Multidimensional measures on Cantor sets

Malin Palö

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Abstract

Cantor sets in $\mathbb{R}$ are common examples of sets on which Hausdorff measures can be positive and finite. However, there exists Cantor sets on which no Hausdorff measure is supported and finite. The purpose of this thesis is to try to resolve this problem by studying an extension of the Hausdorff measures μ_h on $\mathbb{R}$ by allowing test functions to depend on the midpoint of the covering intervals instead of only on the diameter. As a partial result a theorem about the Hausdorff measure of any regular enough Cantor set, with respect to a chosen test function, is obtained.

Acknowledgements

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Contents

Nomenclature

0^m	The binary word which consists of m zeros	7
1^m	The binary word which consists of m ones	7
a_j	The left endpoint of a basic interval I_j	9
b_j	The right endpoint of a basic interval I_j	9
δ_j	The diameter of a basic interval I_j	9
$\dim[C](\xi)$	The local Hausdorff dimension of the set C at the point $\xi \in C$	11
$\dim_H(E)$	The Hausdorff dimension of the set E	10
$d[\nu](\xi)$	The local dimension of a measure ν at the point ξ	11
$D[\sigma](w)$	The upper density of a measure σ at a point w	42
$\Delta_h[\sigma](w)$	The upper density of a measure σ with respect to a test function at a point w	42
G_j	The gap removed from the basic interval I_j in the construction of a Cantor set $C \sim \{I_j\}$	8
I_j	An interval from the construction of a Cantor set, where j is a binary word specifying which interval I_j represents	7
$I(w, \delta)$	The interval with diameter δ and midpoint in w	6
$j_1 j_2$	The concatenation of the two binary words j_1 and j_2	7
$j\|k$	The binary word consisting of the first k digits of the binary word j	7
$j\|-k$	The binary word which is the binary word j with the last k digits removed	7
$m_{\alpha(w)}$	The Hausdorff measure of dimension $\alpha(w)$	7
μ_h	The Hausdorff measure associated with the test function h	6
ν	The Cantor measure associated with some Cantor set	10
ν_p	The p-Cantor measure associated with some Cantor set	10
$\sim$	$C \sim \{c_j\}$ means C is the Cantor set associated with the sequence C. $\{c_j\} \sim \{I_j\}$ means that the two sequences are associated with the same Cantor set C	10
w_j	The midpoint of a basic interval I_j	9

1 Introduction

The purpose of this thesis is to give an extension of the set of measures usually called Hausdorff measures on $\mathbb{R}$. Throughout this thesis we will investigate this new class of measures as well as the connection between these measures and the Cantor sets on which they have support.

The connection between Cantor sets and Hausdorff measures associated to some testfunction $h(\delta)$ have been investigated by several authors. This has been done with the purpose of categorising the set of all Cantor sets according to which Hausdorff measures, in their classical sense, give them nonzero and finite measure ([4] and [3]). This has also been done in order to be able to estimate or calculate a Hausdorff measure of some specific Cantor set ([2] and [8]). In both these cases, the tool which Hausdorff measures constitute has the drawback that for many Cantor sets there exists no Hausdorff measure which gives it a finite and positive measure. Furthermore, for many Cantor sets there exists no Hausdorff measure whose restriction to the Cantor set is absolutely continuous to the Cantor measure of the set, or equivalently, is a mass distribution on the set. The extension considered in this thesis aims to resolve these issues by considering a larger class of measures.

The contents of this thesis will be structured as follows:

In the next section the basic concepts of this thesis will be defined.

In section 3 we define sets $\mathcal{C}_h$ of Cantor sets and give bounds for $\mu_h(C)$ for all $C \in \mathcal{C}_h$ and all nice enough test functions h. Given further restrictions on h, we prove a theorem which can be used to calculate the Hausdorff measure of any given regular enough Cantor set. We then use this theorem to calculate the Hausdorff measure of three Cantor sets studied in [8] and [2].

In section 4 we consider the Hausdorff measures associated to test functions h of exponential type and show that $\mathcal{C}_h$ is nonempty for such test functions. We also show that the assumptions of the theorems in section 3 generally hold in this case.

In section 5 we show that some of the properties which are known to hold for ordinary Hausdorff measures also holds for the type of Hausdorff measures considered in this thesis.

2 Definitions

2.1 Hausdorff measures

Felix Hausdorff, in his paper *Dimension und äußeres Maß* from 1918, as translated by Sawhill, Edgar and Olson in the book *Classics on Fractals* [5], defined the following class of measures:

Definition 2.1 (theorem)**:** Let $\mathcal{U}$ be a system of bounded sets U in a q-dimensional space having the property that one can cover any set A with an at most countable number of sets U from $\mathcal{U}$ having arbitrarily small diameters $|U|$. Let $h : \mathcal{U} \to [0, \infty)$ be a set function. Denote by

$$m^{\delta}_{\mathcal{U},h}(A) = \inf \sum h(U_n)$$

where the infinum runs over all countable subsets $\{U_n\}$ of $\mathcal{U}$ such that $\cup U_n$ covers A and $|U_n| < \delta$ for all n. If $\mathcal{U}$ is the Borel sets then $\mu_{\mathcal{U},h}(A) = \lim_{\delta\to 0} m^{\delta}_{\mathcal{U},h}(A)$ is a measure. If $h(U)$ is continuous or $h(U) = h(\bar{U})$, then $\mu_{\mathcal{U},h}$ is an outer measure.

From this quite general definition, a common restriction is the class of Hausdorff measures which one gets by considering set functions dependent only of the diameter of the set. It is also often required that the decrease of the set function is bounded is the following sense.

Definition 2.2: An increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is *doubling* if there exists a constant C such that $C \cdot h(s) > h(2s)$ for all $s > 0$.

Using definition 2.2, we can formulate to more common definition of Hausdorff measures, using $|E|$ to denote the diameter of a set E:

Definition 2.3: Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, increasing and doubling function such that $h(0) = 0$. Then the h-Hausdorff measure of the set E is defined by

$$\mu_h(E) = \liminf_{\delta\to 0} \left\{ \sum h(|E_j|), \text{ where } \{E_j\} \text{ is a } \delta\text{-covering of } E \right\}$$

The elements in the sequence; $\inf\{\sum h(|E_j|), \text{ where } \{E_j\} \text{ is a } \delta\text{-covering of } E\}$, will be denoted by μ^{δ}_h. The function h will be called the *test function* associated with the measure μ_h. Additionally, any function h with these properties will be called a test function.

When the sets we want to measure lie in $\mathbb{R}$, which will be our primary focus of study, we get an equivalent definition if considering only coverings by intervals.

We will use $I(w, \delta)$ to denote the interval with midpoint w and diameter δ. Using this notation we can formulate the definition of Hausdorff measures with which we will be concerned in this thesis.

Definition 2.4: Let $h : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function with $\lim_{\delta \to 0} h(w, \delta) = 0$ for all $w \in [0, 1]$ which is increasing and doubling in the second argument. Then the Hausdorff measure of the set $E \subseteq \mathbb{R}$ with respect to the test function h is defined by

$$\mu_h(E) = \lim_{\delta \to 0} \inf \left\{ \sum h(w_k, \delta_k), \quad \text{where } \{I(w_k, \delta_k)\} \text{ is a } \delta\text{-covering of } E \right\}$$

The function h will be called the *test function* associated with the measure μ_h and μ_h will be called the Hausdorff measure associated with the test function h.

It can be shown ([10]) that the resulting measure does not depend on whether the sets considered in the covering in the definition above is open or closed. In this thesis we will mainly consider coverings by closed sets.

If the test function h is of the form $h(w, \delta) = \delta^{\alpha(w)}$ and there is no risk of confusion, we will write $\mu_h = m_{\alpha(w)}$.

2.2 Cantor sets

A Cantor set in $\mathbb{R}$ is a compact, perfect and totally disconnected set of Lebesgue measure zero. A more constructive, but equivalent, definition is stated below. To formulate this definition, we need the following notations:

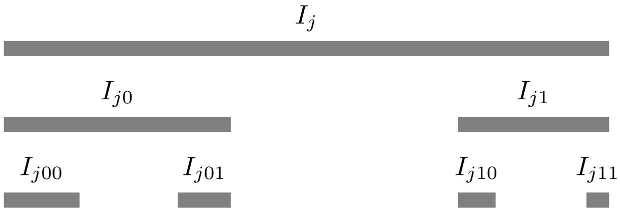
If j is a binary word of finite or infinite length, we write $j|k$, where $k \in \mathbb{N}$, to denote the word consisting of the first k digits of j . Similarly, $j| - k$ will be used to denote the binary word which is the binary word j with the last k digits removed. Also, 0^m will be used throughout this text to denote the binary word which consists of m zeros. 1^m is defined analogously. When j_1 and j_2 are two binary words, $j_1 j_2$ will denote their concatenation.

We now proceed to our definition of a Cantor set.

Definition 2.5: Let $\{I_j\}_{j \in \{0,1\}^n,\, n=0,1,2,3,\ldots}$ be a sequence of closed intervals such that for all binary words j

1. I_j is non-empty
2. $I_{j0} \cap I_{j1} = \emptyset$
3. $I_{j0}, I_{j1} \subseteq I_j$ and
4. I_{j0} and I_j have the same left endpoint and I_{j1} and I_j have the same right endpoint.

Set $C^{(n)} = \bigcap\limits_{k=1}^{n} \bigcup_{j \in \{0,1\}^k} I_j$ and $C = \lim\limits_{n \to \infty} C^{(n)}$. This limit is called the Cantor set associated with the sequence $\{I_j\}$.



The intervals I_j appearing in the construction of a Cantor set C will be called the *basic intervals* associated with C. Moreover, the intervals whose left endpoint is the left endpoint of a basic interval and whose right endpoint is a the right endpoint of a basic interval will be called the *near basic intervals* associated with C.

We will now state a couple of definitions, all equivalent to definition 2.5.

Denote for any interval I and any $c \in (0,1)$ by $c \cdot_L I$ the leftmost c-proportion of the set I, and analogously by $c \cdot_R I$ the rightmost c-proportion of the set I. Note that this implies that $1 \cdot_L I = I$, $1 \cdot_R I = I$, $0 \cdot_L I = \emptyset$ and $0 \cdot_R I = \emptyset$.

Definition 2.6: Let $\{c_j\}_{j\in\{0,1\}^n,\, n=1,2,3,\ldots}$ be a sequence of strictly positive numbers such that $0 < c_{j0} + c_{j1} < 1$ for all binary words j and let $I_\emptyset$ be a closed interval.

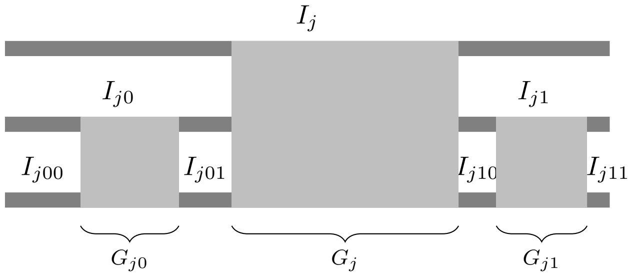
For any binary word j set $I_{j0} = c_{j0} \cdot_L I_j$ and $I_{j1} = c_{j1} \cdot_R I_j$.

Define $C^{(n)} = \bigcap\limits_{k=1}^{n} \bigcup_{j\in\{0,1\}^k} I_j$ and $C = \lim\limits_{n\to\infty} C^{(n)}$. This limit is called the Cantor set associated with the sequence $\{c_j\}$ and the interval $I_\emptyset$. If $I_\emptyset = [0,1]$, we say that C is the Cantor set associated with the sequence $\{c_j\}$.

Where the last definitions defined a Cantor set more or less directly through the intervals kept at each step of the construction, the following definitions instead defines it through the intervals removed at each step. The idea is that the set of all intervals which will be removed from a specific basic interval uniquely defines the basic interval.

Definition 2.7: Let $\{G_j\}_{j\in\{0,1\}^n,\, n=0,1,2,3,\ldots}$ be a sequence of disjoint open intervals such that the closure of $\cup_{k\in\{0,1\}^n,\, n=1,2,3,\ldots} G_{jk}$ is an interval for any binary word j.

Let $I_j = \overline{\bigcup_{k\in\{0,1\}^n,\, n=1,2,3,\ldots} G_{jk}}$, set $C^{(n)} = \bigcap\limits_{k=1}^{n} \bigcup_{j\in\{0,1\}^k} I_j$ and $C = \lim\limits_{n\to\infty} C^{(n)}$. This limit is called the Cantor set associated with the sequence $\{G_j\}$.



The intervals G_j in the definition above will be called the gaps associated with a Cantor set.

As in the definition above the next definition defines a Cantor set through the intervals in its complement. However, this definition specifies only the length of these intervals as their exact position is uniquely determined by the position of the leftmost point in the Cantor set.

Definition 2.8: Let $\{|G_j|\}_{j\in\{0,1\}^n,\, n=0,1,2,3,\ldots}$ be a sequence of positive real numbers such that $\sum |G_j|$ is finite and let $a_\emptyset$ be any real number. Let $<$ be the lexicographical ordering of binary words. Let

$$\begin{cases} b_\emptyset = a_\emptyset + \sum |G_j| \\ a_{j0} = a_j \\ a_{j1} = a_\emptyset + \sum_{j'<j1} |G_{j'}| \\ b_{j0} = a_{j1} - |G_j| \\ b_{j1} = b_j \end{cases}$$

and set $I_j = [a_j, b_j]$. Define $C^{(n)} = \bigcap_{k=1}^{n} \bigcup_{j\in\{0,1\}^k} I_j$ and $C = \lim_{n\to\infty} C^{(n)}$. This limit is called the Cantor set associated with the sequence $\{|G_j|\}$ and the interval $I_\emptyset = [a_\emptyset, b_\emptyset]$. If the exact position of the Cantor set is of no importance we say that C is the Cantor set associated with the sequence $\{|G_j|\}$.

Remark 2.9*:* In general, we will let a_j denote the left endpoint and b_j denote the right endpoint of a basic interval I_j associated with some Cantor set so that $I_j = [a_j, b_j]$. Similarly, we will use w_j to denote the midpoint and δ_j to denote the diameter of a basic interval I_j so that $I_j = I(w_j, \delta_j)$.

Remark 2.10*:* In this thesis we will almost exclusively use binary words to enumerate the elements of the construction of a Cantor set. However, two other commonly used notations should be mentioned:

Some authors write I_l^k to represent the lth interval in the kth construction step. If j is a binary word and we let j_{10} be the integer we get if converting j when considered as a binary number to base 10, we can convert between the two notations by $I_j = I_{j_{10}}^{|j|}$. Similarly $c_j = c_{j_{10}}^{|j|}$ and $G_j = G_{j_{10}}^{|j|}$. We will only use this notation in example 3.8 and example 3.9.

Another common notation is to enumerate the intervals by natural numbers instead of binary words. We can convert between notations by $I_j = I_{2^{|j|}+j_{10}}$. Similarly $I_j = I_{2^{|j|}+j_{10}}$, $c_j = c_{2^{|j|}+j_{10}}$

and $G_j = G_{2^{|j|}+j_{10}}$.

We write $C \sim \{c_j\}$ when C is the unique Cantor set associated with a sequence of proportions $\{c_j\}$. Similarly we write $C \sim \{I_j\}$, $C \sim \{G_j\}$ and $C \sim \{|G_j|\}$. We use the same symbol to indicate that two sequences are associated with the same Cantor set; e.g. $\{I_j\} \sim \{G_j\}$.

An interval I_j will be called *older* than another interval I_k if I_j appear in an earlier step of the construction of C than I_k, i.e. if $|j| < |k|$. Analogously, we say that a gap G_{j_1} is older than a gap G_{j_1} if $|j_1| < |j_2|$.

To each Cantor set $C \sim \{I_j\}$ there is an associated measure:

Definition 2.11: Let $C \sim \{I_j\}$ be any Cantor set. The unique probability measure ν_p satisfying $\nu(I_{j0}) = p\,\nu(I_j)$ and $\nu(I_{j1}) = (1-p)\,\nu(I_j)$ for all binary words j is called the *p-Cantor measure* associated with the Cantor set C.

To simplify notations, we write ν instead of ν_p when $p = \frac{1}{2}$ and say that ν is the *Cantor measure* associated with C.

That the p-Cantor measure is a well defined measure follows by proposition 1.7 in [7].

Example 2.12: One of the most frequently mentioned Cantor sets C is the so called ternary Cantor set; the Cantor set associated with the sequence $\{c_j\}$ for which $c_j = c = \frac{1}{3}$ for all binary words j. The Cantor measure associated with this set is the restriction of the Hausdorff measure $m_{\frac{\log 2}{\log 3}} = m_{\frac{\log 2}{-\log c}}$ to C. Analogous results exists for all constant sequences $\{c_j\}$ where $c_j = c$ for some $c \in (0, 0.5)$.

For a test function h, a Cantor set C is said to be *h-regular* if μ_h is finite and supported on C. A measure which, given a set E, is finite and supported on E is called a *mass distribution* on E. A Cantor set C is said to be *singledimensional* if there exists a test function $h(\delta)$ such that C is h-regular.

