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Value-at-Risk and Expected Shortfall

Managing risk for an equity portfolio

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Abstract

This thesis intends to examine a risk measure used for estimating a potential future loss. The risk measure Value-at-Risk, is widely used throughout the world of financial risk management. We will examine different approaches to computing Value-at-Risk for two equity portfolios, one univariate portfolio and one multivariate portfolio. We assume that portfolio losses have a certain distribution. Even though Value-at-Risk is widely used and accepted within financial management, Value-at-Risk is not a coherent risk measure. We will therefore include another risk measure in our thesis, the so-called Expected Shortfall. What we find is that our assumption considering portfolio losses are not valid for all methods of computing Value-at-Risk. Methods investigated in this thesis are not suitable for capturing more extreme losses that occur during periods of market turbulences.

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1. Introduction

In the last thirty or so years the world has seen many crises occur on the financial markets and managing the risk, of for example an equity portfolio, is therefore of vital importance for financial institutions such as banks, funds and insurance companies. Large crises have had substantial impact on financial markets. In just one day the North American market experienced a large crash and the Dow Jones Industrial Average fell by more than 22 percent, this incident is called Black Monday, which took place on the 19th of October in 1987. Another example is the Dot.com Bubble, which took place in 2000, when the financial markets suffered from yet another blow and during a period of 10 days, the market lost 10 percent of its value. Also, in 2008-2009 the financial markets experienced turmoil in the aftermath of the Lehman Brothers default, due to the sub-prime mortgage crash. The above-mentioned events surely raise the question of portfolio risk and the need for efficient measures of these risks.

Risk relates to uncertainties, e.g. the uncertainty of how the value of an asset will change in the future. Banks and other financial institutions face market risk, i.e. the risk of changes in components that affect underlying value of a financial asset. Market uncertainties affect assets, and this kind of uncertainty show the need for a risk measure that financial institutions could use to decide which amount of funds is needed to withstand a future potential loss. Global financial markets are connected, and a large loss for one financial institution might affect other institutions causing a chain reaction. Therefore the kind of market turbulence for example, as described above, has led to the construction of both rules as well as guidelines for the financial institutions to prevent large repercussions of market volatility. Large financial crises raise the question of how to quantify risk and it is the purpose of this thesis.

In this thesis we intend to examine two of the most popular measures of risk, Value-at-Risk (VaR) and Expected shortfall (ES). We will study different methods to compute VaR and ES for an equity portfolio consisting of stocks, which is different from estimating VaR and ES for other kinds of portfolios. VaR attempts to measure the portfolio risk that a financial institution could be exposed to, and in this paper we will focus on the one day ahead VaR for an equity portfolio. This measure of risk has gained popularity by expanding previous risk measures to include a

confidence level, which tells us how much the bank could lose in a worst-case scenario with a certain probability. The popularity is also due to the simplicity of obtaining one value or percentage, for worst-case losses, which are easy to understand in boardrooms where complicated reports can be misinterpreted. Within the Basel accords, that provide rules and guidelines for financial institutions, the VaR measurement has been given an important role and all banks are required to estimate VaR on their portfolios. Under the Basel accords, banks have to set aside regulatory capital, a “buffer”, which will absorb potential losses and prevent liquidity problems for the financial institution (Hull, 2011).

Another risk measure closely related to VaR is ES, which actually is more preferred over VaR to many risk managers in practice, partly because it is a coherent measure. A coherent risk measure is a function that satisfies certain properties that we will discuss further in Subsection 2.3, where we will give a detailed description of the measure’s properties and how to compute ES for an equity portfolio.

The rest of the thesis is organized as follows: In section 2 we present our methodologies for the different ways to estimate both VaR and ES. The methods we have chosen are a selection of many different ways to estimate these risk-measures. There are a vast number of approaches for estimating VaR and ES, but due to restrictions we will only include a selection of these methods. The methods chosen are historical VaR, VaR under normal distribution, VaR under student’s t-distribution, and the Monte Carlo simulation under the assumption that the losses are normally or t-distributed. We will also include a brief overview of Stressed VaR. In Section 3 we present results from our VaR and ES estimates on our equity portfolio. In Section 4 we present our conclusion.

2. Methods for computing Value-at-Risk and Expected Shortfall

In this section we will present some popular methods to evaluate Value-at-Risk for an equity portfolio. The methods used will be described in more detail in each subsection. It is noteworthy that it is possible to estimate VaR for a wide range of portfolios including credit portfolios and options portfolios. We will give a short description of the differences between estimating VaR for other kinds of portfolios; however, in this thesis we will present results of estimations for an equity portfolio consisting of one thousand Volvo stocks as well as an equity portfolio consisting of 26 stocks drawn from the Stockholm stock exchange OMXS30. We also will give a general definition of Expected Shortfall and present methods for estimating ES for an equity portfolio. This section will be organized as follows; in Subsection 2.1, we present a general approach for computing VaR and we continue to Subsection 2.2, which includes an introduction of the general approach for computing ES. In Subsection 2.3 we discuss losses for an equity portfolio and when moving on to Subsection 2.4, we present how to estimate VaR using Historical Simulation. Within Subsection 2.5 we present how to estimate VaR under normal distribution. For Subsection 2.6 we will present how to estimate VaR under student's t-distribution. In Subsection 2.7 we will present how to compute VaR using a Monte Carlo simulation. Later, in Subsection 2.8 we introduce the topic of stressed VaR. Finally, in Subsection 2.9 we will present how to perform VaR estimations under a multivariate normal setting.

2.1 General approach for estimating Value-at-Risk

In this subsection we will give a brief presentation of the birth of VaR and the meaning of risk, we will also give a rigorous definition of the general concept of VaR as well as show this method formally. We give a brief presentation of other kinds of portfolios and the steps taken to estimate VaR for these portfolios. For the formal presentation in this subsection we will closely follow the notation of McNeil, Frey & Embrechts (2005).

In 1993, the risk measure VaR became official when G-30 published a seminal report to address derivatives in a systematic way. However, the idea of having just one simple number to present before corporate executives was developed during the same time. At J.P.Morgan the chief

executive officer required the staff to daily hand over a one-page short daily summary of the market risk that the bank was facing. At this time there was a noticeable need for risk management of derivatives within the banking industry. This gave way for the Value-at-Risk measure to rise as a market risk measure (McNeil, et al., 2005). “Formally, VaR measures the worst expected loss over a given horizon under normal market conditions at a given confidence level” (Jorion, 2001, p. xxii).

Consider an equity portfolio; if we knew the future outcomes of this portfolio we would not have any risk. Since this is never the case in reality, it must be that the portfolio’s future outcome is due to randomness and this needs to be quantified if we want to estimate future outcomes.

Consider the above mentioned equity portfolio again; in order to quantify the risks of the future outcomes we need to define our one-period loss in the portfolio which we denote by L . Thus L is the potential loss of tomorrow. Since tomorrow’s value is uncertain, we need to assume that L can take any value from negative infinity to positive infinity. Furthermore most of the modeling of L concerns with its distribution function, which is the probability of a loss worse than l by the end of the period, that is $P[L \leq l]$ where P is a probability measure used in our model and where l represents the possible values that L can take. Note that a negative loss of L is a gain, meaning that when the portfolio yields a positive return, L will be negative (McNeil, et al., 2005).

One might say that risk measurement is mainly a statistical issue; we base estimations on historical observations, using a specific model, and a statistical estimate of the change in value of an asset or a position. Financial risk consists mainly of three types of risk; market risk, credit risk and operational risk. In this thesis we will only focus on market risk, which is the risk of a change in the value of a financial position. The managing of risk is essential when facing an uncertain world. For bankers it means using techniques to create portfolios with minimized risk while maximizing profits (McNeil, et al., 2005).

With the previously explained concept of loss, and the brief description of Value-at-Risk as an easy to interpret risk measure, we proceed by explaining how to both estimate and interpret VaR. To estimate VaR on our portfolio with random loss L we choose a confidence level $\alpha \in (0,1)$. When estimating the VaR of our equity portfolio at our confidence level α , we obtain a number for our loss L , that is the VaR_α , where the probability of L to exceed VaR_α is smaller or equal to $(1 - \alpha)$ during a period T . In other words, if we choose $\alpha = 0.95$, our estimation of VaR_α

will provide us with a number that represents the potential loss with a certain probability. Our realized loss will only exceed our estimate with a probability of 0.05, meaning that with a period of 200 days and $\alpha = 0.95$, our realized loss would exceed our estimated VaR in 10 of these 200 days. Typical values for α are 0.95, 0.975, 0.99, and 0.999. The time horizon T for the estimated VaR of an equity portfolio is usually 1 or 10 days. Note that when estimating VaR for a credit portfolio the typical time horizon is one year (Hull, 2011).

The above definition can be formalized as follows. For a portfolio with loss L over the period T, and a given confidence level α , we define VaR_α as

$$\begin{aligned}
 VaR_\alpha &= \inf[l \in \mathbb{R}: P(L > l) \leq 1 - \alpha] & (1) \\
 &= \inf[l \in \mathbb{R}: 1 - P(L \leq l) \leq 1 - \alpha] \\
 &= \inf[l \in \mathbb{R}: P(L \leq l) \geq \alpha] \\
 &= \inf[l \in \mathbb{R}: F_L(l) \geq \alpha].
 \end{aligned}$$

The VaR of the portfolio is thus for a certain α , given by the smallest number l , which is a real number \mathbb{R} , such that the probability that the loss L does not exceed l is larger than α . Note that VaR_α is thus the α -quantile of the loss L.

If L is a continuous random variable, then VaR_α simplifies to

$$VaR_\alpha = F_L^{-1}(\alpha) \quad (2)$$

where $F_L^{-1}(\alpha)$ is the inverse of the distribution function for the loss L (McNeil, et al., 2005, p. 38).

When estimating VaR on portfolios consisting of for example forward contracts, swaps, options, and loans, we first need to identify market rates and prices that could affect the value of our portfolio. In other words, we need to evaluate the market factors and their probability distributions. Usually one must begin with breaking down the instruments so we can relate them to basic market risk factors, depending on our position in for example a future we would potentially need; current spot price, foreign interest rates, domestic interest rates or other factors affecting our derivative. We use formulas to determine the current mark-to-market value of the position that affects underlying value of our assets. After that we need to estimate the statistical

distribution of our potential future value of these market factors and determine potential changes in the future that would change the value of our portfolio. VaR becomes the measure of these potential future changes in portfolio value (Pearson, 2000).

2.2 General approach for estimating Expected Shortfall

In this subsection we present different methods for the general approach to estimating Expected Shortfall. When formally describing the method we will closely follow the notation and structure of McNeil, et.al.,(2005).

The risk measure ES is closely related to VaR and is actually more preferred in practice by many risk managers, this is due to ES being a coherent risk-measure while VaR is not. The properties that need to be fulfilled for coherence are monotonicity, sub-additivity, homogeneity, and translational invariance. Monotonicity implies that if we have two portfolios with losses L_1 & L_2 where one always has greater loss (is more risky), i.e. $L_1 \leq L_2$, then it will follow that $VaR_\alpha(L_1) \leq VaR_\alpha(L_2)$ is always true. Translational invariance means that if we add or subtract an amount l from a portfolio, and this l is independent of the volatility of this portfolio, then we have altered the capital requirements by l . Depending on whether l is a loss or profit, l is added or subtracted accordingly. Homogeneity in this context means that the measure is applicable whether the portfolio's underlying assets are one Euro or one thousand Euro, the potential loss is a percentage of this amount and is not altered unless the volatility changes. For example, if a is a constant then it follows that $VaR_\alpha(aL) = aVaR_\alpha(L)$. However, the most important property to fulfill is sub-additivity, so that when combining two portfolios, the risk is smaller or equal in the combined portfolio than the risk is for the separate portfolios. This is in accordance with the principle of diversification for reduction of risk. The VaR measure is not a coherent measure since it does not fulfill the sub-additivity property, i.e. $VaR_\alpha(L_1 + L_2) \not\leq VaR_\alpha(L_1) + VaR_\alpha(L_2)$, and this may create problems when adding two or more VaR estimates, since the combined VaR may be higher than for the separate measures (McNeil, et al., 2005).

The formal definition of ES is that given a loss L with distribution function $F_L(x)$ and a confidence level $\alpha \in (0, 1)$, where u represents the quantile, is

$$ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L) du.$$

When L is a continuous variable with a distribution $F_L(x)$ with the inverse $F_L^{-1}(x)$, then

$$ES_\alpha = \frac{1}{1-\alpha} \int_L^1 F_L^{-1}(u) du$$

furthermore, when L is a continuous variable we can show that (from McNeil et.al. (2005), pp. 45)

$$ES_\alpha(L) = E[L|L \geq VaR_\alpha(L)]. \quad (3)$$

From the formal definition in Equation (3) it is clear that ES is the expected loss given that the loss is larger than or equal to the loss estimated by VaR (McNeil, et al., 2005, pp. 44-45).

2.3 Value-at-Risk using Historical Simulation

When doing the so-called historical simulation of VaR, we use past events to estimate VaR. Assume a sample size of 500 losses; this would give us 500 possible scenarios for tomorrow's return. With $\alpha = 0.99$ we order our historical losses from best to worst and then find the fifth largest historical loss. This method gives us an empirical distribution of the portfolio losses and when VaR is historically estimated we will only obtain risk estimation on the worst scenarios from the past, the future could possibly involve larger volatility, and this will not be included in the forecast. Note that volatility is defined as the standard deviation of losses which is the square root of the variance of losses (Hull, 2011).

