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Hedging European Options under a Jump-diffusion Model with Transaction Cost

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ABSTRACT

This thesis investigates the performance of hedging strategies when the underlying asset is governed by Merton (1976)'s jump-diffusion model. We hedge a written European call option and analyse the performance through simulation of stock prices. We find that delta hedging is costly and poorly performing regardless of rebalancing frequency and that the performance is improved when an option is used instead of the underlying asset. The Gauss-Hermite quadratures strategy is an improvement to the delta hedging strategies. It is found to require a wide range of strike prices but that its performance is only moderately affected by restrictions on the strikes available. The Least squares hedge is the best performing strategy for all number of options included and the range of strikes required is relatively narrow. We find that this strategy performs equally well with five options as the Gauss-Hermite quadratures hedge does with 15 options. Both of the latter strategies are treated as static and found to be relatively cheap due to the limited number of transactions.

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1. Introduction

All over the world practitioners and academics in finance use the Black-Scholes model to value European options. The model has had an enormous impact in the world of option pricing but some of the underlying assumptions are unrealistic in practice. First, it is impossible to trade in continuous time, and even if it were possible it would be tremendously expensive because of transaction costs. Second, the assumption about normally distributed returns has been proven wrong with evidence from individual stocks, such as Enron and Lehman Brothers, as well as whole markets in 1929, 1987 and 2008 when the world's stock markets tumbled. The volatility smile shows that investors and traders are aware of this misspecification. Real world stock prices do clearly not evolve as a purely diffusive process (Wilmott, 2006). Instead, they tend to exhibit discontinuities from time to time as new information arrives and is priced by the market.

In 1976, an article written by Robert C. Merton was published in which the implication of these violated assumptions is discussed. He claims that the impossibility to trade continuously is not of major concern as long as the price evolution has a continuous path. He proceeds by claiming that the validity of Black-Scholes instead rests upon whether the stock price in a short time interval can change by only a small amount or if there is a non-zero probability for a larger movement, a "jump". As already mentioned, there is empirical evidence that points to the latter. For this reason Merton (1976) developed a process that incorporates the possibility of discontinuities by adding a jump-term to the traditional geometric Brownian motion and a valuation model for options following such a process.

The model is an extension to the Black-Scholes formula and shares its attributes in terms of being relatively easy to apply even though it requires three additional parameters for the properties of the jumps, namely their variance, expected amplitude and number of jumps per year. However, hedging under a jump-diffusion process is more cumbersome even in the absence of transaction costs. As explained by Kennedy et al. (2009), a continuously rebalanced delta hedge will not lead to a completely risk-free portfolio. Such a hedge is only capable of capturing the diffusive parts of the process and will lead to a loss if a jump occurs regardless of the direction of the jump or the rebalancing frequency.

There are both dynamic strategies and strategies that are rebalanced on an infrequent basis that can handle these jumps more successfully. Such strategies are Gauss-Hermite quadrature (GHQ) hedging developed by Carr & Wu (2014) and Least squares hedging developed by He, et al. (2006) among others. Both of these aims to replicate the payoff from an option by holding a portfolio of instruments that protects against movements in the underlying asset regardless of their magnitude. One common factor is that the underlying asset is not solely used, but is combined with options or completely excluded.

Previous work by Hinde (2006) has evaluated the performance of similar strategies and finds that the semi-static Least squares strategy weighted by the transition PDF replicates the payoff of the target well. However, transaction costs was left outside the analysis and there was no optimization regarding rebalancing frequency or the number of options to include. In addition, there is limited focus on the impact of restrictions on the available strike prices in the market.

This thesis aims to compare the performance of the strategies mentioned above with and without transaction costs. We use Monte Carlo simulation to construct sample paths for stock prices under the jump-diffusion framework. The simulated data is used for application of the hedging strategies and the analysis of their performance. The performance is observed for different restrictions on the availability of hedging instruments, which is crucial in less liquid markets. The delta hedge is applied on a less frequent basis than that of Carr & Wu (2014) where intra-daily rebalancing is used. We apply the delta suggested by Merton (1976) and compare the outcome from using the underlying and an option respectively. In the Least squares strategy, the expectation regarding the distribution of future stock prices is taken into account through a uniform weighting function. This weighting function is not theoretically optimal, but is more robust to lack of knowledge regarding the process followed by the stock price. Finally, the best performing hedging strategy among the studied is found by analysing the mean, standard deviation and percentile ranges of the hedging errors.

The analysis is based on both constant volatility and interest rate. In a framework where these are treated as stochastic the result will differ. An advantage of delta hedging with the underlying asset is that it can eliminate the delta-risk without altering the risk exposure associated with any of the other Greek letters. In the analysis of the strategies including options it is therefore important to note that the risks associated with varying volatility and interest rates are left outside our work.

The thesis is structured as follows. Section 2 outlines previous literature on the topic jump-diffusion and relevant hedging strategies. Section 3 describes the process of simulating the data used for the analysis and how the hedges are set up. It also describes the measures used for evaluation. Section 4 presents the results for each strategy and ends with a comparison. Finally, the findings are summarized in the concluding section.

2. Literature Review and Theory

2.1 Literature Review

In 1973, Fisher Black and Myron Scholes derived the well-known Black-Scholes formula for valuation of European options. They showed that a continuously rebalanced portfolio consisting of the underlying asset and bonds can replicate the payoff from an option. In the absence of arbitrage opportunities, the value of an option has to be equal to the cost of performing this replication. Because the model does not require any knowledge about expected returns or other investor specific beliefs it was quickly adopted by professionals and is still widely used.

The Black-Scholes model assumes that the stock price follows a geometric Brownian motion (GBM) process. This process yields a continuous path with constant drift and variance resulting in a log-normal distribution of stock prices. One of the critiques to the model is the discrepancy between the stock returns produced by a GBM and those observed in the market. Hinde (2006) highlights this issue by comparing actual market returns from the DJIA with simulated returns from a GBM. It is obvious from the comparison that the GBM produces far too few extreme events. This implies that the distribution from actual market returns has fatter tails than the Gaussian distribution. The difference between empirical and theoretical returns is well known and visible in option markets where it is often referred to as a volatility smile or volatility skew, which shows that the volatility used to price an option is varying with its moneyness. This phenomenon is consistent with a higher probability of extreme movements, i.e. it is more likely that a call option deep out of the money will expire in the money than the Black-Scholes model predicts.

Several researchers, including Merton (1976), Heston (1993) and Kou (2002) have developed alternative models to solve the issue regarding the erroneous distribution of returns assumed in the Black-Scholes model. Merton modifies the GBM by adding a jump term to the diffusive process which allows the stock price to move discontinuously at discrete times to replicate the extreme events empirically observed. The jumps are described as abnormal vibrations that are due to events or announcements of great importance for the particular stock or industry, such as profit warnings or reports not meeting expectations.

Merton derives a pricing formula for options following this modified GBM, assuming that the jump size is log-normally distributed and that the number of jumps occurring during any given period follow a Poisson distribution. One interesting aspect is that the Black-Scholes

implied volatilities of these option values produces volatility smiles similar to those observed in option markets.

With the modifications of the GBM it is no longer obvious how to hedge the risk-exposure associated with an option. Wilmott (2006) highlights the problem of hedging under a jump-diffusion, stating that even if it were possible to trade in continuous time, it would be impossible to hedge away the risk of random sized jumps occurring at discrete times. This issue makes the use of traditional dynamic hedging strategies questionable since the hedger will be exposed through the jump due to the linear payoff from hedging using the underlying asset.

In 2014, Carr & Wu showed that rebalancing on a higher frequency than once per day does not increase the performance of a delta hedging strategy. They rebalance their hedge up to ten times per day but do not find any improvements in terms of variance in the hedging errors. An alternative hedging strategy based on ideas presented in an article by Breeden & Litzenberger (1978) is therefore developed. The article demonstrated how the risk associated with a path independent option can be eliminated by a combination of options with the same maturity. Being limited to only use options with the same maturity has a negative impact on the application in reality where the range of options with a certain maturity is limited. Upon this finding, Carr & Wu developed a strategy based on the no-arbitrage theorem that an option can be perfectly hedged using a continuum of shorter-term options. The theorem is converted into a hedging strategy in which the Gauss-Hermite quadrature rule is used to approximate the continuum with a finite number of options. Given the maturity of the hedging options, the method calibrates the strike prices of the hedging options and their weight in the hedging portfolio.

Based on the work by Carr & Wu, He, et al. (2006) developed a least squares method that aims to minimize the squared difference between the target option and the hedging portfolio at some future point in time. In their work, they emphasize that the availability of strike prices is limited in reality. The strike prices are therefore not calibrated by the model, but required as an input. It enables the hedger to use a weighting function to express the expectation regarding the distribution of future stock prices. Similar to the method using the Gauss-Hermite quadratures, the strategy is semi-static, i.e. rebalanced only infrequently.

The traditional dynamic delta hedging strategies using the underlying asset is simple to implement and does not face any liquidity issues in most situations. On the other hand, it involves a large number of transactions which will have a negative impact on the payoff from

the strategy in the presence of transaction costs. Thus, it is desirable to find a strategy that includes only a few transactions and performs satisfactory.

Even though the method developed by Carr & Wu solves the problem of only one maturity and is rebalanced infrequently, their method might require a wide range of strike prices for the hedging options (Balder & Mahayni, 2006), which is unrealistic to find in many markets. It is therefore of interest to study how this method performs as strike prices are restricted to a limited range. We will compare its performance to that of the Least squares strategy where the hedger manually chooses the strike prices.

2.2 Merton's jump-diffusion

This section outlines the framework behind Merton's jump-diffusion process in terms of stock price dynamics and option pricing. The theory is presented together with fundamentals of the Black-Scholes model, which the Merton model is based upon.

2.2.1 Stock price dynamics

In 1976, Merton derived an extension to the Black-Scholes model that incorporates the possibility of discontinuities in the process followed by the stock price. As in the original model there is a diffusive part of the process that captures normal vibrations in the stock price on an ordinary day without any extraordinary events occurring. The arrival of extraordinary information is assumed to be firm or industry specific and is modelled though a jump-part of the process. The model is a modified GBM with a third term that creates these discontinuous returns occurring at random, discrete points in time. In a risk neutral environment, the process is defined as:

$$\frac{dS}{S} = (r - \lambda\kappa)dt + \sigma dZ(t) + (Y_t - 1)dN_t \quad (2.1)$$

$$dS = (r - \lambda\kappa)Sdt + \sigma SdZ(t) + (Y_t - 1)SdN_t \quad (2.2)$$

The variable determining the jump amplitude, Y_t , is random and independent of the diffusive part of the process. The random jumps are log-normally distributed, $\ln(Y) \sim N(\mu, \delta^2)$. This implies that the jumps cannot result in a negative stock price but have the possibility to take on any positive value. Due to the properties of a log-normal distribution the expected jump size, κ , is:

$$E[(Y_t - 1)] = e^{\mu + \frac{1}{2}\delta^2} - 1 = \kappa$$

With this in mind it is clear that equation (2.1) becomes a standard GBM if the jump size is zero, i.e. the process is identical to that of the Black-Scholes model in the absence of jumps. The interpretation of the parameters is (Hinde, 2006):

- S = Stock price.
- r = Risk-free rate.
- σ = Volatility of the diffusion process.
- λ = Average number of jumps per year.
- κ = Expected jump size.
- μ = Mean jump size in terms of $\ln(S)$.
- δ^2 = Variance of the jump size in terms of $\ln(S)$.

The probability of a jump occurring during any short time interval dt is determined by a Poisson process dN_t , with constant intensity λ (Merton, 1976) (Sideri, 2013):

$$\text{Prob. \{the event does not occur in time interval } dt \} = 1 - \lambda dt + O(dt)$$

$$\text{Prob. \{the event occurs once in time interval } dt \} = \lambda dt + O(dt)$$

$$\text{Prob. \{the event occurs more than once in time interval } dt \} = O(dt)$$

As the time steps becomes smaller, the probability of more than one jump $O(dt)$ during dt approaches zero and in continuous time no more than one jump can occur during any instant.