2.3 Definitions of dimensions

Definition 2.13: For any set E there exists a unique positive number α such that $m_\beta(E) = 0$ for all $\beta > \alpha$, and $m_\beta(E) = \infty$ for all $\beta < \alpha$. This unique number will be called the *Hausdorff dimension* of the set E, and will be denoted by $\dim_H(E)$.

Note that a set E having Hausdorff dimension α does not guarantee neither that m_α is a mass distribution on E nor that there exists any test function $h(\xi, \delta)$ such that μ_h is a mass distribution on E, even though this is true for certain sets. An example a set for which this is true is the Cantor set $C \sim \{c_j\}$, where $c_j = c$ for all binary words j, which has Hausdorff dimension $\alpha = -\frac{\log 2}{\log c}$ and $m_\alpha(C_\alpha) = 1$.

The Hausdorff dimension of a set could be described as a measure of how *dense* the set is. Such densities could, however, be non-constant on the set. A simple example of such a set is the set $\{0\} \cup [1,2]$. The set $\{0\}$ has Hausdorff dimension zero whereas the set $[1,2]$ has Hausdorff dimension one. The Hausdorff dimension of their union would, however, be one; i.e. the largest Hausdorff measure obtained on any of its subsets. To better be able to describe the dimension of a set, we would thus need a more local definition of the dimension of a set. To be able to state one such common definition we first need to make the following observation:

Observation 2.14*:* Let C be a Cantor set and $\xi \in C$. Then there exists a unique binary sequence j such that $\xi \in I_{j|k}$ for all $k \in \mathbb{N}$. If $\xi \in I_{j|k}$ for all $k \in \mathbb{N}$ we write $\xi = \xi_j$.

Definition 2.15: Let C be a Cantor set and let $\xi = \xi_j$ be a point in C. The *local Hausdorff dimension* of $\mathcal{C}$ at ξ is the unique number α such that

$$\lim_{k\to\infty} \prod_{k=0}^{n} \left(c^{\beta}_{(j|k)0} + c^{\beta}_{(j|k)1} \right) = \begin{cases} \infty \text{ if } \beta < \alpha \\ 0 \text{ if } \beta > \alpha \end{cases}$$

The local Hausdorff dimension at $\xi \in \mathcal{C}$ will be denoted by $\dim[C](\xi)$.

To be able to assign a dimension to each measure, which tells something about the dimension of the sets they measure, we will use the following definition:

Definition 2.16: Let ν be a mass distribution on a set E and let $w \in E$. Then the *local dimension* of ν at w in is defined by

$$d[\nu](w) = \limsup_{\delta\to 0} \frac{\log \nu\left(I(w,\delta)\right)}{\log \delta}$$

Note that neither the local dimension of a measure, nor the local Hausdorff dimension of a set, is a fixed number but rather a function in $w \in C$.

3 Multidimensional Cantor sets in the support of Hausdorff measures

In this section we will define a set $\mathcal{C}_h$ of Cantor sets for any test function h. We will then show that given some restrictions of h, μ_h is a mass distribution on all $C \in \mathcal{C}_h$. We will also show that given some additional assumptions on the test function, the restriction of μ_h to any set C in $\mathcal{C}_h$ is equivalent to the Cantor measure on C.

3.1 Cantor sets associated with a test function

In this section we will define the sets $\mathcal{C}_h$ and then continue by giving our first upper limit of the Hausdorff measure μ_h of any $C \in \mathcal{C}_h$. In general, due to the infinum in the definition of Hausdorff measures, it is much easier to find an upper limit of the Hausdorff measure of a set C than a lower limit, and we will later see that the upper limit given below is sharp in most cases, which our first lower limit will not be.

Definition 3.1: Let $C \sim \{I_j\}$ be any Cantor set and let h be a test function. If

$$h(I_j) = \nu(I_j) = 2^{-|j|}$$

for all long enough binary words j we say C is associated with h and write $C \in \mathcal{C}_h$. If the test function is of the form $h(w, \delta) = \delta^{\alpha(w)}$, and there is no risk of misunderstanding, we write $\mathcal{C}_{\alpha(w)}$ instead of $\mathcal{C}_h$.

Note that the test function is not uniquely determined by a Cantor set C, since the set neither determines the value of h for intervals which are not basic intervals associated with C nor for any large δ.

In general, it is not obvious that there exist any Cantor set $C \in \mathcal{C}_h$. When the test function h depends on δ concavity alone is enough to guarantee the existence of Cantor sets $C \in \mathcal{C}_h$. The corresponding condition, given a general test function $h(w, \delta)$, is that

$$h(w - t_0, 2t_0) + h(w + t_1, 2t_1) \geq h(w, 2t_0 + 2t_1)$$

for all w, t_0 and t_1. This is equivalent of saying that the test function h, when considered as an interval function, is subadditive. However, as in the singledimensional case, this is a very much stronger condition than needed. The question of whether or not $\mathcal{C}$ is nonempty will be dealt with in detail in the special case of test functions on the form $h(w, \delta) = \delta^{\alpha(w)}$ in the next section.

Lemma 3.2: *Let $C \sim \{I_j\}$ be any Cantor set and let h be any test function defined on the basic intervals associated with C such that $h(I_j) = \nu(I_j)$ for any basic interval I_j. Then $\mu_h(E \cap C) \leq \nu(E)$*

for any interval E.

Proof. For each $\delta > 0$ we can find $n \in \mathbb{N}$ such that $|I_j| < \delta$ for all binary words j of length $|j| \geq n$. Then $\{I_j\}_{|j|=n}$ is a disjoint δ-covering of C, and thus

$$\mu_h^\delta(E \cap C) \leq \sum_{j\in\{0,1\}^n \text{ and } E\cap I_j\neq\emptyset} h(I_j) = \sum_{j\in\{0,1\}^n \text{ and } E\cap I_j\neq\emptyset} \nu(I_j) =$$
$$\nu(\bigcup_{j\in\{0,1\}^n \text{ and } E\cap I_j\neq\emptyset} I_j) \leq \nu(E) + \sum_{j\in\{0,1\}^n \text{ and } \partial E\cap I_j\neq\emptyset} \nu(I_j)$$

As at most two basic intervals from any step can contain the endpoints of E and $\nu(I_j) = 2^{-|j|}$ for any basic interval, the last sum can be bounded from above by $2 \cdot 2^{-|j|} = 2 \cdot 2^{-n}$. We thus get

$$\mu_h^\delta(E \cap C) \leq \nu(E) + \sum_{j\in\{0,1\}^n \text{ and } \partial E\cap I_j\neq\emptyset} \nu(I_j) \leq \nu(E) + 2 \cdot 2^{-n}$$

By letting $\delta \to 0$ we get $\mu_h(E \cap C) \leq \nu(E)$. □

3.2 The mass distribution principle

Several proofs in this and the succeeding sections will use what is called the *mass distribution principle*:

The mass distribution principle: Let ν be a mass distribution on a set E, $h(\xi, \delta)$ a test function and $D, \delta_0 > 0$ positive numbers such that

$$h(I) \geq D \cdot \nu(I)$$

for all intervals I with diameter less that δ_0 contained in $(1 + \delta_0)E$. Then

$$\mu_h(E \cap C) \geq D \cdot \nu(E)$$

Proof of the mass distribution principle. Fix $\delta < \delta_0$ and let $\{I_k\}_{k\in K}$ be an arbitrarily chosen δ-covering of E. Then

$$\sum_{k\in K} h(I_k) \geq \sum_{k\in K} D \cdot \nu(I_k) \geq D \cdot \nu(E)$$

since $E \subset \bigcup_{k\in K} I_k$. By letting $\delta \to 0$, we get $\mu_h(E \cap C) \geq D \cdot \nu(E)$. □

3.3 Hausdorff measures as mass distributions on Cantor sets

Given that $\mathcal{C}_h$ is non-empty and h is sufficiently nice, the following theorem shows that μ_h is a mass distribution on any $C \in \mathcal{C}_h$ with $c_j \leq 0.5$ for all binary words j.

Theorem 3.3: *Let h be any test function which is increasing as an interval function for all small enough intervals and let D be the doubling constant associated with h. Let $C \sim \{c_j\} \in \mathcal{C}_h$ and assume $c_j \leq 0.5$. Then μ_h is a mass distribution on C. Further, for any interval $J \subseteq [0,1]$, $\frac{1}{2D^2} \cdot \nu(J) \leq \mu_h(J \cap C) \leq \nu(J)$, where ν is the Cantor measure associated with C.*

Proof. Note first that the upper limit of $\mu_h(J \cap C)$ follows directly from lemma 3.2. The claim of the lower limit will be proved using the mass distribution principle. We thus need to show that $h(I) \geq \frac{1}{2D^2}\nu(I)$ for all small enough intervals I.

Let $\Delta > 0$ be small enough for h to be increasing for all intervals with diameter less than 2Δ and to have $h(I_j) = \nu(I_j)$ for all basic intervals associated with C with diameter less than 2Δ. Pick any open interval $I \subset [0,1]$ with diameter smaller that Δ. We may assume that $I \cap C \neq \emptyset$, since otherwise $h(I) > 0 = \nu(I)$ in which case we are finished. Since h is increasing, we can also assume that I is a near basic interval.

Since $I \subseteq [0,1] = I_\emptyset$, there exists at least one basic interval in which I is contained. Since any two basic intervals are either disjoint or one is a subset of the other, and two disjoint intervals cannot both be supersets of I, the basic intervals containing I are totally ordered by inclusion, i.e. form a sequence $[0,1] \supset I_{j_1}, \supset I_{j_2} \supset I_{j_k} \supset \cdots$. Since I have strictly positive length, and the length of the basic intervals tend to zero as $k \to \infty$, this sequence must eventually stop. Thus there exists a unique shortest basic interval I_j containing I. Note that since $I \subseteq I_j$ we get that

$$I \cap C = I_j \cap (I \cap C) = (I_{j0} \cup I_{j1}) \cap (I \cap C)$$

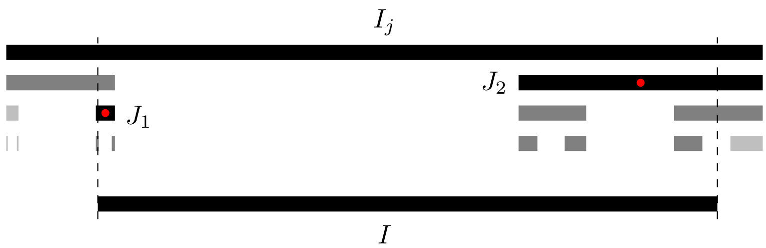
Now let J_1 be the shortest basic interval such that $I_{j0} \cap (I \cap C) = J_1 \cap (I \cap C)$ and J_2 be the shortest basic interval such that $I_{j1} \cap (I \cap C) = J_2 \cap (I \cap C)$.

That these intervals exist and are unique follows by analogous reasoning as above and are thus omitted here.

We then have

$$I \cap C \subseteq I_j \cap C = (I_{j0} \cup I_{j1}) \cap C = (J_1 \cup J_2) \cap C$$

We will now argue that $J_1 \cup J_2 \subseteq 4I$.



Suppose that $J_1 \not\subseteq I$ and note that J_1 is the leftmost of J_1 and J_2. Since $c_j < 0.5$ for all binary words j and J_1 is the shortest interval whose union with J_2 contain $I \cap C$, the midpoint and the right endpoint of J_1 lies in I. This implies that $J_1 \subseteq 4I$. Since analogous arguments hold for J_2, we also get $J_2 \subseteq 4I$.

Since h is increasing for all small intervals, we have $h(4I) \geq h(J_1)$ and $h(4I) \geq h(J_1)$ which directly implies

$$2 \cdot h(4I) \geq h(J_1) + h(J_2)$$

Since $D \cdot h(I) \geq h(2I)$, we get

$$2D^2 \cdot h(I) \geq h(J_1) + h(J_2)$$

Let ν be the Cantor measure associated with C. As $h(I_j) = \nu(I_j)$ for all basic intervals contained in $2I$; $h(J_1) = \nu(J_1)$ and $h(J_2) = \nu(J_2)$. Thus

$$2D^2 \cdot h(I) \geq h(J_1) + h(J_2) = \nu(J_1) + \nu(J_2) = \nu(J_1 \cup J_2) \geq \nu(I \cap C) = \nu(I)$$

since $I \cap C \subseteq J_1 \cup J_2$. This proves the theorem. □

3.4 A result concerning the measure of Cantor sets

We will now prove the main theorem om this section, which is concerned with finding the exact measure of a Cantor set.

Theorem 3.4: *Let $J \subseteq [0,1]$ be any closed interval and let $\varepsilon > 0$ be a small positive number.*

Let h be a test function. For any fixed w and δ, set $f(t_0, t_1) = h(w - t_0 + t_1, \delta + 2t_0 + 2t_1)$ and assume $\frac{\partial f}{\partial t_0} \geq 0$, $\frac{\partial f}{\partial t_1} \geq 0$, $\frac{\partial^2 f}{\partial t_0 \partial t_1} \leq 0$ and $\frac{\partial^2 f}{\partial t_1^2} \leq 0$ for all small enough δ, t_0 and t_1 with $I(w - t_0 + t_1, \delta + 2t_0 + 2t_1) \subseteq (1 + \varepsilon) \cdot J$

Let $C \sim \{I_j\}$ be a Cantor set and assume that there exists two positive numbers q and r such that

$$q \cdot \nu(I_j) \leq h(I_j) \leq r \cdot \nu(I_j)$$

for all small enough basic intervals I_j contained in $(1 + \varepsilon) \cdot J$ for some $\varepsilon > 0$. Further assume

$$\rho \cdot \nu(I_{j1}) \geq \nu(\rho \cdot_L (G_j \cup I_{j1})) \tag{1}$$

for all long enough binary words j with $I_j \subseteq (1 + \varepsilon) \cdot J$ and all $\rho \in [0, 1]$. Then

$$\big(q - (r - q)\big) \cdot \nu(J) \leq \mu_h(J \cap C) \leq r \cdot \nu(J) \tag{2}$$

Proof. For the upper bound on $\mu_h(J \cap C)$, consider the covering of $J \cap C$ with the basic intervals I_j from some fixed step k of the construction which intersects J, i.e. all basic intervals I_j for which

$I_j \cap J \neq \emptyset$ and $|j| = k$. Then

$$\mu_h(J \cap C) \leq \lim_{k\to\infty} \sum_{\substack{|j|=k \\ I_j\cap J\neq\emptyset}} h(I_j) \leq \lim_{k\to\infty} \sum_{\substack{|j|=k \\ I_j\cap J\neq\emptyset}} r\cdot\nu(I_j) =$$

$$\lim_{k\to\infty} r\cdot\nu\left(\bigcup_{\substack{|j|=k \\ I_j\cap J\neq\emptyset}} I_j\right) \leq \lim_{k\to\infty} r\cdot\nu\,(J) + r\cdot\nu\left(\bigcup_{\substack{|j|=k \\ I_j\cap \partial J\neq\emptyset}} I_j\right)$$

As at most two basic intervals from any fixed step k of the construction can intersect ∂J, and $\nu(I_j) = 2^{-|j|}$ for any basic interval, we get

$$\mu_h(J\cap C) \leq \lim_{k\to\infty} r\cdot\nu\,(J) + r\cdot\nu\Big(\bigcup_{\substack{|j|=k \\ I_j\cap \partial J\neq\emptyset}} I_j\Big) \leq \lim_{k\to\infty} r\cdot\nu\,(J) + r\cdot 2\cdot 2^{-k} = r\cdot\nu\,(J)$$

We will now show that the lower limit in equation (2) holds. To show that $\mu_h(J\cap C) \geq \big(q-(r-q)\big)\cdot\nu(J)$ we will use the mass distribution principle, i.e. we will show that $h(I) \geq \big(q-(r-q)\big)\cdot\nu(I)$ for all interval $I \subseteq J(1+\varepsilon)$ with $|I| < \Delta$ for some small $\Delta > 0$.

Pick Δ small enough for the assumptions of the theorem to hold when $\delta + 2t_0 + 2t_1 < \Delta$.

Since $h(I)$ is increasing, it is enough to consider the case when I is a near basic interval. Let I be any near basic interval associated with C with $|I| < \Delta$.

Let G_j be the oldest gap which is a subset of I. Since G_j is the oldest gap in I and I is a near basic interval, $I \subseteq I_j$. Set $J_1 = I \cap I_{j0}$ and $J_2 = I \cap I_{j1}$ and note that $I \cap \mathcal{C} \subseteq J_1 \cup J_2$.

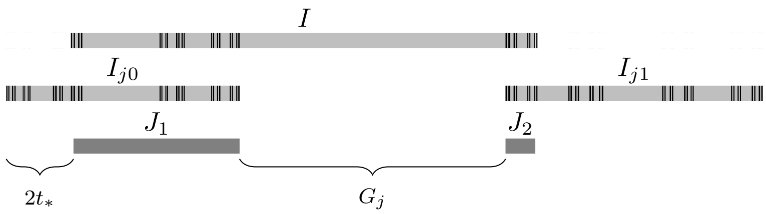


Figure 1: The image above shows some of the elements of the proof. The black parts inside the light grey intervals are some of the basic intervals of the Cantor set. Note that the endpoints of I coincide with the endpoints of basic intervals and also that I must be contained in I_j since if it was not, G_j would not be the oldest gap in I. Note also that the right endpoint of I_{j0} and J_1 coincide.

