Thesis for the Degree of Doctor of Philosophy

Tight maps, a classification

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Abstract

This thesis concerns the classification of tight totally geodesic maps between Hermitian symmetric spaces of noncompact type.

In Paper I we classify holomorphic tight maps. We introduce a new criterion for tightness of Hermitian regular subalgebras. Following the classification of holomorphic maps by Ihara and Satake we go through the lists of (H2)-homomorphisms and Hermitian regular subalgebras and determine which are tight.

In Paper II we show that there are no nonholomorphic tight maps into classical codomains (except the known ones from the Poincaré disc). As the proof relies heavily on composition arguments we investigate in detail when a composition of tight maps is tight. We develop a new criterion for nontightness in terms of how complex representations of Hermitian Lie algebras branches when restricted to certain subalgebras. Using this we prove the result for a few low rank cases which then extends to the full result by composition arguments.

The branching method in Paper II fails to encompass exceptional codomains. We treat one exceptional case using weighted Dynkin diagrams and the other by showing that there exists an unexpected decomposition of homomorphisms in Paper III. Together these three papers yield a full classification of tight maps from irreducible domains.

Keywords: Tight maps, Tight homomorphisms, Bounded Kähler class, Maximal representations, Toledo invariant, Hermitian symmetric spaces, Bounded cohomology

Preface

This thesis consists of the following papers.

- ▷ Oskar Hamlet,
  "*Tight holomorphic maps, a classification*",
  in *J. Lie Theory* **23** (2013), no. 3, 639–654.
- ▷ Oskar Hamlet,
  "*Tight maps and holomorphicity*",
  accepted for publication in *Transformation Groups*.
- ▷ Oskar Hamlet & Takayuki Okuda,
  "*Tight maps and holomorphicity, exceptional spaces*",
  preprint.

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Finishing this thesis I find myself contemplating the past five years. More so than any other period in my life it has been a time of very high highs and really low lows. I owe a lot of people thanks for providing me with a fun and stimulating environment and for supporting me through the hardships.

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Oskar Hamlet,
Göteborg, August 2014

Contents

Tight maps, a classification

Oskar Hamlet

Part I

INTRODUCTION

Introduction

This thesis concerns the classification of tight totally geodesic maps between Hermitian symmetric spaces of noncompact type. For the non-expert this first sentence probably contains a lot of unfamiliar concepts. This is typical, but unfortunate, when presenting mathematical research.

In an attempt to counteract this I will devote the first section of the introduction to a crash course on Hermitian symmetric spaces. The theory of symmetric spaces is a very rich one, through which we will navigate quickly to reach our goal of defining tight maps and some of the tools for understanding them. I will sometimes sacrifice full rigour in favour of accessibility and brevity. To give the reader some intuitive feel for the subject and these spaces I will illustrate their properties using examples rather than giving abstract proofs of general results. For the curious reader there are many good books on the subject, see for example [**H4**], [**S1**].

The second section is intended as a complement to the papers. Here I will work through some examples and try to convey some of the ideas behind the proofs of the results. I will also discuss some of the technicalities arising when considering nonholomorphic maps. In the third section I will briefly present the mathematical context into which tight maps fit and some applications of the results.

1. A brief introduction to Hermitian Symmetric spaces

Geometry is the study of concepts such as angles, distances and areas and how these relate to each other for various geometric configurations like triangles, circles etc. The classical geometers explored this in the plane and in three-dimensional space. In modern geometry we generalize the setting in which we study configurations to so called manifolds. Manifolds are the natural generalization of curves and surfaces to higher dimension.

An n-dimensional manifold is a space that locally "looks like" a piece of $\mathbb{R}^n$. The formal definition is as follows.

Definition 1. A topological space $\mathcal{X}$ is called an n-dimensional (topological) *manifold* if there exists an open covering $\{U_\alpha\}_{\alpha\in A}$ of $\mathcal{X}$ paired with homeomorphisms $\phi_\alpha\colon U_\alpha \to V_\alpha \subset \mathbb{R}^n$ for open subsets V_α.

The pairs (U_α, ϕ_α) are called *charts* and the set of pairs $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ is called an *atlas*. If the *transition maps* $\phi_\beta \circ \phi_\alpha^{-1}\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are differentiable for all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$ we say that $\mathcal{X}$ is a *differentiable manifold*. If we replace $\mathbb{R}^n$ by $\mathbb{C}^n$ in the definition and require the transition maps to be holomorphic we say that $\mathcal{X}$ is an n-dimensional *complex manifold*.

The most obvious example of a manifold is of course $\mathbb{R}^n$ itself. The manifold structure is given by one chart consisting of the set $\mathbb{R}^n$ paired with the identity map.

An example that better illustrates the definition is $S^2 := \{x \in \mathbb{R}^3 : ||x|| = 1\}$. We get the manifold structure on S^2 using the covering $\{U_+, U_-\}$, where $U_\pm = S^2 \setminus \{(\pm 1, 0, 0)\}$. We define the chart maps $\phi_\pm\colon U_\pm \to \mathbb{R}^2$ by $\phi_\pm(x) = \frac{1}{\pm 1 - x_1}(x_2, x_3)$. We have inverses $\phi_\pm^{-1}(y) = (\pm 1, 0, 0) + \frac{2}{||y||^2+1}(\mp 1, y_1, y_2)$, and we thus get the transition maps $\phi_\pm \circ \phi_\mp^{-1}(y) = \frac{1}{||y||^2} y$.

We observe that the transition map distorts any Euclidean geometry put on the charts. If we want to do geometry on a manifold we will have to introduce something new.

Before we do that, let us recall how we do differential geometry in Euclidean space. We will denote Euclidean space by $\mathbb{E}^n$ rather than $\mathbb{R}^n$, considering the former as having a geometric structure and the latter having none (this will soon be made more precise when we have introduced Riemannian metrics). Suppose we have a curve $\gamma = (\gamma_1, \gamma_2)\colon [0, T] \to \mathbb{E}^2$. Let us denote the time derivate of curves by a dot, i.e. $\dot{\gamma} := \frac{d\gamma}{dt}$. We calculate the length of γ by

$$l(\gamma) = \int_0^T \sqrt{\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2} dt.$$

The idea behind this calculation is that $\dot{\gamma}(t)$ gives us a velocity vector, the norm of that vector gives us the speed and the integral of the speed gives us the distance traveled, i.e. the length of the curve.

What distinguishes the geometry as Euclidean in the above calculation is how we measure the length of tangent vectors. The generalization we do in Riemannian geometry is that we allow our inner product to vary between points in $\mathbb{R}^n$. We call such a varying inner product a *Riemannian metric on* $\mathbb{R}^n$. We could thus view a Riemannian metric as a matrix valued function $A(x)$. Calculating the length of a curve $\gamma\colon [0,T] \to \mathbb{R}^n$ with respect to A would then be done as:

$$l_A(\gamma) = \int_0^T \sqrt{\dot{\gamma}^t(t)A(\gamma(t))\dot{\gamma}(t)}dt.$$

Euclidean space, $\mathbb{E}^n$, is the space $\mathbb{R}^n$ equipped with the constant metric $A(x) = I_n$, where I_n is the n by n identity matrix. Let us for a moment look at a slightly more formal way of defining and denoting Riemannian metrics. This will prove to be useful as we do calculations and investigate the properties of metrics.

For each point $x \in \mathbb{R}^n$ we denote the tangent space at x by $T_x\mathbb{R}^n$. We denote the collection of all tangent spaces by $T\mathbb{R}^n = \bigcup_x T_x\mathbb{R}^n$. We thus have $T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$ where we interpret a point $(x,v) \in T\mathbb{R}^n$ as the tangent vector v at x. For each tangent space $T_x\mathbb{R}^n$ we also define the dual space of $T_x\mathbb{R}^n$, called the space of *cotangent* vectors and denoted by $T_x^*\mathbb{R}^n$. We choose a basis $\{dx_i\}$ for $T_x^*\mathbb{R}^n$ defined by $dx_i(v) = v_i$ for $v = (v_1, ..., v_n) \in T_x\mathbb{R}^n$. We also form the space $T^*\mathbb{R}^n = \bigcup_x T_x^*\mathbb{R}^n$.

With this notation in place let us return to our Riemannian metric. Rather than writing it as a matrix valued function $x \mapsto (a_{ij}(x))$ and thinking of it as an inner product we can now write it in the formally correct way $\boldsymbol{g}(x) = \sum_{i,j} a_{ij}(x)dx_i \otimes dx_j$. We require an inner product to be positive definite and symmetric. The latter condition implies that $a_{ij} = a_{ji}$, any Riemannian metric is thus of the form $\boldsymbol{g}(x) = \sum_{i\leq j} a_{ij}(x)(dx_i \otimes dx_j + dx_j \otimes dx_i)$. We write this as $2\sum_{i\leq j} a_{ij}(x)dx_i \odot dx_j$, using the convenient symmetric product $dx_i \odot dx_j := \frac{1}{2}(dx_i \otimes dx_j + dx_j \otimes dx_i)$. With this notation Euclidean space, $\mathbb{E}^n$, is the space $\mathbb{R}^n$ equipped with the constant metric $\boldsymbol{g}(x) = \sum_i dx_i \otimes dx_i = \sum_i dx_i \odot dx_i$.

Recall that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ induces a linear map for tangent vectors $f_*\colon T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^n$, $f_*(v) = (\frac{\partial f_i}{\partial x_j})v$, where $(\frac{\partial f_i}{\partial x_j})$ is the n by n matrix with entry $\frac{\partial f_i}{\partial x_j}$ at the (i,j)-th position. We define a linear map

$f^*\colon T^*_{f(x)}\mathbb{R}^n \to T^*_x\mathbb{R}^n$ from the relation $\alpha(f_*v) = (f^*\alpha)(v)$. We have

$$f^*dx_i(v) = dx_i(f_*v) = dx_i((\frac{\partial f_j}{\partial x_k})v) = \sum_k \frac{\partial f_i}{\partial x_k} v_k = \sum_k \frac{\partial f_i}{\partial x_k} dx_k(v),$$

i.e. $f^*dx_i = \sum_k \frac{\partial f_i}{\partial x_k} dx_k$. Introducing the notationally convenient operator $d\colon \mathcal{C}^\infty(\mathbb{R}^n) \to T^*\mathbb{R}^n$, $dh := \sum_j \frac{\partial h}{\partial x_j} dx_j$, we can rewrite this as $f^*dx_i = df_i$.

Applying these transformation rules to a Riemannian metric $\boldsymbol{g} = \sum a_{ij}(x)dx_i \odot dx_j$ we get

$$\begin{aligned}(f^*\boldsymbol{g})(x)(v,w) &= \boldsymbol{g}(f(x))(f_*v, f_*w)\\ &= \sum_{i\leq j} a_{ij}(f(x))dx_i \odot dx_j\Big(\Big(\frac{\partial f_k}{\partial x_l}(x)\Big)v, \Big(\frac{\partial f_r}{\partial x_s}(x)\Big)w\Big)\\ &= \sum_{i\leq j} a_{ij}(f(x))df_i(x) \odot df_j(x)(v,w),\end{aligned}$$

i.e. $f^*\boldsymbol{g}(x) = \sum a_{ij}(f(x))df_i(x) \odot df_j(x)$. Knowing how metrics transform under differentiable maps we are ready to define Riemannian metrics for manifolds.

Definition 2. Let $(\mathcal{X}, \{(U_\alpha, \phi_\alpha)\}_{\alpha\in A})$ be a differentiable manifold. A *Riemannian metric on the manifold* $\mathcal{X}$ is a collection of Riemannian metrics $\boldsymbol{g} = \{\boldsymbol{g}_\alpha\}$ on $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ such that $(\phi_\beta \circ \phi_\alpha^{-1})^*\boldsymbol{g}_\alpha = \boldsymbol{g}_\beta$ for all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$. A differentiable manifold paired with a Riemannian metric is called a *Riemannian manifold.*

Let us return to the example of S^2. Equip the charts $U_\pm$ with the metrics $\boldsymbol{g}_\pm(y) := \frac{1}{(||y||^2+1)^2}\sum dy_i \otimes dy_i$. Let us denote the transition map $\phi_+ \circ \phi_-^{-1}$ by f and calculate $f^*\boldsymbol{g}_-$,

$$\begin{aligned}f^*\boldsymbol{g}_-(y) &= \frac{1}{(||f(y)||^2+1)^2}\sum_i df_i(y) \otimes df_i(y)\\ &= \frac{1}{\big(||\frac{1}{||y||^2}y||^2+1\big)^2}\Big(\frac{(y_2^2-y_1^2)dy_1 - 2y_1y_2dy_2}{||y||^4} \otimes \frac{(y_2^2-y_1^2)dy_1 - 2y_1y_2dy_2}{||y||^4}\\ &+ \frac{(y_1^2-y_2^2)dy_2 - 2y_1y_2dy_1}{||y||^4} \otimes \frac{(y_1^2-y_2^2)dy_2 - 2y_1y_2dy_1}{||y||^4}\Big)\end{aligned}$$

$$= \frac{1}{||y||^4(||y||^2+1)^2}\Big(((y_2^2-y_1^2)^2+4y_1^2y_2^2)dy_1\otimes dy_1$$
$$+((y_2^2-y_1^2)^2+4y_1^2y_2^2)dy_2\otimes dy_2\Big)$$
$$= \frac{1}{||y||^4(||y||^2+1)^2}\Big(||y||^4 dy_1\otimes dy_1+||y||^4 dy_2\otimes dy_2\Big)$$
$$= \frac{1}{(||y||^2+1)^2}\sum dy_i\otimes dy_i = \boldsymbol{g}_+.$$

We see that the metrics $\boldsymbol{g}_+$ and $\boldsymbol{g}_-$ agree under the transition map. They thus give us a well-defined Riemannian metric for S^2.

Sometimes two seemingly different Riemannian manifolds actually encode the exact same geometry. We say that they are different models of the same geometry or that they are isometric.

Definition 3. Let $(\mathcal{X}_1, \boldsymbol{g}_1)$ and $(\mathcal{X}_2, \boldsymbol{g}_2)$ be Riemannian manifolds and $f\colon \mathcal{X}_1 \to \mathcal{X}_2$ a diffeomorphism. We say that f is an *isometry* and that $(\mathcal{X}_1, \boldsymbol{g}_1)$ and $(\mathcal{X}_2, \boldsymbol{g}_2)$ are *isometric* if $f^*\boldsymbol{g}_2 = \boldsymbol{g}_1$.

We are often interested in isometries $f\colon \mathcal{X} \to \mathcal{X}$. It is easily seen that the isometries of a Riemannian manifold form a group which we call the *isometry group* of $\mathcal{X}$. The isometries preserve all geometric information of objects in $\mathcal{X}$.

The isometries of $\mathbb{E}^n$ are given by translations and rotations, together forming the group $\mathbb{R}^n \rtimes O(n, \mathbb{R})$. As a set this groups is given by pairs (v, A), where $v \in \mathbb{R}^n$ and $A = (a_{ij})$ is a real n by n matrix satisfying $A^tA = I_n$. This condition is equivalent to that $\sum_i a_{ij}a_{ik} = \delta_{jk}$ for all j, k, where δ_{jk} is the Kronecker delta. The group multiplication is given by $(v, A)\cdot(v', A') := (v + Av', AA')$. A group element (v, A) defines an isometry ϕ, $\phi(x) = Ax + v$, of $\mathbb{E}^n$. We see that this indeed is an isometry from the calculation:

$$\begin{aligned}\phi^*(\sum_i dx_i\odot dx_i) &= \sum_i d\phi_i\odot d\phi_i = \sum_i(\sum_j a_{ij}dx_j)\odot(\sum_k a_{ik}dx_k)\\ &= \sum_{i,j,k} a_{ij}a_{ik}dx_j\odot dx_k = \sum_{j,k}(\sum_i a_{ij}a_{ik})dx_j\odot dx_k\\ &= \sum_{j,k}\delta_{jk}dx_j\odot dx_k = \sum_j dx_j\odot dx_j.\end{aligned}$$

Let us now delve a little deeper into a new example, the hyperbolic plane, which is one of the simplest examples of non-Euclidean geometry. Here we will observe properties similar to those of Euclidean geometry as well as things that are radically different. The hyperbolic plane is defined as $\mathbb{H} := (\{x + iy = z \in \mathbb{C} : y > 0\}, \boldsymbol{g}(x, y) = \frac{dx\otimes dx+dy\otimes dy}{y^2})$. The isometries of $\mathbb{H}$ are given by maps $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Composing two such maps the coefficients transform like the matrix multiplication

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix}.$$

We can thus identify the isometry group of $\mathbb{H}$ with $SL(2, \mathbb{R})$, the group of real two by two matrices with determinant one.

