha\in\Delta_k(L)} ||t^{\psi}_{\alpha,k}||^2,$$

since $t^{\psi}_{\alpha,k}$ constitute an orthogonal basis. Also since the lowest term of s_i is $z^{k\alpha_i}$ we must have that $a_{ij} = 0$ for $j < i$ and $a_{ii} = 1$. Hence $\det A = 1$, and consequently

$$\det\left(\int_X s_i \bar{s}_j e^{-k\psi} d\nu\right)_{ij} = \prod_{\alpha\in\Delta_k(L)} ||t^{\psi}_{\alpha,k}||^2.$$

From equation (1.15) we get that

$$\mathcal{L}_k(\varphi,\psi) = \frac{n!}{k^n} \sum_{\alpha\in\Delta_k(L)} (c_k[\psi](\alpha) - c_k[\varphi](\alpha)).$$

For all k let $\tilde{c}_k[\psi]$ denote the function on $\Delta(L)^\circ$ assuming the value of $c_k[\psi]$ in the nearest lattice point of $\Delta_k(L)$ (or the mean of the values if there are multiple lattice points at equal distance). Then

$$\frac{n!}{k^n} \sum_{\alpha\in\Delta_k(L)} (c_k[\psi](\alpha) - c_k[\varphi](\alpha)) = n! \int_{\Delta(L)^\circ} (\tilde{c}_k[\psi] - \tilde{c}_k[\varphi]) d\lambda + \epsilon(k),$$

where the error term $\epsilon(k)$ goes to zero as k tends to infinity since by Khovanskii's theorem we have that $\Delta_k(L)$ fills out more and more of $\Delta(L)^\circ \cap$

$((1/k)\mathbb{Z})^n$. By Propositions 29 and 32 we can thus use dominated convergence to conclude that

$$\lim_{k\to\infty} \mathcal{L}_k(\varphi,\psi) = n! \int_{\Delta(L)^\circ} (c[\psi] - c[\varphi]) d\lambda.$$

Combined with Theorem 34 this proves the theorem. □

1.9 Previous results

Some instances of formula (1.10) are previously known. Here follows three such instances.

1.9.1 The volume as a Monge-Ampère energy

We consider the case where we let $\varphi = \psi + 1$. It is easy to see that this means that $P(\varphi) - P(\psi) = 1$, thus

$$\mathcal{E}(\varphi,\psi) = \frac{1}{n+1}\int_\Omega \mathrm{MA}_n(P(\varphi), P(\psi)). \tag{1.18}$$

Furthermore it has been shown by Berman-Boucksom (see e.g. [1]) that for any n-tuple of psh weights ψ_i with minimal singularities it holds that

$$\int_\Omega dd^c\psi_1 \wedge ... \wedge dd^c\psi_n = \mathrm{vol}(L), \tag{1.19}$$

where Ω denotes the dense Zariski-open set where the weights ψ_i are all locally bounded. Equations (1.18) and (1.19) together yields that

$$\mathcal{E}(\varphi,\psi) = \mathrm{vol}(L). \tag{1.20}$$

Any minimizing section with respect to $\int |.|^2 e^{-k\psi}$ will also minimize the norm

$$\int |.|^2 e^{-k(\psi+1)} = \int |.|^2 e^{-k\varphi}.$$

It follows that $c[\psi] - c[\varphi]$ is identically one. Therefore

$$\int_{\Delta(L)^\circ} (c[\psi] - c[\varphi]) d\lambda = \mathrm{vol}_{\mathbb{R}^n}(\Delta(L)). \tag{1.21}$$

Equations (1.20) and (1.21) and Theorem 27 then gives us that

$$\text{vol}(L) = n!\text{vol}_{\mathbb{R}^n}(\Delta(L)).$$

We have thus recovered Theorem A in [9].

1.9.2 Chebyshev constants and the transfinite diameter

Let K be a regular compact set in $\mathbb{C}$. We let $||.||_K$ denote the norm which takes the supremum of the absolute value on K. Let P_k denote the space of polynomials in z with z^k as highest degree term. Let for any k

$$Y_k(K) := \inf\{||p||_K : p \in P_k\}.$$

One defines the Chebyshev constant $C(K)$ of K as the following limit

$$C(K) := \lim_{k\to\infty} (Y_k(K))^{1/k}.$$

Let $\{x_i\}_{i=1}^k$ be a set of k points in K. Let $d_k(\{x_i\})$ denote the product of their mutual distances, i.e.

$$d_k(\{x_i\}) := \prod_{i<j} |x_i - x_j|.$$

One calls the points $\{x_i\}$ Fekete points if among the set of k-tuples of points in K they maximize the function d_k. Define $T_k(K)$ as $d_k(\{x_i\})$ for any set of Fekete points $\{x_i\}_{i=1}^k$. Then the transfinite diameter $T(K)$ of K is defined as

$$T(K) := \lim_{k\to\infty} (T_k(K))^{1/\binom{k}{2}}.$$

We will now think of $\mathbb{C}$ as imbedded in the complex projective space $\mathbb{P}^1$. Let Z_0, Z_1 be a basis for $H^0(\mathcal{O}(1))$, therefore $[Z_0, Z_1]$ are homogeneous coordinates for $\mathbb{P}^1$. Let

$$z := \frac{Z_1}{Z_0} \qquad \text{and} \qquad w := \frac{Z_0}{Z_1}.$$

Let p denote the point at infinity

$$[0, 1].$$

Then w is a holomorphic coordinate around p, and Z_1 is a local trivialization of the line bundle $\mathcal{O}(1)$ around p. Thus we will identify a section $Z_0^\alpha Z_1^{k-\alpha} \in H^0(\mathcal{O}(k))$ with the polynomial w^α as well as with $z^{k-\alpha}$. This means that the Okounkov body $\Delta(\mathcal{O}(1))$ of $\mathcal{O}(1)$ is the unit interval $[0,1]$ in $\mathbb{R}$. We observe that a section $s \in H^0(\mathcal{O}(k))$ lies in P_i as a polynomial in z if and only if

$$s = w^{k-i} + \text{ higher order terms.}$$

For a section s let $\tilde{s}$ denote the corresponding polynomial in z. Consider the weight $P_K(\ln|Z_0|^2)$. It will be continuous since K is assumed to be regular (see e.g. [1]). Then we have the following lemma.

LEMMA 35. *For any* $\alpha \in [0,1]$, *i.e. that lies in the Okounkov body of* $\mathcal{O}(1)$, *we have that*

$$c[P_K(\ln|Z_0|^2)](\alpha) = 2(1-\alpha)\ln C(K).$$

Proof. By basic properties of the projection operator P_K (see [1]) it holds that for for any section $s \in H^0(\mathcal{O}(k))$

$$\sup_K\{|s|^2 e^{-k\ln|Z_0|^2}\} = \sup_{\mathbb{P}^1}\{|s|^2 e^{-kP_K(\ln|Z_0|^2)}\}. \tag{1.22}$$

Since the conversion to the z-variable means letting Z_0 be identically one, we also have that

$$\sup_K\{|s|^2 e^{-k\ln|Z_0|^2}\} = \sup_K\{|\tilde{s}|^2\} = ||\tilde{s}||_K^2. \tag{1.23}$$

We see that $s \in A_{\alpha,k}$ iff $\tilde{s} = z^{k-k\alpha} +$ lower order terms. Hence

$$F[P_K(\ln|Z_0|^2)](k\alpha, k) = 2\ln Y_{k\alpha-k}(K),$$

and

$$c[P_K(\ln|Z_0|^2)](\alpha) = \lim_{k\to\infty}\frac{F[P_K(\ln|Z_0|^2)](k\alpha,k)}{k} =$$
$$= \lim_{k\to\infty}\frac{2}{k}\ln Y_{k\alpha-k}(K) = \lim_{k\to\infty} 2(1-\alpha)\ln(Y_{k-k\alpha}(K))^{k-k\alpha} =$$
$$= 2(1-\alpha)\ln C(K).$$

∎

Let K and K' be two regular compact subsets of $\mathbb{C}$. From Theorem 27 and Lemma 35 we get that

$$\mathcal{E}(P_{K'}(\ln|Z_0|^2), P_K(\ln|Z_0|^2)) =$$

$$= \int_{(0,1)} (c[P_K(\ln|Z_0|^2)] - c[P_{K'}(\ln|Z_0|^2)])d\lambda(\alpha)$$

$$= \int_{(0,1)} (2(1-\alpha)\ln C(K) - 2(1-\alpha)\ln C(K'))\, d\lambda(\alpha) =$$

$$= \ln C(K) - \ln C(K').$$

On the other hand it follows from Corollary A in [1] that

$$\ln T(K) - \ln T(K') = \mathcal{E}(P_{K'}(\ln|Z_0|^2), P_K(\ln|Z_0|^2)). \tag{1.24}$$

Thus by Theorem 27, using Lemma 35 and equation (1.24) we get that

$$\ln T(K) - \ln T(K') = \ln C(K) - \ln C(K').$$

In fact it is easy to check that for the unit disc D, $T(D) = C(D) = 1$, so we recover the classical result in potential theory that the transfinite diameter $T(K)$ and the Chebyshev constant $C(K)$ are equal.

For a thorough exposition on the subject of the transfinite diameter and capacities of compacts in $\mathbb{C}$ we refer the reader to the book [13] by Saff-Totik.

1.9.3 Invariant weights on toric varieties

Let X be a smooth projective toric variety. We will view X as a compactified $(\mathbb{C}^*)^n$, such that the torus action on X via this identification corresponds to the usual torus action on $(\mathbb{C}^*)^n$. As is well-known, there is a polytope Δ naturally associated to the embedding $(\mathbb{C}^*)^n \subseteq X$. We assume that Δ lies in the non-negative orthant of $\mathbb{R}^n$. There is a line bundle L_Δ with a trivialization on $(\mathbb{C}^*)^n$ such that

$$\Delta_k(L_\Delta) = \Delta \cap (\frac{1}{k}\mathbb{Z})^n,$$

and any section $s \in H^0(kL_\Delta)$ can in fact be written as a linear combination of the monomials z^α where

$$\alpha \in k\Delta \cap \mathbb{Z}^n.$$

Let dV be a smooth volume form on X invariant under the torus action. Then it holds that for any torus invariant weight ψ,

$$\int_X z^\alpha \bar{z}^\beta e^{-k\psi} dV = 0$$

when $\alpha \neq \beta$. This follows from Fubini since trivially the monomials are orthogonal with respect to the Lebesgue measure on e.g. tori. Because of this for any torus invariant weight ψ the minimizing sections $t^\psi_{a,k}$ are given by $z^{k\alpha}$, and consequently

$$c_k[\psi, dV](\alpha) = \frac{1}{k} \ln \int_X |z^{k\alpha}|^2 e^{-k\psi} dV.$$

Assume for simplicity that ψ is positive.

LEMMA 36. *For any strictly positive torus invariant weight ψ we have that*

$$c[\psi](\alpha) = \ln \big(\sup_{z \in \mathbb{C}^n} \{|z^\alpha|^2 e^{-\psi(z)}\}\big).$$

Proof. We have that

$$\int_X |z^{k\alpha}|^2 e^{-k\psi} dV \leq dV(X) \sup_X \{|z^{k\alpha}|^2 e^{-k\psi}\} =$$
$$= dV(X)\big(\sup_{z \in X} \{|z^\alpha|^2 e^{-\psi(z)}\}\big)^k,$$

which yieds the inequality

$$c[\psi](\alpha) \leq \ln \big(\sup_{z \in X} \{|z^\alpha|^2 e^{-\psi(z)}\}\big).$$

By the Bernstein-Markov property of dV with respect to ψ we get that

$$\int_X |z^{k\alpha}|^2 e^{-k\psi} dV \geq Ce^{-\varepsilon k} \sup_{z \in X} \{|z^{k\alpha}|^2 e^{-k\psi(z)}\} =$$
$$= Ce^{-\varepsilon k}\big(\sup_{z \in X} \{|z^\alpha|^2 e^{-\psi(z)}\}\big)^k.$$

Using Proposition 29 it follows from this that

$$c[\psi](\alpha) = \ln \big(\sup_{z \in X} \{|z^\alpha|^2 e^{-\psi}\}\big).$$

Since ψ is a weight on L_Δ it obeys certain growth conditions in $\mathbb{C}^n$. In fact for α lying in the interior of $\Delta = \Delta(L_\Delta)$ it holds that

$$\sup_X \{|z^\alpha|^2 e^{-\psi(z)}\} = \sup_{z\in\mathbb{C}^n} \{|z^\alpha|^2 e^{-\psi(z)}\},$$

and the lemma follows. □

Remark. If we do not assume that the weight ψ is strictly positive, the lemma still holds if we in the supremum replace ψ with the projection $P(\psi)$.

Let Θ denote the map from $(\mathbb{C}^*)^n$ to $\mathbb{R}^n$ that maps z to $(\ln|z_1|, ..., \ln|z_n|)$. Since we assumed ψ to be torus invariant, the function $\psi \circ \Theta^{-1}$ is well-defined on $\mathbb{R}^n$. We will denote $\psi\circ\Theta^{-1}$ by ψ_Θ. Since ψ was assumed to be psh, it follows that ψ_Θ will be convex on $\mathbb{R}^n$. Recall the definition of the Legendre transform. Given a convex function g on $\mathbb{R}^n$ the Legendre transform of g, denoted g^*, evaluated in a point $p \in \mathbb{R}^n$ is given by

$$g^*(p) := \sup_{x\in\mathbb{R}^n} \{\langle p, x\rangle - g(x)\}.$$

Observe that

$$\ln\left((|z^\alpha|^2 e^{-\psi}) \circ \Theta^{-1}(x)\right) = 2\langle\alpha, x\rangle - \psi_\Theta(x). \tag{1.25}$$

Thus by equation (1.25) and Lemma 36 we get that

$$c[\psi](\alpha) = 2\left(\frac{\psi_\Theta}{2}\right)^*(\alpha).$$

The function on the right is called the symplectic potential of ψ, denoted by u_ψ. Theorem (27) now gives us that for any two invariant weights ψ and φ on L it holds that

$$\mathcal{E}(\psi, \varphi) = n! \int_{\Delta^\circ} (u_\varphi - u_\psi) d\lambda,$$

which is well-known in toric geometry. In fact this can be derived from the fact that the real Monge-Ampère measure of a convex function is the pullback of the Lebegue measure with respect to the gradient of the convex function times a constant.

1.10 The Chebyshev transform on the zero-fiber

Let us assume that

$$z_1 = 0$$

is a local equation around p for an irreducible variety which we denote by Y. Let $H^0(X|Y, kL)$ denote the image of the restriction map from $H^0(X, kL)$ to $H^0(Y, kL_{|Y})$, and let $\Gamma(X|Y, L)$ denote the semigroup

$$\cup_{k\geq 0}\left(v(H^0(X|Y, kL)) \times \{k\}\right) \subset \mathbb{N}^n.$$

Note that since $z_2, ..., z_n$ are local coordinates on Y, $v(H^0(X|Y, kL)$ will be a set of vectors in $\mathbb{N}^{n-1}$.

DEFINITION 14. *The restricted Okounkov body $\Delta_{X|Y}(L)$ is defined as the Okounkov body of the semigroup $\Gamma(X|Y, L)$.*

LEMMA 37. *If Y is not contained in the augmented base locus $B_+(L)$, then $\Gamma(X|Y, L)$ generates $\mathbb{Z}^n$ as a group.*

This is part of Lemma 2.16 in [9].

Remark. The augmented base locus $B_+(L)$ of L is defined as the base locus of any sufficiently small perturbation $L - \varepsilon A$, where A is some ample line bundle. Here we are only interested in the case where L is ample, and then it is easy to see that the augmented base locus $B_+(L)$ always is empty.

Assume now that L is ample. We will show that the Chebyshev transform $c[\psi]$ can be defined not only in the interior of the Okounkov body but also on the zero fiber,

$$\Delta(L)_0 := \Delta(L) \cap \left(\{0\} \times \mathbb{R}^{n-1}\right).$$

From Theorem 4.24 in [9] we get the following fact,

$$\Delta(L)_0 = \Delta_{X|Y}(L). \tag{1.26}$$

Note that since the Okounkov body lies in the positive orthant of $\mathbb{R}^n$, $\Delta(L)_0$ is a part of the boundary of $\Delta(L)$, hence the Chebyshev transform of a continuous weight is a priori not defined on the zero-fiber. Nevertheless, we want to show that one can extend the Chebyshev transform to the interior of zero-fiber $\Delta(L)_0$. To do this, we need to know how Γ behaves near this boundary, something which Theorem 5 does not tell us anything about.

LEMMA 38. *Assume L to be ample, and p any point in the interior of $\Delta(L)_0$. Let $\Sigma^{\mathbb{Z}}_{n+1}$ denote the unit simplex in $\mathbb{Z}^{n+1}$, $\Sigma^{\mathbb{R}}_{n-1}$ the unit simplex in $\mathbb{R}^{n-1}$, and let S denote the simplex $\{0\} \times \Sigma^{\mathbb{R}}_{n-1} \times \{0\}$. Then $\Gamma(L)$ contains a translated unit simplex $(\alpha, k) + \Sigma_{n+1}$ such that (kp, k) lies in the interior of the $(n-1)$-simplex*

$$(\alpha, k) + S$$

(i.e interior with respect to the $\mathbb{R}^{n-1}$ topology).

Proof. The augmented base locus of L is empty since L is ample, thus by Lemma 37 we may use Lemma 6 in combination with equation (1.26) to reach the conclusion that for large k, there are sections s_k such that (p, k) lies in the interior of $(v(s_k), k) + S$ with respect to the $\mathbb{R}^{n-1}$ topology. We may write L as a difference of two very ample divisors A and B. We may choose B such that $\Delta_1(B)$ contains Σ_n in $\mathbb{Z}^n$, and A such that $\Delta_1(A)$ contains origo. Now

$$kL = B + (kL - B).$$

Since L is ample, for k large we can find sections $s'_k \in H^0(kL - B)$ such that $v(s'_k) = v(s_k)$. We get that

$$(v(s_k), k) + \Sigma_n \subseteq \Gamma(L),$$

by multiplying s'_k by the sections of B corresponding to the points in the unit simplex $\Sigma_n \subseteq \Delta_1(B)$. Also observe that

$$(k + 1)L = A + (kL - B).$$

Now by multiplying s'_k with the section of A corresponding to origo in $\Delta_1(A)$ we get

$$(v(s'_k), k) + (0, ..., 0, 1) \subseteq \Gamma(L).$$

Since

$$\Sigma_n \times \{0\} \cup (0, ..., 0, 1) = \Sigma_{n+1}$$

we get

$$(v(s'_k), k) + \Sigma_{n+1} \subseteq \Gamma(L).$$

□

Remark. The proof is very close to the argument in [9] which shows the existence of a unit simplex in $\Gamma(L)$, when L is big. The difference here is that we need to control the position of the unit simplex, but the main trick of writing L as a difference of two very ample divisors is the same.

LEMMA 39. *Let p be as in the statement of Lemma 38. Then there exists a neighbourhood U of p such that if we denote the intersection $U \cap \Delta(L)$ by $\tilde{U}$, for k large it holds that*

$$(k\tilde{U}, k) \cap \mathbb{Z}^{n+1} \subseteq \Gamma(L).$$

Proof. Let $(\alpha, m) + \Sigma^{\mathbb{Z}}_{n+1} \subseteq \Gamma(L)$ be as in the statement of Lemma 38, and let $D^{\mathbb{Z}} \subseteq \Gamma(L)$ denote the set

$$D^{\mathbb{Z}} := (\alpha, m) + \Sigma^{\mathbb{Z}}_n \times \{0\} = (\alpha + \Sigma^{\mathbb{Z}}_n) \times \{m\}.$$

Let also $D^{\mathbb{R}}$ denote the set

$$D^{\mathbb{R}} := (\alpha + \Sigma^{\mathbb{R}}_n) \times \{m\}.$$

Since trivially

$$\underbrace{\Sigma^{\mathbb{Z}}_n + ... + \Sigma^{\mathbb{Z}}_n}_{k} = (k\Sigma^{\mathbb{R}}_n) \cap \mathbb{Z}^n,$$

we have that

$$(kD^{\mathbb{R}}, km) \cap \mathbb{Z}^{n+1} = \underbrace{D^{\mathbb{Z}} + ... + D^{\mathbb{Z}}}_{k} \subseteq \Gamma(L).$$

Therefore the lemma holds when k is a multiple of m. Furthermore, since m and $m+1$ are relatively prime, if k is greater than $m(m+1)$ we can write

$$k = k_1 m + k_2(m+1),$$

where both k_1 and k_2 are non-negative, and $k_2 \leq m$. Thus we consider the set

$$\underbrace{D^{\mathbb{Z}} + ... + D^{\mathbb{Z}}}_{k_1} + k_2(\alpha, m+1) \subseteq \Gamma(L).$$

Because of the bound $k_2 \leq m$, and since $(\alpha, m+1)$ lies on the zero fiber, for a neighbourhood $\tilde{U}$ of p, when k gets large we must have that

$$(k\tilde{U}, k) \cap \mathbb{Z}^{n+1} \subseteq \underbrace{D^{\mathbb{Z}} + ... + D^{\mathbb{Z}}}_{k_1} + k_2(\alpha, m+1) \subseteq \Gamma(L).$$

□

COROLLARY 40. *Assume L is ample, then the chebyshev function $c[\psi]$ is well-defined on the interior of the zero-fiber, $\Delta(L)_0$, and it is continuous and convex on its extended domain $\Delta(L)^\circ \cup \Delta(L)_0^\circ$.*

Proof. The proof goes exactly as for the case of an interior point, now using Lemma 39 instead of Theorem 5. □

LEMMA 41. *Assume L is ample, and ψ is a continuous weight. Then for any regular compact set K it holds that the projection $P_K(\psi)$ also is continuous. In particular, since X is regular, $P(\psi)$ is continuous when L is ample.*

Proof. See e.g. [1]. □

We will have use for the Ohsawa-Takegoshi extension theorem. We choose to record one version (see e.g. [6]).

THEOREM 42. *Let L be a holomorphic line bundle and let S be a divisor. Assume that L and S have metrics Ψ_L and Ψ_S respectively satisfying*

$$dd^c\Psi_L \geq (1+\delta)dd^c\Psi_S + dd^c\Psi_{K_X},$$

where Ψ_{K_X} is some smooth metric on the canonical bundle K_X. Assume also that

$$dd^c\Psi_L \geq dd^c(\Psi_S + \Psi_{K_X}).$$

Then any holomorphic section $\tilde{t}$ of the restriction of L to S extends holomorphically to a section t of L over X satisfying

$$\int_X |t|^2 e^{-\Psi_L} \omega_n \leq C_\delta \int_S |\tilde{t}|^2 e^{-\Psi_L} \frac{dS}{|ds|^2 e^{-\Psi_S}}.$$

Here ω_n is a smooth volume form on X and dS is a smooth volume form on S.

LEMMA 43. *Suppose L is ample. Let A be an ample line bundle, with a holomorphic section s such that locally $s = z_1$. Also assume that the zero-set of s, which we will denote by Y, is a smooth submanifold. Then for all $\alpha \in \Delta_{X|Y}(L)$ we have that*

$$c_X[\varphi](0, \alpha) = c_Y[P(\varphi)_{|Y}](\alpha). \tag{1.27}$$

Proof. We may choose $\tilde{z}_1 = z_2, ..., \tilde{z}_{n-1} = z_n$ as holomorphic coordinates on Y around p. We consider the discrete Chebyshev transforms of the restrictions of $P(\varphi)$ and $P(\psi)$ to Y. Since L is ample, by Lemma 41 $P(\varphi)$ and $P(\psi)$ are continuous, therefore the restrictions will also be continuous psh-weights on $L_{|Y}$, therefore the Chebyshev transforms $c_Y[P(\varphi)_{|Y}]$ and $c_Y[P(\psi)_{|Y}]$ are well-defined.

We note that if $t \in H^0(X, kL)$ and

$$t = z^{k(0,\alpha)} + \text{higher order terms},$$

the restriction of t to Y will be given by

$$t_{|Y} = \tilde{z}^{k\alpha} + \text{higher order terms}.$$

Furthermore

$$\sup_Y \{|t_{|Y}|^2 e^{-kP(\varphi)}\} \leq \sup_X \{|t|^2 e^{-kP(\varphi)}\}.$$

This gives the inequality

$$c_X[\varphi](0, \alpha) \geq c_Y[P(\varphi)_{|Y}](\alpha),$$

by taking t to be some minimizing section with respect to the supremum norm on X.

For the opposite inequality we use Proposition 29 which says that one can use Bernstein-Markov norms to compute the Chebyshev transform.

If $\tilde{t} \in H^0(Y, kL_{|Y})$,

$$\tilde{t} = \tilde{z}^{k\alpha} + \text{higher order terms},$$

then if k is large enough there exists a section $t \in H^0(X, kL)$ such that $t_{|Y} = \tilde{t}$. This is because we assumed L to be ample, so we have extension properties (by e.g. Ohsawa-Takegoshi). We observe that any such extension must look like

$$t = z^{k(0,\alpha)} + \text{higher order terms},$$

because if we had that

$$t = z^{k(\beta_1,\beta)} + \text{higher order terms}$$

with $\beta_1 > 0$, then since all higher order terms also restrict to zero,

$$t_Y = 0,$$

which is a contradiction.

Let Ψ be some smooth strictly positive weight on L. Then for some m

$$dd^c m\Psi > (1+\delta)dd^c\Psi_A + dd^c\Psi_{K_X}$$

and

$$dd^c m\Psi > dd^c\Psi_A + dd^c\Psi_{K_X},$$

where Ψ_A and Ψ_{K_X} are weights on A and K_X respectively. We have that $dd^c P(\varphi) \geq 0$, hence

$$dd^c((k-m)P(\varphi) + m\Psi) > (1+\delta)dd^c\Psi_A + dd^c\Psi_{K_X}$$

and

$$dd^c((k-m)P(\varphi) + m\Psi) > dd^c\Psi_A + dd^c\Psi_{K_X}$$

for all $k > m$. Since $P(\varphi)$ is continuous hence locally bounded, we also have that for some constant C,

$$\Psi - C < P(\varphi) < \Psi + C.$$

We can apply Theorem 42 to these weights, and get that for large k, given a $\tilde{t} \in H^0(Y, kL_{|Y})$ there exists an extension $t \in H^0(X, kL)$ such that

$$\int_X |t|^2 e^{-kP(\varphi)}\omega_n \leq e^{mC}\int_X |t|^2 e^{-(k-m)P(\varphi)-m\Psi}d\mu$$
$$\leq e^{mC}C_\delta \int_Y |\tilde{t}|^2 e^{-(k-m)P(\varphi)-m\Psi}d\nu \leq e^{2mC}C_\delta \int_Y |\tilde{t}|^2 e^{-kP(\varphi)}d\nu,$$

where C_δ is constant only depending on δ and $d\nu$ is a smooth volume form on Y. By letting $\tilde{t}$ be the minimizing section with respect to $\int_Y |.|^2 e^{-kP(\varphi)}d\nu$ and using Proposition 29 we get that

$$c_X[\varphi](0,\alpha) \leq c_Y[P(\varphi)_{|Y}](\alpha),$$

since

$$\int_X |t|^2 e^{-k\varphi}\omega_n \leq \int_X |t|^2 e^{-kP(\varphi)}\omega_n.$$

□

PROPOSITION 44. *Let L, A and Y be as in the statement of Lemma 43. Then we have that*

$$\mathcal{E}_Y(P(\varphi)_{|Y}, P(\psi)_{|Y}) = (n-1)!\int_{\Delta(L)_0}(c[\psi]-c[\varphi])(0,\alpha)d\alpha.$$

Proof. The proposition follows from Lemma 43 by integration of equality (1.27) over the interior of the zero-fiber, and Theorem 27 which says that

$$\mathcal{E}_Y(P(\varphi)_{|Y}, P(\psi)_{|Y}) = (n-1)!\int_{\Delta(L_{|Y})} c_Y[P(\psi)_{|Y}] - c_Y[P(\varphi)_{|Y}]d\lambda.$$

□

We will cite Proposition 4.7 from [1] which is a recursion formula relating the Monge-Ampère energy and the restricted energy.

PROPOSITION 45. *Suppose L is ample, let $s \in H^0(L)$, and let Y be the smooth submanifold defined by s. Let ψ and φ be two continuous weights. Then*

$$(n+1)\mathcal{E}_X(\psi,\varphi) - n\mathcal{E}_Y(P(\psi)_{|Y}, P(\varphi)_{|Y}) =$$
$$= \int_X (\ln|s|^2 - P(\varphi))MA(P(\varphi)) - \int_X (\ln|s|^2 - P(\psi))MA(P(\psi)).$$

In particular, combining Theorem 27, Proposition 44 and Proposition 45 we get the following.

PROPOSITION 46. *Let L, s and Y be as in Proposition 45. Then it holds that*

$$\int_{\Delta(L)^\circ}(c_X[\varphi]-c_X[\psi])d\lambda_n = \frac{1}{n+1}\int_{\Delta(L)_0^\circ}(c_X[\varphi]-c_X[\psi])d\lambda_{n-1} + \\ +\frac{1}{(n+1)!}\int_X(\ln|s|^2 - P(\varphi))MA(P(\varphi)) - \\ -\frac{1}{(n+1)!}\int_X(\ln|s|^2 - P(\psi))MA(P(\psi)).$$

1.11 Directional Chebyshev constants in $\mathbb{C}^n$

In [3] Bloom-Levenberg define the weighted version of the directional Chebyshev constants originally introduced by Zaharjuta in [14]. In this section we will describe how this relates to the Chebyshev transforms we have introduced.

The setting in [3] is as follows. Let $<_1$ be the order on $\mathbb{N}^n$ such that $\alpha <_1 \beta$ if $|\alpha| < |\beta|$, or if $|\alpha| = |\beta|$ and $\alpha <_{\text{lex}} \beta$. Let P_α denote the set of polynomials $p(z_1, ..., z_n)$ in the variables z_i such that

$$p = z^\alpha + \text{ lower order terms.}$$

Observe that here we want lower order terms, and not higher order terms. Let K be a compact set and h an admissible weight function on K. For any $\alpha \in \mathbb{N}^n$ they define the weighted Chebyshev constant $Y_3(\alpha)$ as

$$Y_3(\alpha) := \inf\{\sup_{z\in K}\{|h(z)^{|\alpha|}p(z)|\} : p \in P_\alpha\}.$$

They then show that the limit

$$\tau^h(K,\theta) := \lim_{\alpha/\deg(\alpha)\to\theta} Y_3(\alpha)^{1/\deg(\alpha)}$$

exists. These limits are called directional Chebyshev constants.