2.2.2 The Partial Differential Equation

One of the main insights provided in the article by Black and Scholes is that if a derivative on an asset is dependent on the same process as the asset itself, where the only source of uncertainty is a common Wiener process, it is possible to construct a portfolio consisting of a long (short) position in the asset and a short (long) position in the derivative that is instantaneously risk-free. Conditional on continuous rebalancing, the payoff can be perfectly replicated until maturity without any hedging error. Merton used this insight and extended it to the jump-diffusion model. Since the jumps are assumed to be firm or industry specific, they are uncorrelated with the market. Assuming that the CAPM holds, the risk is diversifiable so that there should be no risk-premium reward from them. As shown in appendix A.2, the Merton PDE is (Merton, 1976; Sideri, 2013):

$$-\frac{\partial F}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - \lambda E[F(SY_t, t) - F(S, t)] + \lambda E[Y_t - 1] \frac{\partial F}{\partial S} S + rF(S, t) - rS \frac{\partial F}{\partial S} = 0 \quad (2.3)$$

The terms involving λ show that the positions in the option and stock will not change by the same amount at the occurrence of a jump. Merton's PDE collapses to the Black-Scholes PDE when $\lambda = 0$, i.e. when there are no jumps expected. Applied to hedging, the presence of jumps complicates the situation since a position in the underlying cannot hedge away the all risk. The market is therefore incomplete and options are non-redundant, i.e. they are non-

replaceable and can play a key role in hedging. When there is a finite number N of possible jump amplitudes, it is possible to set up a perfect hedge using $N + 1$ options and the underlying. When the jump size is continuous, an infinite number of derivatives on the underlying assets would be necessary for a perfect hedge to be possible. Since the number of derivatives available is clearly finite and the transaction costs arising from trading would be tremendous it is impossible to completely hedge the jump risk under a jump-diffusion (Wilmott, 2006).

2.2.3 Option pricing

Let $f(S, K, \sigma_n, r_n, \tau)$ be the time t price of the Black-Scholes option maturing at T . Given that the jump size is log-normally distributed, so that the stock price follows the process outlined in section 2.2.1, Merton's option price of a European option, $F(S, \tau)$, can be calculated as:

$$F(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau} (\lambda'\tau)^n}{n!} f(S, K, \sigma_n, r_n, \tau) \quad (2.4)$$

Each Black-Scholes option is valued assuming that exactly n jumps occur during the life of the option. Since the number of jumps that will occur is unknown at time t , but the Poisson probability of n jumps occurring is known, each option value is weighted with this probability. The Black-Scholes price is calculated with a specific risk-free rate and volatility that corresponds to n . Since a jump will increase the volatility of the stock price, the variance used to value each option will increase in n , and is defined as:

$$\sigma_n^2 \equiv \sigma^2 + \frac{n}{\tau} \delta^2$$

Similarly, the risk free rate is adjusted for the return arising from the jumps:

$$r_n \equiv r + \frac{n}{\tau} \left(\mu + \frac{1}{2} \delta^2 \right) - \lambda \kappa$$

The Poisson probability of n jumps occurring:

$$\mathbb{P}[N(t) = n] = e^{-\lambda'\tau} \frac{(\lambda'\tau)^n}{n!}$$

where $\lambda' = \lambda(1 + \kappa)$ is the intensity of the process.

2.3 Hedging techniques

The holder or writer of an option carries risks associated with the parameters affecting its value. An action taken to reduce this risk exposure is referred to as hedging. The risk can be completely eliminated by taking the opposite position in an identical option (Hull, 2012), but as has already been mentioned it is impossible to create a perfect hedge under the considered jump-diffusion framework in any other way. Writing an option and immediately buying an identical might not be possible if the target option is tailor-made for a certain customer.

Because of this, there is a need for additional risk management tools. In a complete world a derivative such as an option can be replicated using the underlying asset so that the total portfolio replicates the payoff from a risk-free investment (Wilmott, 2007). As outlined in section 2.2.2, this replication is not possible in a market with jumps, which complicates the situation. Other complications in reality are the presence of transaction costs and the impossibility of trading continuously which will result in risk exposure.

Hedging techniques are often divided into static and dynamic strategies. A static hedge is not rebalanced during the hedging period, whereas a dynamic hedge can be rebalanced with any frequency. Under the Black-Scholes framework, a more frequent rebalanced hedge will outperform a less frequent rebalanced in the absence of transaction costs, since the Greeks (see section 2.3.1) are not constant and will be outdated after some time. A third way to hedge is referred to as semi-static hedging. While a dynamic hedge is rebalanced frequently to replicate the payoff from the target option, a semi-static hedge replicates the payoff at some specific future time through infrequent trading in a portfolio of options (Carr, 2001). Depending on the desired hedging period and the maturities of the options available in the market, the hedge may need to be rolled over repeatedly, which makes the strategy semi-static (He, et al. 2006). As an example, imagine that we at time t write an option with maturity T that we would like to hedge. In the market there are only options available with maturity u , $u < T$, i.e. we can only use shorter dated options to hedge our longer dated option. Because of this limitation, we create a semi-static hedge at time t which aims to replicate the payoff at time u . At this time we have the possibility to set up a new hedge or to close the position using an identical option, if such an option is now available.

The hedging approaches considered in this thesis are dynamic and semi-static. All techniques are described from the perspective of hedging a short position in an option.

2.3.1 Hedging using the Greek letters

The sensitivity of the value of a derivative with respect to a parameter is often referred to as a Greek letter. These are the partial derivatives of the value function with respect to any parameter of interest. Three of the most common Greeks are delta (Δ), gamma (Γ) and theta (Θ).

$$\begin{aligned} \text{Delta } (\Delta) &= \frac{\partial F}{\partial S} \\ \text{Gamma } (\Gamma) &= \frac{\partial^2 F}{\partial S^2} \\ \text{Theta } (\Theta) &= \frac{\partial F}{\partial t} \end{aligned}$$

Delta shows the sensitivity of the option value with respect to changes in the stock price. For a long European call it is bound between zero and one. As an option moves deep into the money, it becomes unlikely that the option will expire out of the money and the delta of the option approaches one. Similarly, a deep out of the money option will have a delta close to zero. The delta risk can be eliminated by taking a certain position in the underlying asset or any other asset with a non-zero delta. This eliminates the risk associated with small movements in the underlying asset for an instant, but since the option delta is a function of all the parameters in the valuation formula it is sensitive to a change in any of these. Thus, the portfolio has to be rebalanced frequently to minimize hedging errors. In the jump-diffusion model, delta is calculated as (Grünewald & Trautman, 1996):

$$\Delta_{F(S,\tau)} = \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau} (\lambda'\tau)^n}{n!} \Delta_{f(S,\tau)}$$

$$\Delta_{f(S,\tau)} = N\left(\frac{\ln\left(\frac{S}{X}\right) + \left(r_n + \frac{\sigma_n^2}{2}\right)\tau}{\sigma_n\sqrt{\tau}}\right) = N(d_1)$$

Where $N(d_1)$ is the cumulative PDF of a standard normal distribution. Gamma is the second order derivative with respect to the stock price and thereby measures the rate of change in delta with respect to changes in the stock price. Hedging the gamma risk will decrease the curvature of the delta which will result in a more sustainable delta hedge conditional on no jump occurring, i.e. the delta position does not have to be rebalanced as often. Similarly to delta, gamma is calculated as (Hinde, 2006):

$$\gamma_{F(S,\tau)} = \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau} (\lambda'\tau)^n}{n!} \left[\frac{N'(d_1)}{S_0 \sigma_n \sqrt{\tau}} \right]$$

Where $N'(d_1)$ is the probability density function for a standard normal distribution function. The value of a European option is decreasing with time which means that, ceteris paribus, an option with shorter time to maturity will have a lower value than one with longer time to maturity. The rate of change in the option value due to the passage of time is measured by theta.

Hedging using the Greek letters is widely popular but has a major drawback in the jump-diffusion setting. The Greek letters are only effective to hedge the diffusive part of the process and since the hedger is not able to rebalance the portfolio through a jump, the positions in the hedging instruments will be erroneous and cause major hedging errors (Wilmott, 2006).

2.3.2 Semi-static hedging using Gauss-Hermite Quadratures

Using Gauss-Hermite Quadratures Carr & Wu (2014) developed a technique to set up a hedge that uses options with maturities shorter than that of the target option. Given the maturities available in the market this technique calibrates the portfolio of hedging options by finding their optimal weights and strike prices. The strategy is developed under the assumptions that stock prices are Markov and that there are no arbitrage possibilities. The value of a European call option at time t with maturity T and strike K can then be perfectly replicated with a portfolio consisting of an infinite number of options with different strike prices \mathcal{K} , weights $w(\mathcal{K})$ and maturity $u < T$:

$$F(S, t; K, T) = \int_0^\infty w(\mathcal{K})F(S, t; \mathcal{K}, u)d\mathcal{K}, \quad u \in [t, T] \quad (2.5)$$

Under the risk neutral measure, the weighting function can be expressed as:

$$w(\mathcal{K}) = \frac{\partial^2 F}{\partial \mathcal{K}^2}(\mathcal{K}, u; K, T)$$

This means that the weight of each option with strike \mathcal{K} is proportional to the gamma of the target option at time u if the stock price at that time is equal to \mathcal{K} . Because of this proportionality, the weighting function will have the same shape as gamma, which is bell shaped around the strike price. The most weight will therefore go to the options with the strikes \mathcal{K} closest to K . As u approaches T , more weight will go to the options with strikes $\mathcal{K} \approx K$ so that the target option is hedged using an option that is identical to the target as $u = T$, i.e. the position is closed.

For the strategy to be applicable, it is necessary to limit the number of options used to a finite number $n \in \mathbb{N}$ and approximate the integral in (2.5):

$$F(S, t; K, T) = \int_0^\infty w(\mathcal{K})F(S, t; \mathcal{K}, u)d\mathcal{K} \approx \sum_{j=1}^n W_j F(S, t; \mathcal{K}_j, u)$$

The value of the target option is then approximated as a weighted sum of the values of a finite number of options with maturity u . For the approximation to have a fit as good as possible, the hedging options must be chosen carefully in terms of strike prices and weights. The method finds these weights and strikes using the Gauss-Hermite quadrature rule. Gaussian quadratures are described in appendix A.3, while this section focuses on Gauss-Hermite quadratures and the application of the hedging strategy.

Gauss-Hermite quadratures are used for the approximation of infinite integrals of the form (Carr & Wu, 2014):

$$\int_{-\infty}^{\infty} e^{-x^2} f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$$

Here, the abscissas x_i are given by the roots of the Hermite polynomial $H_n(x)$ which satisfies the recursive relation:

$$\begin{aligned} H_{n+1}(x) &= 2xH_n - H'_n(x) \\ H_0(x) &= 1 \end{aligned}$$

where

The associated weights are given by (Abramowitz & Stegun, 1972):

$$w_i = \frac{2^{n-1} n! \sqrt{\pi}}{n^2 [H'_{n-1}(x_i)]^2}$$

For lower order of n , the abscissas and weight factors for Hermite integration can be found in Abramowitz & Stegun (1972), p 924.

The quadrature rule is applied to hedging using functions that maps the optimal strikes and weights of the hedging options to approximate the integral in (2.5). Carr & Wu (2014) choose these strikes as:

$$\mathcal{K}_j = K e^{x_j v \sqrt{2(T-u)} - (r + \frac{v^2}{2})(T-u)} \quad (2.6)$$

where x_j is given from the Hermite polynomial and v is the annualized standard deviation:

$$v^2 = \sigma^2 + \lambda ((\mu_j)^2 + \sigma_j^2)$$

The weight of each option is calculated using:

$$W_j(\mathcal{K}) = \frac{w(\mathcal{K}_j) \mathcal{K}_j v \sqrt{2(T-t)}}{e^{-x_j^2}} w_j \quad (2.7)$$

where

$$w(\mathcal{K}) = e^{-r(T-u)} \sum_{n=0}^{\infty} \Pr(n) e^{(r_n)(T-u)} \frac{n(d_{1n}(\mathcal{K}, u; K, T))}{\mathcal{K} \sigma_n \sqrt{T-u}}$$

As mentioned above, the weight of each option is related to the gamma of the target option. Under the jump-diffusion model, $w(\mathcal{K})$ is therefore motivated by the Merton gamma which is calculated in a similar way to the option value by weighting the gamma of the option conditional on n jumps with the Poisson probability of n jumps occurring, $\Pr(n)$. The hedging portfolio is then set up using options with strikes given by (2.6), each with a unique weight from (2.7).

It is important to note that as the optimal weights are unaffected by the passage of time, the weights will remain constant regardless of the movements of the underlying asset until time u . As stated by Carr & Wu (2014), no arbitrage implies that the hedging portfolio will have the same value as the target option for all times until u . Thus, it is theoretically possible to hedge away all risk even in the presence of jumps.