2.4 Equity portfolio losses

The outline represented in Subsections 2.1-2.3 holds for any type of portfolio and thus any kind of portfolio loss L . In the rest of this thesis we will focus on an equity portfolio consisting of only stocks. We closely follow the notation and structure of McNeil et.al. (2005) and Herbertsson (2013).

Hence, consider a portfolio consisting of d different stocks with α_1 stocks of company 1, α_2 stocks of company 2 etc. Furthermore, we denote the price of the stock from company i at day n by $S_{n,i}$. Then the total value of the portfolio at day n , denoted by V_n , is defined as

$$V_n = \sum_{i=1}^d \alpha_i S_{n,i}. \quad (4)$$

The loss L_{n+1} of the portfolio is then given by the change in price of the portfolio between day n and day $n+1$ is given by

$$L_{n+1} = -\left(\sum_{i=1}^d \alpha_i S_{n+1,i} - \sum_{i=1}^d \alpha_i S_{n,i}\right). \quad (5)$$

Equation (4) and Equation (5) are enough to calculate the historical loss for a portfolio, though sometimes it is necessary to model the portfolio loss L_{n+1} and this is shown below.

$S_{n,i}$ can also be modelled as $S_{n,i} = e^{Z_{n,i}}$ where $Z_{n,i}$ is a random variable for each n and i . Now, let $X_{n+1,i}$ be the log-returns between day n and $n+1$ of the stock price such that

$$X_{n+1,i} = \ln\left(\frac{S_{n+1,i}}{S_{n,i}}\right) = \ln S_{n+1,i} - \ln S_{n,i} = Z_{n+1,i} - Z_{n,i}$$

from which we get that that

$$S_{n+1,i} = S_{n,i} e^{X_{n+1,i}}. \quad (6)$$

To find the loss in the period from n to $n+1$ for the portfolio, we need to follow the steps below where $L_{n+1} = -(V_{n+1} - V_n)$ is given by

$$L_{n+1} = -\left(\sum_{i=1}^d \alpha_i S_{n+1,i} - \sum_{i=1}^d \alpha_i S_{n,i}\right) = -\left(\sum_{i=1}^d \alpha_i e^{Z_{n+1,i}} - \sum_{i=1}^d \alpha_i e^{Z_{n,i}}\right)$$

$$= - \sum_{i=1}^d \alpha_i (e^{Z_{n+1,i}} - e^{Z_{n,i}}). \quad (7)$$

So combining Equation (6) with Equation (7) yields

$$L_{n+1} = - \sum_{i=1}^d \alpha_i (e^{Z_{n,i} X_{n+1,i}} - e^{Z_{n,i}}) = - \sum_{i=1}^d \alpha_i e^{Z_{n,i}} (e^{X_{n+1,i}} - 1)$$

From earlier we know that $e^{Z_{n,i}} = S_{n,i}$ and the portfolio loss is thus given by

$$L_{n+1} = - \sum_{i=1}^d \alpha_i S_{n,i} (e^{X_{n+1,i}} - 1) \quad (8)$$

The loss L_{n+1} in Equation (8) can often be approximated by its linear counterpart. More specific since e^x has the Taylor-expansion (Sydsaeter, 1991) given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \quad (9)$$

so for small x , Equation (9) yields that $e^x \approx 1 + x$ since $\frac{x^n}{n!} \approx 0$ for large n and small x . Thus, if we combine Equation (8) and Equation (9) we can approximate the portfolio loss L_{n+1} with the linearized loss L_{n+1}^{Δ} by

$$L_{n+1}^{\Delta} = - \sum_{i=1}^d \alpha_i S_{n,i} X_{n+1,i}$$

By letting \mathbf{X} denote the vector $\mathbf{X} = (x_{n+1,1}, \dots, x_{n+1,d})$, then the linearized loss can be rewritten as

$$L_{n+1}^{\Delta} = -\mathbf{w}^T \mathbf{X} \quad (10)$$

where $\mathbf{w}^T = (\alpha_1 S_{n,1}, \alpha_2 S_{n,1}, \dots, \alpha_i S_{n,d})$ is a vector of weights for the portfolio.

Also note that if \mathbf{X} only contains small changes, we can assume that $L_{n+1}^{\Delta} \approx L_{n+1}$ and use e.g. Equation (5) to calculate the loss.

2.5 Normally distributed losses

In this subsection we will present an approach to estimating VaR and ES assuming that the loss is normally distributed with mean μ and variance σ^2 and $\alpha \in (0,1)$. For the formal presentation of VaR under assumed normal distribution, we will closely follow the notation of McNeil, et.al.,(2005).

Let L be a stochastic variable with distribution function $F_L(x)$, i.e. $F_L(x) = P[L \leq x]$, then it follows from Equation (2) that $VaR_\alpha(L) = F_L^{-1}(\alpha)$ since $F_L(x)$ is a continuous function because L is normally distributed. If $L \sim N(\mu, \sigma^2)$, the distribution function $F_L(x)$ is given by

$$F_L(x) = P[L \leq x] = P\left[\frac{L-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right] = N\left(\frac{x-\mu}{\sigma}\right) \quad (11)$$

where $N(x)$ is the standard normal distribution. The inverse to $F_L(x)$ is defined as the function $F_L^{-1}(x)$ which solves the equation $F_L(x) = y$, i.e $x = F_L^{-1}(y)$. Hence, to find $F_L^{-1}(y)$ we need to isolate x and express it as a function of y . Hence, we have that

$$F_L(x) = y \Leftrightarrow x = F_L^{-1}(y). \quad (12)$$

From Equation (11) we know that $F_L(x) = N\left(\frac{x-\mu}{\sigma}\right)$ so $F_L(x) = y \Leftrightarrow N\left(\frac{x-\mu}{\sigma}\right) = y$. It follows that $N\left(\frac{x-\mu}{\sigma}\right) = y \Leftrightarrow N^{-1}\left(N\left(\frac{x-\mu}{\sigma}\right)\right) = N^{-1}(y) \Leftrightarrow \frac{x-\mu}{\sigma} = N^{-1}(y) \Leftrightarrow x = \mu + \sigma N^{-1}(y)$.

Hence, we have that

$$x = \mu + \sigma N^{-1}(y). \quad (13)$$

However, if we now combine what we know from Equation (10) and (11) we get $F_L^{-1}(y) = \mu + \sigma N^{-1}(y)$. We also know from Equation (2) that $VaR_\alpha(L) = F_L^{-1}(\alpha)$, therefore we get that

$$VaR_\alpha(L) = \mu + \sigma N^{-1}(\alpha) \quad (14)$$

where $N^{-1}(y)$ is the inverse to $N(y)$. To prove this we can show that $F_L(VaR_\alpha) = \alpha$ since

$$P(L \leq VaR_\alpha) = P[L \leq \mu + \sigma N^{-1}(\alpha)] = P\left(\frac{L-\mu}{\sigma} \leq N^{-1}(\alpha)\right) = N(N^{-1}(\alpha)) = \alpha.$$

Computing ES when assuming that the loss distribution F_L is normally distributed with mean μ and variance σ^2 and $\alpha \in (0,1)$ we get

$$ES_\alpha = \mu + \sigma \frac{\phi(N^{-1}(\alpha))}{1-\alpha} \quad (15)$$

where ϕ is the density of the standard normal distribution and $N^{-1}(\alpha)$ is the inverse of the standard normal distribution. To prove this, first note that

$$ES_\alpha = \mu + \sigma E\left(\frac{L-\mu}{\sigma} \mid \frac{L-\mu}{\sigma} \geq q_\alpha\left(\frac{L-\mu}{\sigma}\right)\right).$$

Now it is enough to compute the ES for the standard normal random variable $ES_\alpha(\tilde{L}) = (L - \mu)/\sigma$. It then follows that

$$ES_\alpha(\tilde{L}) = \frac{1}{1-\alpha} \int_{N^{-1}(\alpha)}^{\infty} l\phi(l)dl = \frac{1}{1-\alpha} [-\phi(l)]_{N^{-1}(\alpha)}^{\infty} = \frac{\phi(N^{-1}(\alpha))}{1-\alpha}. \quad (16)$$

When assuming that losses are normally distributed it is possible to transform a one-day VaR estimate to a k-day VaR. For example, transforming a one-day VaR to a 10-day VaR is needed when estimating regulatory capital, and this transformation is done by multiplying the one-day VaR with the square root of k. The proof for this equation is found in Appendix 2.

$$VaR_\alpha(L_{n+k}^\Delta) = \sqrt{k} \cdot VaR_\alpha(L_{n+1}^\Delta). \quad (17)$$

Hence, under the assumption that the log-returns are i.i.d and normally distributed with zero mean, we know that we can calculate the k-day VaR by multiplying with the one-day VaR with the \sqrt{k} and thus motivates $VaR_\alpha^{10} \approx \sqrt{10}VaR_\alpha^1$. Where VaR_α^{10} represents the 10-day VaR for our portfolio with a confidence level of $\alpha\%$, and similarly VaR_α^1 represents the one day VaR (Herbertsson, 2013).

2.6 Value-at-risk under Student's t-distribution

In this subsection we will present an approach to estimating VaR with the assumption that the portfolio loss L has a student's t-distribution. The approach is very similar to the model shown in Subsection 2.3, however for the student's t-distribution we need to decide on what degrees of freedom to use. Again, for the formal presentation of VaR, assuming a student's t-distribution of losses, we will closely follow the notation and structure of McNeil, et.al.,(2005).

Let L be our loss and assume that $\frac{L-\mu}{\sigma}$ is a random variable which has a student's t-distribution with ν degrees of freedom where μ is given by $E[L] = \mu$ and σ is a constant. We get that the variance V is given by

$$V\left(\frac{L-\mu}{\sigma}\right) = \frac{\nu}{\nu-2} \quad (18)$$

since $\frac{L-\mu}{\sigma} \sim t(\nu)$. We also know that $V\left(\frac{L-\mu}{\sigma}\right) = \frac{1}{\sigma^2} V(L - \mu)$ and since μ is a constant we get

$$\frac{1}{\sigma^2} V(L - \mu) = \frac{1}{\sigma^2} V(L). \quad (19)$$

Thus, combining Equation (18) and (19) yields

$$V(L) = \frac{\sigma^2 \nu}{\nu-2} \quad (20)$$

From Equation (13) we have the standard deviation for the t-distribution $\sigma = \sqrt{V(L)} = \sqrt{\frac{\sigma^2 \nu}{\nu-2}}$ where $\nu > 2$ (McNeil et.al. 2005). The following equations follow the same steps taken in Equations (11) to (13),

$$F_L(x) = P[L \leq x] = P\left[\frac{L-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right] = t_\nu\left(\frac{x-\mu}{\sigma}\right) \quad (21)$$

where $t_\nu(x)$ is the distribution function for the student's t-distribution with ν degrees of freedom. Since L is a continuous random variable $\left(\frac{L-\mu}{\sigma}\right)$ is student t-distributed then $VaR_\alpha(L) = F_L^{-1}(\alpha)$. Thus we need to find $F_L^{-1}(x)$ when $F_L(x)$ is given by (α) . Similar calculations as in Subsection 2.3 together with Equation (18), then this yields that $F_L^{-1}(x)$ is given by

$$x = \mu + \sigma t_v^{-1}(y) \quad (22)$$

If we combine what we know from Equation (4) and (10) we get that $F_L^{-1}(y) = \mu + \sigma t_v^{-1}(y)$. We also know from Equation (2) that $VaR_\alpha(L) = F_L^{-1}(\alpha)$, therefore we get that $VaR_\alpha(L) = \mu + \sigma t_v^{-1}(\alpha)$ where $t_v^{-1}(y)$ is the inverse to $t_v(y)$ (McNeil, et al., 2005).

When computing ES for a student's t-distribution assuming that L is distributed so that $\tilde{L} = (L - \mu)/\sigma$ has a standard t distribution with ν degrees of freedom, one can easily show that

$$ES_\alpha = \mu + \sigma ES_\alpha(\tilde{L}).$$

Therefore when computing ES for t-distribution we use:

$$ES_\alpha(\tilde{L}) = \frac{g_\nu(t_v^{-1}(\alpha))}{1-\alpha} \left(\frac{\nu + (t_v^{-1}(\alpha))^2}{\nu-1} \right) \quad (23)$$

where t_ν is the distribution function to the student's t-distribution with ν degrees of freedom, and g_ν is the density of the student's t-distribution (McNeil, et al., 2005).

To be able to estimate VaR with a student's t-distribution we need to know the degrees of freedom of the distribution. The t-distribution is different from the normal distribution in the sense that the tails are fatter, as displayed in Figure 1. We see that for small degrees of freedom the area in the tail is much greater than for the normal distribution but already at 15 degrees of freedom we barely see a difference between the normal and t-distribution. There is a convergence of the t-distribution towards the normal distribution as the degrees of freedom increases. In Figure 2 we illustrate how a VaR estimate, assuming t-distributed losses, converges towards the normal distribution as the degrees of freedom approaches infinity.