In our setting we wish to view $\mathbb{C}^n$ as an affine space lying in $\mathbb{P}^n$. Also, polynomials in z_i can be interpreted as sections of multiples of the line bundle

$\mathcal{O}(1)$ on $\mathbb{P}^n$ in the following sense. Let $Z_0, ..., Z_n$ be a basis for $H^0(\mathcal{O}(1))$ on $\mathbb{P}^n$, and identify them with the homogeneous coordinates $[Z_0, ..., Z_n]$. We can choose

$$p := [1 : 0 : ... : 0]$$

to be our base point, and let $z_i := \frac{Z_i}{Z_0}$ be holomorphic coordinates around p. We also let Z_0 be our local trivialization of the bundle. Given a section $s \in H^0(\mathcal{O}(k))$we represent it as a function in z_i by dividing by a power of Z_0

$$\frac{s}{Z_0^k} = \sum a_\alpha z^\alpha.$$

Therefore we see that

$$Z^{(\alpha_0,\alpha_1,\ldots,\alpha_n)} \mapsto z^{(\alpha_1,\ldots,\alpha_n)}.$$

We could also choose a different set of coordinates. Let

$$q := [0 : ... : 0 : 1]$$

be our new base point, and let $w_i := \frac{Z_i}{Z_n}$ be coordinates around q. Let Z_n be the local trivialization around q. Given a section $s \in H^0(\mathcal{O}(k))$ we represent it as a function in w_i by dividing by a power of Z_n

$$\frac{s}{Z_n^k} = \sum b_\alpha w^\alpha.$$

Hence

$$Z^{(\alpha_0,\alpha_1,\ldots,\alpha_n)} \mapsto w^{(\alpha_0,\ldots,\alpha_{n-1})}.$$

To define Chebyshev transforms we need an additive order on $\mathbb{N}^n$. Since the semigroup $\Gamma(\mathcal{O}(1))$ will not depend on the order, we are free to choose any additive order. Let $<_2$ be the order which corresponds to inverting the order $<_1$ with respect to the z_i variables, i.e.

$$(\alpha_0, ..., \alpha_{n-1}) <_2 (\beta_0, ..., \beta_{n-1})$$

iff

$$(\beta_1, ..., \beta_n) <_1 (\alpha_1, ..., \alpha_n).$$

Therefore

$$z^{(\alpha_1,\ldots,\alpha_n)} + \text{ lower order terms} = w^{(\alpha_0,\ldots,\alpha_{n-1})} + \text{ higher order terms}. \quad (1.28)$$

We may identify the weight function h with a metric $h = e^{-\psi/2}$ on $\mathcal{O}(1)$. Consider the weight $P_K(\psi)$. For simplicity assume that K is regular. Since $\mathcal{O}(1)$ is ample from Lemma 41 it follows that $P_K(\psi)$ is continuous, therefore the Chebyshev transform $c[P_K(\psi)]$ is well-defined. It is a simple fact that

$$\sup_{z\in K}\{|s(z)|^2 e^{-k\psi(z)}\} = \sup_{z\in\mathbb{P}^n}\{|s(z)|^2 e^{-kP_K(\psi)(z)}\}. \quad (1.29)$$

Let $\alpha_0 = 0$, and let $k = \sum_1^n \alpha_i$. By (1.28) we see that $s \in A_{(\alpha_0,\ldots,\alpha_{n-1}),k}$ iff it is on the form

$$z^{(\alpha_1,\ldots,\alpha_n)} + \text{ lower order terms}.$$

By (1.29) it follows that

$$\ln Y_3(\alpha_1, ..., \alpha_n) = F[P_K(\psi)](k\alpha, k).$$

Thus we get that for $\theta = (\theta_1, ..., \theta_n) \in \Sigma^0$

$$c[P_K(\psi)](0, \theta_1, ..., \theta_{n-1}) = 2\ln \tau^h(\theta_1, ..., \theta_n). \quad (1.30)$$

Observe that the order $<_2$ we used to defined the Chebyshev transform has the property that $(0, \alpha) <_2 (\beta_1, \beta)$ when $\beta_1 > 0$. It was this property of the lexicographic order we used in the proof of Proposition 44. Therefore the theorem holds also for Chebyshev transforms defined using $<_2$ instead of $<_{\text{lex}}$. Let (K', h') be another weighted set in $\mathbb{C}^n$, and let ψ' be the corresponding weight on $\mathcal{O}(1)$ associated to h'. Then integrating (1.30) gives us that

$$\frac{1}{\text{meas}(\Sigma^0)}\int_{\Sigma^0} \ln\tau^h(K,\theta) - \ln\tau^{h'}(K',\theta)d\theta =$$
$$= \frac{(n-1)!}{2}\int_{\Delta(\mathcal{O}(1))_0} c[P_K(\psi)] - c[P_{K'}(\psi')]d\theta, \quad (1.31)$$

where $Y := \{Z_0 = 0\}$. Here we used that $\Delta(\mathcal{O}(1))_0$ is a $(n-1)$-dimensional unit simplex, and thus

$$\text{meas}(\Delta(\mathcal{O}(1))_0) = \frac{1}{(n-1)!}.$$

Bloom-Levenberg define a weighted transfinite diameter $d^h(K)$ of K which is given by

$$d^h(K) := \exp\left(\frac{1}{\text{meas}(\Sigma^0)}\int_{\Sigma^0} \ln \tau^h(K,\theta)d\theta\right).$$

There is also another transfinite diameter, $\delta^h(K)$, which is defined as a limit of certain Vandermonde determinants. By Corollary A in [1] we have that

$$\ln \delta^h(K) - \ln \delta^{h'}(K') = \frac{(n+1)}{2n}\mathcal{E}(P_{K'}(\psi'), P_K(\psi)).$$

Then by Theorem 27, equation (1.31) and Proposition 46 we get that

$$\ln \delta^h(K) - \ln \delta^{h'}(K') =$$
$$= \ln d^h(K) - \ln d^{h'}(K') + \frac{1}{n}\int_{\mathbb{P}^n} \frac{1}{2}(\ln|Z_0|^2 - P_K(\psi))\text{MA}(P_K(\psi)) -$$
$$-\frac{1}{n}\int_{\mathbb{P}^n} \frac{1}{2}(\ln|Z_0|^2 - P_{K'}(\psi'))\text{MA}(P_{K'}(\psi')).$$

In fact, the positive measure $\text{MA}(P_K(\psi))$ has support on K, and $P_K(\psi) = \psi$ a.e. with respect to $\text{MA}(P_K(\psi))$. In the notation of [3], $(\psi - \ln|Z_0|^2)/2$ is denoted Q, and $\text{MA}(P_K(\psi))$ is denoted $(dd^cV^*_{K,Q})^n$. Thus in their notation

$$\ln \delta^h(K) - \ln \delta^{h'}(K') =$$
$$= \ln d^h(K) - \ln d^{h'}(K') - \frac{1}{n}\int_K Q(dd^cV^*_{K,Q})^n + \frac{1}{n}\int_{K'} Q'(dd^cV^*_{K',Q'})^n.$$

For the unit ball B, with $h \equiv 1 \equiv |Z_0|^2$ and therefore $Q_h = 0$, it is straightforward to show that we have an equality

$$\delta^h(B) = d^h(K).$$

Using this we get that

$$\ln \delta^h(K) = \ln d^h(K) - \frac{1}{n}\int_K Q(dd^cV^*_{K,Q})^n.$$

By taking the exponential we have derived the formula of Theorem 2.9 in [3].

1.12 Chebyshev transforms of weighted $\mathbb{Q}$- and $\mathbb{R}$-divisors

Because of the homogeneity of Okounkov bodies, one may define the Okounkov body $\Delta(D)$ of any big $\mathbb{Q}$-divisor D. Set

$$\Delta(D) := \frac{1}{p}\Delta(pD)$$

for any p that clears the denominators in D. In [9] Lazarsfeld-Mustaţă show that this mapping of a $\mathbb{Q}$-divisor to its Okounkov body has a continuous extension to the class of $\mathbb{R}$-divisors.

In Proposition 26 we saw that Chebyshev transforms also are homogeneous under scaling. Therefore we may define the Chebyshev transform of a $\mathbb{Q}$-divisor D with weight ψ, by letting

$$c[\psi](\alpha) = \frac{1}{p}c[p\psi](p\alpha), \qquad \alpha \in \Delta(D)^{\circ}, \tag{1.32}$$

for any p clearing the denominators in D. We wish to show that this can be extended continuously to the class of weighted $\mathbb{R}$-divisors.

We will use the construction introduced in [9]. Let $D_1, ..., D_r$ be divisors such that every divisor is numerically equivalent to a unique sum

$$\sum a_i D_i, \qquad a_i \in \mathbb{Z}.$$

Lazarsfeld-Mustaţă show that for effective divisors the coefficients a_i may be chosen non-negative.

DEFINITION 15. *The semigroup of* X, $\Gamma(X)$, *is defined as*

$$\Gamma(X) := \bigcup_{a \in \mathbb{N}^r} \Big(v(H^0(\mathcal{O}_X(\sum a_i D_i))) \times \{a\} \Big) \subseteq \mathbb{Z}^{n+r},$$

where v *stands for the usual valuation,*

$$s = z^{\alpha} + \text{ higher order terms} \qquad \Rightarrow \qquad s \mapsto \alpha.$$

Lazarsfeld-Mustaţă show in [9] that $\Gamma(X)$ generates $\mathbb{Z}^{n+r}$ as a group.

Let $\Sigma(\Gamma(X))$ denote the closed convex cone spanned by $\Gamma(X)$, and let for $a \in \mathbb{N}^r$

$$\Delta(a) := \Sigma(\Gamma(X)) \cap (\mathbb{R}^n \times \{a\}).$$

By [9] for any big $\mathbb{Q}$-divisor $D = \sum a_i D_i$,

$$\Delta(a) = \Delta(D), \qquad a = (a_1, ..., a_r).$$

Let for each $1 \leq i \leq r$ ψ_i be a continuous weights on D_i. Then for $a \in \mathbb{N}^r$, $\sum a_i \psi$ is a continuous weight on $\sum a_i D_i$. For an element $(\alpha, a) \in \Gamma(X)$, let $A_{\alpha,a} \subseteq H^0(\sum a_i D_i)$ be the set of sections of the form

$$z^\alpha + \text{ higher order terms.}$$

DEFINITION 16. *The discrete global Chebyshev transform $F[\psi_1, ..., \psi_r]$ is defined by*

$$F[\psi_1, ..., \psi_r](\alpha, a) := \inf\{\ln ||s||^2_{\alpha,a} : s \in A_{\alpha,a}\}$$

for $(\alpha, a) \in \Gamma(X)$.

LEMMA 47. *$F[\psi_1, ..., \psi_r]$ is subadditive on $\Gamma(X)$.*

Proof. If $s \in H^0(\mathcal{O}_X(\sum a_i D_i))$,

$$s = z^\alpha + \text{ higher order terms,}$$

and $t \in H^0(\mathcal{O}_X(\sum b_i D_i))$,

$$t = z^\beta + \text{ higher order terms,}$$

then $st \in H^0(\mathcal{O}_X(\sum (a_i + b_i) D_i))$ and

$$st = z^{\alpha+\beta} + \text{ higher order terms.}$$

Thus the subadditivity of $F[\psi_1, ..., \psi_r]$ follows exactly as for $F[\psi]$ in Lemma 20. $\square$

LEMMA 48. *$F[\psi_1, ..., \psi_r]$ is locally linearly bounded from below.*

Proof. Let $(\alpha, a) \in \Sigma(\Gamma(X))^\circ$. Let $\psi_{i,p}$ be the trivializations of the weights ψ_i, then

$$\sum a_i \psi_{i,p}$$

is the trivialization of $\sum a_i \psi_i$. Let D be as in the proof of Lemma 21, and choose A such that

$$e^{-\sum a_i \psi_{i,p}} > A.$$

Since the inequality

$$e^{-\sum b_i \psi_{i,p}} > A$$

holds for all b in a neighbourhood of a, the lower bound follows as in the proof of Lemma 21. □

DEFINITION 17. *The global Chebyshev transform* $c[\psi_1, ..., \psi_r]$*of the* r*-tuple* $(\psi_1, ..., \psi_r)$ *is defined as the convex envelope of* $F[\psi_1, ..., \psi_r]$ *on* $\Sigma(\Gamma(X))^\circ$.

PROPOSITION 49. *For any sequence* $(\alpha(k), a(k)) \in \Gamma(X)$ *such that* $|(\alpha(k), a(k))| \to \infty$ *and*

$$\frac{(\alpha(k), a(k))}{|(\alpha(k), a(k))|} \to (p, a) \in \Sigma(\Gamma(X))^\circ$$

it holds that

$$\lim_{k\to\infty} \frac{F[\psi_1, ..., \psi_r](\alpha(k), a(k))}{|(\alpha(k), a(k))|} = c[\psi_1, ..., \psi_r](p, a).$$

Proof. By Lemma 47 and Lemma 48 we can use Theorem 14, which gives us the proposition. □

PROPOSITION 50. *For rational a, i.e* $a = (a_1, ..., a_r) \in \mathbb{Q}^r$, *the global Chebyshev transform* $c[\psi_1, ..., \psi_r](p, a)$ *coincides with* $c\left[\sum a_i \psi_i\right](p)$, *where the Chebyshev transform of the* $\mathbb{Q}$*-divisor* $\sum a_i D_i$ *as defined by (1.32).*

Proof. By construction it is clear that for all $(\alpha, a) \in \Gamma(X)$ we have that

$$F[\psi_1, ..., \psi_r](\alpha, ka) = F\left[\sum a_i \psi_i\right](\alpha, k).$$

Choose a sequence $(\alpha(k), ka) \in \Gamma(X)$ such that

$$\lim_{k\to\infty} \frac{(\alpha(k), ka))}{|(\alpha(k), ka))|} = \frac{(p, a)}{|(p, a)|},$$

where we only consider those k such that ka is an integer. Then by Proposition 49 we have that

$$c[\psi_1, ..., \psi_r](p, a) = \lim_{k\to\infty} |(p, a)| \frac{F[\psi_1, ..., \psi_r](\alpha(k), ka)}{|(\alpha(k), ka)|} =$$

$$= \lim_{k\to\infty} |(p, a)| \frac{F\left[\sum a_i\psi_i\right](\alpha(k), k)}{|(\alpha(k), ka)|} = \lim_{k\to\infty} \left(\frac{|(p, a)|k}{|(\alpha(k), ka)|}\right) c\left[\sum a_i\psi_i\right](p) =$$

$$= c\left[\sum a_i\psi_i\right](p).$$

□

Now that we have defined the Chebyshev transform for weighted $\mathbb{R}-$divisors we wish to show that the formula of Theorem 27 holds true also in this case. First we need some preliminary lemmas.

LEMMA 51. *The function* $\mathcal{E}(t\psi, t\varphi)$ *is* $(n+1)$*-homogeneous in t for* $t > 0$*, i.e.*

$$\mathcal{E}(t\psi, t\varphi) = t^{n+1}\mathcal{E}(\psi, \varphi).$$

Proof. For weights with minimal singularities ψ' and φ', by definition of the Monge-Ampère energy we have that

$$\mathcal{E}(t\psi, t\varphi) = \frac{1}{n+1}\int_\Omega (t\psi' - t\varphi')\mathrm{MA}_n(t\psi', t\varphi') =$$

$$= \frac{t^{n+1}}{n+1}\int_\Omega (\psi' - \varphi')\mathrm{MA}_n(\psi', \varphi') = t^{n+1}\mathcal{E}(\psi, \varphi). \tag{1.33}$$

We also observe that $t\psi'$ is a psh weight on tL iff ψ' is a psh weight on L. Therefore we get that

$$P(t\psi) = tP(\psi). \tag{1.34}$$

Combining (1.33) and (1.34) the lemma follows. □

LEMMA 52. *Assume that L is ample. Let ψ and ψ' be two continuous weights on L, and let φ and φ' be two continuous weights on some other big line bundle L'. Then the function*

$$\mathcal{E}(\psi + t\varphi, \psi' + t\varphi')$$

is continuous in t for t such that $L + tL'$ is ample.

Proof. We show continuity at $t = 0$. Since L is ample, for some $\varepsilon > 0$

$$L + \varepsilon L'$$

will be ample. Furthermore the Monge-Ampère energy is homogeneous. We may write

$$L + t\varepsilon L'$$

as

$$(1-t)(L + \frac{t}{1-t}(L + \varepsilon L')),$$

thus without loss of generality we can assume that L' is ample. By the cocycle property of the Monge-Ampère energy we have that for any continuous weight $\tilde{\varphi}$ on L'

$$\mathcal{E}(\psi+t\varphi, \psi'+t\varphi') = \mathcal{E}(\psi+t\varphi, \psi+t\tilde{\varphi})+\mathcal{E}(\psi+t\tilde{\varphi}, \psi'+t\tilde{\varphi})+\mathcal{E}(\psi'+t\tilde{\varphi}, \psi'+t\varphi').$$

Thus it suffices to consider two special cases. The first where we assume that $\psi = \psi'$. In the second case we instead assume that $\varphi = \varphi'$ and that φ is psh.

First assume that $\psi = \psi'$. Since $\mathcal{E}(\psi, \psi) = 0$, we must show that $\mathcal{E}(\psi + t\varphi, \psi + t\varphi')$ tends to zero when t tends to zero. Lemma 1.14 in [1] tells us that the projection operator is 1-Lipschitz continuous. In our case this means that

$$\sup_X |P(\psi + t\varphi) - P(\psi + t\varphi')| \leq t \sup_X |\varphi - \varphi'|.$$

We get that

$$|\mathcal{E}(\psi + t\varphi, \psi + t\varphi')| =$$

$$= \frac{1}{n+1} |\int_X (P(\psi + t\varphi) - P(\psi + t\varphi'))\mathrm{MA}_n(P(\psi + t\varphi), P(\psi + t\varphi'))| \leq$$

$$\leq t \sup_X |\varphi - \varphi'| \frac{1}{n+1} \int_X \mathrm{MA}_n(P(\psi + t\varphi), P(\psi + t\varphi')) =$$

$$= t \sup_X |\varphi - \varphi'| \mathrm{vol}(L + tL').$$

Since the volume is continuous (see e.g. [1]), we get continuity in this case.

Now we intead assume that $\varphi = \varphi'$ and that φ is psh. We first show right-continuity. Since φ is psh, for all $r \leq t$ we have that

$$P(\psi + r\varphi) + (t - r)\varphi$$

is psh and it is clearly dominated by $\psi + t\varphi$, thus by the definition of the projection operator

$$P(\psi + t\varphi) \geq P(\psi + r\varphi) + (t - r)\varphi.$$

It follows that $P(\psi + t\varphi) - t\varphi$ is increasing in t. Also

$$dd^c(P(\psi + t\varphi) - t\varphi) \geq -tdd^c\varphi,$$

thus by standard results in potential thoery we have that

$$dd^c \lim_{t\to 0}(P(\psi + t\varphi) - t\varphi) \geq 0.$$

This gives us that

$$\lim_{t\to 0}(P(\psi + t\varphi) - t\varphi) = P(\psi).$$

The same holds for

$$P(\psi' + t\varphi) - t\varphi.$$

We now write $P(\psi + t\varphi)$ as

$$(P(\psi + t\varphi) - t\varphi) + t\varphi$$

and $P(\psi' + t\varphi)$ as

$$(P(\psi' + t\varphi) - t\varphi) + t\varphi$$

in the expression for

$$\mathcal{E}(\psi + t\varphi, \psi' + t\varphi)$$

and the right-continuity follows from the fact that mixed Monge-Ampère operators are continuous along pointwise decreasing sequences of psh or quasi-psh weights converging to a weight with minimal singularities (see [5]). For the left-continuity we use the homogeneity of the Monge-Ampère energy exactly as above to reduce to the case of right-continuity already considered. □

We are now ready to prove our main theorem in the setting of weighted ample $\mathbb{R}$-divisors.

THEOREM 53. *For ample $\mathbb{R}$-divisors $\sum a_i D_i$ we have that*

$$\mathcal{E}(\sum a_i\psi_i, \sum a_i\varphi_i) =$$
$$= n!\int_{\Delta(\sum a_i D_i)} (c[\varphi_1, ..., \varphi_r](p,a) - c[\psi_1, ..., \psi_r](p,a))d\lambda(p). \qquad (1.35)$$

Proof. First we show that (1.35) holds when $a \in \mathbb{Q}^r$. By the homogeneity of the Okounkov body and the Chebyshev transform we have that

$$n!\int_{\Delta(tL)^\circ} (c[t\psi] - c[t\varphi])d\lambda = t^{n+1}n!\int_{\Delta(L)^\circ} (c[\psi] - c[\varphi])d\lambda =$$
$$= t^{n+1}\mathcal{E}(\varphi, \psi) = \mathcal{E}(t\varphi, t\psi),$$

where the last equality follows from Lemma 51. Then by Proposition 50, (1.35) holds for $a \in \mathbb{Q}^r$. Therefore by the continuity of the Monge-Ampère energy, the continuity of the global Chebyshev transform, and the fact that equation (1.35) holds for rational a, the proposition follows. □

1.13 Differentiability of the Monge-Ampère energy

We wish to understand the behaviour of the Monge-Ampère energy $\mathcal{E}(\psi_t, \varphi_t)$ when the weights ψ_t and φ_t vary with t. In [1] Berman-Boucksom study the case where ψ_t and φ_t are weights on a fixed line bundle or more generally a $\mathbb{R}$-divisor. We are interested in the case where the underlying $\mathbb{R}$-divisor is allowed to vary as well. In [9] Lazarsfeld-Mustaţă prove the differentiability of the volume by studying the variation of the Okounkov bodies. Since our Theorem 27 and Theorem 53 states that the Monge-Ampère energy is given by the integration of the difference of Chebyshev transforms on the Okounkov body, we wish to use the same approach as Lazarsfeld-Mustaţă did in [9]. The situation becomes a bit more involved, since we have to consider not only the variation of the Okounkov bodies but also the variation of the Chebyshev transforms.

In this section we will assume that L is an ample $\mathbb{R}$-divisor.

To account for the variation of the Chebyshev transform when the underlying line bundle changes it becomes necessary to consider not only continuous weights but also weights with singularities. Specifically weights of the form

$$\psi - t\ln|s|^2,$$

where ψ is a continuous weight on L, s is some section of an ample line bundle A, and t is positive. Observe that these weights only have $+\infty$ singularities.

In fact, by general approximation arguments one can show that the results that we have established for continuous weights also hold for weights that are lower semicontinuous and only have $+\infty$ singularities. But for completeness we include arguments proving this for $\psi - t\ln|s|^2$.

Let Ψ be some fixed continuous positive weight on A. For any number R we denote by $\ln|s|^2_{+R}$ the weight

$$\ln|s|^2_{+R} := \max(\ln|s|^2, \Psi - R).$$

LEMMA 54. *For $R \gg 0$ we have that*

$$P(\psi - t\ln|s|^2_{+R}) = P(\psi - r\ln|s|^2).$$

Proof. That

$$P(\psi - t\ln|s|^2_{+R}) \leq P(\psi - t\ln|s|^2)$$

is clear since

$$\psi - t\ln|s|^2_{+R} \leq \psi - t\ln|s|^2.$$

$P(\psi - t\ln|s|^2)$ is psh, therefore upper semicontinuous by definition, which means that it is locally bounded from above. Thus locally we can find $R \gg 0$ such that

$$\psi - t(\Psi - R) \geq P(\psi - t\ln|s|^2).$$

But we have assumed that our manifold X is compact, so there exists an R such that $\psi - t(\Psi - R)$ dominates $P(\psi - t\ln|s|^2)$ on the whole of X. The same must be true for $\psi - t\ln|s|^2_{+R}$. By definition $P(\psi - t\ln|s|^2_{+R})$ dominates all psh weights less or equal to $\psi - t\ln|s|^2_{+R}$, in particular it must dominate $P(\psi - r\ln|s|^2)$. □

LEMMA 55. *If L is integral, i.e. a line bundle, then for $R \gg 0$ such that*

$$P(\psi - t\ln|s|^2_{+R}) = P(\psi - t\ln|s|^2),$$

we have that $F[\psi - t\ln|s|^2_{+R}] = F[\psi - t\ln|s|^2]$.

Proof. This follows the fact that for all weights φ and all sections s it holds that

$$\sup_{x\in X}\{|s(x)|^2 e^{-\varphi(x)}\} = \sup_{x\in X}\{|s(x)|^2 e^{-P(\varphi)(x)}\},$$

see e.g. [1]. □

From Lemma 55 it follows that the Chebyshev transform $c[\psi - t\ln|s|^2]$ is well-defined, also for $\mathbb{R}-$divisors, and that Proposition 22 holds in this case. The formula for the Monge-Ampère energy as the integral of Chebyshev transforms will also still hold.

PROPOSITION 56. *For any continuous weight φ on $L - tA$ it holds that*

$$\mathcal{E}(\psi - t\ln|s|^2, \varphi) = \tag{1.36}$$

$$= n!\int_{\Delta(L-tA)^\circ} c[\varphi] - c[\psi - t\ln|s|^2]d\lambda. \tag{1.37}$$

Proof. For integral L, choose an $R \gg 0$ such that

$$P(\psi - t\ln|s|^2_{+R}) = P(\psi - t\ln|s|^2).$$

Then (1.36) follows in this case from Theorem 27 and Lemma 55. By homogeneity (1.36) holds for rational L, and by continuity for arbitrary ample $\mathbb{R}$-divisors. □

Theorem B in [1] states that the Monge-Ampère energy is differentiable when the weights correspond to a fixed big line bundle. By the comment in the beginning of section 4 in [1] this holds more generally for big $(1,1)$ cohomology classes, e.g. $\mathbb{R}$-divisors. We thus have the following.

THEOREM 57. *Let ψ_t be a smooth family of weights on a big $\mathbb{R}$-divisor D, and φ any psh-weight with minimal singularities. Then the function*

$$f(t) := \mathcal{E}(\psi_t, \varphi)$$

is differentiable, and

$$f'(0) = \int_\Omega uMA(P(\psi_0)),$$

where $u = \frac{d}{dt}_{|0}\psi_t$.

We also need to consider the case where

$$\psi_t = \psi_0 + t(\Phi - \ln|s|^2),$$

where Φ is some continuous weight on A.

LEMMA 58. *For every* ε *there exists a* $R \gg 0$ *such that*

$$P(\psi_0 + t(\Phi - \ln|s|^2_{+R})) = P(\psi_0 + t(\Phi - \ln|s|^2))$$

for $t \geq \varepsilon$.

Proof. Recall that $\ln|s|^2_{+R}$ was defined as $\max\{\Psi - R, \ln|s|^2\}$ for some continuous weight Ψ on A. That

$$P(\psi_0 + t(\Phi - \ln|s|^2_{+R})) \leq P(\psi_0 + t(\Phi - \ln|s|^2))$$

is clear since

$$\psi_0 + t(\Phi - \ln|s|^2_{+R}) \leq \psi_0 + t(\Phi - \ln|s|^2)$$

and the projection operator is monotone. When

$$R \geq \frac{P(\psi_0 + t(\Phi - \ln|s|^2)) - \psi_0 - t\Phi}{t} + \Psi$$

we get that

$$P(\psi_0 + t(\Phi - \ln|s|^2_{+R})) = P(\psi_0 + t(\Phi - \ln|s|^2))$$

because for such R

$$\psi_0 + t(\Phi - \ln|s|^2_{+R}) \geq \psi - t(\Psi - R) \geq P(\psi_0 + t(\Phi - \ln|s|^2))$$

and the same is true for the projection. By the homogeneity of the projection operator we have that

$$\frac{P(\psi_0 + t(\Phi - \ln|s|^2)) - \psi_0 - t\Phi}{t} + \Psi =$$
$$= P(\frac{\psi_0}{t} + \Phi - \ln|s|^2) - \frac{\psi_0}{t} - \Phi + \Psi.$$

We also have that for $t > r$

$$P(\frac{\psi_0}{t} + \Phi - \ln|s|^2) - \frac{\psi_0}{t} \leq P(\frac{\psi_0}{t} + \Phi - \ln|s|^2) - \frac{P(\psi_0)}{t} \leq$$
$$\leq P(\frac{\psi_0}{r} + \Phi - \ln|s|^2) - \frac{P(\psi_0)}{r}$$

by the same arguments as in the proof of Lemma 52. $P(\frac{\psi_0}{r} + \Phi - \ln|s|^2)$ is psh and therefore upper semicontinuous, and since L is ample, $P(\psi_0)$ is continuous. This yields that

$$P(\frac{\psi_0}{r} + \Phi - \ln|s|^2) - \frac{P(\psi_0)}{r} - \Phi + \Psi$$

is an upper semicontinuous function on the compact space X, so it has an upper bound. The lemma follows by setting $r = 1/\varepsilon$ and choosing R larger than

$$P(\frac{\psi_0}{r} + \Phi - \ln|s|^2) - \frac{P(\psi_0)}{r} - \Phi + \Psi.$$

□

We state and prove a slight variation of Lemma 3.1 in [2].

LEMMA 59. *Let f_k be a sequence of concave functions on the unit interval, and let g be a function on $[0,1]$ such that f_k converges to g pointwise. It follows that*

$$g'(0) \leq \liminf_{k\to\infty} f_k'(0).$$

Proof. Since f_k is concave we have that

$$f_k(0) + f_k'(0)t \geq f_k(t)$$

hence

$$\liminf_{k\to\infty} t f_k'(0) \geq g(t) - g(0).$$

The lemma follows by letting t tend to zero. □

We now prove that Theorem 57 holds true also in our singular setting.

LEMMA 60. *The function*

$$f(t) := \mathcal{E}(\psi_0 + t(\Phi - \ln|s|^2), \varphi)$$

is right-differentiable at zero and

$$\frac{d}{dt}_{|0+} f(t) = \int_\Omega (\Phi - \ln|s|^2) MA(P(\psi_0)).$$

Proof. Let us denote $\Phi - \ln|s|^2$ by u, and let

$$u_k := \Phi - \ln|s|^2_{+k}.$$

Let f_k denote the function

$$f_k(t) := \mathcal{E}(\psi_0 + tu_k, \varphi).$$

By e.g. [1] the functions f_k are concave, and by Theorem 57 they are differentiable. By Lemma 58 we get that for any $\varepsilon > 0$ there exists a k such that $f = f_k$ on $(\varepsilon, 1)$. Therefore it follows that f is concave and that

$$f_k \to f$$

pointwise. Since f is concave it is right-differentiable. We also have that

$$f_k'(0) = \int_\Omega u_k \mathrm{MA}(P(\psi_0))$$

by Theorem 57. Thus from Lemma 59 we get that

$$f'(0) \leq \int_\Omega u \mathrm{MA}(P(\psi_0)).$$

Since f is concave the derivative is decreasig, for all $\varepsilon > 0$

$$f'(0) \geq f'(\varepsilon) = \lim_{k\to\infty} \int_\Omega u_k \mathrm{MA}(P(\psi_0 + \varepsilon u_k)) = \int_\Omega u \mathrm{MA}(P(\psi_0 + \varepsilon u)),$$

where the last step follows by monotone convergence since

$$\mathrm{MA}(P(\psi_0 + \varepsilon u_k)) = \mathrm{MA}(P(\psi_0 + \varepsilon u))$$

for large k by Lemma 58. The projection operator is 1-Lipschitz continuous, therefore we get that $P(\psi_0 + \varepsilon u_k)$ will converge to $P(\psi_0)$ uniformly. The Monge-Ampère operator is continuous along sequences of psh weights with minimal singularities converging uniformly (see [5]), hence

$$\lim_{\varepsilon\to 0}\int_\Omega u\mathrm{MA}(P(\psi_0+\varepsilon u)) = \int_\Omega u\mathrm{MA}(P(\psi_0)),$$

and the lemma follows. □

We will also need an integration by parts formula involving $\ln|s|^2$.

LEMMA 61. *Let φ and φ' be continuous weights on an ample $\mathbb{R}$-divisor L. Let ψ be a continuous psh weight on an ample line bundle A, and let $s \in H^0(A)$ be a section such that its zero set variety Y is a smooth submanifold. Then it holds that*

$$\int_X (\psi - \ln|s|^2) dd^c(P(\varphi) - P(\varphi')) \wedge MA_{n-1}(P(\varphi), P(\varphi')) =$$
$$= \int_X (P(\varphi) - P(\varphi'))dd^c\psi \wedge MA_{n-1}(P(\varphi), P(\varphi')) - n\mathcal{E}_Y(P(\varphi)_{|Y}, P(\varphi')_{|Y}).$$

Proof. The lemma will follow by the Lelong-Poincare formula as soon as we establish that

$$\int_X (\psi - \ln|s|^2) dd^c(P(\varphi) - P(\varphi')) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) =$$
$$= \int_X (P(\varphi) - P(\varphi'))dd^c(\psi - \ln|s|^2) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')),$$

which is an integration by parts formula. By [5] we may integrate by parts when the functions are differences of quasi-psh weights with minimal singularities. We denote by u_k the quasi-psh weight with minimal singularities $\psi - \ln|s|^2_{+k}$ and get that

$$\int_X u_k dd^c(P(\varphi) - P(\varphi')) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) =$$
$$= \int_X (P(\varphi) - P(\varphi'))dd^c u_k \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')).$$

Since $P(\varphi)$ and $P(\varphi')$ are both continuous, by the Chern-Levine-Nirenberg inequalities (see e.g. [6]) we get that

$$\int_X |(\psi - \ln|s|^2)| dd^c P(\varphi) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) \leq C \int_X |(\psi - \ln|s|^2)| dV$$

and

$$\int_X |(\psi - \ln|s|^2)| dd^c P(\varphi') \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) \leq C' \int_X |(\psi - \ln|s|^2)| dV$$

for some constants C and C' and some smooth volume form dV. By standard results $\ln|s|^2$ is locally integrable, thus both integrals are finite. This means that we can use monotone convergence to conclude that the LHS will converge to

$$\int_X (\psi - \ln|s|^2) dd^c (P(\varphi) - P(\varphi')) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi'))$$

when k goes to infinity. A special case of Proposition 4.9 in [6], chapter 3, is that monotone convergence for Monge-Ampère expressions holds when one of the terms has analytic singularities and the others are locally bounded. By this it follows that the LHS will converge to

$$\int_X (P(\varphi) - P(\varphi')) dd^c (\psi - \ln|s|^2) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')),$$

and we are done. □

Assume that we have chosen our coordinates $z_1, ..., z_n$ centered at p such that

$$z_1 = 0$$

is a local equation for an irreducible variety Y. Assume also that Y is the zero-set of a holomorphic section $s \in H^0(A)$ of an ample line bundle A. Then by Theorem 4.24 in [9] the Okounkov bodies of L and $L + tA$ with respect to these coordinates are related in the following way

$$\Delta(L) = (\Delta(L + tA) - te_1) \cap (\mathbb{R}_+)^n.$$

There is also correspondence between the Chebyshev transforms of weights on L and $L + tA$.

