# Environmental Taxation in Airline Markets

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**Abstract**

Over the last two decades many airline markets have been deregulated, resulting in increased competition and use of different types of networks. At the same time there has been an intense discussion on environmental taxation of airline traffic. It is likely that an optimal environmental tax and the effects of a tax differ between different types of aviation markets. In this paper we derive optimal environmental taxes for different types of airline markets. The first type of market is a multiproduct monopoly airline operating either a point-to-point network or a hub-and-spoke network. The optimal tax is shown to be similar in construction to an optimal tax for a monopolist. We also compare the environmental impact of the two types of networks. Given no differences in marginal damages between airports we find that an airline will always choose the network with the highest environmental damages. The second type of market we investigate is a multiproduct duopoly, where two airlines compete in both passengers and flights. The formulation of the optimal tax is similar to the optimal tax of a single product oligopoly. However, we also show that it is, because of strategic effects, difficult to determine the effects of the tax on the number of flights.

**Key words:** *Environmental taxation, Multiproduct duopoly, Aviation, Networks.*

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## 1. Introduction[1]

In this paper we discuss the issue of optimal environmental taxation for different types of aviation markets. In the standard perfect competition model the optimal prescription is a tax equal to marginal external damages, a so-called Pigouvian tax. However, airline markets has two properties that affect the optimal environmental tax: imperfect competition and network effects. Most airline markets consists of only a few actors and in many cases only one or two airlines operate on a particular route. Network effects occur because aviation markets (connections) in many cases are related on the demand and/or cost side. One interesting development of airline markets has been the increased use of hub-and-spoke operations (see e.g. Borenstein 1992). The main explanation for the formation of hub-and-spoke networks is probably economies of traffic density (e.g. Caves et al. 1984), but there could also be positive effects on the demand side since hubbing can result in more frequent flights to a larger number of cities (market presence).

The environmental impact from the aviation sector depends on the number of flights, types of aircraft engines that are used, and the location of the airports. In this paper we take technology as given, even though this is probably an equally important factor for environmental improvements. Instead we focus on the number of flights as the environmental impact. In the model presented an airline has two choice variables, number of passengers and number of flights. We follow the standard approach (see e.g. DeVany 1975, Schmalensee 1977) and assume that demand is increasing in capacity (number of flights) since delay costs are decreasing in capacity. Consequently an airline has incentives to increase the number of flights in order to increase demand for air travel.

In the first part of the paper we elaborate on the model presented in Nero and Black (1998). They analyse a monopoly airline and the differences in environmental impact between a point-to-point network and hub-and-spoke network. Here we extend this model to cover non-symmetric equilibrium, thereby allowing for different effects on the number of flights at different connections. We derive optimal environmental taxes for the two types of networks, and compare the environmental impacts of the two networks,

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by comparing the number of flights. In the second part of the paper we extend the discussion to a market with two airlines. The airlines make decisions both about the number of passengers and the number flights, which means that the airlines are multiproduct duopolists. The two "products", passengers and flights, are related through the demand function; where market demand depends on the aggregate number of passengers and flights. Given this model we derive the optimal environmental tax and the comparative statics of the tax.

## 2. General outline of the model

In order to derive useful result we will work with a rather simple model. For any city pair *ij* the cost function for an airline is given by:

$$C_{ij} = cQ_{ij} + (b + t_{ij})F_{ij}, \tag{1}$$

where $Q_{ij}$ is the number of passengers, $F_{ij}$ the number of flights, c is the marginal passenger cost, b is the marginal flight cost, net of any environmental tax, and $t_{ij}$ is the environmental tax. The cost function is the same as in Nero and Black (1998),[2] and admittedly the assumption of separability and the absence of economies of scope are restrictive.[3] The demand on city pair *ij* is a function of price, number of flights and a market-specific demand shift parameter, $\Omega_{ij}$. We assume the following inverse demand:

$$P_{ij} = \Omega_{ij}^{1/\varepsilon} F_{ij}^{\alpha/\varepsilon} Q_{ij}^{-1/\varepsilon} = \Omega_{ij}^{\sigma} F_{ij}^{\beta} Q_{ij}^{-\sigma}. \tag{2}$$

Throughout the paper we restrict the absolute value of the price-elasticity, $\varepsilon$, to be larger than unity, and the flight elasticity, $\alpha$, to be lower than unity.[4] For the inverse

[2] Their model in turn builds on the models by DeVany (1975) and Schmalensee (1977).

[3] The assumption is made for analytical convenience, but it should be noted that the comparison between the two types of network could be affected by changing the functional form of the cost function.

[4] The restrictions of the elasticities have some support by empirical findings. Summarising major survey results on price elasticities Oum et al. (1992) finds that most studies show values of the price elasticity between 0.8 and 2.0. However, it should be noted that some studies indicate that business passengers are less price elastic, with values around 1.0 (see for example Oum et al. 1986 and Oum et al. 1993). DeVany (1972) estimate the flight elasticity for domestic US flights to around 1.2, while Morrison and Winston (1986) find flight elasticities for business passengers to be around 0.2 and for leisure travel roughly 0.05. Berechman and de Wit (1996), using European data, estimate the flight elasticity to 0.7 for business passengers and 0.3 for leisure passengers.

demand function the absolute value of the inverse price-elasticity is given by $\sigma$, where $s < 1$, and the flight-elasticity is given by $\beta$ times $\varepsilon$.

For the monopoly model we assume a network with three cities: *h, 1, 2,* where city *h* is a potential hub. In the point-to-point network the airline operates the network with traffic on all city-pairs, i.e. there are three direct connections *h1, h2*, and *12*. In the hub-and-spoke network, where city *h* is the hub airport, the airline operates the network with traffic on connection *h1* and *h2*, and passengers travelling between city *1* and *2* are funnelled through the hub *h*. For the oligopoly model we only consider competition on one connection.

The regulator has only one instrument available: the environmental tax. The goal for the regulator is to set an optimal tax in the sense that the tax maximises social welfare, and the regulator determines the taxes before the airlines make their decisions. It should be noted that we do not model the possible effect of the tax on investments in cleaner technology; an effect which could affect the environmental impact and the optimal tax.

## 3. Monopoly network market and environmental taxation

In this section we derive optimal taxes for the point-to-point network and the hub-and-spoke network. Let $f = \{h1, h2, 12\}$ denote the set of all city-pairs and let $k = \{h1, h2\}$ denote the set of city-pairs with a direct connection in a hub-and-spoke network.