The easiest way to see that these maps are indeed isometries is by allowing complex coefficients for our cotangent vectors and introducing the following notation:

$$\frac{\partial}{\partial z_i} := \frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}), \qquad \frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}),$$
$$dz_i := dx_i + idy_i, \qquad d\bar{z} := dx_i - idy_i.$$

These satisfy

$$\frac{\partial}{\partial z_i}(z_j) = \delta_{ij}, \frac{\partial}{\partial \bar{z}_i}(z_j) = 0, \frac{\partial}{\partial \bar{z}_i}(z_j) = 0, \frac{\partial}{\partial \bar{z}_i}(\bar{z}_j) = \delta_{ij},$$
$$df = \sum_i \frac{\partial f}{\partial x_i}dx_i + \frac{\partial f}{\partial y_i}dy_i = \sum_i \frac{\partial f}{\partial z_i}dz_i + \frac{\partial f}{\partial \bar{z}_i}d\bar{z}_i.$$

Returning to $\mathbb{H}$ we rewrite our metric as $\boldsymbol{g} = \frac{dx\otimes dx+dy\otimes dy}{y^2} = \frac{dz\odot d\bar{z}}{(z-\bar{z})^2}$. For a map $f(z) = \frac{az+b}{cz+d}$ we get

$$f^*\boldsymbol{g} = f^*\Big(\frac{dz \odot d\bar{z}}{(z - \bar{z})^2}\Big) = \frac{df \odot d\bar{f}}{(f(z) - \bar{f}(z))^2} = \frac{d\frac{az+b}{cz+d} \odot d\frac{a\bar{z}+b}{c\bar{z}+d}}{\big(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\big)^2}$$
$$= \frac{\frac{dz}{(cz+d)^2} \odot \frac{d\bar{z}}{(c\bar{z}+d)^2}}{\big(\frac{(az+b)(c\bar{z}+d)-(a\bar{z}+b)(cz+d)}{|cz+d|^2}\big)^2} = \frac{dz \odot d\bar{z}}{(adz + bc\bar{z} - ad\bar{z} - bcz)^2} = \frac{dz \odot d\bar{z}}{(z - \bar{z})^2}.$$

Before we can consider geometric configurations such as triangles for general Riemannian manifolds we must generalize one of the most fundamental concepts, the straight line. When generalizing the straight line we take as the defining property that a line is a curve that (locally) minimizes the path-length between two points on the curve.

Definition 4. Let $(\mathcal{X}, \boldsymbol{g})$ be a Riemannian manifold and $\gamma\colon [0,T] \to \mathcal{X}$ a smooth curve. We say that γ is a *geodesic* if for any $x \in \gamma([0,T])$ there is an open neighbourhood U of x such that γ is the shortest path between x and y for any point $y \in U \cap \gamma([0,T])$.

Let us find the geodesics in $\mathbb{H}$. We begin by trying to find a geodesic between the points i and iy_0. Start with an arbitrary smooth curve $\gamma = \gamma_1 + i\gamma_2\colon [0,1] \to \mathbb{H}$ fulfilling $\gamma(0) = i$ and $\gamma(1) = iy_0$. The length of γ is

$$l(\gamma) = \int_0^1 \boldsymbol{g}(\dot{\gamma}(t), \dot{\gamma}(t))^{\frac{1}{2}} dt = \int_0^1 \frac{(\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2)^{\frac{1}{2}}}{\gamma_2(t)} dt.$$

As we do not need to move in the x-direction a first step towards minimizing $l(\gamma)$ is to choose $\gamma_1(t) \equiv 0$. We arrive at

$$l(\gamma) = \int_0^1 \frac{|\dot{\gamma}_2(t)|}{\gamma_2(t)} dt.$$

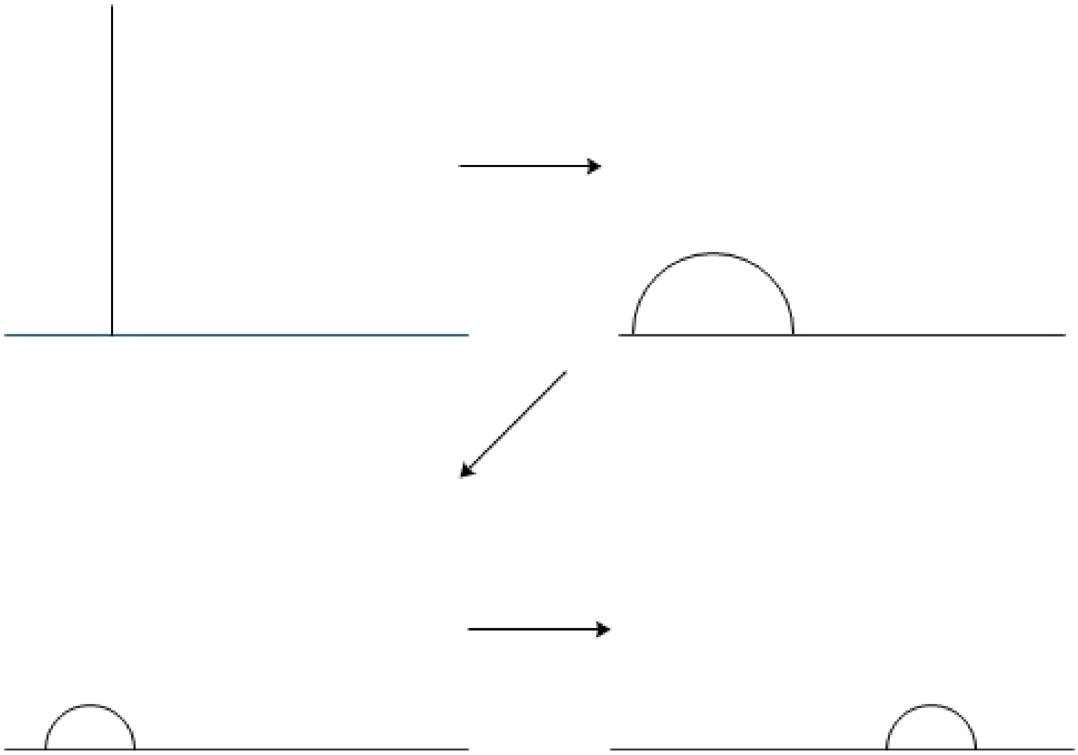
Travelling back and forth adds unnecessary distance, we can thus conlude that γ_2 should be monotone. Using that $|\dot{\gamma}_2| = \operatorname{sgn}(\dot{\gamma}_2)\dot{\gamma}_2$ and that for a monotone γ_2 we have $\operatorname{sgn}(\dot{\gamma}_2) = \operatorname{sgn}(\log(y_0))$ we get the length:

$$\begin{aligned} l(\gamma) &= \operatorname{sgn}(\dot{\gamma}_2) \int_0^1 \frac{\dot{\gamma}_2(t)}{\gamma_2(t)} dt = \operatorname{sgn}(\dot{\gamma}_2) \int_0^1 \frac{d}{dt} \log(\gamma_2(t)) dt \\ &= \operatorname{sgn}(\dot{\gamma}_2)(\log\gamma_2(1) - \log\gamma_2(0)) = \operatorname{sgn}(\log(y_0))(\log(y_0) - \log(1)) \\ &= |\log(y_0)|. \end{aligned}$$

This calculation holds for any y_0 and so we can conclude that $t \mapsto it$, $t > 0$ is an infinite geodesic. Trying to approach general geodesics in this way is harder but we have one trick up our sleeve. Namely, if we apply isometries to a geodesic it will transform into new geodesics. Consider the following three isometries of $\mathbb{H}$:

$$\begin{aligned} z &\mapsto \frac{\sqrt{2} + \sqrt{2}z}{\sqrt{2} - \sqrt{2}z} = \frac{1+z}{1-z}, \\ z &\mapsto \frac{\lambda^{\frac{1}{2}} z}{\lambda^{-\frac{1}{2}} z} = \lambda z, \lambda \in \mathbb{R}, \\ z &\mapsto z + \mu, \mu \in \mathbb{R}. \end{aligned}$$

The first isometry maps our geodesic to a half-circle centred at zero. The second isometry changes the radius of the halfcircle and the third translates it in the x-direction.



Varying the parameters λ, μ we can transform our vertical geodesic into a half-circle with center at an arbitrary point on the real line and an arbitrary radius. Two arbitrary points in $\mathbb{H}$ determine such a half-circle. We have thus found geodesics passing through any pair of points.

Let ϕ be an isometry that sends the geodesic $t \mapsto it$ to a halfcircle connecting two fixed points $z, w \in \mathbb{H}$. Suppose there is another geodesic γ connecting z and w. Then $\phi^{-1}(\gamma)$ is a geodesic connecting $\phi^{-1}(z) \in i\mathbb{R}$ and $\phi^{-1}(w) \in i\mathbb{R}$. As geodesics between points on $i\mathbb{R}$ are unique up to parametrization by our previous calculations, γ can not differ from the half-circle geodesic. We thus know all the geodesics of $\mathbb{H}$ and that there is a unique (up to parametrization) geodesic connecting any pair of points.

Having familiarized us a bit with geodesics we are ready to shed some light on the first sentence of this thesis.

Definition 5. Let $(\mathcal{X}_1, \boldsymbol{g}_1)$ and $(\mathcal{X}_2, \boldsymbol{g}_2)$ be Riemannian manifolds. A map $f\colon \mathcal{X}_1 \to \mathcal{X}_2$ is called *totally geodesic* is $f(\gamma(t))$ is a geodesic in $\mathcal{X}_2$ for every geodesic $\gamma(t)$ in $\mathcal{X}_1$. In this context we consider a constant curve a geodesic.

For general pairs of Riemannian manifolds typically only constant totally geodesic maps exist. An example of a nonconstant totally geodesic map is $f\colon \mathbb{E}^2 \to \mathbb{H}$, $(x, y) \mapsto ie^x$. We will learn how to construct more totally geodesic maps later on.

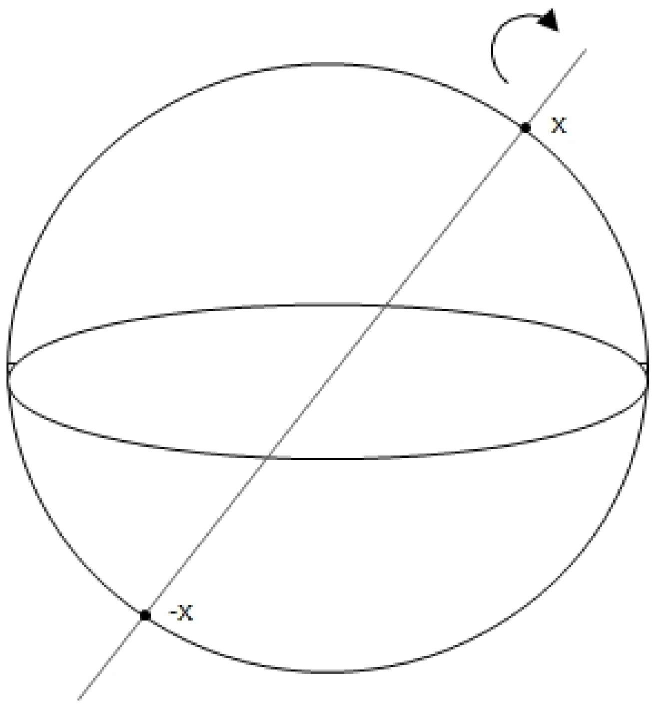
When finding the geodesics of $\mathbb{H}$ the isometries proved very useful. A key property of $\mathbb{H}$ allowing us to use them the way we did is that $\mathbb{H}$ has "many" isometries. In fact, $\mathbb{H}$ is an example of a certain class of Riemannian manifolds called symmetric spaces.

Definition 6. A Riemannian manifold $(\mathcal{X}, \boldsymbol{g})$ is called a *symmetric space* if for every point $x \in \mathcal{X}$ there exists an isometry ϕ_x of $\mathcal{X}$ such that

(1) $\phi_x^2 = \mathrm{Id}$,
(2) $\phi_x \neq \mathrm{Id}$,
(3) x is an isolated fix-point of ϕ_x.

A diffeomorphism satisfying (1) and (2) is called an *involution* or an *involutive* diffeomorphism.

We are actually already familiar with a lot of symmetric spaces. A complete list of the two dimensional spaces are $\mathbb{E}^2$, S^2 and $\mathbb{H}$. For $\mathbb{E}^2$ the involutions ϕ_x are given by $\phi_x(y) = 2x - y$. In $\mathbb{H}$ we have the involutions $\phi_z(w) = \frac{(z+\bar{z})w - 2z\bar{z}}{2w - (z+\bar{z})}$. In S^2 the involutive isometry ϕ_x is given by a 180 degree rotation around the axis through x and $-x$.



We say that a symmetric space is *irreducible* if it can not be decomposed into a product of smaller symmetric spaces. Irreducible symmetric spaces comes in three types. We say that the Euclidean spaces $\mathbb{E}^n$ are of *Euclidean type.* The remaining irreducible symmetric spaces are divided into *compact* and *noncompact type.* We say that a non-Euclidean symmetric space $\mathcal{X}$ is (non-) compact if the isometry group of $\mathcal{X}$ is (non-) compact.

Before we define Hermitian symmetric spaces we return to complex manifolds. For a complex n-manifold $\mathcal{X}$ with a chart U_α there is a natural identification of T_zU_α with $\mathbb{C}^n$. From this identification we get a *complex structure* $J_z\colon T_zU_\alpha \to T_zU_\alpha$. The complex structure is simply given by multiplication by i under the identification of T_zU_α with $\mathbb{C}^n$. Piecing together the J_z:s we get a map $J\colon T\mathcal{X} \to T\mathcal{X}$. This is well-defined, independent of which chart we choose, since the transition maps are required to be holomorphic. A Hermitian symmetric space is a symmetric space and a complex manifold where the metric and complex structure are compatible, more precisely:

Definition 7. A complex Riemannian manifold $(\mathcal{X}, \boldsymbol{g}, J)$ is called a *Hermitian symmetric space* if

(1) $\boldsymbol{g}(Jv, Jw) = \boldsymbol{g}(v, w)$,
(2) for every point $x \in \mathcal{X}$ there exists a holomorphic involutive isometry ϕ_x of $\mathcal{X}$ with x as an isolated fix-point.