PROPOSITION 62. *Let A and s be as above. Suppose also that we have chosen the holomorphic coordinates so that $z_1 = s$ locally. Then for $a \geq r$ it holds that*

$$c_L[\psi](a,\alpha) - c_L[\varphi](a,\alpha) = \\ = c_{L-rA}[\psi - r\ln|s|^2](a-r,\alpha) - c_{L-rA}[\varphi - r\ln|s|^2](a-r,\alpha). \quad (1.38)$$

Proof. First assume that L is integral. Since we have that locally $s = z_1$, for $t \in H^0(kL)$,

$$t = z^{k(a,\alpha)} + \text{higher order terms},$$

if and only if

$$\frac{t}{s^{rk}} = z^{k(a-r,\alpha)} + \text{higher order terms}.$$

We also have that

$$\sup_{x\in X}\{|t(x)|^2 e^{-k\varphi(x)}\} = \sup_{x\in X}\{\frac{|t(x)|^2}{|s^{rk}(x)|^2} e^{-k(\varphi(x) - r\ln|s(x)|^2)}\}.$$

Thus (1.38) holds for integral L. By the homogeneity and continuity of the Chebyshev transform it will therefore hold for ample $\mathbb{R}$-divisors. □

We are now ready to state and prove our generalization of Theorem 57 in the ample setting, where the underlying $\mathbb{R}$-divisor is allowed to vary within the ample cone.

THEOREM 63. *Let A_i, $i = 1, ..., m$ be a finite collection of ample line bundles, and for each i let φ_i and φ_i' be two continuous weights on A_i. Let O denote the open cone in $\mathbb{R}^m$ such that $a \in O$ iff $\sum a_i A_i$ is an ample $\mathbb{R}$-divisor. Then the function*

$$f(a) := \mathcal{E}_{\sum a_i A_i}(\sum a_i\varphi_i, \sum a_i\varphi_i')$$

is $\mathcal{C}^1$ on O.

Proof. Let a be a point in O, and denote $\sum a_i A_i$ by L. Denote $\sum a_i\varphi$ by φ and $\sum a_i\varphi_i'$ by φ'. We want to calculate the partial derivatives of F at a. Thus we let L' be an ample line bundle, let ψ and ψ' be two continuous metrics on L' and we consider the function

$$f(t) := \mathcal{E}_{L+tL'}(\varphi + t\psi, \varphi' + t\psi').$$

We claim that f is differentiable at $t = 0$, and that the derivative varies continuously with L, φ and φ'.

We may assume that L' has a non-trivial section s such that $Y := \{s = 0\}$ is a smooth manifold, since otherwise because of the homogeneity we may just as well consider some large multiple of L' instead. We choose local holomorphic coordinates such that $z_1 = s$. Recall that the Okounkov bodies of L and $L+tL'$ are related in the following way

$$\Delta(L) = (\Delta(L + tL') - te_1) \cap (\mathbb{R}_+)^n. \tag{1.39}$$

Let $\Delta(L)_r$ denote the fiber over r of the projection of the Okounkov body down to the first coordinate, i.e.

$$\Delta(L)_r := \Delta(L) \cap (\{r\} \times \mathbb{R}^{n-1}).$$

Then one may write equation (1.39) as

$$\Delta(L + tL') = \cup_{0\leq r\leq t}\Delta(L + tL')_r \cup (\Delta(L) + te_1). \tag{1.40}$$

Furthermore the energy is given by integration of the Chebyshev transforms over the Okounkov bodies. Using (1.40) and Proposition 62 we get that

$$\begin{aligned}
&\mathcal{E}_{L+tL'}(\varphi + t\psi_t, \varphi' + t\psi'_t) = \\
&= n! \int_{\Delta(L+tL')^\circ} c[\varphi' + t\psi'_t] - c[\varphi + t\psi_t] d\lambda = \\
&= n! \int_{r=0}^{t} \int_{\Delta(L+tL')_r^\circ} c[\varphi' + t\psi'_t](r,\alpha) - c[\varphi + t\psi_t](r,\alpha) d\alpha dr + \\
&+ n! \int_{\Delta(L)^\circ} c[\varphi' + t(\psi'_t - \ln|s|^2)] - c[\varphi + t(\psi_t - \ln|s|^2)] dp = \\
&= n! \int_{r=0}^{t} \int_{\Delta(L+tL')_r^\circ} c[\varphi' + t\psi'_t](r,\alpha) - c[\varphi + t\psi_t](r,\alpha) d\alpha dr + \\
&+ \mathcal{E}_L(\varphi + t(\psi_t - \ln|s|^2), \varphi' + t(\psi'_t - \ln|s|^2)).
\end{aligned}$$

Hence by Theorem 57 and the fundamental theorem of calculus it follows that this function is right-differentiable. We also want to calculate the right-derivative.

We get that

$$\frac{d}{dt}_{|0+}\mathcal{E}_{L+tL'}(\varphi+t\psi_t,\varphi'+t\psi'_t)=$$
$$=n!\int_{\Delta(L)_0^\circ}c[\varphi'](0,\alpha)-c[\varphi](0,\alpha)d\alpha+$$
$$+\frac{d}{dt}_{|0+}\mathcal{E}_L(\varphi+t(\psi_t-\ln|s|^2),\varphi'+t(\psi'_t-\ln|s|^2))=$$
$$=n\mathcal{E}_Y(P(\varphi')_{|Y},P(\varphi)_{|Y})+\frac{d}{dt}_{|0+}\mathcal{E}_L(\varphi+t(\psi_t-\ln|s|^2),\varphi'+t(\psi'_t-\ln|s|^2)),$$

using Proposition 44 in the last step. Since in the second term the divisor L does not change with t, we may use Theorem 57. Also, because of the cocycle property of the Monge-Ampère energy, we only need to consider two cases, one where $\varphi=\varphi'$, and the other one where we let $\varphi\neq\varphi'$ but instead assume that $\psi_t=\psi'_t=\psi$ is some fixed smooth positive metric on L'.

First assume that $\varphi=\varphi'$. The first term disappears and we get that

$$\frac{d}{dt}_{|0+}\mathcal{E}_{L+tL'}(\varphi+t\psi_t,\varphi+t\psi'_t)=$$
$$=\frac{d}{dt}_{|0}\mathcal{E}_L(\varphi+t(\psi_t-\ln|s|^2),\varphi+t(\psi'_t-\ln|s|^2))=$$
$$=\int_X(\psi_0-\ln|s|^2)\mathrm{MA}(P(\varphi))-\int_X(\psi'_0-\ln|s|^2)\mathrm{MA}(P(\varphi))=$$
$$=\int_X(\psi_0-\psi'_0)\mathrm{MA}(P(\varphi)).\quad(1.41)$$

Here we used Lemma 60.

This term depends continuously on the weight φ.

Now let $\varphi\neq\varphi'$ but instead assume that $\psi_t=\psi'_t=\psi$ is some fixed smooth positive metric on L'. Then we have that

$$
\begin{aligned}
&\frac{d}{dt}_{|0+}\mathcal{E}_{L+tL'}(\varphi+t\psi,\varphi'+t\psi)= \\
&= n\mathcal{E}_Y(P(\varphi)_{|Y},P(\varphi')_{|Y})+ \qquad (1.42)\\
&+\frac{d}{dt}_{|0}\mathcal{E}_L(\varphi+t(\psi-\ln|s|^2),\varphi'+t(\psi-\ln|s|^2))= \\
&= n\mathcal{E}_Y(P(\varphi)_{|Y},P(\varphi')_{|Y})+\int_X(\psi-\ln|s|^2)\mathrm{MA}(P(\varphi))- \\
&-\int_X(\psi-\ln|s|^2)\mathrm{MA}(P(\varphi'))= \\
&= n\mathcal{E}_Y(P(\varphi)_{|Y},P(\varphi')_{|Y})+ \\
&+\int_X(\psi-\ln|s|^2)dd^c(P(\varphi)-P(\varphi'))\wedge\mathrm{MA}_{n-1}(P(\varphi),P(\varphi'))= \\
&= \int_X(P(\varphi)-P(\varphi'))dd^c\psi\wedge\mathrm{MA}_{n-1}(P(\varphi),P(\varphi')). \qquad (1.43)
\end{aligned}
$$

In the last step we used Lemma 61.

This will also depend continuously on the pair (φ,φ') exactly as in Lemma 52.

By definition a $\mathbb{R}-$divisor can be written as a finite positive sum of ample line bundles, thus since we have shown that the Monge-Ampère energy is continuously partially right-differentiable in the ample integral directions it follows that the function f is right-differentible when L' is any ample $\mathbb{R}$-divisor. Since the derivatives we have calculated for ample line bundles are linear, the same formulas hold for arbitrary $\mathbb{R}$-divisors.

Now we consider the question of left-differentiability By Lemma 51 the Monge-Ampère energy is $(n+1)-$homogeneous. For some possibly large k, $kL-L'$ is ample. Because of the homogeneity of the Monge-Ampère energy, without loss of generality, we may assume that $L-L'$ is ample, otherwise just change L to kL. Also

$$\frac{1}{1-t}(L-tL')=L+\frac{t}{1-t}(L-L').$$

Using this and the homogeneity we get that

$$\mathcal{E}_{L-tL'}(\varphi - t\psi_t, \varphi' - t\psi'_t) =$$
$$= (1-t)^{n+1}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi + \frac{t}{1-t}(\varphi - \psi_t), \varphi' + \frac{t}{1-t}(\varphi' - \psi'_t)). \quad (1.44)$$

The left-differentiability thus follows from the previous case by equation (1.44) and the chain rule.

To show the differentiability of f then, we only need to calculate the left-derivative to make sure it coincides with the right-derivative. Recall that because of the cocycle property we only needed to consider two cases. First assume that $\varphi = \varphi'$. Equations (1.44) and (1.41) now yields that

$$\frac{d}{dt}_{|0-}\mathcal{E}_{L+tL'}(\varphi + t\psi_t, \varphi + t\psi'_t) = -\frac{d}{dt}_{|0+}\mathcal{E}_{L-tL'}(\varphi - t\psi_t, \varphi - t\psi'_t) =$$
$$-\frac{d}{dt}_{|0+}(1-t)^{n+1}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi + \frac{t}{1-t}(\varphi - \psi_t), \varphi + \frac{t}{1-t}(\varphi - \psi'_t)) =$$
$$= -\frac{d}{dt}_{|0+}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi + \frac{t}{1-t}(\varphi - \psi_t), \varphi + \frac{t}{1-t}(\varphi - \psi'_t)) =$$
$$-\int_X((\varphi - \psi_0) - (\varphi - \psi'_0))\mathrm{MA}(P(\varphi)) = \int_X(\psi_0 - \psi'_0)\mathrm{MA}(P(\varphi)) =$$
$$= \frac{d}{dt}_{|0+}\mathcal{E}_{L+tL'}(\varphi + t\psi_t, \varphi + t\psi'_t).$$

Now let $\varphi \neq \varphi'$ but instead assume that $\psi_t = \psi'_t = \psi$ is some smooth positive weight on L'. By the cocycle property we may also assume that φ and $\varphi - \psi$ are smooth and positive. By equation (1.42) we get that

$$\frac{d}{dt}_{|0-}\mathcal{E}_{L+tL'}(\varphi+t\psi,\varphi'+t\psi)=$$

$$=-\frac{d}{dt}_{|0+}(1-t)^{n+1}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi+\frac{t}{1-t}(\varphi-\psi),\varphi'+\frac{t}{1-t}(\varphi'-\psi))=$$

$$=(n+1)\mathcal{E}_L(\varphi.\varphi')-$$

$$-\frac{d}{dt}_{|0+}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi+\frac{t}{1-t}(\varphi-\psi),\varphi'+\frac{t}{1-t}(\varphi'-\psi))=$$

$$=(n+1)\mathcal{E}_L(\varphi.\varphi')-$$

$$-\int_X(P(\varphi)-P(\varphi'))dd^c(\varphi-\psi)\wedge\mathrm{MA}_{n-1}(P(\varphi),P(\varphi'))-$$

$$-\int_X((\varphi-\psi)-(\varphi'-\psi))\mathrm{MA}(P(\varphi'))=$$

$$=\int_X(P(\varphi)-P(\varphi'))dd^c(\psi)\wedge\mathrm{MA}_{n-1}(P(\varphi),P(\varphi'))=$$

$$=\frac{d}{dt}_{|0+}\mathcal{E}_{L+tL'}(\varphi+t\psi,\varphi'+t\psi).$$

We used that $\varphi'=P(\varphi')$ a.e. with respect to $\mathrm{MA}(P(\varphi'))$ (see e.g. [1]). We also used the observation that

$$dd^c\varphi\wedge\mathrm{MA}_{n-1}(P(\varphi),P(\varphi'))+\mathrm{MA}(P(\varphi'))=\mathrm{MA}_n(P(\varphi),P(\varphi')),$$

and that by definition

$$\int(P(\varphi)-P(\varphi'))\mathrm{MA}_n(P(\varphi),P(\varphi'))=(n+1)\mathcal{E}_L(\varphi,\varphi').$$

The differentiability of f follows, and we saw that the derivative depended continuously on L, φ and φ'. Hence the function F is $\mathcal{C}^1$ on O. □

Note that in the special case where $\psi_t=\psi_0+t\Psi$ and $\varphi_t=\varphi_0+t\Psi$ for some fixed positive weight Ψ on L', our calculations show that

$$f'(0)=\sum_{j=0}^{n-1}\int_X(P(\psi_0)-P(\varphi_0))dd^c\Psi\wedge(dd^cP(\psi_0))^j\wedge(dd^cP(\varphi_0))^{n-j-1}.$$

Let $\psi_0 = \psi_1 + 1$ and as Ψ choose $\ln|s|^2$ where s is a defining section for an ample divisor A. Then $f(t) = \mathrm{vol}_X(L + tA)$ and using the Lelong-Poincare formula on the RHS of equation above one recovers the classical result that

$$\frac{d}{dt}_{|0} \mathrm{vol}_X(L + tA) = n\mathrm{vol}_A(L_{|A}).$$

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PAPER II

David Witt Nyström
Test configurations and Okounkov bodies

2

Test configurations and Okounkov bodies

ABSTRACT

We associate to a test configuration for a polarized variety a filtration of the section ring of the line bundle. Using the recent work of Boucksom-Chen we get a concave function on the Okounkov body whose law with respect to Lebesgue measure determines the asymptotic distribution of the weights of the test configuration. We show that this is a generalization of a well-known result in toric geometry. As an application, we prove that the pushforward of the Lebesgue measure on the Okounkov body is equal to a Duistermaat-Heckman measure of a certain deformation of the manifold. Via the Duisteraat-Heckman

formula, we get as a corollary that in the special case of an effective $\mathbb{C}^{\times}$-action on the manifold lifting to the line bundle, the pushforward of the Lebesgue measure on the Okounkov body is piecewise polynomial.

2.1 Introduction

2.1.1 Okounkov bodies

In [14] Okounkov introduced a way to associate a convex body in $\mathbb{R}^n$ to any ample divisor on a n-dimensional projective variety. This procedure was later shown to work in a more general setting by Lazarsfeld-Mustaţă in [12] and by Kaveh-Khovanskii in [8] and [9].

Let L be a big line bundle on a complex projective manifold X of dimension n. The Okounkov body of L, denoted by $\Delta(L)$, is a convex subset of $\mathbb{R}^n$, constructed in such a way so that the set-valued mapping

$$\Delta : L \longmapsto \Delta(L)$$

has some very nice properties (for the explicit construction see Section 2.2). It is homogeneous, i.e. for any $k \in \mathbb{N}$

$$\Delta(kL) = k\Delta(L).$$

Here kL denotes the the k:th tensor power of the line bundle L. Secondly, the mapping is convex, in the sense that for any big line bundles L and L', and any $k, m \in \mathbb{N}$, the following holds

$$\Delta(kL + mL') \supseteq k\Delta(L) + m\Delta(L'),$$

where the plus sign on the right hand side refers to Minkowski addition, i.e.

$$A + B := \{x + y : x \in A, y \in B\}.$$

Recall that the volume of a line bundle L, denoted by vol(L), is defined by

$$\text{vol}(L) := \limsup_{k\to\infty} \frac{\dim H^0(kL)}{k^n/n!}.$$

By definition L is big if $\mathrm{vol}(L) > 0$. The third and crucial property, which makes Okounkov bodies useful as a tool in birational geometry, is that for any L

$$\mathrm{vol}(L) = n!\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)).$$

where the volume of the Okounkov body is measured with respect to the standard Lesbesgue measure on $\mathbb{R}^n$.

2.1.2 Test configurations

Given an ample line bundle L on X, a class of algebraic deformations of the pair (X, L), called test configurations, were introduced by Donaldson in [5], generalizing a previous notion of Tian [21] in the context of Fano manifolds. In short, a test configuration consists of:

(i) a scheme $\mathcal{X}$ with a $\mathbb{C}^\times$-action ρ,

(ii) an $\mathbb{C}^\times$-equivariant line bundle $\mathcal{L}$ over $\mathcal{X}$,

(iii) and a flat $\mathbb{C}^\times$-equivariant projection $\pi : \mathcal{X} \to \mathbb{C}$ such that $\mathcal{L}$ restricted to the fiber over 1 is isomorphic to rL for some $r > 0$.

To a test configuration $\mathcal{T}$ there are associated discrete weight measures $\tilde{\mu}(\mathcal{T}, k)$ (see Section 2.4 for the definition). The asymptotics of the first moments of these measures, together with the Hilbert polynomial, is used to define the Futaki invariant (see Section 2.4). This in turn is used to formulate stability conditions, such as K-stability, on the pair (X, L). These conditions are conjectured to be equivalent to the existence of a constant scalar curvature metric with Kähler form in $c_1(L)$, a conjecture which is sometimes called the Yau-Tian-Donaldson conjecture. This is one of the big open problems in Kähler geometry. By the works of e.g. Yau, Tian and Donaldson, a lot of progress has been made, in particular in the case of Kähler-Einstein metrics, i.e. when L is a multiple of the canonical bundle. For more on this, we refer the reader to the expository article [15] by Phong-Sturm.

When L is assumed to be a toric line bundle on a toric variety with associated polytope P, it was shown by Donaldson in [5] that a torus equivariant test configuration is equivalent to specifying a concave rationally piecewise affine function on the polytope P. This has made it possible to translate algebraic stability conditions on L into geometric conditions on P, which has proved very useful.

Specifically, Donaldson has a formula for the Futaki invariant which only involves the moment polytope and the piecewise affine function (see [5]).

Heuristically, the relationship between a general line bundle L and its Okounkov body is supposed to mimic the relationship between a toric line bundle and its associated polytope. Therefore, one would hope that one could translate a general test configuration into some geometric data on the Okounkov body. The main goal of this article is to show that this in fact can be done, thus presenting a generalization of the well-known toric picture referred to above, and described in greater detail in Section 2.7.

In this article we show how to get a concave function on the Okounkov body, which generalizes the toric picture. Using the concave function one can compute the leading order term in the asymptotic expansion of the first moments. However, the Okounkov body and the concave function on it does not in general determine the Futaki invariant, since it also involves the second-order terms in the expansions. What is special about the toric case is that there the moment polytope and the piecewise affine function determine the full asymptotics of the Hilbert polynomial and the first moments of the weight measures.

2.1.3 The concave transform of a test configuration

By a filtration $\mathcal{F}$ of the section ring $\oplus_k H^0(kL)$ we mean a vector space-valued map from $\mathbb{R} \times \mathbb{N}$,

$$\mathcal{F} : (t, k) \longmapsto \mathcal{F}_t H^0(kL),$$

such that for any k, $\mathcal{F}_t H^0(kL)$ is a family of subspaces of $H^0(kL)$ that is decreasing and left-continuous in t. $\mathcal{F}$ is said to be multiplicative if

$$(\mathcal{F}_t H^0(kL))(\mathcal{F}_s H^0(mL)) \subseteq \mathcal{F}_{t+s} H^0((k+m)L),$$

it is left-bounded if for all k

$$\mathcal{F}_{-t}H^0(kL) = H^0(kL) \qquad \text{for} \qquad t \gg 1,$$

and is said to linearly right-bounded if there exist a C such that

$$\mathcal{F}_t H^0(kL) = \{0\} \qquad \text{for} \qquad t \geq Ck.$$

The filtration $\mathcal{F}$ is called admissible if it has all the above properties.

Given a filtration $\mathcal{F}$, one may associate discrete measures $\nu(\mathcal{F}, k)$ on $\mathbb{R}$ in the following way

$$\nu(\mathcal{F}, k) := \frac{1}{k^n}\frac{d}{dt}(-\dim \mathcal{F}_{tk}H^0(kL)),$$

where the differentiation is done in the sense of distributions.

In their article [2] Boucksom-Chen show how any admissible filtration $\mathcal{F}$ of the section ring $\oplus_k H^0(kL)$ of L gives rise to a concave function $G[\mathcal{F}]$ on the Okounkov body $\Delta(L)$ of L. $G[\mathcal{F}]$ is called the concave transform of $\mathcal{F}$. The main result of [2], Theorem A, states that the discrete measures $\nu(\mathcal{F}, k)$ converge weakly as k tends to infinity to $G[\mathcal{F}]_* d\lambda_{|\Delta(L)}$, the push-forward of the Lebesgue measure on $\Delta(L)$ with respect to the concave transform of $\mathcal{F}$.

Let $\mathcal{T}$ be a test configuration on (X, L). Given a section $s \in H^0(kL)$, there is a unique invariant meromorphic extension to configuration scheme $\mathcal{X}$. Using the vanishing order of this extension along the central fiber of $\mathcal{X}$ we define a filtration of the section ring $\oplus_k H^0(kL)$, which we show has the property that for any k

$$\tilde{\mu}(\mathcal{T}, k) = \nu(\mathcal{F}, k).$$

We will denote the associated concave transform by $G[\mathcal{T}]$. Combined with Theorem A of [2] we thus get our first main result.

THEOREM 1. *Given a test configuration $\mathcal{T}$ of L there is a concave function $G[\mathcal{T}]$ on the Okounkov body $\Delta(L)$ such that the measures $\tilde{\mu}(\mathcal{T}, k)$ converge weakly as k tends to infinity to the measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$.*

We embed our test configuration into $\mathbb{C}$ times a projective space $\mathbb{P}^N$, so that the associated action comes from a $\mathbb{C}^\times$-action on $\mathbb{P}^N$. This we can always do (see e.g. [19]). The manifold X lies embedded in $\mathbb{P}^N$, and we thus via the action get a family X_τ of submanifolds. As τ tends to 0, X_τ converges in the sense of currents to an algebraic cycle $|X_0|$ (see [6]). We let ω_{FS} denote the Fubini-Study on $\mathbb{P}^N$. Restricted to X_τ the (n,n)-form $\omega_{FS}^n/n!$ defines a positive measure, that as τ goes to zero converges to a positive measure $d\mu_{FS}$, the Fubini-Study volume form on $|X_0|$. There is also a Hamiltonian function h for the S^1-action. Using a result of Donaldson in [6] and Theorem 1 we can relate this picture with the concave transform by the following Corollary.

COROLLARY 2. *Assume that we have embedded the test configuration $\mathcal{T}$ in some $\mathbb{P}^N \times \mathbb{C}$, let h denote the corresponding Hamiltonian and $d\mu_{FS}$ the positive measure on $|X_0|$ defined above. Then we have that*

$$h_* d\mu_{FS} = G[\mathcal{T}]_* d\lambda_{|\Delta(L)}.$$

If $|X_0|$ is a smooth manifold, on which the S^1-action is effective, the measure $h_* d\mu_{FS}$ is the sort of measure studied by Duistermaat-Heckman in [7]. They prove that such a Duistermaat-Heckman measure is piecewise polynomial, i.e. the distribution function with respect to Lebesgue measure on $\mathbb{R}$ is piecewise polynomial. For a product test configuration, $|X_0| \cong X$, therefore we can apply the result of Duistermaat-Heckman to get the following.

COROLLARY 3. *Assume that there is a $\mathcal{C}^\times$-action on X which lifts to L, and that the corresponding S^1-action is effective. If we denote the associated product test configuration by $\mathcal{T}$, the concave transform $G[\mathcal{T}]$ is such that the push-forward measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$ is piecewise polynomial.*

We also consider the case of a product test configuration, which means that there is an algebraic $\mathbb{C}^\times$-action ρ on the pair (X, L). We let φ be a positive S^1-invariant metric on L. Using the action ρ, we get a geodesic ray φ_t of positive metrics on L such that $\varphi_1 = \varphi$. Let us denote the t derivative at the point one by $\dot{\varphi}$. It is a real-valued function on X. There is also a natural volume element, given by $dV_\varphi := (dd^c\varphi)^n/n!$. By the function $\dot{\varphi}/2$ we can push forward the measure dV_φ to a measure on $\mathbb{R}$, which we denote by μ_φ. This measure does not depend on the particular choice of positive S^1-invariant metric φ. In fact, we have the following.

THEOREM 4. *If we denote the product test configuration by $\mathcal{T}$, and the corresponding concave transform by $G[\mathcal{T}]$, then for any positive S^1-invariant metric φ it holds that*

$$\mu_\varphi = G[\mathcal{T}]_* d\lambda_{|\Delta(L)}.$$

The proof uses Theorem 1 combined with the approach of Berndtsson in [1], but is simpler in nature.

Phong-Sturm have in their articles [15] and [17] shown that the pair of a test configuration $\mathcal{T}$ and a positive metric φ on L canonically determines a $C^{1,1}$ geodesic ray of positive metrics on L emanating from φ. We conjecture that the analogue of Theorem 4 is true also in that more general case.

In [14] Okounkov considered the case of a connected reductive group G acting on a projective variety, and there used the concept of an Okounkov body to prove that in the classical limit the law describing the multiplicities as a funciton of their respective highest weight was log-concave. The case $G = S^1$ corresponds to what we have called a product test configuration. However Okounkov, for his purposes, chooses a flag which is invariant under the group action, while we let the flag to be chosen independently of the action, focusing on the resulting concave function on the Okounkov body. See also [10] where Kaveh-Khovanskii extend the previous work of Okounkov in [14], building a theory on Okounkov bodies associated to graded G-algebras, obtaining among other things general results on log-concavity of the accompanying Duistermaat-Heckman measures.

2.1.4 Organization of the paper

The definition of Okounkov bodies and some fundamental results concerning them is in Section 2.2, using [12] by Lazarsfeld-Mustaţă as our main reference.

Section 2.3 is devoted to describing the setup, definitions and main results of the article [2] by Boucksom-Chen on the concave transform of filtrations.

Section 2.4 contains a brief introduction to test configurations, following mainly Donaldson in [5] and [6].

We discuss embeddings of test configurations in Section 2.5, and link it to certain Duistermaat-Heckman measures.

In Section 2.6 we show how to construct the associated filtration to a test configuration, and prove Theorem 1, Corollary 2 and Corollary 3.

Section 2.7 concerns toric test configurations. We show that what we have done is a generalization of the toric picture, by proving that in the toric case, the concave transform is identical to the function on the polytope considered by Donaldson in [5].

Relying on the work of Ross-Thomas in [18] and [19], we obtain in Section 2.8 an explicit description of the concave transforms corresponding to a special class of test configurations, namely those arising from a deformation to the normal cone with respect to some subscheme.

In Section 2.9 we study the case of product test configurations, and relate it to geodesic rays of positive hermitian metrics. Hence we prove Theorem 4.

2.1.5 Acknowledgements

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2.2 The Okounkov body of a line bundle

Let Γ be a subset of $\mathbb{N}^{n+1}$, and suppose that it is a semigroup with respect to vector addition, i.e. if α and β lie in Γ, then the sum $\alpha+\beta$ should also lie in Γ. We denote by $\Sigma(\Gamma)$ the closed convex cone in $\mathbb{R}^{n+1}$ spanned by Γ.

DEFINITION 1. *The Okounkov body* $\Delta(\Gamma)$ *of* Γ *is defined by*

$$\Delta(\Gamma) := \{\alpha : (\alpha, 1) \in \Sigma(\Gamma)\} \subseteq \mathbb{R}^n.$$

Since by definition $\Sigma(\Gamma)$ is convex, and any slice of a convex body is itself convex, it follows that the Okounkov body $\Delta(\Gamma)$ is convex.

By $\Delta_k(\Gamma)$ we will denote the set

$$\Delta_k(\Gamma) := \{\alpha : (k\alpha, k) \in \Gamma\} \subseteq \mathbb{R}^n.$$

It is clear that for all non-negative k,

$$\Delta_k(\Gamma) \subseteq \Delta(\Gamma) \cap ((1/k)\mathbb{Z})^n.$$

We will explain the procedure, which is due to Okounkov (see [14]), of associating a semigroup to a big line bundle.

Let X be a complex compact projective manifold of dimension n, and L a holomorphic line bundle, which we will assume to be big. Suppose we have chosen a point p in X, and local holomorphic coordinates $z_1, ..., z_n$ centered at p, and let $e_p \in H^0(U, L)$ be a local trivialization of L around p. If we divide a section $s \in H^0(X, kL)$ by e_p^k we get a local holomorphic function. It has an unique represention as a convergent power series in the variables z_i,

$$\frac{s}{e_p^k} = \sum a_\alpha z^\alpha,$$

which for convenience we will simply write as

$$s = \sum a_\alpha z^\alpha.$$

We consider the lexicographic order on the multiindices α, and let $v(s)$ denote the smallest index α such that $a_\alpha \neq 0$.

DEFINITION 2. *Let* $\Gamma(L)$ *denote the set*

$$\{(v(s), k) : s \in H^0(kL), k \in \mathbb{N}\} \subseteq \mathbb{N}^{n+1}.$$

It is a semigroup, since for $s \in H^0(kL)$ *and* $t \in H^0(mL)$

$$v(st) = v(s) + v(t).$$

The Okounkov body of L, denoted by $\Delta(L)$*, is defined as the Okounkov body of the associated semigroup* $\Gamma(L)$*.*

We write $\Delta_k(\Gamma(L))$ simply as $\Delta_k(L)$.

Remark. Note that the Okounkov body $\Delta(L)$ of a line bundle L in fact depends on the choice of point p in X and local coordinates z_i. We will however supress this in the notation, writing $\Delta(L)$ instead of the perhaps more proper but cumbersome $\Delta(L, p, (z_i))$.

From the article [12] by Lazarsfeld-Mustaţă we recall some results on Okounkov bodies of line bundles.

LEMMA 5. *The number of points in $\Delta_k(L)$ is equal to the dimension of the vector space $H^0(kL)$.*

LEMMA 6. *We have that*

$$\Delta(L) = \overline{\cup_{k=1}^{\infty} \Delta_k(L)}.$$

LEMMA 7. *The Okounkov body $\Delta(L)$ of a big line bundle is a bounded hence compact convex body.*

DEFINITION 3. *The volume of a line bundle L, denoted by vol(L), is defined by*

$$vol(L) := \limsup_{k\to\infty} \frac{\dim H^0(kL)}{k^n/n!}.$$

The most important property of the Okounkov body is its relation to the volume of the line bundle, described in the following theorem.