Let w be the midpoint of J_1 and $\delta = |J_1|$ and consider the function

$$f(t_0, t_1) = h(w - t_0 + t_1, \delta + 2t_0 + 2t_1)$$

Since $J_1 = I \cap I_{j0}$ and I_{j0} have their right end point in common and $J_1 \subseteq I_{j0}$ there exists a unique number $t_* \in \mathbb{R}_+$ such that $I(w - t_*, \delta + 2t_*) = I_{j0}$.

Set $f(t) = f(0, \frac{t}{2})$ and $f_*(t) = f(t_*, \frac{t}{2})$. Then

$$f_*(t) = f(t_*, \frac{t}{2}) = h(w - t_* + \frac{t}{2}, \delta + 2t_* + t) = h(w_{j0} + \frac{t}{2}, |I_{j0}| + t)$$

which implies $f_*(0) = h(I_{j0}) = h(I_{j1})$ and $f_*(|G_j| + |I_{j1}|) = h(I_j)$. Similarly, $f(0) = h(J_1)$.

Since $\frac{\partial}{\partial t_1} f(t_0, t_1)$ decreases as t_0 increases for all t_1 by assumption, we have $f_*'(t) \leq f'(t)$ for all t which in turn implies $f(t) - f(0) \geq f_*(t) - f_*(0)$ for all t.

Set $T = |G_j| + |J_{j1}|$. Then

$$\begin{aligned} f_*(T) - f_*(0) = h(I_j) - h(I_{j0}) \geq \\ q \cdot \nu(I_j) - r \cdot \nu(I_{j0}) = q \cdot \big(\nu(I_j) - \nu(I_{j0})\big) - (r-q) \cdot \nu(I_{j0}) = \\ q \cdot \nu(I_{j1}) - (r-q) \cdot \nu(I_{j1}) = \big(q - (r-q)\big) \cdot \nu(I_{j1}) \end{aligned} \tag{3}$$

Since $\frac{\partial^2 f}{\partial t_1^2} \leq 0$ and $\frac{\partial f}{\partial t_1} \geq$ by assumption, $f_*'(t)$ is positive and decreasing. Using this we get

$$\begin{aligned} f(\rho T) - f(0) = \int_0^{\rho T} f'(t)\, dt \geq \int_0^{\rho T} f_*'(t)\, dt \overset{f_*' decreasing}{\geq} \\ \rho \cdot \big(f_*(T) - f_*(0)\big) \overset{(3)}{\geq} \big(q - (r-q)\big) \cdot \rho \cdot \nu(I_{j1}) \end{aligned} \tag{4}$$

for any $\rho \in [0,1]$. Now fix $\rho \in [0,1]$ as the unique number such that $\rho T = |G_j| + |J_2|$. Then $\rho \cdot_L (G_j \cup I_{j1}) = G_j \cup J_2$. Using equation (1) we then get

$$\begin{aligned} h(I) = f(\rho) \overset{(4)}{\geq} f(0) + \big(q - (r-q)\big) \cdot \rho \cdot \nu(I_{j1}) = h(J_1) + \big(q - (r-q)\big) \cdot \rho \cdot \nu(I_{j1}) \overset{(1)}{\geq} \\ h(J_1) + \big(q - (r-q)\big) \cdot \nu(\rho \cdot_L (G_j \cup I_{j1})) = h(J_1) + \big(q - (r-q)\big) \cdot \nu(G_j \cup J_2) = \\ h(J_1) + \big(q - (r-q)\big) \cdot \nu(J_2) \end{aligned}$$

Since we can repeat this procedure with J_1 instead of I arbitrary many times and $h(I) \to 0$ as $|I| \to 0$ we can conclude that

$$h(I) \geq \big(q - (r-q)\big) \cdot \nu(I)$$

This proves the theorem. □

Remark 3.5*:* The symmetric theorem also holds, i.e. we can assume $\frac{\partial^2 f}{\partial w^2} \leq 0$ and $\rho \cdot \nu(I_{j0}) \geq \nu(\rho \cdot_R (G_j \cup I_{j0}))$ instead of assuming $\frac{\partial^2 f}{\partial t^2} \leq 0$ and $\rho \cdot \nu(I_{j1}) \geq \nu(\rho \cdot_L (G_j \cup I_{j1}))$.

Remark 3.6*:* When $h(w, \delta) = h(\delta)$, the conditions of theorem 3.4 is equivalent to h being increasing and concave.

The only assumption of theorem 3.4 which is not straightforward to verify given a Cantor set C is equation (1), which states that we must have

$$\rho \cdot \nu(I_{j1}) \geq \nu(\rho \cdot_L (G_j \cup I_{j1}))$$

for all long enough binary words j and all $\rho \in [0,1]$. Geometrically this mean that a translated copy of a part of the cumulative distribution function of ν must lie below a certain straight line. The following proposition simplifies the verification of this property.

Proposition 3.7: *Let $C \sim \{I_j\} \sim \{G_j\}$ be a Cantor set. Then the following claims are equivalent:*

(i) *For all long enough binary words j and all $\rho \in [0,1]$*

$$\nu\big(\rho \cdot_L (G_j \cup I_{j1})\big) \leq \rho \cdot \nu(I_{j1}) \tag{5}$$

(ii) *For all long enough binary words j and all $m \in \mathbb{N}$*

$$\frac{1}{2^m} \leq \frac{|G_j| + |I_{j10^m}|}{|G_j| + |I_{j1}|}$$

Proof. We first show that (i) implies (ii).

Let j be any binary word which is long enough for (i) to hold and let $m \in \mathbb{N}$. Set $\rho = \frac{|G_j|+|I_{j10^m}|}{|G_j|+|I_{j1}|}$ such that

$$\rho \cdot_L (G_j \cup I_{j1}) = \frac{|G_j| + |I_{j10^m}|}{|G_j| + |I_{j1}|} \cdot_L (G_j \cup I_{j1}) = G_j \cup I_{j10^m}$$

Then

$$\nu\left(\rho \cdot_L (G_j \cup I_{j1})\right) = \nu\left(G_j \cup I_{j10^m}\right) = \nu\left(I_{j10^m}\right) = \frac{1}{2^m}\nu\left(I_{j1}\right) \tag{6}$$

by the definition of the Cantor measure. Using (i) we get

$$\frac{1}{2^m}\,\nu\left(I_{j1}\right) \overset{(6)}{=} \nu\left(\rho \cdot_L (G_j \cup I_{j1})\right) \overset{(i)}{\leq} \rho \cdot \nu(I_{j1}) = \frac{|G_j| + |I_{j10^m}|}{|G_j| + |I_{j1}|} \cdot \nu(I_{j1})$$

Dividing by $\nu(I_{j1})$ gives us (ii).

We will now show that the reverse implication holds, i.e. that (ii) implies (i).

To show that (i) holds, we need to show that, given (ii), the graph in figure 2 corresponding to $\nu\big(\rho \cdot_L (G_j \cup I_{j1})\big)$ lies below the line $\rho \cdot \nu(I_{j1})$ for any large enough j.

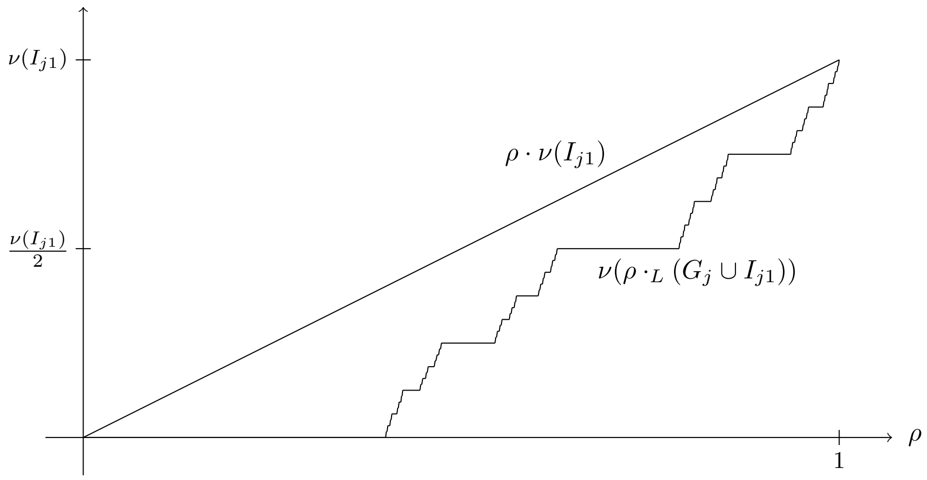


Figure 2: The setting for the last part of the proof of proposition 3.7.

To do this, fix any binary word j which is long enough for (ii) to hold and pick any $\rho_0 \in [0,1]$. Then there is a unique point $w \in G_j \cup I_{j1}$ such that w is the right endpoint of $\rho_0 \cdot_L (G_j \cup I_{j1})$.

If $\rho = 0$, then both sides of equation (5) are equal to zero. If $\rho = 1$, then both sides of equation (5) equal $\nu(I_{j1})$. If $w \in G_j$, then the left hand side of equation (5) is zero and the right hand side is positive. Thus in all these three cases, equation (5) holds, and we can thus assume $w \in I_{j1}^{\circ}$.

We will consider three different cases which together cover all remaining possibilities:

1. $w = b_{j1l}$ for some binary word l
2. $w \in G_{j1l}$ for some binary word l
3. $w = \lim_{k\to\infty} b_{j1(l|k)}$ for some binary sequence l

We will begin by dealing with the last two cases by showing that (ii) implies (i) in these cases given that (ii) implies (i) in the first case, and then end by showing that (ii) implies (i) in the first case:

Case 2: In this case we have $w \in G_{j1l}$ for some binary word l. Then b_{j1l0} is the point in C lying closest to w to the left. Let ρ_{j1l0} be the unique point in $[0, \rho_0]$ such that the right endpoint of $\rho_{j1l0} \cdot_L (G_j \cup I_{j1})$ is b_{j1l0}. By the first case;

$$\nu\left(\rho_{j1l0} \cdot_L (G_j \cup I_{j1})\right) \leq \rho_{j1l0} \cdot \nu(I_{j1})$$

Since the right hand side of equation (5) is constant for ρ between ρ_{j1l0} and ρ_0 and the left hand side of equation (5) is increasing in ρ, we get

$$\nu\left(\rho_0 \cdot_L (G_j \cup I_{j1})\right) = \nu\left(\rho_{j1l0} \cdot_L (G_j \cup I_{j1})\right) \leq \rho_{j1l0} \cdot \nu\left(I_{j1}\right) \leq \rho_0 \cdot \nu(I_{j1})$$

Thus the second case follows from the first case.

Case 3: Let $\{b_{j1(l|k)}\}_{k=1,2,3,...}$ be any sequence with $b_{j1(l|k)} \to w$ as $k \to \infty$. Let $\rho_{j1(l|k)}$ be the unique point in $[0,1]$ such that $b_{j1(l|k)}$ is the right endpoint of $\rho_{j1(l|k)} \cdot_L (G_j \cup I_{j1})$. Then by the first case, for all $k \in \mathbb{N}\backslash\{0\}$, we have

$$\nu\left(\rho_{j1(l|k)} \cdot_L (G_j \cup I_{j1})\right) \leq \rho_{j1(l|k)} \cdot \nu(I_{j1}) \tag{7}$$

As $\rho_{j1(l|k)} \to \rho_0$ as $k \to \infty$ and both sides of equation (7) are continuous in ρ, we get

$$\nu\big(\rho_0 \cdot_L (G_j \cup I_{j1})\big) \leq \rho_0 \cdot \nu(I_{j1})$$

We now only need to show that (i) follows from (ii) in the first case.

During the rest of the proof we will use $\rho_{j_1}^{(j_2)}$ to denote the unique number in $[0,1]$ for which b_{j_1} is the right endpoint of $\rho_{j_1}^{(j_2)} \cdot_L (G_{j_2} \cup I_{j_2 1})$ for any binary words j_1 and j_2 where $j_2 = j_1|k$ for some $k \in \mathbb{N}$.

Case 1: Now again let j be a fixed binary word. Let $\hat{l}$ be any binary word and consider $\rho = \rho_{j1\hat{l}}^{(j)}$. If $\hat{l}$ is the empty word then $\rho = 1$ and we get equality in equation (7), so we can assume $\hat{l} \neq \emptyset$.

Since $b_{j1\hat{l}} = b_{j1\hat{l}1^k}$ for any $k \in \mathbb{N}$ and any binary word $\hat{l}$ we can assume that $\hat{l}$ ends with at least one zero and write $\hat{l} = l0^k$ for some binary word l which ends with a one and some $k \in \mathbb{N}$. We will now use induction on the length of l to show that equation (5) holds for $\rho_{j1l0^k}^{(j)}$ for any binary word l which is either empty or ends with a one and any $k \in \mathbb{N}\backslash\{0\}$.

To finish the proof in this case, and thus to finish the theorem, we need to show that

$$\nu\left(\rho_{j1l0^k}^{(j)} \cdot_L (G_j \cup I_{j1})\right) \leq \rho_{j1l0^k}^{(j)} \cdot \nu(I_{j1}) \tag{8}$$

for any long enough binary word j, any binary word k ending with a one and any $k \in \mathbb{N}$.

Suppose first that $l = \emptyset$ so that $|l| = 0$. Then by equation (8) we have

$$\frac{1}{2^k} \leq \frac{|G_j| + |I_{j10^k}|}{|G_j| + |I_{j1}|} = \rho_{j10^k}^{(j)}$$

and thus

$$\nu\left(\rho_{j10^k}^{(j)} \cdot (G_j \cup I_{j1})\right) = \nu\left(G_j \cup I_{j10^k}\right) = \nu\left(I_{j10^k}\right) = \frac{1}{2^k} \cdot \nu\left(I_{j1}\right) \overset{(ii)}{\leq} \rho_{j10^k}^{(j)} \cdot \nu\left(I_{j1}\right)$$

i.e. equation (7) holds.

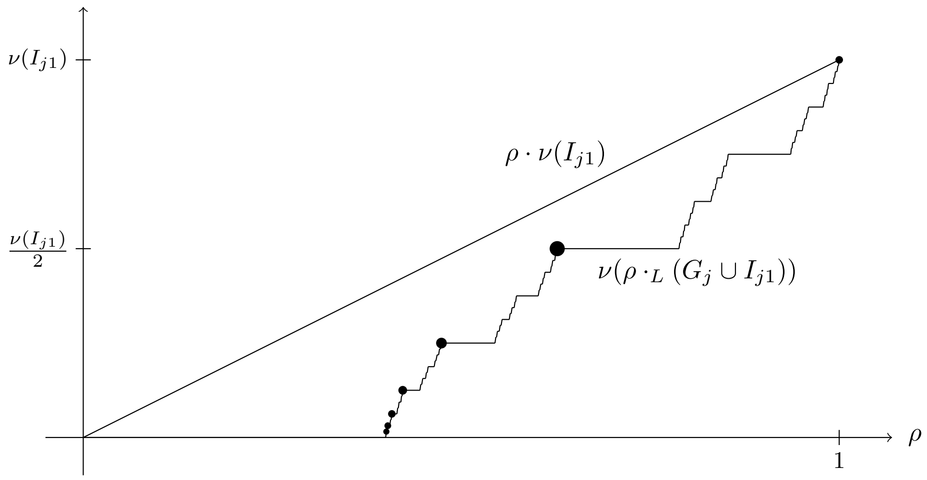


Figure 3: The black points $\big(\rho, \nu(\rho \cdot_L (G_j \cup I_{j1}))\big)$, where $\rho = \rho_{jl0^k}$ for some $k \in \mathbb{N}\backslash\{0\}$ and $|l| = 0$, are the points first considered in the first part of the proof of the first case. Here k increases when we move from one black point to any point to the left of it.

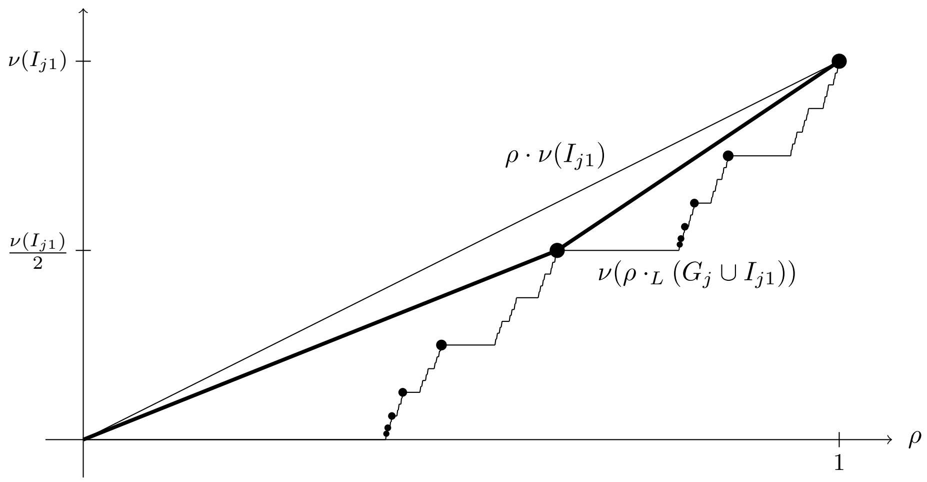


Figure 4: The figure above, together with figure 5, shows the basic idea of the rest of the proof; the same arguments which show that the black points in figure 3 lie below the straigh line shows that the black points in this figure lies below the bold line. As both endpoints of this line lies below the thinner line by the previous step (induction in general, and the case $|l| = 0$ in this particular case), all the black points lie below the thin black line.