2.3.3 Least squares minimization of hedging errors

Based on ideas developed by Carr & Wu (2014) (see section 2.3.2), He, et al. (2006) propose a strategy that minimizes the squared change in the portfolio value at some specific time in the future. They emphasize that the availability of options with a particular maturity in real markets is restricted to a few strikes. Since no other options can be used in the application, the strategy minimizes the hedging error using these strikes for a continuum of possible future stock prices, each weighted by some PDF. Through the PDF, the hedger has the possibility to express the importance of minimizing the hedging error arising for specific values of Y .

The minimization problem can be expressed as:

$$\min_{\phi, w_S} \int_0^{\infty} [\Pi_{t+1} - \Pi_t]^2 W(Y) dY \quad (2.8)$$

Where Π_t denotes the time t value of the portfolio for the stock price S_t and $W(Y)$ is the weighting function. The difference between the portfolio values is a result from the stock price moving from S_t to $S_{t+1} = S_t Y$ over the hedging period. The solution to equation (2.8) then yields the optimal weights of the underlying asset and the options in the portfolio given the time to maturity and strike price of each option. He, et al. (2006) finds the minimization problem as follows:

At time 0, the value of the replicating portfolio is set up to be equal to the value of the target option, F_0 :

$$F_0 = \phi_0 \cdot I_0 + wS_0 + B_0$$

Where ϕ is a vector containing the weights in each available hedging option and vector I contains the values of the corresponding options. The weight in the underlying asset S is denoted w and the amount initially invested in bonds is B_0 . The time t value of the portfolio satisfies the self-financing condition:

$$\phi_t \cdot I_t + e_t S_t + B_t = \phi_{t-1} \cdot I_t + w_{t-1} S_t + B_{t-1} e^{r \cdot dt} \quad (2.9)$$

The left side of equation (2.9) represents the value of the portfolio an instant after rebalancing and the right represents the value an instant before rebalancing. This condition states that the change in value of the portfolio completely arises from changes in value of the instruments in it. Thus, there is no withdrawal or insertion of money in the portfolio at rebalancing. Assuming a short position in the target option, the value of the total portfolio at times t and $t + 1$ are:

$$\Pi_t = -F_t + \phi_{t-1} \cdot I_t + w_{t-1} S_t + B_{t-1} e^{r \cdot dt} \quad (2.10)$$

$$\Pi_{t+1} = -F_{t+1} + \phi_t \cdot I_{t+1} + w_t S_{t+1} + B_t e^{r \cdot dt} \quad (2.11)$$

Manipulating equation (2.11) through substitution of equations (2.9) and (2.10) the minimization problem becomes:

$$\min_{\phi, w_s} \mathbb{E}[(F_{t+1} - F_t e^{r^* dt} - \phi(I_{t+1} - I_t e^{r^* dt}) - w_s(S_{t+1} - S_t e^{r^* dt}))^2] \quad (2.12)$$

Ideally, the difference in values should be zero, which would imply a perfect hedge where the target option is exactly replicated. The expectation operator \mathbb{E} is taken into account through the weighting function $W(Y)$ in equation (2.8). The choice of PDF depends on the knowledge of the distribution of stock prices and the time over which the option should be hedged. Common choices of weighting functions are the PDF of the jump-amplitude and the transition PDF. If there is no expectation regarding the distribution of future stock prices it is possible to use a uniform distribution.

A full derivation of equation (2.12) can be found in appendix A.4.

3. Data and Methodology

We use simulations to study and analyse the performance of different hedging techniques under Merton's jump diffusion framework. This section describes the approach used to simulate the stock price, how the options and their sensitivities are calculated and how the different hedges are set up and evaluated.

3.1 Stock Price Simulation

The daily stock returns are simulated using the jump-diffusion model with log-normally distributed jumps described by Merton (1976).

$$dS = (r - \lambda\kappa)Sdt + \sigma SZ(t) + (Y_t - 1)dN_t$$

$$\ln Y_t \sim N(\mu, \delta)$$

The simulation approach described here can be found in Glasserman (2004). To simplify the calculations, the stock price that is to be simulated is converted to the natural logarithm, $x = \ln S$. The stock price is simulated on a daily basis for one year. Assuming 256 trading days per year, we generate a total of 256 draws, $Z(t) \sim N(0,1)$, from a standard normal distribution for the Wiener process. The GBM is then simulated for the entire time period:

$$x_{GBM} = x_{t-1} + \left(r - \frac{1}{2}\sigma^2 - \lambda\kappa\right)dt + \sigma Z(t)\sqrt{dt}$$

At this stage, the simulation of the diffusive part is completed. Since the logarithm of the asset price is used, the effect of the jumps are addable and can be simulated separately. The time interval between jump times τ_j and τ_{j+1} are calculated as:

$$\tau_{j+1} - \tau_j = -\frac{\ln(U)}{\lambda}$$

Where U is a random variable, $U \sim \mathcal{U}(0,1)$, so that the time between jumps is exponentially distributed. With λ expected jumps per year, the time between every jump will on average be $1/\lambda$ years. For each jump, we generate $Z_j(t) \sim N(0,1)$ and calculate the random jump size as:

$$\ln(Y_j) = \mu + \delta Z_j(t)$$

Finally we add the cumulative impact from the jumps to the GBM:

$$x_t = x_{GBM} + \sum_{j=1}^{sup_j \tau_j < t} \ln(Y_j)$$

Where sup_j is the supremum of $\tau_j < t$, i.e. the highest τ_j until time t . The stock price is then converted to its standard form, $S_t = e^{x_t}$.

The parameters used in the simulations are given by He, et al. (2006) where market data from S&P 500 is used to calibrate values for the process. The parameter values are summarized in table 3.1.

Parameter	Value
r	5%
σ	20%
λ	0.1
μ	-92%
δ	42.5%
dt	1/256

Table 3.1 – Parameter values for the jump-diffusion model used for simulation and valuation.

With these parameters the expected jump size becomes -56.4%. The presence of discontinuities is clear in figure 3.1, where a sample of simulated price paths for one year is shown.

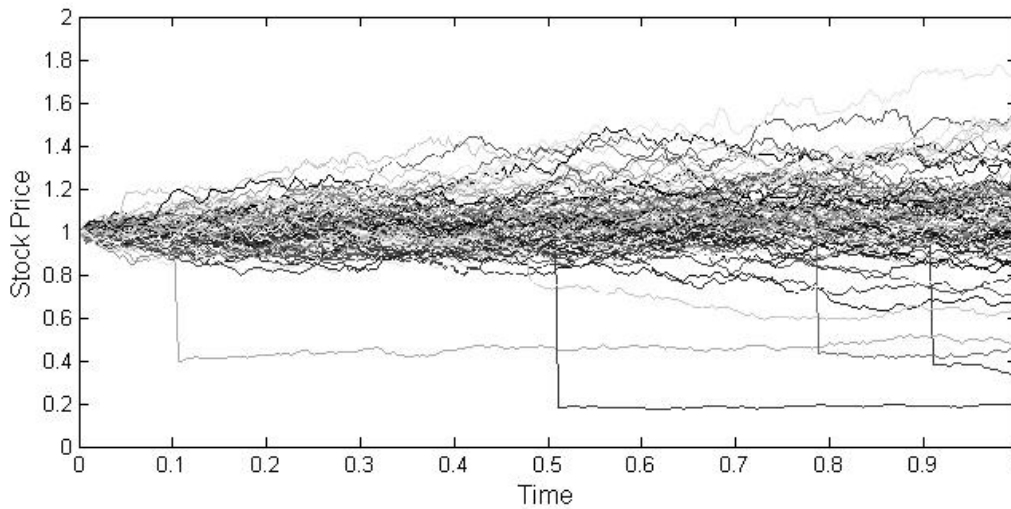


Figure 3.1 – 100 simulated stock price paths for the parameters in table 3.1. The initial stock price is set to 1 and the time period is one year.

3.2 Calculation of Option Prices

Formula (3.1) is the Merton valuation formula used under the jump diffusion process:

$$F(S, \tau) = \sum_{n=0}^{10} \frac{e^{-\lambda\tau} (\lambda\tau)^n}{n!} f(S, K, \sigma_n, r_n, \tau) \quad (3.1)$$

An issue with the Merton valuation formula is that it theoretically requires an infinite number of Black-Scholes values and Poisson probabilities. Similar to Sideri (2013) we truncate the calculation at $N = 10$, which is reasonable considering that the probability for 10 jumps to occur when $\lambda = 0.1$ is negligible.

3.3 Hedging of the option

For all hedging strategies we hedge a written European call option from a banks perspective. Imagine that a customer wants to buy a call option with maturity T whereas the market consists of shorter dated options only. As suggested by Carr & Wu (2014) this might be an attractive situation for the bank where they have the possibility of selling this longer term

option at a relatively high premium. The bank then has to hedge the option until the time $u \leq T$ when an identical option is available and it is possible to close the position.

The target option is assumed to have a maturity of two years and the time at which the position can be closed, u is set to one year. The semi-static strategies are not rolled over, which means that they will be applied as static in our analysis. At the time when the option is written the proceeds from the sale are used to finance the hedging portfolio. We also assume that it is possible to take long and short positions in the risk-free asset for purposes of financing the hedging portfolio. As in He, et al. (2006) we assume that the options in the market have strike prices available in $0.05S_0$ intervals, all with maturity u . The transaction cost for stocks TC_s is set to 1% as proposed by Zakamouline (2006) and Clewlow & Hodges (1997) while the transaction cost for options TC_o is set to 2%, consistent with a range between 1% and 4% used by Choi, et al. (2004).

3.3.1 Delta hedge

The hedge is applied in two ways, one where the underlying asset is used and one where an option is used. When using the underlying asset, a position is immediately taken when the target option is sold. The number of shares bought is:

$$w_0 = \frac{dF(S, \tau)}{dS} = \Delta_{F(S, \tau)}$$

Thus, the number of shares in the hedge portfolio is equal to the delta of a long position in the written option. At this time the total portfolio is instantaneously immune to diffusive movements in S , but as time passes the delta will evolve which will result in a hedging error. To once again achieve delta-neutrality the hedge portfolio has to be rebalanced. The number of days between each rebalancing considered is $n = \{1, 4, 16, 64, 128, 256\}$. At each rebalancing time $i \in \left(1, \frac{256}{n}\right)$, the position is adjusted with the change in delta:

$$dw_i = \Delta_{F(S_i, \tau_i)} - \Delta_{F(S_{i-1}, \tau_{i-1})}$$

At any time i the value of the total portfolio is:

$$\Pi_i = B_i + w_i S_i - F(S_i, \tau_i)$$

Where $w_i S_i$ is the value of the stock position, $F(S_i, \tau_i)$ is the value of the target option and B_i is the amount invested in bonds to finance the hedge portfolio and transaction costs including continuously compounded interest.

$$\begin{aligned} B_i &= B_{i-1} e^{r(\tau_i - \tau_{i-1})} - (dw_i + TC_s |dw_i|) S_i \\ B_0 &= F(S_0, T) - (w_0 + TC_s |w_0|) S_0 \end{aligned}$$

When an option is used for the delta hedge, the size of the option position that makes the portfolio delta-neutral is the ratio of the target option delta $\Delta_{F(S_i, \tau_i)}$ to the hedging option delta $\Delta_{H(S_i, \tau_i)}$:

$$w_i = \frac{\Delta_{F(S_i, \tau_i)}}{\Delta_{H(S_i, \tau_i)}}$$

The portfolio is then set up in the same way as when the underlying is used. When half of the hedging period has passed, the hedging instrument is replaced with a new option. The initial hedging option has a strike equal to S_0 , and the second a strike that is as close to the current stock price as possible. The reason for this rebalancing is that as an option approaches maturity, its delta becomes unstable and improper for hedging purposes.

3.3.2 Hedging using Gauss-Hermite Quadratures

To set up the hedge, we start by collecting the Gauss-Hermite quadrature abscissas, x_i and weights, w_i for the desired number of hedging options. The strike price, \mathcal{K}_j for each option is calculated using formula (2.6) and the parameters specified in section 3.1. Using these strike prices we calculate the corresponding weights $W_j(\mathcal{K})$ for the options using formula (2.7). Each strike price is rounded to the closest available in the universe outlined in section 3.3. The weights are not adjusted for these rounded strike prices. Finally, the value of each hedging option $I_j(S_0, u)$ is calculated and the amount invested in each option is then:

$$W_j(\mathcal{K})I_j(S_0, u)$$

The amount that is invested in bonds to finance the hedge, B_0 , is the difference between the proceeds from writing the option and the value of the hedge position:

$$B_0 = F(S_0, T) - \left(\sum_{j=1}^N W_j I_j(S_0, u) \right) (1 + TC_o)$$

And the value of the portfolio at the end of the period, time u , is equal to:

$$\Pi_u = B_0 e^{ru} - F(S_u, T - u) + \sum_{j=1}^N W_j I_j(S_u, 0)$$

The number of options considered is in the range 3 to 20.