THEOREM 8. *For any big line bundle it holds that*

$$vol(L) = n! vol_{\mathbb{R}^n}(\Delta(L)),$$

where the volume of the Okounkov body is measured with respect to the standard Lesbesgue measure on $\mathbb{R}^n$.

For the proof see [12].

2.3 The concave transform of a filtered linear series

In this section, we will follow Boucksom-Chen in [2].

First we recall what is meant by a filtration of a graded algebra.

DEFINITION 4. *By a filtration $\mathcal{F}$ of a graded algebra $\oplus_k V_k$ we mean a vector space-valued map from $\mathbb{R} \times \mathbb{N}$,*

$$\mathcal{F} : (t, k) \longmapsto \mathcal{F}_t V_k,$$

such that for any k, $\mathcal{F}_t V_k$ is a family of subspaces of V_k that is decreasing and left-continuous in t.

In [2] Boucksom-Chen consider certain filtrations which behaves well with respect to the multiplicative structure of the algebra.

They give the following definition.

DEFINITION 5. *Let $\mathcal{F}$ be a filtration of a graded algebra $\oplus_k V_k$. We shall say that*

(i) $\mathcal{F}$ is multiplicative if

$$(\mathcal{F}_t V_k)(\mathcal{F}_s V_m) \subseteq \mathcal{F}_{t+s} V_{k+m}$$

for all $k, m \in \mathbb{N}$ and $s, t \in \mathbb{R}$.

(ii) $\mathcal{F}$ is pointwise left-bounded if for each k $\mathcal{F}_t V_k = V_k$ for some t.

(iii) $\mathcal{F}$ is linearly right-bounded if there exist a constant C such that for all k, $\mathcal{F}_{kC} V_k = \{0\}$.

A filtration $\mathcal{F}$ is said to be admissible if it is multiplicative, pointwise left-bounded and linearly right-bounded.

Given a line bundle L on X, its section ring $\oplus_k H^0(kL)$ is a graded algebra.

Boucksom-Chen in [2] show how an admissible filtration on the section ring $\oplus_k H^0(kL)$ of a big line bundle L gives rise to a concave function on the Okounkov body $\Delta(L)$. We will review how this is done.

First let us define the following set

$$\Delta_{k,t}(L, \mathcal{F}) := \{v(s)/k : s \in \mathcal{F}_t H^0(kL)\} \subseteq \mathbb{R}^n,$$

where as before $v(s) = \alpha$ if locally

$$s = Cz^{\alpha} + \text{higher order terms},$$

C being some nonzero constant. From the definition it is clear that

$$\Delta_{k,t}(L, \mathcal{F}) \subseteq \Delta_k(L),$$

since

$$\Delta_k(L) = \{v(s)/k : s \in H^0(kL)\}$$

and $\mathcal{F}_t H^0(kL) \subseteq H^0(kL)$. Similarly as in Lemma 5, from [12] we get that

$$|\Delta_{k,t}(L, \mathcal{F})| = \dim \mathcal{F}_t H^0(kL), \tag{2.1}$$

where $|.|$ denotes the cardinality of the set.

For each k we may define a function G_k on $\Delta_k(L)$ by letting

$$G_k(\alpha) := \sup\{t : \alpha \in \Delta_{k,t}(L, \mathcal{F})\}.$$

From the assumption that $\mathcal{F}$ is both left- and right-bounded it follows that G_k is well-defined and real-valued.

LEMMA 9. *If we denote by $\nu_k(L)$ the sum of dirac measures at the points of $\Delta_k(L)$, i.e.*

$$\nu_k(L) := \sum_{\alpha \in \Delta_k(L)} \delta_\alpha,$$

then we have that

$$G_{k*}\nu_k(L) = \frac{d}{dt}(-\dim \mathcal{F}_t H^0(kL)).$$

Proof. By equation (2.1) and the definition of G_k we have that

$$\dim \mathcal{F}_t H^0(kL) = |\Delta_{k,t}(L, \mathcal{F})| = \int_{\{G_k \geq t\}} d\nu_k(L) = \int_t^{\infty} (G_k)_*(\nu_k(L)). \tag{2.2}$$

The lemma now follows by differentiating the equation (2.2). □

On the union $\cup_{k=1}^{\infty}\Delta_k(L)$ one may define the function

$$G[\mathcal{F}](\alpha) := \sup\{G_k(\alpha)/k : \alpha \in \Delta_k(L)\}.$$

By Boucksom-Chen in [2], or Witt Nyström in [22], one then gets that the function $G[\mathcal{F}]$ extends to a concave and therefore continuous function on the interior of $\Delta(L)$. In fact one gets that $G[\mathcal{F}]$ is not only the supremum but also the limit of G_k/k, i.e. for any $p \in \Delta(L)^\circ$

$$G[\mathcal{F}](p) = \lim_{k\to\infty} G_k(\alpha_k)/k,$$

for any sequence α_k converging to p.

Remark. To show how this fits into the framework of [22], we note that if we let

$$\tilde{G}(\alpha, k) := G_k(\alpha/k),$$

then $\tilde{G}$ is a function on $\Gamma(L)$. By the multiplicity of $\mathcal{F}$ it follows that $\tilde{G}$ is superadditive, and by the linear right-boundedness, $\tilde{G}$ is going to be linearly bounded from above. Thus one may apply the results of [22] to this function.

The main result of [2], Theorem A, is that we also have weak convergence of measures.

THEOREM 10. *The measures*

$$\frac{1}{k^n}((G_k/k)_*\nu_k(L))$$

converge weakly to the measure

$$G[\mathcal{F}]_* d\lambda_{|\Delta(L)}$$

as k tends to infinity, where $d\lambda_{|\Delta(L)}$ denotes the Lebesgue measure on $\mathbb{R}^n$ restricted to $\Delta(L)$.

2.4 Test configurations

We will give a very brief introduction to the subject of test configurations. Our main references are the articles [5] and [6] by Donaldson.

First the definition of a test configuration, as introduced by Donaldson in [5].

DEFINITION 6. *A test configuration $\mathcal{T}$ for an ample line bundle L over X consists of:*

(i) a scheme $\mathcal{X}$ with a $\mathbb{C}^\times$-action ρ,

(ii) an $\mathbb{C}^\times$-equivariant line bundle $\mathcal{L}$ over $\mathcal{X}$,

(iii) and a flat $\mathbb{C}^\times$-equivariant projection $\pi : \mathcal{X} \to \mathbb{C}$ where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by multiplication, such that $\mathcal{L}$ is relatively ample, and such that if we denote by $X_1 := \pi^{-1}(1)$, then $\mathcal{L}_{|X_1} \to X_1$ is isomorphic to $rL \to X$ for some $r > 0$.

By rescaling we can for our purposes without loss of generality assume that $r = 1$ in the definition.

A test configuration is called a product test configuration if there is a $\mathbb{C}^\times$-action ρ' on $L \to X$ such that $\mathcal{L} = L \times \mathbb{C}$ with ρ acting on L by ρ' and on $\mathbb{C}$ by multiplication. A test configuration is called trivial if it is a product test configuration with the action ρ' being the trivial $\mathbb{C}^\times$-action.

Since the zero-fiber $X_0 := \pi^{-1}(0)$ is invariant under the action ρ, we get an induced action on the space $H^0(kL_0)$, also denoted by ρ, where we have denoted the restriction of $\mathcal{L}$ to X_0 by L_0. Specifically, we let $\rho(\tau)$ act on a section $s \in H^0(kL_0)$ by

$$(\rho(\tau)(s))(x) := \rho(\tau)(s(\rho^{-1}(\tau)(x))). \tag{2.3}$$

Remark. Some authors refer to the inverted variant

$$(\rho(\tau)(s))(x) := \rho^{-1}(\tau)(s(\rho(\tau)(x)))$$

as the induced action. This is only a matter of convention, but one has to be aware that all the weights as defined below changes sign when changing from one convention to the other.

Any vector space V with a $\mathbb{C}^\times$-action can be split into weight spaces V_{η_i} on which $\rho(\tau)$ acts as multiplication by τ^{η_i}, (see e.g. [5]). The numbers η_i with non-trivial weight spaces are called the weights of the action. Thus we may write $H^0(kL_0)$ as

$$H^0(kL_0) = \oplus_\eta V_\eta$$

with respect to the induced action ρ.

In [15], Lemma 4, Phong-Sturm give the following linear bound on the absolute value of the weights.

LEMMA 11. *Given a test configuration there is a constant C such that*

$$|\eta_i| < Ck$$

whenever $\dim V_{\eta_i} > 0$.

There is an associated weight measure on $\mathbb{R}$:

$$\mu(\mathcal{T}, k) := \sum_{\eta=-\infty}^{\infty} \dim V_\eta \delta_\eta,$$

and also the rescaled variant

$$\tilde{\mu}(\mathcal{T}, k) := \frac{1}{k^n} \sum_{\eta=-\infty}^{\infty} \dim V_\eta \delta_{k^{-1}\eta}. \tag{2.4}$$

The first moment of the measure $\mu(\mathcal{T}, k)$, which we will denote by w_k, thus equals the sum of the weights η_i with multiplicity $\dim V_{\eta_i}$. It can also be seen as the weight of the induced action on the top exterior power of $H^0(kL_0)$. The total mass of $\mu(\mathcal{T}, k)$ is $\dim H^0(kL_0)$, which we will denote by d_k. By the flatness of π it follows that for k large it will be equal to $\dim H^0(kL)$ (see e.g. [18]). One is interested in the asymptotics of the weights, and from the equivariant Riemann-Roch theorem one gets that there is an asymptotic expansion in powers of k of the expression w_k/kd_k (see e.g. [5]),

$$\frac{w_k}{kd_k} = F_0 - k^{-1}F_1 + O(k^{-2}). \tag{2.5}$$

F_1 is called the Futaki invariant of $\mathcal{T}$, and will be denoted by $F(\mathcal{T})$.

DEFINITION 7. *A line bundle L is called K-semistable if for all test configurations $\mathcal{T}$ of L over X, it holds that $F(\mathcal{T}) \geq 0$. L is called K-stable if it is K-semistable and furthermore $F(\mathcal{T}) = 0$ iff $\mathcal{T}$ is a product test configuration.*

Donaldson has conjectured that L being K-stable is equivalent to the existence of a positive constant scalar curvature hermitian metric with Kähler form in $c_1(L)$ (see [5], [6] and the expository article [16]).

2.5 Embeddings of test configurations

One way to construct a test configuration of a pair (X, L) is by using a Kodaira embedding of (X, L) into $(\mathbb{P}^N, \mathcal{O}(1))$ for some N. If ρ is a $\mathbb{C}^\times$-action on $\mathbb{P}^N$, this gives rise to a product test configuration of $(\mathbb{P}^N, \mathcal{O}(1))$. If we restrict to the image of ρ's action on (X, L), we end up with a test configuration of (X, L). A basic fact (see e.g. [19]) is that all test configurations arise this way, so that one may embed $\mathcal{X}$ into $\mathbb{P}^N \times \mathbb{C}$ for some N, the action ρ coming from a $\mathbb{C}^\times$-action on $\mathbb{P}^N$.

Let $\mathcal{T}$ be a test configuration, and assume that we have chosen an embeddding as above. Let z_i be homogeneous coordinates on $\mathbb{P}^N$, and let us define the following functions

$$h_{ij} := \frac{z_i \bar{z}_j}{||z||^2}.$$

We assume that we have chosen our coordinates so that the metric $||z||^2$ is invariant under the corresponding S^1-action on $\mathbb{C}^{N+1}$. Then the infinitesimal generator of the action ρ is given by a hermitian matrix A. We define a real-valued function h on $\mathbb{P}^N$ by

$$h := \sum A_{ij} h_{ij}.$$

It is a Hamiltonian for the S^1-action (see [6]). Let ω_{FS} denote the Fubini-Study form on $\mathbb{P}^N$. The zero-fiber X_0 of the test configuration can via the embedding be identified with subsheme of $\mathbb{P}^N$, invariant under the action of ρ. By $|X_0|$ we will denote the corresponding algebraic cycle, and we let $[X_0]$ denote its integration current. The wedge product of $[X_0]$ with the positive (n, n)-form $\omega_{FS}^n/n!$ gives a positive measure, $d\mu_{FS}$, with $|X_0|$ as its support. We have the following proposition.

PROPOSITION 12. *In the setting as above, the normalized weight measures $\tilde{\mu}(\mathcal{T}, k)$ of the test configuration converges weakly as k tends to infinity to the pushforward of the measure $d\mu_{FS}$ with respect to the Hamiltonian h,*

$$\tilde{\mu}(\mathcal{T}, k) \to h_* d\mu_{FS}.$$

Proof. This is essentially just a reformulation of a result by Donaldson in [6]. Using the weight measures $\tilde{\mu}(\mathcal{T}, k)$, Equation (20) in the proof of Proposition 3 in [6] says that

$$\int_{\mathbb{R}} x^r d\tilde{\mu}(\mathcal{T}, k) = \int_{|X_0|} h^r d\mu_{FS} + o(1).$$

for any positive integer r. In other words, for all such r, the r-moments of the measures $\tilde{\mu}(\mathcal{T}, k)$ converge to the r-moment of the pushforward measure $h_* d\mu_{FS}$. But then it is classical that this implies weak convergence of measures. □

The measure $h_* d\mu_{FS}$ is the sort of measure studied by Duistermaat-Heckman in [7]. They consider a smooth symplectic manifold M with symplectic form σ, and an effective Hamiltonian torus action on M. This gives rise to a moment mapping J, which is a map from M to the dual of the Lie algebra of the torus, which we can naturally identify with $\mathbb{R}^k$, k being the dimension of the torus (we refer the reader to [7] for the definitions). There is a natural volume measure on M, given by $\sigma^n/n!$, called the Liouville measure. The pushforward of the Liouville measure with the moment map J, $J_*(\sigma^n/n!)$, is called a Duistermaat-Heckman measure. They prove that it is absolute continuous with respect to Lebesgue measure on $\mathbb{R}^k$, and provide an explicit formula, in the literature referred to as the Duistermaat-Heckman formula, for the density function f. As a corollary they get the following.

THEOREM 13. *The density function f of the measure $J_*(\sigma^n/n!)$ is a polynomial of degree less that the dimension of M on each connected component of the set of regular values of the moment map J.*

In our setting the Liouville measure is given by $d\mu_{FS}$, and the moment map J is simply given by the Hamiltonian h. Thus when all components of the algebraic cycle $|X_0|$ are smooth manifolds, and the action is effective, we can apply Theorem 13 to our measure $h_* d\mu_{FS}$ and conclude that it is a piecewise polynomial measure on $\mathbb{R}$. In general of course some components of $|X_0|$ may have singularities. However, one case where we know that X_0 is a smooth manifolld is when we have a product test configuration, because then $X_0 = X$. Hence we get the following.

PROPOSITION 14. *For a product test configuration, with a corresponding effective S^1-action, it holds that the law of the asymptotic distribution of its weights is piecewise polynomial.*

Proof. By Proposition 12 the law of the asymptotic distribution of weights is given by the measure $h_* d\mu_{FS}$ and by the remarks above we can use Theorem 13 to conclude that $h_* d\mu_{FS}$ is piecewise polynomial. □

2.6 The concave transform of a test configuration

Given a test configuration $\mathcal{T}$ of L we will show how to get an associated filtration $\mathcal{F}$ of the section ring $\oplus_k H^0(kL)$.

First note that the $\mathbb{C}^\times$-action ρ on $\mathcal{L}$ via the equation (2.3) gives rise to an induced action on $H^0(\mathcal{X}, k\mathcal{L})$ as well as $H^0(\mathcal{X} \setminus X_0, k\mathcal{L})$, since $\mathcal{X} \setminus X_0$ is invariant.

Let $s \in H^0(kL)$ be a holomorphic section. Then using the $\mathbb{C}^\times$-action ρ we get a canonical extension $\bar{s} \in H^0(\mathcal{X} \setminus X_0, k\mathcal{L})$ which is invariant under the action ρ, simply by letting

$$\bar{s}(\rho(\tau)x) := \rho(\tau)s(x) \tag{2.6}$$

for any $\tau \in \mathbb{C}^\times$ and $x \in X$.

We identify the coordinate t with the projection function $\pi(x)$, and we also consider it as a section of the trivial bundle over $\mathcal{X}$. Exactly as for $H^0(\mathcal{X}, k\mathcal{L})$, ρ gives rise to an induced action on sections of the trivial bundle, using the same formula (2.3). We get that

$$(\rho(\tau)t)(x) = \rho(\tau)(t(\rho^{-1}(\tau)x) = \rho(\tau)(\tau^{-1}t(x)) = \tau^{-1}t(x), \tag{2.7}$$

where we used that ρ acts on the trivial bundle by multiplication on the t-coordinate. Thus

$$\rho(\tau)t = \tau^{-1}t,$$

which shows that the section t has weight -1.

By this it follows that for any section $s \in H^0(kL)$ and any integer η, we get a section $t^{-\eta}\bar{s} \in H^0(\mathcal{X} \setminus X_0, k\mathcal{L})$, which has weight η.

LEMMA 15. *For any section $s \in H^0(kL)$ and any integer η the section $t^{-\eta}\bar{s}$ extends to a meromorphic section of $k\mathcal{L}$ over the whole of $\mathcal{X}$, which we also will denote by $t^{-\eta}\bar{s}$.*

Proof. It is equivalent to saying that for any section s there exists an integer η such that $t^{\eta}\bar{s}$ extends to a holomorphic section $S \in H^0(\mathcal{X}, k\mathcal{L})$. By flatness, which was assumed in the definition of a test configuration, the direct image bundle $\pi_*\mathcal{L}$ is in fact a vector bundle over $\mathbb{C}$. Thus it is trivial, since any vector bundle over $\mathbb{C}$ is trivial. By e.g. Lemma 2 in [15] any complex vector bundle over $\mathbb{C}$ with a $\mathbb{C}^{\times}$-action has an equivariant trivialization. The trivialization consits of global sections S_i, giving a basis at each point t, and with the additional property that

$$\rho(\tau)S_i = \sum f_{ij}(\tau)S_j, \tag{2.8}$$

where the $f_{ij}s$ are holomorphic. The action restricts to the fiber over zero, and thus we have a decomposition in a finite number of weight spaces V_η. By restricting equation (2.8) to this fiber it follows that the functions f_{ij} are Laurent polynomials in τ whose degrees are bounded from above by the maximum of the weights η and from below by the minimum of the weights.

Consider a section $s \in H^0(kL)$. We can write it as $s = \sum a_i S_i(1)$. It follows that

$$\bar{s}(t) = (\sum a_i \rho(t) S_i)(t) = (\sum a_i f_{ij}(t) S_j)(t). \tag{2.9}$$

Since we observed that the degrees of the Laurent polynomials $f_{ij}s$ were bounded from below, equation (2.9) tells us that $\bar{s}(t)$ extends holomorphically after multiplying by t raised to some large power. □

DEFINITION 8. *Given a test configuration $\mathcal{T}$ we define a vector space-valued map $\mathcal{F}$ from $\mathbb{Z} \times \mathbb{N}$ by letting*

$$(\eta, k) \longmapsto \{s \in H^0(kL) : t^{-\eta}\bar{s} \in H^0(\mathcal{X}, k\mathcal{L})\} =: \mathcal{F}_\eta H^0(kL).$$

It is immediate that $\mathcal{F}_\eta$ is decreasing since $H^0(\mathcal{X}, k\mathcal{L})$ is a $\mathbb{C}[t]$-module. We can extend $\mathcal{F}$ to a filtration by letting

$$\mathcal{F}_\eta H^0(kL) := \mathcal{F}_{\lceil \eta \rceil} H^0(kL)$$

for non-integers η, thus making $\mathcal{F}$ left-continuous. Since

$$t^{-(\eta+\eta')}\overline{ss'} = (t^{-\eta}\bar{s})(t^{-\eta'}\bar{s'}) \in H^0(\mathcal{X}, k\mathcal{L})H^0(\mathcal{X}, m\mathcal{L}) \subseteq H^0(\mathcal{X}, (k+m)\mathcal{L})$$

whenever $s \in \mathcal{F}_\eta H^0(kL)$ and $s' \in \mathcal{F}_{\eta'} H^0(kL)$, we see that

$$(\mathcal{F}_\eta H^0(kL))(\mathcal{F}_{\eta'} H^0(mL)) \subseteq \mathcal{F}_{\eta+\eta'} H^0((k+m)L),$$

i.e. $\mathcal{F}$ is multiplicative. Furthermore, by Lemma 39 it follows that $\mathcal{F}$ is left-bounded and right-bounded.

PROPOSITION 16. *For $k \gg 0$*

$$\mu(\mathcal{T}, k) = \frac{d}{d\eta}(-\dim \mathcal{F}_\eta H^0(kL)).$$

Proof. Recall that we had the decomposition in weight spaces

$$H^0(kL_0) = \oplus_\eta V_\eta,$$

and that

$$\mu(\mathcal{T}, k) := \sum_{\eta=-\infty}^{\infty} \dim V_\eta \delta_\eta.$$

We have the following isomorphism:

$$(\pi_* k\mathcal{L})_{|\{0\}} \cong H^0(\mathcal{X}, k\mathcal{L})/tH^0(\mathcal{X}, k\mathcal{L}),$$

the right-to-left arrow being given by the restriction map, see e.g. [19]. Also, for $k \gg 0$, $(\pi_* k\mathcal{L})_{|\{0\}} = H^0(kL_0)$, therefore we get that for large k

$$H^0(kL_0) \cong H^0(\mathcal{X}, k\mathcal{L})/tH^0(\mathcal{X}, k\mathcal{L}), \tag{2.10}$$

We also have a decomposition of $H^0(\mathcal{X}, k\mathcal{L})$ into the sum of its invariant weight spaces W_η. By Lemma 39 it is clear that a section $S \in H^0(\mathcal{X}, k\mathcal{L})$ lies in W_η

if and only if it can be written as $t^{-\eta}\bar{s}$ for some $s \in H^0(kL)$, in fact we have that $s = S_{|X}$. Thus we get that

$$W_\eta \cong \mathcal{F}_\eta H^0(kL),$$

and by the isomorphism (3.38) then

$$V_\eta \cong \mathcal{F}_\eta H^0(kL)/\mathcal{F}_{\eta+1} H^0(kL).$$

Thus we get

$$\dim \mathcal{F}_\eta H^0(kL) = \sum_{\eta' \geq \eta} \dim V_{\eta'}, \tag{2.11}$$

and the lemma follows by differentiating with respect to η on both sides of the equation (3.39). □

PROPOSITION 17. *The filtration associated to a test configuration $\mathcal{T}$ is always admissible. If we let $G_k[\mathcal{T}]$ denote the functions on $\Delta_k(L)$ associated to the filtration $\mathcal{F}(\mathcal{T})$ as previously definied, then we have that*

$$\mu(\mathcal{T}, k) = G_k[\mathcal{T}]_* \nu_k(L) \tag{2.12}$$

and

$$\tilde{\mu}(\mathcal{T}, k) = \frac{1}{k^n}((G_k[\mathcal{T}]/k)_*(\nu_k(L))). \tag{2.13}$$

Proof. The equality of measures (2.12) follows immediately from combining Lemma 9 and Proposition 16, and (2.13) is just a rescaling of (2.12). Since by Lemma 38 the weights of a test configuration is linearly bounded, by (2.12) we get that the same holds for the functions $G_k[\mathcal{T}]$, i.e. the filtration $\mathcal{F}$ is linearly left- and right-bounded. It is hence admissible, since the other defining properties had already been checked. □

THEOREM 18. *With the setting as in the proposition above, we have the following weak convergence of measures as k tends to infinity*

$$\tilde{\mu}(\mathcal{T}, k) \to G[\mathcal{T}]_* d\lambda_{|\Delta(L)}.$$

Proof. Follows from Theorem 10 together with Proposition 17. □

COROLLARY 19. *In the asymtotic expansion*

$$\frac{w_k}{kd_k} = F_0 - k^{-1}F_1 + O(k^{-2})$$

we have that

$$F_0 = \frac{n!}{vol(L)} \int_{\Delta(L)} G(\mathcal{T})d\lambda.$$

Proof. Recall that in Section 2.4 we defined w_k by

$$w_k := \int_{\mathbb{R}} x d\mu(\mathcal{T}, k),$$

i.e. in other words

$$w_k = \sum \eta \dim V_\eta,$$

$\oplus_\eta V_\eta$ being the weight space decomposition of $H^0(kL_0)$.Thus Theorem 18 implies that

$$\lim_{k\to\infty} \frac{w_k}{k^{n+1}} = \lim_{k\to\infty} \int_{\mathbb{R}} x\tilde{\mu}(\mathcal{T}, k) = \int_{\mathbb{R}} x(G[\mathcal{T}])_*(d\lambda_{|\Delta(L)}) = \int_{\Delta(L)} G(\mathcal{T})d\lambda, \tag{2.14}$$

using the weak convergence and the definition of the push forward of a measure. (2.14) together with the standard expansion

$$d_k := \dim H^0(kL) = k^n vol(L)/n! + o(k^n)$$

yields the corollary. □

Another consequence of Theorem 18 is that it relates the Okounkov body $\Delta(L)$ with the central fibre X_0, and therefore X, in the sense of the following corollary.

COROLLARY 20. *Assume that we have embedded the test configuration $\mathcal{T}$ in some $\mathbb{P}^N \times \mathbb{C}$, let h denote the corresponding Hamiltonian and $d\mu_{FS}$ the Fubini-Study volume measure on $|X_0|$ as in Section 2.4. Then we have that*

$$G[\mathcal{T}]_* d\lambda_{|\Delta(L)} = h_* d\mu_{FS}.$$

Proof. Follows immediately from combining Proposition 12 and Theorem 18. □

As in Section 2.5, if we restrict to the case of product test configurations where the S^1-action is effective, we can apply the Duistermaat-Heckman theorem to these measures, and get the following.

COROLLARY 21. *Assume that there is a $\mathbb{C}^\times$-action on X which lifts to L, and that the corresponding S^1-action is effective. If we denote the associated product test configuration by $\mathcal{T}$, the concave transform $G[\mathcal{T}]$ is such that the push-forward measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$ is piecewise polynomial.*

Proof. Follows from combining Proposition 14 and Corollary 20. □

2.7 Toric test configurations

We will cite some basic facts of toric geometry, all of which can be found in the article [5] by Donaldson. Let $L_P \to X_P$ be a toric line bundle with corresponding polytope $P \subseteq \mathbb{R}^n$. Thus for every k there is a basis for $H^0(kL_P)$ such that there is a one-one correspondence between the basis elements and the integer lattice points of kP. We write this as

$$\alpha \in kP \cap \mathbb{Z}^n \leftrightarrow z^\alpha \in H^0(kL_P).$$

In [5] Donaldson describes the relationship between toric test configurations and the geometry of polytopes. Let g be a positive concave rational piecewise affine function defined on P. One may define a polytope Q in $\mathbb{R}^{n+1}$ with P as its base and the graph of g as its roof, i.e.

$$Q := \{(x, y) : x \in P, y \in [0, g(x)]\}.$$

That g is rational means precisely that the polytope Q is rational, i.e. it is the convex hull of a finite set of rational points in $\mathbb{R}^n$. In fact, by scaling we can without loss of generality assume that Q is integral, i.e. the convex hull of a finite set of integer points. Then by standard toric geometry this polytope Q corresponds to a toric line bundle L_Q over a toric variety X_Q of dimension $n + 1$. We may write the correspondence between integer lattice points of kQ

and basis elements for $H^0(kL_Q)$ as

$$(\alpha, \eta) \in kQ \cap \mathbb{Z}^{n+1} \leftrightarrow t^{-\eta} z^{\alpha} \in H^0(kL_Q). \tag{2.15}$$

There is a natural $\mathbb{C}^{\times}$-action ρ given by multiplication on the t-variable. We also get a projection π of X_Q down to $\mathbb{P}^1$, by letting

$$\pi(x) := \frac{t^{-\eta+1} z^{\alpha}(x)}{t^{-\eta} z^{\alpha}(x)}$$

for any η, α such that this is well defined. Donaldson shows in [5] that if one excludes $\pi^{-1}(\infty)$, then the triple L_Q, ρ and π is in fact a test configuration, so π is flat and the fiber over 1 of (X_Q, L_Q) is isomorphic to (X_P, L_P).

It was shown by Lazarsfeld-Mustaţă in [12], Example 6.1, that if one choses the coordinates, or actually the flag of subvarieties, so that it is invariant under the torus action, the Okounkov body of a toric line bundle is equal to its defining polytope, up to translation. Thus we may assume that $P = \Delta(L_P)$ and

$$v(z^{\alpha}) = \alpha.$$

The invariant meromorphic extension of the section $z^{\alpha} \in H^0(kL_P)$ is $z^{\alpha} \in H^0(kL_Q)$, where we have identified X_P with the fiber over 1. By our calculations in Section 2.6, equation (3.36), the weight of $t^{-\eta} z^{\alpha}$ is η. Thus we see that

$$G_k(\alpha) = \sup\{\eta : t^{-\eta} z^{k\alpha} \in H^0(kL_Q)\} = kg(\alpha),$$

by the correspondence (2.15) and the fact that g is the defining equation for the roof of Q. We get that G_k/k is equal to the function g restricted to $\Delta_k(L)$, and thus by the convergence of G_k/k to $G[\mathcal{T}]$, that

$$G[\mathcal{T}] = g.$$

We see that our concave transform $G[\mathcal{T}]$ is a proper generalization of the well-known correspondence between test configurations and concave functions in toric geometry.

It is thus clear that, as was shown for product test configurations in Proposition 21, for toric test configurations it holds that the pushforward measure

$$G[\mathcal{T}]_* d\lambda_{|\Delta(L_P)} = g_* d\lambda_{|P}$$

is the sum of a piecewise polynomial measure and a multiple of a dirac measure, simply because P is a polytope and g is piecewise affine (the dirac measure part coming the top of the roof).

2.8 Deformation to the normal cone

One interesting class of test configurations is the ones which arise as a deformation to the normal cone with respect to some subscheme. This is described in detail by Ross-Thomas in [18] and [19], and we will only give a brief outline here.

Let Z be any proper subscheme of X. Consider the blow up of $X \times \mathbb{C}$ along $Z \times \{0\}$, and denote it by $\mathcal{X}$. Hence we get a projection π to $\mathbb{C}$ by composition $\mathcal{X} \to X \times \mathbb{C} \to \mathbb{C}$. We let P denote the exceptional divisor, and for any positive rational number c we get a line bundle

$$\mathcal{L}_c := \pi^* L - cP.$$

By Kleimans criteria (see e.g. [11]) it follows that $\mathcal{L}_c$ is relatively ample for small c. The action on $(X \times \mathbb{C}, L \times \mathbb{C})$ given by multiplication on the $\mathbb{C}$-coordinate lifts to an action ρ on $(\mathcal{X}, \mathcal{L}_c)$, since both $Z \times \{0\}$ and $L \times \mathbb{C}$ are invariant under the action downstairs. Ross-Thomas in [18] show that this data defines a test configuration.

From the proof of Theorem 4.2 in [18] we get that

$$H^0(\mathcal{X}, k\mathcal{L}_c) = \bigoplus_{i=1}^{ck} t^{ck-i} H^0(X, kL \otimes \mathcal{J}_Z^i) \oplus t^{ck}\mathbb{C}[t]H^0(kL), \tag{2.16}$$

for k sufficiently large and $ck \in \mathbb{N}$. Here $\mathcal{J}_Z$ denotes the ideal sheaf of Z, and the sections of kL are being identified with their invariant extensions. From the expression (2.16) we can read off the associated filtration $\mathcal{F}$ of $H^0(kL)$. That

$$t^{ck} H^0(kL) \subseteq H^0(\mathcal{X}, k\mathcal{L}_c)$$

means that

$$\mathcal{F}_{-ck} H^0(kL) = H^0(kL).$$

Furthermore, for $0 \leq i \leq ck$ and any $s \in H^0(kL)$ we get that $t^{ck-i}s \in H^0(\mathcal{X}, k\mathcal{L}_c)$ iff $s \in H^0(kL \otimes \mathcal{J}_Z^i)$. This implies that for $-ck \leq \eta \leq 0$,

$$\mathcal{F}_\eta H^0(kL) = H^0(kL \otimes \mathcal{J}_Z^{ck+\eta}).$$

Also, when $\eta > 0$ we get that $\mathcal{F}_\eta H^0(kL) = \{0\}$. In summary, if we let $g_{c,k}$ be defined by

$$g_{c,k}(\eta) := \lceil \max(\eta + ck, 0) \rceil$$

for $\eta \in (-\infty, 0]$ and let $g_{c,k} \equiv \infty$ on $(0, \infty)$, then by our calculations

$$\mathcal{F}_\eta H^0(kL) = H^0(kL \otimes \mathcal{J}_Z^{g_{c,k}(\eta)}). \tag{2.17}$$

Thus this natural class of filtrations can be seen as coming from test configurations.