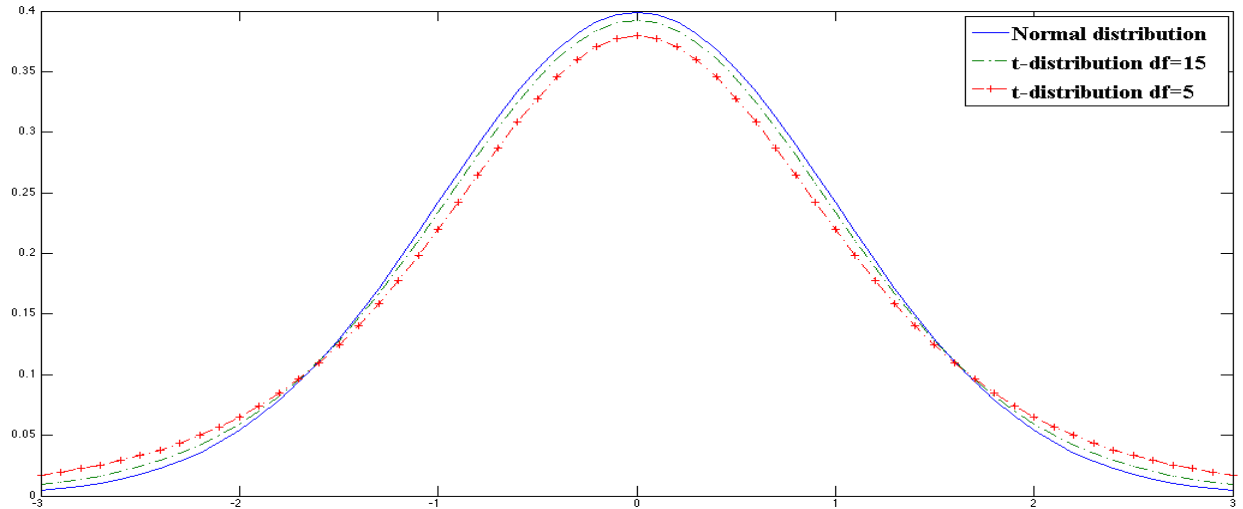


Figure 1. Density function of the t-distribution for $\nu = 5$ and $\nu = 15$ degrees of freedom and a normal distribution.

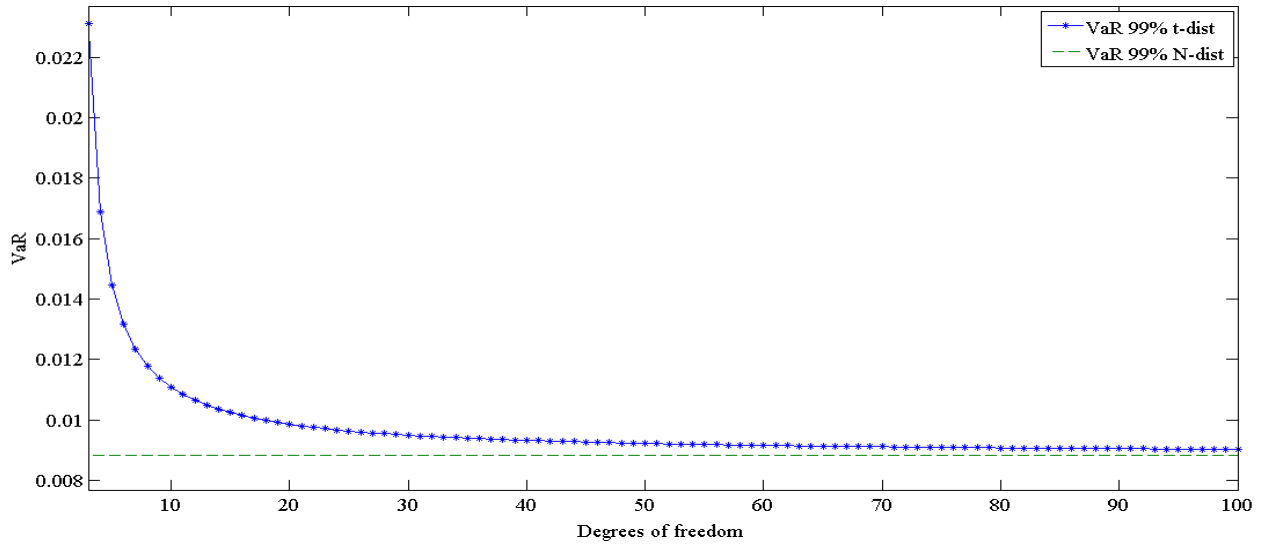


Figure 2. Convergence, when increasing degrees of freedom, of the t-distributed VaR estimate with $\alpha = 0.95$ towards the normally distributed VaR with $\alpha = 0.95$ estimate based on the first 250 days.

2.7 Monte Carlo Simulation

In this subsection we will present the Monte Carlo simulation as a method for estimating VaR of our equity portfolio. We will closely follow the notation and structure of McNeil, et al.,(2005).

When performing a Monte Carlo simulation we choose a distribution that we believe represents the changes in market factors that would affect the portfolio. A random number generator is used to generate hypothetical changes in the chosen market factors. These hypothetical changes are

then used to create thousands of different (theoretical) losses for each stock in the portfolio, and then the simulated losses are ordered from smallest loss to largest loss. Using this order of hypothetical profits and losses it is possible to estimate VaR at the preferred confidence level α using the empirical distribution function for any simulated loss data. When constructing VaR estimations using normal- or t-distributions, the distributions are given. The freedom to choose a distribution that one sees fit for the available historical data is an advantage with the Monte Carlo method (Pearson, 2000).

Since we will perform a Monte Carlo Simulation on a well-diversified equity portfolio, our market factors will consist of the general market risk.

Firstly, one needs to choose a model and estimate the model to historical data. Then, let a random number-generator generate m changes of the risk-factors for a future time period, which are denoted by $\tilde{X}_{t+1}^{(1)}, \dots, \tilde{X}_{t+1}^{(m)}$. A loss function is obtained and then applied to the simulated vectors to obtain simulated realizations of the loss, where the value $\tilde{L}_{t+1}^{(1)}$ gives the loss when the simulated change is $\tilde{X}_{t+1}^{(1)}$. Thus, $\tilde{L}_{t+1}^{(1)} = l_{[t]}(\tilde{X}_{t+1}^{(1)})$ where $l_{[t]}$ is the so-called loss function (McNeil et.al., 2005, pp)

When performing a Monte Carlo (MC) simulation with an i.i.d sample with a random variable X , the Law of large numbers (LLN) implies that MC estimates will converge towards the corresponding estimates for expected value $E(X)$, which in our case are for the normal and the student's t-distributions. Due to the Law of Large numbers (LLN), as n increases one can expect to get a convergence of the MC estimates and the corresponding estimates for the normal and t-distribution.

2.8 Stressed Value-at-Risk

In this subsection we will give a brief presentation of the Basel Accords and introduce Stressed-VaR. Estimation of stressed VaR is beyond the scope of this thesis but we will present the purpose and use of this method to inform the reader of its existence and future importance.

When talking about regulations for banks and financial institutes, and with the mention of the Basel accords, we want to give a brief presentation of these regulations. The Basel Accords are written by the Basel Committee of Banking, and their purpose is to give recommendations on banking regulations. The first Basel accord was introduced 1988 and was focused on credit risk, i.e the risk that arises from lending. The second Basel accord was first presented in year 2001, and the focus was on credit risk and operational risk, this version was published in year 2004. In the light of the most recent financial crises, the Basel Committee on Banking Supervision has agreed upon a revised version of the Basel II. In the new Basel III we see a change toward stricter capital requirements and tighter regulations concerning the methods used when measuring risk. Since we are examining the differences between several methods of estimating VaR, a discussion about the new banking regulations regarding these tests are relevant. Within Basel III, the Stressed VaR measurement will become a requirement. This measure is used to replicate a VaR measure if market factors are experiencing periods of stress. Full implementation of Basel III is not estimated to occur until 2023, but parts of the new regulation will be introduced earlier (Latham & Watkins, 2011).

The purpose of general stress-testing is to see how the portfolio would endure large losses due to crises, and to evaluate weaknesses. When performing a Stress-test one estimates how well a portfolio would have performed during financial crises and during periods of relevant stress. Stress-testing is performed by various financial institutions and companies as a complement to estimating VaR, however, the Stressed VaR is its own measure. Even though the estimated probabilities would tell us that large financial crises are rare, we see that large crises arise every 5 to 10 years (Hull, 2011).

The Stressed VaR is used to simulate effects on current portfolios when different market factors are under stress, meaning when markets are affected by events that cause increased volatility. Banks are required to estimate a Stressed VaR using previous events that have led to crises in the past, such as the subprime crash of 2007/2008, Black Monday of 1987 and many more. Since crises are difficult to predict, there is a need to test how well the financial institution would handle large losses and to determine how to improve the financial institution's ability to handle a financial crisis (Latham & Watkins, 2011).

2.9 Multivariate setting

In this section we will give a very brief introduction to the multivariate normal setting. When computing VaR for a portfolio with multiple assets we cannot use the univariate setting introduced in the previous subsections. We will demonstrate the how to compute VaR when assuming multivariate normally distributed losses. Other multivariate distributions, such as the multivariate student's t-distribution is outside the scope of this thesis.

Remember from Subsection 2.1, Equation (10) that we defined the loss for the portfolio as

$$L_{n+1}^\Delta = -\mathbf{w}^T \mathbf{X}$$

where $\mathbf{w}^T = (\alpha_1 S_{n,1}, \alpha_2 S_{n,1}, \dots, \alpha_i S_{n,d})$ and $\mathbf{X} = (X_{n+1,1}, \dots, X_{n+1,d})$.

The vector \mathbf{X} is multivariate normally distributed. By properties of the multivariate normal random variable the one-dimensional random variable $\mathbf{w}^T \mathbf{X}$ will also be a one-dimensional normal random variable with mean $\mathbf{w}^T \boldsymbol{\mu}$ and variance $\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$, that is $\mathbf{w}^T \mathbf{X} \sim N(\mathbf{w}^T \boldsymbol{\mu}, \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w})$ where \mathbf{w} is the vector of weights defined above. Calculating the mean and variance of the portfolio is more complex than for a single stock. One needs to know how the stocks are weighted in the portfolio in order to estimate both the mean and variance correctly. When calculating the variance, one needs to know the weights of the stocks in the portfolio as well as keep track of how the stocks are correlated with each other. Larger correlations between stocks increase the risk of the portfolio.

By using the historical values in \mathbf{X} we find the point estimates to create the mean vector $\hat{\boldsymbol{\mu}}$ and covariance matrix $\hat{\boldsymbol{\Sigma}}$ to \mathbf{X} . We can now use the fact that $\mathbf{w}^T \mathbf{X} \sim N(\mathbf{w}^T \boldsymbol{\mu}, \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w})$ and combine this with Equation (12) which is $VaR_\alpha(L) = \mu + \sigma N^{-1}(\alpha)$ to get

$$VaR_\alpha(L) = -\mathbf{w}^T \hat{\boldsymbol{\mu}} + \sqrt{\mathbf{w}^T \hat{\boldsymbol{\Sigma}} \mathbf{w}} N^{-1}(\alpha). \quad (24)$$

If we combine Equation (15) which is $ES_\alpha = \mu + \sigma \frac{\phi(N^{-1}(\alpha))}{1-\alpha}$ assuming that

$\mathbf{w}^T \mathbf{X} \sim N(\mathbf{w}^T \boldsymbol{\mu}, \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w})$, we get

$$\widehat{ES}_\alpha(L) = -\mathbf{w}^T \hat{\boldsymbol{\mu}} + \sqrt{\mathbf{w}^T \hat{\boldsymbol{\Sigma}} \mathbf{w}} \frac{\phi(N^{-1}(\alpha))}{1-\alpha} \quad (25)$$

where $\varphi(x)$ is the density for a standard normal random variable (McNeil, et al., 2005).

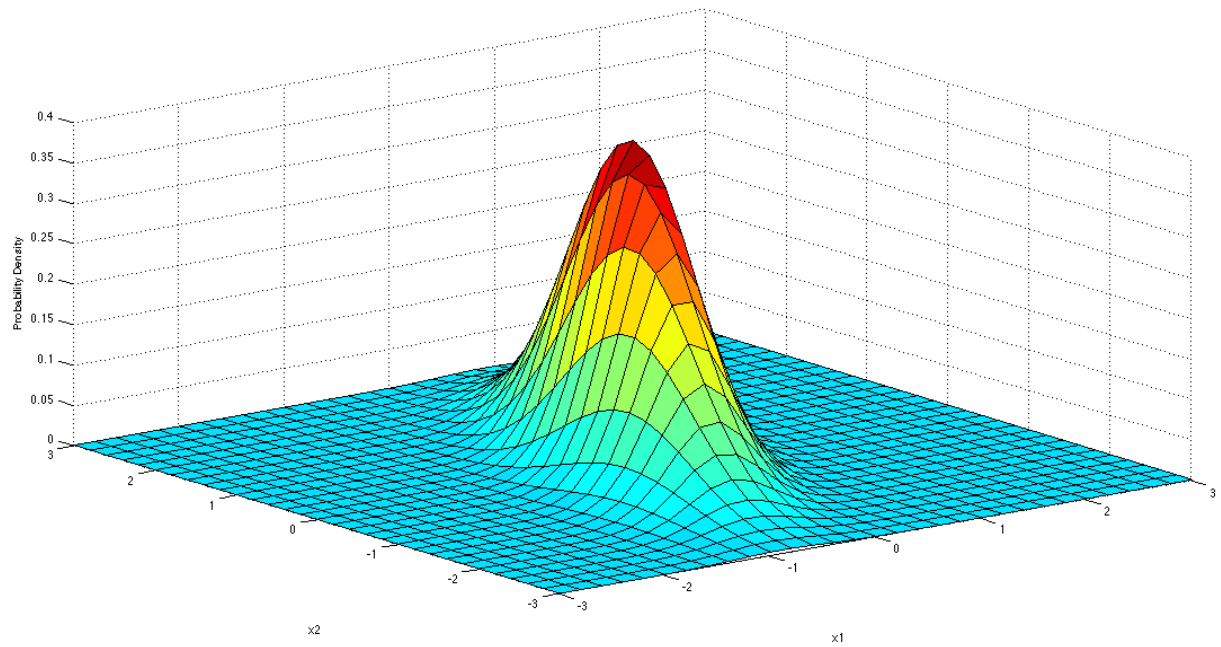


Figure 3. An illustration of a multivariate normal distribution.