Hermitian symmetric spaces come equipped with a differential two-form known as the Kähler form. A *differential two-form* is a skew-symmetric tensor

$$\alpha(x) = \frac{1}{2}\sum_{i<j} a_{ij}(x)(dx_i \otimes dx_j - dx_j \otimes dx_i) =: \sum_{i<j} a_{ij}(x) dx_i \wedge dx_j$$

which can be interpreted to measure area in a submanifold $\mathcal{X} \subset \mathbb{R}^n$. For two-forms in $U \subset \mathbb{R}^2$ we define the integral $\int_U a(x)dx_1 \wedge dx_2 := \int_U a(x)dx_1dx_2$, where the right hand side is the usual integral. If we let $\psi\colon U \to V$, $V \subset \mathbb{R}^2$, be a diffeomorphism we have

$$\int_U \psi^*(a dx_1 \wedge dx_2)(x) = \int_U a(\psi(x)) d\psi_1(x) \wedge d\psi_2(x)$$
$$= \int_U a(\psi(x))\Big(\frac{\partial \psi_1}{\partial x_1}(x)dx_1 + \frac{\partial \psi_1}{\partial x_2}(x)dx_2\Big) \wedge \Big(\frac{\partial \psi_2}{\partial x_1}(x)dx_1 + \frac{\partial \psi_2}{\partial x_2}(x)dx_2\Big)$$

$$= \int_U a(\psi(x))\Big(\frac{\partial \psi_1}{\partial x_1}(x)\frac{\partial \psi_2}{\partial x_2}(x) - \frac{\partial \psi_2}{\partial x_1}(x)\frac{\partial \psi_1}{\partial x_2}(x)\Big) dx_1 \wedge dx_2$$

The factor $(\frac{\partial \psi_1}{\partial x_1}(x)\frac{\partial \psi_2}{\partial x_2}(x) - \frac{\partial \psi_2}{\partial x_1}(x)\frac{\partial \psi_1}{\partial x_2}(x))$ is the familiar Jacobian determinant appearing when we change coordinates in $\mathbb{R}^2$. The rules for coordinate change is thus "built into" the tensor, i.e.

$$\int_U \psi^*\alpha = \int_V \alpha. \tag{1.1}$$

Let $\mathcal{S} \subset \mathcal{X}$ be a surface with a parametrization $\rho\colon V \to \mathcal{S}$, we define $\int_{\mathcal{S}} \alpha := \int_V \rho^*\alpha$. If $\eta\colon U \to \mathcal{S}$ is another parametrization, we have

$$\int_U \eta^*\alpha = \int_U \eta^*(\rho^{-1})^*\rho^*\alpha = \int_V \rho^*\alpha,$$

where we have used (1.1) in the last equality. The definition of $\int_{\mathcal{S}} \alpha$ is thus independent of the choice of parametrization.

The *Kähler form* associated to a Hermitian symmetric space $(\mathcal{X}, \boldsymbol{g}, J)$ is defined as:

$$\omega(v, w) := \boldsymbol{g}(Jv, w).$$

This is indeed a a differential form, i.e. antisymmetric, since

$$\begin{aligned}\omega(v, w) &= \boldsymbol{g}(Jv, w) = \boldsymbol{g}(J^2v, Jw) = \boldsymbol{g}(-v, Jw)\\ &= -\boldsymbol{g}(Jw, v) = -\omega(w, v).\end{aligned}$$

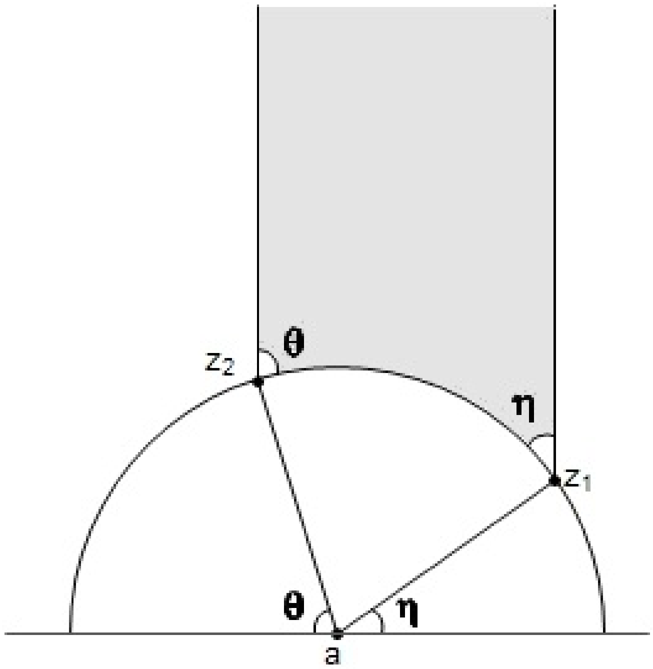
The Kähler form is invariant under holomorphic isometries (since $\boldsymbol{g}$ and J are). Let us also note that we can recover the metric from the Kähler form and the complex structure,

$$\omega(v, Jw) = \boldsymbol{g}(Jv, Jw) = \boldsymbol{g}(v, w).$$

Let us calculate the Kähler form for $\mathbb{H}$ and investigate its behaviour. The metric is $\boldsymbol{g} = \frac{dx\otimes dx + dy \otimes dy}{y^2}$, we get

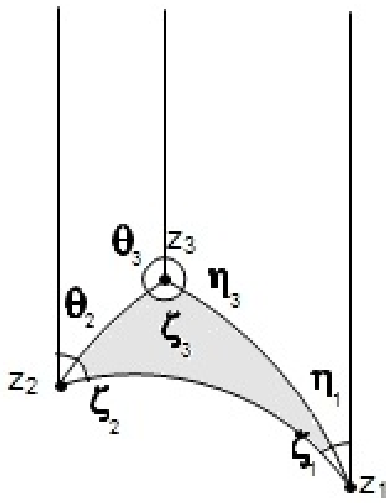
$$\begin{aligned}\omega((v_1, v_2), (w_1, w_2)) &= \boldsymbol{g}(J(v_1, v_2), (w_1, w_2)) = \boldsymbol{g}((-v_2, v_1), (w_1, w_2))\\ &= \frac{-v_2 w_1 + v_1 w_2}{y^2} = \frac{dx \otimes dy - dy \otimes dx}{y^2}((v_1, v_2), (w_1, w_2))\\ &= \frac{dx \wedge dy}{y^2}((v_1, v_2), (w_1, w_2)).\end{aligned}$$

Let us play around a bit with the Kähler form. We start by calculating the area above a geodesic segment between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, as depicted in the picture below.



We set $r := |z_1 - a| = |z_2 - a|$ and calculate :

$$\int_{x_2}^{x_1}\int_{\sqrt{r^2-(x-a)^2}}^{\infty}\frac{1}{y^2}dydx = \int_{x_2}^{x_1}\frac{1}{\sqrt{r^2-(x-a)^2}}dx = \int_{\frac{x_1-a}{r}}^{\frac{x_2-a}{r}}\frac{1}{\sqrt{1-t^2}}dt$$
$$= \sin^{-1}(\frac{x_1-a}{r}) - \sin^{-1}(\frac{x_2-a}{r}) = \pi - \theta - \eta.$$



Putting three such "strips" in the same figure as above we can calculate the area of the geodesic triangle $\Delta(z_1, z_2, z_3)$. We have

$$\begin{aligned}\text{Area}(\Delta) &= \pi - (\zeta_1 + \eta_1) - (\zeta_2 + \theta_2) - (\pi - \eta_1 - \eta_3) - (\pi - \theta_2 - \theta_3)\\ &= (\theta_3 + \eta_3 - \pi) - \zeta_1 - \zeta_2 = \pi - \zeta_1 - \zeta_2 - \zeta_3.\end{aligned}$$

The (Kähler) area of a triangle is just a function of the angles! Moreover, the area of any triangle is bounded by π. This property is shared by all Hermitian symmetric spaces of noncompact type:

Theorem 1.1 ([**DT**],[**CØ**]). *Let $\mathcal{X}$ be a Hermitian symmetric space of noncompact type with (a suitably normalized*[1]*) Kähler form ω. Then*

$$sup_{\Delta\subset\mathcal{X}} \int_\Delta \omega = rank(\mathcal{X})\pi.$$

The *rank* of a symmetric space is the dimension of the largest Euclidean space that can be totally geodesically embedded in it. All the symmetric spaces we consider in this section are of rank one.

A natural question to ask is where in $\mathcal{X}$ do we find the largest triangles? It turns out that the supremum is not realised by any triangle in $\mathcal{X}$. However, if we allow triangles with points at the boundary, so called *ideal* triangles, the supremum can be realised. The area above the geodesic segment in the calculation above is a triangle with one vertex at infinity. If we let z_1 and z_2 tend to the real line we will reach an ideal triangle with all angles equal to zero and an area of π.

A similar question is if there are any subspaces of $\mathcal{X}$ containing the largest triangles? More precisely, which subspaces $\mathcal{Y} \subset \mathcal{X}$ fulfills

$$\sup_{\Delta\subset\mathcal{Y}} \int_\Delta \omega|_{\mathcal{Y}} = \sup_{\Delta\subset\mathcal{X}} \int_\Delta \omega \ ?$$

For this question to be well-defined we must require that $\mathcal{Y}$ is a totally geodesic submanifold of $\mathcal{X}$, i.e. that the inclusion map $\iota\colon \mathcal{Y} \to \mathcal{X}$ is a totally geodesic map. In general, for a totally geodesic map $\rho\colon \mathcal{Y} \to \mathcal{X}$ (possibly not injective) we have:

$$\sup_{\Delta\subset\mathcal{Y}} \int_\Delta \rho^*\omega = \sup_{\Delta\subset\rho(\mathcal{Y})} \int_\Delta \omega|_{\rho(\mathcal{Y})} \leq \sup_{\Delta\subset\mathcal{X}} \int_\Delta \omega. \tag{1.2}$$

[1]For readers familiar with the concept, the Kähler form is normalized such that the minimal holomorphic curvature is -1.

Definition 8. A totally geodesic map $\rho\colon \mathcal{Y} \to \mathcal{X}$ between Hermitian symmetric spaces of noncompact type is called *tight* if we have equality in (1.2).

With that definition we check off the last word of the first sentence of the thesis. To see some examples of tight maps we need to introduce some more Hermitian symmetric spaces of noncompact type. Let us start by defining another model of $\mathbb{H}$, the so called Poincaré disc:

$$\mathbb{D} := (\{z \in \mathbb{C} : |z| < 1\}, \boldsymbol{g} = \frac{dz \odot d\bar{z}}{(1-|z|^2)^2}),$$
$$\omega_{\mathbb{D}} = \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}.$$

This space is isometric to $\mathbb{H}$ via the isometry $f\colon \mathbb{H} \to \mathbb{D}$ given by $f(z) = i\frac{z-i}{z+i}$. The new space we introduce is the unit ball in $\mathbb{C}^2$:

$$\mathbb{B} := (\{z = (z_1, z_2) \in \mathbb{C}^2 : |z| < 1\}, \boldsymbol{g} = \frac{dz_1 \odot d\bar{z}_1 + dz_2 \odot d\bar{z}_2}{(1-|z|^2)}$$
$$+ \sum_{i,j=1}^{2} \frac{z_i \bar{z}_j dz_j \odot d\bar{z}_i}{(1-|z|^2)^2}), \omega_{\mathbb{B}} = \frac{dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2}{(1-|z|^2)} + \sum_{i,j=1}^{2} \frac{z_i \bar{z}_j dz_j \wedge d\bar{z}_i}{(1-|z|^2)^2}.$$

There are essentially three totally geodesic maps $\upsilon, \rho, \eta\colon \mathbb{D} \to \mathbb{B}$, defined by

$$\upsilon(z) = (0, 0),$$
$$\rho(z) = (z, 0),$$
$$\eta(z) = \tfrac{\sqrt{2}}{1+|z|^2}(z, \bar{z}).$$

Calculating the pullbacks of $\omega_{\mathbb{B}}$ we get

$$\upsilon^* \omega_{\mathbb{B}} = 0,$$
$$\rho^* \omega_{\mathbb{B}} = \frac{d\rho_1 \wedge d\bar{\rho}_1 + d\rho_2 \wedge d\bar{\rho}_2}{(1-|\rho(z)|^2)} + \sum_{i,j=1}^{2} \frac{\rho_i(z)\bar{\rho}_j(z) d\rho_j \wedge d\bar{\rho}_i}{(1-|\rho(z)|^2)^2}$$
$$= \frac{dz \wedge d\bar{z}}{(1-|z|^2)} + \frac{z\bar{z} dz \wedge d\bar{z}}{(1-|z|^2)^2} = \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} = \omega_{\mathbb{D}}.$$

Before attempting to calculate $\eta^* \omega_{\mathbb{B}}$ we observe that $\eta_1 = \bar{\eta}_2$. This implies

$$d\eta_1 \wedge d\bar{\eta}_2 = d\eta_1 \wedge d\eta_1 = 0,$$

$$d\eta_1 \wedge d\bar{\eta}_1 + d\eta_2 \wedge d\bar{\eta}_2 = d\eta_1 \wedge d\bar{\eta}_1 + d\bar{\eta}_1 \wedge d\eta_1 = 0.$$

Keeping this in mind we can easily calculate:

$$\eta^*\omega_{\mathbb{B}} = \frac{d\eta_1 \wedge d\bar{\eta}_1 + d\eta_2 \wedge d\bar{\eta}_2}{(1 - |\eta(z)|^2)} + \sum_{i,j=1}^{2} \frac{\eta_i(z)\bar{\eta}_j(z)d\eta_j \wedge d\bar{\eta}_i}{(1 - |\eta(z)|^2)^2} = 0$$

Hence υ and η are not tight. By applying Theorem 1.1 twice we get

$$\begin{aligned} \sup_{\Delta\subset\mathbb{D}} \int_\Delta \rho^*\omega_{\mathbb{B}} = \sup_{\Delta\subset\mathbb{D}} \int_\Delta \omega_{\mathbb{D}} &= \pi, \\ \sup_{\Delta\subset\mathbb{B}} \int_\Delta \omega_{\mathbb{B}} &= \pi, \end{aligned}$$

i.e. ρ is tight.

We have already seen that isometry groups can be very useful when working with symmetric spaces. To get a better understanding of these groups and to see how much of the geometric information that is contained in them we need to introduce some basic Lie theory.

Definition 9. A group is called a *Lie group* if it has the structure of a differentiable manifold such that the group operation and inversion both are differentiable maps.

The type of Lie groups that appear in connection with symmetric spaces are the so called *semisimple* ones. These are in some sense "well-behaved" and have a rich theory. We will restrict our attention to matrix groups to make the presentation more concrete, this is not a serious restriction since most Lie groups can be studied using matrix realisations. Let $M_{n,m}(\mathbb{F})$ denote the space of n by m matrices with entries in the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $M_n = M_{n,n}$, and denote by I_n the identity matrix of size n. Examples of matrix groups are:

$$\begin{aligned} SL(n,\mathbb{F}) &= \{g \in M_n(\mathbb{F}) : \det(g) = 1\}, \\ SU(p,q) &= \{g \in SL(p+q,\mathbb{C}) : g^* \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} g = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}\}, \\ Sp(2n,\mathbb{R}) &= \{g \in SL(2n,\mathbb{R}) : g^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}. \end{aligned}$$

An algebraic object intimately related to a Lie group that often is easier to work with is its Lie algebra.

Definition 10. A *Lie algebra* $\mathfrak{g}$ is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) with a bilinear product $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying:

(1) $[X,Y]=-[Y,X]$,
(2) $[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0$.