*Point-to-point network*

We assume that an airline maximises its profits with respect to passengers and flights, and that the profit for an airline operating a point-to-point network is:

$$\pi^P = \sum_{ij\in f} (P_{ij} - c)Q_{ij} - (b + t_{ij})F_{ij} \; . \qquad (3)$$

First order conditions and the equilibrium levels of passengers and flights are therefore:

$$\frac{\partial \pi^P}{\partial Q_{ij}} = (P_{ij} - c) + \frac{\partial P_{ij}}{\partial Q_{ij}} Q_{ij} = 0 \;\Rightarrow\; Q_{ij} = \Big(\frac{c}{(1-\sigma)\Omega_{ij}^{\sigma} F_{ij}^{\beta}}\Big)^{-1/\sigma}, \qquad (4)$$

and, substituting in for $Q_{ij}$ and using the fact that $\alpha = \beta / \sigma$:

$$\frac{\partial \pi^P}{\partial F_{ij}} = Q_{ij}\frac{\partial P_{ij}}{\partial F_{ij}} - b - t_{ij} = 0 \Rightarrow F_{ij} = \left(\frac{\beta c \Omega_{ij}\left(\frac{c}{1-\sigma}\right)^{-1/\sigma}}{(b+t_{ij})(1-\sigma)}\right)^{\frac{1}{1-\alpha}}. \tag{5}$$

We assume that the regulator maximises the unweighted sum of consumer- and producer surplus net of damage costs. The objective function for the regulator is therefore:

$$W = \sum_{ij}\int_0^{Q_{ij}}\int_0^{F_{ij}} P_{ij}(Q,F)dFdQ - C_{ij} - D_{ij}(F_{ij}) + t_{ij}F_{ij} \ , \tag{6}$$

where $D_{ij}$ is the external damage cost, which is a function only of flights; we assume that external damages are strictly increasing in $F_{ij}$. The regulator sets three environmental taxes, one for each connection. Differentiating the regulators objective function with respect to the environmental tax $t_{ij}$, substituting in for $b$ from the first order condition in (5), and solving for $t_{ij}$ we have:

$$t_{ij} = \frac{\partial D_{ij}}{\partial F_{ij}} - \frac{\left(\frac{F_{ij}}{\beta+1}P_{ij} - c\right)\frac{dQ_{ij}}{dt} + Q_{ij}P_{ij}\left(\frac{1}{1-\sigma} - \frac{\beta}{F_{ij}}\right)\frac{dF_{ij}}{dt}}{\frac{dF_{ij}}{dt}}. \tag{7}$$

The optimal environmental tax thus consists of two parts, the marginal damage cost of flights (which would correspond to the standard Pigouvian tax), and a correction part due to the monopoly situation at the connection. This formulation of the optimal environmental tax problem resembles the optimal environmental tax of a single product monopolist (Barnett 1980). The difference is that the second part of the tax expression consists of two welfare effects: the effect on output and the effect on flights. Substituting in the price- and flight elasticity and using the fact that $P_{ij} = \frac{c\varepsilon}{\varepsilon - 1}$, the optimal tax can be written:

$$t_{ij} = \frac{\partial D_{ij}}{\partial F_{ij}} - \frac{\left(\frac{F_{ij}\varepsilon}{\alpha+\varepsilon}\frac{\varepsilon}{\varepsilon-1}c - c\right)\frac{dQ_{ij}}{dt} + Q_{ij}\frac{\varepsilon}{\varepsilon-1}c\left(\frac{\varepsilon}{\varepsilon-1} - \frac{\alpha}{\varepsilon}\frac{1}{F_{ij}}\right)\frac{dF_{ij}}{dt}}{\frac{dF_{ij}}{dt}}. \tag{8}$$

Differentiating the second part of the tax expression it can be shown that, for a given change in output and flights, the second part is decreasing in both the price- and flight elasticity. The intuition behind this is the same as in the monopoly case; the more elastic demand is the smaller is the welfare loss associated with the reduction in output and flights. In order to determine the sign of the second part of the tax expression we need to know the sign of the tax effects on passengers and flights. Since:

$$\frac{dF_{ij}}{dt_{ij}} = -\frac{1}{(1-\alpha)(b+t_{ij})}F_{ij} < 0 \text{ and } \frac{dQ_{ij}}{dt_{ij}} = \frac{\alpha Q_{ij}}{F_{ij}}\frac{dF_{ij}}{dt_{ij}} < 0 \tag{9}$$

the second part of the optimal tax is non-negative.[5] Consequently the optimal tax is lower than the corresponding Pigouvian tax. The reason for this is that the tax has a negative effect on output and flights, and since the monopolist is already supplying a less than optimal level there is a negative welfare effect of this reduction. The magnitude of the second part depends on the elasticities and the magnitude of the effects on output and flights.

*Hub-and-spoke network*

Now the airline operates a hub-and-spoke network, where city $h$ is the hub. The airline's cost function for city pair $hj$ is:

$$C_{hj} = cQ_{hj} + cQ_{12} + (b+t_{hj})F_{hj}, \; hj \in k. \tag{10}$$

The inverse demand for connecting passengers, i.e. passengers travelling between city 1 and city 2 through the hub, is assumed to have the following form:

[5] The expressions in the parentheses in equation (8) are non-negative since these are the marginal willingness to pay minus the marginal cost for passengers and flights.

$$P_{12} = \Omega_{12}^{\sigma} F_{12}^{\beta} Q_{12}^{-\sigma}; F_{12} = \min\{F_{h1}, F_{h2}\}. \tag{11}$$

The reason for this assumption is that we want to catch the effect of "the weakest link". Even if the number of flights are high at one connection, connecting passengers also have to rely on the number of flights at the other connection. In order to make the model simple we therefore assume that only flights on the connection with the lowest number of flights affect demand.[6] For the other markets we assume the same inverse demand function as before. The profit function for the airline can now be written:

$$\pi^{H} = \sum_{ij \in f} P_{ij}(Q_{ij})Q_{ij} - \sum_{hj \in k} C_{hj} . \tag{12}$$

Again the airline maximises its profit with respect to passengers and flights. First order conditions and the equilibrium number of passengers are:

$$\begin{aligned} (P_{hj} - c) + \frac{\partial P_{hj}}{\partial Q_{hj}} Q_{hj} = 0 \Rightarrow Q_{hj} = \left(\frac{c}{(1-\sigma)\Omega_{hj}^{\sigma} F_{hj}^{\beta}}\right)^{-1/\sigma}, \\ (P_{12} - 2c) + \frac{\partial P_{12}}{\partial Q_{12}} Q_{12} = 0 \Rightarrow Q_{12} = \left(\frac{2c}{(1-\sigma)\Omega_{12}^{\sigma} F_{12}^{\beta}}\right)^{-1/\sigma}. \end{aligned} \tag{13}$$