3.3.3 Hedging using Least Squares

The integral in equation (2.8) requires an infinite number of stock prices at time u for the minimization problem. Similar to Hinde (2006) we approximate this integral by discretising it to a finite number of nodes. In our analysis the nodes consists of possible stock prices in the range $0.01S_0 - 3S_0$, which captures the most likely realizations for the given parameters. The range is justified by the negative expected jump, which makes it essential to cover very small

values of S_u . We divide the range into $0.005S_0$ intervals, i.e. there is 298 possible values of S_u in the optimization. Except for these limits, the hedge is set up without any subjective beliefs about the future stock price distribution using a uniform PDF as weighting function.

The strikes are chosen so that the first option used has a strike equal to that of the target option. As the number of options in the portfolio is increased, it will cover a broader range up to $0.5S_0 - 1.5S_0$. For the first 11 options the strike prices are chosen in $0.1S_0$ intervals and for more options the gaps are filled up with $0.05S_0$ strikes.

Using the desired number of hedging options we find the optimal weights by solving the least squares minimization problem (2.12) without any constraints. We consider hedging portfolios consisting of 1 to 20 options. At the end of the hedging period, the value of the portfolio is:

$$\begin{aligned}\Pi_u &= -F_u + \boldsymbol{\phi}_0 \cdot \mathbf{I}_u + w_0 S_u + B_0 e^{ru} \\ B_0 &= F(S_0, T) - (w_0 + TC_s |w_0|) S_0 - (\boldsymbol{\phi}_0 + TC_o |\boldsymbol{\phi}_0|) \cdot \mathbf{I}_u\end{aligned}$$

3.4 Relative Profit and Loss

For each simulation the performance of the hedge is measured by the relative profit and loss as suggested by He, et al. (2006).

$$\text{Relative P\&L} = e^{-ru} \frac{\Pi_u}{F(S_0, T)}$$

The expected relative P&L for any hedging strategy is zero in the absence of transaction costs. To systematically achieve a perfect hedge in an incomplete market is impossible which means that there will be deviations from this expectation. The deviations will be of varying magnitudes depending on the applied hedging technique. In order to evaluate the performance of the considered strategies we simulate 20 000 stock price paths. This results in a smooth distribution of yearly hedging errors from which we calculate the risk and mean of any given strategy.

As in He, et al. (2006) we calculate percentiles for the tails of the distribution. We choose to use the 1st, 10th, 90th and 99th percentiles. The 1st and 99th percentiles capture the ability of any hedging strategy to handle the extreme events that occur under a jump-diffusion, while the 10th and 90th percentiles capture the diffusive and less extreme realizations. Kennedy, et al. (2009) focus mainly on the mean and standard deviation when deciding on which of the techniques to use in the presence of transaction costs. We follow a similar approach but also keep focus on the percentiles.

When comparing the different strategies we calculate the ratio of the inter-percentile ranges for the hedged to the unhedged portfolios to create a normalized measure:

$$\text{Relative percentile range} = \frac{[Prct_U - Prct_L]_{Hedged}}{[Prct_U - Prct_L]_{Unhedged}} \quad (3.2)$$

Where $Prct_U$ and $Prct_L$ are the upper and lower percentiles respectively. We calculate a similar measure for the standard deviations:

$$\text{Relative standard deviation} = \frac{\sigma_{Hedged}}{\sigma_{Unhedged}} \quad (3.3)$$

For the unhedged position, equations (3.2) and (3.3) will take on the value 100% and can be expected to decrease for a hedged position, with a lower limit of 0%.

4. Results & Analysis

In this section we present and analyse the performance of the hedging strategies. We show the performance of each strategy with and without transaction costs. Further we compare the strategies in order to find the best performing strategy and hedging frequency for the given conditions.

4.1. Delta hedging using the underlying asset

The distributions of the relative P&L from a delta hedge using the underlying asset with and without transaction costs are shown in figures 4.1 and 4.2. The performance in the absence of transaction costs is consistent with findings from previous work by Carr & Wu (2014), who analyse varying intraday rebalancing frequencies. They find that the standard deviation is unchanged regardless of the hedging frequency, and from table 4.1 it is evident that this finding extends to less frequent rebalancing.

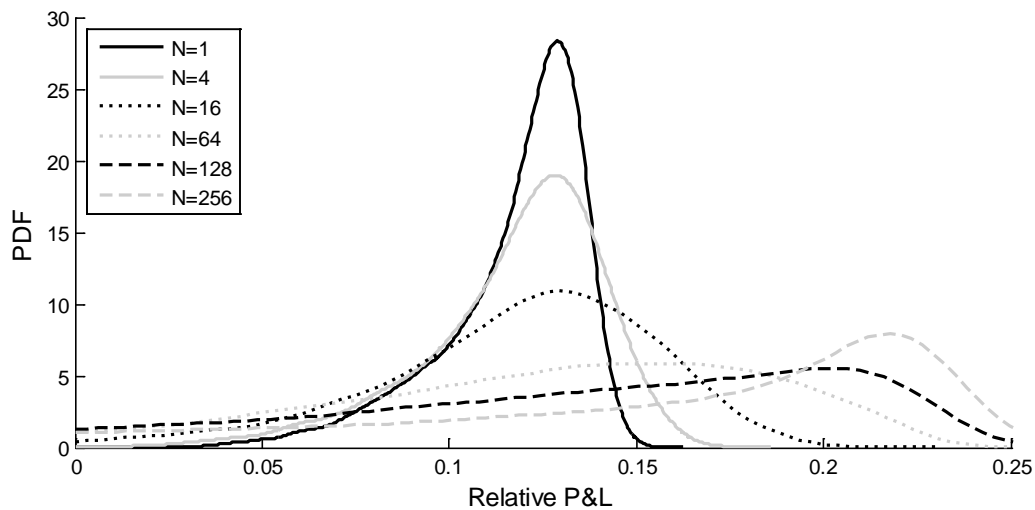


Figure 4.1 - Relative P&L distribution of a delta hedge using the underlying asset for one year in the absence of transaction costs. N is the number of days between rebalancing.

Frequency (N)	Percentiles						
	Mean	Std. dev.	1st	10th	50th	90th	99th
1	0.2%	41.0%	-192.1%	5.7%	12.1%	13.6%	14.4%
4	0.2%	41.0%	-190.5%	4.3%	12.0%	14.3%	15.8%
16	0.1%	41.1%	-190.7%	0.5%	11.9%	16.0%	18.4%
64	0.1%	41.4%	-191.4%	-9.8%	12.2%	19.2%	21.9%
128	0.2%	41.3%	-185.1%	-17.7%	12.9%	21.3%	22.7%
256	0.1%	42.4%	-187.3%	-29.9%	15.3%	22.6%	22.8%
No hedging	-0.7%	86.2%	-268.6%	-118.2%	18.0%	95.0%	100%

Table 4.1 - Descriptive statistics of the relative P&L distribution of a delta hedge using the underlying asset in the absence of transaction costs. The statistics corresponds to figure 4.1.

From the descriptive statistics it is clear that the P&L distributions are skewed with a long left tail similarly to findings by Xiao (2010) and Kennedy, et al. (2009). The 1st percentiles are of extreme magnitudes compared to the 99th, which are relatively close to the median and highlights the skewness. The largest losses occur at jump times regardless of the direction of the jump since the hedge position underestimates the effect on the option value for large movements in the underlying asset. Consistent losses are also seen in work by He, et al. (2006). With the parameters employed a jump will occur once every tenth year on average and as long as there is no jump in the underlying, the delta hedge performs similarly to a delta hedge in the Black-Scholes framework with the exception of not being centered around zero. In contrast to what could be the impression from figure 4.1, the mean relative P&L is close to zero for all strategies and the deviations are only a result of the randomness in the simulation. The skewness is due to that the written option is priced with a premium to compensate for the negative jumps which means that when the stock price does not jump during the hedging period, a small profit is received on average. Since the hedge portfolio remains static through the jump regardless of the hedging frequency, the strategy is unable to capture the effect of a jump. As a result of this, the performance at jump times is poor. A more frequent rebalancing provides a higher peak with more of the hedging errors in a smaller range, but the first percentile does not change. The percentiles in table 4.1 show that in 80% of the observations the relative P&L for the daily rebalancing lies in the range 5.7%-13.6%. As the rebalancing frequency decreases this range increases consistently.

Once the transaction costs are imposed for trading in the underlying asset the situation changes drastically. Even though the standard deviations are almost unchanged, the shape of the distributions have changed. Due to the transaction costs there is a negative shift for all mean payoffs. Since more frequent rebalancing involves more transactions the shift is larger for these strategies. The platykurtic distribution of the P&L for daily rebalancing implies that the payoff is hard to predict and it does not seem to exist any benefits as the mean changes considerably while the performance is close to unchanged. This finding stands in sharp contrast to the other frequencies where the peaks are still present, although considerably lower. Considering the reduction in risk, it seems beneficial to hedge on a less frequent basis.

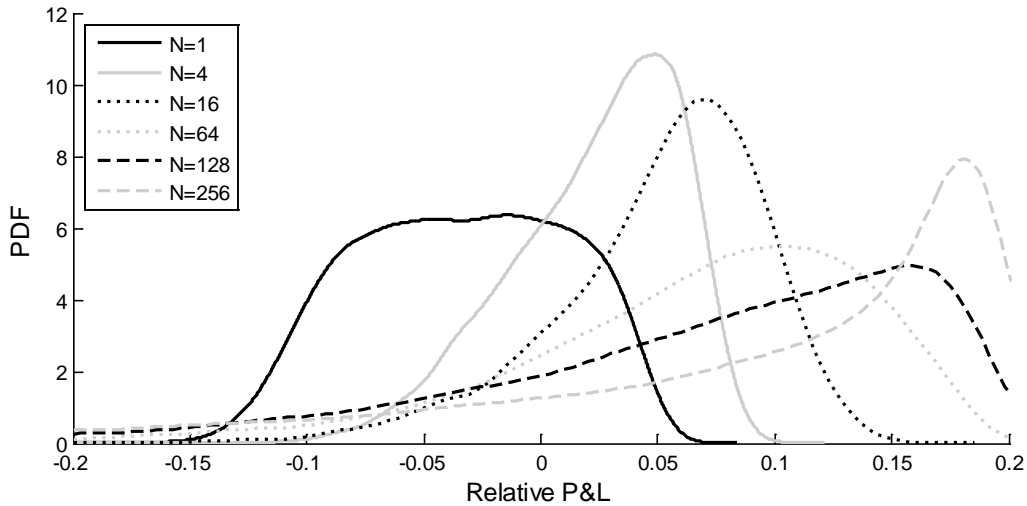


Figure 4.2 - Relative P&L distribution of a delta hedge using the underlying asset for one year in the presence of transaction costs. N is the number of days between rebalancing.

Frequency (N)	Percentiles						
	Mean	Std. dev.	1st	10th	50th	90th	99th
1	-14.7%	40.1%	-201.4%	-12.3%	-4.1%	2.4%	4.5%
4	-9.2%	40.7%	-198.0%	-6.9%	2.5%	6.3%	8.1%
16	-6.4%	41.1%	-196.8%	-7.8%	5.5%	10.0%	12.9%
64	-5.1%	41.6%	-197.2%	-16.2%	7.0%	14.6%	17.8%
128	-4.6%	41.5%	-189.7%	-23.4%	8.1%	17.2%	18.9%
256	-3.6%	42.4%	-190.9%	-33.5%	11.7%	18.9%	19.2%
No hedging	-0.7%	86.2%	-268.6%	-118.2%	18.0%	95.0%	100%

Table 4.2 - Descriptive statistics of the relative P&L distribution of a delta hedge using the underlying asset in the presence of transaction costs. The statistics corresponds to figure 4.2.

The performance of this strategy illustrates the impact of violating the continuous path assumption in the Black-Scholes model. As stock prices in real markets do exhibit discontinuities, a delta hedge might not be the most attractive alternative since it does not offer sufficient protection at jump times. Figure 4.3 illustrates that the hedging errors to a large extent arise not from the discretization of time, but from the discontinuity of the stock price. It shows the development of the relative P&L during the hedging period for 200 simulations assuming no transaction costs and it is clear that the major deviations arise when there is a jump in the underlying.