Let us assume that Z is an ample divisor with a defining holomorphic section $s \in H^0(Z)$, i.e. $Z = \{s = 0\}$. Let a be a number between zero and c, then $L - aZ$ is still ample. Using multiplication with s^{ka} we can embed $H^0(k(L - aZ))$ into $H^0(kL)$. With respect to this identification of $H^0(k(L - aZ))$ as a subspace of $H^0(kL)$ for all k, we can identify the Okounkov body of $L - aZ$ with a subset of $\Delta(L)$. By vanishing theorems (see e.g. [12]), for large k

$$H^0(k(L - aZ)) = H^0(kL \otimes \mathcal{J}_Z^{ka}), \tag{2.18}$$

and therefore by (2.17)

$$H^0(k(L - aZ)) = \mathcal{F}_{k(a-c)} H^0(kL).$$

It follows that the part of $\Delta(L)$ where $G[\mathcal{T}]$ is greater or equal to $a - c$ coincides with $\Delta(L - aZ)$.[1]

Recall that by Theorem 8

$$\mathrm{vol}_{\mathbb{R}^n} \Delta(L - aZ) = \frac{\mathrm{vol}(L - aZ)}{n!}.$$

[1]We thank Julius Ross for poining this out to us.

By this, a direct calculation yields that the pushforward measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$ can be written as

$$\frac{\mathrm{vol}(L-cZ)}{n!}\delta_0 - \chi_{[-c,0]}\frac{d}{dx}\left(\frac{\mathrm{vol}(L-(x+c)Z)}{n!}\right)dx,$$

where δ_0 denotes the dirac measure at zero and $\chi_{[-c,0]}$ the indicator function of the interval $[-c,0]$. Since for any ample (or even nef) line bundle the volume is given by integration of the top power of the first Chern class,

$$\mathrm{vol}(L) = \int_X c_1(L)^n,$$

it follows that the volume function is polynomial of degree n in the ample cone. Thus the measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$ is a sum of a polynomial measure of degree less than n and a dirac measure.

Let again Z be an arbitrary subscheme of X. Consider the blow up of X along Z, and let E denote the exceptional divisor. If E is irreducible we may introduce local holomorphic coordinates (z_i) on the blow up, such that locally E is given by the equation $z_1 = 0$. Using these coordinates we get an associated Okounkov body $\Delta(L')$ where $L' = \mu^* L$, and μ denotes the projection from the blow up down to X. However, since all sections of L' and its multiples are lifts of sections of L and its multiples, it is customary to think of $\Delta(L')$ as an Okounkov body of L (see [12]). We will do that from here on. For $s \in H^0(kL)$, the first coordinate of $v(s)$ is equal to the vanishing order of s along Z, i.e. the largest integer r such that $s \in H^0(kL \otimes \mathcal{J}_Z^r)$. Thus by (2.17) we get that

$$\Delta_{k,\eta}(L) = \{v(s)/k : s \in \mathcal{F}_\eta H^0(kL)\} = \Delta_k(L) \cap \{x_1 \geq g_{c,k}(\eta)/k\}.$$

Furthermore

$$G_k(\alpha) = \sup\{\eta : \alpha \in \Delta_{k,\eta}(L)\} =$$
$$= \sup\{\eta : \alpha_1 \geq g_{c,k}(\eta)/k\} = k\min(\alpha_1 - c, 0),$$

and therefore

$$G[\mathcal{T}](x) = \min(x_1 - c, 0).$$

2.9 Product test configurations and geodesic rays

There is an interesting interplay between on the one hand test configurations and geodesic rays in the space of metrics on the other (see e.g. [15] and [17]). The model case is when we have a product test configuration.

Let $\mathcal{H}_L$ denote the space of positive hermitian metrics ψ of a positive line bundle L over X. The tangent space of $\mathcal{H}_L$ at any point ψ is naturally identified with the space of smooth real-valued functions on X. The works of Mabuchi, Semmes and Donaldson (see [13], [20] and [4]) have shown that there is a natural Riemannian metric on $\mathcal{H}_L$, by letting the norm of a tangent vector u at a point $\psi \in \mathcal{H}_L$ be defined by

$$||u||_{\psi}^2 := \int_X |u|^2 dV_{\psi},$$

where $dV_{\psi} := (dd^c\psi)^n$. Let ψ_t be a ray of metrics, $t \in (0, \infty)$. We may extend it to complex valued t in $\mathbb{C}^{\times}$ if we let ψ_t be independent on the argument of t. We say that ψ_t is a geodesic ray if

$$(dd^c\psi_t)^{n+1} = 0 \tag{2.19}$$

on $X \times \mathbb{C}^{\times}$. The equation (2.19) is the geodesic equation with respect to the Riemannian metric on $\mathcal{H}_L$ (see e.g. [17]).

Let $\mathcal{T}$ be a product test configuration. That means that there is a $\mathbb{C}^{\times}$-action ρ on the original pair (X, L). Restriction of ρ to the unit circle gives a S^1-action. Let φ be an S^1-invariant positive metric on L. We get a $\mathbb{C}^{\times}$ ray $\tau \longmapsto \varphi_{\tau} \in \mathcal{H}_L$ of metrics by letting for any $\xi \in L$

$$|\xi|_{\varphi_{\tau}} := |\rho(\tau)^{-1}\xi|_{\varphi}. \tag{2.20}$$

Similarly we get corresponding rays $k\varphi_{\tau}$ in $\mathcal{H}_{kL}$. Since φ was assumed to be S^1-invariant, φ_{τ} only depends on the absolute value $|\tau|$. Also because the action ρ is holomorphic, it follows that

$$(dd^c\varphi_{\tau})^{n+1} = 0,$$

therefore φ_{τ} is a geodesic ray.

In [1] Berndtsson introduces sequences of spectral measures on $\mathbb{R}$ arising naturally from a geodesic segment of metrics, and shows that they converge weakly to a certain pushforward of a volume form on X. Inspired by his result, we consider the analogue in our setting.

Let $\dot{\varphi}$ denote the derivative of φ_τ at 1, so $\dot{\varphi}$ is a smooth real-valued function on X. We consider the positive measure on $\mathbb{R}$ we get by pushing forward the volume form $dV_\varphi := (dd^c\varphi)^n$ on X with this function divided by two,

$$\mu_\varphi := (\dot{\varphi}/2)_* dV_\varphi.$$

The measure μ_φ does not does not depend on the choice of S^1-invariant metric φ. In fact, we have the following result.

THEOREM 22. *Let $G[\mathcal{T}]$ denote the concave transform of the product test configuration. We have an equality of measures*

$$\mu_\varphi = G[\mathcal{T}]_* d\lambda_{|\Delta(L)}.$$

Proof. We will use one of the main ideas in the proof of the main result of Berndtsson in [1], Theorem 3.3. However, in our setting where the geodesic comes from a $\mathbb{C}^\times$-action things are much simpler since we do not need the powerful estimates used in [1].

Let dV be some fixed smooth volume form on X. We will introduce two families of scalar products on $H^0(kL)$, parametrized by τ, $||.||_{\tau,1}$ and $||.||_{\tau,2}$. First we let for any $s \in H^0(kL)$

$$||s||^2_{\tau,1} := \int_X |s|^2_{k\varphi_\tau} dV,$$

while we let

$$||s||^2_{\tau,2} := \int_X |\rho(\tau)^{-1}s|^2_{k\varphi} dV = ||\rho(\tau)^{-1}s||^2_{1,1}.$$

Direct calculations yield that

$$\frac{d}{d\tau}||s||^2_{\tau,1} = \frac{d}{d\tau}\int_X |s|^2_{k\varphi_\tau} dV = \int_X (-k\dot{\varphi}_\tau)|s|^2_{k\varphi_\tau} dV = (T_{-k\dot{\varphi}_\tau} s, s)_{\tau,1}, \tag{2.21}$$

where $T_{-k\dot{\varphi}_\tau}$ denotes the Toeplitz operator with symbol $-k\dot{\varphi}_\tau$.

Differentiating $||.||_{\tau,2}$ with respect to τ we get that

$$\frac{d}{d\tau}||s||^2_{\tau,2} = \frac{d}{d\tau}(\rho(\tau)^{-1}s, \rho(\tau)^{-1}s)_{1,1} = ((\frac{d}{d\tau}\rho(\tau)^{-2})s, s)_{1,1}. \tag{2.22}$$

On the other hand

$$||s||^2_{\tau,1} = \int_X |s(x)|^2_{k\varphi_\tau} dV(x) = \int_X |\rho(\tau)^{-1}(s(x))|^2_{k\varphi} dV(x) =$$
$$= \int_X |(\rho(\tau)^{-1}s)(x)|^2_{k\varphi} dV(\rho(\tau)x) = \int_X |\rho(\tau)^{-1}s|^2_{k\varphi} dV_\tau, \tag{2.23}$$

where $dV_\tau(x) := dV(\rho(\tau)x)$ thus denotes the resulting volume form after the τ-action. Since $dV_\tau(x)$ depends smoothly on τ, using (2.23) we get that

$$\left|\frac{d}{d\tau}_{|\tau=1}||s||^2_{\tau,1} - \frac{d}{d\tau}_{|\tau=1}||s||^2_{\tau,2}\right| = \tag{2.24}$$
$$= \left|\frac{d}{d\tau}_{|\tau=1}\int_X |\rho(\tau)^{-1}s|^2_{k\varphi}(dV_\tau - dV)\right| \leq$$
$$\leq \int_X |\frac{d}{d\tau}_{|\tau=1} dV_\tau| \int_X |s|^2_{k\varphi} dV = C||s||^2_{1,1}, \tag{2.25}$$

where thus C is a uniform constant independent of s and k. Therefore letting $\tau = 1$ in equations (2.21) and (2.22), and using (2.24) we get that

$$\frac{d}{d\tau}\rho(\tau)_{|\tau=1} = T_{k\dot{\varphi}/2} + E_k, \tag{2.26}$$

where the error term E_k is uniformly bounded, $||E_k|| < C'$.

Let A be a self-adjoint operator on a N-dimensional Hilbert space, and let λ_i denote the eigenvalues of A, which therefore are real, counted with multiplicity. The spectral measure of A, denoted by $\nu(A)$, is defined as

$$\nu(A) := \sum_i \delta_{\lambda_i}.$$

We consider the normalized spectral measure of $T_{k\dot{\varphi}/2}$,

$$\nu_k := \frac{1}{k^n}\nu(T_{k\dot{\varphi}/2}/k).$$

By Theorem 3.2 in [1], which is a variant of a theorem of Boutet de Monvel-Guillemin (see [3]), we get that the measures ν_k converge weakly as k tends to infinity to the measure μ_φ.

Let $H^0(kL) = \sum_\eta V_\eta$ be the decomposition in weight spaces, and let P_η denote the projection to V_η. Then

$$\rho(\tau) = \sum_\eta \tau^\eta P_\eta,$$

and thus

$$\frac{d}{d\tau}\rho(\tau)_{|\tau=1} = \sum \eta P_\eta. \tag{2.27}$$

From (2.27) we see that the normalized spectral measures of $\frac{d}{d\tau}\rho(\tau)_{|\tau=1}$, which we denote by μ_k, coincides with the previously defined weight measure

$$\tilde{\mu}(\mathcal{T}, k) = \frac{1}{k^n}\sum_{\eta=-\infty}^{\infty} \dim V_\eta \delta_{k^{-1}\eta}.$$

According to Theorem 18 the sequence $\tilde{\mu}(\mathcal{T}, k)$, and therefore μ_k, converges weakly to the measure $G[\mathcal{T}]_* d\lambda_{|\Delta(L)}$.

Lastly, by the the min-max principle, when perturbing an operator A by an operator E with small norm $||E|| < \varepsilon$, then each eigenvalue is perturbed at most by ε. Thus from (2.26) it follows that $\nu_k - \mu_k$ converges weakly to zero, and the theorem follows. □

We will relate this result to our previous discussion on Duistermaat-Heckman measures in Section 2.5 and 2.6, by showing that the map $\dot{\varphi}/2$ is a Hamiltonian for the S^1-action when the symplectic form is given by $dd^c\varphi$. This is of course well-known (see e.g. [4]), but we include it here for the benefit of the reader.

Let V be the holomorphic vector field on X generating the action ρ. Hence, the imaginary part ImV of V generates the S^1-action. By definition, $\dot{\varphi}/2$ is a Hamiltonian if it holds that

$$\mathrm{Im}V \rfloor dd^c\varphi = d\dot{\varphi}/2, \tag{2.28}$$

where $\rfloor$ denotes the contraction operator.

If we can show that

$$-iV\rfloor dd^c\varphi = \bar{\partial}\dot{\varphi}/2,$$

equation (2.28) will follow by taking the real part on both sides. We calculate locally with respect to some trivialization and without loss of generality we may assume that

$$V = \frac{\partial}{\partial z_1}.$$

Recall that by definition

$$dd^c\varphi = \frac{i}{2}\sum \frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

Hence we get that

$$-iV\rfloor dd^c\varphi = \frac{1}{2}\sum \frac{\partial^2\varphi}{\partial z_1 \partial \bar{z}_j} d\bar{z}_j = \frac{1}{2}\bar{\partial}\frac{\partial\varphi}{\partial z_1}.$$

Since $V = \partial/\partial z_1$ generates the action, it follows that locally $\partial/\partial z_1\varphi = \dot{\varphi}$, and we are done.

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PAPER III

Julius Ross and David Witt Nyström
Analytic test configurations and geodesic rays

3

Analytic test configurations and geodesic rays

ABSTRACT

Starting with the data of a curve of singularity types, we use the Legendre transform to construct weak geodesic rays in the space of locally bounded metrics on an ample line bundle L over a compact manifold. Using this we associate weak geodesics to suitable filtrations of the algebra of sections of L. In particular this works for the natural filtration coming from an algebraic test configuration, and we show how this in the non-trivial case recovers the weak geodesic ray of Phong-Sturm.

3.1 Introduction

Let $\mathcal{H}(L)$ be the space of smooth strictly positive hermitian metrics on an ample line bundle L over a compact manifold X. Then, by the work of Mabuchi, Semmes and Donaldson (see [24], [32], [18]), $\mathcal{H}(L)$ has the structure of an infinite dimensional symmetric space with a canonical Riemannian metric. Thus a natural way to study this space is through its geodesics, an approach that has been taken up by a number of authors (e.g. Chen-Tian, Donaldson, Phong-Sturm, Mabuchi and Semmes among others).

In this paper we give a general method for constructing weak geodesics in the space of locally bounded positive metrics on L. The initial data consists of a fixed smooth positive metric ϕ and a curve of singular positive metrics ψ_λ on L for $\lambda \in \mathbb{R}$ that is concave in λ. We are really only interested in the singularity type of ψ_λ, so we consider the equivalence class of ψ_λ under the relation $\psi_\lambda \sim \psi'_\lambda$ if $\psi_\lambda - \psi'_\lambda$ is bounded globally on X. We define the *maximal envelope* of this data to be

$$\phi_\lambda := \sup\{\psi : \psi \leq \phi \text{ and } \psi \sim \psi_\lambda\}^*$$

where the supremum is over positive metrics ψ with the same singularity type as ψ_λ, and the star denotes the operation of taking the upper-semicontinuous regularization.

THEOREM 1. *Suppose ψ_λ is a test curve (as defined in (11)) and $\phi \in \mathcal{H}(L)$, and consider the Legendre transform of its maximal envelope ϕ_λ given by*

$$\widehat{\phi}_t := \sup_\lambda\{\phi_\lambda + \lambda t\}^* \quad \textit{for } t \in [0,\infty).$$

Then $\widehat{\phi}_t$ is a weak geodesic ray in the space of locally bounded positive metrics on L that emanates from ϕ.

We recall what is meant by a weak geodesic. Let $A := \{e^a < |z| < e^b\}$ be an annulus and π be the projection $X \times A \to X$. Given a curve ϕ_t, $a < t < b$, of positive metrics, consider the metric $\Phi(x, w) := \phi_{\log|w|}(x)$ on $\pi^*(L)$. Then a simple calculation reveals that if the ϕ_t are smooth then the geodesic equation for ϕ_t is equivalent to the degenerate homogeneous Monge-Ampère equation

$$\Omega^{n+1} = 0 \quad \text{on } X \times A, \tag{3.1}$$

where $\Omega = \pi^*\omega_0 + dd^c\Phi$ and ω_0 is the curvature of the initial metric. A curve of locally bounded positive metrics is said to be a *weak geodesic* if it solves (3.1) in sense of currents.

The first step in our approach to Theorem 1 is showing that the Monge-Ampère measure of the maximal envelope ϕ_λ satisfies

$$MA(\phi_\lambda) = \mathbf{1}_{\{\phi_\lambda=\phi\}} MA(\phi_\lambda) \quad \text{for all } \lambda, \tag{3.2}$$

where $\mathbf{1}_S$ denotes the characteristic function of a set S. We say that a positive metric ϕ_λ bounded by ϕ and having property (3.2) is *maximal* with respect to ϕ (see Definition 10), and a test curve ϕ_λ where ϕ_λ is maximal with respect to ϕ for all λ is referred to as a *maximal test curve*. We show that the Aubin-Mabuchi energy of the Legendre transform of a maximal test curve is linear in t, which is well known to be equivalent to (3.1) once it is established the curve is a subgeodesic.

A now standard conjecture, originally due to Yau, states that for a smooth projective manifold it should be possible to detect the existence of a constant scalar curvature Kähler metric algebraically. Through ideas developed by many authors (e.g. Chen, Donaldson, Mabuchi, Tian) a general picture has emerged in which such metrics appear as critical points of certain energy functionals that are convex along smooth geodesics. The input from algebraic geometry arises through Donaldson's notion of a test configuration which, roughly speaking, is a one-parameter algebraic degeneration of our original projective manifold.

In a series of papers, Phong-Sturm show how one can naturally associate a weak geodesic ray to a test configuration [25, 26, 28]. (See also [1] by Arezzo-Tian, [11, 12] by Chen, [14] by Chen-Tang and [13] by Chen-Sun for other constructions of geodesic rays related to test configurations.) We show how the geodesic constructed above can be viewed as a generalization of the geodesic of Phong-Sturm.

Generalizing slightly, suppose that $\mathcal{F}_{k,\lambda}$, for $k \in \mathbb{N}, \lambda \in \mathbb{R}$ is a multiplicative filtration of the graded algebra $\oplus_k H^0(X, kL)$. Using our underlying smooth positive metric ϕ we have an L^2-inner product on each $H^0(X, kL)$, and thus can consider the associated Bergman metric

$$\phi_{k,\lambda} = \frac{1}{k} \ln \sum_{\alpha} |s_\alpha|^2$$

where $\{s_\alpha\}$ is an orthonormal basis for $\mathcal{F}_{k,\lambda k} \subset H^0(X, kL)$.

THEOREM 2. *Suppose that $\mathcal{F}_{k,\lambda}$ is left continuous and decreasing in λ and bounded (see* (18)*). Then there is a well-defined limit*

$$\phi_\lambda^{\mathcal{F}} = (\lim_{k \to \infty} \phi_{k,\lambda})^*.$$

Furthermore this limit is maximal except possibly for one critical value of λ, and its Legendre transform is a weak geodesic ray.

In particular this applies to a natural filtration associated to a test configuration, and thus we have associated a weak geodesic to any such test configuration. We prove that, in the case when the analytic test configuration we get is non-trivial, we recover the construction of Phong-Sturm. Hence one interpretation of Theorem 1 is that in the problem of finding weak geodesics, the algebraic data of a test configuration can be replaced with a curve of singularity types which we thus refer to as an *analytic test configuration*.

It should be stressed that in the problem of finding constant scalar curvature metrics it is important to have control of the regularity of geodesics under consideration. By using approximations to known regularity results of solutions of Monge-Ampère equations, Phong-Sturm prove that their weak geodesic is in fact $C^{1,\alpha}$ for $0 < \alpha < 1$ (see [28]). It is interesting to ask whether such regularity holds more generally, which is a topic we hope to address in a future work.

Organization: We start in Section 3.2 with some motivation from convex analysis, and Section 3.3 contains preliminary material on the space of singular metrics, the Monge-Ampère measure and the Aubin-Mabuchi functional. The real work starts in Section 3.4 where we consider the maximal envelopes associated to a given singularity type. Along the way we prove a generalization of a theorem of Bedford-Taylor which says that such envelopes are maximal (Theorem 20). This is then extended to the case of a test curve of singularities, and in Section 3.6 we discuss the Legendre transform and prove Theorem 1.

Following these analytic results, we move on to the algebraic picture. In Section 3.7 we associate a test curve to a suitable filtration of the coordinate ring of (X, L), and prove Theorem 2. We then recall how such filtrations arise from test configurations, and in Section 3.9 show how this agrees with the construction of Phong and Sturm.

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3.2 Convex motivation

This section contains some motivation from convex analysis in the study of the homogeneous Monge-Ampère equation. Much of this material is standard; our main references are the two papers [30] and [31] by Rubinstein-Zelditch. Although this is logically independent of the rest of the paper, the techniques used are very similar: we shall presently see how solutions to this equation can be found using the Legendre transform in two different, but ultimately equivalent, ways.

Let $\mathrm{Conv}(\mathbb{R}^n)$ denote the space of convex functions on $\mathbb{R}^n$. We take the convention that the function identically equal to $-\infty$ is in $\mathrm{Conv}(\mathbb{R}^n)$.

DEFINITION 1. *Let ϕ be a C^2 convex function on an open subset of $\mathbb{R}^n$. The* (real) Monge-Ampère measure *of ϕ, denoted by $MA(\phi)$, is the Borel measure* *defined as*

$$MA(\phi) := d\frac{\partial\phi}{\partial x_1} \wedge ... \wedge d\frac{\partial\phi}{\partial x_{n+1}}.$$

Furthermore MA has an unique extension to a continuous operator on the cone of (finite-valued) convex functions (see [31] for references). If ϕ is C^2 then

$$MA(\phi) = det(\nabla^2\phi)dx = (\nabla\phi)^* dx, \tag{3.3}$$

i.e. the Monge-Ampère measure is the pullback of the Lebesgue measure under the gradient map.

If $\phi \in \mathrm{Conv}(\mathbb{R}^n)$, let Δ_ϕ denote the set of subgradients of ϕ, i.e. the set of points y in $\mathbb{R}^n$ such that the convex function $\phi - x \cdot y$ is bounded from below. So, if ϕ is differentiable then Δ_ϕ is simply the image of $\nabla\phi$. One can easily check that Δ_ϕ is convex, that if $r > 0$ then $\Delta_{r\phi} = \Delta_\phi$ and $\Delta_{\phi+\psi} \subseteq \Delta_\phi + \Delta_\psi$.

When ϕ is C^2 it follows from equation (3.3) that the total mass of the Monge-Ampère measure $MA(\phi)$ equals the Lebesgue volume of the set of gradients Δ_ϕ. An important fact [31] is that this is true for all convex functions on $\mathbb{R}^n$ with linear growth, i.e.

$$\int_{\mathbb{R}^n} MA(\phi) = vol(\Delta_\phi). \tag{3.4}$$

We say two convex functions ϕ and ψ are *equivalent* if $|\phi - \psi|$ is bounded, and denote this by $\phi \sim \psi$. Since for two equivalent convex functions ϕ and ψ with linear growth we clearly have that

$$\Delta_\phi = \Delta_\psi,$$

it follows from (3.4) that

$$\int_{\mathbb{R}^n} MA(\phi) = \int_{\mathbb{R}^n} MA(\psi) \quad \text{whenever } \phi \sim \psi.$$

DEFINITION 2. *Let $\phi \in \mathrm{Conv}(\mathbb{R}^n)$ and let $\dot{\phi}$ be a bounded continuous function on $\mathbb{R}^n$. A curve ϕ_t in $\mathrm{Conv}(\mathbb{R}^n)$, $t \in [a, b]$, is said to solve the* Cauchy problem *for the homogeneous real Monge-Ampère equation, abbreviated as HRMA, with* *initial data $(\phi, \dot{\phi})$, if the function $\Phi(x,t) := \phi_t(x)$ is convex on $\mathbb{R}^n \times [a,b]$, and satisfies the equation*

$$MA(\Phi) = 0 \quad \textit{on the strip} \quad \mathbb{R}^n \times (a,b),$$

with initial data

$$\phi_0 = \phi, \qquad \frac{\partial}{\partial t}_{|t=0^+} \phi_t = \dot{\phi}.$$

Let ϕ_0 and ϕ_1 be two equivalent convex functions with linear growth, and ϕ_t be the affine curve between them. The *energy* of ϕ_1 relative to ϕ_0, denoted by $\mathcal{E}(\phi_1, \phi_0)$ is defined as

$$\mathcal{E}(\phi_1, \phi_0) := \int_{t=0}^{1} \left(\int_{\mathbb{R}^n} (\phi_1 - \phi_0) MA(\phi_t) \right) dt.$$

We observe that by the linear growth assumption it follows that the relative energy $\mathcal{E}(\phi_1, \phi_0)$ is finite. This energy has a cocycle property, namely if ϕ_0, ϕ_1 and ϕ_2 are equivalent with finite energy then

$$\mathcal{E}(\phi_2, \phi_0) = \mathcal{E}(\phi_2, \phi_1) + \mathcal{E}(\phi_1, \phi_0),$$

which is easily seen to be equivalent to the fact that

$$\frac{\partial}{\partial t} \mathcal{E}(\phi_t, \phi) = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \phi_t MA(\phi_t).$$

The energy along a smooth curve ϕ_t of convex functions with linear growth is related to the Monge-Ampère measure of $\Phi(x,t) := \phi_t(x)$ by the identity

$$\int_{\mathbb{R}^n \times [a,b]} MA(\Phi) = \frac{\partial}{\partial t}_{|t=b} \mathcal{E}(\phi_t, \phi_a) - \frac{\partial}{\partial t}_{|t=a} \mathcal{E}(\phi_t, \phi_a). \tag{3.5}$$

Thus a smooth curve ϕ_t of equivalent convex functions of linear growth solves the HRMA equation if and only if Φ is convex and the energy $\mathcal{E}(\phi_t, \phi_a)$ is linear in t.

As is noted in [30] the Cauchy problem is not always solvable. Nevertheless there is a standard way to produce solutions ϕ_t with $t \in [0, \infty)$ to the homogeneous Monge-Ampère equation with given starting point $\phi_0 = \phi$ using the Legendre transform. We give a brief account of this.

For simplicity assume from now on that ϕ is differentiable and strictly convex. Recall that the Legendre transform of ϕ, denoted by ϕ^*, is the function on Δ_ϕ defined as

$$\phi^*(y) := \sup_x \{x \cdot y - \phi(x)\}$$

(which we can also think of as being defined on the whole of $\mathbb{R}^n$, by being $+\infty$ outside of Δ_ϕ). Since ϕ^* is defined as the supremum of the linear functions $x \cdot y - \phi(x)$, it is convex. In fact, one can show that ϕ being differentiable and strictly convex implies that ϕ^* is also differentiable and strictly convex.

For a given $y \in \Delta_\phi$, the function $x \cdot y - \phi(x)$ is strictly concave, and is maximized at the point where the gradient is zero. Thus we get that

$$\phi^*(y) = x \cdot y - \phi(x) \quad \text{where} \quad \nabla\phi(x) = y, \tag{3.6}$$

and hence

$$\nabla\phi^*(y) = x \quad \text{where} \quad \nabla\phi(x) = y.$$

The Legendre transform is an involution. For using the above formula for ϕ^{**} we deduce that $\nabla\phi^{**}(x) = y$, for x such that $\nabla\phi^*(y) = x$ which holds when $\nabla\phi(x) = y$, i.e.

$$\nabla\phi^{**}(x) = \nabla\phi(x).$$

If $\nabla\phi(x) = y$, then $\phi^*(y) = x \cdot y - \phi(x)$, therefore

$$\phi^{**}(x) = x \cdot y - \phi^*(y) = x \cdot y - (x \cdot y - \phi(x)) = \phi(x),$$

and hence $\phi^{**} = \phi$.

LEMMA 3. *If ϕ_t is a curve of convex functions, then for any point $y \in \Delta_{\phi_t}$*

$$\frac{\partial}{\partial t}\phi_t^*(y) = -\frac{\partial}{\partial t}\phi_t(x),$$

where x is the point such that $\nabla\phi(x) = y$.

Proof. Let x_t be the solution to the equation $\nabla\phi_t(x_t) = y$. By the implicit function theorem x_t varies smoothly with t. By equation (3.6) we know

$$\frac{\partial}{\partial t}\phi_t^*(y) = \frac{\partial}{\partial t}(x_t \cdot y - \phi_t(x)) = \frac{\partial}{\partial t}(x_t \cdot y - \phi(x)) - \frac{\partial}{\partial t}\phi_t(x).$$

Since $x_t \cdot y - \phi(x)$ is maximized at $x = x_0$ the derivative of that part vanishes at $t = 0$, so we get the lemma for $t = 0$, and similarly for all t. □

This leads us to the following formula relating the energy with the Legendre transform,

LEMMA 4. *We have that*

$$\mathcal{E}(\phi_t, \phi) = \int_{\Delta_\phi} (\phi^* - \phi_t^*) dy. \tag{3.7}$$

Proof. We noted above that the derivative with respect to t of the left-hand side of (3.7) is equal to

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} \phi_t MA(\phi_t).$$

On the other hand, differentiating the right-hand side yields

$$\frac{\partial}{\partial t} \int_{\Delta_\phi} (\phi^* - \phi_t^*) dy = - \int_{\Delta_\phi} \frac{\partial}{\partial t} \phi_t^* dy = \int_{\Delta_\phi} \frac{\partial}{\partial t} \phi_t (\nabla \phi_t^{-1}(y)) dy =$$

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} \phi_t (\nabla \phi_t)^* dy = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \phi_t MA(\phi_t),$$

where we used Lemma 3 and the fact that $(\nabla \phi_t)^* dy = MA(\phi_t)$. Since both sides of the equation (3.7) is zero when $\phi_t = \phi$ and the derivatives coincide, we get that they must be equal for all t. □

Now fix a smooth bounded strictly concave function u on Δ_ϕ and let

$$\tilde{\phi}_t := (\phi^* - tu)^*.$$

By the involution property of the Legendre transform $(\tilde{\phi}_t)^* = \phi^* - tu$.

PROPOSITION 5. *The curve $\tilde{\phi}_t$, $t \in [0, \infty)$ solves the HRMA equation.*

To see this note that from (3.7) it follows that

$$\mathcal{E}(\tilde{\phi}_t, \phi) = \int_{\Delta_\phi} (\phi^* - \tilde{\phi}_t^*) dy = \int_{\Delta_\phi} (\phi^* - \phi^* + tu) dy = t \int_{\Delta_\phi} u dy,$$

which is linear in t. The convexity of $\tilde{\Phi}(t, x) = \tilde{\phi}_t(x)$ can of course be shown directly, but it also follows from another characterization of $\tilde{\phi}_t$ that also involves a Legendre transform, but in the t-coordinate instead of in the x-coordinates which we now discuss.