For illustration purposes, we display the density of a multivariate (two dimensional) normal distribution in Figure 3. This in order to get a sense of the difference between a univariate distribution and a multivariate distribution, where more dimensions are added.

3. Empirical investigation of measures

In this section we will present the results of our Value-at-Risk and expected shortfall estimates. We have performed estimations for two different portfolios, a univariate portfolio and a multivariate that portfolio, and we will present a data description for both of these portfolios in Subsection 3.1.

The presentation of our empirical investigation will be presented within Subsection 3.2 through Subsection 3.5. In Subsection 3.1 we give a description of our data. When moving on to Subsection 3.2 we show our findings for the univariate portfolio assuming both a normal and a student's t-distribution. For Subsection 3.3 we present the findings for the multivariate portfolio assuming normal distribution. Subsection 3.4 includes the Monte Carlo simulation for the multivariate portfolio. Finally, in Subsection 3.5 we will display a historically simulated VaR for a multivariate portfolio.

In the Subsections we will show results from backtesting, which we use when we investigate how well our VaR estimates performed. This is done by counting how often losses exceed the estimated VaR and then divide the total amount of exceedances by the length of the period. We expect to see $1 - \alpha$ percent exceedances (McNeil, 2005, pg. 55).

3.1 Data description

The data used when estimating a univariate VaR i.e. the one-stock portfolio, consists of a portfolio with one thousand shares of Volvo stock, thus $d = 1$, where $x_1 = 1000$. The choice of stock and the number of shares is an arbitrary amount and selection. We want to show how the models are applied when using a univariate portfolio. The data used when estimating the multivariate VaR is a 26-stock portfolio that consists of daily prices for 26 stocks listed on the Stockholm Stock Exchange, and all of them are included in the OMXS30 index. Again, the stocks chosen are an arbitrary selection and the portfolio is constructed for the purpose of testing the models. In Table A3 in the Appendix we have a list of stocks included in the multivariate

portfolio. For the purpose of estimating multivariate VaR we have chosen to hold one share of each stock in the portfolio, so that $\alpha_1 = \alpha_2 = \dots = \alpha_{26}$.

The data for both portfolios is sampled daily and it has been divided into two periods, a normal- and a crisis period. The crisis period will consist of data from 2006-01-01 to 2009-12-31 and the less volatile period will consist of data from 2010-01-01 to 2013-12-31. The first period is referred to as Period 1 and the second period is referred to as Period 2. The large crisis included Period 1 is the 2007/2008 sub-prime crisis. We have divided the data into two periods since it is of interest to investigate whether VaR performs better in periods of less volatility. Throughout this report all estimations, graphs, and tables have been done with MATLAB and Excel.

In Table 1 we present some descriptive statistics for the portfolio with one thousand Volvo shares, during both periods. The period Jan-06 to Dec-09 consists of 1003 days and the period Jan-10 to Dec-13 consists of 1005 days. In Jan-06 to Dec-09 we notice that the standard deviation (which is the square root of the variance) of the daily losses is larger than in Jan-10 to Dec-13. We also observe that the range between losses is greater in the first period. The change in the value of the portfolio is the relative difference in portfolio value between the first and last day of each period. The initial value of the portfolio was 75200kr in Period 1 and 63150kr in Period 2.

Descriptive statistics for daily losses for the one-stock portfolio		
	Jan06-Dec09	Jan10-Dec13
Total change in value	-18,28%	33,73%
Std.dev	2 043 kr	1 800 kr
Min loss	-13 000 kr	-8 250 kr
Max loss	13 000 kr	7 700 kr
Days in period	1003	1005
Initial value of portfolio	75 200 kr	63 150 kr

Table 1. Descriptive statistics for the losses for the one-stock portfolio in Period 1 and Period 2.

Figure 4 shows the development of the value of the one-stock portfolio during the period Jan-06 through Dec-13. We observe a substantial decrease in value of the one-stock portfolio which begun in June 2007 and the value kept decreasing until late 2008. This decrease was due to the sub-prime crash of 2007/2008.

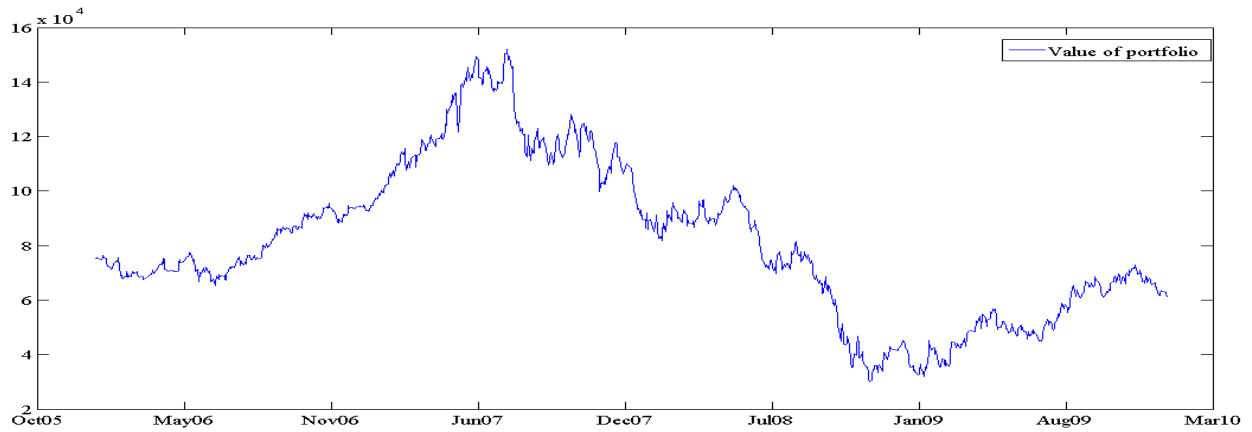


Figure 4. Value of one-stock portfolio during Jan-06 through Dec-09.

Figure 5 shows the value of the one-stock portfolio during the period Jan-10 through Dec-13. We see a sudden decrease in value of the one-stock portfolio during the summer of 2011, which was due to the Greek debt crisis. We observe an increase in value of the one-stock portfolio towards the end of 2011.

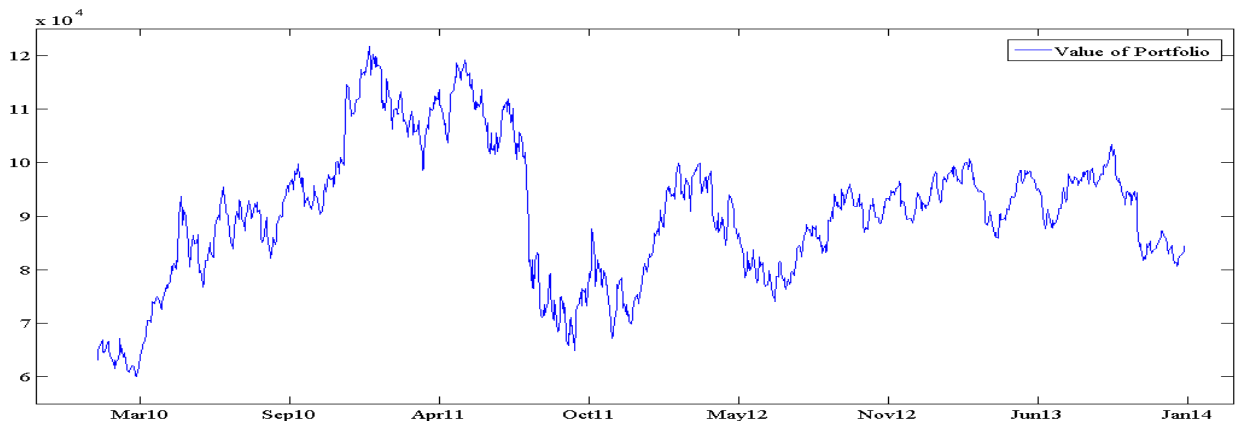


Figure 5. Value of one-stock portfolio during Jan-10 through Dec-13.

Note that Figures 6 and Figure 7 will be displayed with losses. Note that a negative loss is a profit.

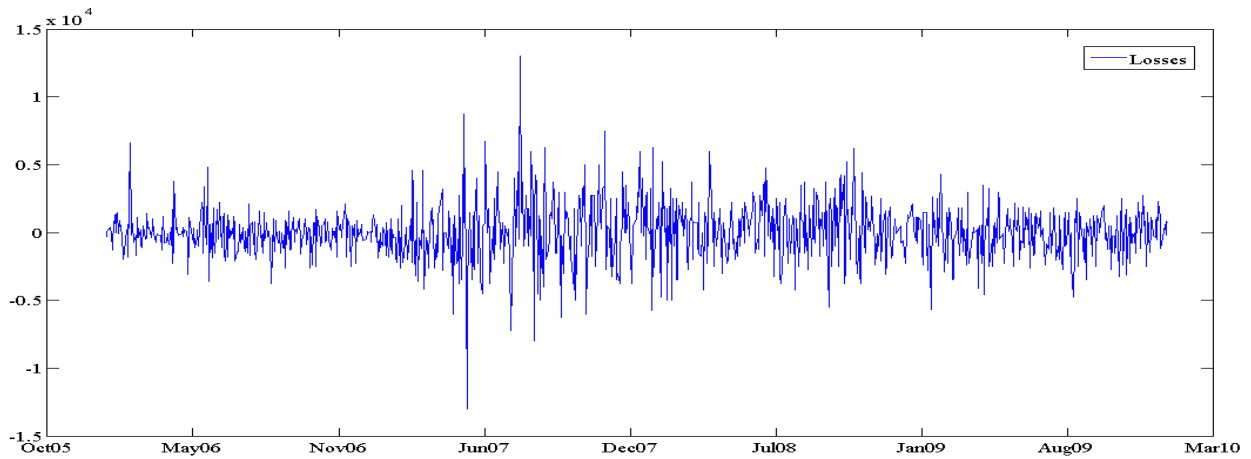


Figure 6. Losses for one-stock portfolio during Jan-06 through Dec-09.

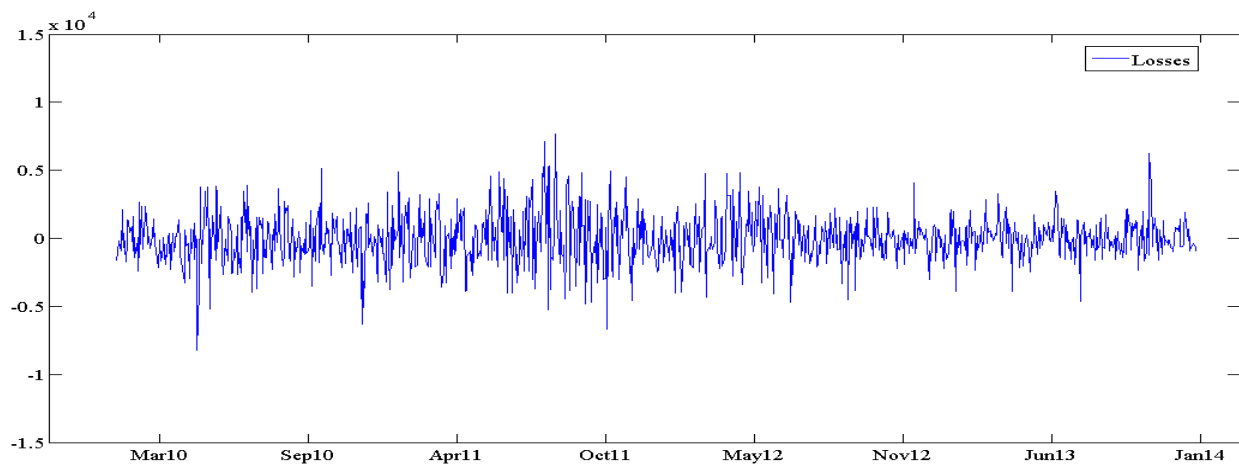


Figure 7. Losses for one-stock portfolio during Jan-10 through Dec-13.

Figure 6 and 7 displays the losses for Jan-06 to Dec-09 and Jan-10 to Dec-13. We observe larger market volatility in Period 1 which is displayed in Figure 6. The spread between losses increase right before the sub-prime crash, compare with graph in Figure 4 for reference. In Figure 7, which displays losses for Period 2, we observe less market volatility. Note the large losses between April-11 and October-11 when the Greek debt crises shook the market.

In Table 2 we present the data used for the multivariate 26-stock portfolio. Notice that our portfolio increased in value in both periods. When measuring the relative difference between the start and end of the period, we see an increase by 6 percent in Period 1 and by 28 percent in Period 2.

Descriptive statistics for daily losses for the 26-stock portfolio		
	Jan06-Dec09	Jan10-Dec13
Total change	5.81 %	28.27 %
Std.dev	51,67 kr	45,24 kr
Min loss	- 219.01 kr	- 201.03 kr
Max loss	202.58 kr	231.17 kr
Days in period	1003	1005
Initial value of portfolio	3 164 kr	3 392 kr

Table 2. Descriptive statistics for the losses for the 26-stock portfolio during both periods.

The standard deviation of daily losses for the portfolio was slightly higher in Jan-06 to Dec-09, meanwhile the spread between portfolio losses, which is calculated using Equation (2) on p.7, is larger in Jan-10 to Dec-13. The initial value of the portfolio was 3164kr in Period 1 and 3392kr in Period 2.