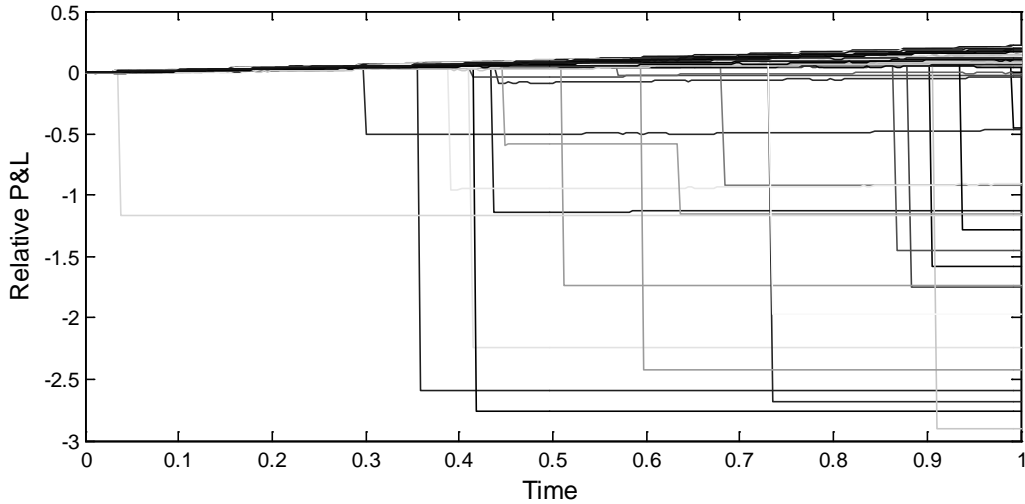


Figure 4.3 - Development of relative P&L under a delta hedging strategy using the underlying for 200 simulations assuming no transaction costs.

4.2 Delta hedging with an option

As shown in section 4.1 the delta hedging strategy using only the underlying asset performs satisfactory as long as the underlying asset price does not jump. At jump times, the major hedging error occurs due to the linear payoff from the hedging portfolio in contrast to the non-linear payoff from the written option. Even though it is impossible to eliminate the jump risk using only one hedging option that is different from the target, it is possible to reduce it since both the target option and the hedging portfolio will have a non-linear payoff. Figure 4.4 shows the performance of a strategy that is identical to 4.1 except that an option is used to impose delta neutrality at each rebalancing time.

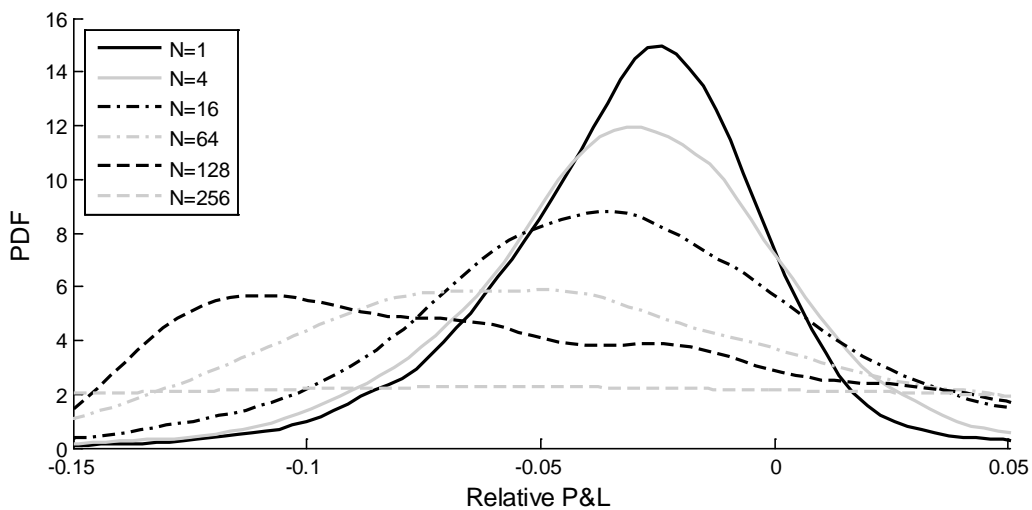


Figure 4.4 - Relative P&L distribution of a delta hedge using an option for one year in the absence of transaction costs. N is the number of days between rebalancing.

Frequency (N)	Percentiles						
	<i>Mean</i>	<i>Std. dev.</i>	<i>1st</i>	<i>10th</i>	<i>50th</i>	<i>90th</i>	<i>99th</i>
1	-0.1%	12.9%	-11.6%	-6.7%	-2.6%	2.0%	49.8%
4	-0.1%	13.0%	-12.7%	-7.2%	-2.7%	3.5%	49.7%
16	-0.1%	13.5%	-15.1%	-8.6%	-3.0%	7.4%	51.8%
64	0.0%	14.8%	-16.2%	-11.2%	-3.8%	14.1%	54.8%
128	0.0%	15.7%	-14.2%	-12.4%	-4.5%	18.5%	56.0%
256	0.0%	16.9%	-26.6%	-21.1%	-2.2%	28.3%	32.6%
No hedging	-0.7%	86.2%	-268.6%	-118.2%	18.0%	95.0%	100%

Table 4.3 - Descriptive statistics of the relative P&L distribution of a delta hedge with an option in the absence of transaction costs. The statistics corresponds to figure 4.4.

The results presented in table 4.3 are considerably different from those in table 4.1. The standard deviations for the different frequencies are lower compared to when using the underlying asset. It is worth noting that instead of large losses occurring on an infrequent basis, there are often profits at jump times. These gains are offset by a small loss occurring on average when there is no jump.

The positive payoffs at jump times are related to the non-zero gamma of the hedging instrument, which is in contrast to the zero gamma of the underlying asset. Given that the hedging instrument has the same (or a similar) strike price as the target option but a shorter maturity, its value function will exhibit a higher curvature around the strike price. For large movements in the underlying asset, the gamma of the hedging instrument will converge to zero faster than that of the target option. As the stock jumps to a lower price, the negative payoff from the hedge portfolio is therefore smaller than the positive payoff from the short position in the target option. In a similar way, the value of hedging portfolio will increase more rapidly when there is a positive jump. This is the opposite scenario to hedging with the underlying and leads to a positive payoff regardless of the direction of the jump. When the properties of the hedging instrument is different this finding might not hold. An option with a lower strike price or a longer time to maturity will behave more similar to the underlying asset. An extreme example is an option with a strike price equal to zero, which will behave identical to the underlying asset with a constant delta of one and therefore yield losses at jump times. With the parameters in our setting and 20 000 simulations, we only observe a few number of negative jumps in the hedged position. The impact of these are very limited, with the hedging error equivalent to an error that can arise from a year without jumps (see appendix A.5). This minor impact is due to that the hedging option has to be replaced with an option with a lower strike to open the possibility for large, negative payoffs. If this has happened, the position in the option that imposes delta neutrality will be relatively small and

only yield small losses at the occurrence of a negative jump. Losses of greater magnitude might occur on the upside, but a positive jump is a relatively rare event as well. The two events that has to coincide to yield a large, negative payoff are therefore rare in combination. Appendix A.6 shows examples of the hedging errors arising from discontinuities of varying magnitudes.

Even though there is a negative shift in the relative P&L once the transaction costs are imposed, the distributions remain similar. The peak in the daily hedging frequency remains in contrast to hedging with the underlying and the differences between the distributions of hedging errors for rebalancing frequencies 1, 4 and 16 are trivial.

Frequency (N)	Percentiles						
	Mean	Std. dev.	1st	10th	50th	90th	99th
1	-7.8%	14.0%	-25.3%	-17.5%	-9.7%	-4.0%	42.6%
4	-5.9%	13.7%	-22.4%	-15.2%	-8.1%	-1.3%	43.8%
16	-5.0%	14.0%	-23.3%	-14.8%	-7.5%	2.9%	46.1%
64	-4.4%	15.1%	-23.3%	-16.4%	-7.8%	9.8%	49.3%
128	-4.3%	15.8%	-19.6%	-17.6%	-8.7%	14.4%	49.6%
256	-1.4%	16.9%	-28.0%	-22.4%	-3.6%	26.9%	31.2%
No hedging	-0.7%	86.2%	-268.6%	-118.2%	18.0%	95.0%	100%

Table 4.4 - Descriptive statistics of the relative P&L distribution of a delta hedge an option in the presence of transaction costs. The statistics corresponds to figure 4.5.

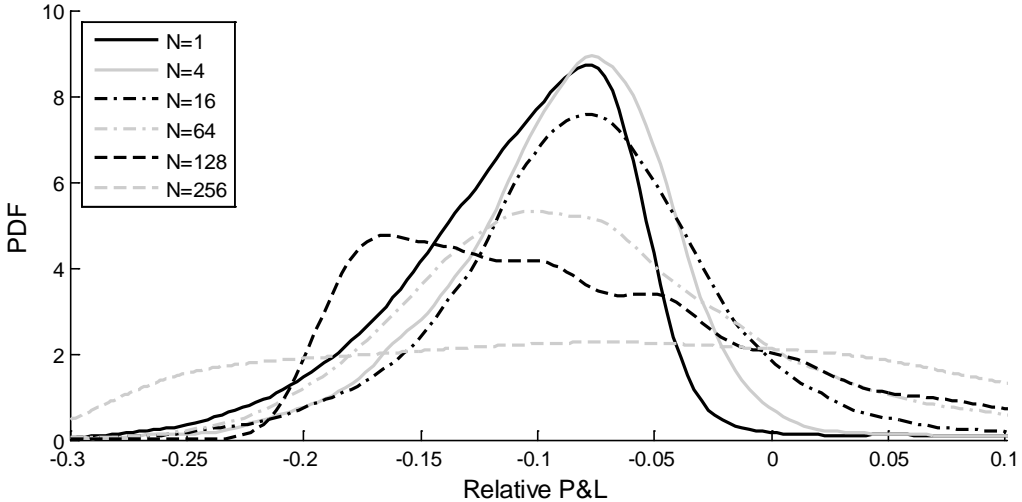


Figure 4.5 - Relative P&L distribution of a delta hedge using an option for one year in the presence of transaction costs. N is the number of days between rebalancing.

Considering the mean P&L to be the cost of hedging, the static hedge strategy ($N = 256$) offers substantial reduction in standard deviation relative to the cost. Since the portfolio is not rebalanced it is not delta neutral at all times, but the strike price of the target and the hedging option will be the same throughout the hedging period. This is of great importance since the impact of a jump is greater when the parameters differ between the target and the hedging

option. However, as the rebalancing frequency is increased the range between the 10th and 90th percentiles is decreasing, indicating a more predictable P&L.

To sum up, delta hedging performs poorly under jump diffusion. Strategies as these cannot handle discontinuities of the underlying asset and at jump times there are significant hedging errors. Using an option improves the performance both in terms of standard deviation and cost even though the technique is in no way claimed to be optimal. A different way of replacing the option and choosing strike price might improve the performance further in terms of both cost and risk. The lower cost is partly a result of the value of the option position being only a fraction of that required in the underlying. The smaller shift in the relative P&L for the daily rebalancing indicates that this result holds even though the transaction costs for options are higher. In addition, the range between the percentiles decreases remarkably so that the hedger is not as exposed to jump risk.

4.3 Hedging using Gauss-Hermite Quadratures

In this section we analyse the Gauss-Hermite strategy in three different settings in order to evaluate its performance in markets with varying liquidity. In the first setting, there is no restriction on the available strike prices in the market. This setting can be expected to yield the smallest hedging errors, but does not take the limited number of options available in the market into account. It might require strike prices that are very different from the initial stock price and thus may be impossible to implement. In the second setting the strike prices are restricted to $0.5S_0-1.5S_0$ with equally spaced $0.05S_0$ intervals. The third setting further restricts the strikes to values between $0.75S_0$ and $1.25S_0$. The performance for each setting with and without transaction costs can be found in tables 4.5-4.10.

The hedging performance of the unrestricted GHQ-strategy is different from the dynamic strategies in several ways. In contrast to the delta hedging strategies where it is of limited use to rebalance on a more frequent basis, the performance improves as the number of hedging options is increased. The marginal risk reduction from including more options is decreasing so that the optimal number of options to include will be subject to the transaction costs. With proportional transaction costs they have a minor impact since all transactions take place the first day and the total value of the hedging portfolio is close or equal to that of the target option, with a small error arising from approximating the infinite integral (2.5), as mentioned in Carr & Wu (2014). The GHQ-strategy does not include short positions for any number of options included. This is due to that the portfolio weight is a function of the strike price, the option gamma and the weight obtained from the solution of the Hermite polynomial. Neither

of these can be negative and thus the portfolio weights will be positive. Because of this, there is an immediate negative shift in the mean P&L and the percentiles that converges to the proportional transaction cost as more options are included and the approximation error vanishes.