Let A_λ be the subset of Δ_ϕ where u is greater than or equal to λ and let ϕ_λ be defined as

$$\phi_\lambda := \sup\{\psi \le \phi : \psi \in \mathrm{Conv}(\mathbb{R}^n), \Delta_\psi \subseteq A_\lambda\}.$$

LEMMA 6. *The curve of functions ϕ_λ is concave in λ and*

$$\{\phi_\lambda = \phi\} = \{x : \nabla\phi(x) \in A_\lambda\}.$$

Proof. Let $\psi_i \le \phi$ be such that $\Delta_{\psi_i} \subseteq A_{\lambda_i}$ with $i = 1, 2$. Let $0 < t < 1$. From our discussion above it follows that $t\psi_1 + (1-t)\psi_2 \le \phi$ and

$$\Delta_{t\psi_1+(1-t)\psi_2} \subseteq t\Delta_{\psi_1} + (1-t)\Delta_{\psi_2} \subseteq tA_{\lambda_1} + (1-t)A_{\lambda_2} \subseteq A_{t\lambda_1+(1-t)\lambda_2},$$

where the last inclusion follows from the fact that u was assumed to be concave. For the second statement, it is easy to see that in fact ϕ_λ is equal to the supremum of affine functions $x \cdot y + C$ bounded by ϕ and y lying in A_λ. □

DEFINITION 3. *For $t \ge 0$ let $\widehat{\phi}_t$ be defined as*

$$\widehat{\phi}_t := \sup_\lambda\{\phi_\lambda + t\lambda\}.$$

Since for each λ the function $(x,t) \mapsto \phi_\lambda(x) + t\lambda$ is convex in all its variables, and the supremum of convex functions is convex, we get $\widehat{\Phi}(x,t) := \widehat{\phi}_t(x)$ is convex.

PROPOSITION 7. *We have that $\tilde{\phi}_t = \widehat{\phi}_t$. In particular this proves that $\tilde{\Phi}$ is convex, thereby proving $\tilde{\phi}_t$ solves the HRMA equation (Proposition 5).*

Proof. We claim

$$\frac{\partial}{\partial t}\widehat{\phi}_t(x) = u(\nabla\widehat{\phi}_t(x)). \tag{3.8}$$

To see this first consider the right-derivative at $t = 0$. As we noted above, the gradient of a Legendre transform is the point where the maximum is attained, thus in this case

$$\frac{\partial}{\partial t}|_{t=0^+}\widehat{\phi}_t(x) = \sup\{\lambda : \phi_\lambda(x) = \phi(x)\}.$$

By the second statement in Lemma 6 it follows that this supremum is equal to $u(\nabla\phi(x))$, and we are done for $t = 0$. On the other hand it is easy to see that

$$\widehat{\phi}_{t_1+t_2} = \widehat{\psi}_{t_2},$$

with $\psi := \widehat{\phi}_{t_1}$. Using this we get that the equation (3.8) holds for all t. Thus by Lemma 3 the Legendre transform of $\widehat{\phi}_t$ is equal to $\phi - tu$, so by the involution property of the Legendre transform $\widehat{\phi}_t$ coincides with $\tilde{\phi}$. □

Now the above discussion can be applied as follows. Let ψ_λ be a concave curve in $\mathrm{Conv}(\mathbb{R}^n)$, with $|\psi_\lambda - \phi|$ bounded for $\lambda < -C$ and $\psi_\lambda \equiv -\infty$ for $\lambda > C$ for some constant C. We call such a curve a test curve. Define ϕ_λ as

$$\phi_\lambda := \sup\{\psi : \psi \leq \phi, \psi \leq \psi_\lambda + o(1), \psi \in \mathrm{Conv}(\mathbb{R}^n)\}.$$

Let also u be the function on Δ_ϕ defined by

$$u(y) := \sup\{\lambda : y \in \Delta_{\psi_\lambda}\}.$$

Since ψ_λ was assumed to be concave it follows that u is concave, so in fact

$$\phi_\lambda = \sup\{\psi \leq \phi : \psi \in \mathrm{Conv}(\mathbb{R}^n), \Delta_\psi \subseteq \{u \geq \lambda\}\}.$$

From Proposition 7, $\widehat{\phi}_t$ solves the homogeneous real Monge-Ampère equation. Thus in order to get solutions to the HRMA, instead of starting with a concave function u on Δ_ϕ we can just as well start with a test curve ψ_λ. In the subsequent sections we will show how this construction carries over in the context of positive metrics on line bundles.

3.3 Preliminary Material

We collect here some preliminary material on the space of positive metrics, the (non pluripolar) Monge-Ampère measure and the Aubin-Mabuchi energy functional. Most of this material is standard, and we give proofs only for those results for which we did not find a convenient reference.

3.3.1 The space of positive singular metrics

Let X be a compact Kähler manifold of complex dimension n, and let L be an ample line bundle on X. A continuous (or smooth) hermitian metric $h = e^{-\phi}$ on L is a continuous (or smooth) choice of scalar product on the complex line L_p at each point p on the manifold. If f is a local holomorphic frame for L on U_f, then one writes

$$|f|_h^2 = h_f = e^{-\phi_f},$$

where ϕ_f is a continuous (or smooth) function on U_f. We will use the convention to let ϕ denote the metric $h = e^{-\phi}$, thus if ϕ is a metric on L, $k\phi$ is a metric on $kL := L^{\otimes k}$.

The curvature of a smooth metric is given by $dd^c\phi$ which is the $(1,1)$-form locally defined as $dd^c\phi_f$, where f is any local holomorphic frame. Here d^c is short-hand for the differential operator

$$\frac{i}{2\pi}(\partial - \bar{\partial}),$$

so $dd^c = i/\pi\partial\bar{\partial}$. A classic fact is that the curvature form of a smooth metric ϕ is a representative for the first Chern class of L, denoted by $c_1(L)$. The metric ϕ is said to be strictly positive if the curvature $dd^c\phi$ is strictly positive as a $(1,1)$-form, i.e. if for any local holomorphic frame f, the function ϕ_f is strictly plurisubharmonic. We let $\mathcal{H}(L)$ denote the space of smooth strictly positive (i.e. locally strictly plurisubharmonic) metrics on L, which is non-empty since we assumed that L was ample.

A positive singular metric ψ is a metric that can be written as $\psi := \phi + u$, where ϕ is a smooth metric and u is a $dd^c\phi$-psh function, i.e. u is upper semicontinuous and $dd^c\psi := dd^c\phi + dd^c u$ is a positive $(1,1)$-current. For convenience we also allow $u \equiv -\infty$. We let $PSH(L)$ denote the space of positive singular metrics on L.

As an important example, if $\{s_i\}$ is a finite collection of holomorphic sections of kL, we get a positive metric $\psi := \frac{1}{k}\ln(\sum |s_i|^2)$ which is defined by letting for any local frame f,

$$e^{-\psi_f} := \frac{|f|^2}{(\sum |s_i|^2)^{1/k}}.$$

We note that $PSH(L)$ is a convex set, since any convex combination of positive metrics yields a positive metric. Another important fact is that if $\psi_i \in PSH(L)$ for all $i \in I$ are uniformly bounded above by some fixed positive metric, then the upper semicontinuous regularization of the supremum denoted by $(\sup\{\psi_i : i \in I\})^*$ lies in $PSH(L)$ as well. If ψ is in $PSH(L)$, then the translate $\psi + c$ where c is a real constant is also in $PSH(L)$. For any $\psi \in PSH(L)$, $dd^c\psi$ is a closed positive $(1,1)$-current, and from the dd^c lemma it follows that any closed positive current cohomologous with $dd^c\psi$ can be written as $dd^c\phi$ for some ϕ in $PSH(L)$. By the maximum principle this ϕ is uniquely determined up to translation.

If there exists a constant C such that $\psi \leq \phi + C$, we say that ψ is more singular than ϕ, and we will write this as

$$\psi \succeq \phi.$$

If both $\psi \succeq \phi$ and $\phi \succeq \psi$ we say that ψ and ϕ are *equivalent*, which we write as $\psi \sim \phi$. Following [9] an equivalence class $[\psi]$ is called a *singularity type*, and we introduce the notation $Sing(L)$ for the set of singularity types. If ψ is equivalent to an element in $\mathcal{H}(L)$ we say that ψ is *locally bounded*.

The singularity locus of a positive metric ψ is the set where ψ is minus infinity, i.e. the set where $\psi_f = -\infty$ when f is a local frame. The unbounded locus of ψ is the set where ψ is not locally bounded. Recall that a set is said to be *complete pluripolar* if it is locally the singularity locus of a plurisubharmonic function. In [9] BEGZ (Boucksom-Eyssidieux-Guedj-Zeriahi) give the following definition.

DEFINITION 4. *A positive metric ψ is said to have* small unbounded locus *if its unbounded locus is contained in a closed complete pluripolar subset of X.*

We note that metrics of the form $\frac{1}{k}\ln(\sum |s_i|^2)$ have small unbounded locus, since they are locally bounded away from the algebraic set $\cup_i\{s_i = 0\}$ which is a closed pluripolar set.

3.3.2 Regularization of positive singular metrics

If f is a plurisubharmonic function on an open subset U of $\mathbb{C}^n$ then using a convolution we can write f as the limit of a decreasing sequence of smooth plurisubharmonic functions on any relatively compact subset of U.

If ψ is a positive singular metric, we can use a partition of unity with respect to some open cover U_{f_i} to patch together the smooth decreasing approximations of ψ_{f_i}. Thus any positive singular metric can be written as the pointwise limit of a decreasing sequence of smooth metrics, but of course because of the patching these smooth approximations will in general not be positive.

A fundamental result due to Demailly [16] is that any positive singular metric can be approximated by metrics of the form $k^{-1}\ln(\sum_i |s_i|^2)$, where s_i are sections of kL. Let $\mathcal{I}(\psi)$ denote the multiplier ideal sheaf of germs of holomorphic functions locally integrable against $e^{-\psi_f}dV$, where f is a local frame for L and dV is an arbitrary volume form. We get a scalar product $(.,.)_{k\psi}$ on the space $H^0(kL \otimes \mathcal{I}(k\psi))$ by letting

$$||s||^2_{k\psi} := \int_X |s|^2 e^{-k\psi} dV.$$

Let $\{s_i\}$ be an orthonormal basis for $H^0(kL \otimes \mathcal{I}(k\psi))$ and set

$$\psi_k := \frac{1}{k}\ln(\sum |s_i|^2).$$

THEOREM 8. *The sequence of metrics ψ_k converge pointwise to ψ as k tends to infinity, and there exists a constant C such that for large k,*

$$\psi \leq \psi_k + \frac{C}{k}.$$

As a reference see [17], but the results of Demailly are in fact much stronger than that stated here, and hold in greater generality [16]. When ψ is assumed to be smooth and strictly positive, a celebrated result by Bouche-Catlin-Tian-Zelditch [7, 10, 35, 39] on Bergman kernel asymptotics implies that the ψ_k in fact converge to ψ in any C^m norm.

Using a variation of this construction Guedj-Zeriahi prove in [21] that any positive singular metric on an ample line bundle is the pointwise limit of a decreasing sequence of smooth positive metrics.

3.3.3 Monge-Ampère measures

Let ψ_i, $1 \leq i \leq n$, be an n-tuple of positive metrics, so for each i, $dd^c\psi_i$ is a positive $(1,1)$-current. If all ψ_i are smooth one can consider the wedge product

$$dd^c\psi_1 \wedge ... \wedge dd^c\psi_n, \tag{3.9}$$

which is a positive measure on X. The fundamental work of Bedford-Taylor shows that one can still take the wedge product of positive currents $dd^c\psi_i$ to get a positive measure as long as the metrics ψ_i are all locally bounded. The Monge-Ampère measure of a locally bounded positive metric ψ, is then defined as the positive measure

$$MA(\psi) := (dd^c\psi)^n.$$

This measure does not put any mass on pluripolar sets (i.e. sets that are locally contained in the unbounded locus of a local plurisubharmonic function). We recall the following important continuity property, proved in [2].

THEOREM 9 (Bedford-Taylor). *If $\psi_{i,k}$, $1 \leq i \leq n+2$, $k \in \mathbb{N}$, are sequences of locally bounded positive metrics such that each $\psi_{i,k}$ decreases to a locally bounded positive metric ψ_i, then the signed measures $(\psi_{1,k} - \psi_{2,k})dd^c\psi_{3,k} \wedge ... \wedge dd^c\psi_{n+2,k}$ converge weakly to $(\psi_1 - \psi_2)dd^c\psi_3 \wedge ... \wedge dd^c\psi_{n+2}$. If each sequence of locally bounded positive metrics $\psi_{i,k}$ instead increase pointwise a.e. to a positive metric ψ_i, then again the measures $(\psi_{1,k} - \psi_{2,k})dd^c\psi_{3,k} \wedge ... \wedge dd^c\psi_{n+2,k}$ converge weakly to $(\psi_1 - \psi_2)dd^c\psi_3 \wedge ... \wedge dd^c\psi_{n+2}$.*

Since the curvature form $dd^c\phi$ of any smooth metric ϕ is a representative of $c_1(L)$, we see that if ϕ_i is any n-tuple of smooth metrics then

$$\int_X dd^c\phi_1 \wedge ... \wedge dd^c\phi_n = \int_X c_1(L)^n \tag{3.10}$$

which is just a topological invariant of L. Since any positive metric can be approximated from above in the manner of Theorem 9 by positive metrics that are smooth, we see that (3.10) still holds if the ϕ_i are merely assumed to be locally bounded instead of smooth.

Recall that a plurisubharmonic function is, by definition, upper semicontinuous, so if ψ is a positive metric then for each local frame f the function ψ_f is upper semicontinuous. The plurifine topology is defined as the coarsest topology in which all local plurisubharmonic functions are continuous; a basis for this topology is given by sets of the form $A \cap \{u > 0\}$, where A is open in the standard topology and u is a local plurisubharmonic function. This topology has the quasi-Lindelöf property [3, Thm 2.7], meaning that an arbitrary union of plurifine open sets differs from a countable subunion by at most a pluripolar set. Any basis set $A \cap \{u > 0\}$ is Borel, so it follows from the quasi-Lindelöf property that the plurifine open (and closed) sets lie in the completion of the Borel σ-algebra with respect to any Monge-Ampère measure [3, Prop 3.1].

DEFINITION 5. *A function f is said to be* quasi-continuous *on a set Ω if for every $\epsilon > 0$ there exists an open set U with capacity less than ϵ so that f is continuous on $\Omega \setminus U$.*

We refer to [2] for the definition of capacity. By [3, Thm 4.9] plurisubharmonic functions are quasi-continuous.

If f_k is a sequence of non-negative continuous functions increasing to the characteristic function of an open set A then the characteristic function of a basis set $A \cap \{u > 0\}$ is the increasing limit of the non-negative quasi-continuous functions

$$kf_k(\max\{u, 0\} - \max\{u - 1/k, 0\}).$$

From this fact and the quasi-Lindelöf property it follows that the characteristic function of any plurifine open set differs from an increasing limit of non-negative quasi-continuous functions at most on a pluripolar set.

A fundamental property of the Bedford-Taylor product is that it is local in the plurifine topology, so if $\psi_i = \psi_i'$ for all i on some plurifine open set O then

$$\mathbf{1}_O dd^c\psi_1 \wedge ... \wedge dd^c\psi_n = \mathbf{1}_O dd^c\psi_1' \wedge ... \wedge dd^c\psi_n',$$

where $\mathbf{1}_O$ denotes the characteristic function of O. We also have that the convergence in Theorem 9 is local in this topology [3, Thm 3.2], i.e. we get convergence when testing against bounded quasi-continuous functions.

LEMMA 10. *Let ψ_k be a sequence of locally bounded positive metrics that decreases pointwise (or increases a.e.) to a locally bounded positive metric ψ, and let O be a plurifine open set. Then*

$$\mathbf{1}_O MA(\psi) \leq \liminf_{k\to\infty} \mathbf{1}_O MA(\psi_k),$$

where the $\liminf$ *is to be understood in the weak sense, i.e. when testing against non-negative continuous functions.*

Proof. Let u_i be a sequence of quasi-continuous functions increasing to $\mathbf{1}_O$ except on a pluripolar set. Let f be a non-negative continuous function. Since $u_i MA(\psi_k)$ converges weakly to $u_i MA(\psi)$, and $MA(\psi_k)$ does not put any mass on a pluripolar set,

$$\int_X f u_i MA(\psi) = \lim_{k\to\infty} \int_X f u_i MA(\psi_k) \leq \liminf_{k\to\infty} \int_O f MA(\psi_k). \quad (3.11)$$

Now u_i increases to the characteristic function of O except possibly on a pluripolar set, so letting i tend to infinity in (3.11) yields

$$\int_O f MA(\psi) \leq \liminf_{k\to\infty} \int_O f MA(\psi_k).$$

□

For singular ϕ_i there is a (non pluripolar) product constructed by Boucksom-Eyssidieux-Guedj-Zeriahi [9], building on a local construction due to Bedford-Taylor [3]. Fix a locally bounded metric ϕ, and consider the auxiliary metrics $\psi_{i,k} := \max\{\psi_i, \phi - k\}$ for $k \in \mathbb{N}$, and the sets

$$O_k := \bigcap_i \{\psi_i > \phi - k\}.$$

The *non-pluripolar product* of the currents $dd^c\psi_i$, here denoted by $dd^c\psi_1 \wedge ... \wedge dd^c\psi_n$ is defined as the limit

$$dd^c\psi_1 \wedge ... \wedge dd^c\psi_n := \lim_{k\to\infty} \mathbf{1}_{O_k} dd^c\psi_{1,k} \wedge ... \wedge dd^c\psi_{n,k}.$$

Since we are assuming that X is compact this limit is well defined [9, Prop. 1.6]. The (non-pluripolar) Monge-Ampère measure of a positive metric is ψ is defined as $MA(\psi) := (dd^c\psi)^n$. Essentially by construction, the non-pluripolar product is local in the plurifine topology [9, Prop. 1.4], and is multilinear [9, Prop 4.4].

Clearly from the definition and (3.10), for any n-tuple of positive metrics ψ_i on L,

$$\int_X dd^c\psi_1 \wedge ... \wedge dd^c\psi_n \leq \int_X c_1(L)^n,$$

however the inequality may well be strict.

Combining Lemma 10 with the fact that the Monge-Ampère measure is local in the plurifine topology yields the following continuity result.

LEMMA 11. *Let ψ_k be a sequence of positive metrics decreasing to a positive metric ψ, and let ϕ be some locally bounded positive metric. If O is a plurifine open set contained in $\{\psi > \phi - C\}$ for some constant C then*

$$\mathbf{1}_O MA(\psi) \leq \liminf_{k\to\infty} \mathbf{1}_O MA(\psi_k), \tag{3.12}$$

where again the $\liminf$ *is to be understood in the weak sense. If ψ_k instead is increasing a.e. to ψ, and O is a plurifine open set contained in $\{\psi_j > \phi - C\}$ for some natural number j and some constant C then once again*

$$\mathbf{1}_O MA(\psi) \leq \liminf_{k\to\infty} \mathbf{1}_O MA(\psi_k).$$

Proof. First assume that ψ_k is decreasing to ψ. Let $\psi'_k := \max\{\psi_k, \phi - C\}$ and $\psi' := \max\{\psi, \phi - C\}$. From Lemma 10 it follows that

$$\mathbf{1}_O MA(\psi') \leq \liminf_{k\to\infty} \mathbf{1}_O MA(\psi'_k),$$

and since by assumption $\psi' = \psi$ and $\psi'_k = \psi_k$ on O the lemma follows from the locality of the non-pluripolar product. The case where ψ_k is increasing a.e. follows by the same reasoning. □

In [9, Thm 1.16] BEGZ prove the following monotonicity property of the non-pluripolar product when restricted to metrics with small unbounded locus.

THEOREM 12. *Let ψ_i, ψ_i' be two n-tuples of positive metrics with small unbounded locus, and suppose that for all i, ψ_i is more singular than ψ_i'. Then*

$$\int_X dd^c\psi_1 \wedge ... \wedge dd^c\psi_n \leq \int_X dd^c\psi_1' \wedge ... \wedge dd^c\psi_n'.$$

BEGZ also prove a comparison principle for metrics with small unbounded locus [9, Cor 2.3] and a domination principle [9, Cor 2.5]. When combined with the comparison principle, the proof of the domination principle in [9] in fact yields a slightly stronger version:

THEOREM 13. *Let ϕ be a positive metric with small unbounded locus and suppose that there exists a positive metric ρ, more singular than ϕ, with small unbounded locus and such that $MA(\rho)$ dominates a volume form. If ψ is a positive metric more singular than ϕ and such that $\psi \leq \phi$ a.e. with respect to $MA(\phi)$, then it follows that $\psi \leq \phi$ on the whole of X.*

3.3.4 The Aubin-Mabuchi Energy

The *Aubin-Mabuchi energy bifunctional* maps any pair of equivalent positive metrics ψ_1 and ψ_2 to the number

$$\mathcal{E}(\psi_1, \psi_2) := \frac{1}{n+1} \sum_{i=0}^{n} \int_X (\psi_1 - \psi_2)(dd^c\psi_1)^i \wedge (dd^c\psi_2)^{n-i}.$$

Observe

$$\mathcal{E}(\psi + t, \psi) = t \int_X MA(\psi).$$

The Aubin-Mabuchi energy restricted to the class of locally bounded metrics has a cocycle property (see, for example, [6, Cor 4.2]), namely if ϕ_0, ϕ_1 and ϕ_2 are locally bounded equivalent metrics then

$$\mathcal{E}(\phi_0, \phi_2) = \mathcal{E}(\phi_0, \phi_1) + \mathcal{E}(\phi_1, \phi_2).$$

In fact the proof in [6] of the cocycle property extends to the case where the equivalent metrics are only assumed to have small unbounded locus, since the integration-by-parts formula of [9] used in the proof holds in that case.

This leads to an important monotonicity property. If ψ_0, ψ_1 and ψ_2 are equivalent with small unbounded locus, and $\psi_0 \geq \psi_1$, then

$$\mathcal{E}(\psi_0, \psi_2) \geq \mathcal{E}(\psi_1, \psi_2)$$

since $\mathcal{E}(\psi_0, \psi_2) = \mathcal{E}(\psi_0, \psi_1) + \mathcal{E}(\psi_1, \psi_2)$, and $\mathcal{E}(\psi_0, \psi_1) \geq 0$ as it is the integral of the positive function $\psi_0 - \psi_1$ against a positive measure.

We also record the following lemma, which comes from the locality of the non-pluripolar product in the plurifine topology.

LEMMA 14. *Let $\psi_1 \sim \psi_2$ be such that $\psi_1 \geq \psi_2$. Let ψ_1' and ψ_2' be two other metrics such that $\psi_1' \sim \psi_2'$ and assume that $\{\psi_1' = \psi_2'\} = \{\psi_1 = \psi_2\}$ and that $\psi_1' = \psi_1$ and $\psi_2' = \psi_2$ on the set where $\psi_1 > \psi_2$. Then*

$$\mathcal{E}(\psi_1', \psi_2') = \mathcal{E}(\psi_1, \psi_2).$$

Following Phong-Sturm in [25] we can relate weak geodesics to the energy functional. Let $A := \{e^a \leq |z| \leq e^b\}$ be an annulus and let π denote the standard projection from $X \times A$ to X. A curve of metrics ϕ_t of L, $a \leq t \leq b$, can be identified with the rotation invariant metric on $\pi^* L$ whose restriction to $X \times \{w\}$ equals $\phi_{\ln |w|}$.

DEFINITION 6. *A curve of positive metrics ϕ_t, $a \leq t \leq b$, is said to be a* weak subgeodesic *if there exists a locally bounded positive metric Φ on $\pi^* L$ that is rotation invariant and whose restriction to $X \times \{w\}$ equals $\phi_{\ln |w|}$. A curve ϕ_t is said to be a* weak geodesic *if it is a weak subgeodesic and furthermore Φ solves the HCMA equation, i.e.*

$$MA(\Phi) = 0$$

on $X \times A^\circ$.

As in the convex setting (3.5) there is a formula [**?**, 6.3] relating the Aubin-Mabuchi energy of a locally bounded subgeodesic ϕ_t with the Monge-Ampère measure of Φ, namely

$$dd_t^c \mathcal{E}(\phi_t, \phi_a) = \pi_*(MA(\Phi)), \tag{3.13}$$

where $\pi_*(MA(\Phi))$ denotes the push-forward of the measure $MA(\Phi)$ with respect to the projection π. From this we immediately get the following lemma.

LEMMA 15. *A curve ϕ_t of locally bounded positive metrics is a weak geodesic if and only if it is a subgeodesic and the Aubin-Mabuchi energy $\mathcal{E}(\phi_t, \phi_a)$ is linear in t.*

3.4 Envelopes and maximal metrics

In studying the Dirichlet problem for the HCMA equation it is often possible to give a solution as an envelope in some space of plurisubharmonic functions (or positive metrics). Such envelopes will be crucial in our setting as well.

DEFINITION 7. *If ϕ is a continuous metric, not necessarily positive, let $P\phi$ denote the envelope*

$$P\phi := \sup\{\psi \leq \phi, \psi \in PSH(L)\}.$$

Since ϕ is assumed to be continuous it follows that $(P\phi)^* \leq \phi$, thus $P\phi = (P\phi)^*$, so $P\phi \in PSH(L)$.

The next theorem is essentially just a reformulation of a local result of Bedford-Taylor [2, Corollary 9.2] in our global setting. It follows as a special case of [6, Prop 1.10] (letting $K = X$).

THEOREM 16. *If ϕ is a continuous metric then $P\phi = \phi$ a.e. with respect to $MA(P\phi)$.*

Recall that if A is a closed set and μ is a Borel measure we say that μ is said to be *concentrated* on A if $\mathbf{1}_A\mu = \mu$, or equivalently $\mu(A^c) = 0$. Thus another way of formulating Theorem 16 is to say that $MA(P\phi)$ is concentrated on $\{P\phi = \phi\}$. We now extend this result to more general envelopes that arise from the additional data of singularity type.

DEFINITION 8. *Given a positive metric $\psi \in PSH(L)$ let P_ψ denote the projection operator on $PSH(L)$ defined by*

$$P_\psi\phi := \sup\{\psi' \leq \min\{\phi, \psi\}, \psi' \in PSH(L)\}.$$

We also define $P_{[\psi]}$ by

$$P_{[\psi]}\phi := \lim_{C\to\infty} P_{\psi+C}\phi = \sup\{\psi' \leq \phi, \psi' \sim \psi, \psi' \in PSH(L)\}.$$

Clearly $P_\psi\phi$ is monotone with respect to both ψ and ϕ. Since $\min\{\phi,\psi\}$ is upper semicontinuous, it follows that the upper semicontinuous regularization of $P_\psi\phi$ is still less than $\min\{\phi,\psi\}$, and thus $P_\psi\phi \in PSH(L)$. By this it follows that $P_\psi(P_\psi\phi) = P_\psi\phi$, i.e. that P_ψ is indeed a projection operator on $PSH(L)$. One also notes that the upper semicontinuous regularization of $P_{[\psi]}\phi$, lies in $PSH(L)$ and is bounded by ϕ.

DEFINITION 9. *The* maximal envelope *of ϕ with respect to the singularity type $[\psi]$ is defined to be*

$$\phi_{[\psi]} := (P_{[\psi]}\phi)^*.$$

DEFINITION 10. *If $\psi \in PSH(L)$, then ψ is said to be* maximal *with respect to a metric ϕ if $\psi \leq \phi$ and furthermore $\psi = \phi$ a.e. with respect to $MA(\psi)$. Similarly, if A is a measurable set, we say that ψ is maximal with respect to ϕ on A if $\psi \leq \phi$ and $\psi = \phi$ a.e. on A with respect to $MA(\psi)$.*

The terminology is justified by a proof below that the maximal envelope of a continuous metric ϕ is maximal with respect to ϕ. Note that we do not know whether the maximal envelope $\phi_{[\psi]}$ is equivalent to ψ. Therefore the method in the proof of Theorem 16 in [6] does not apply, so instead we will use an approximation argument. The reason for the use of the word maximal is motivated by the following property:

PROPOSITION 17. *Let ψ be maximal with respect to a metric ϕ. Suppose also that there exists a positive metric $\rho \succeq \psi$ with small unbounded locus and such that $MA(\rho)$ dominates a volume form. Then for any $\psi' \sim \psi$ with $\psi \leq \phi$ we have $\psi' \leq \psi$.*

Proof. Since $\psi' \leq \phi$, the maximality assumption yields $\psi' \leq \psi$ a.e. with respect to $MA(\psi)$, so the proposition thus follows from the domination principle (Theorem 13). □

The next two lemmas are the main steps in showing that maximal envelopes are maximal.

LEMMA 18. *Let ψ_k be a sequence of positive metrics increasing a.e. to a positive metric ψ, and assume that all ψ_k are maximal with respect to a fixed continuous metric ϕ on some plurifine open set O. Then ψ is maximal with respect to ϕ on O.*

Proof. Since ϕ was assumed to be continuous, $\psi \leq \phi$. Now, for all k

$$\{\psi_k = \phi\} \subseteq \{\psi = \phi\}$$

and thus by the the maximality of ψ_k, we know $\mathbf{1}_O MA(\psi_k)$ is concentrated on $\{\psi = \phi\}$. Since $\psi \leq \phi$ we have that $\{\psi = \phi\} = \{\psi \geq \phi\}$, and since ϕ is continuous this is a closed set. Let C be a constant. The set $O \cap \{\psi_1 > \phi - C\}$ is plurifinely open, so by Lemma 11 it follows that

$$\mathbf{1}_O \mathbf{1}_{\{\psi_1 > \phi - C\}} MA(\psi) \leq \liminf_{k\to\infty} \mathbf{1}_O \mathbf{1}_{\{\psi_1 > \phi - C\}} MA(\psi_k). \tag{3.14}$$

It is easy to see that if μ_k is a sequence of measures all concentrated on a closed set A, and

$$\mu \leq \liminf_{k\to\infty} \mu_k$$

in the weak sense, then μ is also concentrated on A. It thus follows from (3.14) that $\mathbf{1}_O \mathbf{1}_{\{\psi_1 > \phi - C\}} MA(\psi)$ is concentrated on $\{\psi = \phi\}$. Since $MA(\psi)$ puts no mass on the pluripolar set $\{\psi_1 = -\infty\}$ the lemma follows by letting C tend to infinity. □

LEMMA 19. *Let $\psi \in PSH(L)$ and let ϕ be a continuous metric. Then the envelope $P_\psi \phi$ is maximal with respect to ϕ on the plurifine open set $\{\psi > \phi\}$.*

Proof. By definition $P_\psi \phi \leq \phi$. Now let ϕ_k be a sequence of continuous metrics decreasing pointwise to $\min\{\phi, \psi\}$, so that $\phi_k \leq \phi$ for all k and $\phi_k = \phi$ on the set $\{\psi > \phi\}$. For example let $\phi_k := \min\{\phi, \psi_k\}$ where ψ_k is a sequence of smooth metrics decreasing pointwise to ψ. From Theorem 16 it follows that $MA(P\phi_k)$ is concentrated on $\{P\phi_k = \phi_k\}$, and since $\phi_k = \phi$ when $\psi > \phi$

we get that $\mathbf{1}_{\{\psi>\phi\}}MA(P\phi_k)$ is concentrated on $\{P\phi_k = \phi\}$. Now $P\phi_k$ is decreasing in k and $\lim_{k\to\infty} P\phi_k \leq \min\{\phi, \psi\}$. At the same time, for any $k \in \mathbb{N}$ we clearly have that $P_\psi\phi \leq P\phi_k$, which taken together means that

$$\lim_{k\to\infty} P\phi_k = P_\psi\phi.$$

Since $P\phi_k \leq \phi$ this implies that $\{P\phi_k = \phi\}$ is decreasing in k and

$$\{P_\psi\phi = \phi\} = \bigcap_{k\in\mathbb{Z}} \{P\phi_k = \phi\}. \tag{3.15}$$

Let O denote the plurifine open set $\{\psi > \phi\} \cap \{P_\psi\phi > \phi - C\}$. By Lemma 11 we know

$$\mathbf{1}_O MA(P_\psi\phi) \leq \liminf_{k\to\infty} \mathbf{1}_O MA(P\phi_k),$$

and thus we conclude that $\mathbf{1}_O MA(P_\psi\phi)$ is concentrated on $\{P\phi_k = \phi\}$ for any k, so by (3.15) we get that $\mathbf{1}_O MA(P_\psi\phi)$ is concentrated on $\{P_\psi\phi = \phi\}$. Since $MA(P_\psi\phi)$ puts no mass on the pluripolar set $\{P_\psi\phi = -\infty\}$, letting C tend to infinity yields the lemma. □

THEOREM 20. *Let $\psi \in PSH(L)$ and let ϕ be a continuous metric. Then $\phi_{[\psi]}$ is maximal with respect to ϕ, i.e. $\phi_{[\psi]} = \phi$ a.e. with respect to $MA(\phi_{[\psi]})$.*

Proof. $P_{[\psi]}\phi = \phi_{[\psi]}$ a.e., and since $P_{\psi+C}\phi$ increases to $P_{[\psi]}\phi$, it thus increases to $\phi_{[\psi]}$ a.e.. By Lemma 19 we get that $P_{\psi+C}\phi$ is maximal with respect to ϕ on the plurifine open set $\{\psi > \phi - C\}$ and thus also on any set $\{\psi > \phi - C'\}$ whenever $C' \leq C$. From Lemma 18 it thus follows that $\phi_{[\psi]}$ is maximal with respect to ϕ on the set $\{\psi > \phi - C\}$ for any C. Since $MA(\phi_{[\psi]})$ puts no mass on $\{\psi = -\infty\}$ the theorem follows. □

EXAMPLE 1. *Consider the case that s is a section of rL that vanishes along a divisor D, and set $\psi = \frac{1}{r}\ln|s|^2$. Then the maximal envelope $\phi_{[\psi]}$ is considered by Berman [5, Sec. 4], and equals*

$$\sup\{\psi' \leq \phi : \psi' \in PSH(L), \nu_D(\psi') \geq 1\}^*$$

where ν_D denotes the Lelong number along D. This metric governs the Bergman kernel asymptotics of sections of kL for $k \gg 0$ that vanish along the divisor

D. The more general case when ψ has analytic singularities is also considered in [5].

The maximal property gives the following bounds on the energy functional which will be crucial for our construction of weak geodesics (Theorem 28).