Note that $Q_{12} = 2^{-1/\sigma} \frac{\Omega_{12}}{\Omega_{hj}} Q_{hj}$ if $F_{hj} = F_{12}$, and in a symmetric equilibrium, where $\Omega_{hj} = \Omega_{12}$, we have that $Q_{12} = 2^{-1/\sigma} Q_{hj}$. Thus, in a symmetric equilibrium the share of connecting passengers depends only on the price elasticity, while in the nonsymmetric equilibrium the share also depends on the demand shift parameters. The first order conditions for the number of flights is:

$$\begin{aligned} Q_{hj} \frac{\partial P_{hj}}{\partial F_{hj}} + Q_{12} \frac{\partial P_{12}}{\partial F_{hj}} - b - t_{hj} = 0 \text{ if } F_{hj} \le F_{hi}, \text{ and} \\ Q_{hj} \frac{\partial P_{hj}}{\partial F_{hj}} - b - t_{hj} = 0 \text{ if } F_{hj} > F_{hi}; hj \ne hi; hj, hi \in k . \end{aligned} \tag{14}$$

Assuming that $F_{hj} \le F_{hi}$, and using the equilibrium solution for the passengers, the first order condition in (14) yields the following equilibrium level of flights:

$$F_{hj}^{H} = \left(\frac{\beta c(\frac{c}{1-\sigma})^{-1/\sigma}(\Omega_{hj} + 2^{-1/\sigma}\Omega_{12}2)}{(b+t_{hj})(1-\sigma)}\right)^{\frac{1}{1-\alpha}} = \left(1+2^{-\varepsilon}2\frac{\Omega_{12}}{\Omega_{hj}}\right)^{\frac{1}{1-\alpha}} F_{hj}^{P}. \tag{15}$$

Consequently, the difference in the number of flights between the point-to-point network and the hub-and-spoke network depends on the price- and flight elasticities and on the demand shift parameters. If $F_{hj}^{H} < F_{hi}^{H}$ then, for a given tax, the number of flights on connection *hi* is the same as in the point-to-point network:

$$F_{hi}^{H} = \left(\frac{\beta c\Omega_{hi}(\frac{c}{1-\sigma})^{-1/\sigma}}{(b+t_{hi})(1-\sigma)}\right)^{\frac{1}{1-\alpha}} = F_{hi}^{P}. \tag{16}$$

Now the regulator maximises the following objective function:

$$W = \sum_{ij\in f}\int_0^{Q_{ij}}\int_0^{F_{ij}} P_{ij}(Q,F)dFdQ - \sum_{hj\in k} C_{hj} - D_{hj}(F_{hj}) + t_{hj}F_{hj}, \tag{17}$$

by setting two different environmental taxes $t_{h1}$ and $t_{h2}$. The difference between the hub-and-spoke network and the point-to-point network is the demand from the connecting passengers. Whether this affects the tax expression or not depends on whether the number of flights at the connection is higher or lower than the number of flights at the other connection. The model is such that for the connection with the highest number of flights, the airline has no incentive to increase the flights in order to increase the demand for the connecting passengers (there is still an effect for passengers travelling directly on that connection). Consequently, for that connection the optimal tax will not be a function of the connecting passengers. We proceed assuming that $F_{hj} < F_{hi}$, which means that the optimal tax for connection $hi$ is the same as in the

[6] This is also a difference from the model in Nero and Black (1998) where the share of connecting passengers is exogenous.

point-to-point case. For connection $hj$, differentiating the regulators objective function with respect to the tax $t_{hj}$, using the fist order conditions in (14) and solving for the tax:

$$t_{hj} = \frac{\partial D_{hj}}{\partial F_{hj}} - \frac{(\frac{F_{hj}}{\beta+1}P_{hj} - c)\frac{dQ_{hj}}{dt_{hj}} + (\frac{F_{hj}}{\beta+1}P_{12} - 2c)\frac{dQ_{12}}{dt_{hj}} + (\frac{1}{1-\sigma} - \frac{\beta}{F_{hj}})\sum_{gj\in hj,12} Q_{qj}P_{gj}\frac{dF_{hj}}{dt_{hj}}}{\frac{dF_{hj}}{dt_{hj}}} \tag{18}$$

Again, the optimal tax consists of two parts, but the second part now involves the effects on demand in two markets; passengers travelling directly between airport *h* and *j*, and passengers travelling between airport *i* and *j* through the hub airport. As in the point-to-point case we can substitute in the price- and flight elasticities, and show that the tax expression is decreasing in both the price- and flight elasticity, for a given change in output and flights. The comparative statics are also easy to establish; both the number of passengers and flights are decreasing in the tax. Consequently the second part of the tax expression is positive. Compared to the point-to-point network, the second part of the tax expression now consists of two additional positive expressions stemming from the effect on connecting passengers. The tax is therefore reduced even more compared to the corresponding Pigouvian tax at the connection with the least number of flights. For the other connection, the tax is the same as in the point-to-point network.

## 4. Environmental effects in the two networks

We can make a direct comparison of the environmental effects of the two types of networks, at least in a simplified fashion, by comparing the number of flights in the two networks. Thus, in the comparison we rule out any differences in marginal damages between airports and any differences in distance between the cities. This is probably not the case in reality, where the marginal damage could be higher at the hub airport. This mainly due to congestion which is a negative externality, but congestion can also increase other external effects such as noise and local emissions (see Carlsson 1999). It is also likely that there are differences in distance between the cities. We distinguish between two cases: (i) non-symmetric equilibrium with different demand-shift parameters and different capacity and (ii) symmetric equilibrium. The comparison between the networks is made on the premise that the tax is the same for both networks (alternatively that there are no environmental taxes). Note that the difference in the number of flights between the two networks depends on output- and flight elasticities and the demand shift parameters. From (5) we also have that:

$$F_{12}^{P} = \left(\Omega_{12}/\Omega_{hj}\right)^{\frac{1}{1-\alpha}} F_{hj}^{P} . \tag{19}$$

*Symmetric equilibrium*

We begin with the symmetric equilibrium where the demand shift parameters are identical; consequently the difference in flights between the networks depends only on the price- and flight elasticities. The following propositions are modified versions of the propositions given in Nero and Black (1998). The difference is that in our model the share of connecting passengers in the hub-and-spoke networks depends on the price elasticity, while in Nero and Black (1998) this share is exogenous.

*Proposition 1a*: *The number of flights at the leg airport is higher under the point-to-point network than under the hub-and-spoke network if* $2^{-\alpha} - 2^{-\varepsilon} > 0.5$.

*Proof*: From the equilibrium levels of flights in (5) and (15), and using (19) and the fact that $\Omega_{12} = \Omega_{ij}$, we have that proposition 1a is true if $2F^{P} > (1 + 2^{-\varepsilon}2)^{\frac{1}{1-\alpha}} F^{P}$.