Including more options in the portfolio results in a wider spread of the calibrated strike prices. A strategy with 3 options require strikes in the interval of $0.46S_0$ - $1.70S_0$, while the corresponding range for 20 options is $0.05S_0$ - $15.75S_0$. Obviously, the latter strategy is impossible to implement and as described in section 2.3.2 and shown in appendix A.8, the range is to a large extent a result of the maturity gap between the target option and the hedging options.

In setting 2 the strikes are narrowed to $0.5S_0 - 1.5S_0$. This is done by rounding the strikes outside this interval to the closest boundary, keeping the optimal weights unchanged as in Carr & Wu (2014). This restriction will reduce the actual number of options used, since several options will be rounded to the same strike, that is, to the same option. As an example, the strategy where the strikes and weights are calculated for 20 options, 16 of these will be rounded to the boundaries. Hinde (2006) claims that rounding the calibrated strike prices will have a negative effect on the hedging performance but does not investigate this further. As shown in tables 4.7 and 4.8, the performance is only moderately affected by this restriction. Since the calibrated strike prices for a few number of options are less spread out than those for a higher number of options, the impact of the restriction is greater for higher N . The reason to the small deterioration can be found through an analysis of the calibrated weights. Since each portfolio weight is proportional to the gamma that the target option will have when the hedging option expires, if the underlying asset price at that time is equal to the calibrated strike price, the weights on options with extreme strike prices are trivial. The result is indeed encouraging in terms of real world application, since it at least in liquid markets might be possible to implement a strategy using strikes in the range considered. However, even strikes in a range as wide as this is an optimistic assumption in less liquid markets, which makes it interesting to study the performance of the model when the strikes are restricted further. Tables 4.9 and 4.10 shows that the performance in setting 3 is still satisfactory, especially when compared to the dynamic strategies. The strikes for the strategy with 20 options are rounded so that only four different options are used with strikes $0.75S_0$, $0.8S_0$, $1S_0$ and $1.25S_0$, where nine options are rounded to each boundary. The standard deviation is larger for all number of options compared to setting 2 and there are more observations outside

the 10th-90th percentile range, but the changes are trivial in comparison to the large restriction on strikes.

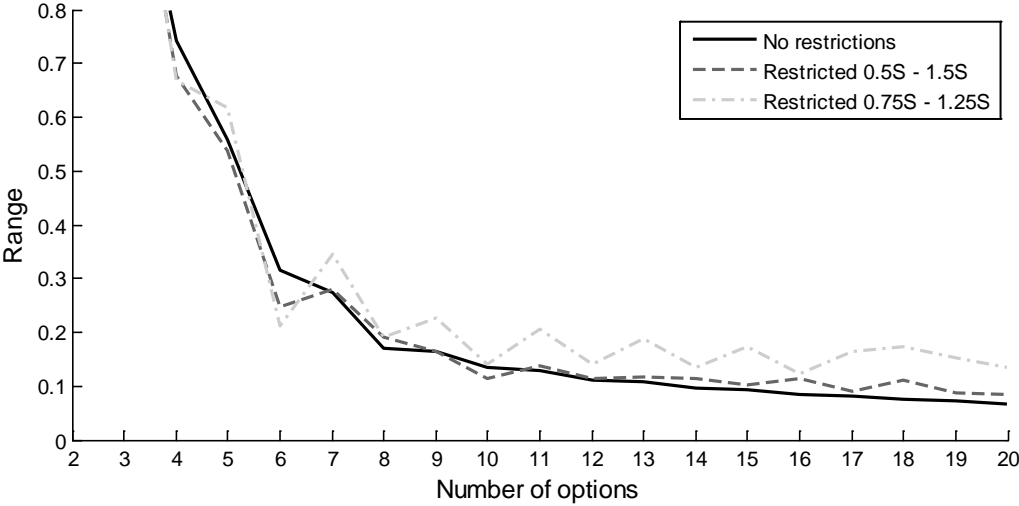


Figure 4.6 – The range between the 1st and 99th percentile for the three different settings.

Figure 4.6 shows the range between the 1st and 99th percentiles of the relative P&L for the different settings. It is clear that the ability to reduce the range suffers clearly when setting 3 is applied compared to the ability in the first two settings.

When the GHQ-method was developed by Carr & Wu (2014) they analysed its performance and found that a strategy using five options outperforms any delta hedge and mention that the performance can be enhanced further by including more options. Focusing on the unrestricted setting, our results support those of Carr & Wu (2014), but also indicate that there is a significant increase in performance when the strategy is expanded to eight options. This number of options provides a relatively good protection from large losses which is persistent throughout the restrictions as well. The standard deviation compared to five options is considerably lower and there is only small improvements beyond this point.

No transaction costs

Transaction costs, Ts=1%, To=2%

Options	Percentiles							Percentiles						
	Mean	Std. dev.	1st	10th	50th	90th	99th	Mean	Std. dev.	1st	10th	50th	90th	99th
3	0.3%	32.0%	-48.0%	-38.3%	-3.5%	43.6%	82.8%	-2.2%	32.0%	-50.4%	-40.7%	-5.9%	41.2%	80.3%
5	0.1%	13.6%	-27.1%	-19.4%	0.7%	16.4%	28.9%	-2.0%	13.6%	-29.2%	-21.5%	-1.5%	14.2%	26.7%
8	0.0%	5.2%	-7.4%	-5.6%	-2.6%	8.8%	9.7%	-2.0%	5.2%	-9.4%	-7.6%	-4.6%	6.8%	7.7%
10	0.0%	3.5%	-6.5%	-4.4%	-0.5%	6.1%	6.9%	-2.0%	3.5%	-8.5%	-6.3%	-2.5%	4.1%	4.9%
15	0.0%	2.0%	-6.4%	-2.3%	0.0%	2.6%	2.8%	-2.0%	2.0%	-8.4%	-4.3%	-2.0%	0.6%	0.8%
20	0.0%	1.5%	-4.0%	-2.3%	0.0%	1.8%	2.7%	-2.0%	1.5%	-6.0%	-4.3%	-2.0%	-0.2%	0.7%

Table 4.5 –Unrestricted strikes

Table 4.6 - Unrestricted strikes

Options	Percentiles							Percentiles						
	Mean	Std. dev.	1st	10th	50th	90th	99th	Mean	Std. dev.	1st	10th	50th	90th	99th
3	0.3%	30.5%	-45.1%	-36.2%	-5.4%	41.7%	83.7%	-2.0%	30.5%	-47.5%	-38.5%	-7.7%	39.4%	81.4%
5	0.1%	12.5%	-25.2%	-18.2%	-0.6%	14.8%	28.7%	-1.9%	12.5%	-27.3%	-20.2%	-2.6%	12.8%	26.7%
8	0.0%	6.2%	-7.9%	-6.1%	-3.3%	10.5%	11.4%	-2.1%	6.2%	-9.9%	-8.1%	-5.3%	8.5%	9.3%
10	0.0%	2.9%	-6.5%	-4.2%	0.4%	4.2%	5.1%	-2.0%	2.9%	-8.4%	-6.2%	-1.6%	2.3%	3.1%
15	0.0%	2.2%	-5.5%	-2.7%	0.0%	2.4%	4.8%	-1.9%	2.2%	-7.4%	-4.7%	-2.0%	0.5%	2.8%
20	0.0%	1.8%	-4.8%	-2.7%	0.3%	2.6%	3.5%	-1.9%	1.8%	-6.8%	-4.7%	-1.6%	0.7%	1.6%

Table 4.7 – Strikes $0.5S_0 - 1.5S_0$

Table 4.8 - Strikes $0.5S_0 - 1.5S_0$

Options	Percentiles							Percentiles						
	Mean	Std. dev.	1st	10th	50th	90th	99th	Mean	Std. dev.	1st	10th	50th	90th	99th
3	0.3%	31.2%	-45.6%	-36.7%	-5.9%	43.2%	86.7%	-2.0%	31.2%	-47.9%	-39.0%	-8.2%	40.9%	84.4%
5	0.1%	14.0%	-26.5%	-19.5%	1.3%	18.2%	35.3%	-1.8%	14.0%	-28.5%	-21.4%	-0.7%	16.2%	33.4%
8	0.0%	6.4%	-9.6%	-7.8%	-1.8%	8.8%	9.7%	-1.9%	6.4%	-11.4%	-9.6%	-3.6%	6.9%	7.8%
10	0.0%	3.4%	-7.3%	-5.0%	0.3%	4.1%	6.8%	-2.0%	3.4%	-9.2%	-7.0%	-1.7%	2.2%	4.9%
15	0.0%	4.1%	-6.9%	-4.1%	-0.5%	6.2%	10.6%	-1.8%	4.1%	-8.7%	-5.9%	-2.3%	4.4%	8.8%
20	0.0%	3.4%	-6.1%	-4.0%	-0.5%	6.1%	7.5%	-1.9%	3.4%	-7.9%	-5.9%	-2.3%	4.3%	5.7%

Table 4.9 – Strikes $0.75S_0 - 1.25S_0$

Table 4.10 - Strikes $0.75S_0 - 1.25S_0$

4.4 Hedging using Least Squares

Table 4.11 and 4.12 shows the performance of the Least squares hedge for different number of options combined with the underlying. Similar to the GHQ-strategy, the Least squares hedge shows a major increase in performance as more options are used. By including only one option in the portfolio, the standard deviation drops by 77% compared to an unhedged position.

Options	Percentiles						
	Mean	Std. dev.	1st	10th	50th	90th	99th
No hedging	-0.7%	86.2%	-268.6%	-118.2%	18.0%	95.0%	100%
1	0.0%	9.2%	-15.4%	-11.4%	-0.8%	15.6%	21.6%
3	0.1%	5.3%	-14.6%	-6.3%	0.6%	7.1%	10.6%
5	0.0%	2.2%	-6.5%	-3.2%	0.5%	2.4%	2.9%
8	0.0%	0.9%	-2.6%	-1.5%	0.2%	0.8%	1.0%
10	0.0%	0.5%	-1.9%	-0.7%	0.1%	0.5%	0.6%
15	0.0%	0.3%	-1.4%	-0.2%	0.1%	0.2%	0.4%
20	0.0%	0.3%	-1.4%	-0.1%	0.1%	0.1%	0.5%

Table 4.11 – Descriptive statistics of the relative P&L from the Least squares hedging strategy for one year. N is the number of options used in combination with the underlying asset.

As more options are included, the standard deviation is constantly decreasing all the way to 0.3%, although the marginal reduction in risk from including more options is sharply decreasing. The practically minded reader must keep in mind that it can be unrealistic to find all the options in this model in reality. It is therefore encouraging to see that the inability to use a large number of options would not be of major concern. The fact that the performance increases for up to 10 options stands in contrast to findings by Hinde (2006), where the improvement in performance is limited beyond five options. Apart from that Hinde uses the transition PDF as a weighting function, the maturities considered are shorter. Since an option with a shorter maturity has a smaller gamma for high and low moneyness, it comes naturally that including options with strikes in a wider spread does not contribute to the performance. As the maturities considered in our analysis are longer, there is a need to cover a wider range. Because of this, it is possible to get a very good fit using the Least squares approach when hedging a longer dated option, even though the performance when using a low number of options might be worse than when the target option has a shorter maturity.

The strategy is allowed to include both long and short positions in the hedging instruments, which implies that even though the total value of the hedging portfolio is equal to the value of the target option, it is not necessarily the case that the transaction costs incurred from constructing the portfolio is proportional to the target option. As three options are used, the

strategy suggested large positions in both directions which results in major transaction costs, visible through the mean of -4.1%. We note that as more options are included, less short positions need to be taken. Hinde (2006) claims that as the number of options approaches infinity, all consistent strategies should converge to a unique combination of options that perfectly replicates the target option. According to Carr & Wu (2014), each weight in this continuum is proportional to the gamma of the target option for a specific price of the underlying asset. Thus, as more options are included in the portfolio, all weights will become positive. This has several advantages; first, the total transaction costs are known to be directly proportional to the value of the target option. Second, the cost of hedging is relatively low, although the proportional transaction cost for options are assumed to be higher than that of stocks.