PROPOSITION 21. *Suppose that ψ is maximal with respect to a positive metric ϕ with small unbounded locus, and let $t > 0$. Then*

$$t\int_X MA(\psi) \le \mathcal{E}(\max\{\psi + t, \phi\}, \phi) \le t\int_X MA(\phi). \tag{3.16}$$

Proof. Since by assumption $\psi \le \phi$ we know $\max\{\psi + t, \phi\} \le \phi + t$, so from the monotonicity of the Aubin-Mabuchi energy it follows that

$$\mathcal{E}(\max\{\psi + t, \phi\}, \phi) \le \mathcal{E}(\phi + t, \phi) = t\int_X MA(\phi)$$

which gives the upper bound. For the lower bound, first choose an ϵ with $0 < \epsilon < t$. Again by monotonicity,

$$\mathcal{E}(\max\{\psi + t, \phi\}, \phi) \ge \mathcal{E}(\max\{\psi + t, \phi\}, \max\{\psi + \epsilon, \phi\}). \tag{3.17}$$

Now clearly

$$\mathcal{E}(\max\{\psi + t, \phi\}, \max\{\psi + \epsilon, \phi\}) \ge (t - \epsilon)\int_{\{\psi+\epsilon>\phi\}} MA(\psi). \tag{3.18}$$

By the assumption that ψ is maximal with respect to ϕ

$$\int_{\{\psi=\phi\}} MA(\psi) = \int_X MA(\psi)$$

and since $\{\psi = \phi\} \subseteq \{\psi + \epsilon > \phi\}$, the combination of (3.17) and (3.18) yields

$$\mathcal{E}(\max\{\psi + t, \phi\}, \phi) \ge (t - \epsilon)\int_X MA(\psi).$$

Since $\epsilon > 0$ was chosen arbitrarily the lower bound in (3.16) follows. □

3.5 Test curves and analytic test configurations

DEFINITION 11. *A map $\lambda \mapsto \psi_\lambda$ from $\mathbb{R}$ to $PSH(L)$ is called a* test curve *if there is a constant C such that*

(i) ψ_λ is equal to some locally bounded positive metric $\psi_{-\infty}$ for $\lambda < -C$,

(ii) $\psi_\lambda \equiv -\infty$ for $\lambda > C$,

(iii) ψ_λ has small unbounded locus whenever $\psi_\lambda \not\equiv -\infty$, and

(iiii) ψ_λ is concave in λ.

Observe that since ψ_λ is concave and constant for λ sufficiently negative it is decreasing in λ.

Note that the set of test curves forms a convex set, by letting

$$(\sum r_i\gamma_i)(\lambda) := \sum r_i\gamma_i(\lambda).$$

It is also clear that any translate $\gamma_a(\lambda) := \gamma(\lambda - a)$ of a test curve γ is a new test curve.

We introduce the notation λ_c for the critical value of a test curve defined as

$$\lambda_c := \inf\{\lambda : \psi_\lambda \equiv -\infty\}.$$

For later use we record here two continuity properties of test curves.

LEMMA 22.

1. *A test curve ψ_λ is left-continuous in λ as long as $\lambda < \lambda_c$.*

2. *Suppose that $\lambda < \lambda_c$ and λ_k is a decreasing sequence that tends to λ. Then*

$$(\lim_{k\to\infty} \psi_{\lambda_k})^* = \psi_\lambda. \tag{3.19}$$

(So a a test curve is right continuous modulo taking an upper semicontinuous regularization.)

Proof. For (1), let λ_k increase to some $\lambda < \lambda_c$, so we need to show that

$$\lim_{k\to\infty} \psi_{\lambda_k} = \psi_\lambda.$$

By our hypothesis there exists a λ' such that $\lambda < \lambda' < \lambda_c$, and thus $\psi_{\lambda'} \not\equiv -\infty$. Since $\psi_\lambda(x)$ is concave in λ it is continuous for all x such that $\psi_{\lambda'}(x) \neq -\infty$. Thus ψ_{λ_k} converges to ψ_λ at least away from a pluripolar set, i.e. a.e. with respect to a volume form. On the other hand we have that ψ_{λ_k} is decreasing in k, so the limit is a positive metric. Now if two positive metrics coincide a.e. with respect to a volume form it follows that they are equal everywhere, because this is true locally for plurisubharmonic function.

The proof of (2) is essentially the same. This time λ_k is a decreasing sequence, so as $\lambda < \lambda_c$ we may as well assume that each $\lambda_k < \lambda'$ and so in particular $\psi_{\lambda_k} \not\equiv -\infty$. Then the ψ_{λ_k} form an increasing sequence so the left hand side of (3.19) is a positive metric. But for the same reason as above, the limit $\lim_{k\to\infty} \psi_{\lambda_k}$ equals ψ_λ away from a pluripolar set, and thus the left and right hand side of (3.19) agree a.e. with respect to a volume form, and thus are equal everywhere. □

DEFINITION 12. *A map γ from $\mathbb{R}$ to $Sing(L)$ is called an* analytic test configuration *if it is the composition of a test curve with the natural projection of $PSH(L)$ to $Sing(L)$.*

We say that an analytic test configuration $[\psi_\lambda]$ is trivial if $[\psi_\lambda] = [\phi]$ for $\lambda < \lambda_c$ and $[\psi_\lambda] = [-\infty]$ for $\lambda > \lambda_c$.

As with the set of test curves, the set of analytic test configurations is convex. We now extend the definition of the maximal envelope (Definition 9) to test curves.

DEFINITION 13. *Let ψ_λ be a test curve and ϕ an element in $\mathcal{H}(L)$. The* maximal envelope *of ϕ with respect to ψ_λ is the map*

$$\lambda \mapsto \phi_\lambda := \phi_{[\psi_\lambda]} = (P_{[\psi_\lambda]}\phi)^*.$$

It is easy to see that ϕ_λ only depends on ϕ and the analytic test configuration $[\psi_\lambda]$, since if $\psi'_\lambda \sim \psi_\lambda$ we trivially have $\phi_{[\psi_\lambda]} = \phi_{[\psi'_\lambda]}$. Observe also that since $\psi_{-\infty}$ is locally bounded, we have $\phi_\lambda = \phi$ for $\lambda < -C$.

DEFINITION 14. *We say that a test curve ψ_λ is* maximal *if for all λ the metric ψ_λ is maximal with respect to $\psi_{-\infty}$.*

Since ψ_λ is decreasing in λ,

$$\{\psi_{\lambda'} = \psi_\lambda\} \supseteq \{\psi_{\lambda'} = \psi_{-\infty}\} \quad \text{if} \quad \lambda \leq \lambda'.$$

It follows that if ψ_λ is a maximal test curve, $\psi_{\lambda'}$ is maximal with respect to ψ_λ whenever $\lambda \leq \lambda'$.

PROPOSITION 23. *The maximal envelope ϕ_λ is a maximal test curve.*

Proof. We first show it is a test curve. Pick a real number C. Let λ and λ' be two real numbers, and let $0 \leq t \leq 1$. By the concavity of ψ_λ,

$$tP_{\psi_\lambda + C}\phi + (1-t)P_{\psi_{\lambda'}+C}\phi \leq t\psi_\lambda + (1-t)\psi_{\lambda'} + C \leq \psi_{t\lambda+(1-t)\lambda'} + C.$$

Thus from the definition of the projection operator,

$$tP_{\psi_\lambda + C}\phi + (1-t)P_{\psi_{\lambda'}+C}\phi \leq P_{\psi_{t\lambda+(1-t)\lambda'}+C}\phi,$$

which means that $P_{\psi_\lambda+C}\phi$ is concave in λ for all C. Since $P_{\psi_\lambda+C}\phi$ increases to $P_{[\psi_\lambda]}\phi$, and an increasing sequence of concave functions is concave, we get that $P_{[\psi_\lambda]}\phi$ is concave, and because of the monotonicity of the upper semicontinuous regularization it follows that $P_{[\psi_\lambda]}\phi^* = \phi_\lambda$ also is concave. The other properties of a test curve are immediate.

Clearly $\phi_{-\infty} = \phi$, so that ϕ_λ is maximal follows from Theorem 20. □

3.6 The Legendre transform and geodesic rays

If f is a convex function in the real variable λ, the set of subderivatives of f, denoted by Δ_f, is the set of $t \in \mathbb{R}$ such that $f(\lambda) - t\lambda$ is bounded from below. If f happens to be differentiable, then the set subderivatives coincides with the image of the derivative of f. By convexity of f, the set of subderivatives is convex, i.e. an interval. Recall that the Legendre transform of f, here denoted by $\widehat{f}$, is the function on Δ_f defined as

$$\widehat{f}(t) := \sup_\lambda \{t\lambda - f(\lambda)\}.$$

Since $\widehat{f}$ is defined as the supremum of the linear functions $t\lambda - f(\lambda)$, it follows that $\widehat{f}$ is convex.

If f is concave instead of convex, then of course $-f$ is convex, and one can define the Legendre transform of f, also denoted by $\widehat{f}$, as the Legendre transform of $-f$, i.e.

$$\widehat{f}(t) := \sup_{\lambda}\{f(\lambda) + t\lambda\},$$

which is thus convex.

DEFINITION 15. *The* Legendre transform *of a test curve ψ_λ, denoted by $\widehat{\psi_t}$, is given by*

$$\widehat{\psi_t} := (\sup_{\lambda\in\mathbb{R}}\{\psi_\lambda + t\lambda\})^*,$$

where $t \in [0, \infty)$.

Recall that the star denotes the operation of taking the upper semicontinuous regularization.

LEMMA 24. *Let ψ_λ be any test curve (not necessarily maximal). Then the Legendre transform $\widehat{\psi_t}$ is locally bounded for all t, and the map $t \mapsto \widehat{\psi_t}$ is a subgeodesic ray emanating from $\psi_{-\infty}$.*

Proof. By assumption, for some λ, ψ_λ is locally bounded, and trivially $\widehat{\psi_t} \geq \psi_\lambda + t\lambda$, thus $\widehat{\psi_t}$ is locally bounded. It is clear that for a fixed λ, the curve $\psi_\lambda + t\lambda$ is a subgeodesic. Clearly $\sup_{\lambda\in\mathbb{R}}\{\psi_\lambda + t\lambda\}$ is convex and Lipschitz in t, and the same is easily seen to hold for $\widehat{\psi_t}$. Thus $\widehat{\psi_t}$ is upper semicontinuous in the directions in X and also Lipschitz in t, which implies that it is upper semicontinuous on the product $X \times \mathbb{R}_{\geq 0}$. Therefore $\widehat{\psi_t}$ (thought of as a function on the product) coincides with the upper semicontinuous regularization of of $\sup_{\lambda\in\mathbb{R}}\{\psi_\lambda + t\lambda\}$.

Now, taking the upper semicontinuous regularization of the supremum of subgeodesics yields a subgeodesic, as long as it is bounded from above. We observed above that $\psi_\lambda \leq \psi_{-\infty}$. Now for some constant C, $\psi_C \equiv -\infty$. It follows that $\widehat{\psi_t} \leq \psi_{-\infty} + tC$, so it is bounded from above and thus it is a subgeodesic.

Finally by definition $\widehat{\psi}_0 = (\sup_\lambda(\psi_\lambda))^*$, which clearly is equal to $\psi_{-\infty}$ since $\psi_\lambda \leq \psi_{-\infty}$ (ψ_λ being decreasing in λ) and $\psi^*_{-\infty} = \psi_{-\infty}$. □

One can also consider the inverse Legendre transform, going from subgeodesic rays to concave curves of positive metrics.

DEFINITION 16. *The Legendre transform of a subgeodesic ray* ϕ_t, $t \in [0, \infty)$, *denoted by* $\widehat{\phi}_\lambda$, $\lambda \in \mathbb{R}$, *is defined as*

$$\widehat{\phi}_\lambda := \inf_{t \in [0,\infty)} \{\phi_t - t\lambda\}.$$

It follows from Kiselman's minimum principle (see [22]) that for any $\lambda \in \mathbb{R}$, $\widehat{\phi}_\lambda$ is a positive metric (we would like to thank Bo Berndtsson for this observation). Furthermore it is clear that $\widehat{\phi}_\lambda$ is concave and decreasing in λ. From the involution property of the (real) Legendre transform it follows that the Legendre transform of $\widehat{\phi}_\lambda$ is ϕ_t, thus any subgeodesic ray is the Legendre transform of a concave curve of positive metrics.

The goal of this section is to prove that if ψ_λ is an *maximal* test curve then the Legendre transform $\widehat{\psi}_t$ of ψ_λ is a weak geodesic ray emanating from $\psi_{-\infty}$. By Lemma 24 we know $\widehat{\psi}_t$ is a subgeodesic ray emanating from $\psi_{-\infty}$. What remains then is to show that if ψ_λ is maximal then the Aubin-Mabuchi energy $\mathcal{E}(\widehat{\psi}_t, \widehat{\psi}_0)$ is linear in t, which we now do with an approximation argument.

For $N \in \mathbb{N}$ consider the approximation $\widehat{\psi}_t^N$ to $\widehat{\psi}_t$, given by

$$\widehat{\psi}_t^N := \sup_{k \in \mathbb{Z}} \{\psi_{k2^{-N}} + tk2^{-N}\}.$$

Since ψ_λ is concave it is continuous in λ at all points such that $\psi_\lambda(x) > -\infty$. From the continuity it follows that $\widehat{\psi}_t^N$ will increase pointwise to $\widehat{\psi}_t$ a.e. as N tends to infinity. Also let $\widehat{\psi}_t^{N,M}$ denote the curve

$$\widehat{\psi}_t^{N,M} := \sup_{k \in \mathbb{Z}, k \leq M} \{\psi_{k2^{-N}} + tk2^{-N}\}.$$

Once again, $\widehat{\psi}_t^N$ and $\widehat{\psi}_t^{N,M}$ are all locally bounded.

LEMMA 25. *Let $M < M'$ be two integers. Then*

$$\widehat{\psi}_t^{N,M'} = \psi_{M'2^{-N}} + tM'2^{-N}$$

implies that

$$\widehat{\psi}_t^{N,M} = \psi_{M2^{-N}} + tM2^{-N}.$$

Proof. Certainly $f(\lambda) := \psi_\lambda(x) + t\lambda$ is concave in λ. If

$$\widehat{\psi}_t^{N,M} > \psi_{M2^{-N}} + tM2^{-N}$$

at x, then f would be strictly decreasing at $\lambda = M2^{-N}$, so by concavity we would get that $f(M'2^{-N}) < f(M2^{-N}) < \widehat{\psi}_t^{N,M}(x)$, which would be a contradiction. □

LEMMA 26. *If ψ_λ is a maximal test curve then*

$$t2^{-N}\int_X MA(\psi_{(M+1)2^{-N}}) \leq \mathcal{E}(\widehat{\psi}_t^{N,M+1}, \widehat{\psi}_t^{N,M}) \leq t2^{-N}\int_X MA(\psi_{M2^{-N}}).$$

Proof. By Lemma 25 it follows that $\widehat{\psi}_t^{N,M} = \psi_{M2^{-N}} + tM2^{-N}$ on the support of $\widehat{\psi}_t^{N,M+1} - \widehat{\psi}_t^{N,M}$ and thus Lemma 14 yields

$$\mathcal{E}(\widehat{\psi}_t^{N,M+1}, \widehat{\psi}_t^{N,M}) = \mathcal{E}(\max\{\psi_{M2^{-N}}, \psi_{(M+1)2^{-N}} + t2^{-N}\}, \psi_{M2^{-N}}). \tag{3.20}$$

Since we assumed that ψ_λ was maximal we get that $\psi_{(M+1)2^{-N}}$ is maximal with respect to $\psi_{M2^{-N}}$, and thus the lemma follows immediately from Lemma 21. □

Let ψ_λ be a maximal test curve, and let $F(\lambda)$ denote the function

$$F(\lambda) := \int_X MA(\psi_\lambda).$$

Whenever $\lambda < \lambda'$, $\psi_{\lambda'} \leq \psi_\lambda$ and therefore it follows from Theorem 12 that $F(\lambda)$ is decreasing in λ, hence $F(\lambda)$ is Riemann integrable.

PROPOSITION 27. *If ψ_λ is a maximal test curve then*

$$\mathcal{E}(\widehat{\psi}_t, \widehat{\psi}_0) = -t\int_{\lambda=-\infty}^{\infty} \lambda dF(\lambda). \tag{3.21}$$

Proof. Suppose first $m \in \mathbb{Z}$ is such that $\psi_m = \psi_{-\infty}$. For a given $N \in \mathbb{N}$ set $M = m2^N$. Then

$$\widehat{\psi}_t^{N,M} = \psi_{-\infty} + tm = \widehat{\psi}_0 + tm.$$

By repeatedly using the cocycle property of the Aubin-Mabuchi energy in combination with Lemma 26 we get that

$$t \sum_{k>M} 2^{-N} F((k+1)2^{-N}) \leq \mathcal{E}(\widehat{\psi}_t^N, \widehat{\psi}_t^{N,M}) \leq t \sum_{k>M} 2^{-N} F(k2^{-N}). \quad (3.22)$$

We noted above that $\widehat{\psi}_t^N$ increases pointwise to $\widehat{\psi}_t$ a.e. as N tends to infinity. By the continuity of the Aubin-Mabuchi energy under a.e. pointwise increasing sequences (11),

$$\mathcal{E}(\widehat{\psi}_t, \widehat{\psi}_0 + tm) = t \int_{\lambda=m}^{\infty} \lambda F(\lambda) d\lambda,$$

since both the left- and the right-hand side of (3.22) converges to this. Again using the cocycle property we get that

$$\mathcal{E}(\widehat{\psi}_t, \widehat{\psi}_0) = \mathcal{E}(\widehat{\psi}_t, \widehat{\psi}_0 + tm) + \mathcal{E}(\widehat{\psi}_0 + tm, \widehat{\psi}_0) =$$
$$= t \int_{\lambda=m}^{\infty} \lambda F(\lambda) d\lambda + tm \int_X MA(\psi_{-\infty}) = t \int_{\lambda=m}^{\infty} F(\lambda) d\lambda + tmF(m) \quad (3.23)$$

Since by our assumption the measure dF is zero on $(-\infty, m)$, integration by parts yields

$$-t \int_{\lambda=-\infty}^{\infty} \lambda dF(\lambda) = -\lambda F(\lambda)|_m^{\infty} + t \int_{\lambda=m}^{\infty} F(\lambda) d\lambda =$$
$$= tmF(m) + \int_{\lambda=m}^{\infty} F(\lambda) d\lambda. \quad (3.24)$$

The proposition follows from combining equation (3.23) and equation (3.24). □

THEOREM 28. *The Legendre transform $\widehat{\psi}_t$ of a maximal test curve ψ_λ is a weak geodesic ray emanating from $\psi_{-\infty}$.*

Proof. That $\widehat{\psi}_t$ is a subgeodesic emanating from $\psi_{-\infty}$ was proved in Lemma 24. According to Proposition 27 the energy $\mathcal{E}(\widehat{\psi}_t, \widehat{\psi}_0)$ is linear in t, and therefore by Lemma 15 we get that $\widehat{\psi}_t$ is a geodesic ray. □

These weak geodesics are continuous in ϕ in the following sense:

PROPOSITION 29. *Let ψ_λ be a test curve and $\phi, \phi' \in \mathcal{H}(L)$. Suppose ϕ_λ is the maximal curve of ϕ (with respect to ψ_λ) and similarly for ϕ'_λ. If $||\phi - \phi'||_\infty < C$ then*

$$||\widehat{\phi}_t - \widehat{\phi'}_t||_\infty < C \quad \textit{for all } t.$$

Proof. We claim that $||\phi_\lambda - \phi'_\lambda||_\infty < C$ for all λ. But this is clear since $\phi \leq \phi'$ implies that $\phi_\lambda \leq \phi'_\lambda$ for all λ. It is also clear that $(\phi + C)_\lambda = \phi_\lambda + C$ when C is a constant. Now we noted above that $\phi \leq \phi'$ implies that $\phi_\lambda \leq \phi'_\lambda$ for all λ, and so it follows that $\widehat{\phi}_t \leq \widehat{\phi'}_t$ for all t. We also noted that $(\phi + C)_\lambda = \phi_\lambda + C$, so consequently $\widehat{\phi + C}_t = \widehat{\phi}_t + C$ which proves the lemma. □

Let $[\psi_\lambda]$ be an analytic test configuration, and let ϕ_λ be an associated maximal test curve. Then $[\phi_\lambda]$ defines a new analytic test configuration. This could possibly differ from $[\psi_\lambda]$, but the following proposition tells us that the associated geodesic rays are the same.

PROPOSITION 30. *Let $\phi' \in \mathcal{H}(L)$. Then the Legendre transform of $\phi'_{[\phi_\lambda]}$ coincides with the Legendre transform of $\phi'_\lambda := \phi'_{[\psi_\lambda]}$.*

Proof. Since $\phi'_\lambda \sim \phi_\lambda$ we get that $\phi'_{[\phi_\lambda]} = \phi'_{[\phi'_\lambda]}$, thus without loss of generality we can assume that $\phi' = \phi$. Recall that the critical value λ_c was defined as

$$\lambda_c := \inf\{\lambda : \phi_\lambda \equiv -\infty\}.$$

If $\lambda < \lambda_c$ there exists a λ' such that $\lambda < \lambda' < \lambda_c$, and thus by the assumption $\phi_{\lambda'}$ has small unbounded locus. Let C be a constant less than λ such that $\phi_C = \phi$. By concavity it follows that

$$\phi_\lambda \geq r\phi + (1 - r)\phi_{\lambda'},$$

where $0 < r < 1$, is chosen such that

$$\lambda = rC + (1 - r)\lambda'.$$

If we let

$$\rho := r\phi + (1 - r)\phi_{\lambda'},$$

by the multilinearity of the Monge-Ampére operator it follows that $MA(\rho)$ dominates the volume form $r^n MA(\phi)$. Furthermore ρ has small unbounded locus and is more singular than ϕ_λ. Thus by Proposition 17 we get that

$$P_{\phi_\lambda + C}\phi \leq \phi_\lambda$$

for any constant C and therefore

$$\phi_{[\phi_\lambda]} = \phi_\lambda, \tag{3.25}$$

whenever $\lambda < \lambda_c$. If $\lambda > \lambda_c$ then clearly equation (3.25) holds as well since both sides are identically equal to minus infinity. It follows that for any $\epsilon > 0$,

$$\phi_\lambda \leq \phi_{[\phi_\lambda]} \leq \phi_{\lambda-\epsilon},$$

which implies that

$$\widehat{(\phi_\lambda)}_t \leq \widehat{(\phi_{[\phi_\lambda]})}_t \leq \widehat{(\phi_{\lambda-\epsilon})}_t = \widehat{(\phi_\lambda)}_t + \epsilon t.$$

Since $\epsilon > 0$ was arbitrary the proposition follows. □

3.7 Filtrations of the ring of sections

First we recall what is meant by a filtration of a graded algebra.

DEFINITION 17. *A* filtration $\mathcal{F}$ *of a graded algebra* $\oplus_k V_k$ *is a vector space-valued map from* $\mathbb{R} \times \mathbb{N}$,

$$\mathcal{F} : (t, k) \longmapsto \mathcal{F}_t V_k,$$

such that for any k, $\mathcal{F}_t V_k$ *is a family of subspaces of* V_k *that is decreasing and left-continuous in* t.

In [8] Boucksom-Chen consider certain filtrations which behaves well with respect to the multiplicative structure of the algebra. They give the following definition.

DEFINITION 18. *Let $\mathcal{F}$ be a filtration of a graded algebra $\oplus_k V_k$. We shall say that*

(i) $\mathcal{F}$ is multiplicative *if*

$$(\mathcal{F}_t V_k)(\mathcal{F}_s V_m) \subseteq \mathcal{F}_{t+s} V_{k+m}$$

for all $k, m \in \mathbb{N}$ and $s, t \in \mathbb{R}$.

(ii) $\mathcal{F}$ is (linearly) bounded *if there exists a constant C such that $\mathcal{F}_{-kC} V_k = V_k$ and $\mathcal{F}_{kC} V_k = \{0\}$ for all k.*

The goal in this section is to associate an analytic test configuration $\phi_\lambda^{\mathcal{F}}$ to any bounded multiplicative filtration of the section ring $R(L) = \oplus_k H^0(kL)$.

Let $\phi \in \mathcal{H}(L)$, and let dV be some smooth volume form on X with unit mass. This gives the L^2-scalar product on $H^0(kL)$ by letting

$$(s,t)_{k\phi} := \int_X s(z)\overline{t(z)} e^{-k\phi(z)} dV(z).$$

For any $\lambda \in \mathbb{R}$ let $\{s_{i,\lambda}\}$ be an orthonormal basis for $\mathcal{F}_{k\lambda} H^0(kL)$ and define

$$\phi_{k,\lambda} := \frac{1}{k} \ln\left(\sum |s_{i,\lambda}|^2\right),$$

which is a positive metric on L.

LEMMA 31. *For any λ, the sequence of metrics $\phi_{k,\lambda}$ converges to a limit as k tends to infinity, and the usc regularization of the limit*

$$\phi_\lambda^{\mathcal{F}} := (\lim_{k\to\infty} \phi_{k,\lambda})^*$$

is a positive metric.

Proof. Since

$$K_\lambda(z,w) := \sum_i s_{i,\lambda}(z)\overline{s_{i,\lambda}(w)}$$

is a reproducing kernel of $\mathcal{F}_{k\lambda} H^0(kL)$ with respect to $(\cdot,\cdot)_{k\phi}$, as for the full Bergman kernel we have the following useful characterization

$$\sum |s_{i,\lambda}|^2 = \sup\{|s|^2 : s \in \mathcal{F}_{k\lambda} H^0(kL), ||s||^2_{k\phi} \leq 1\}. \tag{3.26}$$

Let $||s||_{\infty}^2 := \sup_{z\in X}\{|s(z)|^2 e^{-k\phi}\}$ and define

$$F_{k,\lambda}(z) := \sup\{|s(z)|^2 : s \in \mathcal{F}_{k\lambda}H^0(kL), ||s||_{\infty}^2 \leq 1\}.$$

We trivially have the upper bound

$$F_{k,\lambda}(z) \leq e^{-k\phi(z)}.$$

It follows that

$$(\frac{1}{k}\ln F_{k,\lambda})^* = (\sup\{\frac{1}{k}\ln |s|^2 : s \in \mathcal{F}_{k\lambda}H^0(kL), ||s||_{\infty}^2 \leq 1\})^*$$

is a positive metric. Let λ be fixed, pick a point $z \in X$, and let for all k, $s_k \in \mathcal{F}_{k\lambda}H^0(kL)$ be such that $||s_k||_{\infty} = 1$ and

$$F_{k,\lambda}(z) = |s_k(z)|^2.$$

Since the product $s_k s_m$ lies in $\mathcal{F}_{(k+m)\lambda}H^0((k+m)L)$ by the multiplicativity of $\mathcal{F}$, and $||s_k s_m||_{\infty} \leq ||s_k||_{\infty}||s_m||_{\infty}$, we get that

$$F_{k+m,\lambda}(z) \geq F_{k,\lambda}(z)F_{m,\lambda}(z), \tag{3.27}$$

i.e. the map $k \mapsto F_{k,\lambda}(z)$ is supermultiplicative. The existence of a limit

$$\lim_{k\to\infty}\frac{1}{k}\ln F_{k,\lambda}(z)$$

thus follows from Fekete's lemma (see e.g. [37]). Since we assumed that dV had unit mass, we get that for any section s

$$||s||_{k\phi}^2 \leq ||s||_{\infty}^2,$$

and thus by equation (3.26)

$$\sum |s_{i,\lambda}(z)|^2 \geq F_{k,\lambda}(z).$$

On the other hand, by the Bernstein-Markov property of any volume form dV we have that for any $\epsilon > 0$ there exists a constant C_ϵ so that

$$||s||_{\infty}^2 \leq C_\epsilon e^{\epsilon k}||s||_{k\phi}^2,$$

and thus

$$\sum |s_{i,\lambda}(z)|^2 \leq C_\epsilon e^{\epsilon k} F_{k,\lambda}(z), \tag{3.28}$$

(see [37]). It follows that the difference $\phi_{k,\lambda}(z) - \frac{1}{k}\ln F_{k,\lambda}(z)$ tends to zero as k tends to infinity, thus the convergence of $\phi_{k,\lambda}$ follows.

By the supermultiplicativity we get that for any $k \in \mathbb{N}$

$$\frac{1}{k}\ln F_{k,\lambda} \leq \lim_{l\to\infty} \frac{1}{l}\ln F_{l,\lambda} = \lim_{l\to\infty} \phi_{l,\lambda},$$

and thus

$$(\frac{1}{k}\ln F_{k,\lambda})^* \leq (\lim_{l\to\infty} \phi_{l,\lambda})^* =: \phi_\lambda^{\mathcal{F}}. \tag{3.29}$$

On the other hand, clearly

$$\lim_{l\to\infty} \phi_{l,\lambda} \leq \sup_k \{(\frac{1}{k}\ln F_{k,\lambda})^*\},$$

and it follows that

$$\phi_\lambda^{\mathcal{F}} = (\sup_k \{(\frac{1}{k}\ln F_{k,\lambda})^*\})^*$$

so $\phi_\lambda^{\mathcal{F}}$ is indeed a positive metric. □

Remark. Since all volume forms dV on X are equivalent, the limit ϕ_λ does not depend on the choice of volume form dV.

LEMMA 32. *We have that*

$$\phi_{k,\lambda} \leq \phi_\lambda^{\mathcal{F}} + \epsilon(k),$$

where $\epsilon(k)$ is a constant independent of λ that tends to zero as k tends to infinity.

Proof. By combining the inequalities (3.28) and (3.29) from the proof of the the previous lemma we get that for any $\epsilon > 0$ there exists a constant C_ϵ independent of λ such that

$$\phi_{k,\lambda} \leq \phi_\lambda^{\mathcal{F}} + \epsilon + (1/k)\ln C_\epsilon.$$

This yields the lemma. □

PROPOSITION 33. *The map $\lambda \mapsto \phi_\lambda^{\mathcal{F}}$ is a test curve.*

Proof. Let λ be such that $F_{k\lambda}H^0(kL) = H^0(kL)$ for all k. Then $\phi_{k,\lambda}$ is the usual Bergman metric, and by the result on Bergman kernel asymptotics due to Bouche-Catlin-Tian-Zelditch (see Section 3.3) we get that $\phi_{k,\lambda}$ converges to ϕ. Trivially we see that if $F_{k\lambda}H^0(kL) = \{0\}$ for all k then $\phi^{\mathcal{F}}_\lambda \equiv -\infty$. By the boundedness of the filtration we thus have $\phi^{\mathcal{F}}_\lambda = \phi$ for $\lambda < -C$ and $\phi^{\mathcal{F}}_\lambda \equiv -\infty$ for $\lambda > C$.