This means that when the flight elasticity is low and the price elasticity is high the number of flights is higher at the leg-airports under the point-to-point network compared to the hub-and-spoke network. When $\alpha \to 0$ this is always true, and when $\varepsilon \to 1$ this is never true. From the proposition we can calculate critical levels of the elasticities, i.e. values where the number of flights are the same in the two networks. For example, when the flight elasticity is 0.5 then the price elasticity would have to be higher than 2.27 in order for this to be the case.

*Proposition 1b*: *The total number of flights at the hub airport is always higher under the hub-and-spoke network.*

*Proof*: From the equilibrium levels of flights in (5) and (15), and using the fact that $\Omega_{12} = \Omega_{ij}$, we have that proposition 1b is true if $2F^P < (1+2^{-\varepsilon}2)^{\frac{1}{1-\alpha}} 2F^P$. This implies that the proposition is true if $(1+2^{-\varepsilon}2)^{\frac{1}{1-\alpha}} > 1$, and since $\alpha > 1$ this is always true.

Proposition 1b is not surprising, but what is interesting is under what conditions the difference in number of flights is small or large. This follows directly from the proposition; there is a lower difference in number of flights when the flight elasticity is low and the price elasticity is high. In that case demand does not increase that much due to the increase in number of flights, and demand decreases much due to the increase in prices.

*Proposition 1c*: *The total number of flights is higher under the point-to-point network if* $1.5^{1-\alpha} - 2^{1-\varepsilon} > 1$.

*Proof*: From the equilibrium levels of flights in (5) and (15), and using (19) and the fact that $\Omega_{12} = \Omega_{ij}$, we have that proposition 1c is true if $6F^P > (1+2^{-\varepsilon}2)^{\frac{1}{1-\alpha}} 4F^P$.

Consequently, the total number of flights is higher in the point-to-point network when the flight elasticity is low and the price elasticity is high. If we assume no differences in marginal damages between airports, we could also conclude that total external damages are higher in a point-to-point network when the flight elasticity is low and the price elasticity is high.[7] However, the elasticities will also determine the airline's choice of network. For the airline's choice of network we have the following proposition.

[7] The flight elasticity would have to be rather low in order for this to be the case. For example if the flight elasticity is 0.3, then the price elasticity would have to be at least 5.3 in order for this to be the case, while if flight elasticity is 0.1, then the price elasticity would have to be at least 2.51.

*Proposition 1d: It is optimal for the airline to operate a hub-and-spoke network when* $1 > 1.5^{1-\alpha} - 2^{1-\varepsilon}$.

*Proof: See Proof 1 in the Appendix.*

The interesting aspect is that this condition is the opposite of the condition in Proposition 1c. This means that the airline will choose the network with the largest number of flights. Given no differences in marginal damage between flights, this also means that the airline will choose the network with the highest environmental damages.

*Nonsymmetric equilibrium*

It is easy to extend the propositions to the nonsymmetric case, although the exact interpretation of them is more complicated. Suppose that $F_{hj} < F_{hi}; hj \neq hi$ in the hub-and-spoke network. A crucial difference between this case and the symmetric case is then that the number of flights is the same at the other leg airport $hi$. In this extreme case, the only effect on this connection is an increased load factor. We can now establish the following propositions.

*Proposition 2a*: *The total number of flights and the number of flights at leg airport j is higher under the point-to-point network if* $\Psi = 1 + (\frac{\Omega_{12}}{\Omega_{hj}})^{\frac{1}{1-\alpha}} - \left(1 + 2^{1-\varepsilon}\frac{\Omega_{12}}{\Omega_{hj}}\right)^{\frac{1}{1-\alpha}} > 0$.

Proof: From the equilibrium levels of flights in (5) and (15), and using (19) we have that Proposition 2a is true if: $(1 + (\frac{\Omega_{12}}{\Omega_{hj}})^{\frac{1}{1-\alpha}})F_{hj}^{P} - \left(1 + 2^{-\varepsilon}2\frac{\Omega_{12}}{\Omega_{hj}}\right)^{\frac{1}{1-\alpha}} F_{hj}^{P} > 0$, since $F_{hi}^{P} = F_{hi}^{H}$.

The difference between the symmetric and non-symmetric equilibrium is the demand-shift parameters. However, as in the symmetric case, the elasticities and the demand shift parameters will affect the airline's choice of network. We therefore have the following proposition.

*Proposition 2b: It is optimal for the airline to operate a hub-and-spoke network when* $\Psi = 1 + (\frac{\Omega_{12}}{\Omega_{hj}})^{\frac{1}{1-\alpha}} - (1 + 2^{1-\varepsilon}\frac{\Omega_{12}}{\Omega_{hj}})^{\frac{1}{1-\alpha}} < 0$.

*Proof:* The proof is similar to the proof of proposition 1d (using the result of proposition 2a).

Again, an airline will choose the network with the largest number of flights and, given the assumptions about the damage function, the highest environmental damages. In Table 1 we present the level of the price elasticity where $\Psi$ is approximately equal to zero (i.e. where total traffic and profits are equal between the networks) for different levels of the demand shift parameter and the flight elasticity.[8] We then see that when $\Omega_{12}$ is larger than $\Omega_{hj}$, total traffic is higher under a point-to-point network for most cases.[9] Furthermore, by looking at the case when the demand shift parameters are equal it is also easy to make a comparison with the symmetric equilibrium. Not surprisingly the critical level of the price elasticity is lower in this case, and the reason for this is of course that in the nonsymmetric case the flights from airport *hi* are the same in both networks.

**Table 1.** *Critical levels of the price elasticity.*

| Demand shift parameters, $\frac{\Omega_{12}}{\Omega_{hj}}$ | Flight elasticity, $\alpha$ | Value of price elasticity, $\varepsilon$, where $\Psi \approx 0$ |
|---|---|---|
| 0.5 | 0.1 / 0.25 / 0.5 / 0.75 | 1.29 / 1.81 / 3.08 / 6.01 |
| 0.75 | 0.1 / 0.25 / 0.5 / 0.75 | 1.24 / 1.65 / 2.59 / 4.40 |
| 1 | 0.1 / 0.25 / 0.5 / 0.75 | 1.21 / 1.55 / 2.27 / 3.40 |
| 1.5 | 0.1 / 0.25 / 0.5 / 0.75 | 1.17 / 1.43 / 1.90 / 2.40 |
| 2 | 0.1 / 0.25 / 0.5 / 0.75 | 1.14 / 1.35 / 1.69 / 1.96 |

Finally, for the hub airport, it is easy to establish a similar condition as for the symmetric case, but where the difference in flights between the networks is low when $\Omega_{12}$ is large compared to $\Omega_{hj}$.

[8] Note that $\Psi$ is always increasing in the price elasticity, $\varepsilon$.