We have chosen to display the correlation matrices, for the first 250 days in both periods in Tables A1 and A2 in the Appendix. Since all stocks are chosen from the same stock index, there are plenty of stocks with high correlations and the highest correlations are found between SKF, Sandvik, and Atlas Copco. This follows from the fact that they are active within the same industries.

In Figure 8 and Figure 9 we will display the value of the 26-stock portfolio for period Jan-06 to Dec-09 and period Jan-10 to Dec-13.

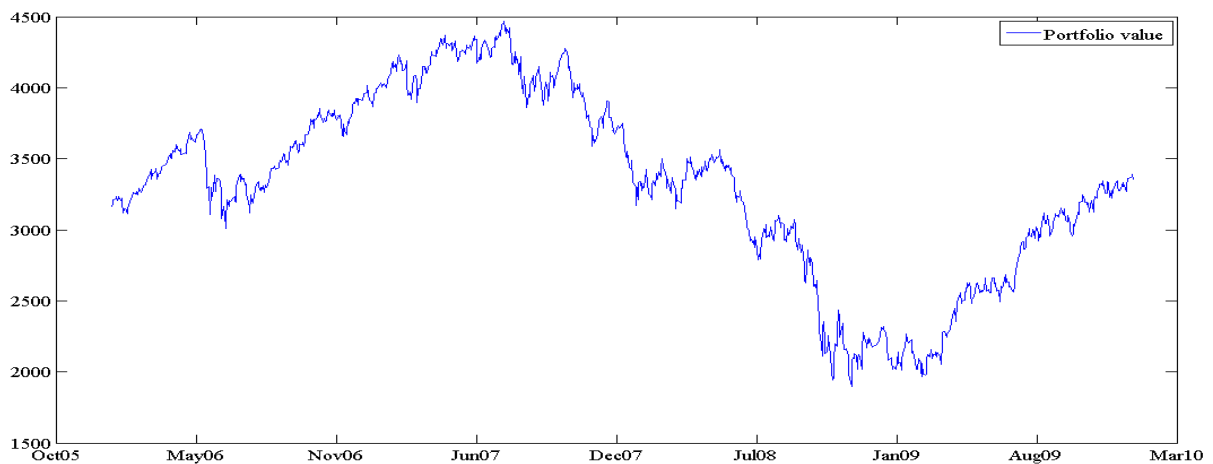


Figure 8. The value of the multivariate portfolio for Jan-06 through Dec-09.

As we can see from Figure 8, our portfolio value decreased during the sub-prime crash that occurred during fall 2007. Towards the end of 2008 the economy picked up and our portfolio value started to increase.

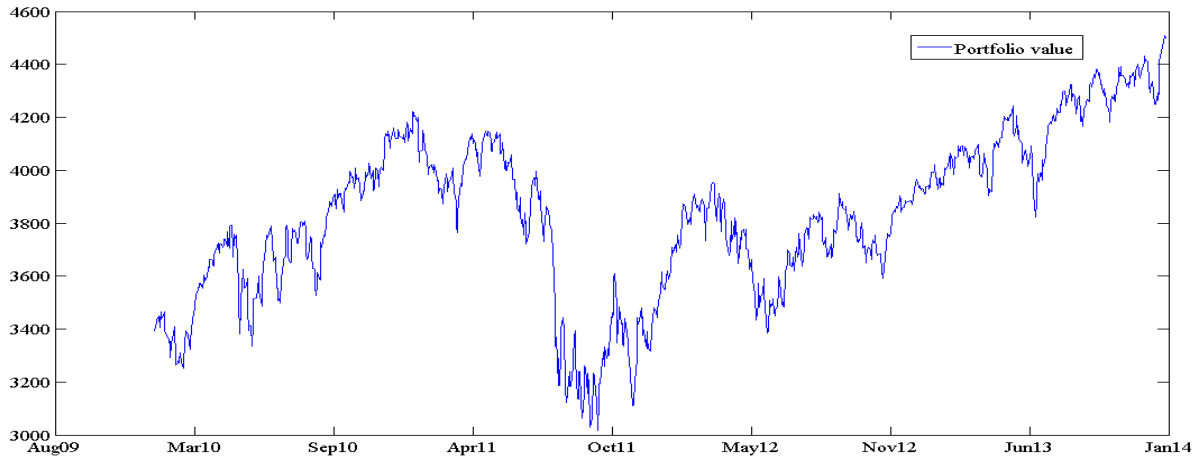


Figure 9. The value of the multivariate portfolio for Jan-10 through Dec-13.

In Figure 9 we see the development of our portfolio value. When the Greek debt crisis occurred, the portfolio rapidly decreased in value. Portfolio value started to increase around October 2011 and continued to increase, with minor dips, throughout the period.

Note that the rest of the figures throughout Section 3 will be presented with losses and that a negative loss is a profit.

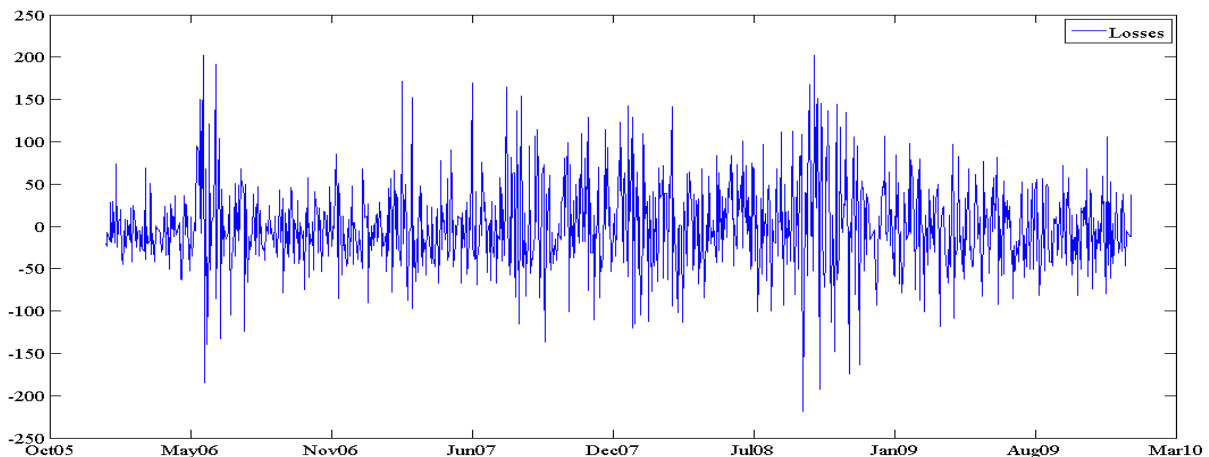


Figure 10. Losses for the multivariate portfolio Jan-06 through Dec-09.

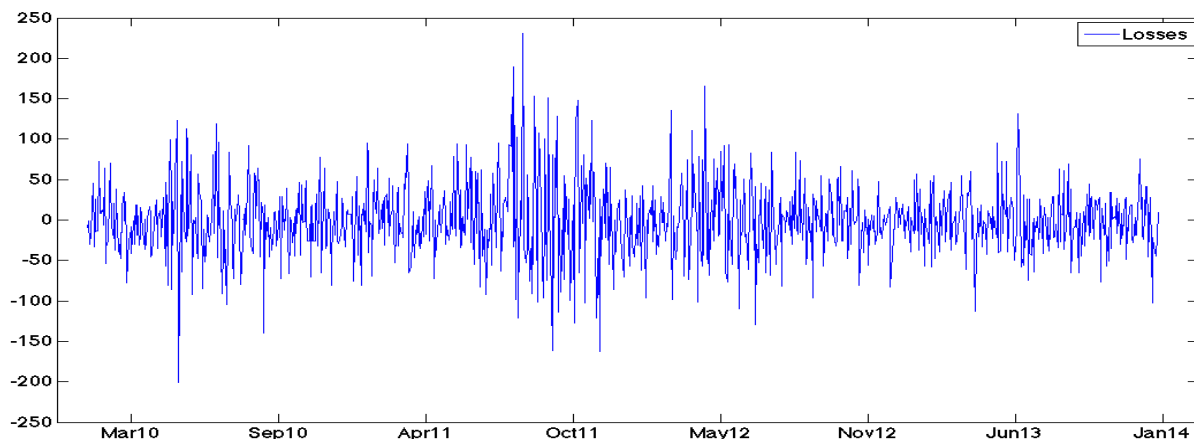


Figure 11. Losses for the multivariate portfolio Jan-10 through Dec-13.

Between Jan-06 and Dec-09 there seems to be more volatility than between Jan-10 and Dec-13, and both periods have clusters of volatility where the markets exhibits more volatility compared to the rest of the period. Period 2 has only a few clusters of high volatility while they are more frequent in Period 1. Since these figures resemble Figure 3 and Figure 4, we will not discuss these in more detail.

3.2 Value-at-Risk and Expected Shortfall computed for the univariate portfolio

In this subsection we will present the numerical Var and ES computations for a one-stock equity portfolio. The portfolio consists of a thousand Volvo shares and we will display our results assuming both normally and student's t-distributed losses. We begin by showing rolling VaR and ES estimates based on the previous 250 days. Rolling VaR and ES means that estimates are based on previous observations and this is repeated day-by-day for a moving window of historical observations, which allows us to plot VaR and ES and illustrate when and where exceedances occur. Since the first 250 days of each period are used to estimate the first VaR and ES, all figures will depict VaR, ES, and losses from the 251st day and forward.

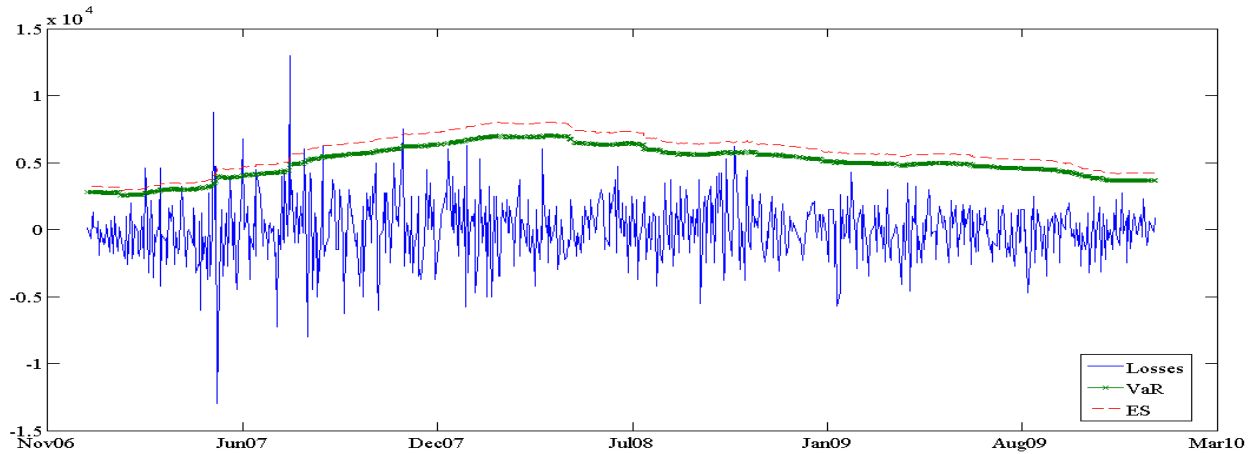


Figure 12. VaR(bold) and ES(dotted line) estimates with $\alpha = 0.99$, assuming normal distribution, based on previous 250 days plotted against losses during Jan-07 through Dec-09.

In Figure 12 we have plotted VaR and ES with $\alpha = 0.99$ under normal distribution for Jan-06 to Dec-09. We observe a substantial amount of exceedances during the sub-prime crash of 2007/2008. Since estimates of μ and σ (see subsection 2.5) are based on past data consisting of 250 days, we observe that it takes some time for VaR and ES estimates to react to the crisis. For a faster reaction one can use fewer observations in the estimation. The largest VaR and ES estimates are observed in February of 2008, even though the crisis struck during the summer of 2007. As we recall from Equation 8, the ES estimate is always larger than or equal to the VaR estimate.

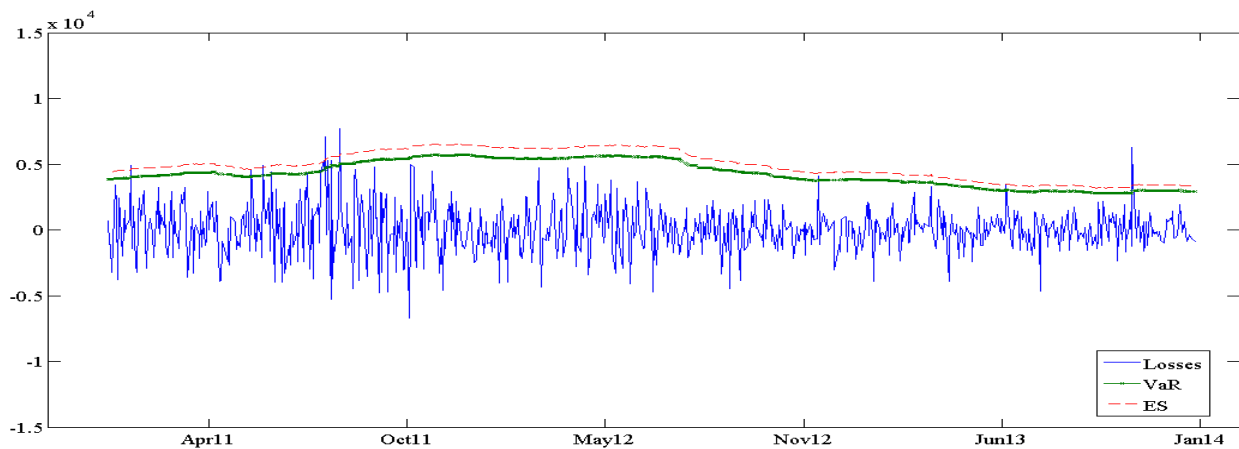


Figure 13. VaR(bold) and ES(dotted line) estimates with $\alpha = 0.99$, assuming normal distribution, based on previous 250 days plotted against losses during Jan-11 through Dec-13.