We relate a Lie algebra to a matrix Lie group G as follows. Let $\mathfrak{g}$ denote the tangent space over the identity, e, of G. We can identify $\mathfrak{g}$ with the set $\{\gamma\colon(-\epsilon,\epsilon)\to G : \gamma(0)=Id\}/\sim$, where $\gamma_1\sim\gamma_2$ if $\dot\gamma_1(0)=\dot\gamma_2(0)$. The identification is simply $\gamma\mapsto\dot\gamma(0)$. We define an action of G on $\mathfrak{g}$ by

$$g\cdot X := \frac{d}{dt}(g\gamma(t)g^{-1})|_{t=0} = g\frac{d}{dt}(\gamma(t))|_{t=0}g^{-1} = gXg^{-1}.$$

Differentation and matrix multiplication commute since the operation of multiplicating a curve γ with a fixed matrix $g\in G$ only involves scalar multiplication and addition. We can also let one curve act on another. Let γ_i, $i=1,2$ be curves through the identity and set $\dot\gamma_i(0)=:X_i$. Define

$$\begin{aligned}[X_1,X_2] &:= \frac{d}{ds}(\gamma_1X_2\gamma_1^{-1})(s)|_{s=0} = (\frac{d\gamma_1}{ds}X_2\gamma_1^{-1}+\gamma_1X_2\frac{d}{ds}\gamma_1^{-1})(s)|_{s=0}\\ &= (\dot\gamma_1(s)X_2\gamma_1^{-1}(s)-\gamma_1(s)X_2\gamma_1^{-1}(s)\dot\gamma_1(s)\gamma_1^{-1}(s))|_{s=0} = X_1X_2-X_2X_1.\end{aligned}$$

A quick calculation shows that this product satisfies Definition 10. We thus have a Lie algebra structure on the tangent space $\mathfrak{g}=T_eG$ that is derived from the group structure. The Lie algebra related to a Lie group in this way is usually denoted by the same letters but in lower case Gothic letters. The Lie algebras of the matrix groups above are:

$$\begin{aligned}\mathfrak{sl}(n,\mathbb{F}) &= \{X\in M_n(\mathbb{F}) : \operatorname{tr}(X)=0\},\\ \mathfrak{su}(p,q) &= \{X\in\mathfrak{sl}(p+q,\mathbb{C}) : X^*\begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix}+\begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix}X=0\}\\ &= \{X=\begin{pmatrix} A & B\\ B^* & C\end{pmatrix} : A\in M_p(\mathbb{C}),\, B\in M_{p,q}(\mathbb{C}),\, C\in M_q(\mathbb{C}),\\ &\qquad A^*=-A,\, C^*=-C,\, \operatorname{tr}(A)+\operatorname{tr}(C)=0\},\qquad (1.3)\\ \mathfrak{sp}(2n,\mathbb{R}) &= \{X\in\mathfrak{sl}(2n,\mathbb{R}) : X^t\begin{pmatrix} 0 & I_n\\ -I_n & 0\end{pmatrix}+\begin{pmatrix} 0 & I_n\\ -I_n & 0\end{pmatrix}X=0\}.\end{aligned}$$

A homomorphism $\rho\colon G_1\to G_2$ between Lie groups induce a linear map between the corresponding Lie algebras $\rho_*\colon T_eG_1\simeq\mathfrak{g}_1\to\mathfrak{g}_2\simeq T_eG_2$. From the definition of the algebraic structure on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ above we can deduce that this linear map is a Lie algebra homomorphism. We also have the *exponential map*, $\exp\colon \mathfrak{g} \to G$, which is given by matrix exponentiation, i.e. $\exp(X) := e + \sum_n \frac{1}{n!} X^n$.

Before we relate Lie theory to symmetric spaces we need one more definition.

Definition 11. Let G be a Lie group and K a compact subgroup. We say that (G, K) is a *symmetric pair* if there exists an involutive automorphism $\sigma\colon G \to G$ such that K is the set of fixed points of σ.

A symmetric pair determines a symmetric space as follows. Let $\mathcal{X} = G/K$ be the space of cosets and choose a metric $\tilde{\boldsymbol{g}}_o$ on the single tangent space $T_o\mathcal{X}$ of $o = eK$. G acts on $\mathcal{X}$ by $g \cdot (hK) = (gh)K$. The action of K thus fixes o. We get a K-invariant metric $\boldsymbol{g}_o$ on $T_o\mathcal{X}$ by averaging $\tilde{\boldsymbol{g}}_o$ over the action of K, i.e. $\boldsymbol{g}_o(v, w) := \frac{1}{\mu(K)} \int_{k \in K} \tilde{\boldsymbol{g}}_o(k_* v, k_* w) d\mu(k)$. Here μ is the finite, K-invariant measure known as the Haar measure (such a measure always exists).

Having a K-invariant metric on $T_o\mathcal{X}$ we can extend it to a G-invariant metric on all of $\mathcal{X}$ using the group action. We define :

$$\boldsymbol{g}_{hK}(v, w) := \boldsymbol{g}_{eK}(h_*^{-1} v, h_*^{-1} w).$$

This is well-defined since if $hK = \tilde{h}K$, then $\tilde{h} = hk$ for some $k \in K$, and

$$\begin{aligned} \boldsymbol{g}_{\tilde{h}K}(v, w) &= \boldsymbol{g}_{eK}(\tilde{h}_*^{-1} v, \tilde{h}_*^{-1} w) = \boldsymbol{g}_{eK}(k_*^{-1} h_*^{-1} v, k_*^{-1} h_*^{-1} w) \\ &= \boldsymbol{g}_{eK}(h_*^{-1} v, h_*^{-1} w) = \boldsymbol{g}_{hK}(v, w) \end{aligned}$$

by the K-invariance of $\boldsymbol{g}_{eK}$. This construction determines a symmetric space $(\mathcal{X}, \boldsymbol{g})$ that is unique up to multiplication of the metric by a constant.

We get an involutive isometry s_o fixing o by $s_o(gK) := \sigma(g)K$. For an arbitrary point $gK \in \mathcal{X}$ we have an involutive isometry $s_{gK} := g \circ s_o \circ g^{-1}$.

Let us try to work through this construction for our example $\mathbb{D}$. The isometry group of $\mathbb{D}$ is

$$G = SU(1,1) := \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}\,, |a|^2 - |b|^2 = 1 \right\}.$$

It acts on $\mathbb{D}$ via fractional linear maps $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$. We choose an involutive isomorphism $\sigma\colon G \to G$, $\sigma(g) = (g^{-1})^*$. This involution fixes the compact subgroup

$$K = S(U_1 \times U_1) := \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : |a|^2 = 1 \right\}$$

$$= \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}.$$

We identify the space of cosets $\mathcal{X} = G/K$ with $\mathbb{D}$ via the diffeomorphism $gK \mapsto g \cdot 0$. We see that this is well-defined since $k \cdot 0 = \frac{a0+0}{\bar{0}0+\bar{a}} = 0$ for

$$k = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}.$$

Next we equip the tangent space $T_0\mathbb{D}$ with an arbitrary metric $\tilde{\boldsymbol{g}}_0 = \alpha dz \odot dz + \beta dz \odot d\bar{z} + \gamma d\bar{z} \odot d\bar{z}$ for some constants α, β and γ. For $k_\theta(z) = \frac{e^{i\theta}z}{e^{-i\theta}} = e^{2i\theta}z$ we get $dk_\theta = e^{2i\theta}dz$ and $d\bar{k}_\theta = e^{-2i\theta}d\bar{z}$.

Next we average $\tilde{\boldsymbol{g}}_0$ using the K-invariant measure $d\mu = \frac{d\theta}{2\pi}$ to get the K-invariant $\boldsymbol{g}_0$:

$$\begin{aligned} \boldsymbol{g}_0 = \int_{k \in K} k^* \tilde{\boldsymbol{g}}_0 d\mu(k) &= \frac{1}{2\pi} \int_0^{2\pi} (\alpha dk_\theta \odot dk_\theta + \beta dk_\theta \odot d\bar{k}_\theta + \gamma d\bar{k}_\theta \odot d\bar{k}_\theta) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\alpha e^{4i\theta} dz \odot dz + \beta dz \odot d\bar{z} + \gamma e^{-4i\theta} d\bar{z} \odot d\bar{z}) d\theta = \beta dz \odot d\bar{z}. \end{aligned}$$

Next we want to translate this metric around all of $\mathbb{D}$. For each $w \in \mathbb{D}$ we choose the element

$$g_w = \frac{1}{\sqrt{1 - |w|^2}} \begin{pmatrix} 1 & w \\ \bar{w} & 1 \end{pmatrix} \in SU(1,1)$$

that satisfies $g_w \cdot 0 = w$. We have the inverse

$$g_w^{-1} = \frac{1}{\sqrt{1 - |w|^2}} \begin{pmatrix} 1 & -w \\ -\bar{w} & 1 \end{pmatrix} \in SU(1,1).$$

As an isometry we have $g_w^{-1}(z) = \frac{z-w}{-\bar{w}z+1}$ with differentials

$$\begin{aligned} d(g_w^{-1}) &= \frac{-\bar{w}z + 1 + (z - w)\bar{w}}{(-\bar{w}z + 1)^2} dz = \frac{1 - |w|^2}{(-\bar{w}z + 1)^2} dz, \\ d(\overline{g_w^{-1}}) &= \frac{-w\bar{z} + 1 + (\bar{z} - \bar{w})w}{(-w\bar{z} + 1)^2} d\bar{z} = \frac{1 - |w|^2}{(-w\bar{z} + 1)^2} d\bar{z}. \end{aligned}$$

We get our Riemannian metric for $\mathbb{D}$:

$$\begin{aligned} \boldsymbol{g}_w(u, v) &:= (g_w^{-1})^* \boldsymbol{g}_0(u, v) = \boldsymbol{g}_0((g_w^{-1})_* u, (g_w^{-1})_* v) \\ &= \beta dz \odot d\bar{z}((g_w^{-1})_* u, (g_w^{-1})_* v) = \beta d(g_w^{-1}) \odot d(\overline{g_w^{-1}})(u, v) \end{aligned}$$

$$= \beta\Big(\frac{-\bar{w}z + 1 + (z-w)\bar{w}}{(-\bar{w}z+1)^2}dz\Big) \odot \Big(\frac{-w\bar{z} + 1 + (\bar{z}-\bar{w})w}{(-w\bar{z}+1)^2}d\bar{z}\Big)\Big|_{z=w}(u,v)$$
$$= \beta\Big(\frac{1-|w|^2}{(1-|w|^2)^2}dz\Big) \odot \Big(\frac{1-|w|^2}{(1-|w|^2)^2}d\bar{z}\Big)(u,v) = \frac{\beta dz \odot d\bar{z}}{(1-|w|^2)^2}(v,w).$$

As expected we get the familiar metric up to a multiplicative constant.

For a symmetric pair (G,K) the involution $\sigma\colon G \to G$ induces an involution $\sigma_*\colon \mathfrak{g} \to \mathfrak{g}$. This involution splits $\mathfrak{g}$ as a sum of eigenspaces of $d\sigma$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the eigenspace of the eigenvalue 1 and is the Lie algebra of K and $\mathfrak{p}$ is the eigenspace of the eigenvalue -1. We call this splitting the *Cartan decomposition* of $\mathfrak{g}$. The projection map $\pi\colon G \to G/K = \mathcal{X}$ induces a linear map $\pi_*\colon T_eG = \mathfrak{g} \to T_o\mathcal{X}$. The kernel of this map is $\mathfrak{k}$ and we can hence identify $T_o\mathcal{X} \simeq \mathfrak{p}$.

The Cartan decomposition contains a lot of geometric information about $\mathcal{X}$. For example, one can show that the composition $\pi \circ \exp\colon \mathfrak{p} \to G \to \mathcal{X}$ is a diffeomorphism between $\mathfrak{p}$ and $\mathcal{X}$ that maps lines through the origin in $\mathfrak{p}$ to geodesics through o in $\mathcal{X}$. Suppose we have a map $\rho\colon \mathcal{X}_1 = G_1/K_1 \to \mathcal{X}_2 = G_2/K_2$ satisfying $\rho(o_1) = o_2$. From the identification $T_{o_i} = \mathfrak{p}_i$ the map ρ induces a linear map $\rho_*\colon \mathfrak{p}_1 \to \mathfrak{p}_2$. In this setting there exists a very practical connection between the algebraic structure of $\mathfrak{g}_i$ and the geometric structure of $\mathcal{X}_i$:

Theorem 1.2. *The map $\rho\colon \mathcal{X}_1 \to \mathcal{X}_2$ is a totally geodesic map if $\rho_*\colon \mathfrak{p}_1 \to \mathfrak{p}_2$ can be extended to a Lie algebra homomorphism $\tilde{\rho}\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. Conversely, any Lie algebra homomorphism $\tilde{\rho}\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ defines a totally geodesic map $\rho\colon \mathcal{X}_1 \to \mathcal{X}_2$.*

Let us see how a Lie algebra homomorphism defines a totally geodesic map in an example. For $SU(p,q)$ we have the involution $\sigma(g) = (g^{-1})^*$ defining the symmetric pairs, this induces the following Cartan decomposition for $\mathfrak{g}_1 = \mathfrak{su}(1,1)$ and $\mathfrak{g}_2 = \mathfrak{su}(2,1)$:

$$\mathfrak{g}_1 = \left\{ X = \begin{pmatrix} k & z \\ \bar{z} & \bar{k} \end{pmatrix} : k,z \in \mathbb{C},\ \mathrm{tr}(X) = 0 \right\}$$
$$= \left\{ \begin{pmatrix} k & 0 \\ 0 & \bar{k} \end{pmatrix} \right\} + \left\{ \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} \right\} =: \mathfrak{k}_1 + \mathfrak{p}_1,$$
$$\mathfrak{g}_2 = \left\{ X = \begin{pmatrix} k_{11} & k_{12} & z_1 \\ -\bar{k}_{12} & k_{22} & z_2 \\ \bar{z}_1 & \bar{z}_2 & k_{33} \end{pmatrix} : k_{ij}, z_i \in \mathbb{C}, \mathrm{Re}(k_{ii}) = 0,\ \mathrm{tr}(X) = 0 \right\}$$

$$= \left\{ \begin{pmatrix} k_{11} & k_{12} & 0 \\ -\bar{k}_{12} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{pmatrix} \right\} + \left\{ \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ \bar{z}_1 & \bar{z}_2 & 0 \end{pmatrix} \right\} =: \mathfrak{k}_2 + \mathfrak{p}_2.$$

We choose a homomorphism $\tilde{\rho}\colon \mathfrak{su}(1,1) \to \mathfrak{su}(2,1)$,

$$\begin{pmatrix} k & z \\ \bar{z} & \bar{k} \end{pmatrix} \mapsto \begin{pmatrix} 2k & 0 & \sqrt{2}z \\ 0 & 2\bar{k} & \sqrt{2}\bar{z} \\ \sqrt{2}\bar{z} & \sqrt{2}z & 0 \end{pmatrix}.$$