[9] Given that city h is the hub airport, it is more likely that the demand shift parameter $\Omega_{hj}$ is larger than $\Omega_{12}$, since these parameters in a sense measure the economic activity (income) in the different cities.

## 5. Duopoly market and environmental taxation

After deregulation, airlines now face competition on many routes. One interesting problem is then how an optimal environmental tax should be designed under competition. We apply a Cournot duopoly model where each airline simultaneously maximises its profit with respect to passengers and flights. Previous research on environmental taxation in duopoly models has focused on single product oligopolies (see e.g. Carlsson 2000, Simpson 1995). The problem with an analysis of multiproduct oligopolies is especially to determine conditions for stability of the equilibrium, and to derive the comparative statics. Here we use the conditions for stability derived by Zhang and Zhang (1996). There have been some papers on multiproduct oligopolies in the case of aviation markets (Brueckner and Spiller 1991 and Oum et al. 1995). However, in those papers the multiproduct nature of the model is that an airline operates a network where travel on each city-pair market is seen as a single product, and where each product (market) is possibly related through the cost function. In our model, the two products, passengers and flights, are related through the demand function. The inverse market demand is a function of total number of passengers, $Q = q_1 + q_2$, where $q_i$ is airline $i$'s number of passengers, and total number of flights, $F = f_1 + f_2$, where $f_i$ is airline $i$'s number of flights. Consequently, passengers do not differentiate between the two airlines' flights; they only care about total number of flights. The inverse market demand function is therefore $P = \Omega^{\sigma} F^{\beta} Q^{-\sigma}$, and the profit for airline $i$ is:

$$\pi^i = [P(Q,F) - c]q_i - (b + t\theta_i) f_i ; \; i = 1,2 . \tag{20}$$

In order to allow for differences between the airlines we impose an exogenous aircraft engine technology, which in turn affects the emissions from a particular flight. Emissions from a flight are equal to $\theta_i f_i$, and if $\theta_1 = \theta_2$ then the airlines use the same technology. Each airline maximises its profit with respect to $q_i$ and $f_i$, given its rivals' choice of these variables, and the choice of $q_i$ and $f_i$ is made simultaneously. First order conditions are therefore:

$$(P-c)+\frac{\partial P}{\partial q_i}q_i=0 \text{ and } q_i\frac{\partial P}{\partial f_i}-b-t\theta_i=0\,. \tag{21}$$

We assume that the regulator cannot differentiate the tax between the airlines, and that he maximises the following objective function:

$$W=\int_0^Q\int_0^F P(z,x)dxdz-\sum_i C_i+t\theta_i f_i-D(E)\,, \tag{22}$$

by setting the environmental tax, t, where $E=\sum_i \theta_i f_i$. Differentiating the objective function with respect to the tax, substituting in for $b$ from the first order conditions in (21) and solving for t we have:

$$t=\frac{\partial D}{\partial E}-\frac{\sum_{i=1}^{2}\left(\frac{F}{\beta+1}P-c\right)\frac{dq_i}{dt}+P\left(\frac{Q}{1-\sigma}-q_i\frac{\beta}{F}\right)\frac{df_i}{dt}}{\sum_{i=1}^{2}\theta_i\frac{df_i}{dt}}=0\,. \tag{23}$$

The resulting optimal tax is similar to an optimal tax for a single product duopoly (Carlsson 2000, Simpson 1995). The difference from a single product duopoly is that there are welfare effects from effects on both passengers and flights. However, the main problem is to determine the sign of the tax effect on the number of passengers and flights. In the single product oligopoly case, output of both firms must not necessarily be decreasing in the tax since the tax can shift production between firms. However, under the condition that marginal costs are increasing in the tax industry output will be decreasing in the tax (see Carlsson 2000). As we will see it is difficult to determine how the tax will affect the number of flights in this model. This implies that we do not know if the optimal tax is lower or higher than the marginal damage of flights.

When determining the sign of the comparative statics we will use the stability condition for multiproduct oligopolies derived by Zhang and Zhang (1996). Let vector $X^i$ denote firm *i*'s passengers and flights, and let $X^i=R^i(X^j)$ denote firm *i*'s reaction function. A *sufficient* condition for stability of the equilibrium point is then that for some matrix norm $\|\cdot\|$,

$$\left\| \frac{\partial R^1}{\partial X^2} \frac{\partial R^2}{\partial X^1} \right\|_p^2 < 1 \text{ and } \left\| \frac{\partial R^2}{\partial X^1} \frac{\partial R^1}{\partial X^2} \right\|_p^2 < 1; \; p = 1,2,\infty . \tag{24}$$

This condition implies that the magnitude of the eigenvalues of the matrices $(\frac{\partial R^1}{\partial X^2} \frac{\partial R^2}{\partial X^1})$ and $(\frac{\partial R^2}{\partial X^1} \frac{\partial R^1}{\partial X^2})$ are less than unity (for a proof of the condition see Zhang and Zhang 1996).[10] We will assume that the equilibrium is stable, and hence impose this condition on the reaction functions. Furthermore, note that the profit function has the following properties:

$$\pi^i_{q_i q_i} < 0 , \; \pi^i_{f_i f_i} = \pi^i_{f_i f_j} < 0 , \; \pi^i_{f_i q_i} = \pi^i_{q_i f_i} = \pi^i_{q_i f_j} > 0 , \; \pi^i_{f_i q_j} < 0 , \; \pi^i_{q_i t} = 0 , \; \pi^i_{f_i t} = -\theta_i , \tag{25}$$

where subscripts denote partial derivatives, $i, j = 1,2$, $i \neq j$. We assume that both passengers and flights are strategic substitutes, i.e. $\pi^i_{f_i f_j} < 0$ and $\pi^i_{q_i q_j} < 0$.[11] Differentiating the first order conditions with respect to the tax we have by matrix notation:

$$\frac{dX^i}{dt} = -(I - R^i_j R^j_i)^{-1} \left( R^i_j (\Pi^j_{jj})^{-1} \Pi^j_{jt} + (\Pi^i_{ii})^{-1} \Pi^i_{it} \right); i, j = 1,2; i \neq j , \tag{26}$$

where $R^i_j = \frac{\partial R^i}{\partial X^j} = -(\Pi^i_{ii})^{-1} \Pi^i_{ij}$, $\Pi^i_{ii} = \begin{bmatrix} \pi^i_{q_i q_i} & \pi^i_{q_i f_i} \\ \pi^i_{f_i q_i} & \pi^i_{f_i f_i} \end{bmatrix}$ and $\Pi^i_{ij} = \begin{bmatrix} \pi^i_{q_i q_j} & \pi^i_{q_i f_j} \\ \pi^i_{f_i q_j} & \pi^i_{f_i f_j} \end{bmatrix}$. Let us write the derivative matrix of firm i's reaction function as:

$$R^i_j = -\frac{1}{\Delta^i} adj\left(\Pi^i_{ii}\right) \Pi^i_{ij} = \begin{bmatrix} r^i_{qq} & r^i_{qf} \\ r^i_{fq} & r^i_{ff} \end{bmatrix}, \tag{27}$$

[10] See Szidarovszky and Li (2000) for an extended discussion of stability of multiproduct oligopolies. They derive stability conditions for a more general case including adaptive expectations. Further, they show that the necessary conditions for stability given in Zhang and Zhang (1996) must not necessarily hold. However, this does not affect the discussion in this paper.