In Figure 13 we display VaR and ES with $\alpha = 0.99$ under normal distribution for Jan-11 and Dec-13. We observe few exceedances during this period. The exceedances observed are due to the Greek debt crisis, which affected our portfolio during summer of 2011. Again VaR and ES

display a lagged reaction to market volatility, this is due to the 250 days needed for estimating the daily VaR and ES here.

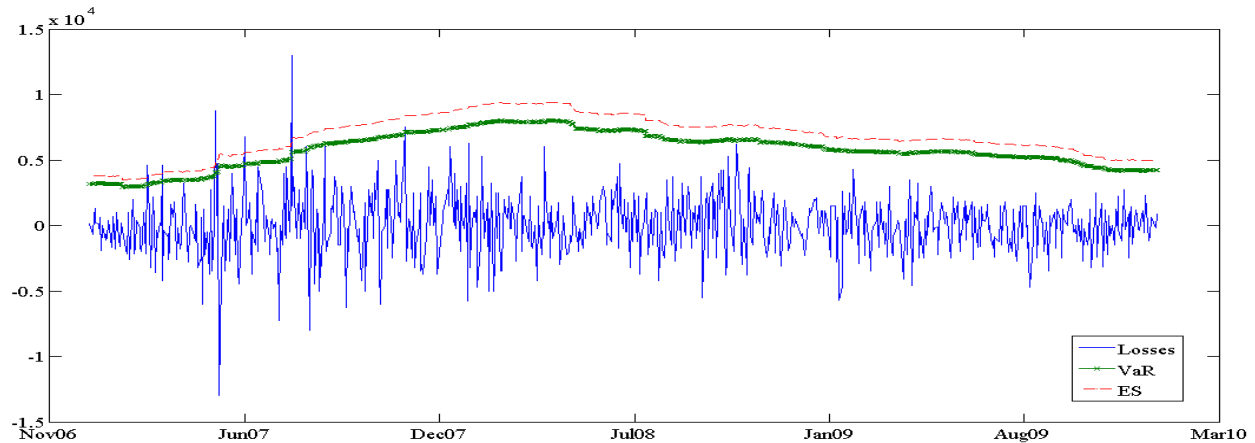


Figure 14. VaR(bold) and ES(dotted line) estimates with $\alpha = 0.99$, assuming t-distribution, based on previous 250 days plotted against losses during Jan-07 through Dec-09.

In Figure 14 and Figure 15 we show VaR and ES with $\alpha = 0.99$ and $\nu=20$ degrees of freedom under student's t-distribution, based on the previous 250 days. Since these figures are very much alike the ones for VaR and ES under normal distribution there is no need for further comments beyond that here the VaR and ES are slightly higher than the VaR and ES under normal distribution. This is due to the fatter tails of the t-distribution where we have a larger area under the tails and therefore obtain larger values as shown in Figure 1 and Figure 2.

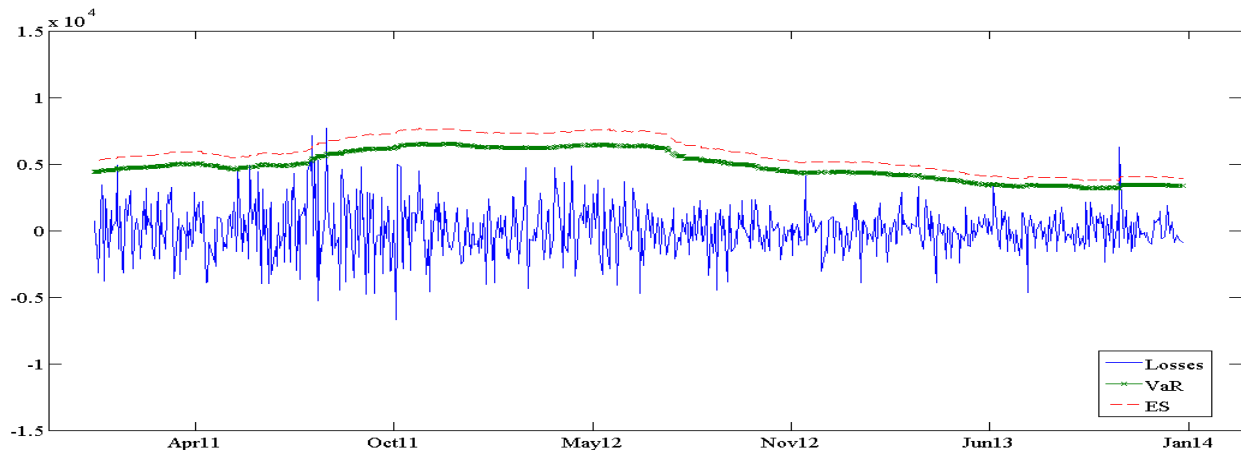


Figure 15. VaR(bold) and ES(dotted line) estimates with $\alpha = 0.99$, t-distribution, based on previous 250 days plotted against losses during Jan-11 through Dec-13.

In Table 3 we present two estimates for VaR during Jan-06 and Dec-09 and Jan-10 and Dec-13. These estimates show the first observed VaR for the corresponding period as well as the largest observed estimate in each period. VaR estimates show us how much we are expecting to lose during the following day, therefore the computed VaR for January 2nd is presented on January 1st. This number can be converted to a percentage of our portfolio value and is valid if we should increase investment in the portfolio, provided we use the same weights as before. The percentage of losses on the portfolio would only change if volatility would change due to a shift in portfolio weights or a general shift in market volatility, as described in the discussion of homogeneity in Subsection 2.3.

Univariate VaR_α				
Jan-06 through Dec-09			Jan-10 through Dec-13	
α	Jan 2 nd -07	Apr 11 th -08	Dec 28 th -10	Nov 23 rd -11
0.95	1 942 kr (2,1%)	4 990 kr (5,1%)	2 646 kr (2,3%)	4 085 kr (5,9%)
0.975	2 328 kr (2,5%)	5 922 kr (6,1%)	3 194 kr (2,7%)	4 842 kr (7,0%)
0.99	2 778 kr (2,9%)	7 006 kr (7,2%)	3 831 kr (3,3%)	5 722 kr (8,2%)
0.999	3 715 kr (3,9%)	9 266 kr (9,6%)	5 159 kr (4,4%)	7 557 kr (10,9%)

Table 3. One-day VaR_α for normal distribution and its value in percent of the portfolio value for different α .

In Table 4 we present two estimates for ES during Jan-06 to Dec-09 and Jan-10 to Dec-13. These estimates show the first observed ES and the highest value for the period at different confidence levels, as well as the percentage of the portfolio value.

ES_α				
Jan-06 through Dec-09			Jan-10 through Dec-13	
α	Jan 2 nd -07	Apr 11 th -08	Dec 28 th -10	Nov 23 rd -11
0.95	2 454 kr (2,6%)	6 226 kr (6,4%)	3 373 kr (2,9%)	5 089 kr (7,3%)
0.975	2 792 kr (2,9%)	7 040 kr (7,3%)	3 851 kr (3,3%)	5 749 kr (8,3%)
0.99	3 193 kr (3,4%)	8 009 kr (8,3%)	4 420 kr (3,8%)	6 536 kr (9,4%)
0.999	4 054 kr (4,3%)	10 085 kr (10,4%)	5 640 kr (4,8%)	8 223 kr (11,8%)

Table 4. ES_α for normal distribution and its value in percent of the portfolio value for different α .

In Table 5 we present two estimates for VaR with $\nu=20$ degrees of freedom for Jan-06 through Dec-09 and Jan-10 through Dec-13. The first estimate is for the 251st day of each period and the second estimate is the largest observed estimate for the period. When comparing these estimates with the Var estimates under normal distribution we find that the student's t-distribution provides us with larger VaR estimates. This is due to the fatter tails of the t-distribution where different alphas cover a larger area in the tails, i.e. the values become larger.

Univariate VaR_α				
Jan-06 through Dec-09			Jan-10 through Dec-13	
α	Jan 2 nd -07	Apr 11 th -08	Dec 28 th -10	Nov 23 rd -11
0.95	2 154 kr (2,3%)	5 502 kr (5,7%)	2 947 kr (2,5%)	4 501 kr (6,5%)
0.975	2 621 kr (2,8%)	6 629 kr (6,8%)	3 609 kr (3,1%)	5 415 kr (7,8%)
0.99	3 193 kr (3,4%)	8 007 kr (8,3%)	4 419 kr (3,8%)	6 535 kr (9,4%)
0.999	4 517 kr (4,8%)	11 200 kr (11,5%)	6 295 kr (5,4%)	9 128 kr (13,1%)

Table 5. VaR_α for student's t-distribution with $\nu=20$ degrees of freedom, and its value in percent of the portfolio value for different α .

In Table 6 we present ES for student's t-distribution with $\nu=20$ degrees of freedom. Recall that the ES estimate is always larger or equal to the VaR, therefore we find that our values are larger. This table follows the same form as previous tables, with the first estimate being the 251st day and the second being the largest observed during the period.

ES_α				
Jan-06 through Dec-09			Jan-10 through Dec-13	
α	Jan 2 nd -07	Apr 11 th -08	Dec 28 th -10	Nov 23 rd -11
0.95	2 797 kr (3,0%)	7 052 kr (7,3%)	3 858 kr (3,3%)	5 760 kr (8,3%)
0.975	3 230 kr (3,4%)	8 096 kr (8,3%)	4 471 kr (3,8%)	6 607 kr (9,5%)
0.99	3 773 kr (4,0%)	9 407 kr (9,7%)	5 241 kr (4,5%)	7 672 kr (11,0%)
0.999	5 072 kr (5,4%)	12 540 kr (12,9%)	7 082 kr (6,1%)	10 216 kr (14,7%)

Table 6. ES_α for student's t-distribution with $\nu=20$ degrees of freedom, and its value in percent of the portfolio value for different α .

In Table 7 we present backtesting of VaR under normal distribution. The values are the percentage of the number of exceedances of our VaR estimates, meaning when losses are larger than the computed VaR for the corresponding day. These exceedances are divided by the length

of the period. We are expecting $(1 - \alpha)100$ exceedances, therefore we have included the expected percentage in the rightmost column in the tables. As we can see, the exceedances are substantially larger than expected when computing VaR with $\alpha = 0.99$ and $\alpha = 0.999$. The estimated VaR is more accurate with lower alpha values, which shows that the model is not fully applicable for larger values of alpha. This suggests that assuming normally distributed losses is not a valid assumption.

Backtesting of univariate VaR_α			
α	Jan06 - Dec09	Jan10 - Dec13	Expected %
0.95	4,77%	4,89%	5%
0.975	3,05%	3,17%	2.5%
0.99	1,99%	1,72%	1%
0.999	0,66%	0,40%	0.1%

Table 7. Backtesting of one-day VaR_α for normal distribution.

In Table 8 we present backtesting of our computed VaR with ν degrees of freedom under student's t-distribution during Jan-06 to Dec-09. As mentioned earlier in Subsection 2.6, the t-distribution converges towards the normal distribution as the degrees of freedom approaches infinity. When analyzing results we find that there are fewer exceedances with smaller degrees of freedom. Even though there are few exceedances we must remember that if the financial institution consequently keeps a larger buffer than needed, potential investment opportunities could be lost due to a pessimistic VaR.

Backtesting of t-distributed VaR_α Jan-07 to Dec-09							
α	$\nu=7$	$\nu=9$	$\nu=11$	$\nu=20$	$\nu=40$	$\nu=300$	Expected %
0.95	1,99%	2,52%	3,05%	3,45%	3,98%	4,77%	5%
0.975	0,93%	1,33%	1,72%	1,99%	2,52%	3,05%	2.5%
0.99	0,66%	0,66%	0,80%	1,33%	1,59%	1,99%	1%
0.999	0,13%	0,27%	0,27%	0,53%	0,66%	0,66%	0.1%

Table 8. Backtesting of t-distributed one-day VaR_α with ν degrees of freedom.

In Table 9 we present backtesting of our estimated VaR with ν degrees of freedom under student's t-distribution during Jan-10 to Dec-13. We observe that with $\alpha = 0.95$ there are more exceedances than during Jan-06 to Dec-13, though with higher values of alpha the exceedances are smaller than in the first period.

Backtesting of t-distributed VaR_α Jan-11 to Dec-13							
α	$\nu=7$	$\nu=9$	$\nu=11$	$\nu=20$	$\nu=40$	$\nu=300$	Expected %
0.95	1,98%	2,25%	3,17%	3,57%	4,50%	4,89%	5%
0.975	0,66%	1,32%	1,59%	1,98%	2,38%	3,17%	2.5%
0.99	0,26%	0,40%	0,40%	0,93%	1,59%	1,72%	1%
0.999	0,00%	0,13%	0,13%	0,13%	0,40%	0,40%	0.1%

Table 9. Backtesting of t-distributed one-day VaR_α with ν degrees of freedom.

3.3 Value-at-Risk and Expected Shortfall computed for the multivariate portfolio

In this subsection we will present the numerical Var and ES computations for a multivariate equity portfolio consisting of 26 stocks from the OMXS30. We will display our findings assuming that the losses are normally distributed. In Figure 16 and Figure 17 we show a rolling VaR and ES estimates based on the previous 250 days.