By the multiplicativity of the filtration we see $\phi_\lambda \equiv -\infty$ if and only if for all k,

$$\mathcal{F}_{k\lambda}H^0(kL) = \{0\}.$$

Pick a λ such that $\phi^{\mathcal{F}}_\lambda \not\equiv -\infty$, then for some k, $\mathcal{F}_{k\lambda}H^0(kL)$ is non-trivial. From Lemma 32 it follows that $\phi^{\mathcal{F}}_\lambda$ has small unbounded locus since $\phi_{k,\lambda}$ has small unbounded locus.

It remains to prove concavity. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and let t be a rational point in the unit interval. Let m be a natural number such that mt is an integer. Given a point $z \in X$, let $s_1 \in \mathcal{F}_{k\lambda_1}H^0(kL)$ and $s_2 \in \mathcal{F}_{k\lambda_2}H^0(kL)$ be two sections with $||s_1||_\infty = ||s_2||_\infty = 1$ such that

$$F_{k,\lambda_1} = |s_1(z)|^2$$

and

$$F_{k,\lambda_2} = |s_2(z)|^2.$$

By the multiplicativity of the filtration,

$$s_1^{mt}s_2^{m(1-t)} \in F_{mk(t\lambda_1+(1-t)\lambda_2)}H^0(mkL),$$

and trivially $||s_1^{mt}s_2^{m(1-t)}||_\infty \leq 1$. It follows that

$$F_{mk,t\lambda_1+(1-t)\lambda_2}(z) \geq F_{k,\lambda_1}(z)^{mt}F_{k,\lambda_2}(z)^{m(1-t)}.$$

Taking the logarithm on both sides, dividing by mk, and taking the limit yields that

$$\phi^{\mathcal{F}}_{t\lambda_1+(1-t)\lambda_2} \geq t\phi^{\mathcal{F}}_{\lambda_1} + (1-t)\phi^{\mathcal{F}}_{\lambda_2} \tag{3.30}$$

except possibly on the pluripolar set where the limits are not equal to their upper semicontinuous regularization. But it is easily seen that if a positive metric is larger than or equal to another except on a pluripolar set then it is in fact larger than or equal on the whole space. Thus we get that (3.30) holds on the whole of X. Recall that t was assumed to be rational. If $\lambda_1 \leq \lambda_2$, the left-hand side of (3.30) is decreasing in t since clearly $\phi_\lambda^{\mathcal{F}}$ is decreasing in λ. The right-hand side of (3.30) is continuous in t, so it follows that the equation (3.30) holds for all $t \in (0,1)$, i.e. $\phi_\lambda^{\mathcal{F}}$ is concave in λ. □

LEMMA 34. *For any two $\phi, \psi \in \mathcal{H}(L)$ and any $\lambda \in \mathbb{R}$ we have $\phi_\lambda^{\mathcal{F}} \sim \psi_\lambda^{\mathcal{F}}$.*

Proof. Assume that $\phi \leq \psi$, then it is immediate that for all k and λ we have that $\phi_{k,\lambda} \leq \psi_{k,\lambda}$, and we thus get that $\phi_\lambda^{\mathcal{F}} \leq \psi_\lambda^{\mathcal{F}}$. Also it is clear that $(\phi + C)_{k,\lambda} = \phi_{k,\lambda} + C$. When combining these two facts we get the lemma. □

DEFINITION 19. *We call the map $\lambda \mapsto [\phi_\lambda^{\mathcal{F}}]$ the* analytic test configuration associated to the filtration $\mathcal{F}$.

So by the previous lemma this analytic test configuration depends only on $\mathcal{F}$ and not on the choice of $\phi \in \mathcal{H}(L)$. Our next goal is to show the curve $\phi_\lambda^{\mathcal{F}}$ is maximal for $\lambda < \lambda_c$, for which we will need a Skoda-type division theorem.

THEOREM 35. *Let L be an ample line bundle. Assume that L has a smooth positive metric ϕ with the property that $dd^c\phi \geq dd^c\phi_{K_X}$ for some smooth metric ϕ_{K_X} on the canonical bundle K_X. Let $\{s_i\}$ be a finite collection of holomorphic sections of L and $m > n+2$ where $n = \dim X$.*

Suppose s is a section of mL such that

$$\int_X \frac{|s|^2}{(\sum |s_i|^2)^m} dV < \infty.$$

Then there exists sections $h_\alpha \in H^0((n+1)L)$ such that

$$s = \sum_\alpha h_\alpha s^\alpha,$$

where α is a multiindex $\alpha = (\alpha_i)$ with $\sum_i \alpha_i = m - n - 1$, and s^α are the monomials $s^\alpha := \Pi_i s_i^{\alpha_i}$.

Proof. Let k be an integer such that $n+2 \leq k \leq m$. Then given a section $t \in H^0(kL)$ with

$$\int_X \frac{|t|^2}{(\sum_i |s_i|^2)^k} dV < \infty$$

an application of the Skoda division theorem [36, Thm. 2.1] yields sections $\{t_i\}$ of $(k-1)L$ such that $t = \sum_i t_i s_i$ and

$$\int_X \frac{|t_i|^2}{(\sum_i |s_i|^2)^{k-1}} dV < \infty.$$

(To apply the cited theorem replace F, E, ψ, η with $kL - K_X, L, k\phi - \phi_{K_X}, \phi$ respectively and replace αq with $k-1 > n+1$.)

Now we first apply the above with $k = m$ to the section s, and then apply again with $k = m-1$ to each of the sections t_i. Repeating this process with $k = m, m-1, \ldots, n+2$ we see that s can be written as a linear sum of monomials in the s_i as required. □

PROPOSITION 36. *For λ less than the critical value λ_c we have that*

$$\phi_\lambda^{\mathcal{F}} = \lim_{k\to\infty} \phi_{[\phi_{k,\lambda}]}.$$

Proof. Let $\phi_k := \phi_{k,-\infty}$, i.e. the Bergman metric $1/k \ln(\sum |s_i|^2)$, where $\{s_i\}$ is an orthonormal basis for the whole space $H^0(kL)$ with respect to $(\cdot,\cdot)_{k\phi}$. By the Bernstein-Markov property of any volume form dV (see e.g. [37]), or simply the maximum principle, we get that

$$\phi_k \leq \phi + \epsilon_k, \tag{3.31}$$

where ϵ_k tends to zero as k tends to infinity. Since $\phi_{k,\lambda}$ is decreasing in λ, the inequality (3.31) still holds when ϕ_k is replaced by $\phi_{k,\lambda}$, i.e. $\phi_{k,\lambda} - \epsilon_k \leq \phi$. Therefore $\phi_{k,\lambda} - \epsilon_k$ belongs to the class of metrics the supremum of which yields $P_{[\phi_{k,\lambda}]}\phi$, and thus clearly

$$\phi_{k,\lambda} \leq P_{[\phi_{k,\lambda}]}\phi + \epsilon_k.$$

Letting k tend to infinity yields

$$\phi_\lambda^{\mathcal{F}} \leq (\lim_{k\to\infty} P_{[\phi_{k,\lambda}]}\phi)^*.$$

For the other inequality it is enough to show that for any constant C,

$$P_{\phi_{k,\lambda}+C}\,\phi \leq \phi_\lambda^{\mathcal{F}}. \tag{3.32}$$

By the assumption that $\lambda < \lambda_c$ we have that $\phi_\lambda^{\mathcal{F}} \not\equiv -\infty$. Let ψ be a positive metric dominated by both $\phi_{k,\lambda}+C$ and ϕ, where k is large enough so that kL fulfills the requirements of Theorem 35. We denote by $\mathcal{J}(k\psi)$ the multiplier ideal sheaf of germs of holomorphic functions locally integrable against $e^{-k\psi}$. Let $\{s_i\}$ be an orthonormal basis of $H^0(kL\otimes\mathcal{J}(k\psi))$, and denote by ψ_k the Bergman metric

$$\psi_k := \frac{1}{k}\ln(\sum|s_i|^2).$$

By Theorem 8 we have that

$$\psi \leq \psi_k + \delta_k$$

where δ_k tends to zero as k tends to infinity, and ψ_k converges pointwise to ψ. If s lies in $H^0(kL\otimes\mathcal{J}(k\psi))$, specifically we must have that

$$\int_X \frac{|s|^2}{\sum|s_{i,\lambda}|^2}dV < \infty,$$

since we assumed that ψ was dominated by $\phi_{k,\lambda}+C = 1/k\ln(\sum|s_{i,\lambda}|^2)+C$. Similarly if s lies in $H^0(kmL\otimes\mathcal{J}(km\psi))$ we have

$$\int_X \frac{|s|^2}{(\sum|s_{i,\lambda}|^2)^m}dV < \infty.$$

From Theorem 35 applied to the sections $\{s_{i,\lambda}\}$ it thus follows that

$$s = \sum h_\alpha s^\alpha,$$

where $h_\alpha \in H^0(k(n+1)L)$, and the s^α are monomials in the $\{s_{i,\lambda}\}$ of degree $m-n-1$. Because of the multiplicativity of the filtration each s^α lies in $\mathcal{F}_{k(m-n-1)\lambda}H^0(k(m-n-1)L)$, and by the boundedness of the filtration we also have that each h_α lies in $F_{-k(n+1)C}H^0(k(n+1)L)$ for some fixed constant C. We thus get that $H^0(kmL\otimes\mathcal{J}(km\psi))$ is contained in

$$(F_{-k(n+1)C}H^0(k(n+1)L))(\mathcal{F}_{k(m-n-1)\lambda}H^0(k(m-n-1)L)) \subseteq \mathcal{F}_{k(m-n-1)\lambda-k(n+1)C}H^0(kmL). \tag{3.33}$$

Since we assumed that $\psi \leq \phi$ we have that ψ_{km} is less than or equal to the Bergman metric using an orthonormal basis for $H^0(kmL \otimes \mathcal{J}(km\psi))$ with respect to ϕ. Because of (3.33) this Bergman metric is certainly less than or equal to $\phi_{km,\lambda'}$, where

$$\lambda' := \frac{1}{km}(k(m-n-1)\lambda - k(n+1)C).$$

Hence

$$\psi_{km} \leq \phi_{km,\lambda'}.$$

On the other hand, by Lemma 32 we have that

$$\phi_{km,\lambda'} \leq \phi^{\mathcal{F}}_{\lambda'} + \epsilon(km),$$

where $\epsilon(km)$ is a constant independent of λ' that tends to zero as km tends to infinity. Since λ' tends to λ as m tends to infinity we get that $\psi \leq \lim_{\lambda' \to \lambda} \phi^{\mathcal{F}}_{\lambda'}$, and thus by Lemma 22 $\psi \leq \phi^{\mathcal{F}}_{\lambda}$. Taking the supremum over all such ψ completes the proof. □

COROLLARY 37. *The test curve $\phi^{\mathcal{F}}_{\lambda}$ is maximal for $\lambda < \lambda_c$ and its Legendre transform is a geodesic ray.*

Proof. Theorem 20 tells us that $\phi_{[\phi_{k,\lambda}]}$ is maximal with respect to $\phi = \phi_{-\infty}$. By Lemma 18 it follows that this is true for the limit $\phi^{\mathcal{F}}_{\lambda} = \lim_{k\to\infty} \phi_{[\phi_{k,\lambda}]}$ as well. Let ϕ_{λ} be the test curve defined by $\phi_{\lambda} := \phi^{\mathcal{F}}_{\lambda}$ for $\lambda < \lambda_c$ and $\phi_{\lambda} \equiv -\infty$ for $\lambda \geq \lambda_c$. Then we get that ϕ_{λ} is a maximal test curve, thus its Legendre transform is a geodesic ray. On the other hand, for every $\epsilon > 0$ we have that

$$\phi_{\lambda} \leq \phi^{\mathcal{F}}_{\lambda} \leq \phi_{\lambda-\epsilon},$$

and therefore

$$\widehat{\phi}_t \leq \widehat{(\phi^{\mathcal{F}})}_t \leq \widehat{\phi}_t + \epsilon t.$$

Since ϵ was arbitrary we get that the Legendre transform of $\phi^{\mathcal{F}}_{\lambda}$ coincides with that of ϕ_{λ}, and thus it is a geodesic ray. □

Remark. Given an analytic test configuration $[\psi_\lambda]$ there is a naturally associated filtration $\mathcal{F}$ of the section ring, defined as

$$\mathcal{F}_{k\lambda} H^0(kL) := H^0(kL \otimes \mathcal{J}(k\psi_\lambda)).$$

This filtration is bounded, but in general not multiplicative.

3.8 Filtrations associated to algebraic test configurations

We recall briefly Donaldson's definition of a test configurations [19, 20]. In order to not confuse them with the our analytic test configurations, we will in this article refer to them as algebraic test configuration.

DEFINITION 20. *An* algebraic test configuration $\mathcal{T}$ *for an ample line bundle* L *over* X *consists of:*

(i) a scheme $\mathcal{X}$ *with a* $\mathbb{C}^\times$*-action* ρ*,*

(ii) a $\mathbb{C}^\times$*-equivariant line bundle* $\mathcal{L}$ *over* $\mathcal{X}$*,*

(iii) and a flat $\mathbb{C}^\times$*-equivariant projection* $\pi : \mathcal{X} \to \mathbb{C}$ *where* $\mathbb{C}^\times$ *acts on* $\mathbb{C}$ *by multiplication, such that* $\mathcal{L}$ *is relatively ample, and such that if we denote by* $X_1 := \pi^{-1}(1)$*, then* $\mathcal{L}_{|X_1} \to X_1$ *is isomorphic to* $rL \to X$ *for some* $r > 0$*.*

By rescaling we can without loss of generality assume that $r = 1$ in the definition. An algebraic test configuration is called a *product test configuration* if there is a $\mathbb{C}^\times$-action ρ' on $L \to X$ such that $\mathcal{L} = L \times \mathbb{C}$ with ρ acting on L by ρ' and on $\mathbb{C}$ by multiplication. An algebraic test configuration is called *trivial* if it is a product test configuration with the action ρ' being the trivial $\mathbb{C}^\times$-action.

Since the zero-fiber $X_0 := \pi^{-1}(0)$ is invariant under the action ρ, we get an induced action on the space $H^0(kL_0)$, also denoted by ρ, where we have denoted the restriction of $\mathcal{L}$ to X_0 by L_0. Specifically, we let $\rho(\tau)$ act on a section $s \in H^0(kL_0)$ by

$$(\rho(\tau)(s))(x) := \rho(\tau)(s(\rho^{-1}(\tau)(x))). \tag{3.34}$$

By standard theory any vector space V with a $\mathbb{C}^\times$-action can be split into weight spaces V_{λ_i} on which $\rho(\tau)$ acts as multiplication by τ^{λ_i}, (see e.g. [19]). The numbers λ_i with non-trivial weight spaces are called the weights of the action. Thus we may write $H^0(kL_0)$ as

$$H^0(kL_0) = \oplus_\lambda V_\lambda$$

with respect to the induced action ρ.

In [26, Lem. 4] Phong-Sturm give the following linear bound on the absolute value of the weights.

LEMMA 38. *Given a test configuration there is a constant C such that*

$$|\lambda_i| < Ck$$

whenever $\dim V_{\lambda_i} > 0$.

In [38] the second author showed how to get an associated filtration $\mathcal{F}$ of the section ring $\oplus_k H^0(kL)$ given a test configuration $\mathcal{T}$ of L which we now recall.

First note that the $\mathbb{C}^\times$-action ρ on $\mathcal{L}$ via the equation (3.34) gives rise to an induced action on $H^0(\mathcal{X}, k\mathcal{L})$ as well as $H^0(\mathcal{X} \setminus X_0, k\mathcal{L})$, since $\mathcal{X} \setminus X_0$ is invariant. Let $s \in H^0(kL)$ be a holomorphic section. Then using the $\mathbb{C}^\times$-action ρ we get a canonical extension $\bar{s} \in H^0(\mathcal{X} \setminus X_0, k\mathcal{L})$ which is invariant under the action ρ, simply by letting

$$\bar{s}(\rho(\tau)x) := \rho(\tau)s(x) \tag{3.35}$$

for any $\tau \in \mathbb{C}^\times$ and $x \in X$.

We identify the coordinate z with the projection function $\pi(x)$, and we also consider it as a section of the trivial bundle over $\mathcal{X}$. Exactly as for $H^0(\mathcal{X}, k\mathcal{L})$,

ρ gives rise to an induced action on sections of the trivial bundle, using the same formula (3.34). We get that

$$(\rho(\tau)z)(x) = \rho(\tau)(z(\rho^{-1}(\tau)x) = \rho(\tau)(\tau^{-1}z(x)) = \tau^{-1}z(x), \qquad (3.36)$$

where we used that ρ acts on the trivial bundle by multiplication on the z-coordinate. Thus

$$\rho(\tau)z = \tau^{-1}z,$$

which shows that the section z has weight -1.

By this it follows that for any section $s \in H^0(kL)$ and any integer λ, we get a section $z^{-\lambda}\bar{s} \in H^0(\mathcal{X} \setminus X_0, k\mathcal{L})$, which has weight λ.

LEMMA 39. *For any section $s \in H^0(kL)$ and any integer λ the section $z^{-\lambda}\bar{s}$ extends to a meromorphic section of $k\mathcal{L}$ over the whole of $\mathcal{X}$, which we also will denote by $z^{-\lambda}\bar{s}$.*

Proof. It is equivalent to saying that for any section s there exists an integer λ such that $z^{\lambda}\bar{s}$ extends to a holomorphic section $S \in H^0(\mathcal{X}, k\mathcal{L})$. By flatness, which was assumed in the definition of a test configuration, the direct image bundle $\pi_*\mathcal{L}$ is in fact a vector bundle over $\mathbb{C}$. Thus it is trivial, since any vector bundle over $\mathbb{C}$ is trivial. Therefore there exists a global section $S' \in H^0(\mathcal{X}, k\mathcal{L})$ such that $s = S'_{|X}$. On the other hand, as for $H^0(kL_0)$, $H^0(\mathcal{X}, k\mathcal{L})$ may be decomposed as a direct sum of invariant subspaces $W_{\lambda'}$ such that $\rho(\tau)$ restricted to $W_{\lambda'}$ acts as multiplication by $\tau^{\lambda'}$. Let us write

$$S' = \sum S'_{\lambda'}, \qquad (3.37)$$

where $S_{\lambda'} \in W_{\lambda'}$. Restricting the equation (3.37) to X gives a decomposition of s,

$$s = \sum s_{\lambda'},$$

where $s_{\lambda'} := S'_{\lambda'|X}$. From (3.35) and the fact that $S'_{\lambda'}$ lies in $W_{\lambda'}$ we get that for $x \in X$ and $\tau \in \mathbb{C}^{\times}$ we have that

$$\bar{s}_{\lambda'}(\rho(\tau)(x)) = \rho(\tau)(s_{\lambda'}(x)) = \rho(\tau)(S'_{\lambda'}(x)) = (\rho(\tau)S'_{\lambda'})(\rho(\tau)(x))) = \\ = \tau^{\lambda'}S'_{\lambda'}(\rho(\tau)(x)),$$

and therefore $\bar{s}_{\lambda'} = \tau^{\lambda'} S'_{\lambda'}$. Since trivially

$$\bar{s} = \sum \bar{s}_{\lambda'}$$

it follows that $t^{\lambda}\bar{s}$ extends holomorphically as long as $\lambda \geq \max -\lambda'$. $\square$

DEFINITION 21. *Given a test configuration $\mathcal{T}$ we define a vector space-valued map $\mathcal{F}$ from $\mathbb{Z} \times \mathbb{N}$ by letting*

$$(\lambda, k) \longmapsto \{s \in H^0(kL) : z^{-\lambda}\bar{s} \in H^0(\mathcal{X}, k\mathcal{L})\} =: \mathcal{F}_\lambda H^0(kL).$$

It is immediate that $\mathcal{F}_\lambda$ is decreasing since $H^0(\mathcal{X}, k\mathcal{L})$ is a $\mathbb{C}[z]$-module. We can extend $\mathcal{F}$ to a filtration by letting

$$\mathcal{F}_\lambda H^0(kL) := \mathcal{F}_{\lceil\lambda\rceil} H^0(kL)$$

for non-integers λ, thus making $\mathcal{F}$ left-continuous. Since

$$z^{-(\lambda+\lambda')}\overline{ss'} = (z^{-\lambda}\bar{s})(z^{-\lambda'}\bar{s'}) \in H^0(\mathcal{X}, k\mathcal{L})H^0(\mathcal{X}, m\mathcal{L}) \subseteq H^0(\mathcal{X}, (k+m)\mathcal{L})$$

whenever $s \in \mathcal{F}_\lambda H^0(kL)$ and $s' \in \mathcal{F}_{\lambda'} H^0(kL)$, we see that

$$(\mathcal{F}_\lambda H^0(kL))(\mathcal{F}_{\lambda'} H^0(mL)) \subseteq \mathcal{F}_{\lambda+\lambda'} H^0((k+m)L),$$

i.e. $\mathcal{F}$ is multiplicative.

Recall that we had the decomposition of $H^0(kL_0)$ into weight spaces V_λ.

LEMMA 40. *For each λ, we have that*

$$\dim F_\lambda H^0(kL) = \sum_{\lambda' \geq \lambda} \dim V_{\lambda'}.$$

Proof. We have the following isomorphism:

$$(\pi_* k\mathcal{L})_{|\{0\}} \cong H^0(\mathcal{X}, k\mathcal{L})/zH^0(\mathcal{X}, k\mathcal{L}),$$

the right-to-left arrow being given by the restriction map, see e.g. [29]. Also, for $k \gg 0$, $(\pi_* k\mathcal{L})_{|\{0\}} = H^0(kL_0)$, therefore we get that for large k

$$H^0(kL_0) \cong H^0(\mathcal{X}, k\mathcal{L})/zH^0(\mathcal{X}, k\mathcal{L}), \tag{3.38}$$

We also had a decomposition of $H^0(\mathcal{X}, k\mathcal{L})$ into the sum of its invariant weight spaces W_λ. By Lemma 39 it is clear that a section $S \in H^0(\mathcal{X}, k\mathcal{L})$ lies in W_λ if and only if it can be written as $z^{-\lambda}\bar{s}$ for some $s \in H^0(kL)$, in fact we have that $s = S_{|X}$. Thus we get that

$$W_\lambda \cong \mathcal{F}_\lambda H^0(kL),$$

and by the isomorphism (3.38) then

$$V_\lambda \cong \mathcal{F}_\lambda H^0(kL)/\mathcal{F}_{\lambda+1} H^0(kL).$$

Thus we get

$$\dim \mathcal{F}_\lambda H^0(kL) = \sum_{\lambda' \geq \lambda} \dim V_{\lambda'}. \tag{3.39}$$

□

Using Lemma 40 together with Lemma 38 shows that the filtration $\mathcal{F}$ is bounded.

3.9 The geodesic rays of Phong and Sturm

In [26] Phong-Sturm show how to construct a weak geodesic ray, starting with a $\phi \in \mathcal{H}(L)$ and an algebraic test configuration $\mathcal{T}$ (see also [33] for how this works in the toric setting). In the previous section we showed how to associate an analytic test configuration $[\phi_\lambda^{\mathcal{F}}]$ to an algebraic test configuration, and thus get a weak geodesic using the Legendre transform of its maximal envelope. Recall by Proposition 30 this geodesic is the same as the Legendre transform of the original test curve $\phi_\lambda^{\mathcal{F}}$. The goal in this section is to prove that for non-trivial analytic test configurations this ray coincides with the one constructed by Phong-Sturm.

To describe what we aim to show, recall that if V is a vector space with a scalar product, and $\mathcal{F}$ is a filtration of V, there is a unique decomposition of V into a direct sum of mutually orthogonal subspaces V_{λ_i} such that

$$\mathcal{F}_\lambda V = \oplus_{\lambda_i \geq \lambda} V_{\lambda_i}.$$

Furthermore we allow for λ_i to be equal to λ_j even when $i \neq j$, so we can assume that all the subspaces V_{λ_i} are one dimensional. This additional decomposition is of course not unique, but it will not matter in what follows.

Let $\phi \in \mathcal{H}(L)$ and $H^0(kL) = \oplus V_{\lambda_i}$ be the decomposition of $H^0(kL)$ with respect to the scalar product $(\cdot,\cdot)_{k\phi}$ coming from the volume form $(dd^c\phi)^n$. Consider next the filtration coming from an algebraic test configuration (note that then the collection of λ_i will depend also on k but we omit that from our notation) and define the normalized weights to be

$$\bar{\lambda}_i := \frac{\lambda_i}{k},$$

which form a bounded family by Lemma 38.

Now if s_i is a vector of unit length in V_{λ_i}, then $\{s_i\}$ will be an orthonormal basis for $H^0(kL)$. Since the filtration $\mathcal{F}$ encodes the $\mathbb{C}^*$-action on $H^0(k\mathcal{L})$ it is easy to see that the basis $\{s_i\}$ is the same one as in [26, Lem 7]. In terms of the notation in the previous sections we have

$$\phi_{k,\lambda} = \sum_{\lambda_i \geq k\lambda} |s_i|^2 \quad \text{and} \quad \phi^{\mathcal{F}}_{\lambda} = (\lim_{k\to\infty} \phi_{k,\lambda})^*.$$

DEFINITION 22. *Let*

$$\Phi_k(t) := \frac{1}{k} \ln(\sum_i e^{t\lambda_i} |s_i|^2)$$

The Phong-Sturm *ray is the limit*

$$\Phi(t) := \lim_{k\to\infty} (\sup_{l\geq k} \Phi_l(t))^*. \tag{3.40}$$

Our goal is the following:

THEOREM 41. *Let $\phi^{\mathcal{F}}$ be the analytic test configuration associated to the filtration $\mathcal{F}$ from a test configuration. Assume that $\phi^{\mathcal{F}}$ is non-trivial. Then*

$$\Phi(t) = \widehat{(\phi^{\mathcal{F}})}_t.$$

LEMMA 42.

$$\Phi(t) = \lim_{k\to\infty} (\sup_{l\geq k} \Phi_l(t))^* = \lim_{k\to\infty} (\sup_{l\geq k} \max_i \{\phi_{l,\bar{\lambda}_i} + t\bar{\lambda}_i\})^*. \tag{3.41}$$

Proof. Our proof will be based on the elementary fact that if $\{a_{l,i} : i \in I_l\}$ is a set of real numbers then

$$\max_{i \in I_l} a_{l,i} \leq \frac{1}{l} \ln \sum_{i \in I_l} e^{l a_{l.i}} \leq \max_{i \in I_l} a_{l,i} + \frac{1}{l} \ln |I_l|. \tag{3.42}$$

Now pick $x \in X$ and $t > 0$. Let

$$a_{l,i} := \frac{1}{l} \ln |s_i(x)|^2 + t\bar{\lambda}_i$$

and I_l be the indexing set for the λ_i. Then $|I_l| = O(l^n)$ and

$$\Phi_l(t) = \frac{1}{l} \ln \left(\sum_i e^{l a_{l,i}} \right).$$

Thus by (3.42)

$$\max_i \{a_{l,i}\} \leq \Phi_l(t) \leq \max_i \{a_{l,i}\} + \frac{|I_l|}{l}. \tag{3.43}$$

Now set

$$b_{l,i} := \phi_{l,\bar{\lambda}_i} + t\bar{\lambda}_i = \frac{1}{l} \ln \sum_{\lambda_j \geq \lambda_i} |s_j(x)|^2 + t\bar{\lambda}_i.$$

For fixed i, pick any j_0 such that

$$\max_{\lambda_j \geq \lambda_i} |s_j(x)|^2 = |s_{j_0}|^2 \quad \text{and} \quad \lambda_{j_0} \geq \lambda_i.$$

Then

$$b_{l,i} \leq \frac{1}{l} \ln(|I_l||s_{j_0}|^2 + t\bar{\lambda}_i \leq \frac{1}{l} \ln |s_{j_0}|^2 + t\bar{\lambda}_{j_0} + \frac{\ln |I_l|}{l} = a_{j_0,l} + \frac{\ln |I_l|}{l}.$$

Clearly $a_{l,i} \leq b_{l,i}$ for all i, so we in fact have

$$\max_i \{a_{l,i}\} \leq \max\{b_{l,i}\} \leq \max_i \{a_{l,i}\} + \frac{\ln |I_l|}{l},$$

which combined with (3.43) yields

$$\max_i \{b_{l,i}\} - \frac{\ln |I_l|}{l} \leq \Phi_l(t) \leq \max_i \{b_{l,i}\} + \frac{\ln |I_l|}{l}.$$

Now taking the supremum over all $l \geq k$ followed by the upper semicontinuous regularization and then the limit as k tends to infinity gives the result since $k^{-1} \ln |I_k|$ tends to zero. □

Proof of Theorem 41. From Lemma 32 we know that there is a constant $\epsilon(l)$ such that

$$\phi_{l,\bar{\lambda}_i} + t\bar{\lambda}_i \le \phi^{\mathcal{F}}_{\bar{\lambda}_i} + t\bar{\lambda}_i + \epsilon(l),$$

where $\epsilon(l)$ is independent of λ_i and tends to zero as l tends to infinity. Thus we certainly have

$$\max_i\{\phi_{l,\bar{\lambda}_i} + t\bar{\lambda}_i\} \le \sup_\lambda\{\phi^{\mathcal{F}}_\lambda + t\lambda\} + \epsilon(l),$$

and so

$$(\sup_{l\ge k}\max_i\{\phi_{l,\bar{\lambda}_i} + t\bar{\lambda}_i\})^* \le (\widehat{\phi^{\mathcal{F}}})_t + \sup_{l\ge k}\epsilon(l).$$

Hence taking the limit as k tends to infinity and using Lemma 42 gives

$$\Phi(t) \le (\widehat{\phi^{\mathcal{F}}})_t.$$

For the opposite inequality, since we assumed that the analytic test configuration was non-trivial, there is a $\lambda' < \lambda_c$ such that $\phi \ne \phi^{\mathcal{F}}_{\lambda'}$. Set

$$\eta = \sup\{\lambda : \phi^{\mathcal{F}}_\lambda = \phi\}$$

so, by assumption, η is strictly less than λ_c.

Consider first λ such that $\phi^{\mathcal{F}}_\lambda \ne \phi$ and $\phi^{\mathcal{F}}_\lambda \not\equiv -\infty$. Then for any $\delta > 0$ there are, for arbitrarily large k, normalized weights $\lambda_{i(k)}$ in $(\lambda - \delta, \lambda)$, since otherwise we would have $\phi^{\mathcal{F}}_{\lambda-\delta} = \phi^{\mathcal{F}}_\lambda$, which is impossible by concavity of $\phi^{\mathcal{F}}$. Thus we have for $k \ge k_0$,

$$\sup_{l\ge k}\max_i\{\phi_{l,\lambda_i} + t\lambda\} \ge \phi_{k,\lambda_{i(k)}}(x) + t\lambda_i(k) \ge \phi_{k,\lambda}(x) + t(\lambda - \delta).$$

Therefore

$$\Phi(t) \ge \phi^{\mathcal{F}}_\lambda + t\lambda \quad \text{if} \quad \phi^{\mathcal{F}}_\lambda \ne \phi, \text{ and } \phi^{\mathcal{F}}_\lambda \not\equiv -\infty, \tag{3.44}$$

and in particular

$$\Phi(t) \ge (\sup_{\lambda>\eta}\{\phi^{\mathcal{F}}_\lambda + t\lambda\})^*.$$

Now if λ is such that $\phi_\lambda^{\mathcal{F}} = \phi$ then using the right continuity part of Lemma 22,

$$\phi_\lambda^{\mathcal{F}} + t\lambda = \phi + t\lambda = (\sup_{\lambda'>\eta}\{\phi_{\lambda'}^{\mathcal{F}}\})^* + t\lambda \leq (\sup_{\lambda'>\eta}\{\phi_\lambda^{\mathcal{F}} + t\lambda'\})^* \leq \Phi(t),$$

which along with (3.44) completes the proof. □

Remark. A natural conjecture would be that the analytic test configuration is trivial iff the algebraic test configuration $\mathcal{T}$ has non-zero norm (see [34]).

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