[11] For a discussion of strategic substitutes and complements see Bulow et al. (1985).

where $\Delta^i$ is the determinant of $\Pi^i_{ii}$, which is greater than zero by the second order condition. It can then be shown that $r^i_{qq}, r^i_{ff} < 0$, and $r^i_{qf} = 0$. The reason for $r^i_{qf}$ being zero is the assumption about the demand function; the passengers do not differentiate between the two airlines' flights. This means that the direct and cross effect of flights on marginal profits will be equal. Furthermore, a sufficient condition for $r^i_{fq} < 0$ is that (see Proof 2 in the Appendix):

$$\beta - \sigma\beta(q_i / Q)^2 + \sigma(\sigma+1)(q_i / Q)^2 - 2\sigma(q_i / Q) < 0. \tag{28}$$

The expression in (28) is negative if either the price- or the flight eleasticity is low; when proceeding we will assume that $r^i_{fq} < 0$. The elements of the matrix:

$$R^i_j R^j_i = \begin{bmatrix} r^i_{qq} r^j_{qq} & 0 \\ r^i_{fq} r^j_{qq} + r^i_{ff} r^j_{fq} & r^i_{ff} r^j_{ff} \end{bmatrix} = \begin{bmatrix} \alpha_i & 0 \\ \gamma_i & \delta_i \end{bmatrix}, \tag{29}$$

can then be shown to be $\alpha_i, \gamma_i, \delta_i > 0$. Note that the matrix $R^i_j R^j_i$ is a triangular matrix, which means that the eigenvalues of $R^i_j R^j_i$ are the entries on the main diagonal. The stability condition then implies that the diagonal elements all are less than unity, i.e. that $r^1_{qq} r^2_{qq} < 1$ and $r^1_{ff} r^2_{ff} < 1$. A sufficient condition for the diagonal elements to be less than unity is that the absolute values of the diagonal elements of the matrices $R^i_j$ and $R^j_i$ are less than unity, i.e. that $\left| r^i_{qq} \right| < 1$ and $\left| r^i_{ff} \right| < 1$. These conditions are similar to stability conditions for the two markets in isolation, i.e. when the two "markets" are not related (Zhang and Zhang 1996). Consequently, we have similar conditions as in the single product case, with the difference that we do not have any restrictions on $r^i_{fq}$, apart from that it should be negative. Given that the equilibrium is stable, which implies that the eigenvalues of the matrices $R^i_j R^j_i$ are all less than unity, we have by the Neumann lemma (Ortega and Rheinboldt 1970, p. 45) that $(I - R^i_j R^j_i)^{-1}$ exists and that:

$$(I - R^i_j R^j_i)^{-1} = \lim_{k\to\infty} \sum_{i=0}^{k} \left(R^i_j R^j_i\right)^i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}. \tag{30}$$

This series must converge to a matrix of the same sign as the matrix $R^i_j R^j_i$, consequently we have that $a_i, c_i, d_i > 0$ and $b_i = 0$. Furthermore, since $R^i_j R^j_i$ is a triangular matrix, we have that $a_1 = a_2 = a$ and $d_1 = d_2 = d$.

Using (27) and (30) we can now calculate the effect of the tax on the number of passengers (see Proof 3 in the Appendix):

$$\frac{dq_i}{dt} = \underbrace{-\frac{a_i \theta_i \pi^i_{f_i q_i}}{\Delta^i}}_{negative} \underbrace{-\frac{a_i r^i_{qq} \theta_j \pi^j_{f_j q_j}}{\Delta^j}}_{positive}. \tag{31}$$

The first part on the right-hand side in (31), representing the own effect, is negative, while the second part, the strategic effect, is positive. Consequently, the effect of the tax on the individual airline's number of passengers is not determined. However, the effect on the total number of passengers is determined:

$$\frac{dQ}{dt} = -\frac{(1 + r^2_{qq}) a \theta_1 \pi^1_{f_1 q_1}}{\Delta^1} - \frac{(1 + r^1_{qq}) a \theta_2 \pi^2_{f_2 q_2}}{\Delta^2} < 0 \text{ if} \left| r^i_{qq} \right| < 1. \tag{32}$$

The total number of passengers is always decreasing in the tax, which means that at least one airline's number of passengers must be decreasing in the tax. Consequently, for a symmetric equilibrium both airlines' number of passengers is decreasing in the tax.[12] Using (27) and (30) we can also calculate the effect of the tax on the number of flights (see Proof 3 in the Appendix):

$$\frac{df_i}{dt} = \underbrace{\frac{\theta_i \left(d_i \pi^i_{q_i q_i} - c_i \pi^i_{f_i q_i}\right)}{\Delta^i}}_{negative} \underbrace{-\frac{\theta_j \left(c_i r^i_{qq} \pi^j_{f_j q_j} + d_i r^i_{fq} \pi^j_{f_j q_j}\right)}{\Delta^j}}_{positive} + \underbrace{\frac{d_i r^i_{ff} \theta_j \pi^j_{q_j q_j}}{\Delta^j}}_{negative}. \tag{33}$$

[12] Note that the equilibrium is symmetric if the airlines use the same technology, i.e. if $\theta_1 = \theta_2$.

The first part on the right-hand side in (33), representing the own effect, is negative, while the sum of the second and third part, the strategic effects, is not determined in sign. Consequently, the effect of the tax on the number of flights for an airline is not determined.[13] The effect on the total number of flights is:

$$\frac{dF}{dt} = \frac{d\theta_1(\pi^1_{q_1q_1} - r^2_{fq}\pi^1_{f_1q_1} + r^2_{ff}\pi^1_{q_1q_1})}{\Delta^1} - \frac{\pi^1_{f_1q_1}\theta_1(c_2r^2_{qq} + c_1)}{\Delta^1} + \frac{d(\pi^2_{q_2q_2} - r^1_{fq}\pi^2_{f_2q_2} + r^1_{ff}\pi^2_{q_2q_2})}{\Delta^2} - \frac{\pi^2_{f_2q_2}(c_1r^1_{qq} + c_2)}{\Delta^2} \tag{34}$$

If we impose the restriction that $\left|r^i_{fq}\right| < 1$, then total number of flights is decreasing in the tax, if $c_2$ is not sufficiently larger than $c_1$ (which is true for a symmetric equilibrium). Consequently, under these conditions both airlines' number of flights is decreasing in the tax under a symmetric equilibrium. This result is in a sense dissatisfying, since it implies that we do not know with certainty how the number of flights will be affected by the environmental tax. Therefore, the environmental effect of the tax is unclear, and we might even have the perverse result of increased flights (emissions) from the environmental tax. It should however be noted that we have ruled out economies of scope in this mode. This could clearly affect the results

## 6. Conclusions

Optimal environmental taxation of airline traffic is complicated by several factors, and in this paper we have discussed networks effects and imperfect competition. The formulation of the optimal environmental tax is similar to the optimal tax for both a monopolist and a duopoly market. The environmental tax depends on both the competitive situation and the type of network that the airline operates. For a monopoly the optimal tax is always lower than the marginal damages, while for a duopoly the result is ambiguous.