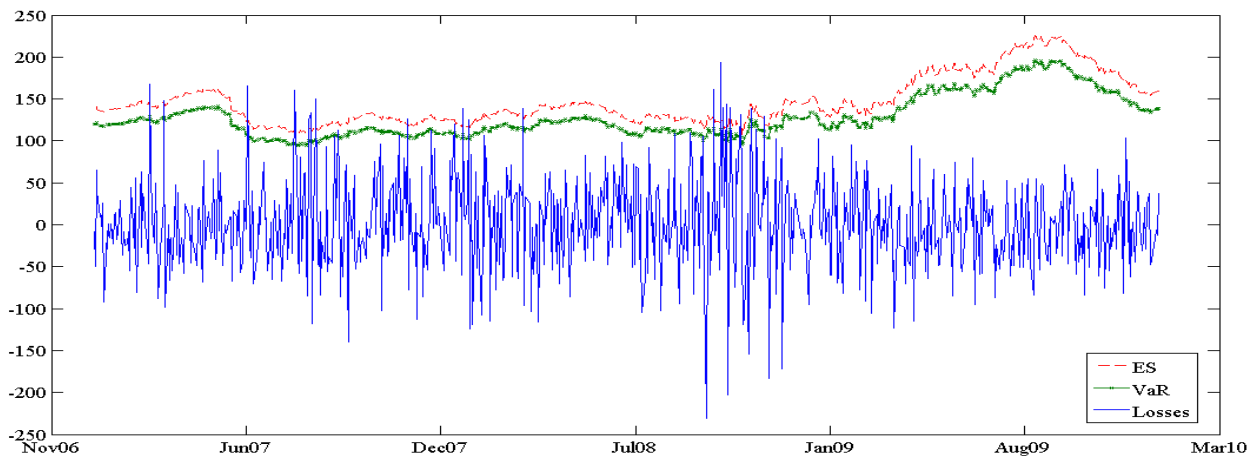


Figure 16. VaR(bold) and ES(dotted line) plotted against losses for the portfolio at $\alpha = 0.99$ under normal distribution for Jan-07 to Dec-09.

In Figure 16 we present VaR and ES with at $\alpha = 0.99$ under normal distribution for Jan-06 to Dec-09. Due to the substantial market volatility during this period we observe exceedances during the sub-prime crash of 2007/2008.

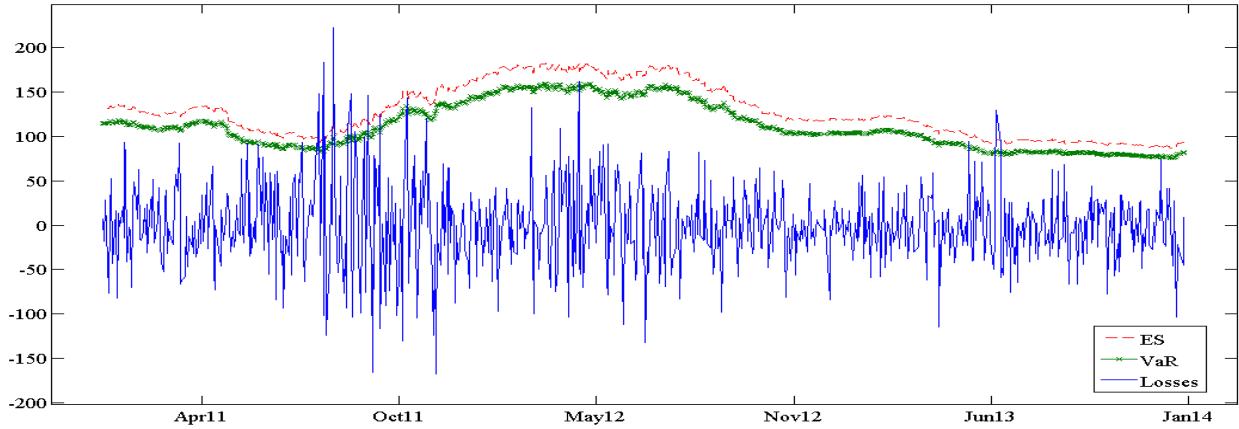


Figure 17. VaR(bold) and ES(dotted line) plotted against losses for the portfolio at $\alpha = 0.99$ under normal distribution for Jan-11 to Dec-13.

In Figure 17 we present VaR and ES with $\alpha = 0.99$ under normal distribution for Jan-10 to Dec-13. With less market volatility than in the first period, we observe a smaller amount of exceedances.

In Table 10 we present estimates for VaR for two days in both periods. The first estimate is the VaR for the 251st day in each period and the second estimate is the largest observed estimate in each period.

Multivariate VaR_α				
α	Jan 2 nd -07	Aug 26 th -09	Dec 28 th -10	Mar 20 th -12
0.95	83 kr (2.1%)	139kr (4.5%)	80kr (1.9%)	113kr (3.2%)
0.975	100kr (2.5%)	166kr (5.3%)	96kr (2.3%)	134kr (3.8%)
0.99	120kr (3.0%)	197kr (6.3%)	115kr (2.8%)	160kr (4.5%)
0.999	160kr (4.0%)	262kr (8.4%)	154kr (3.7%)	212kr (6.0%)

Table 10. One-day VaR_α for the multivariate portfolio under normal distribution and its value in percent of the portfolio value for different α .

In Table 11 we present estimates for ES for two days in both periods. The first estimate is the ES for the 251st day in each period and the second estimate is the largest observed estimate in each period. Again, these estimates are larger than the VaR estimates for corresponding days.

ES_α				
α	Jan 2 nd -07	Aug 26 th -09	Dec 28 th -10	Mar 20 th -12
0.95	106kr (2.7%)	174kr (5.6%)	101kr (2.4%)	141kr (4.0%)
0.975	120kr (3.0%)	198kr (6.4%)	115kr (2.8%)	160kr (4.5%)
0.99	138kr (3.5%)	226kr (7.3%)	132kr (3.2%)	183kr (5.1%)
0.999	175kr (4.5%)	285kr (9.1%)	168kr (4.1%)	230kr (6.5%)

Table 11. One-day ES_α for the multivariate portfolio under normal distribution and its value in percent of the portfolio value for different α .

In Table 12 we present backtesting of VaR for both periods. We observe exceedances above the expected amount for all values of alpha except for VaR with $\alpha = 0.95$ in Jan-10 to Dec-13. Again, this shows that the model is not applicable for these values.

Backtesting of multivariate VaR_α			
α	Jan06 - Dec09	Jan10 - Dec13	Expected %
0.95	6,10%	4,63%	5%
0.975	4,38%	3,57%	2.5%
0.99	2,92%	2,38%	1%
0.999	1,06%	0,79%	0.1%

Table 12. Backtesting of one-day VaR_α for the multivariate portfolio under normal distribution.

3.4 Monte Carlo simulation

In Table 13, we present the multivariate, Monte Carlo simulated, VaR estimate. We assume that the risk factor changes are multivariate normally distributed $X \sim N(\mu, \Sigma)$ where the mean μ and covariance matrix Σ are estimated using the previous 250 days.

Monte Carlo simulated multivariate VaR_α		
α	Jan 2 nd -07	Dec 28 th -10
0.95	83 kr (2.1%)	80kr (1.9%)
0.975	100kr (2.5%)	96kr (2.3%)
0.99	120kr (3.0%)	115kr (2.8%)
0.999	160kr (4.0%)	154kr (3.7%)

Table 13. Simulated one-day multivariate VaR_α with 10^6 simulations for normal distribution and its value in percent of the portfolio value for different α .

The VaR estimates obtained from the Monte Carlo simulation are almost identical (due to the law of large numbers) to the estimates for the 251st day for both Period 1 and Period 2, as presented in Table 10. Monte Carlo simulation is a useful tool when one does not have an analytical expression to compute VaR. For the multivariate normal distribution we do have a closed form expression to compute VaR presented in Equation (24). There are distributions where it is not possible to get a closed form expression for the VaR computations and one must therefore rely on Monte Carlo simulations to compute VaR.

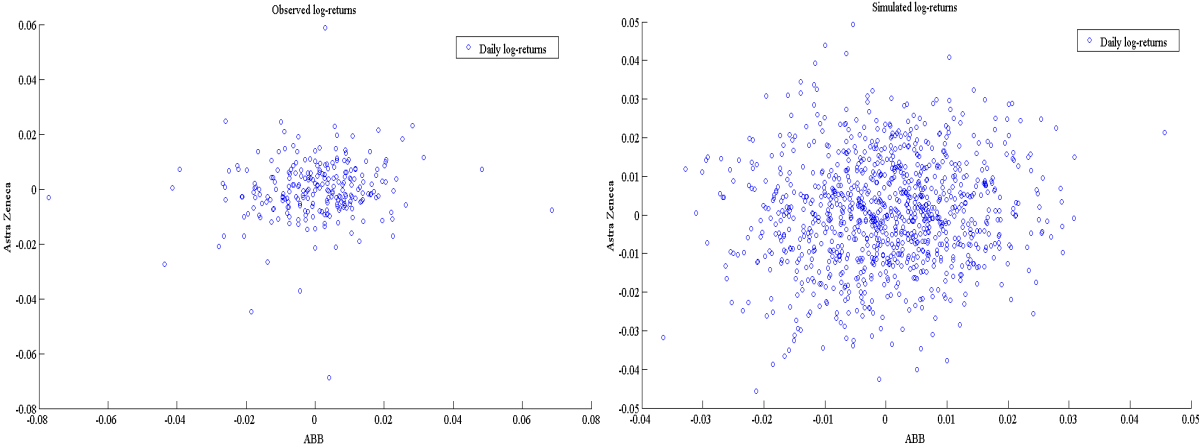


Figure 18. Scatter plot between ABB and Astra Zeneca. Actual values are displayed on the left and simulated values to the right. Risk factor changes are the daily log-returns.

In Figure 18 we display two scatter plots for risk factor changes, which is the daily log-return, for Astra Zeneca and ABB. We can see that the values are scattered with a low correlation. We want to show that the correlation structure is the same in the simulation. In Figure 19 we display two scatter plots for risk factor changes for Sandvik and Atlas Copco. We can see that they are highly correlated.

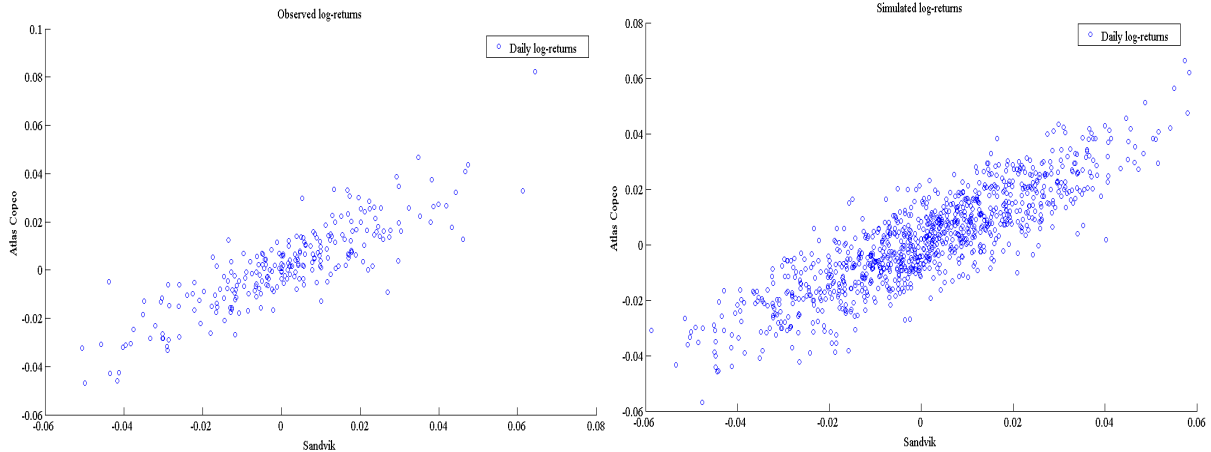


Figure 19. Scatter plot between Sandvik and Atlas Copco. Observed log-returns are displayed on the left and simulated values to the right.

3.5 Historical Simulation

We have plotted a rolling historically simulated VaR and these are displayed in Figure 18 and in Figure 19. The historically estimated VaR only takes into account historical observations and does not assume any distribution. From looking at Figure 20 and Figure 21, we can see that the Historically simulated VaR reacts relatively fast to clusters of volatility. When the financial crisis strikes it only takes a few large losses until the VaR estimate has adjusted to the current market conditions. However, this method of estimating VaR is also relatively slow to adjust to less volatile market conditions after a period of high volatility and we in Figure 20 we observe only one exceedance after January 2009.