We have diffeomorphisms $\pi_1 \circ \exp_1\colon \mathfrak{p}_1 \to G_1 \to \mathbb{D}$ and $\pi_2 \circ \exp_2\colon \mathfrak{p}_2 \to G_2 \to \mathbb{B}$:

$$\pi_1 \circ \exp_1 \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} = \pi_1 \begin{pmatrix} \cosh(|z|) & \frac{\sinh(|z|)}{|z|} z \\ \frac{\sinh(|z|)}{|z|} \bar{z} & \cosh(|z|) \end{pmatrix} = \frac{\tanh(|z|)}{|z|} z,$$

$$\pi_2 \circ \exp_2 \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ \bar{z}_1 & \bar{z}_2 & 0 \end{pmatrix} =$$

$$\pi_2 \begin{pmatrix} \frac{\cosh(|\boldsymbol{z}|)}{|\boldsymbol{z}|^2}|z_1|^2 & \frac{\cosh(|\boldsymbol{z}|)}{|\boldsymbol{z}|^2} z_1\bar{z}_2 & \frac{\sinh(|\boldsymbol{z}|)}{|\boldsymbol{z}|} z_1 \\ \frac{\cosh(|\boldsymbol{z}|)}{|\boldsymbol{z}|^2} z_2\bar{z}_1 & \frac{\cosh(|\boldsymbol{z}|)}{|\boldsymbol{z}|^2}|z_2|^2 & \frac{\sinh(|\boldsymbol{z}|)}{|\boldsymbol{z}|} z_2 \\ \frac{\sinh(|\boldsymbol{z}|)}{|\boldsymbol{z}|} \bar{z}_1 & \frac{\sinh(|\boldsymbol{z}|)}{|\boldsymbol{z}|} \bar{z}_2 & \cosh(|\boldsymbol{z}|) \end{pmatrix} = \frac{\tanh(|\boldsymbol{z}|)}{|\boldsymbol{z}|} \boldsymbol{z},$$

where $\boldsymbol{z} = (z_1, z_2)$. We calculate the totally geodesic map $\rho = \pi_2 \circ \exp_2 \circ \tilde{\rho} \circ (\pi_1 \circ \exp_1)^{-1}\colon \mathbb{D} \to \mathbb{B}$,

$$\begin{aligned} \rho(w) &= \pi_2 \circ \exp_2 \circ \tilde{\rho} \circ (\pi_1 \circ \exp_1)^{-1}(w) \\ &= \pi_2 \circ \exp_2 \circ \tilde{\rho}(\tanh^{-1}(|w|) \begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix}) \\ &= \pi_2 \circ \exp_2(\tanh^{-1}(|w|) \begin{pmatrix} 0 & 0 & \sqrt{2}\frac{w}{|w|} \\ 0 & 0 & \sqrt{2}\frac{\bar{w}}{|w|} \\ \sqrt{2}\frac{\bar{w}}{|w|} & \sqrt{2}\frac{w}{|w|} & 0 \end{pmatrix}) \\ &= \frac{\tanh(2\tanh^{-1}(|w|))}{\sqrt{2}|w|}(w, \bar{w}) \\ &= \frac{2\tanh(\tanh^{-1}(|w|))}{\sqrt{2}|w|(1 + \tanh(\tanh^{-1}(|w|))^2)}(w, \bar{w}) = \frac{\sqrt{2}}{1 + |w|^2}(w, \bar{w}). \end{aligned}$$

As we see it is quite a bit of work moving between the Lie algebra homomorphisms and the totally geodesic maps in practice, especially if the homomorphisms become more complicated. Fortunately, we can mostly stay on the Lie algebra side for most of our concerns. If we want to know if a homomorphism $\tilde{\rho}\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ corresponds to a holomorphic map $\rho\colon \mathcal{X}_1 \to \mathcal{X}_2$ we use the characterization that ρ is holomorphic if $\rho_* \circ J_1 = J_2 \circ \rho_*\colon T_x\mathcal{X}_1 \to T_{\rho(x)}\mathcal{X}_2$ for all $x \in \mathcal{X}_1$. For symmetric spaces and totally geodesic maps it is enough if this is satisfied for one point $x \in \mathcal{X}_1$. By choosing our realisations $\mathcal{X}_i = G_i/K_i$ carefully we can assume that $\rho(o_1) = o_2$. We thus have that ρ is holomorphic if $\tilde{\rho}_* \circ J_1 = J_2 \circ \tilde{\rho}_*\colon T_{o_1}\mathcal{X}_1 \simeq \mathfrak{p}_1 \to \mathfrak{p}_2 \simeq T_{o_2}\mathcal{X}_2$. Further, for Hermitian Lie algebras $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ there is an element $Z \in \mathfrak{k}$ such that $[Z, \cdot] = J\colon \mathfrak{p} \to \mathfrak{p}$. The homomorphism $\tilde{\rho}\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ thus corresponds to a holomorphic map if $\tilde{\rho}([Z_1, X]) = [Z_2, \tilde{\rho}(X)]$ for all $X \in \mathfrak{p}_1$.

Let us return to our example to observe this. For $\mathfrak{g}_1 = \mathfrak{su}(1,1)$ and $\mathfrak{g}_2 = \mathfrak{su}(2,1)$ with Cartan decompositions as above we have

$$Z_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ Z_2 = \frac{1}{3}\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$

For the homomorphism $\tilde{\rho}$ in the example above we have

$$\begin{aligned}
\tilde{\rho}([Z_1, X]) &= \tilde{\rho}\Big[\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}\Big] = \tilde{\rho}\begin{pmatrix} 0 & iz \\ -i\bar{z} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \sqrt{2}iz \\ 0 & 0 & -\sqrt{2}i\bar{z} \\ -\sqrt{2}i\bar{z} & \sqrt{2}iz & 0 \end{pmatrix},
\end{aligned}$$

while

$$\begin{aligned}
[Z_2, \tilde{\rho}(X)] &= \Big[\frac{1}{3}\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, \begin{pmatrix} 0 & 0 & \sqrt{2}z \\ 0 & 0 & \sqrt{2}\bar{z} \\ \sqrt{2}\bar{z} & \sqrt{2}z & 0 \end{pmatrix}\Big] \\
&= \begin{pmatrix} 0 & 0 & \sqrt{2}iz \\ 0 & 0 & \sqrt{2}i\bar{z} \\ -\sqrt{2}i\bar{z} & -\sqrt{2}iz & 0 \end{pmatrix}.
\end{aligned}$$

The totally geodesic map corresponding to $\tilde{\rho}$ is thus not holomorphic which agrees with our previous calculations where we found the map to be $w \mapsto \frac{\sqrt{2}}{1+|w|^2}(w, \bar{w})$.

Before finishing this introductory part let me mention that there are also ways of determining whether a Lie algebra homomorphism corresponds to a tight map. Explaining these methods is beyond the scope of this introduction though. Let us just note that we can study totally geodesic maps using Lie algebra homomorphisms without having to do the cumbersome translations into totally geodesic maps. It is mainly from the perspective of Lie algebras and their homomorphisms I have studied tight maps. For the reader curious to learn more about Lie algebras there are many good books, I warmly recommend [**FH**] and for a more introductory level [**H1**]. Let me finish this part by stating the main result of this thesis.

Theorem 1.3. *Let $\mathcal{X}$ be an irreducible Hermitian symmetric space of noncompact type that is not isometric to $\mathbb{D}$. Then any tight map $\rho\colon \mathcal{X} \to \mathcal{Y}$ must be (anti-) holomorphic.*

This is shown for classical codomains in Paper II and for exceptional codomains in Paper III. Further, the tight holomorphic maps are fully classified in Paper I. The theorem above implies that this is a full classfication for irreducible domains, the only exception being the tight nonholomorphic maps from $\mathbb{D}$, but these are classfied in [**BIW2**].

2. Summary of the papers

This section is meant as a complement to the papers, to be read in conjunction with them. In this section I will try to highlight some of the main ideas and work through some examples of the more general methods in the papers. Before we start I would like to state two guiding principles for how to think about compositions and products of maps with respect to tightness:

"Lemma" 2.1. *Let $\rho\colon \mathcal{X}_1 \to \mathcal{X}_2$ and $\eta\colon \mathcal{X}_2 \to \mathcal{X}_3$ be totally geodesic maps between Hermitian symmetric spaces of noncompact type. The composition $\eta \circ \rho$ is tight if and only if both ρ and η are tight.*

"Lemma" 2.2. *Let $\rho = \rho_1 \times ... \times \rho_n \colon \mathcal{Y} \to \mathcal{X} = \mathcal{X}_1 \times ... \times \mathcal{X}_n$ be a totally geodesic map between Hermitian symmetric spaces of noncompact type. The map ρ is tight if and only if all the ρ_i are tight.*

These "lemmas" are almost fully valid in the holomorphic case, but as we start considering nonholomorphic maps we run into trouble. We will return to these "lemmas" in the subsection about nonholomorphic tight maps to see where they go wrong and what we can do about it.

2.1. The classification of holomorphic tight maps. In the first paper we classify all the holomorphic tight maps. The classification is fairly straightforward; holomorphic maps were classified long ago, [**S2**], [**I**], and there is a nice criterion for tightness in terms of Lie algebra homomorphisms due to Burger, Iozzi and Wienhard:

Theorem 2.3 ([**BIW2**]). *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Hermitian Lie algebras and $d_i\colon \mathfrak{su}(1,1) \to \mathfrak{g}_i$ diagonal discs for $i = 1, 2$. Further let $Z_{\mathfrak{su}(1,1)}$ (resp. Z_2) denote the elements of $\mathfrak{su}(1,1)$ (resp. $\mathfrak{g}_2$) inducing the complex structure on the corresponding symmetric spaces. A homomorphism $\rho\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ corresponds to a tight and positive map if and only if*

$$\langle \rho(d_1(Z_{\mathfrak{su}(1,1)})) - d_2(Z_{\mathfrak{su}(1,1)}), Z_2 \rangle = 0.$$

Here the brackets denote the Killing form of $\mathfrak{g}_2$.

At a first glance it seems like all we have to do is go through the list of holomorphic maps using this criterion. Let us illustrate with two examples how the classification is done and why having a second criterion might be convenient.

Before we start we recall that as a part of the classification of holomorphic maps Satake and Ihara proved that for each homomorphism $\rho\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ that corresponds to a holomorphic map there is a decomposition $\rho = \iota \circ \tilde{\rho}\colon \mathfrak{g}_1 \to \mathfrak{g}_3 \to \mathfrak{g}_2$ such that

(1) $\mathfrak{g}_3$ is a Hermitian regular subalgebra of $\mathfrak{g}_2$ containing the image of ρ,
(2) the homomorphism $\tilde{\rho}$, which is ρ with a restricted codomain, satisfies the condition $(H2)$.

The classification of holomorphic maps in Satake and Ihara consists of a list of all $(H2)$-homomorphisms and a list of all Hermitian regular subalgebras. Any holomorphic map can then be constructed as a composition.

Say we want to classify which homomorphisms $\rho\colon \mathfrak{su}(3,1) \to \mathfrak{su}(9,3)$ correspond to tight holomorphic maps. We begin by listing the possible Hermitian regular subalgebras of $\mathfrak{su}(9,3)$, they are all of the form

$$\mathfrak{g} = \sum \mathfrak{su}(p_i, q_i) \text{ such that } \sum_i p_i \leq 9, \sum_i q_i \leq 3.$$

Next we check the lists for $(H2)$-homomorphisms from $\mathfrak{su}(3,1)$ into these $\mathfrak{g}$:s. The list of relevant $(H2)$-homomorphisms from $\mathfrak{su}(3,1)$ into simple Hermitian Lie algebras consists of

$$\begin{aligned}\rho_{100}\colon \mathfrak{su}(3,1) &\to \mathfrak{su}(3,1),\\ \rho_{010}\colon \mathfrak{su}(3,1) &\to \mathfrak{su}(3,3),\end{aligned}$$

where ρ_{ijk} denotes the restriction of a complex representation of highest weight (i,j,k). The homomorphism ρ_{100} is just the identity homomorphism, ρ_{010} is defined from a skewsymmetric tensor product of power two. As the $(H2)$-property is preserved under sums of homomorphisms we arrive at the following four homomorphisms:

$$\begin{aligned}\iota_1 \circ \rho_{100}\colon \mathfrak{su}(3,1) &\to \mathfrak{su}(3,1) \to \mathfrak{su}(9,3),\\ \iota_2 \circ \rho_{100}^{\oplus 2}\colon \mathfrak{su}(3,1) &\to \mathfrak{su}(3,1)^{\oplus 2} \to \mathfrak{su}(9,3),\\ \iota_3 \circ \rho_{100}^{\oplus 3}\colon \mathfrak{su}(3,1) &\to \mathfrak{su}(3,1)^{\oplus 3} \to \mathfrak{su}(9,3),\\ \iota_4 \circ \rho_{010}\colon \mathfrak{su}(3,1) &\to \mathfrak{su}(3,3) \to \mathfrak{su}(9,3).\end{aligned}$$

We begin by checking which $(H2)$-homomorphisms are tight. The homomorphism ρ_{100} is tight since it is just the identity. Tightness of $\rho_{100}^{\oplus 2}$ and $\rho_{100}^{\oplus 3}$ follows by "Lemma" 2.2. To figure out if ρ_{010} is tight we use Theorem 2.3. Using the matrix models in (1.3) we define diagonal discs:

$$d_1\colon \mathfrak{su}(1,1) \to \mathfrak{su}(3,1),\ d_2\colon \mathfrak{su}(1,1) \to \mathfrak{su}(3,3),$$

$$d_1\begin{pmatrix} k & z \\ \bar{z} & \bar{k}\end{pmatrix} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & z \\ 0 & \bar{z} & \bar{k}\end{pmatrix},$$

$$d_2\begin{pmatrix} k & z \\ \bar{z} & \bar{k}\end{pmatrix} := \begin{pmatrix} k & 0 & 0 & 0 & 0 & z \\ 0 & k & 0 & 0 & z & 0 \\ 0 & 0 & k & z & 0 & 0 \\ 0 & 0 & \bar{z} & \bar{k} & 0 & 0 \\ 0 & \bar{z} & 0 & 0 & \bar{k} & 0 \\ \bar{z} & 0 & 0 & 0 & 0 & \bar{k}\end{pmatrix}.$$

Next we have to calculate $\rho_{010}(d_1(Z_{\mathfrak{su}(1,1)})$. To do this we need to recall how skewsymmetric tensor representations are defined. This is most easily done by abstracting away from the matrix models for a moment.

Let $(V, F_{3,1})$ be a complex vector space of dimension four paired with a Hermitian form of signature $(3,1)$. The Lie algebra $\mathfrak{su}(3,1)$ is defined as the Lie algebra of endomorphisms $X: V \to V$ satisfying $F_{3,1}(Xv, w) + F_{3,1}(v, Xw) = 0$. If we take the wedge product $V \wedge V$ we get a complex vector space of dimension six. The Hermitian form $F_{3,1}$ has a natural extension $\tilde{F}_{3,1}$ to $V \wedge V$ by

$$\tilde{F}_{3,1}(v \wedge w, v' \wedge w') := F_{3,1}(v, v')F_{3,1}(w, w') - F_{3,1}(v, w')F_{3,1}(w, v').$$

The endomorphisms $X \in \mathfrak{su}(3,1)$ induce endomorphisms of $V \wedge V$ by $X(v \wedge w) := Xv \wedge w + v \wedge Xw$. A simple calculation shows that $\tilde{F}_{3,1}(X(v \wedge w), v' \wedge w') + \tilde{F}_{3,1}(v \wedge w, X(v' \wedge w')) = 0$. The signature of $\tilde{F}_{3,1}$ is $(3,3)$ and the wedge product thus defines a homomorphism $\rho_{010}: \mathfrak{su}(3,1) \to \mathfrak{su}(3,3)$. Let $\{e_i\}_1^4$ be an orthonormal basis for $(V, F_{3,1})$ with $F_{3,1}(e_i, e_i) = 1$ for $i = 1, 2, 3$ and $F_{3,1}(e_4, e_4) = -1$. With respect to this basis we have

$$d_1(Z_{\mathfrak{su}(1,1)})e_i = \begin{cases} 0 \text{ if } i = 1, 2, \\ \frac{i}{2}e_3 \text{ if } i = 3, \\ -\frac{i}{2}e_4 \text{ if } i = 4. \end{cases}$$

The set $\{e_i \wedge e_j\}_{1 \le i < j \le 4}$ defines an orthonormal basis for $(V \wedge V, \tilde{F}_{3,1})$ with $\tilde{F}_{3,1}(e_i \wedge e_4, e_i \wedge e_4) = -1$ and $\tilde{F}_{3,1}(e_i \wedge e_j, e_i \wedge e_j) = 1$ for $j \neq 4$. We get that

$$\rho_{010} \circ d_1(Z_{\mathfrak{su}(1,1)})(e_i \wedge e_j) = \begin{cases} 0 \text{ if } (i,j) = (1,2) \text{ or } (3,4), \\ \frac{i}{2}e_i \wedge e_j \text{ if } (i,j) = (1,3) \text{ or } (2,3), \\ -\frac{i}{2}e_i \wedge e_j \text{ if } (i,j) = (1,4) \text{ or } (2,4). \end{cases}$$

Fixing the ordered basis $\{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_4, e_3 \wedge e_4\}$, the Lie algebra $\mathfrak{su}(3,3)$ is realised as our standard model (1.3) and hence

$$d_2(Z_{\mathfrak{su}(1,1)})(e_i \wedge e_j) = \begin{cases} \frac{i}{2}e_i \wedge e_j \text{ if } j \neq 4, \\ -\frac{i}{2}e_i \wedge e_j \text{ if } j = 4. \end{cases}$$

We thus have $\rho_{010} \circ d_1(Z_{\mathfrak{su}(1,1)}) - d_2(Z_{\mathfrak{su}(1,1)}) = \frac{i}{2}\text{diag}(-1, 0, 0, 0, 0, 1)$. The complex structure for $\mathfrak{su}(3,3)$ is $Z_2 = \frac{i}{2}\text{diag}(1, 1, 1, -1, -1, -1)$. Hence ρ_{010} does not satisfy the condition in Theorem 2.3 and is not tight.