Another interesting result in the paper is that a monopoly airline will choose the type of network with the highest number of flights. This implies that if there are no

[13] However, if we have the extreme case that the competing airline uses a clean technology, i.e. if $\theta_j = 0$, then the number of flights is decreasing in the tax.

differences in marginal damages between flights, an airline will choose the network with the higher environmental damages. Of course, in reality, the marginal damages differ between different connections, especially in the case of local externalities. One implication of adopting a hub-and-spoke network is that the number of flights will increase at the hub airport, given the construction of our model. Since there are reasons to believe that the external damages are higher at the hub airport (perhaps mainly due to the simple fact that the traffic volume is larger), this implies that a regulator, from an environmental perspective, should pay attention to the adoption of hub-and-spoke network. However, we have ruled out certain effects of a hub-and-spoke system such as increased load factors, which could affect the results. This would be a natural extension of the model presented here. We also showed that there are significant differences between a symmetric and a non-symmetric equilibrium. Especially we presented an extreme case where the number of flights on one route was not affected by the introduction of a hub-and-spoke network, which of course was in favour for the hub-and-spoke network.

Even with the seemingly simple model of two competing airlines as multiproduct duopolists, the analysis became rather complicated. We can show that for a symmetric equilibrium both airlines' number of passengers is decreasing in the tax, and that the total number of passengers is always decreasing in the tax. The total number of flights is decreasing in the tax if some additional, reasonable, conditions are imposed. This implies that there might be perverse effects of an environmental tax, since at least one airlines' number of flights can be increasing in tax.

The formulation of competing airlines as multiproduct firms is interesting, and a natural development of this type of model is to link this with the model of Oum *et al.* (1995), and thus to include several connections, where the connections are linked through the cost function. Another interesting development is to introduce adaptive expectations and use the stability conditions in Szidarovszky and Li (2000).

## References


Barnett, A. (1980), The pigouvian tax rule under monopoly, *American Economic Review* **70**, 1037-1041.

Berechman, J. and J. de Wit (1996), An analysis of the effects of European aviation deregulation in airline's network structure and choice of a primary West European hub airport, *Journal of Transport Economics and Policy* **30**, 251-274.

Borenstein, S. (1992), The evolution of U.S. airline competition, *Journal of Economic Perspectives* **6**, 45-73.

Brueckner, J. and P. Spiller (1991), Competition and mergers in airline networks, *International Journal of Industrial Organization* **9**, 323-342.

Bulow, J., J. Geanakoplos, and P. Klemperer (1985), Multimarket oligopoly: Strategic substitutes and complements, *Journal of Political Economy* **93**, 488-511.

Carlsson, F. (1999), Incentive-based environmental regulation of domestic civil aviation in Sweden, *Transport Policy* **6**, 75-82.

Carlsson, F. (2000), Environmental taxation and strategic commitment in duopoly models, *Environmental and Resource Economics* **15**, 243-256.

Caves, D., L. Christensen and M. Tretheway (1984), Economics of density versus economies of scale: Why trunk and local service airline costs differ, *Rand Journal of Economics* **15**, 471-489.

DeVany, A. (1975), The effect of price and entry regulation on airline output, capacity and efficiency, *Bell Journal of Economics* **6**, 327-345.

Morrison, S. and C. Winston (1986), *The Economics Effects of Airline Deregulation*, Brookings Institution, Washington.

Nero, G. and J. Black (1998), Hub-and-spoke networks and the inclusion of environmental costs on airport pricing, *Transportation Research D* **4**, 275-296.

Ortega, J.M. and W.C. Rheinboldt (1970), *Iterative Solutions of Nonlinear Equations in Several Variables*, Academic Press, New York.

Oum, T., D. Gillen and S. Noble (1986), Demand for fareclasses and pricing in airline markets, *Logistics and Transportation Review* **22**, 195-222.

Oum, T., A. Zhang and Y. Zhang (1993), Inter-firm rivalry and firm-specific price elasticities in deregulated airline markets, *Journal of Transport Economics and Policy* **27**, 171-192.

Oum, T., W.G. Waters II and J-S. Yong (1992), Concepts of price elasticities of transport demand and recent empirical estimates, *Journal of Transport Economics and Policy* **26**, 164-169.

Oum, T., A. Zhang and Y. Zhang (1995), Airline network rivalry, *Canadian Journal of Economics* **95**, 836-857.

Schmalensee, R. (1977), Comparative static properties of regulated airline oligopolies, *Bell Journal of Economics* **8**, 565-576.

Szidarovszky, F. and W. Li (2000), A note on the stability of Cournot-Nash equilibrium: the multiproduct case with adaptive expectations, *Journal of Mathematical Economics* **33**, 101-107.

Simpson, R. (1995), Optimal pollution taxation in a Cournot duopoly, *Environmental and Resource Economics* **6**, 359-369.

Zhang and Zhang (1995), Stability of a Cournot-Nash equilibrium: The multiproduct case, *Journal of Mathematical Economics* **26**, 441-462.