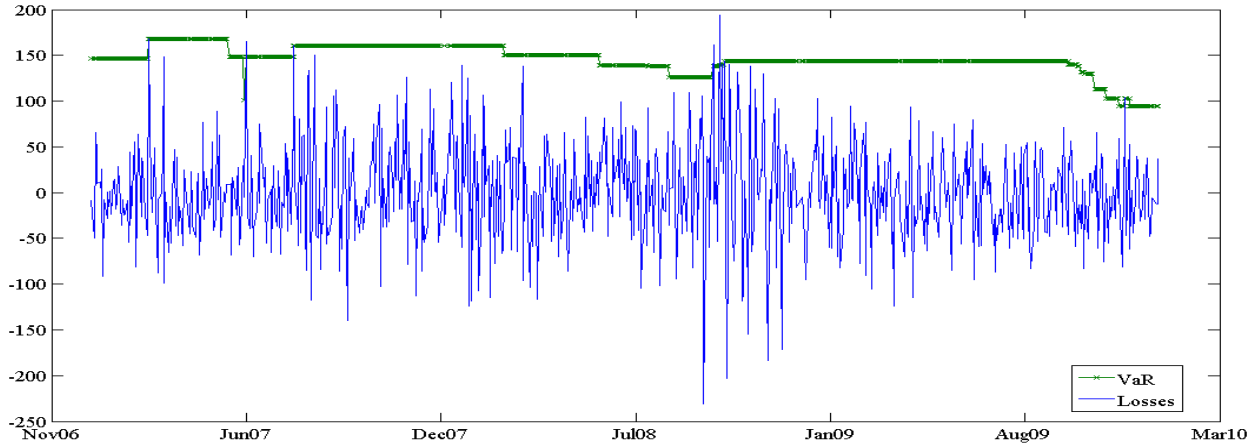


Figure 20. Rolling Var using Historical simulation for $\alpha = 0.99$ for Period 1.

As we can see in Figure 21, we find that VaR responds similarly to the estimated VaR values in Figure 20. In this period we observe a smaller amount of exceedances due to the relatively smaller amount of volatility.

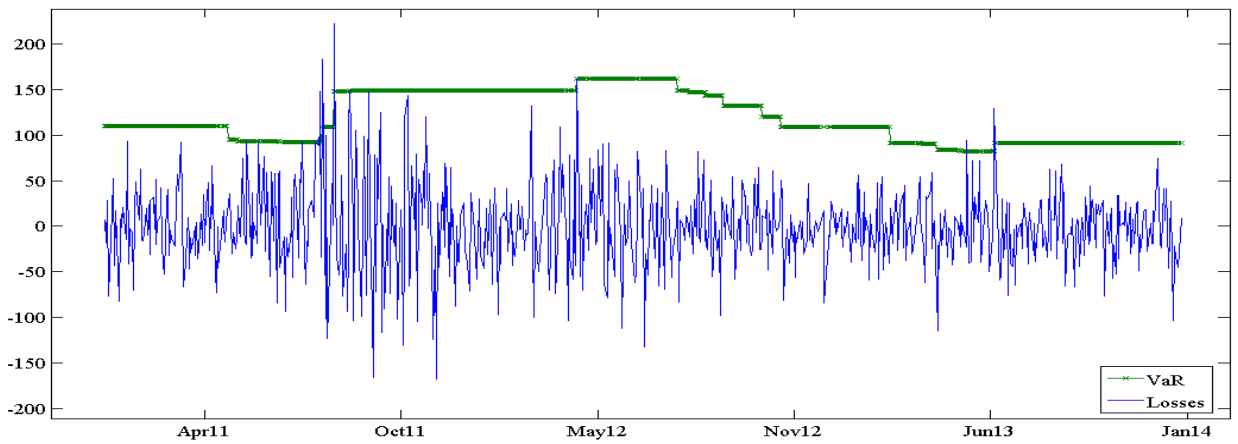


Figure 21. Rolling Var using Historical simulation for $\alpha = 0.99$ for Period 2.

4. Conclusion

In this thesis we have examined how different methods of VaR and ES estimates. We divided our time series into two periods and created two fictive portfolios, one consisting of one thousand Volvo stocks and one consisting 26 stocks chosen from the OMXS30 index.

By using backtesting we have examined the accuracy of VaR estimates for our portfolios. What we have found is that, during periods of relatively low volatility on the market, VaR estimations will in general provide a more accurate forecast of future losses. Our results indicate that, for the models we have used, there are difficulties in producing reliable estimates for high levels of alpha. For the normal distribution, backtesting displays that for $\alpha \geq 0.99$ the VaR estimates are exceeded far more often than expected. Regarding the student's t-distribution, we find that the accuracy of the VaR estimates depend heavily on chosen degrees of freedom.

The methods presented in Subsections 2.5 and 2.6 assume symmetrically distributed losses, however, during periods of substantial volatility the losses are most likely not symmetrically distributed. Even if the losses were symmetrically distributed, the models tested were not able to capture the extreme losses and this issue proves the need for a method to estimate more extreme values.

Issues that arise from less accurate VaR forecasts are that financial institutions will have a buffer that is not large enough to withstand large losses during periods of significant volatility. Also, issues arise when financial institutions set aside buffers that are too large during periods of more stable markets, since these funds could be used to make further investments.

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Appendix 1

Correlation matrices for the multivariate portfolio

	ATCA	ATCB	ATC	tra	Zene	Boliden	ectrolux	ricsson	jettinge	H&M	investor	FMTC	B	Nordea	Sandvik	SCA	B	Scania	B	SEB	A	ndelbank	SKAB	SKFB	SSAB	SWEDA	SWMA	Tele	Z	B	TLSN	Volvo	B			
1																																				
0.9803	1																																			
0.3439	0.3418	1																																		
0.5491	0.5515	0.2830	1																																	
0.5754	0.5663	0.3845	0.4505	1																																
0.4700	0.4427	0.3520	0.3628	0.5318	1																															
0.4473	0.4394	0.3152	0.4262	0.3631	0.4247	1																														
0.5170	0.5136	0.3146	0.4731	0.4217	0.4847	0.4086	1																													
0.6938	0.6782	0.4483	0.5630	0.6207	0.6593	0.5414	0.5940	1																												
0.5213	0.5197	0.2983	0.5179	0.4937	0.3068	0.4571	0.4332	0.5357	1																											
0.6336	0.6185	0.4266	0.4972	0.5298	0.5515	0.5025	0.5020	0.7074	0.5140	1																										
0.7981	0.7953	0.2882	0.5357	0.4877	0.4370	0.4630	0.5099	0.6537	0.4691	0.5889	1																									
0.4316	0.4286	0.3134	0.5003	0.3693	0.3502	0.4557	0.3898	0.5011	0.4315	0.5095	0.4284	1																								
0.3480	0.3275	0.2140	0.2599	0.3353	0.2940	0.2499	0.3404	0.5041	0.2762	0.3544	0.3262	0.2677	1																							
0.6406	0.6256	0.3580	0.5333	0.5308	0.5203	0.4589	0.5300	0.7085	0.4790	0.7926	0.6208	0.5339	0.3791	1																						
0.5892	0.5615	0.2202	0.5163	0.5222	0.4441	0.4726	0.5306	0.6736	0.4304	0.6781	0.5781	0.4674	0.3407	0.7250	1																					
0.5368	0.5249	0.3524	0.5192	0.4852	0.4927	0.4763	0.6047	0.6724	0.4486	0.5956	0.5647	0.4379	0.3964	0.6194	0.6035	1																				
0.8031	0.7869	0.3145	0.5404	0.5596	0.4985	0.4869	0.5038	0.7091	0.4957	0.6305	0.8086	0.4505	0.3955	0.6293	0.6003	0.6127	1																			
0.5645	0.5603	0.1788	0.6197	0.3899	0.2830	0.3166	0.4343	0.5126	0.5597	0.5157	0.5865	0.4015	0.2573	0.5536	0.5946	0.4675	0.5735	1																		
0.5187	0.5149	0.2913	0.4556	0.4904	0.4350	0.4371	0.4421	0.6610	0.4107	0.6891	0.4672	0.4797	0.3315	0.7422	0.7506	0.5490	0.5280	0.4373	1																	
0.2979	0.2902	0.2321	0.2630	0.2017	0.2952	0.2620	0.2886	0.3795	0.2913	0.4111	0.3182	0.3055	0.1680	0.3687	0.3536	0.3783	0.3511	0.2467	0.2547	1																
0.3808	0.3725	0.2954	0.3780	0.3683	0.4494	0.3870	0.3503	0.4955	0.3166	0.4668	0.3869	0.3701	0.2160	0.4521	0.4158	0.3907	0.4123	0.2663	0.4253	0.2412	1															
0.4149	0.4171	0.2632	0.2863	0.2630	0.3199	0.3446	0.3761	0.4344	0.3135	0.5013	0.3892	0.3287	0.2353	0.4718	0.4180	0.4011	0.3860	0.3016	0.4800	0.2563	0.3638	1														
0.5371	0.5515	0.2444	0.4724	0.4315	0.4441	0.3847	0.4710	0.6185	0.4334	0.5858	0.5715	0.4472	0.4068	0.5399	0.4488	0.4878	0.5576	0.4807	0.4693	0.2540	0.3620	0.3954	1													

Table A1. Correlation Matrix for the 250 first days of period from Jan-06 through Dec-09

Stocks in the multivariate 26-stock portfolio

ABB	Assa Abloy B
Atlas Copco A	Atlas Copco B
Astra Zeneca	Boliden
Electrolux B	Ericsson B
Getinge B	H&M B
Investor B	MTG B
Nordea	Sandvik
SCA B	Scania B
SEB A	Handelsbanken
Skanska B	SKF B
SSAB	Swedbank A
Swedish Match	Tele2
Telia Sonera	Volvo B

Table A3. The stock selection included in the 26-stock portfolio. All stocks selected from the OMXS30.

Appendix 2

Mathematical derivations

First, let us assume a simple portfolio of one stock. The stock price at time t_n is given by $S_{n+1} = S_n e^{X_{n+1}}$ where X_{n+1} is a random variable and $S_n > 0$ and a is the number of units of the stock. We know that

$$L_{n+1} = -(S_{n+1} - S_n) \quad (\text{B1})$$

since a negative loss is a profit and note that this is a special case of L_{n+1} given in Equation (5). So, the loss of the portfolio is given by

$$L_{n+1} = -aS_n(e^{X_{n+1}} - 1)$$

and the linearized loss L_{n+1}^Δ is given by

$$L_{n+1}^\Delta = -aS_n X_{n+1} = aS_n \cdot (-X_{n+1})$$

since $e^x \approx 1 + x$ for small x . We also assume that $L_{n+1} \approx L_{n+1}^\Delta$ so that $\text{VaR}_\alpha(L_{n+1}) \approx \text{VaR}_\alpha(L_{n+1}^\Delta)$. By linearity of VaR (see Subsection 2.3), and also imposing $a > 0$ we get that

$$\text{VaR}_\alpha(L_{n+1}^\Delta) = aS_n \text{VaR}_\alpha(-X_{n+1}).$$

Remember that $S_{n+1} = S_n e^{X_{n+1}}$, and for any time t_n , we will for any integer k have that

$$S_{n+k} = S_{n+k-1} e^{X_{n+k}} = S_{n+k-2} e^{X_{n+k-1} + X_{n+k}} = \dots = S_n e^{X_{n+1} + \dots + X_{n+k}}$$

that is

$$S_{n+k} = S_n e^{X_{n+1} + \dots + X_{n+k}},$$

it then follows that the linearized loss over the period of t_n to t_{n+k} is given by

$$L_{n+k}^\Delta = aS_n (-1)(X_{n+1} + \dots + X_{n+k}).$$

Note that we assume that $L_{n+k} \approx L_{n+k}^\Delta$ so that $\text{VaR}_\alpha(L_{n+k}) \approx \text{VaR}_\alpha(L_{n+k}^\Delta)$, we then get that

$$\text{VaR}_\alpha(L_{n+k}^\Delta) = aS_n \text{VaR}_\alpha(-\sum_{i=1}^k X_{n+i}).$$

Next we assume that our log-returns, i.e. $\ln\left(\frac{S_{n+1}}{S_n}\right)=X_n$ for each n are independent with identical distribution (i.i.d.), and if we assume that $X_n \sim N(0, \sigma^2)$ for each $n = 1, 2, \dots$

If X_{n+1}, \dots, X_{n+k} are i.i.d where $X_i \sim N(0, \sigma^2)$ for each i then we can define W_k as $W_k = X_{n+1} + \dots + X_{n+k}$ satisfies $W_k \sim N(0, k\sigma^2)$. We also know that a normal random variable Z with mean 0 is symmetric, meaning that the distribution for Z and $-Z$ is the same. Hence, if $-X_{n+1}$ has the same distribution as X_{n+1} it follows that $-W_k$ has the same distribution as W_k .

Given that the distribution is the same, it follows that

$$VaR_\alpha(-X_{n+1}) = VaR_\alpha(X_{n+1})$$

and

$$VaR_\alpha(-\sum_{i=1}^k X_{n+i}) = VaR_\alpha(\sum_{i=1}^k X_{n+i}) = VaR_\alpha(W_k).$$

Since a normally distributed random variable is a continuous random variable we know that

$$VaR_\alpha(X_{n+1}) = F_{X_{n+1}}^{-1}(\alpha) = \sigma N^{-1}(\alpha) \quad (B2)$$

and since $W_k \sim N(0, k\sigma^2)$

$$VaR_\alpha(W_k) = F_{W_k}^{-1}(\alpha) = \sqrt{k}\sigma N^{-1}(\alpha). \quad (B3)$$

since,

$$F_{W_k}(x) = P[W_k \leq x] = P\left[\frac{W_k}{\sqrt{k}\sigma} \leq \frac{x}{\sqrt{k}\sigma}\right] = N\left(\frac{x}{\sqrt{k}\sigma}\right)$$

implying Equation (16). Finally, comparing Equation (B2) and Equation (B3) we conclude that

$$VaR_\alpha(W_k) = \sqrt{k}\sigma N^{-1}(\alpha) = \sqrt{k} VaR_\alpha(X_{n+1})$$

So, combining Equation (B2) and (B3) we get

$$VaR_\alpha(L_{n+k}^\Delta) = \sqrt{k} \cdot VaR_\alpha(L_{n+1}^\Delta). \quad (B4)$$