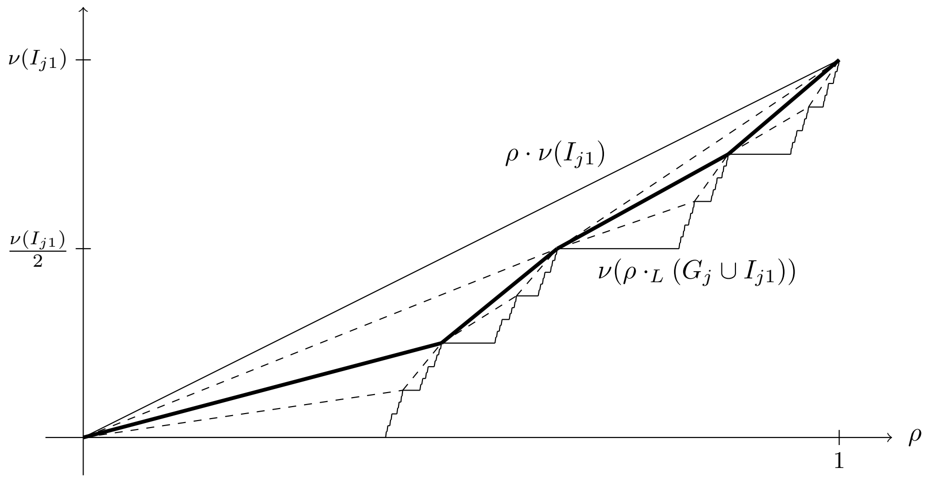


Figure 5: This figure shows how the induction progresses through all points considered in the first case; by showing that a certian subset of the points lies below line segments between points which by the induction assumption lies below the topmost line. The dashed line above the bold line shows the previous step in the induction (in this case; $|l| = 1$), the bold line the current step ($|l| = 0$) and the dashed line below the bold line the next step ($|l| = 2$).

Now instead suppose that $|l| = m$ and that the equation (7) is true for all $k \in \mathbb{N}\backslash\{0\}$ when $|l| < m$. Set $j' = j1(l|m-1)$ so that $j1l0^k = j'10^k$. Then by the case $|l| = 0$ we get

$$\nu\left(\rho^{(j')}_{j'10^k} \cdot_L (G_{j'} \cup I_{j'1})\right) \leq \rho^{(j')}_{j'10^k} \cdot \nu(I_{j'1}) \tag{9}$$

By adding $\nu([a_{j1}, b_{j1l0}])$ to both sides of equation (9) we get

$$\nu\left([a_{j1}, b_{j'0}]\right) + \nu\left(\rho^{(j')}_{j'10^k} \cdot_L (G_{j'} \cup I_{j'1})\right) \leq \nu\left([a_{j1}, b_{j'0}]\right) + \rho^{(j')}_{j'10^k} \cdot \nu\left(I_{j'1}\right) \tag{10}$$

The left hand side of equation (10) can be rewritten as

$$\begin{aligned} \nu\left([a_{j1}, b_{j'0}]\right) + \nu\left(\rho^{(j')}_{j'10^k} \cdot_L (G_{j'} \cup I_{j'1})\right) = \nu\left([a_{j1}, b_{j'0}]\right) + \nu\left([b_{j'0}, b_{j'10^k}]\right) = \\ \nu\left([a_{j1}, b_{j1l0^k}]\right) = \nu\left([b_{j0}, b_{j1l0^k}]\right) = \nu\left(\rho^{(j)}_{j1l0^k} \cdot_L (G_j \cup I_{j1})\right) \end{aligned} \tag{11}$$

The right hand side of equation (10) is a point on the line segment between the two points

$$\left(\rho^{(j)}_{j1(l|-1)0},\ \nu\left(\rho^{(j)}_{j1(l|-1)0} \cdot_L (G_j \cup I_{j1})\right)\right)$$

and

$$\left(\rho^{(j)}_{j1l},\ \nu\left(\rho^{(j)}_{j1l} \cdot_L (G_j \cup I_{j1})\right)\right)$$

Since the binary word l ends with a 1, the last of these points can also be written as

$$\Big(\rho^{(j)}_{j1(l|-1)},\ \nu\big(\rho^{(j)}_{j1(l|-1)} \cdot_L (G_j \cup I_{j1})\big)\Big)$$

Both end points of the line segment are thus points on the graph of $\nu\left(\rho \cdot_L (G_j \cup I_{j1})\right)$, which lie below the line $\rho \cdot \nu\left(I_{j1}\right)$ by induction since $|(l| - 1)| = |l| - 1 < m$. Thus all points on this line must also lie below the line $\rho \cdot \nu(I_{j1})$, which implies

$$\nu([a_{j1}, b_{j'0}]) + \rho^{(j')}_{j'10^k} \cdot \nu(I_{j'1}) \leq \rho^{(j)}_{j1l0^k} \cdot \nu(I_{j1}) \tag{12}$$

Combining equation (10), equation (11) and equation (12) gives

$$\nu(\rho^{(j)}_{j1l0^k} \cdot_L (G_j \cup I_{j1})) \overset{(11)}{=} \nu([a_{j1}, b_{j'0}]) + \nu(\rho^{(j')}_{j'10^k} \cdot_L (G_{j'} \cup I_{j'1})) \overset{(10)}{\leq}$$

$$\nu([a_{j1}, b_{j'0}]) + \rho^{(j')}_{j'10^k} \cdot \nu(I_{j'1}) \overset{(12)}{\leq} \rho^{(j)}_{j1l0^k} \cdot \nu(I_{j1})$$

As this finishes the proof in the first case, we have proved the theorem. □

3.5 Examples

We will end this section with a few examples which shows the usefulness of theorem 3.4 by calculating the exact measure of some Cantor sets studied in [8] and [2] and for which the measure (to the authors knowledge) was previously unknown.

Example 3.8: Consider the Cantor set $C_{(p)}$ associated with the sequence of gap lengths

$$|G^k_l| = \frac{1}{(2^k + l)^p}$$

where p is any real number which is strictly larger than one.

In [2] (theorem 1.1), Cabrielli, Molter, Paulauskas and Shonkwiler showed that

$$\frac{1}{8}\left(\frac{2^p}{2^p - 2}\right)^{1/p} \leq m_{1/p}(C_{(p)}) \leq \left(\frac{1}{p-1}\right)^{1/p}$$

We will show that we by using theorem 3.4 can compute the exact value of $m_{1/p}(C_{(p)})$ for any $p > 1$. To do this we will need the following result from [2].

$$\frac{2^p}{2^p - 2} \cdot \left(\frac{1}{2^k + l + 1}\right)^p \leq |I^k_l| \leq \frac{2^p}{2^p - 2} \cdot \left(\frac{1}{2^k + l}\right)^p \tag{13}$$

Let I^k_l and $I^{k'}_{l'}$ be any two basic intervals associated with $C_{(p)}$ with $I^{k'}_{l'} \subseteq I^k_l$. Then $l' \geq l \cdot 2^{k'-k}$

and thus

$$\frac{l'}{2^{k'}} \geq \frac{l}{2^k} \tag{14}$$

Let $h(w,\delta) = \delta^{1/p}$. Then

$$h(I_{l'}^{k'}) = |I_{l'}^{k'}|^{1/p} \overset{(13)}{\leq} \frac{2}{(2^p-2)^{1/p}} \cdot \frac{1}{2^{k'}+l'} = \frac{2}{(2^p-2)^{1/p}} \cdot \frac{1}{1+\frac{l'}{2^{k'}}} \cdot \frac{1}{2^{k'}} =$$

$$\frac{2}{(2^p-2)^{1/p}} \cdot \frac{1}{1+\frac{l'}{2^{k'}}} \cdot \nu(I_{l'}^{k'}) \overset{(14)}{\leq} \frac{2}{(2^p-2)^{1/p}} \cdot \frac{1}{1+\frac{l}{2^k}} \cdot \nu\left(I_{l'}^{k'}\right)$$

Completely analogously, we get the lower limit

$$\frac{2}{(2^p-2)^{1/p}} \cdot \frac{1}{1+\frac{l+1}{2^k}} \cdot \nu(I_{l'}^{k'}) \leq h(I_{l'}^{k'})$$

We thus have

$$\frac{2}{(2^p-2)^{1/p}} \cdot \frac{1}{1+\frac{l+1}{2^k}} \cdot \nu(I_{l'}^{k'}) \leq h(I_{l'}^{k'}) \leq \frac{2}{(2^p-2)^{1/p}} \cdot \frac{1}{1+\frac{l}{2^k}} \cdot \nu(I_{l'}^{k'}) \tag{15}$$

We will now calculate a lower bound for $\frac{|G_j|+|I_{j10^m}|}{|G_j|+|I_{j1}|}$ for all long enough binary words j and all $m \in \mathbb{N}\backslash\{0\}$. To do this, fix any long enough binary word j. Then there exists $l,k \in \mathbb{N}$ such that $I_j = I_l^k$. Using this, we get

$$\frac{|G_j|+|I_{j10^m}|}{|G_j|+|I_{j1}|} = \frac{|G_l^k|+|I_{2^m(2l+1)}^{k+m+1}|}{|G_l^k|+|I_{2l+1}^{k+1}|} \overset{(15)}{\geq} \frac{\frac{1}{(2^k+l)^p} + \frac{2^p}{(2^p-2)} \cdot \frac{1}{(2^{k+m+1}+2^m(2l+1)+1)^p}}{\frac{1}{(2^k+l)^p} + \frac{2^p}{(2^p-2)} \cdot \frac{1}{(2^{k+1}+(2l+1))^p}} =$$

$$\frac{(2^p-2) + 2^p \cdot \frac{(2^k+l)^p}{(2^k+l+\frac{2^m+1}{2^{m+1}})^p} \cdot \left(\frac{1}{2^{m+1}}\right)^p}{(2^p-2) + 2^p \cdot \frac{(2^k+l)^p}{(2^k+l+\frac{1}{2})^p} \cdot \left(\frac{1}{2}\right)^p} = \frac{(2^p-2) + \frac{(2^k+l)^p}{(2^k+l+\frac{2^m+1}{2^{m+1}})^p} \cdot \left(\frac{1}{2^m}\right)^p}{(2^p-2) + \frac{(2^k+l)^p}{(2^k+l+\frac{1}{2})^p}} >$$

$$\frac{(2^p-2)+(1-\varepsilon)\cdot\left(\frac{1}{2^m}\right)^p}{(2^p-2)+(1-\varepsilon)} = \frac{(2^p-2)+(1-\varepsilon)\cdot\frac{1}{2^{pm}}}{(2^p-2)+(1-\varepsilon)} = 1-(1-\varepsilon)\cdot\frac{1-\frac{1}{2^{pm}}}{(2^p-2)+(1-\varepsilon)}$$

As $p \to 1$, this expressions tends to $\frac{1}{2^m}$. To show that $\frac{|G_j|+|I_{j10^m}|}{|G_j|+|I_{j1}|} \geq 2^{-m}$ for all $p > 1$, it would be enough to show that the last expression in example 3.8 is increasing in p.

To simplify notations somewhat, set $x = 2^p$ and define

$$g(x) = 1-(1-\varepsilon)\cdot\frac{1-x^{-m}}{x-1-\varepsilon}$$

Then

$$g'(x) = -(1-\varepsilon)\cdot\left(\frac{mx^{-(m+1)}}{x-1-\varepsilon} - \frac{1-x^{-m}}{(x-1-\varepsilon)^2}\right) =$$
$$-\frac{(1-\varepsilon)}{(x-1-\varepsilon)^2}\cdot\left(mx^{-(m+1)}(x-1-\varepsilon)-1+x^{-m}\right) =$$
$$\frac{(1-\varepsilon)}{(x-1-\varepsilon)^2 x^{m+1}}\cdot\left(x^{m+1}-(m+1)x+m(1+\varepsilon)\right)$$

which is positive since $x > 2$. We can thus conclude that $\frac{|G_j|+|I_{j10^m}|}{|G_j|+|I_{j1}|} \geq 2^{-m}$ for all $p > 1$.

Now set $f(w,t) = h(w - t_0 + t_1, \delta + 2t_0 + 2t_1) = (\delta + 2t_0 + 2t_1)^{1/p}$. Then clearly $\frac{\partial f}{\partial t_1} \geq 0$, $\frac{\partial f}{\partial t_0} \geq 0$, $\frac{\partial^2 f}{\partial t_1^2} \leq 0$ and $\frac{\partial^2 f}{\partial t_0 \partial t_1} \leq 0$ for all t_0 and t_1.

Using the properties now shown to hold for $C^{(p)}$ we are now, according to by proposition 3.7, set up to use theorem 3.4. Using theorem 3.4 we get

$$m_{1/p}(C_{(p)}) = \sum_{l=0}^{2^k-1} m_{1/p}(C_{(p)}\cap I_l^k) \overset{(15)}{\leq}$$
$$\sum_{l=0}^{2^k-1}\frac{2}{(2^p-2)^{1/p}}\cdot\frac{1}{1+\frac{l}{2^k}}\cdot\nu(I_l^k) = \frac{2}{(2^p-2)^{1/p}}\cdot\sum_{l=0}^{2^k-1}\frac{1}{1+\frac{l}{2^k}}\cdot\frac{1}{2^k}$$

Since this is true for all k, and

$$\lim_{k\to\infty}\sum_{l=0}^{2^k-1}\frac{1}{1+\frac{l}{2^k}}\cdot\frac{1}{2^k} = \int_0^1\frac{dx}{1+x} = [\log(1+x)]_0^1 = \log 2$$

we get

$$m_{1/p}(C_{(p)}) \leq \frac{2\log 2}{(2^p-2)^{1/p}} \tag{16}$$

Similarly for the lower limit;

$$m_{1/p}(C_{(p)}) = \sum_{l=0}^{2^k-1} m_{1/p}\left(C_{(p)}\cap I_l^k\right) \overset{(15)}{\geq}$$
$$\sum_{l=0}^{2^k-1}\frac{2}{(2^p-2)^{1/p}}\cdot\left(\frac{1}{1+\frac{l+1}{2^k}}-\left(\frac{1}{1+\frac{l}{2^k}}-\frac{1}{1+\frac{l+1}{2^k}}\right)\right)\cdot\nu(I_l^k) \geq$$
$$\frac{2}{(2^p-2)^{1/p}}\cdot\sum_{l=0}^{2^k-1}\left(\frac{1}{1+\frac{l+1}{2^k}}-\frac{1}{2^k}\right)\cdot\frac{1}{2^k} = \tag{17}$$
$$\frac{2}{(2^p-2)^{1/p}}\cdot\left(\sum_{l=0}^{2^k-1}\frac{1}{1+\frac{l+1}{2^k}}\cdot\frac{1}{2^k}-\frac{1}{2^k}\right)$$

which tends to $\dfrac{2}{(2^p-2)^{1/p}} \cdot \int_0^1 \dfrac{1}{1+x}dx = \dfrac{2\log 2}{(2^p-2)^{1/p}}$ as $k \to \infty$. This gives the lower limit

$$m_{1/p}(C_{(p)}) \geq \frac{2\log 2}{(2^p-2)^{1/p}} \tag{18}$$

By combining equation (16) and equation (18) we can conclude that

$$m_{1/p}(C_{(p)}) = \frac{2\log 2}{(2^p-2)^{1/p}}$$

Example 3.9: As a small variation of the Cantor sets $C_{(p)}$ where $p > 1$, we can consider the Cantor set $C_{(p,x)}$, where $p > 1$ and $x > 2$, associated with the sequence of gap lengths $|G_l^k| = \frac{1}{([x^k]+l)^p}$. This set was also studied in [2] where Cabrielli, Molter, Mendevil, Paulauskas and Shonkwiler gave the bounds

$$c \cdot \left(\frac{x^p}{x^p-2}\right)^{1/p} \leq m_{1/p}(C_{(p,x)}) \leq \left(\frac{4^p}{2^p-2}\right)^{\frac{\log 2}{p\log x}}$$

where c is some constant depending on p and x. We will calculate the measure of $C_{(p,x)}$ using theorem 3.4.

$$|I_l^k| = \sum_{h=0}^{\infty}\sum_{j=0}^{2^h-1}\left|G_{2^h l+j}^{k+h}\right| \leq \sum_{0}^{\infty}\frac{2^h}{(\lfloor x^{k+h}\rfloor + l\cdot 2^h)^p} \leq \frac{1}{x^{kp}}\cdot\sum_{h=0}^{\infty}\frac{2^h}{\left(x^h-\frac{1}{x^k}\right)^p} \leq$$
$$\frac{1}{x^{kp}}\cdot(1+\varepsilon_k^{(1)})\cdot\sum_{h=0}^{\infty}\frac{2^h}{x^{ph}} = \frac{1}{x^{kp}}\cdot\left(1+\varepsilon_k^{(1)}\right)\cdot\frac{1}{1-\frac{2}{x^p}}$$

where $\varepsilon_k^{(1)}$ is a small positive number which tends to zero as $k \to \infty$.