Options	Percentiles						
	Mean	Std. dev.	1st	10th	50th	90th	99th
No hedging	-0.7%	86.2%	-268.6%	-118.2%	18.0%	95.0%	100%
1	-2.4%	9.2%	-17.7%	-13.8%	-3.1%	13.2%	19.2%
3	-4.1%	5.3%	-18.7%	-10.4%	-3.5%	3.0%	6.4%
5	-2.4%	2.2%	-8.9%	-5.6%	-1.9%	0.0%	0.5%
8	-2.1%	0.9%	-4.7%	-3.6%	-1.8%	-1.3%	-1.1%
10	-2.0%	0.5%	-3.9%	-2.7%	-1.9%	-1.5%	-1.5%
15	-2.0%	0.3%	-3.4%	-2.2%	-1.9%	-1.8%	-1.6%
20	-2.0%	0.3%	-3.4%	-2.1%	-2.0%	-1.9%	-1.6%

Table 4.12 - Descriptive statistics of the relative P&L from the Least squares hedging strategy for one year. N is the number of options used in combination with the underlying asset.

Table 4.12 is constructed from the assumption that the proportional transaction costs are constant without penalizing more options in the portfolio. In reality, the transaction costs might vary with the number of options in the portfolio. If it becomes more expensive to trade in a variety of options, the mean P&L will be lower as more options are used. It is however clear from table 4.12 that the transaction cost would have to increase severely for the strategy to be more expensive than the dynamic strategies and would still outperform them in terms of risk. This claim is in line with Kennedy et al. (2009) where a dynamic strategy including several hedging options is used to hedge a European straddle. A higher number of hedging options is penalized by a larger bid-ask spread but as more options are included in the portfolio, the mean payoff is decreasing only slightly.

4.5 Comparison of the semi-static strategies

The findings in the previous sections show that the traditional dynamic strategies are not only performing poorly in the presence of jumps, but also that a higher rebalancing frequency is

notoriously cost ineffective. This is not true for the static strategies. In this section we therefore exclude the dynamic strategies and focus on the semi-static. To generalize the comparison and minimize the sensitivity to our assumptions about the availability of options in the market, the GHQ-strategy is presented without restrictions.

N	<i>Least squares</i>				<i>GHQ (Unrestricted)</i>			
	Rel. 1-99	Rel. 10-90	Rel. Std.	Mean	Rel. 1-99	Rel. 10-90	Rel. Std.	Mean
1	10,0%	12,6%	10,7%	-2.4%	-	-	-	-
3	6.8%	6.3%	6.2%	-4.1%	35.5%	38.4%	37.1%	-2.2%
5	2.5%	2.6%	2.5%	-2.4%	15.2%	16.8%	15.8%	-2.0%
8	1.0%	1.1%	1.0%	-2.1%	4.6%	6.8%	6.0%	-2.0%
10	0.7%	0.5%	0.6%	-2.0%	3.6%	4.9%	4.1%	-2.0%
15	0.5%	0.2%	0.3%	-2.0%	2.5%	2.3%	2.4%	-2.0%
20	0.5%	0.1%	0.3%	-2.0%	1.8%	1.9%	1.7%	-2.0%

Table 4.13 – Relative performance of the static strategies outlined in the previous sections in the presence of transaction costs. N is the number of options included.

When analysing the relative performance measures in table 4.13, the reader must keep in mind that the underlying asset is included in the Least squares hedge for all N . Thus, there is a payoff from the hedge portfolio at time u for all $S > 0$. It turns out that as N increases and a wider range of strikes is included, the weight in the underlying approaches zero. In the GHQ-strategy where no underlying asset is included, the hedge portfolio will have a zero payoff for all values of S that are smaller than the lowest calibrated strike price. The fact that the target option has a non-zero value for all S_u is an intuitive explanation to why the GHQ-strategy calibrates strikes in a wide range with small weights in the tails.

Carr & Wu (2014) notes that the performance of this strategy suffers as the maturity gap increases when applied to real world data. This deterioration is interpreted as a sign of other factors, such as stochastic volatility, affecting the value of the options. Since our analysis does not include stochastic volatility, we claim that the dispersion of calibrated strikes is a contributing factor to this finding. As the maturity gap increases, the calibrated strikes will be in an even wider range which makes it necessary to include more options to make the performance persistent. For a small number of options, it therefore fails to capture the range of the value function where the most curvature is and places the strikes at stock prices of minor importance. However, the performance does improve as more options are used, but we note that for 20 options, the most extreme strike price calibrated is $15.75S_0$ and that eight of them are greater than $1.7S_0$. With this spread in mind it is interesting to see that when five options are included the GHQ-hedge outperforms any delta hedge in terms of standard deviation and performance in extreme events.

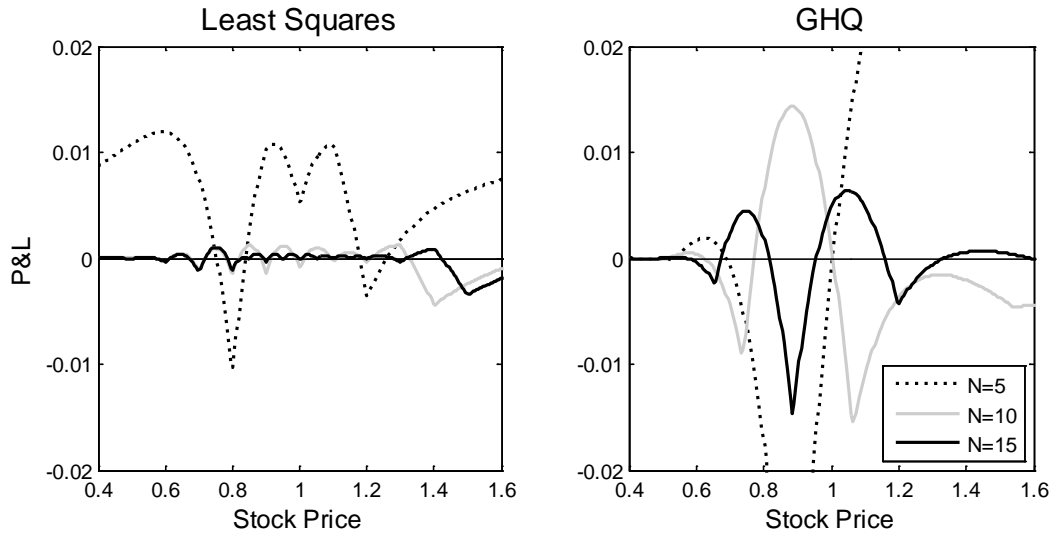


Figure 4.7 – Difference between the value of the hedge portfolio with N options and the target option at time u for a range of stock prices. The value of the target option at time 0 is 0.2089.

Since neither of the strategies is path dependent it is possible to analyse the performance at time u for different stock prices. Figure 4.7 shows that both strategies improves significantly as the number of options included is increased from five to ten. It also shows that for $N = 5$, the Least squares strategy performs worse than GHQ for stock prices in the lower range. This result is because of the previously mentioned position in the underlying asset, which yields a linear payoff for all values of S . Unfortunately for the Least squares strategy using $N = 5$ there are major hedging errors in the lower range where the stock price is likely to close when there is a jump during the hedging period (see appendix A.5). These errors are probably a result of using a uniform weighting function that does not take the distributional properties of the simulated stock price into account and therefore puts equal effort into minimizing hedging errors at stock prices of minor importance than those more likely to occur. Our results are in contrast to those of Hinde (2006) who uses the transition PDF as weighting function which has the same shape as the figure in appendix A.7. The transition PDF is shown to yield very small hedging errors for the lower values of S when $N = 5$. In fact, in this lower range the GHQ strategy performs best regardless of the number of options included. For the same number of options and higher values of S , the P&L for the GHQ strategy is large and exceeds the axis by far, while Least squares performs relatively well.

When comparing the strategies using table 4.13 and figure 4.7 it is clear that the Least squares method is best performing regardless of the number of options included. The satisfying performance of this strategy has also been shown in He, et al. (2006) and Kennedy, et al. (2009) and through our analysis it is clear that the performance is persistent when using

a uniform weighting function as well. Having the possibility of choosing strike prices in the areas with the most curvature is an advantage to the Least squares method over GHQ.

Theoretically it is without doubt preferable to include as many options as possible, but when it comes to suggesting one particular number that is realistic to use it is important to consider that a strategy including more options might be difficult or impossible to set up due to liquidity restrictions in the market. Hence we suggest the Least squares strategy with five to eight options depending on the availability of strikes and the cost incurred from trading options that are deep in and out of the money. The strategy offers substantial risk reduction and the cost effectiveness is robust to lack of knowledge regarding the underlying stock price dynamics. Even though the GHQ method has been outperformed in our analysis it has the attractive feature of calibrating the strike prices used for hedging and it is likely to perform better when the maturity gap is not very wide. It might also be an attractive alternative when hedging complex derivatives such as barrier options.

5. Conclusions

This thesis investigates the performance of hedging strategies when the underlying asset follows Merton's jump-diffusion process. The hedging strategies applied are delta hedging using the underlying asset, delta hedging using an option, Gauss-Hermite Quadratures (GHQ) hedging and Least squares hedging. The first two of these are dynamic strategies which aims to replicate the payoff of the target option through frequent rebalancing, while the latter two are semi-static that are rebalanced infrequently. The GHQ-strategy is based on a spanning relation, stating that the value of an option at any time can be replicated by a continuum of shorter dated options. By approximating the continuum with a finite number of options, it calibrates the optimal strikes and weights in the hedging portfolio. In contrast, the Least squares hedging takes liquidity constraints into account by taking the strikes as given and calibrates the weights to minimize the squared hedging error at a future point in time. The data for the analysis is generated through simulation of the underlying asset path. For all strategies, we hedge a short European call option with two years to maturity during one year, assuming that it is possible to close the position at that time. We also assume that the options available for hedging purposes have a maturity of one year.

We find that delta hedging using the underlying asset performs poorly at jump times. Similar to Kennedy, et al. (2009) the payoff is found to be negative regardless of the direction of the jump. We do not find any improvements in terms of standard deviation for higher rebalancing frequencies, consistent with findings by Carr & Wu (2014) and Hinde (2006). We do however find decreasing hedging error ranges for the 10th to 90th percentiles as the rebalancing frequency is increased, implying that the performance is only increased for times when there is no jump. Further, we find that the strategy suffers greatly once transaction costs are imposed due to the large positions and high number of transactions. With more frequent rebalancing, the strategy becomes more costly with close to a non-existent risk reduction compared to a static delta hedge. Out of the four strategies analysed this strategy is clearly the least attractive.

The results from delta hedging using an option highlight the importance of using non-linear instruments in the hedging portfolio when there is jump risk. We find that the standard deviation for all frequencies are remarkably lower compared to hedging using the underlying. However, we do not find any dramatic improvements as frequency is increased. It is worth noting that due to the non-linearity and shorter maturity of the hedging option considered, the impact of extreme events is not as severe and that the hedger will receive a profit when a

jump occurs conditional on the options having similar strike prices. This profit is offset by a small negative payoff occurring on average when the underlying does not jump, yielding a zero mean payoff. Due to the smaller value of the hedging portfolio, the strategy is not as costly as the standard delta hedge even though the proportional transaction costs are higher. In addition, we find that the distributions of the hedging errors are similar with and without transaction costs, which makes it different from the spot hedge.

Delta hedging under a jump-diffusion has been proven cumbersome in previous work. In addition to the poor performance in the presence of jumps, this thesis highlights the sensitivity to imposing transaction costs. Delta hedging in this setting is indeed costly, which can be generalized to the standard Black-Scholes framework as well.

The semi-static strategies are clear improvements from the dynamic strategies considered. Our findings support those of Carr & Wu (2014), claiming that a GHQ-hedge with five options is preferred to any standard delta hedge. The results are strengthened further when transaction costs are imposed. The fact that the GHQ-strategy is financed with the proceeds of the written option makes it less expensive than the delta hedge. In addition, the transaction costs have only a minor impact on the distribution of hedging errors since all transactions take place at the first day. We find that the GHQ-strategy requires a wide range of strikes due to the difference in maturities between the target and the hedging options. To increase the applicability in reality, we restrict the strikes to certain intervals and values. The calibrated strikes are then rounded to the closest available value without altering the weights. Hinde (2006) claims that such a restriction would have an unfavourable effect on the performance of the strategy, but we find only marginal support for this. Regardless of the restriction, the standard deviation is decreasing as the calibration is done for a higher number of options.

The Least squares hedge is the best performing strategy examined. We use a uniform weighting function and find that the strategy has the smallest standard deviations and percentile ranges for all number of options used in the hedging portfolio. He, et al (2006) also finds the hedging strategy to be very successful in the presence of jumps, even though different weighting functions are applied. The superior performance of this strategy compared to GHQ is also found in Hinde (2006), where the maturities are shorter and the transition PDF is used as a weighting function. Similarly to the GHQ-strategy, the Least squares hedge is cheap compared to the dynamic strategies and quickly reduces risk with a few number of options. We find that the Least squares hedge performs equally well with five options combined with the underlying as the unrestricted GHQ hedge does with 15 options. Based on

the performance measures, we favour a Least squares hedge with five to eight options since it is both feasible regarding the range of strike prices and the improvement in performance beyond this point is limited.