## Appendix

**Proof 1: Proof of Proposition 1d**

Using the equilibrium levels of passengers in (4) and (13), and the equilibrium levels of flights in (5), (15) and (16), the reduced forms of the profit functions can be written:

$$\pi^P = 3(b+t)\frac{\sigma-\beta}{\beta}F^P$$

$$\pi^H = 2(b+t)\frac{\sigma-\beta}{\beta}F^H$$

Since $F^H = \left[1+2^{-\varepsilon}2\right]^{\frac{1}{1-\alpha}}F^P$ we have that profits are higher in the hub-and-spoke network if:

$$2(b+t)\frac{\sigma-\beta}{\beta}\left[1+2^{-\varepsilon}2\right]^{\frac{1}{1-\alpha}}F^P > 3(b+t)\frac{\sigma-\beta}{\beta}F^P$$

$$\left[1+2^{-\varepsilon}2\right]^{\frac{1}{1-\alpha}} > \frac{3}{2} \Rightarrow 1+2^{1-\varepsilon} > (\frac{3}{2})^{1-\alpha}$$

**Proof 2: Sign of elements in the derivative matrix of firm i's reaction function**

Firm 1's derivative matrix of the reaction function, equation (27), is:

$$R_2^1 = -\frac{1}{\Delta^1}adj\left(\Pi_{ii}^i\right)\Pi_{ij}^i = -\frac{1}{\Delta^1}\begin{bmatrix}\pi^1_{f_1f_1} & -\pi^1_{q_1f_1}\\ -\pi^1_{f_1q_1} & \pi^1_{q_1q_1}\end{bmatrix}\begin{bmatrix}\pi^1_{q_1q_2} & \pi^1_{q_1f_2}\\ \pi^1_{f_1q_2} & \pi^1_{f_1f_2}\end{bmatrix}$$

$$R_2^1 = -\frac{1}{\Delta^1}\begin{bmatrix}\pi^1_{f_1f_1}\pi^1_{q_1q_2} - \pi^1_{q_1f_1}\pi^1_{f_1q_2} & \pi^1_{f_1f_1}\pi^1_{q_1f_2} - \pi^1_{q_1f_1}\pi^1_{f_1f_2}\\ -\pi^1_{f_1q_1}\pi^1_{q_1q_2} + \pi^1_{q_1q_1}\pi^1_{f_1q_2} & -\pi^1_{f_1q_1}\pi^1_{q_1f_2} + \pi^1_{q_1q_1}\pi^1_{f_1f_2}\end{bmatrix} = -\frac{1}{\Delta^1}\begin{bmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{bmatrix}$$

Since $\pi^1_{f_1f_1}, \pi^1_{q_1q_1}, \pi^1_{q_1q_2}, \pi^1_{f_1q_2} < 0$ and $\pi^1_{f_1q_1} = \pi^1_{q_1f_1} > 0$, it follows directly that $\alpha_1, \gamma_1 > 0$. Further $\beta_1 = 0$ since $\pi^1_{f_1f_1} = \pi^1_{f_1f_2}$ and $\pi^1_{q_1f_2} = \pi^1_{q_1f_1}$. Finally,

$$\delta_1 = -\left(\beta PF^{-1}(1-\sigma\frac{q_i}{Q})\right)^2 + \sigma PQ^{-1}((\sigma+1)\frac{q_i}{Q}-2)(\beta-1)\beta PF^{-2}q_i$$

$$\delta_1 = -\beta P^2F^{-2}\left(\beta(1-\sigma\frac{q_i}{Q})^2 - \sigma\frac{q_i}{Q}(\beta-1)((\sigma+1)\frac{q_i}{Q}-2)\right)$$

The expression in the inner parenthesis can be written:

$$\beta(1+\sigma^2(\frac{q_i}{Q})^2 - 2\sigma\frac{q_i}{Q}) - (\sigma\frac{q_i}{Q}\beta - \sigma\frac{q_i}{Q})((\sigma+1)\frac{q_i}{Q}-2)$$

$$\beta + \beta\sigma^2(\frac{q_i}{Q})^2 - 2\beta\sigma\frac{q_i}{Q} - \sigma(\frac{q_i}{Q})^2\beta(\sigma+1) + 2\sigma\frac{q_i}{Q}\beta + \sigma(\sigma+1)(\frac{q_i}{Q})^2 - 2\sigma\frac{q_i}{Q}$$

$$\beta - \sigma(\frac{q_i}{Q})^2\beta + \sigma(\sigma+1)(\frac{q_i}{Q})^{22} - 2\sigma\frac{q_i}{Q}.$$

This expression is negative, and consequently $\delta_1 > 0$, if $\sigma$ and $\beta$ are sufficiently low.

**Proof 3: Comparative statics for the duopoly market**

Using (27) and (30) the comparative statics for firm 1 can be written:

$$\begin{bmatrix} \frac{dq_1}{dt} \\ \frac{df_1}{dt} \end{bmatrix} = -\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \left( \begin{bmatrix} r^1_{qq} & r^1_{qf} \\ r^1_{fq} & r^1_{ff} \end{bmatrix} \frac{1}{\Delta^2} \begin{bmatrix} \pi^2_{f_2 f_2} & -\pi^2_{f_2 q_2} \\ -\pi^2_{q_2 f_2} & \pi^2_{q_2 q_2} \end{bmatrix} \Pi^2_{2t} + \frac{1}{\Delta^1} \begin{bmatrix} \pi^1_{f_1 f_1} & -\pi^1_{f_1 q_1} \\ -\pi^1_{q_1 f_1} & \pi^1_{q_1 q_1} \end{bmatrix} \Pi^1_{1t} \right)$$

Since, $\Pi^1_{1t} = \begin{bmatrix} \pi^1_{q_1 t} \\ \pi^1_{f_1 t} \end{bmatrix} = \begin{bmatrix} 0 \\ -\theta_1 \end{bmatrix}$ and $\Pi^2_{2t} = \begin{bmatrix} \pi^2_{q_2 t} \\ \pi^2_{f_2 t} \end{bmatrix} = \begin{bmatrix} 0 \\ -\theta_2 \end{bmatrix}$, we have:

$$\frac{dX^1}{dt} = -\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \left( \begin{bmatrix} r^1_{qq} & r^1_{qf} \\ r^1_{fq} & r^1_{ff} \end{bmatrix} \frac{1}{\Delta^2} \begin{bmatrix} \theta_2 \pi^2_{f_2 q_2} \\ -\theta_2 \pi^2_{q_2 q_2} \end{bmatrix} + \frac{1}{\Delta^1} \begin{bmatrix} \theta_1 \pi^1_{f_1 q_1} \\ -\theta_1 \pi^1_{q_1 q_1} \end{bmatrix} \right)$$

Finally, since $b_1, r^1_{qf} = 0$, we have:

$$\frac{dX^1}{dt} = -\begin{bmatrix} a_1 & 0 \\ c_1 & d_1 \end{bmatrix} \frac{1}{\Delta^2} \begin{bmatrix} r^1_{qq} \theta_2 \pi^2_{f_2 q_2} \\ r^1_{fq} \theta_2 \pi^2_{f_2 q_2} - r^1_{ff} \theta_2 \pi^2_{q_2 q_2} \end{bmatrix} - \begin{bmatrix} a_1 & 0 \\ c_1 & d_1 \end{bmatrix} \frac{1}{\Delta^1} \begin{bmatrix} \theta_1 \pi^1_{f_1 q_1} \\ -\theta_1 \pi^1_{q_1 q_1} \end{bmatrix}$$

Solving this yields (31) and (33).