Similarly, but by somewhat more tedious calculations, we get

$$|I_l^k| = \sum_{h=0}^{\infty}\sum_{j=0}^{2^h-1}\left|G_{2^h l+j}^{k+h}\right| \geq \sum_{h=0}^{\infty}\frac{2^h}{(\lfloor x^{k+h}\rfloor + l\cdot 2^h + 2^h - 1)^p} \geq \sum_{h=0}^{\infty}\frac{2^h}{(x^{k+h}+(l+1)\cdot 2^h)^p} \geq$$
$$\frac{1}{x^{kp}}\cdot\sum_{h=0}^{\infty}\frac{2^h}{x^{hp}\cdot\left(1+\frac{l+1}{x^k}\cdot\frac{2^h}{x^h}\right)^p} \geq \frac{1}{x^{kp}}\cdot\sum_{h=0}^{\infty}\frac{2^h}{x^{hp}\cdot\left(1+\frac{2^k}{x^k}\cdot\frac{2^h}{x^h}\right)^p} \geq$$
$$\frac{1}{x^{kp}}\cdot\sum_{h=0}^{\infty}\frac{2^h}{x^{hp}}\cdot\frac{1}{\left(1+\frac{2^k}{x^k}\right)^p} \geq \frac{1}{x^{kp}}\cdot\frac{1}{1-\frac{2}{x^p}}\cdot\frac{1}{\left(1+\frac{2^k}{x^k}\right)^p} = \frac{1}{x^{kp}}\cdot\frac{1}{1-\frac{2}{x^p}}\cdot\left(1-\varepsilon_k^{(2)}\right)$$

where $\varepsilon_k^{(2)}$ is a small positive number which tends to zero as $k \to \infty$.

To verify that $\dfrac{|G_j|+|I_{j10^m}|}{|G_j|+|I_{j1}|} = \dfrac{|G_l^k|+|I_{2^m(2l+1)}^{k+m+1}|}{|G_l^k|+|I_{2l+1}^{k+1}|} \geq \dfrac{1}{2^m}$ for all large enough $|j|$ and all $m \in \mathbb{N}\backslash\{0\}$ (or equivalently for all large enough k) requires a calculation very similar to the analogous calculation of the previous example and is therefore omitted here.

Set $h(w,\delta) = \delta^{\frac{\log 2}{p \log x}}$. Then

$$\left(\left(1-\varepsilon_k^{(2)}\right)\cdot\frac{1}{1-\frac{2}{x^p}}\right)^{\frac{\log 2}{p \log x}}\cdot\nu(I_l^k)\le h(I_l^k)\le\left(\left(1+\varepsilon_k^{(1)}\right)\cdot\frac{1}{1-\frac{2}{x^p}}\right)^{\frac{\log 2}{p \log x}}\cdot\nu(I_l^k)$$

Using theorem 3.4 we thus get

$$\left(\left(1-\varepsilon_k^{(2)}\right)^{\frac{\log 2}{p \log x}}-\left(\left(1+\varepsilon_k^{(1)}\right)^{\frac{\log 2}{p \log x}}-\left(1-\varepsilon_k^{(2)}\right)^{\frac{\log 2}{p \log x}}\right)\right)\cdot\left(\frac{1}{1-\frac{2}{x^p}}\right)^{\frac{\log 2}{p \log x}}\le m_{\frac{\log 2}{p \log x}}(C_{(p,x)})$$

and

$$m_{\frac{\log 2}{p \log x}}(C_{(p,x)})\le\left(1+\varepsilon_k^{(1)}\right)^{\frac{\log 2}{p \log x}}\cdot\left(\frac{1}{1-\frac{2}{x^p}}\right)^{\frac{\log 2}{p \log x}}$$

By letting k tend to infinity in these both equations, we get

$$m_{\frac{\log 2}{p \log x}}(C_{(p,x)})=\left(\frac{1}{1-\frac{2}{x^p}}\right)^{\frac{\log 2}{p \log x}}$$

The method for calculating the Hausdorff measure of a given Cantor set used above can with small modifications be used also to calculate the measure of the third and last Cantor set mentioned in [2], namely the Cantor set $C_{(p)}^{(n)} \sim \{G_l^k\}$, where $|G_l^k| = \frac{1}{(2^k+l)^p}$ but where $(n-1)$ open intervals are removed from each remaining interval in each step of the construction of the Cantor set instead of one. Small adjustments to theorem 3.4 and its proof and similar calculations as the calculations in example 3.8 and example 3.9, although omitted here, gives

$$m_{1/p}(C_{(p)}^{(n)})=\frac{n\log n}{(n^p-n)^{1/p}}\cdot\frac{1}{n-1}$$

4 Cantor sets associated with test functions of exponential type

In this section we will consider Hausdorff measures associated with test functions of the form $h(w,\delta) = \delta^{\alpha(w)}$. This case is interesting since in provides a direct analogy to the well studied case $h(\delta) = \delta^{\alpha}$ where α is the Hausdorff dimension of any set E for which $m_\alpha(E)$ is finite and non-zero. The function $\alpha(w)$ in the exponent of the test function $h(w,\delta) = \delta^{\alpha(w)}$ will due to this analogy be called the *dimension function* of the Hausdorff measure $m_{\alpha(w)}$.

We will start this section by giving conditions on $\alpha(w)$ which guarantees that $\mathcal{C}_h = \mathcal{C}_{\alpha(w)}$, as defined by definition 3.1, is non-empty. We will then show that when h fulfils some continuity conditions we have that μ_h is a mass distribution on all $C \in \mathcal{C}_{\alpha(w)}$ and the restriction of $m_{\alpha(w)}$ to C is the Cantor measure on C for any $C \in \mathcal{C}_{\alpha(w)}$.

4.1 An existence result

To be able to state our existence result we first need to show that the test function, when seen as an interval function, is increasing.

Lemma 4.1: *Let $\alpha(w) : [0,1] \to (0,1)$ be a continuously differentiable function. Then the interval function $h(I) = h(w,\delta) = \delta^{\alpha(w)}$ is increasing for all small enough δ.*

Proof. Let $I = I(w_*, \delta) \subseteq [0,1]$ be any interval of length δ with midpoint w_*.

Define $f_0(t_0) = h(w_* - t_0, \delta + 2t_0)$ and $f_1(t_1) = h(w + t_1, \delta + 2t_1)$. To show that the $h(I) = \delta^{\alpha(w)}$ is increasing it is enough to show that the two functions $f_0(t)$ and $f_1(t)$ are increasing in t for all fixed w_* and δ when t and δ is small enough.

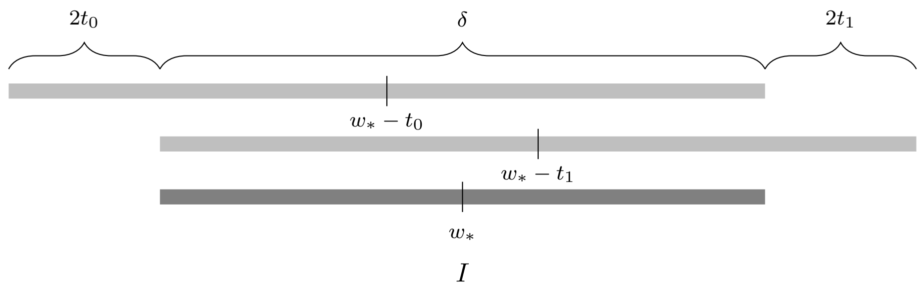


Figure 6: The setting for the proof of lemma 4.1

To show that $f_1(t)$ is increasing we will show that $f_1'(t) \geq 0$.

$$\begin{aligned} f_1'(t) = f(t)\left(\alpha'(w_* + t)\log(\delta + 2t) + \frac{2\alpha(w_*)}{\delta + 2t}\right) = \\ \frac{f(t)}{\delta + 2t} \cdot (\alpha'(w_* + t) \cdot (\delta + 2t)\log(\delta + 2t) + 2\alpha(w_* + t)) \end{aligned} \tag{19}$$

Since $\alpha'(w)$ is continuous on a compact interval, $\alpha'(w)$ is bounded. Similarly since $\alpha(w)$ is strictly positive and continuously differentiable on a compact interval, it is uniformly bounded away from zero.

As $(\delta + 2t)\log(\delta + 2t) \to 0$ as $\delta + 2t \to 0$ and $\alpha'(w)$ is bounded, the first term in equation (19) can be made arbitrarily small by chosing δ and t small enough. Since $\alpha(w)$ is uniformly bounded away from zero, we can choose δ and t small enough to have $f_1'(t) \geq 0$ for all w_* and δ. Thus $f_1(t)$ is increasing for small δ and t.

The proof that $f_2(t)$ is increasing with t for small t and δ is completely analogous and is therefore omitted here. □

We can now proceed to the theorem which shows that $\mathcal{C}_{\alpha(w)}$ is nonempty if h is sufficiently nice.

Theorem 4.2: *Let $\alpha(w) : [0,1] \to (0,1)$ be continuously differentiable and set $h(w,\delta) = \delta^{\alpha(w)}$. Then $\mathcal{C}_{\alpha(w)}$ is non-empty, i.e. there exists at least one Cantor set $C \sim \{I_j\}$ such that $h(I_j) = \nu(I_j)$ for all large enough $|j|$.*

Proof. Since $\alpha(w)$ is continuously differentiable and $[0,1]$ is compact, $\alpha(w)$ is Lipschitz continuous on $[0,1]$, i.e. there exists a constant $\lambda > 0$ such that $|\alpha(w_1) - \alpha(w_2)| \leq \lambda|w_1 - w_2|$ for all $w_1, w_2 \in [0,1]$. Pick Δ small enough to have

$$(2x)^{\lambda x} > 2^{\max \alpha(w) - 1} \text{ for all } x \leq \Delta \tag{20}$$

Since the expression of the left hand side tends to one as $\Delta \to 0$ and the right hand side is strictly smaller than one and does not depend on x, such Δ exists. We can assume that Δ is small enough to imply that $h(\delta)$ is increasing for all $\delta < \Delta$.

Let $n \in \mathbb{N}$ be such that $|j| \geq n$ implies $4|I_j| < \Delta$. Since

$$2^n \cdot h(w, \frac{1}{2^n}) = \frac{2^n}{2^{\alpha(w)n}} > 1 \text{ for all } w \in [0,1] \text{ and } n \in \mathbb{N}$$

we can pick the basic intervals $\{I_j\}_{|j|=n}$ at level n as any 2^n disjunct intervals in $[0,1]$ satisfying $h(I_j) = 2^{-n}$.

For any $k \in \mathbb{N}$ with $k < n$, we define the basic intervals I_j with $|j| = k$ by setting $I_j = [a_{j0}, b_{j1}]$, where a_{j0} is the left endpoint of I_{j0} and b_{j1} is the right endpoint of I_{j1}. In this way, all basic intervals I_j with $|j| \leq n$ are defined.

We will now show that for any binary word j with $|j| \geq n$ we can pick two smaller disjoint intervals I_{j0} and I_{j1} contained in I_j such that the left endpoint of I_j and I_{j0} coincide and the right endpoint of I_j and the right endpoint of I_{j1} coincide and $2h(I_{j0}) = 2h(I_{j1}) = h(I_j)$. Given this last condition it follows by induction that $h(I_{j0}) = \frac{1}{2} \cdot h(I_j) = 2^{-(|j|+1)} = 2^{-|j0|}$ and analogously that $h(I_{j1}) = \frac{1}{2} \cdot h(I_j) = 2^{-(|j|+1)} = 2^{-|j1|}$, and therefore that $h(I_j) = 2^{-|j|}$ for any $|j| > n$. This implies the existence of at least one set $C \in \mathcal{C}_{\alpha(w)}$. By the definition of the Cantor measure, we get $\nu(I_j) = 2^{-|j|} = h(I_j)$ for the Cantor measure ν associated with C.

Thus let $I_j = I(w_j, \delta_j)$ be any basic interval already chosen with $|j| \geq n$. By the choice of n we then have $|I_j| < \Delta$, or equivalently, that $\delta_j < \Delta$.

Let $\tilde{I}_{j0}$ be the left half of I_j, i.e. set

$$\tilde{I}_{j0} = \frac{1}{2} \cdot_L I_j = I(w_j - \frac{\delta_j}{4}, \frac{\delta_j}{2}) = I(w_0, 2\delta)$$

for $w_0 = w_j - \frac{\delta_j}{4}$.

Since $\delta_j < \Delta$ we also have $\frac{\delta_j}{2} < \Delta$ and thus equation (20) implies

$$\left(\frac{\delta_j}{2}\right)^{\alpha(w_0)-\alpha(w_j)} \geq \left(\frac{\delta_j}{2}\right)^{\lambda \frac{\delta_j}{2}} > 2^{\max \alpha(w)-1} > 2^{\alpha(w_j)-1} \tag{21}$$

Using this, we get

$$2h(\tilde{I}_{j0}) = 2 \cdot h(w_0, \frac{\delta_j}{2}) = 2 \cdot \left(\frac{\delta_j}{2}\right)^{\alpha(w_0)} = 2 \cdot \left(\frac{\delta_j}{2}\right)^{\alpha(w_0)-\alpha(w_j)} \cdot \left(\frac{\delta_j}{2}\right)^{\alpha(w_j)} \overset{21}{\geq}$$
$$2 \cdot 2^{\alpha(w_j)-1} \cdot \left(\frac{\delta_j}{2}\right)^{\alpha(w_j)} = (\delta_j)^{\alpha(w_j)} = h(I_j)$$

Since h is continuous and increasing, there exists an interval $I_{j0} \subseteq \tilde{I}_{j0}$ with left endpoint in common with $\tilde{I}_{j0}$ (and thus also in common with I_j) such that $2 \cdot h(I_{j0}) = h(I_j)$. By completely analogous arguments we can find an interval $I_{j1} \subset I_j$ with right endpoint in common with I_j such that $2 \cdot h(I_{j1}) = h(I_j)$. Since $2|I_{j0}| < |I_j|$ and $2|I_{j1}| < |I_j|$ we have $I_{j0} \cap I_{j1} = \emptyset$.

Since this construction works for all binary words j with $|j| \geq n$, thus inductively defines a Cantor set $C \sim \{I_j\}$ for which $h(I_j) = \nu(I_j)$ for all $|j| \geq n$. This completes the proof. □

4.2 Mass distribution and Cantor measure

We will now state and prove a couple of theorems which provide conditions given which $m_{\alpha(w)}$ is a mass distribution on $C \in \mathcal{C}_{\alpha(w)}$. We will also give conditions given which $m_{\alpha(w)}|_C \equiv \nu_C$. All theorems in this subsection are consequences of the corresponding theorems in the previous section, and the majority of the content of this section are therefore arguments showing that the needed assumptions hold when $h(w, \delta) = \delta^{\alpha(w)}$ for some sufficiently nice dimension function $\alpha(w)$.