The next step is to determine which regular subalgebras are tightly embedded. Before we can do that we need to set some notation. We use our usual matrix model

$$\mathfrak{su}(9,3) = \left\{ \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \right\}$$

where A, B, C are block matrices as in (1.3). We choose the Cartan decomposition

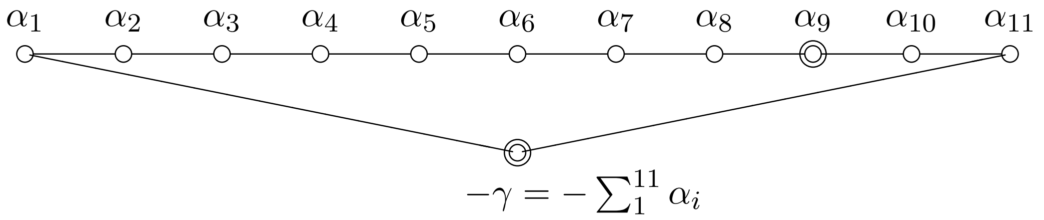
$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \right\}, \mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \right\}.$$

We choose a maximal abelian subalgebra $\mathfrak{h} := \{\text{diagonal matrices}\} \subset \mathfrak{k}$. We note that $Z = \frac{i}{12}\text{diag}(3I_9, -9I_3) \in \mathfrak{h}$. Next we complexify $\mathfrak{su}(9,3)$ to get $\mathfrak{sl}(12, \mathbb{C})$. Now $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{sl}(12, \mathbb{C})$ and we have a root space decomposition $\mathfrak{sl}(12, \mathbb{C}) = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$. Since $Z \in \mathfrak{h} \subset \mathfrak{h}^{\mathbb{C}}$, $Z \in \text{center}(\mathfrak{k}^{\mathbb{C}})$ and $\text{ad}^2(Z)(X) = -X$ for all $X \in \mathfrak{p}^{\mathbb{C}}$ we have that $\alpha(Z) = 0, i$ or $-i$ for all $\alpha \in \Delta$. We say that α is compact in the first case and noncompact in the two latter. We want to choose a set of simple roots Γ for Δ such that $\alpha(Z) \neq -i$ for all $\alpha \in \Gamma$. Let $E_{i,j}$ denote the matrix with entry one at the (i,j)-th position and zeros elsewhere and $E^*_{k,l}$ the basis of the dual space of $M_{12}(\mathbb{C})$, i.e. the linear maps defined by $E^*_{k,l}(E_{i,j}) = \delta_{ki}\delta_{lj}$. The set $\Gamma = \{\alpha_i = E^*_{i,i} - E^*_{i+1,i+1}\}_{i=1}^{11}$ forms a set of simple roots with the desired property. We have

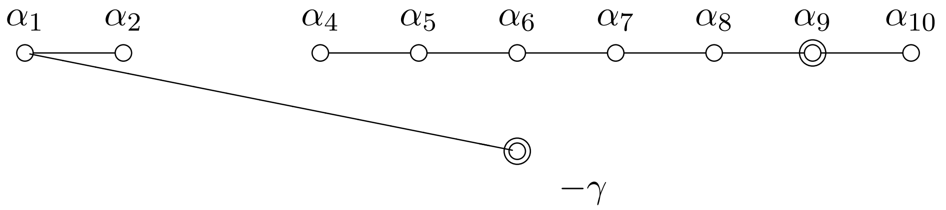
$$\alpha_j(Z) = \begin{cases} 0 \text{ if } j \neq 9, \\ i \text{ if } j = 9. \end{cases}$$

With some notation in place we are ready to construct our Hermitian regular subalgebras. This is done by choosing a subroot system $\Delta' \subset \Delta$. The smallest subalgebra of $\mathfrak{sl}(12, \mathbb{C})$ containing $\sum_{\alpha \in \Delta'} \mathfrak{g}_\alpha$, denoted $\mathfrak{g}^{\mathbb{C}}(\Delta')$, is then a (complex) regular subalgebra of $\mathfrak{sl}(12, \mathbb{C})$. We define a Hermitian regular subalgebra $\mathfrak{g}(\Delta') \subset \mathfrak{su}(9,3)$ by taking the intersection $\mathfrak{g}(\Delta') := \mathfrak{g}^{\mathbb{C}}(\Delta') \cap \mathfrak{su}(9,3)$. A practical way of constructing the subroot systems is via *elementary operations* on the Dynkin diagram. An elementary operation consists of first adding the lowest root to the diagram and then removing some roots of our choosing. We can repeat the process on the components of the resulting Dynkin diagram.

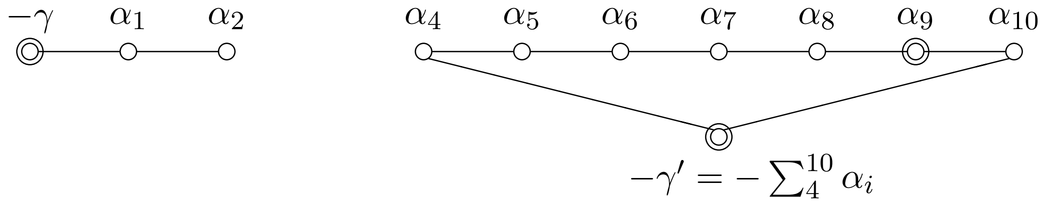
To get the Hermitian regular subalgebra $\mathfrak{su}(3,1)^{\oplus 3} \subset \mathfrak{su}(9,3)$ we start with the Dynkin diagram of $\mathfrak{sl}(12, \mathbb{C})$ where we have added the lowest root.



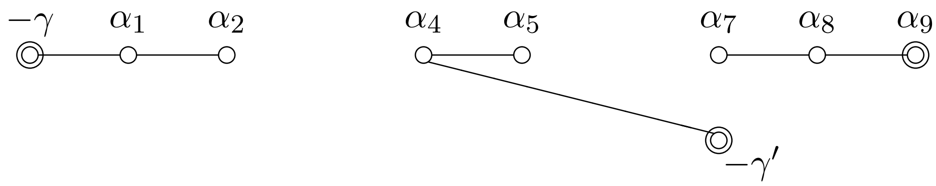
We have marked the noncompact roots with a circle. We start by removing the roots α_3 and α_{11}:



We repeat the process on the right component by first adding the lowest root $-\gamma' = -\sum_4^{10} \alpha_i$:



We then remove the roots α_6 and α_{10} to arrive at:



This is the Dynkin diagram of $\mathfrak{sl}(4,\mathbb{C})^{\oplus 3} \subset \mathfrak{sl}(12,\mathbb{C})$. Having a distinguishment between compact and noncompact roots we can see in the diagram that the real form we get when intersecting with $\mathfrak{su}(9,3)$ is $\mathfrak{su}(3,1)^{\oplus 3}$. The subroot systems for the copies of $\mathfrak{su}(3,1)$ have their sets of simple roots:

$$\Delta_1 = \{-\gamma, \alpha_1, \alpha_2\}, \Delta_2 = \{-\gamma', \alpha_4, \alpha_5\}, \Delta_3 = \{\alpha_7, \alpha_8, \alpha_9\}.$$

Rather than calculating the concrete inclusion homomorphism we will use the subroot system structure to figure out whether this inclusion is tight. Let us for a moment switch our viewpoint to the corresponding Hermitian symmetric spaces:

$$\mathcal{X}_{3,1} \times \mathcal{X}_{3,1} \times \mathcal{X}_{3,1} \subset \mathcal{X}_{9,3}.$$

Let $\boldsymbol{g}_{3,1}$ denote the normalized Riemannian metric of $\mathcal{X}_{3,1}$ and $\boldsymbol{g}_{9,3}$ that of $\mathcal{X}_{9,3}$. Being symmetric subspaces we have that the restriction satisfies $\boldsymbol{g}_{9,3}|_{\mathcal{X}_{3,1}} = c_i \boldsymbol{g}_{3,1}$ for some positive constant c_i for the i:th copy of $\mathcal{X}_{3,1}$. Since the inclusion of a regular subalgebra corresponds to a holomorphic embedding of the corresponding spaces this implies that $\omega_{9,3}|_{\mathcal{X}_{3,1}} = c_i \omega_{3,1}$ for the same constant c_i as for the metric.

To figure out whether the embedding is tight we have to determine the c_i:s. The normalization of the metric is determined by the minimal holomorphic curvature. The holomorphic curvature of $\mathcal{X}_{9,3}$ is minimized by vectors $X \in \mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma} \cap \mathfrak{su}(9,3)$ where γ is the highest root. The same is true for the subalgebras (for their own highest roots). After a little algebraic manipulation we soon arrive at $c_i = \frac{\langle \gamma, \gamma \rangle}{\langle \gamma_i, \gamma_i \rangle}$ where γ_i is the highest root of Δ_i and the brackets denotes the Killing form of $\mathfrak{su}(9,3)$. Just using the definition of tightness and Theorem 1.1 we get that the inclusion is tight if $\sum \frac{\langle \gamma, \gamma \rangle}{\langle \gamma_i, \gamma_i \rangle} r_i = r$, where r_i is the real rank $\mathfrak{g}(\Delta_i) = \mathfrak{su}(3,1)$ and r is the real rank of $\mathfrak{su}(9,3)$. In our case all roots are of the same length and the rank of $\mathfrak{su}(p,q) = \min(p,q)$. Hence

$$\sum \frac{\langle \gamma, \gamma \rangle}{\langle \gamma_i, \gamma_i \rangle} r_i = 1 + 1 + 1 = 3 = r$$

and we can conlcude that this inclusion is tight.

For the regular subalgebras $\mathfrak{su}(3,1)^{\oplus 2} \subset \mathfrak{su}(9,3)$ and $\mathfrak{su}(3,1) \subset \mathfrak{su}(9,3)$ we have less terms in the sum and we thus do not get equality. The inclusions of these regular subalgebras are thus not tight. Summarizing, there is only one homomorphism that corresponds to a tight holomorphic map, namely

$$\iota_3 \circ \rho_{100}^{\oplus 3} \colon \mathfrak{su}(3,1) \to \mathfrak{su}(3,1)^{\oplus 3} \to \mathfrak{su}(9,3).$$

In the above example we briefly introduced and applied the tools used in the first paper. In this example it is not that hard to see that the Hermitian regular subalgebras are block subalgebras. We could rather easily have used Theorem 2.3 for the Hermitian regular subalgebras in this case. Let us do one more (short) example where the new criterion really comes in handy.

Consider the problem of classifying which $\rho\colon \mathfrak{e}_{6(-14)} \to \mathfrak{e}_{7(-25)}$ correspond to tight holomorphic maps. In [**I**] we learn that there is only one $(H2)$-map from $\mathfrak{e}_{6(-14)}$, the identity homomorphism to itself. We also see that $\mathfrak{e}_{6(-14)}$ is a regular subalgbra of $\mathfrak{e}_{7(-25)}$. There is thus just one homomorphism that corresponds to a holomorphic map, the inclusion homomorphism of $\mathfrak{e}_{6(-14)}$ as a regular subalgebra of $\mathfrak{e}_{7(-25)}$. As all roots in the root system of $\mathfrak{e}_{7,\mathbb{C}}$ are of the same length we do not even have to know the subroot system defining this inclusion. The highest root of the subsystem will be of the same length as the highest root of the root system of $\mathfrak{e}_{7,\mathbb{C}}$. For any subroot system $\Delta' \subset \Delta$ we thus get

$$\frac{\langle \gamma, \gamma \rangle}{\langle \gamma', \gamma' \rangle} \operatorname{rank}(\mathfrak{e}_{6(-14)}) = 2 \neq 3 = \operatorname{rank}(\mathfrak{e}_{7(-25)})$$

and can thus conclude that the inclusion is not tight by just comparing ranks.

2.2. The nonexistence of tight nonholomorphic maps. In the second and third paper we show the nonexistence of tight non-holomorphic maps. In this section I will try to walk through the proof, not in the order it is presented in the papers, but rather chronologically following how the ideas grew. Starting with a simple case and a rather simple idea we will see how this case generalizes and why some technicalities line up along the way. I hope this will shed some light on the idea and intuition behind the proof.

We begin by considering homomorphisms $\rho\colon \mathfrak{su}(2,1) \to \mathfrak{su}(p,q)$. Fixing a matrix model of $\mathfrak{su}(p,q)$ the homomorphism ρ defines an action of $\mathfrak{su}(2,1)$ on $\mathbb{C}^{p+q}$, i.e. a complex representation. It is the theory of finite dimensional complex representations that will be our main tool. We will frequently switch between the viewpoints of ρ as a homomorphism between abstract real Lie algebras and ρ as a complex representation.

Before we start we recall two things from the classification of tight holomorphic maps. First, out of all irreducible representations of $\mathfrak{su}(3,1)$ there was just two that were holomorphic and of those only one was tight. Proving that nonholomorphic homomorphisms are not tight is thus roughly speaking equivalent to showing that an arbitrary representation is not tight. Second, we saw in the previous section that it required a fair bit of work to calculate $\rho_{010}(d_1(Z_{\mathfrak{su}(1,1)}))$ for the skew-symmetric represention of $\mathfrak{su}(3,1)$. For a general (i,j)- highest weight representation $\rho_{ij}\colon \mathfrak{su}(2,1) \to \mathfrak{su}(p,q)$ the calculations quickly get out of hand. We need a strategy that does not rely on such an explicit description of our homomorphism.