Theorem 4.3: *Let $\alpha(w) : [0, 1] \to (0, 1)$ be a continuously differentiable function. Then $\mathcal{C}_{\alpha(w)}$ is non-empty and $m_{\alpha(w)}$ is a mass distribution on all sets $C \in \mathcal{C}_{\alpha(w)}$. Further, for any interval $I \subseteq [0, 1]$*

$$0.25 \cdot \nu(I) \leq m_{\alpha(w)}(C \cap I) \leq \nu(I)$$

Proof. By theorem 4.2, $\mathcal{C}_{\alpha(w)}$ is non-empty. Moreover, as

$$h(w, 2\delta) = (2\delta)^{\alpha(w)} = 2^{\alpha(w)} \cdot \delta^{\alpha(w)} = 2^{\alpha(w)} \cdot h(w, \delta) \leq 2 \cdot h(w, \delta)$$

the doubling constant of h is smaller than or equal to 2. The theorem thus follows by theorem 3.3. □

Theorem 4.4: *Let $\alpha(w) \in C^2([0, 1], (0, 1))$ be increasing. Then $\mathcal{C}_{\alpha(w)}$ is non-empty and the assumptions of theorem 3.4 is fulfilled for the test function $h(w, \delta) = \delta^{\alpha(w)}$ and any $C \in \mathcal{C}_{\alpha(w)}$. Further $m_{\alpha(w)}(C) = 1$ for all $C \in \mathcal{C}_{\alpha(w)}$ and the restriction of $m_{\alpha(w)}$ to C is the Cantor measure on C.*

Corollary 4.5: *Let $\alpha(w) \in C^2([0, 1], (0, 1))$ be decreasing. Then $\mathcal{C}_{\alpha(w)}$ is non-empty and the assumptions of theorem 3.4 is fulfilled for the test function $h(w, \delta) = \delta^{\alpha(w)}$ and any $C \in calC_{\alpha(w)}$. Further $m_{\alpha(w)}(C) = 1$ for all $C \in \mathcal{C}_{\alpha(w)}$ and the restriction of $m_{\alpha(w)}$ to C is the Cantor measure on C.*

Proof of corollary 4.5. This corollary follows directly from theorem 4.4 by symmetry □

Corollary 4.6: *Let $\alpha(w) \in C^2([0, 1], (0, 1))$. Then $\mathcal{C}_{\alpha(w)}$ is non-empty and $m_{\alpha(w)}(C) = 1$ for all $C \in \mathcal{C}_{\alpha(w)}$. Further, for any $C \in \mathcal{C}_{\alpha(w)}$, the restriction of $m_{\alpha(w)}$ to C is the Cantor measure on C.*

Proof of corollary 4.6. Let J be any open interval. Since $\alpha(w) \in C^2([0, 1], (0, 1))$, $\alpha'(w)$ can change sign at most countably many times in J, say in the points of the set S. Since S is at most countable, $[0, 1]\backslash S$ is the union of a sequence of disjoint open intervals $\{J_k\}_{k=1,2,3,...}$. By theorem 4.4 and corollary 4.5, the assumptions of theorem 3.4 and its reverse (see remark 3.5) is satisfied for any closed set $\hat{J}_k$ contained in J_k for any $k = 1, 2, 3, ...$ Thus for any such closed set we have $\nu(\hat{J}_k) = m_{\alpha(w)}(\hat{J}_k)$. Since both ν and $m_{\alpha(w)}$ are positive measures, this implies

$$\nu(J_k) = m_{\alpha(w)}(J_k \cap C)$$

Now

$$\nu(J \cap C) = \nu\left((S \cap C) \cup \bigcup_{k=1,2,3,...} J_k \cap C\right) = m_{\alpha(w)}(S \cap C) + \sum_{k=1,2,3,...} m_{\alpha(w)}(J_k \cap C)$$

where the last equality follows since the unions are disjoint. Now since S is countable, we have $m_{\alpha(w \cap C)}(S) = 0 = \nu(S \cap C)$ and we thus get

$$m_{\alpha(w)}(J \cap C) = m_{\alpha(w)}(S \cap C) + \sum_{k=1,2,3,...} m_{\alpha(w)}(J_k \cap C) =$$
$$\nu(S \cap C) + \sum_{k=1,2,3,...} m_{\alpha(w)}(J_k \cap C) = \nu(S \cap C) + \sum_{k=1,2,3,...} \nu(J_k \cap C) = \nu(J \cap C) = \nu(J)$$

Since this holds for any open set J, the corollary follows. □

We will now begin to prove theorem 4.4. That $\mathcal{C}_{\alpha(w)}$ is non-empty follows directly from theorem 4.2. To prove the rest of the claims of the theorem we will show that the conditions of theorem 3.4 holds, which is the purpose of the remaining lemmas of this section.

Recall that we, for any binary word j and integer k, use $j|k$ to denote the binary word consisting of the k first digits of j. When $I_{j|k}$ is a basic interval, $w_{j|k}$ denotes the midpoint of the basic interval $I_{j|k}$.

Lemma 4.7: *Let $\alpha : [0, 1] \to (0, 1)$ be continuously differentiable and let j be any binary sequence. Let $w_j \in C \in \mathcal{C}_{\alpha(w)}$ be the unique point in C such that $w \in I_{j|k}$ for all $k \in \mathbb{Z}_+$. Then*

$$\lim_{k\to\infty} c_{j|k} = c_j = 2^{-1/\alpha(w_j)}$$

uniformly in j.

Proof. Since $C \in \mathcal{C}_{\alpha(w)}$ we have $|I_{j|k}|^{\alpha(w_{j|k})} = \nu(I_{j|k}) = 2^{-k}$ for all large enough $k \in \mathbb{N}$. This implies $|I_{j|k}| = 2^{-\frac{k}{\alpha(w_{j|k})}}$ for all large enough $k \in \mathbb{N}$. This implies

$$c_{j|k+1} = \frac{|I_{j|k+1}|}{|I_{j|k}|} =$$
$$\frac{1}{2^{\frac{k+1}{\alpha(w_{j|k+1})}}} \Big/ \frac{1}{2^{\frac{k}{\alpha(w_{j|k})}}} = 2^{\frac{k}{\alpha(w_{j|k})} - \frac{k+1}{\alpha(w_{j|k+1})}} = 2^{-\frac{1}{\alpha(w_{j|k+1})}} \cdot 2^{k\cdot\left(\frac{1}{\alpha(w_{j|k})} - \frac{1}{\alpha(w_{j|k+1})}\right)}$$

Since $\alpha(w)$ is continuous on $[0,1]$, which is compact, $\alpha(w)$ is uniformly continuous on $[0,1]$, which implies $2^{-\frac{1}{\alpha(w_{j|k+1})}} \to 2^{-1/\alpha(w)}$ uniformly in j when $k \to \infty$. To prove the claims of the theorem it is therefore enough to show that $k \cdot \big(\frac{1}{\alpha(w_{j|k})} - \frac{1}{\alpha(w_{j|k+1})}\big) \to 0$ uniformly in j when $k \to \infty$.

As $\alpha(w)$ is continuously differentiable, $\alpha(w)$ is Lipschitz continuous on $[0,1]$. Let λ be the Lipschitz constant. Then

$$\begin{aligned}\left|k \cdot (\frac{1}{\alpha(w_{j|k})} - \frac{1}{\alpha(w_{j|k+1})})\right| = \frac{1}{\alpha(w_{j|k}) \cdot \alpha(w_{j|k+1})} \cdot k \cdot \left|\alpha(w_{j|k+1}) - \alpha(w_{j|k})\right| \leq \\ \frac{1}{\alpha(w_{j|k}) \cdot \alpha(w_{j|k+1})} \cdot k \cdot \lambda \cdot \left|w_{j|k+1} - w_{j|k}\right|\end{aligned} \tag{22}$$

As $I_{j|k+1} \subseteq I_{j|k}$ both $w_{j|k} \in I_{j|k}$ and $w_{j|k+1} \in I_{j|k}$, and therefore

$$\left|w_{j|k+1} - w_{j|k}\right| \leq |I_{j|k}| \tag{23}$$

Using this we get

$$\begin{aligned}\left|k \cdot (\frac{1}{\alpha(w_{j|k})} - \frac{1}{\alpha(w_{j|k+1})})\right| \overset{22}{\leq} \frac{1}{\alpha(w_{j|k}) \cdot \alpha(w_{j|k+1})} \cdot k \cdot \lambda \cdot \left|w_{j|k+1} - w_{j|k}\right| \overset{23}{\leq} \\ \frac{1}{\alpha(w_{j|k}) \cdot \alpha(w_{j|k+1})} \cdot k \cdot \lambda \cdot |I_{j|k}| = \frac{1}{\alpha(w_{j|k}) \cdot \alpha(w_{j|k+1})} \cdot k \cdot \lambda \cdot 2^{\frac{-k}{\alpha(w_{j|k})}} \leq \\ (\min_{w \in [0,1]} \alpha(w))^{-2} \cdot k \cdot \lambda \cdot 2^{\frac{-k}{\max_{w \in [0,1]} \alpha(w)}} \to 0\end{aligned}$$

when $k \to \infty$, uniformly in j. □

Lemma 4.8: *Let $\alpha : [0,1] \to (0,1)$ be continuously differentiable and let $C \in \mathcal{C}_{\alpha(w)}$. Let j be any binary word not containing only zeros. Define $I_\rho = \rho \cdot_L (G_j \cup I_{j1})$. Then for all $\rho \in [0,1]$ and all $|j|$ large enough we have*

$$\rho \cdot \nu(I_{j1}) \geq \nu(I_\rho) \tag{24}$$

Proof. Pick $\varepsilon > 0$ with $\varepsilon < \frac{\delta}{2}$. Then by lemma 4.7 , there exists $N \in \mathbb{N}$ such that $|c_{j0} - c_{jk}| < \varepsilon$ for all binary words j of length at least N and all binary words k.

Fix any such long enough binary word j and let $m \in \mathbb{N} \backslash \{0\}$. Then

$$\frac{|G_j| + |I_{j10^m}|}{|G_j| + |I_{j1}|} = \frac{(|I_j| - |I_{j0}| - |I_{j1}|) + |I_{j10^m}|}{(|I_j| - |I_{j0}| - |I_{j1}|) + |I_{j1}|} = \frac{|I_j| - |I_{j0}| - |I_{j1}| + |I_{j10^m}|}{|I_j| - |I_{j0}|} =$$

$$\frac{|I_j| - c_{j0}|I_j| - c_{j1}|I_j| + c_{j1}(c_{j10} \cdot c_{j100} \cdots c_{j10^m})|I_j|}{|I_j| - c_{j0}|I_j|}$$

If we divide by $|I_j|$ in the numerator and the denominator we get

$$\frac{|G_j| + |I_{j10^m}|}{|G_j| + |I_{j1}|} = \frac{1 - c_{j0} - c_{j1} + c_{j1}(c_{j10} \cdot c_{j100} \cdots c_{j10^m})}{1 - c_{j0}} = 1 - c_{j1} \cdot \frac{(1 - c_{j10} \cdot c_{j100} \cdot c_{j10^m})}{1 - c_{j0}}$$

As $c_{j1} \leq \frac{1}{2}$ and $c_{j10^k} < (c_{j0} - \varepsilon)$ for $k = 1, 2, 3, ..., m$;

$$\frac{|G_j| + |I_{j10^m}|}{|G_j| + |I_{j1}|} > 1 - \frac{1}{2} \cdot \frac{(1 - (c_{j0} - \varepsilon)^m)}{1 - c_{j0}} = 1 - \frac{1}{2} \sum_{l=0}^{m-1} (c_{j0} - \varepsilon)^l \cdot \frac{1 - (c_{j0} - \varepsilon)}{1 - c_{j0}}$$

Since $c_{j0} - \varepsilon \leq c_{j0} \leq \frac{1}{2}$ we get

$$\frac{|G_j| + |I_{j10^m}|}{|G_j| + |I_{j1}|} > 1 - \frac{1}{2} \sum_{l=0}^{m-1} (\frac{1}{2})^l \cdot \frac{1 - (c_{j0} - \varepsilon)}{1 - c_{j0}} = 1 - \sum_{l=1}^{m} \frac{1}{2^l} \cdot \frac{1 - (c_{j0} - \varepsilon)}{1 - c_{j0}}$$

As we can pick ε arbitrarily small, we get

$$\frac{|G_j| + |I_{j10^m}|}{|G_j| + |I_{j1}|} \geq \frac{1}{2^m}$$

By proposition 3.7 this is equivalent to the claims of the lemma. $\square$

Lemma 4.9: *Let $\alpha(w) \in C^2([0,1],(0,1))$ be increasing and define $h(w,\delta) = \delta^{\alpha(w)}$. Set, for fixed δ and w;*

$$f(t_0,t_1) = h(w+t_1-t_0, \delta+2t_0+2t_1)$$

where t_0 and t_1 is such that $w+t_0-t_1 \in [0,1]$ and $\delta+2t_0+2t_1 \in [0,1]$. Then $\frac{\partial}{\partial t_0} f(t_0,t_1) \geq 0$, $\frac{\partial}{\partial t_1} f(t_0,t_1) \geq 0$, $\frac{\partial^2}{\partial t_1^2} f(t_0,t_1) \leq 0$ and $\frac{\partial}{\partial t_0}\frac{\partial}{\partial t_1} f(t_0,t_1) \leq 0$ for all small enough δ, t_0 and t_1.

Proof. We first calculate $\frac{\partial}{\partial t_1} f(t_0,t_1)$:

$$\frac{\partial}{\partial t_1} f\left(t_0,t_1\right) = \left(\delta+2t_0+2t_1\right)^{\alpha\left(w-t_0+t_1\right)} \cdot \left(\frac{\partial}{\partial t_1}\alpha\left(w-t_0+t_1\right) \cdot \log\left(\delta+2t_0+2t_1\right) + 2\alpha\left(w-t_0+t_1\right) \cdot \frac{1}{\delta+2t_0+2t_1}\right) =$$

$$\left(\delta+2t_0+2t_1\right)^{\alpha\left(w-t_0+t_1\right)-1} \cdot \left(\alpha'\left(w-t_0+t_1\right)\left(\delta+2t_0+2t_1\right)\log\left(\delta+2t_0+2t_1\right) + 2\alpha\left(w-t_0+t_1\right)\right)$$

Since $\alpha(w)$ is continuously differentiable and $[0,1]$ is a compact set, $\alpha(w)$ is uniformly bounded away from zero. As $(\delta+2t_0+2t_1)\log(\delta+2t_0+2t_1) \to 0$ when $(\delta+2t_0+2t_1) \to 0$ and $\alpha(w)$ is uniformly bounded away from zero for all $w-t_0+t_1 \in [0,1]$, we get $\frac{\partial}{\partial t_1} f(t_0,t_1) > 0$ for all small enough t_0, t_1 and δ. Similarly, we get

$$\frac{\partial}{\partial t_0} f\left(t_0,t_1\right) =$$

$$\left(\delta+2t_0+2t_1\right)^{\alpha\left(w-t_0+t_1\right)} \cdot \left(\frac{\partial}{\partial t_1}\alpha\left(w-t_0+t_1\right) \cdot \log\left(\delta+2t_0+2t_1\right) + 2\alpha\left(w-t_0+t_1\right) \cdot \frac{1}{\delta+2t_0+2t_1}\right) =$$

$$\left(\delta+2t_0+2t_1\right)^{\alpha\left(w-t_0+t_1\right)-1} \cdot \left(-\alpha'\left(w-t_0+t_1\right)\left(\delta+2t_0+2t_1\right)\log\left(\delta+2t_0+2t_1\right) + 2\alpha\left(w-t_0+t_1\right)\right) > 0$$

since $(\delta+2t_0+2t_1)\log(\delta+2t_0+2t_1) \to 0$ when $(\delta+2t_0+2t_1) \to 0$, $\alpha'(w)$ is uniformly bounded and $\alpha(w)$ is uniformly bounded away from zero for all $w \in [0,1]$.

We now consider the second derivative with respect to t_1:

$$
\begin{aligned}
\frac{\partial^2}{\partial t_1^2} f(t_0, t_1) = \\
(\delta + 2t_0 + 2t_1)^{\alpha(w-t_0+t_1)-1} \cdot \\
\Big(\frac{\partial^2}{\partial t_1^2} \alpha(w - t_0 + t_1) \cdot (\delta + 2t_0 + 2t_1) \log(\delta + 2t_0 + 2t_1) + \\
\alpha'(w - t_0 + t_1) \cdot 2 \log(\delta + 2t_0 + 2t_1) + \\
\alpha'(w - t_0 + t_1)(\delta + 2t_0 + 2t_1) \frac{2}{\delta + 2t_0 + 2t_1} + 2 \frac{\partial}{\partial t_1} \alpha(w - t_0 + t_1)\Big) + \\
(\delta + 2t_0 + 2t_1)^{\alpha(w-t_0+t_1)-2} \cdot \\
\Big(\frac{\partial}{\partial t_1} \alpha(w - t_0 + t_1) \cdot (\delta + 2t_0 + 2t_1) \log(\delta + 2t_0 + 2t_1) + (\alpha(w - t_0 + t_1) - 2)\Big) \cdot \\
(\alpha'(w - t_0 + t_1)(\delta + 2t_0 + 2t_1) \log(\delta + 2t_0 + 2t_1) + 2\alpha(w - t_0 + t_1))
\end{aligned}
$$

This expression can be rewritten as

$$
\begin{aligned}
(\delta + 2t_0 + 2t_1)^{\alpha(w-t_0+t_1)-1} \cdot \\
\Big(\alpha''(w - t_0 + t_1) \cdot (\delta + 2t_0 + 2t_1) \log(\delta + 2t_0 + 2t_1) + \\
2\alpha'(w - t_0 + t_1)(2 + \log(\delta + 2t_0 + 2t_1)) + \\
(\delta + 2t_0 + 2t_1) \cdot \\
(\alpha'(w - t_0 + t_1) \cdot (\delta + 2t_0 + 2t_1) \cdot \log(\delta + 2t_0 + 2t_1) + (\alpha(w - t_0 + t_1) - 2)) \cdot \\
(\alpha'(w - t_0 + t_1) \cdot (\delta + 2t_0 + 2t_1) \cdot \log(\delta + 2t_0 + 2t_1) + 2\alpha(w - t_0 + t_1))\Big)
\end{aligned}
\tag{25}
$$

Since $\alpha''(w)$ is continuous on the compact interval $[0, 1]$, it is bounded on $[0, 1]$. Also, we know that $(\delta + 2t_0 + 2t_1) \log(\delta + 2t_0 + 2t_1) \to 0$ as $(\delta + 2t_0 + 2t_1) \to 0$. Thus the summand on the second row tends to zero as $\delta + 2t_0 + 2t_1 \to 0$.

Since both $\alpha'(w)$ and $\alpha(w)$ are bounded and $(\delta + 2t_0 + 2t_1) \log(\delta + 2t_0 + 2t_1) \to 0$ as $(\delta + 2t_0 + 2t_1) \to 0$, the product of the last two rows is also bounded, which implies that the complete product of the last three rows can be made arbitrarily small by choosing δ, t_0 and t_1 small enough.

The only summand in the paranthesis over the last five rows not yet accounted for is the term $2\alpha'(w - t_0 + t_1)(2 + \log(\delta + 2t_0 + 2t_1))$ which cannot be chosen arbitrarily small. On the contrary, if $\alpha'(w - t_0 + t_1) > 0$, it tends to negative infinity as $\delta + 2t_0 + 2t_1 \to 0$ which causes the complete expression, and thus also $\frac{\partial^2}{\partial t_1^2} f(t_0, t_1)$ to be negative.

If $\alpha'(w - t_0 + t_1) = 0$ then equation (25) reduces to

$$
(\delta + 2t_0 + 2t_1)^{\alpha(w-t_0+t_1)-1} \cdot (\delta + 2t_0 + 2t_1) \cdot (\alpha(w - t_0 + t_1) - 2) \cdot (2\alpha(w - t_0 + t_1))
$$

which is negative since $\alpha(w - t_0 + t_1) < 1$ implies $(\alpha(w - t_0 + t_1) - 2) < 0$.

Analogously, and therefore omitted here,

$$\frac{\partial}{\partial t_0}\frac{\partial}{\partial t_1} f(t_0, t_1) \leq 0$$

for all small enough δ, t_0 and t_1.

$\square$

4.3 Examples

We will end this section with two examples. The first example is a simple application of the results of this section and gives a rough graphical explanation of how the dimension function affects the distribution of the points in the Cantor set.

Example 4.10: Figure 7 shows the first steps of the construction of a Cantor set $C \in \mathcal{C}_{\alpha(w)}$ for $\alpha(w) = 0.2 \cdot \sin(10\pi w) + 0.75$. As $\alpha(w)$ is two times continuously differentiable, the measure of C is exactly one by corollary 4.6. Moreover $m_{\alpha(w)}|_{C_{\alpha(w)}} \equiv \nu_{C_{\alpha(w)}}$.

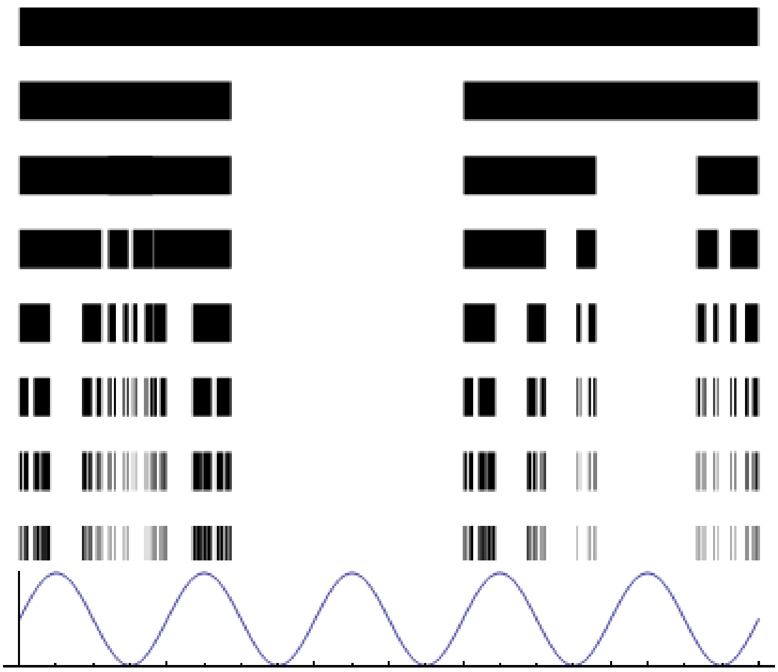


Figure 7: The image above shows the first steps of the construction of a set $C \in \mathcal{C}_{\alpha(w)}$, where $\alpha(w) = 0.2 \cdot \sin(10\pi w) + 0.75$, together with the graph of $\alpha(w)$. Note especially that the density of points are higher where the value of the dimension function $\alpha(w)$ is larger.

The second and last example of this section shows that not all Cantor sets can be measured by exponential test functions and also that there exists dimension functions $\alpha(w)$ and $C \in \mathcal{C}_{\alpha(w)}$ such that $m_{\alpha(w)}$ is not even a mass distribution on C. This later consequence show that we need some more condition than continuity for μ_h to be a mass distribution on all $C \in \mathcal{C}_h$.

Example 4.11: Set $c_j = \frac{1}{2}\left(1 - \frac{1}{|j|+3}\right)$ for any binary word j and define $C \sim \{c_j\}$.

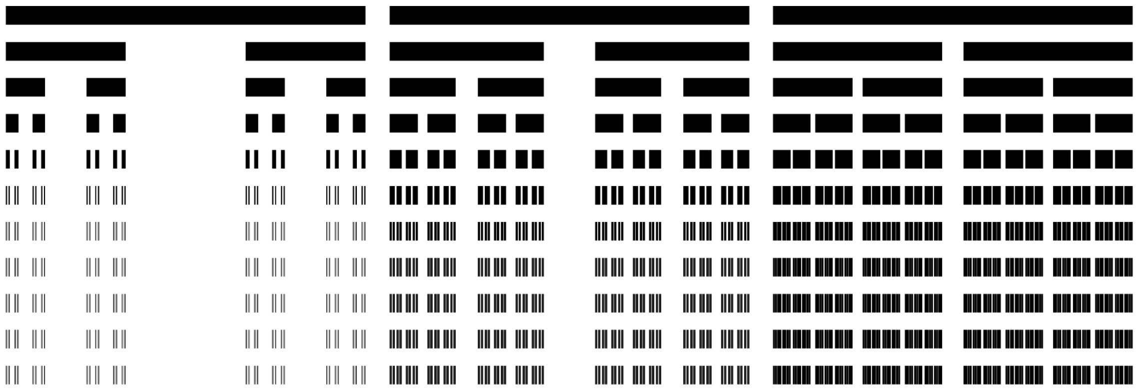


Figure 8: The figure above displays the first construction steps of the cantor set $C \sim \{c_j\}$ where $c_j = \frac{1}{2}\left(1 - \frac{1}{|j|+3}\right)$. The leftmost image shows the first ten construction steps, the image in the middle hos the construction then progresses from one of the basic intervals I_j with $|j| = 10$ and the right most image shows how the construction progresses from one of the basic intervals I_j with $|j| = 20$. On a large scale (at the earliest construction steps), the set is very thin due to the first steps removing a large proportion of the interval. On a very small scale however (the later construction steps), the set is very dense, giving it local dimension one everywhere.

Fix $N \in \mathbb{N}\backslash\{0\}$. Let $\hat{\alpha}_N(w)$ be the function defined at the midpoint w_j of any basic interval I_j with $|j| \geq N$ by

$$\hat{\alpha}_N(w_j) = \frac{|j| \log 2}{(|j| - 1) \log 2 + \log(|j| + 2)}$$

Then let $\alpha_N(w)$ be any continuous extension of this function to all $w \in [0, 1]$.

Let w be any point in C. Then there exists a binary sequence j such that $\lim_{k\to\infty} w_{j|k} = w$. Using this we get

$$\alpha(w) = \lim_{k\to\infty} \alpha(w_{j|k}) = \lim_{k\to\infty} \frac{k \log 2}{(k-1) \log 2 + \log(k+2)} = 1$$

for all $w \in C$.

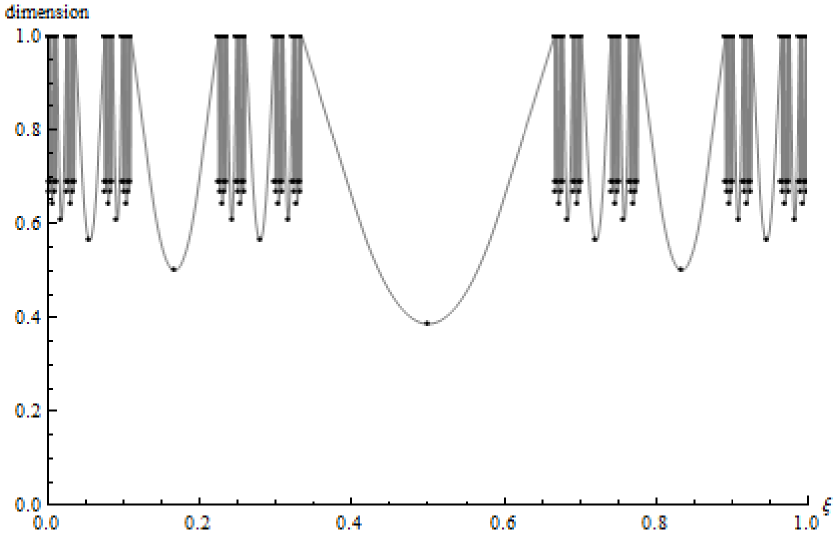


Figure 9: A continuous extension $\alpha_1(w)$ of the dimension function $\hat{\alpha}_1(w)$ defined by its values in the points w_j for all binary words j (in the figure above marked with black points).

If we try to calculate the derivative of $\alpha_N(w)$ at any point $w_j \in C$, where $w_j = \lim_{k\to\infty} w_{j|k}$ we get

$$\alpha'_N(w) = \lim_{k\to\infty} \frac{\alpha_N(w_{j|k}) - \hat{\alpha}_N(w)}{w_{j|k} - w_j} = \lim_{k\to\infty} \frac{\frac{k\log 2}{(k-1)\log 2 + \log(k+2)} - 1}{\frac{1}{2}\cdot |I_{j|k}|} =$$
$$\lim_{k\to\infty} \frac{\frac{k\log 2}{(k-1)\log 2 + \log(k+2)} - 1}{\frac{1}{2}\cdot \frac{1}{2^{k-1}}\cdot \frac{1}{k+2}} = \lim_{k\to\infty} (k+2)\cdot 2^k \cdot \frac{\log 2 - \log(k+2)}{(k-1)\log 2 + \log(k+2)} =$$
$$\lim_{k\to\infty} 2^k \cdot \frac{\log 2 - \log(k+2)}{\frac{(k-1)\log 2}{k+2} + \frac{\log(k+2)}{k+2}} = -\infty$$

i.e no such continuous extension $\alpha_N(w)$ of $\hat{\alpha}_N(w)$ can be differentiable at any point in C. Thus no continuous extension of $\hat{\alpha}_N(w)$ to $[0,1]$ can fulfil the conditions of theorem 3.3. However $C \in C_{\alpha_N(w)}$ for any extension $\alpha_N(w)$ since

$$h(I_j) = |I_j|^{\alpha_N(w_j)} = \Big(\prod_{k=1}^{|j|} c_{j|k}\Big)^{\alpha(w_j)} = \Big(\frac{1}{2^{|j|}}\cdot\frac{2}{|j|+2}\Big)^{\alpha(w_j)} = \frac{1}{2^{|j|}} = \nu(I_j)$$

for all binary words j with $|j| \geq N$.

The question is then whether or not $m_{\alpha_N(w)}$ can be a mass distribution on C for any extension of $\alpha_N(w)$ or any N even though the assumptions of theorem 3.3 do not hold. To show that this is not the case we will construct a sequence of coverings $\{I_k^{(n)}\}$, $n = 1, 2, 3, ...$ of C such that $\sum_k |I_k^{(n)}|^{\alpha(w_k)} \to 0$ as $n \to \infty$ and $C \in \bigcup_k I_k^{(n)}$ for all n.

For each $n \in \mathbb{N}$ with $n \geq N$ consider the covering of C with $\{I(a_j, 2|I_j|)\}_{|j|=n}$, where $\{I_j\}$ are the basic intervals associated with C and a_j is the left endpoint of I_j. Then, since all intervals from the same construction step of C have equal length, any interval $I(a_j, 2|I_j|)$ in the covering have length

$$|I(a_j, 2|I_j|)| = 2 \cdot \prod_{l=1}^{n} c_{j|l} = \frac{1}{2^n} \cdot \frac{2}{n+2}$$

and we can thus cover C with 2^n intervals of length $\frac{1}{2^n} \cdot \frac{2}{n+2}$ centred at points w where $\alpha_N(w) = 1$. This implies

$$m_{\alpha_N(w)} \leq \sum_{|j|=n} |I(a_j, 2|I_j|)|^{\alpha_N(a_j)} = \sum_{|j|=n} |I(a_j, 2|I_j|)|^1 = \sum_{|j|=n} \frac{1}{2^n} \cdot \frac{2}{n+2} = 2^n \cdot \frac{1}{2^n} \cdot \frac{2}{n+2} = \frac{2}{n+2}$$

which tends to zero as $n \to \infty$. We thus get $m_{\alpha_N(w)}(C) = 0$.

5 Properties of multidimensional Hausdorff measures

In this section we will show that the commonly used density results for Hausdorff measures associated with test functions $h(\delta)$ transfer with only small modifications to the multidimensional case, i.e. to Hausdorff measures associated with test functions $h(w, \delta)$. The proofs from the first three subsections of this section are all adaptations of their analogues for test functions of the type $h(\delta) = \delta^\alpha$ for some fixed exponent α as presented in [9].

5.1 A covering theorem of Vitali type

In this subsection we show that the Vitali covering theorem holds for Hausdorff measures μ_h.

Theorem 5.1: *Let h be a test function and let E be an open subset of $\mathbb{R}$ with $0 < \mu_h(E) < \infty$. Further let $\mathcal{Q}$ be a family of closed intervals such that each point $\xi \in E$ is the centre of arbitrarily small intervals $I(\xi, \delta) \in \mathcal{Q}$. Then there exist a sequence of disjoint intervals $\{I_j\}$ in $\mathcal{Q}$ such that either*

$$\mu_h\left(E \backslash \bigcup_j I_j\right) = 0 \quad \text{or} \quad \sum_j h\left(I_j\right) = \infty$$

Proof. Let I_1 be any member of $\mathcal{Q}$. We will find U_2, I_3, .. by using induction. Suppose therefore that $I_1, ..., I_m$ have already been chosen. Let

$$d_m = \sup\{\delta : \exists I(w, \delta) \in \mathcal{Q} \text{ with } I(w, \delta) \cap (I_1 \cup ... \cup I_m) = \emptyset\}$$

If $d_m = 0$ the sequence $I_1, ..., I_m$ have the desired properties and we are then finished. If $d_m > 0$, let $I_{m+1} = I(w_{m+1}, \delta_{m+1})$ be any interval in $\mathcal{Q}$ disjoint from $I_1 \cup ... \cup I_m$ with $\delta_{m+1} > \frac{d_m}{2}$.

Suppose that the process never stops and that $\sum h(I_j) < \infty$. We claim that then

$$E \backslash \bigcup_{j=1}^{m} I_j \subset \bigcup_{j=m+1}^{\infty} 4I_j$$

To prove this claim; let $\xi \in E \backslash \bigcup_{j=1}^{m} I_j$ and let I be an interval in $\mathcal{Q}$ containing ξ which is disjoint from $I_1, ..., I_m$. Since $\delta_j \to 0$ as $j \to \infty$, we can find n such that $|I| > 2|I_n|$. This implies I must have nonempty intersection with some interval I_j for $m < i < n$ for which $|I| \leq 2|I_j|$. But then $I \subset 4I_j$, and the claim thus follows.

Let D be the doubling constant for the test function $h(w, \delta)$. Then for any $\delta > 0$:

$$m_h^\delta(E \setminus \bigcup_{j=1}^{\infty} I_j) \leq m_h^\delta(E \setminus \bigcup_{j=1}^{m} I_j) \leq m_h^\delta(\bigcup_{j=m+1}^{\infty} 4I_j) \leq$$
$$\sum_{j=m+1}^{\infty} h(4I_j) \leq \sum_{j=m+1}^{\infty} h(4I_j) \leq D^2 \sum_{j=m+1}^{\infty} h(I_j)$$

if m is large enough to make $2\delta_j < \delta$ when $j > m$. Since this sum tends to zero as $m \to \infty$, this proves the theorem. □

5.2 Bounds for the local density

In this section we will give upper and lower bounds for upper local densities of Hausdorff measures, often given by the following definition.

Definition 5.2: The upper density of a measure σ with respect to a Hausdorff dimension α at a point $w \in \mathbb{R}$ is defined by

$$D_\alpha^*[\sigma](w) = \limsup_{\delta \to 0} \frac{\sigma(I(w, \delta))}{\delta^\alpha}$$

Since $h(\delta) = \delta^\alpha$ is the test function for the measure m_α, we could extend this definition to be able to calculate the density of a measure with respect to any test function $h(w, \delta)$:

Definition 5.3: The upper density of a measure σ with respect to a test function $h(w, \delta)$ at a point $w \in \mathbb{R}$ is defined by

$$\Delta_h^*[\sigma](w) = \limsup_{\delta \to 0} \frac{\sigma(I(w, \delta))}{h(w, \delta)}$$

Similarly, we can define the lower density of a measure with respect to a test function:

Definition 5.4: The lower density of a measure σ with respect to a test function $h(w, \delta)$ at a point $w \in \mathbb{R}$ is defined by

$$\underline{\Delta}_h[\sigma](w) = \liminf_{\delta \to 0} \frac{\sigma(I(w, \delta))}{h(w, \delta)}$$

The next two theorems gives bounds for the upper density.

Theorem 5.5: *Suppose $E \subseteq \mathbb{R}$ and let h be a test function such that $\mu_h(E)$ is finite. Then*

$$\Delta_h^*[\mu_h|_E](\xi) \leq 1$$

for μ_h-almost all $\xi \in \mathbb{R}$.