For irreducible representations $\rho\colon \mathfrak{su}(1,1) \to \mathfrak{su}(p,q)$ there is a nice characterisation of tightness due to Burger et al., [**BIW2**], namely that ρ is tight if and only if it is of odd highest weight. Now consider the composition $\rho_{ij} \circ \iota\colon \mathfrak{su}(1,1) \to \mathfrak{su}(2,1) \to \mathfrak{su}(p,q)$, where ι is the standard (tight and holomorphic) inclusion. By "Lemma" 2.1 this composition is tight if and only if ρ_{ij} and ι are both tight. Having fixed a tight ι we thus get that ρ_{ij} is tight if and only if $\rho_{ij} \circ \iota$ is tight. The composition $\rho_{ij} \circ \iota$ will never be irreducible. As a representation we have a branching into irreducible representations

$$\rho_{ij} \circ \iota = \sum n_i \rho_i. \tag{2.1}$$

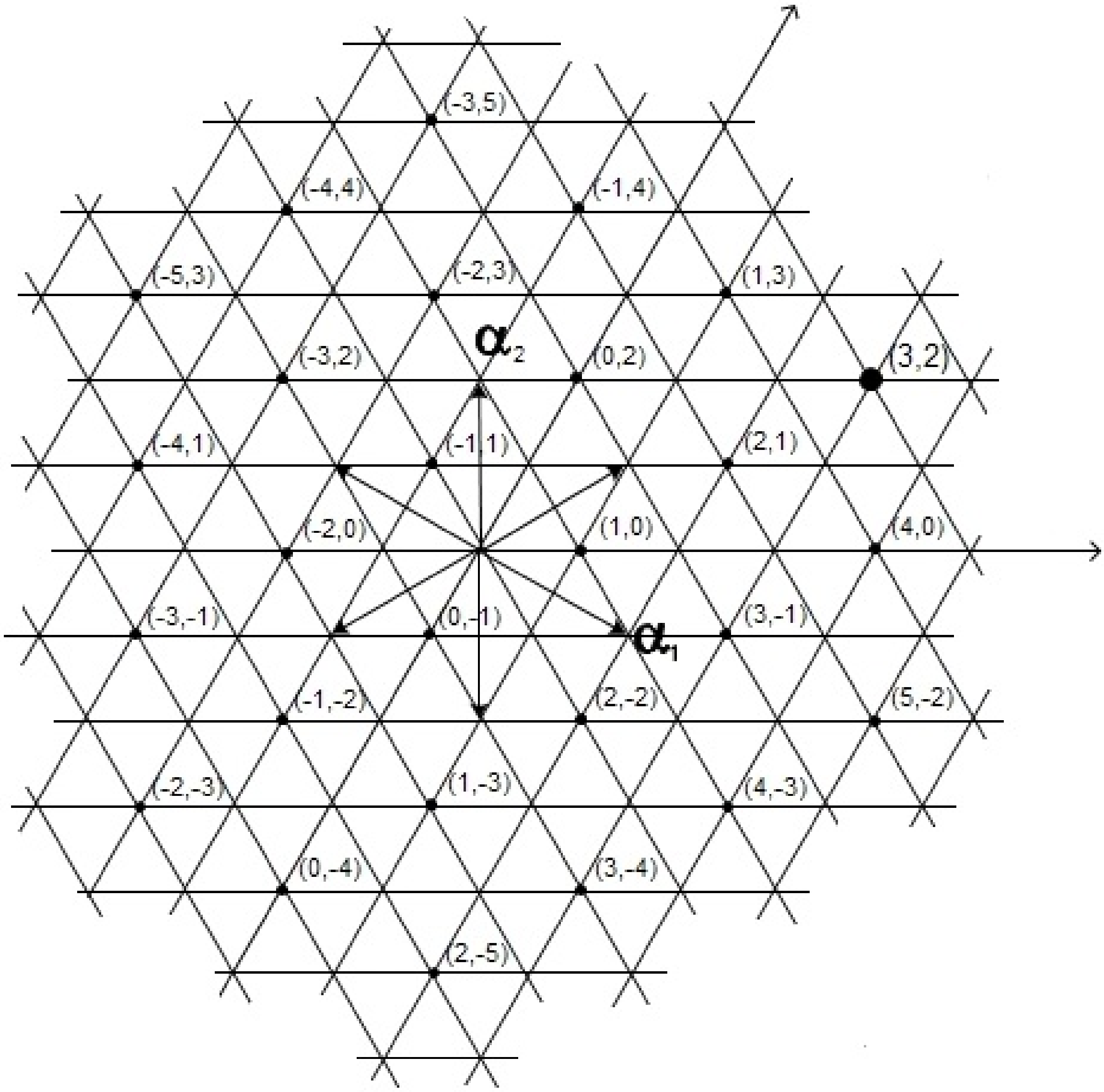
This implies that as a homomorphism $\rho_{ij} \circ \iota$ can be decomposed as

$$\iota' \circ \oplus n_i \rho_i\colon \mathfrak{su}(1,1) \to \bigoplus \mathfrak{su}(p_i,q_i)^{\oplus n_i} \to \mathfrak{su}(p,q).$$

By "Lemma" 2.2 we have that $\oplus n_i \rho_i$ fails to be tight if one ρ_i fails to be so. In turn this imples that $\iota' \circ \oplus n_i \rho_i = \rho_{ij} \circ \iota$ is nontight which implies that ρ_{ij} is nontight by two applications of "Lemma" 2.1. To show that ρ_{ij} is nontight it thus suffices to show that one ρ_i is nontight, or equivalently that one ρ_i is of an even (nonzero) highest weight. Let us look a bit at representations of $\mathfrak{su}(2,1)$ and see why we should expect this to happen.

Any complex representation of $\mathfrak{su}(2,1)$ is a restriction of a complex representation of $\mathfrak{sl}(3,\mathbb{C})$. For a complex representation $\rho\colon \mathfrak{sl}(3,\mathbb{C}) \to \mathfrak{gl}(V)$ the image of both $H_1 := E_{1,1} - E_{2,2}$ and $H_2 := E_{2,2} - E_{3,3}$ will always (with respect to an appropriate basis of V) be diagonal matrices with integer entries. The vector space V splits into weight spaces (simultaneous eigenspaces) as $V = \bigoplus_{(k,l)\in\mathcal{I}} V_{(k,l)}$, where we have $H_1 v = kv$ and $H_2 v = lv$ for $v \in V_{(k,l)}$. Putting a reasonable partial ordering on pairs of integers (i,j), say $(k,l) > (k',l')$ if $k + l > k' + l'$, the representation is completely determined by its highest weight.

Given a highest weight (i,j) we can visualize the weight spaces in the weight diagram. To do this we place a dot at the (i,j)-th position in our skewed coordinate system. We reflect it along the roots $\{\alpha_i\}$ to get six dots. We then put a dot in any position that is inside the convex hull of the six dots and differs from our highest weight by $n_1\alpha_1 + n_2\alpha_2$ for some pair of integers (n_1, n_2). We arrive at a diagram like the one below, which is the weight diagram for highest weight $(3,2)$.



A dot in the (k, l)-th position in the diagram implies that $\dim V_{(k,l)} \geq 1$. Looking at a typical diagram as the one above we observe that in a column of dots either all of them have their second coordinate an odd number or all them have an even one. We also note that every other column is odd and every other is even. Thus one of these columns correspond to even weights for the subalgebra $\mathfrak{su}(1,1)^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}H_2 + \mathbb{C}E_{2,3} + \mathbb{C}E_{3,2}$. Since an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ has either only odd or only even highest weight we can conclude that we get even highest weight representations in the decomposition in (2.1). Hence we do not have tightness for $\rho = \rho_{ij}$ by our previous reasoning. The exceptions are the small weight diagrams of ρ_{10} and ρ_{01} that are tight and (anti-) holomorphic.

Having proved that nonholomorphic maps are not tight for one domain and one class of codomains has a lot of consequences. Assume that $\rho\colon \mathfrak{su}(n,1) \to \mathfrak{su}(p,q)$ is tight and nonholomorphic. Compose with the tight and holomorphic standard inclusion $\iota\colon \mathfrak{su}(2,1) \to \mathfrak{su}(n,1)$. The composition $\rho \circ \iota\colon \mathfrak{su}(2,1) \to \mathfrak{su}(p,q)$ is then tight and nonholomorphic by "Lemma" 2.1, which is a contradiction. Hence there can not exist a tight nonholomorphic homomorphism $\rho\colon \mathfrak{su}(n,1) \to \mathfrak{su}(p,q)$. We can also extend the result to other codomains. There is a tight and holomorphic homomorphism $\iota\colon \mathfrak{sp}(2n,\mathbb{R}) \to \mathfrak{su}(n,n)$. If we consider compositions $\iota \circ \rho\colon \mathfrak{su}(2,1) \to \mathfrak{sp}(2n,\mathbb{R}) \to \mathfrak{su}(n,n)$ we can again argue that ρ can not be tight and nonholomorphic using "Lemma" 2.1.

Playing around with compositions with tight holomorphic maps we can cover a lot more cases. We can not cover all cases using our $\mathfrak{su}(2,1)$ result, we need to prove a few more low rank cases. The smallest set of such low rank cases turns out to be homomorphisms $\rho\colon \mathfrak{sp}(4,\mathbb{R}) \to \mathfrak{su}(p,q)$ and $\rho\colon \mathfrak{sp}(4,\mathbb{R}) \oplus \mathfrak{su}(1,1) \to \mathfrak{su}(p,q)$.

So far the argument seems pretty straight forward. Let us turn our attention to the hidden problems in the simplified picture above.

First off, equivalence for complex representations and equivalence for real Lie algebra homomorphisms are *not* the same. Given two homomorphisms $\rho,\eta\colon \mathfrak{su}(1,1) \to \mathfrak{su}(1,1)$ we say that they are equivalent, as homomorphisms between real Lie algebras, if they differ by an inner automorphism of $\mathfrak{su}(1,1)$. When we use representation theory we take the same homomorphisms ρ and η, fix a matrix model for the codomain, and consider them as homomorphisms $\rho,\eta\colon \mathfrak{su}(1,1) \to \mathfrak{gl}(2,\mathbb{C})$ whose images happen to be contained in $\mathfrak{su}(1,1) \subset \mathfrak{gl}(2,\mathbb{C})$. Homomorphisms into $\mathfrak{gl}(2,\mathbb{C})$ are equivalent if they differ by an inner automorphism of $\mathfrak{gl}(2,\mathbb{C})$. Even if we require the image to stay in a fixed copy of $\mathfrak{su}(1,1) \subset \mathfrak{gl}(2,\mathbb{C})$ we get that non-equivalent homomorphisms become equivalent representations. An example of this is

$$\begin{pmatrix} k & z \\ \bar{z} & \bar{k} \end{pmatrix} \mapsto \begin{pmatrix} k & z \\ \bar{z} & \bar{k} \end{pmatrix} \sim^{rep} \begin{pmatrix} k & z \\ \bar{z} & \bar{k} \end{pmatrix} \mapsto \begin{pmatrix} \bar{k} & \bar{z} \\ z & k \end{pmatrix},$$

$$\text{since} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} k & z \\ \bar{z} & \bar{k} \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} \bar{k} & \bar{z} \\ z & k \end{pmatrix}.$$

Fortunately, what we observed above is sort of the worst case scenario. An equivalence class of an irreducible representation $\rho\colon \mathfrak{su}(p,q) \to \mathfrak{su}(p',q')$ contains at most two equivalence classes of homomorphisms. Still, this forces us to be a bit careful. For a reducible representation such as a diagonal homomorphism $d\colon \mathfrak{su}(1,1) \to \mathfrak{su}(1,1)^{\oplus 2}$, $X \mapsto (X,X)$ we have that d is equivalent to $X \mapsto (X,\bar{X})$, $X \mapsto (\bar{X},X)$ and $X \mapsto (\bar{X},\bar{X})$ as a representation. Even though d is tight, $X \mapsto (X,\bar{X})$ and $X \mapsto (\bar{X},X)$ are not. The notion of tightness is thus *not* well-defined for equivalence classes of representations. At first this seems really bad, using methods from representation theory we only get information up to equivalence. Fortunately, tightness is well-defined for irreducible representations as the cancellation happening in a homomorphism like $X \mapsto (X,\bar{X})$ is the only thing that can go wrong and this will not happen with just one term.

The second problem is to narrow down the precise conditions for turning our "lemmas" into lemmas. Let us look at some examples to see where they fail in their current form. Consider the following three maps:

$$
\begin{aligned}
\rho_i = \rho_{i,1} \times \rho_{i,2}\colon \mathbb{D} \to \mathbb{D} \times \mathbb{D},\\
\rho_1(z) = (z,z),\\
\rho_2(z) = (z,0),\\
\rho_3(z) = (z,\bar{z}),\\
\rho_4(z) = (\bar{z},\bar{z}).
\end{aligned}
$$

Let ω denote the Kähler form of $\mathbb{D}$ and to distinguish the forms belonging to different copies we denote the Kähler form of $\mathbb{D} \times \mathbb{D}$ by $\omega_1 + \omega_2$. We get

$$
\rho_i^*(\omega_1+\omega_2) = \rho_{i,1}^*\omega_1 + \rho_{i,2}^*\omega_2 = \begin{cases} \omega+\omega = 2\omega,\ i=1,\\ \omega + 0 = \omega,\ i=2,\\ \omega - \omega = 0,\ i=3,\\ -\omega-\omega = -2\omega,\ i=4. \end{cases}
$$

We have that ρ_1 and ρ_4 are tight while ρ_2 and ρ_3 are not, for example by applying Theorem 1.1. We observe that "Lemma" 2.1 is valid for the holomorphic and antiholomorphic maps, but not when we "mix" the two in ρ_3. What we need is not necessarily (anti-) holomorphicity but that all pullbacks share the same sign, that they are all *positive* or all *negative*. This is the version of the "lemma" that is [**H2**, Lemma 3.1] and

[**H3**, Lemma 3.7]. We also observe that the "only if" part of "Lemma" 2.1 is always valid.

The other "lemma" is more troublesome to narrow down, let us consider the following maps:

$$\begin{aligned} \eta_i\colon \mathbb{D}\times\mathbb{D} &\to \mathbb{D}\times\mathbb{D}\times\mathbb{D} \\ \eta_1(z,w) &= (z,z,z), \\ \eta_2(z,w) &= (z,z,\bar{w}), \\ \eta_3(z,w) &= (z,w,0). \end{aligned}$$

We denote the Kähler class of the codomain by ω'. Let us look at a few $\eta_j \circ \rho_i$ -combinations:

$$(\eta_1 \circ \rho_2)^*\omega' = \rho_2^* 3\omega_1 = 3\omega, \tag{2.2}$$

$$(\eta_2 \circ \rho_1)^*\omega' = \rho_1^*(2\omega_1 - \omega_2) = 2\omega - \omega = \omega, \tag{2.3}$$

$$(\eta_2 \circ \rho_3)^*\omega' = \rho_3^*(2\omega_1 - \omega_2) = 2\omega + \omega = 3\omega, \tag{2.4}$$

$$(\eta_3 \circ \rho_1)^*\omega' = \rho_1^*(\omega_1 + \omega_2) = \omega + \omega = 2\omega, \tag{2.5}$$

$$(\eta_3 \circ \rho_3)^*\omega' = \rho_3^*(\omega_1 + \omega_2) = \omega - \omega = 0. \tag{2.6}$$

In (2.2) we observe the composition of a tight and a nontight map resulting in a tight map. The villain in this setting is the non-injectivity of η_1; if we require injectivity of the second map our "lemma" is valid in the holomorphic setting, [**H2**, Lemma 3.2]. Without holomorphicity the problems with cancellations return in (2.3); here η_2 and ρ_1 are tight but the composition is not. The situation where we often want to use "Lemma" 2.1 is when we have chosen a tight ρ and want to deduce that η is nontight by showing that the composition $\eta \circ \rho$ is nontight. To get a suitable lemma for this situation we will have to vary the Kähler form of the middle space. Recall that when we define tightness for a map $\rho\colon \mathbb{D} \to \mathbb{D}\times\mathbb{D}$ we do this with respect to a fixed choice of Kähler form of $\mathbb{D}\times\mathbb{D}$. But there are four possible Kähler forms for $\mathbb{D}\times\mathbb{D}$:

$$\omega_1 + \omega_2,\ \omega_1 - \omega_2,\ -\omega_1 + \omega_2,\ -\omega_1 - \omega_2.$$

Which maps are tight depend heavily on this choice, ρ_3 is tight with respect to $\pm(\omega_1 - \omega_2)$ and ρ_1 is tight with respect to $\pm(\omega_1 + \omega_2)$. To deduce that η_i is nontight we have to show that both compositions $\rho_1 \circ \eta_i$ and $\rho_3 \circ \eta_i$ are nontight. If this is true the nontightness of the compositions can not in both cases be due to cancellations after ρ_j^* is applied, but must be due to nontightness of η_i. This argument becomes an important lemma for showing nonexistence of nonholomorphic tight maps, [**H3**, Lemma 3.6]. The converse statement, that η is tight if $\eta \circ \rho$ is tight is always true. We can observe that both compositions (2.5) and (2.6) are nontight while only one of (2.3) and (2.4) is nontight. Thus η_2 is tight while η_3 is not.