Proof. Note first that we may assume that E is a Borel set. Fix $t > 1$ and set

$$B = \{\xi \in E : \Delta_h^*[\mu_h|_E](\xi) > t\}$$

It is then enough to show that $\mu_h(B) = 0$ for any $t > 1$. Pick $\varepsilon > 0$ and $\delta > 0$. Then we can find an open set U containing B such that

$$\mu_h|_E(U) < \mu_h|_E(B) + \varepsilon$$

For each $\xi \in B$ we can find arbitrarily small intervals $I(\xi, r)$ such that $0 < r < \delta/2$, $I(\xi, r) \subset U$ and $\mu_h(I(\xi, r) \cap E) > th(\xi, r)$.

By the Vitali covering lemma we can pick a sequence $I_1, I_2, ...$ of such intervals for which

$$\mu_h|_E\left(B \backslash \bigcup_j I_j\right) = 0$$

This implies

$$\mu_h|_E(B) + \varepsilon > \mu_h|_E(U) \geq \sum_j \mu_h|_E(I_j) > \sum t\, h(I_j) \geq t\, \mu_h^\delta|_E\left(B \cap \bigcup_j I_j\right) = t\, \mu_h^\delta|_E(B)$$

By letting $\varepsilon, \delta \to 0$ and using $t > 1$, we get $\mu_h|_E(B) = 0$. □

We will now state and prove a theorem which gives a lower bound for the upper density with respect to a test function. Before we begin, we will need the following simple consequence of the existance of a doubling constant.

Lemma 5.6: *Let h be a test function with doubling constant D. Then*

$$\left|\frac{h(w_1, \delta)}{h(w_2, \delta)}\right| < D^2$$

for all small enough $\delta > 0$ and all w_1, w_2 with $|w_1 - w_2| < \delta$.

Proof. Since $|w_1 - w_2| < \delta$ we have $I(w_1, \delta) \subseteq I(w_2, 4\delta)$. Since h is increasing as an interval function for all small enough δ, this implies

$$\left|\frac{h(w_1, \delta)}{h(w_2, \delta)}\right| \leq \left|\frac{h(w_2, 4\delta)}{h(w_2, \delta)}\right| \leq \left|\frac{D^2 \cdot h(w_2, \delta)}{h(w_2, \delta)}\right| = D^2$$

□

Theorem 5.7: *Let $h(\xi, \delta)$ be a test function with doubling constant D. Further let E be a subset of $\mathbb{R}$ on which μ_h is positive and finite. Then*

$$\frac{1}{D^3} \leq \Delta_h[\mu_h|_E](\xi)$$

μ_h-almost all $\xi \in E$.

Proof. The set B of all $\xi \in E$ such that $\Delta_h^*[\mu_h|E](\xi) < c$ is the union of the sets

$$B_k = \{w \in E : \mu_h|_E(I(w,\delta)) < \frac{ck}{k+1} \cdot h(I(w,\delta)),\ 0 < \delta < \frac{1}{k}\},\ k = 1,2,3,...$$

To prove the claims of the theorems it is enough to show that $\mu_h(B_k) = 0$ for all $k \in \mathbb{N}\backslash\{0\}$.

Fix k, set $t = \frac{k}{k+1}$ and let $\varepsilon > 0$. We can then find a covering with intervals $I_1, I_2, ...$ of B_k, such that $|I_j| < \frac{1}{k}$, $B_k \cap I_j \neq \emptyset$ and

$$\sum_j h(I_j) \leq \mu_h(B_k) + \varepsilon$$

For each j, pick $\xi_j \in B_k \cap I_j$ and let $\delta_j = 2|I_j|$. Then $B_k \cap I_j \subset E \cap I(\xi_j, \delta_j)$. Then

$$\begin{aligned}\mu_h(B_k) \leq \sum_j \mu_h|_E(B_k \cap I(\xi_j,\delta_j)) \leq \sum_j \mu_h|_E(E \cap I(\xi_j,\delta_j) <\\ \sum_j c \cdot t \cdot h(\xi_j,\delta_j) = \sum_j c \cdot t \cdot h(\xi_j, 2|I_j|) \leq\\ \sum_j c \cdot t \cdot h(\xi_j,\delta_j) = \sum_j c \cdot t \cdot D \cdot h(\xi_j,|I_j|)\end{aligned}$$

Denote the midpoint of I_j by w_j. Then $|\xi_j - w_j| < \delta$. By lemma 5.6, this implies

$$\begin{aligned}\mu_h(B_k) \leq \sum_j \mu_h|_E(B_k \cap I(\xi_j,\delta_j)) \leq \sum_j c \cdot t \cdot D \cdot h\,(\xi_j,|I_j|) =\\ \sum_j c \cdot t \cdot D \cdot h(w_j,|I_j|)\frac{h(\xi_j,|I_j|)}{h(w_j,|I_j|)} \leq \sum_j c \cdot t \cdot D \cdot h(w_j,|I_j|) \cdot D^2 =\\ c \cdot tD^3 \cdot \sum_j h(w_j,|I_j|) = c \cdot tD^3 \cdot \sum_j h(I_j) \leq c \cdot tD^3 \cdot (\mu_h(B_k) + \varepsilon)\end{aligned}$$

Since this holds for any c, we can choose to set $c = D^{-3}$. Then as $\varepsilon \to 0$ we get

$$\mu_h(B_k) \leq t\mu_h(B_k)$$

As $\mu_h(B_k) \leq \mu_h(E) < \infty$ and $t < 1$ this implies $\mu_h(B_k) = 0$. □

5.3 Frostman's lemma

In this section we present a proof of Frostman's lemma for multidimensional Hausdorff measures.

Lemma 5.8 (Frostman's lemma)**:** *Let* σ *be a finite positive measure on* $\mathbb{R}$, $E \subseteq \mathbb{R}$ *and* $\lambda \in (0, \infty)$. *Then*
(1) if $\Delta_h^*[\sigma](\xi) \geq \lambda$ *for all* $\xi \in E$, *then* $\sigma(E) \geq \lambda \cdot \mu_h(E)$
Further, if $\alpha(w)$ *is Hölder continuous:*
(2) if $\Delta_h^*[\sigma](\xi) \leq \lambda$ *for all* $\xi \in E$ *then* $\sigma(E) \leq 2\lambda D^2 \cdot \mu_h(E)$

Proof. (1) Consider the set

$$B = \{\xi \in E : \Delta_h^*[\sigma](\xi) > \lambda\}$$

Pick $\varepsilon > 0$. Then we can find an open set U containing B such that $\nu(U) \leq \nu(B) + \varepsilon$. By the definition of B, for each $\xi \in B$ we can find arbitrarily small intervals $I(\xi, \delta)$ such that

$$\sigma(I(\xi, \delta)) \geq \lambda h(\xi, \delta)$$

By the Vitali covering theorem, we can find a disjoint covering of E by such balls $\{I_j\}$ for which $\mu_h(U \setminus \cup I_j) = 0$. If we let ξ_j be the midpoint of I_j and $\delta_j = |I_j|$ we get

$$\sigma(B) + \varepsilon \geq \sigma(E \cap U) \geq \sum \sigma(I(\xi_j, \delta_j)) \geq \sum \lambda h(\xi_j, \delta_j) \geq \lambda \sum \mu_h^\delta(B \cap I_j) = \lambda \mu_h^\delta(B)$$

Letting $\varepsilon, \delta \to 0$, this implies

$$\sigma(B) \geq \lambda \mu_h(B)$$

(2) Set

$$B_k = \{x \in E : \sigma\left(I(x, \delta)\right) < \frac{\lambda k}{k+1} \cdot h(x, \delta),\ 0 < \delta < \frac{1}{k}\},\ k = 1, 2, 3, ...$$

Then the set $B = \{x \in E$ such that $\Delta_h[\sigma](x) < \lambda\}$ is the union of the sets B_k. We will show that $\sigma(B_k) < 2\lambda C \cdot \mu_h(B_k)$ for all k.

Fix k, set $t = \frac{k}{k+1}$ and let $\varepsilon > 0$. Let $\{I_j\}$ be a covering of B_k with $|I_j| < \frac{1}{k}$, $B_k \cap I_j \neq \emptyset$ and

$$\sum h(I_j) \leq \mu_h(B_k) + \varepsilon, \text{ where } w_j \text{ is the midpoint of } I_j$$

Then for each I_j, pick $\xi_j \in B_k \cap I_j$ and set $\delta_j = 2|I_j|$. Then $B_k \cap I_j \subseteq I(\xi_j, \delta_j)$ and

$$\sigma(I(\xi_j, \delta_j)) \leq \lambda t \cdot h(\xi_j, \delta_j)$$

since $\xi_j \in B_k$.

This gives

$$\sigma(B_k) \le \sum_j \sigma(B_k \cap I_j) \le \sum_j \sigma(I(\xi_j, \delta_j)) \le \sum_j \lambda t \cdot h(\xi_j, \delta_j) \le$$
$$\sum_j 2\lambda t \cdot h(\xi_j, |I_j|) = \sum_j 2\lambda t \cdot h(w_j, |I_j|) \cdot \frac{h(\xi_j, |I_j|)}{h(w_j, |I_j|)} \le \qquad \text{[by lemma 5.6]}$$
$$\sum_j 2\lambda t \cdot h(w_j, |I_j|) \cdot D^2 = 2\lambda t D^2 \sum_j h(w_j, |I_j|) =$$
$$2\lambda t D^2 \sum_j h(I_j) \le 2\lambda t D^2 \left(\mu_h(B_k) + \varepsilon\right)$$

Letting $\varepsilon \to 0$ we get

$$\sigma(B_k) \le 2\lambda t D^2 \cdot \mu_h(B_k)$$

Since this holds for all k, and $B = \bigcup_k B_k$ we get

$$\sigma(B) \le 2\lambda D^2 \cdot \mu_h(B)$$

□

5.4 Local dimension and multidimensional Hausdorff measures

In this section we will study the local dimension of measures and show its connection with the dimension function of Hausdorff measures $m_{\alpha(w)}$ of exponential type. We first recall the definition of the local dimension of a measure σ.

Definition 5.9: Let σ be a finite measure on $\mathbb{R}$. Then the *local dimension* of σ at a point $w \in \text{support}(\sigma)$ is defined by

$$d[\sigma](w) = \limsup_{\delta \to 0} \frac{\log \sigma(I(w, \delta))}{\log \delta}$$

The following proposition shows that the local dimension of a Hausdorff measure of exponential type equals its dimension function if the local density is finite and strictly positive.

Proposition 5.10: *Let $\alpha(w)$ be a dimension function and E a subset of support(σ) on which $m_{\alpha(w)}$ is finite. Then $d[m_{\alpha(w)}|_E](\xi) \ge \alpha(\xi)$ for $m_{\alpha(w)}$-almost all $\xi \in E$. Further, if $\underline{D}[m_{\alpha(w)}|_E](\xi) > 0$ for all $\xi \in E$ then $d[m_{\alpha(w)}|_E](\xi) = \alpha(\xi)$ $m_{\alpha(w)}$-almost all $\xi \in E$.*

Proof. Let $\varepsilon > 0$. Since $D^*_{\alpha(w)}[m_{\alpha(w)}|_E] \leq 1$ by theorem 5.5 we can find $\Delta > 0$ such that

$$\frac{m_{\alpha(w)}|_E(I(\xi,\delta))}{\delta^{\alpha(\xi)}} \leq 1 + \varepsilon$$

for all $\delta < \Delta$. This implies

$$d[m_{\alpha(w)}](\xi) = \limsup_{\delta\to 0} \frac{\log m_{\alpha(w)}|_E(I(\xi,\delta))}{\log \delta} \geq \limsup_{\delta\to 0} \frac{1+\varepsilon}{\log \delta} + \alpha(\xi)$$

Since $1 + \varepsilon$ is bounded, the first term on the right hand side tends to 0 as $\delta \to 0$, i.e. we get $d[m_{\alpha(w)}|_E](\xi) \geq \alpha(\xi)$.

If $\underline{D}[m_{\alpha(w)}](\xi) > 0$ for all $\xi \in E$, we can find $c > 0$ such that

$$c < \frac{m_{\alpha(w)}|_E(I(\xi,\delta))}{\delta^{\alpha(\xi)}}$$

for all $\delta < \Delta$. This gives

$$d[m_{\alpha(w)|_E}](\xi) \leq \limsup_{\delta\to 0} \frac{\log c}{\log \delta} + \alpha(\xi) = \alpha(\xi)$$

□

Proposition 5.10 motivates the following proposition:

Proposition 5.11: *Let σ be a finite measure on $\mathbb{R}$ and define $\alpha(w) = d[\sigma](w)$ for all $w \in$ support(σ). Suppose $\alpha(w)$ can be extended to an upper L, λ-Hölder continuous function $\alpha(w)$ on $\mathbb{R}$ σ-a.e. and that $D_{\alpha(w)}[\sigma](\xi) < \infty$ for σ-almost all $\xi \in$ support(σ). Then $\sigma \ll m_{\alpha(w)}$.*

Proof. Let M be any set such that $m_{\alpha(w)}(M) = 0$. By applying Frostman's lemma to the set $\{\xi \in M : D_{\alpha(w)}[\nu](\xi) < t\}$ we get

$$\sigma(\{\xi \in M : D_{\alpha(w)}[\nu](\xi) < t\}) = 0$$

Since $M = \bigcup_{k\in\mathbb{N}}\{\xi \in M : D_{\alpha(w)}[\sigma](\xi) < k\}$, this implies $\sigma(M) = 0$, and thus $\sigma \ll m_{\alpha(\xi)}$. □

Example 5.12: Consider the measure $\sigma = m_1 + m_0|_{\{0\}}$ on $[0,1]$. As $\sigma(\{0\}) = 1$ and $m_1(\{0\}) = 0$;

$$\sigma = m_1 + m_0|_{\{0\}} \not\ll m_{1-\chi_{\{0\}}} \equiv m_1$$

As the local dimension of σ is $d[\sigma](\xi) = 1 - \chi_{\{0\}}$ which is not upper continuous at $\xi = 0$, this does not contradict proposition 5.11.

Example 5.13: We will now return to the Cantor set $C \sim \{g_j\}$, where $g_j = \frac{1}{|j|+3}$ for any binary word j, which we studied in example 4.11. We will show that the Cantor measure ν of this set has local dimension one at all points in its support even though $\nu \not\ll m_1$. We will also show that the Cantor measure does not fulfil the assumptions of proposition 5.11 since the density with respect to the test function $h(w, \delta) = \delta$ of this measure is infinite at all points in its support.

Recall that the length of I_j is

$$|I_j| = 1 \cdot \frac{1}{2}(1 - \frac{1}{0+3}) \cdot \frac{1}{2}(1 - \frac{1}{1+3}) \cdot ... \cdot \frac{1}{2}(1 - \frac{1}{(|j|-1)+3}) = \frac{1}{2^{|j|}} \cdot \frac{1}{|j|+2}$$

Since C is self similar, we can use this to calculate the local dimension of the Cantor measure ν associated with C at any point $\xi \in C$:

$$\begin{aligned} d[\nu](\xi) = d[\nu](0) = \lim_{n\to\infty} \frac{\log \nu(I_{0^n})}{\log |I_{0^n}|} = \lim_{n\to\infty} \frac{\log \nu\Big([0, \frac{2^{-n}\cdot 2}{n+2}]\Big)}{\log \frac{2^{-n}\cdot 2}{n+2}} = \lim_{n\to\infty} \frac{\log 2^{-n}}{\log \frac{2^{-n}\cdot 2}{n+2}} = \\ = \lim_{n\to\infty} \frac{1}{1 + \frac{\log 2}{\log 2^{-n}} - \frac{\log(n+2)}{\log 2^{-n}}} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{n} + \frac{\log(n+2)}{n \log 2}} = 1 \end{aligned}$$

However;

$$\begin{aligned} m_1(C) \leq \lim_{n\to\infty} 2^n \cdot \Big((1 - \frac{1}{3}) \cdot \frac{1}{2}\Big) \cdot \Big((1 - \frac{1}{3}) \cdot \frac{1}{4}\Big) \cdots \Big((1 - \frac{1}{n+2}) \cdot \frac{1}{2}\Big) = \\ \lim_{n\to\infty} (1 - \frac{1}{3}) \cdot (1 - \frac{1}{4}) \cdots (1 - \frac{1}{n+2}) = \\ \lim_{n\to\infty} \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n+1}{n+2} = \\ \lim_{n\to\infty} \frac{2}{n+2} = 0 \end{aligned}$$

As this contradicts the conclusions of proposition 5.11, we can conclude that $D_1[\sigma](\xi) = \infty$ for all $\xi \in C$, which is confirmed by the following calculation.

$$D_1[\nu](\xi) = D_1[\nu](0) \geq \lim_{n\to\infty} \frac{\nu\Big([0, \frac{2^{-n}\cdot 2}{n+2}]\Big)}{\frac{2^{-n}\cdot 2}{n+2}} = \lim_{n\to\infty} \frac{2^{-n}}{\frac{2^{-n}\cdot 2}{n+2}} = \lim_{n\to\infty} \frac{n+2}{2} = \infty$$

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