Finally, there is one more (big!) problem with this approach. There are no tight holomorphic homomorphisms $\mathfrak{e}_{6(-14)} \to \mathfrak{su}(p,q)$ or $\mathfrak{e}_{7(-25)} \to \mathfrak{su}(p,q)$. The composition arguments thus fail to encompass exceptional codomains. To get the full result of nonexistence of tight nonholomorphic maps we have to disprove the existence of three more tight nonholomorphic homomorpisms:

$$\begin{aligned} \mathfrak{su}(2,1) &\to \mathfrak{e}_{6(-14)}, \\ \mathfrak{sp}(4,\mathbb{R}) &\to \mathfrak{e}_{7(-25)}, \\ \mathfrak{sp}(6,\mathbb{R}) &\to \mathfrak{e}_{7(-25)}. \end{aligned}$$

From these three we can again apply composition arguments to disprove the existence of any homomorphisms into exceptional codomains.

Let us take a quick look at the methods used. The methods are rather ad hoc, so let us try to give some intuition to why one would expect them to work here when they do not generalize particularly well to other cases.

Let us begin with the $\mathfrak{e}_{6(-14)}$ case. The main tool used here is weighted Dynkin diagrams. The weighted Dynkin diagram is a full invariant of complex homomorphisms $\rho_{\mathbb{C}} \colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$ defined as follows. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be a fixed root space decomposition, Γ a set of simple roots for Δ and

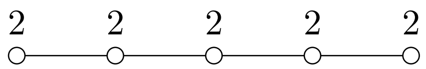
$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C}).$$

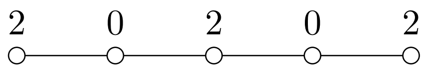
By choosing an appropriate representative from the equivalence class of homomorphisms containing $\rho_{\mathbb{C}}$ we can assume that $H' := \rho_{\mathbb{C}}(H)$ satisfies $H' \in \mathfrak{h}$ and $\alpha(H') \geq 0$ for all $\alpha \in \Gamma$. The weighted Dynkin diagram of $\rho_{\mathbb{C}}$ is constructed by putting the number $\alpha(H')$ next to each simple root $\alpha \in \Gamma$ in the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$. These numbers always belong to the set $\{0,1,2\}$.

The argument for disproving nonholomorphic tight homomorphisms $\mathfrak{su}(2,1) \to \mathfrak{e}_{6(-14)}$ is actually rather short but relies on a rather long calculation. The proof can be summarized in three steps:

(1) Calculate the weighted Dynkin diagram of the complexification of the tight nonholomorphic homomorphism $\rho\colon \mathfrak{su}(1,1) \to \mathfrak{e}_{6(-14)}$ and note that it contains a 2.
(2) Observe that there are two homomorphisms $\iota^1_{\mathbb{C}}, \iota^2_{\mathbb{C}}\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(3,\mathbb{C})$, fulfilling $\iota^2_{\mathbb{C}}(H) = 2\iota^1_{\mathbb{C}}(H)$, where $\iota^1_{\mathbb{C}}$ is the complexification of the tight and holomorphic homomorphism $\iota^1\colon \mathfrak{su}(1,1) \to \mathfrak{su}(2,1)$.
(3) Assume that $\eta\colon \mathfrak{su}(2,1) \to \mathfrak{e}_{6(-14)}$ is tight and nonholomorphic, then $\eta \circ \iota^1\colon \mathfrak{su}(1,1) \to \mathfrak{e}_{6(-14)}$ is tight and nonholomorphic by "Lemma" 2.1. Thus the weighted Dynkin diagram of $\eta_{\mathbb{C}} \circ \iota^1_{\mathbb{C}}$ contains a 2. Since $\eta_{\mathbb{C}} \circ \iota^2_{\mathbb{C}}(H) = 2\eta_{\mathbb{C}} \circ \iota^1_{\mathbb{C}}(H)$ the weighted Dynkin diagram of $\eta_{\mathbb{C}} \circ \iota^2_{\mathbb{C}}$ must contain a 4. This is a contradiction, hence η can not exist.

This method of disproving tight nonholomorphic homomorphisms $\mathfrak{su}(2,1) \to \mathfrak{e}_{6(-14)}$ can be extended to other codomains but not many other domains since step (2) is invalid if we consider for example the domain $\mathfrak{sp}(4,\mathbb{C})$. The calculation in step (1) is a bit long but the result is not unexpected. If we look at weighted Dynkin diagrams of (complexifications of) other tight nonholomorphic homomorphisms $\rho\colon \mathfrak{su}(1,1) \to \mathfrak{g}$ we see that 2:s are appearing frequently. Below are the weighted Dynkin diagrams of two nonholomorphic tight homomorphisms $\mathfrak{su}(1,1) \to \mathfrak{su}(3,3)$.





Let us turn to the $\mathfrak{e}_{7(-25)}$ case next. The method here may seem a bit surprising as we show that any homomorphism $\mathfrak{sp}(2n,\mathbb{R}) \to \mathfrak{e}_{7(-25)}$, $n > 1$, factors through a Hermitian regular subalgebra of $\mathfrak{e}_{7(-25)}$, even though we do not require holomorphicity. This clearly does not generalize well to other cases. We began down this path after observing in the tables of [**D**] that in the complex case, all larger subalgebras of $\mathfrak{e}_{7,\mathbb{C}}$, among them

$\mathfrak{sp}(2n, \mathbb{C})$, $n > 1$, factor through a complex regular subalgebra of $\mathfrak{e}_{7,\mathbb{C}}$. Translating this to a stronger result about the Hermitian real forms and Hermitian regular subalgebras proved hard at first. In the end a chance observation about the size of the centralizers proved to be the key to finishing this case and completing the classification that is the topic of this thesis.

3. Relations to maximal representations

Tight maps were introduced as a tool for studying maximal representations. Maximal representations is a part of what is called *higher Teichmüller spaces.* (Ordinary) Teichmüller space $\mathcal{T}_g$ is the space of marked complex structures, or equivalently, marked hyperbolic structures on a surface Σ_g of genus $g \geq 2$. A hyperbolic structure defines (up to conjugation) a representation $\rho\colon \pi_1(\Sigma_g) \to PSU(1,1)$. We can thus view $\mathcal{T}_g$ as a subspace

$$\mathcal{T}_g \subset \mathrm{Hom}(\pi_1(\Sigma), PSU(1,1))//PSU(1,1) =: \mathcal{R}(\pi_1(\Sigma), PSU(1,1))$$

and study $\mathcal{T}_g$ via this representation variety. However, not all representations in $\mathcal{R}(\pi_1(\Sigma), PSU(1,1))$ correspond to hyperbolic structures. An important question is how to distinguish which parts of $\mathcal{R}(\pi_1(\Sigma), PSU(1,1))$ correspond to hyperbolic structures. In his thesis Goldman gave a characterization of this in terms of Euler numbers, [**G1**].

The higher Teichmüller spaces generalize this picture by replacing $PSU(1,1)$ with a simple Lie group G. In this new setting we want to find parts of $\mathcal{R}(\pi_1(\Sigma), G)$ which share algebraic and geometric properties with Teichmüller space. This has been studied for split real groups, the so called Hitchin representations, [**H6**], but more important for this thesis, for Hermitian Lie groups G using the Toledo invariant.

An important tool for studying the Toledo invariant is bounded cohomology. Bounded cohomology was popularized by Gromov, who among other things, used it to give a new proof of Mostow rigidity, [**G2**], [**G3**]. Bounded cohomology differs from ordinary cohomology in that we require cochains to be bounded. The supremum norm on the cochains then descends to a seminorm on cohomology classes.

Our interest lies in the Kähler class $\kappa_G \in H^2_{cb}(G; \mathbb{R})$. The Kähler class is the cohomology class of the cocycle $c_\omega\colon G \times G \times G \to \mathbb{R}$,

$$c_\omega(g_0, g_1, g_2) := \int_{\Delta(g_0 \cdot o, g_1 \cdot o, g_2 \cdot o)} \omega,$$

where $\Delta(g_0 \cdot o, g_1 \cdot o, g_2 \cdot o)$ is a geodesic triangle in $(\mathcal{X}, \omega)$, the Hermitian symmetric space of noncompact type associated to G, with vertices in $g_0 \cdot o, g_1 \cdot o, g_2 \cdot o$ for some point $o \in \mathcal{X}$. This cocycle was shown to be bounded in [**DT**], [**CØ**].

Given a homomorphism, $\rho\colon G_1 \to G_2$, the induced homomorphism $\rho^*\colon H^\bullet_{cb}(G_2;\mathbb{R}) \to H^\bullet_{cb}(G_1;\mathbb{R})$ is always seminorm nonincreasing, i.e. $||\rho^*\alpha||_1 \leq ||\alpha||_2$ for all $\alpha \in H^\bullet_{cb}(G_2;\mathbb{R})$. If G_2 is a Hermitian Lie group we say that ρ is tight if $||\rho^*\kappa_{G_2}||_1 = ||\kappa_{G_2}||_2$. If G_1 is a Hermitian Lie group as well, the homomorphism defines a totally geodesic map. Then the homomorphism is tight precisely when the totally geodesic map is tight.

From the Kähler class we define the invariant of surface group representations $\rho\colon \pi_1(\Sigma) \to G$ known as the Toledo invariant. Starting with the Kähler class $\kappa_G \in H^2_{cb}(G;\mathbb{R})$, we pull it back to $\rho^*\kappa_G \in H^2_b(\pi_1(\Sigma);\mathbb{R})$. Via the isomorphism $i\colon H^2_b(\pi_1(\Sigma);\mathbb{R}) \to H^2_b(\Sigma;\mathbb{R})$, [**G2**], we get a bounded singular cohomology class $i\rho^*\kappa_G \in H^2_b(\Sigma;\mathbb{R})$. Pairing this class with the fundamental class of Σ we get the Toledo invariant

$$T(\rho) := \langle i\rho^*\kappa_G, [\Sigma]\rangle.$$

The Toledo invariant has finite range and is constant on connected components of $\mathcal{R}(\pi_1(\Sigma), G)$. The representations with maximal Toledo invariant, the maximal representations, exhibit several interesting properties:

Theorem 3.1 ([**BIW1**])**.** *Let $\boldsymbol{G}$ be the connected semisimple algebraic group defined over $\mathbb{R}$ such that $G = \boldsymbol{G}(\mathbb{R})^\circ$ is of Hermitian type. Let Σ be a compact connected oriented surface of genus at least two. If $\rho\colon \pi_1(\Sigma) \to G$ is a maximal representation, then*

(1) *ρ is injective with discrete image;*
(2) *the Zariski closure $\boldsymbol{H} < \boldsymbol{G}$ of the image of ρ is reductive;*
(3) *the reductive Lie group $H := \boldsymbol{H}(\mathbb{R})^\circ$ has compact centralizer in G, and the symmetric space $\mathcal{Y}$ associated to H is Hermitian of tube type, furthermore the inclusion of $\mathcal{Y}$ into $\mathcal{X}$, the symmetric space associated to G, is tight;*
(4) *$\rho(\pi_1(\Sigma))$ stabilizes a maximal tube type subdomain $\mathcal{T} \subset \mathcal{X}$.*

The significance of maximal representations was first observed by Toledo, [**T**], who showed part (4) for $G = PSU(n,1)$. He also noted that in the case $n = 1$ a maximal Toledo invariant coincides with Goldmans characterization using Euler numbers, i.e. the component of maximal representations in $\mathcal{R}(\pi_1(\Sigma), PSU(1,1))$ coincides with Teichmülller space.

After Toledos result there was some gradual generalization in [**H5**], [**BGPG1**], [**BGPG2**], [**BILW**], and several others culminating two decades later in the work by Burger, Iozzi and Wienhard, [**BIW1**], where they proved the theorem above in full generality. They also considered surfaces with boundary for which the theorem above is valid[2] but some other properties differ.

Maximal representations are closely tied to tight homomorphisms, in fact, maximal representations are tight. Knowing more about tight maps and homomorphisms sheds light on maximal representations. A first application of the classification in this thesis would be to strengthen part (3) of Theorem 3.1. An improved version would be:

(3) the reductive Lie group $H := \boldsymbol{H}(\mathbb{R})^\circ$ has compact centralizer in G, and the symmetric space $\mathcal{Y}$ associated to H is Hermitian of tube type, furthermore the inclusion of $\mathcal{Y}$ into $\mathcal{X}$, the symmetric space associated to G, is tight. The inclusion is holomorphic in the following instances:
  (a) $\mathcal{Y}$ is irreducible and not isomorphic to $\mathbb{D}$,
  (b) $\mathcal{Y}$ does not contain any factors isometric to $\mathbb{D}$ and $\mathcal{X}$ is classical.

The case (3a) follows directly from the results in this thesis. The case (3b) follows by a generalization of Theorem 1.3 to reducible domains by Pozzetti, [**P**]. Pozzetti only considered the codomain $\mathfrak{su}(m,n)$ but her result easily generalizes to classical codomains of tube type by composition arguments.

A second application is that given a fixed $\boldsymbol{G}$ we can use the the classification to determine[3] which $\boldsymbol{H}$:s can appear as the Zariski closures of a maximal representation.

There are several generalizations of maximal representations and the Toledo invariant. There is the notion of weakly maximal representations

[2]To be more precise, we replace the condition on genus by requiring that a surface with boundary satisfies $\chi(\Sigma) \leq -1$.

[3]With some limitations since the classification is not complete for reducible domains.

introduced in [**BSBH**$^+$]. The notion of weak maximality separates tightness from maximality in the sense that a representation is maximal if and only if it is tight and weakly maximal.

Toledo and García-Prada defined an analogue of the Toledo invariant for representations of complex hyperbolic lattices in quaternionic Lie groups, [**GPT**]. In this setting an invariant four-form is defined from the metric and the quaternionic structure. From this four-form a Toledo invariant is defined in a fashion similar to the Hermitian case. In their paper they show that the action of a maximal representation on quaternionic hyperbolic space preserves a copy of complex hyperbolic space.

The Toledo invariant has also been defined for representations of complex hyperbolic lattices in Hermitian Lie groups, [**BI**]. In [**P**] Pozzetti considered Zariski dense maximal representations into $PU(m,n)$ and showed that they should be superrigid:

Theorem 3.2. *Let Γ be a lattice in $SU(1,p)$ with $p > 1$. If m is different from n then every Zariski dense maximal representation of Γ in $PU(m,n)$ is a restriction of a representation of $SU(1,p)$.*

Using the classification of tight maps, and the partial generalization of Theorem 1.3 to reducible domains mentioned above, she got the following corollary.

Corollary 3.3. *There are no Zariski dense maximal representations $\rho\colon \Gamma \to PU(m,n)$ for $1 < m < n$.*

In light of the generalization by Pozzetti, let me finish by stating the following, not very bold, conjecture:

Conjecture 3.4. *Let $\rho\colon \mathcal{X} = \mathcal{X}_1 \times ... \times \mathcal{X}_n \to \mathcal{X}'$ be a tight map. Then the restricted map $\rho|\colon \mathcal{X}_i \to \mathcal{X}'$ is holomorphic or antiholomorphic for any $\mathcal{X}_i$ not isometric to the Poincaré disc.*

A proof of this will hopefully appear in [**HP**].

4. Corrections

Paper I has been published and Paper II has been accepted for publication. These papers appear in the version in which they were or are to be published. Since the publication a few errors in Paper I, none of which affect the results, have come to my attention. These are:

(1) The proof of Lemma 3.4 treats only the simple case. The same lemma appears again in Paper II with a proof of the full statement.
(2) The root β_1 used to define the regular subalgebras of $\mathfrak{e}_{7(-25)}$ is erroneously defined. The correct definition is $\beta_1 = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7$.
(3) The (H2)- homomorphisms $\mathfrak{so}(p,2) \to \mathfrak{so}(p',2)$, $p' > p$, defined in [**I**, pp. 292-295], were forgotten in the classification of tight (H2)-homomorphisms. These are immediatly seen to be tight by an application of Corollary 8.5 in [**BIW2**].

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