The effect of transaction costs included in our work is intuitive and easy to implement. Using proportional transaction costs for stocks might be a valid approximation of reality. On the contrary, it does not penalize the number of options used in the hedging portfolio, only the absolute value of the options. Possible areas of future research might focus on more sophisticated ways to incorporate transaction costs. Such ways can be option pricing in the presence of transaction costs that has been developed by Leland (1985) and extended to Merton's jump diffusion by Mocioalca (2007). Another alternative is to use a bid/ask spread and let them vary with the moneyness of the options as in Kennedy, et al. (2009). This will lead to higher transaction costs for options that are out of the money, thereby increasing the realism regarding varying liquidity. Another area that could be analysed is the impact on performance when the volatility is stochastic since it would lead to varying optimal weights in the hedge portfolios calibrated by the semi-static strategies. The research could also be extended to real world data instead of simulated.

Appendix

A.1 Itô's Lemma & Geometric Brownian Motion

An Itô process is an extension of a generalized Wiener process where the drift and variance rates are functions of the underlying variable S and time, t .

$$dS = a(S, t)dt + b(S, t)dZ(t) \quad (\text{A.1})$$

If S is the price of a stock, then (A.1) generates the path followed by S . One of the most common processes in financial literature is the geometric Brownian motion, which is an extension to the Itô process:

$$\frac{dS}{S} = \mu dt + \sigma dZ \quad (\text{A.2})$$

Let F be the price of a financial derivative on the underlying stock, then via Itô's lemma it is possible to derive the value function of that derivative. Using a Taylor series expansion, Itô showed that a function F of the same variables S and t will follow a process that has a deterministic and a stochastic part:

$$dF = \left[\frac{\delta F}{\delta S} a + \frac{\delta F}{\delta t} + \frac{1}{2} \frac{\delta^2 F}{\delta S^2} b^2 \right] dt + \frac{\delta F}{\delta S} b dZ \quad (\text{A.3})$$

Where dZ is the same Wiener process as in (A.1).

Using Ito's lemma it is possible to find the expected stock price at the end of any period. Substituting a and b in equation (A.3) for μS and σS gives the GBM version of Itô's lemma:

$$dF = \left[\frac{\delta F}{\delta S} \mu S + \frac{\delta F}{\delta t} + \frac{1}{2} \frac{\delta^2 F}{\delta S^2} \sigma^2 S^2 \right] dt + \frac{\delta F}{\delta S} \sigma S dZ \quad (\text{A.4})$$

Defining a function F to be the natural logarithm of the stock price and integrating with respect to time, it can be shown that the stock price at time T is:

$$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma Z(T)}$$

Thus, the stock price has a lognormal distribution.

A.2 Derivation of Merton's Jump Diffusion PDE

Let $F(S, t)$ be the time t value of a derivative. Consider a portfolio Π consisting of one short option and a long position of $\partial F / \partial S$ shares of the underlying stock:

$$\Pi = -F(S, t) + \frac{\partial F}{\partial S} S \quad (\text{A.5})$$

As time moves from t to $(t + \Delta t)$, the value of the portfolio changes by:

$$\Delta \Pi = -\Delta F(S, t) + \frac{\partial F}{\partial S} \Delta S \quad (\text{A.6})$$

Transforming (A.2) and (A.4) to discrete form and substitute into (A.6) gives:

$$\Delta\Pi = -\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right)\Delta t \quad (\text{A.7})$$

In (A.7) the Wiener process has cancelled out and thus the uncertainty of the portfolio has vanished. The portfolio is now risk-free, and any risk-free portfolio should yield the risk-free rate for no arbitrage opportunities to exist. Thus,

$$r\Pi\Delta t = \Delta\Pi \quad (\text{A.8})$$

Substituting (A.5) and (A.7) into (A.8) and simplifying results in the Black-Scholes PDE:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS \frac{\partial F}{\partial S} - rF(S, t) = 0$$

Merton shows that by applying Itô's lemma for the diffusion part of the process in equation (A.3) and an analogous lemma for the jump part, the relationship becomes (Merton, 1976; Sideri, 2013):

$$dF(S, t) = \left[(r - \lambda\kappa)S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right]dt + \sigma S \frac{\partial F}{\partial S} dZ_t + [F(SY_t, t) - F(S, t)]dN_t \quad (\text{A.9})$$

Substituting the dynamics of the jump-diffusion process (A.2 & A.9) into the delta-neutral portfolio we get:

$$\begin{aligned} d\Pi &= -dF + \frac{\partial F}{\partial S} dS \\ d\Pi &= -\left[\left((r - \lambda\kappa)S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right)dt + \sigma S \frac{\partial F}{\partial S} dZ(t) + [F(SY_t, t) - F(S, t)]dN_t\right] \\ &\quad + \frac{\partial F}{\partial S} ((r - \lambda\kappa)Sdt + \sigma SdZ(t) + (Y_t - 1)SdN_t) \\ d\Pi &= -\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right)dt - \left(F(SY_t, t) - F(S, t) - (Y_t - 1)\frac{\partial F}{\partial S}S\right) dN_t \end{aligned} \quad (\text{A.10})$$

Since the jumps are diversifiable, we can still argue that the hedged portfolio will grow at the risk free rate, so that:

$$E[d\Pi_t] = r\Pi_t dt$$

Substituting $d\Pi_t$ for equation A.10 and Π_t for A.5 gives:

$$E\left[-\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right)dt - \left(F(SY_t, t) - F(S, t) - (Y_t - 1)\frac{\partial F}{\partial S}S\right) dN_t\right] = r\left(-F(S, t) + \frac{\partial F}{\partial S}S\right)dt$$

Substituting $E[dN_t] = \lambda dt$ and rearranging yields:

$$-\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right)dt - E\left[F(SY_t, t) - F(S, t) - (Y_t - 1)\frac{\partial F}{\partial S}S\right] \lambda dt = r\left(-F(S, t) + \frac{\partial F}{\partial S}S\right)dt$$

Dividing by dt :

$$-\frac{\partial F}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - \lambda E \left[F(SY_t, t) - F(S, t) - (Y_t - 1) \frac{\partial F}{\partial S} S \right] = r \left(-F(S, t) + \frac{\partial F}{\partial S} S \right)$$

We arrive at the PDE under Merton's jump diffusion:

$$-\frac{\partial F}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - \lambda E[F(SY_t, t) - F(S, t)] + \lambda E[Y_t - 1] \frac{\partial F}{\partial S} S + rF(S, t) - rS \frac{\partial F}{\partial S} = 0$$

A.3 Gaussian Quadratures

Gaussian quadrature is a numerical method to estimate the value of a finite integral from values -1 to 1. The estimation is done by dividing the integral into n segments and calculating the area of each of the segments. The approximate value of the integral is then the weighted sum of these areas. The method can be generalized to approximate any finite integral from a to b by manipulating the integral to satisfy the conditions of the Gaussian quadrature rule:

$$\int_a^b f(y) dy = \frac{b-a}{2} \int_{-1}^1 f(y) dy \approx \frac{b-a}{2} \sum_{i=1}^n w_i f\left(\frac{b-a}{2} x_i + \frac{b+a}{2}\right)$$

where x_i are the abscissas given by the i^{th} root of the Legendre polynomial $P_n(x)$ and the weights w_i are given by (Abramowitz & Stegun, 1972):

$$w_i = \frac{2}{(1-x_i^2)[P'_n(x_i)]^2}$$

As n increases the error of the approximation diminishes quickly. For lower order of n , the weights and abscissas can be found in Abramowitz & Stegun (1972), p 916.

A.4 Derivation of the Least Squares minimization

For $t = 1$, the self-financing condition becomes:

$$\boldsymbol{\phi}_1 \cdot \mathbf{I}_1 + w_1 S_1 + B_1 = \boldsymbol{\phi}_0 \cdot \mathbf{I}_1 + w_0 S_1 + B_0 e^{r^* dt} \quad (\text{A.11})$$

The value of the total portfolio at $t = 1$ before rebalancing is:

$$\Pi_1 = -F_1 + \boldsymbol{\phi}_0 \cdot \mathbf{I}_1 + w_0 S_1 + B_0 e^{r^* dt} \quad (\text{A.12})$$

Similarly, at $t = 2$:

$$\Pi_2 = -F_2 + \boldsymbol{\phi}_1 \cdot \mathbf{I}_2 + w_1 S_2 + B_1 e^{r^* dt} \quad (\text{A.13})$$

From equation (A.11), we have:

$$B_1 = \boldsymbol{\phi}_0 \cdot \mathbf{I}_1 + w_0 S_1 + B_0 e^{r^* dt} - \boldsymbol{\phi}_1 \cdot \mathbf{I}_1 - w_1 S_1$$

Substitute this into equation (A.13):

$$\Pi_2 = -F_2 + \boldsymbol{\phi}_1 \cdot \mathbf{I}_2 + w_1 S_2 + [\boldsymbol{\phi}_0 \cdot \mathbf{I}_1 + w_0 S_1 + B_0 e^{r^* dt} - \boldsymbol{\phi}_1 \cdot \mathbf{I}_1 - w_1 S_1] e^{r^* dt}$$

From equation (A.12), we have:

$$B_0 e^{e^* dt} = \Pi_1 + F_1 - \boldsymbol{\phi}_0 \cdot \mathbf{I}_1 - w_0 S_1$$

Substitute this into the recently found expression for Π_2 :

$$\begin{aligned}\Pi_2 &= -F_2 + \boldsymbol{\phi}_1 \cdot \mathbf{I}_2 + w_1 S_2 + [\boldsymbol{\phi}_0 \cdot \mathbf{I}_1 + w_0 S_1 + \Pi_1 + F_1 - \boldsymbol{\phi}_0 \cdot \mathbf{I}_1 - w_0 S_1 - \boldsymbol{\phi}_1 \cdot \mathbf{I}_1 - w_1 S_1]e^{r^*dt} \\ &\rightarrow \Pi_2 = -F_2 + \boldsymbol{\phi}_1 \cdot \mathbf{I}_2 + w_1 S_2 + [\Pi_1 + F_1 - \boldsymbol{\phi}_1 \cdot \mathbf{I}_1 - w_1 S_1]e^{r^*dt}\end{aligned}$$

Rearranging:

$$\Pi_2 = -(F_2 - F_1) + \boldsymbol{\phi}_1 \cdot (\mathbf{I}_2 - \mathbf{I}_1) + w_1(S_2 - S_1) + (F_1 - \boldsymbol{\phi}_1 \cdot \mathbf{I}_1 - w_1 S_1)(e^{r^*dt} - 1) + \Pi_1 e^{r^*dt}$$

We want to minimize the difference between the portfolios at times 1 and 2:

$$\begin{aligned}& (\Pi_2 - \Pi_1 e^{r^*dt}) \\ = & -(F_2 - F_1) + \boldsymbol{\phi}_1 \cdot (\mathbf{I}_2 - \mathbf{I}_1) + w_1(S_2 - S_1) + (F_1 - \boldsymbol{\phi}_1 \cdot \mathbf{I}_1 - w_1 S_1)(e^{r^*dt} - 1) + \Pi_1 e^{r^*dt} - \Pi_1 e^{r^*dt} \\ = & -(F_2 - F_1) + \boldsymbol{\phi}_1 \cdot (\mathbf{I}_2 - \mathbf{I}_1) + w_1(S_2 - S_1) + (F_1 - \boldsymbol{\phi}_1 \cdot \mathbf{I}_1 - w_1 S_1)(e^{r^*dt} - 1) \\ = & -F_2 + \boldsymbol{\phi}_1 \cdot \mathbf{I}_2 + w_1 S_2 + (F_1 - \boldsymbol{\phi}_1 \cdot \mathbf{I}_1 - w_1 S_1)(e^{r^*dt})\end{aligned}$$

By rearranging and squaring we arrive at the minimization problem:

$$\min_{\boldsymbol{\phi}, w} [(\Pi_2 - \Pi_1 e^{r^*dt})^2] = \min_{\boldsymbol{\phi}, w} [(-F_2 + F_1 e^{r^*dt}) + \boldsymbol{\phi}_1 \cdot (\mathbf{I}_2 - \mathbf{I}_1 e^{r^*dt}) + w_1(S_2 - S_1 e^{r^*dt})]^2]$$

A.5 Development of hedging error for delta hedging with option

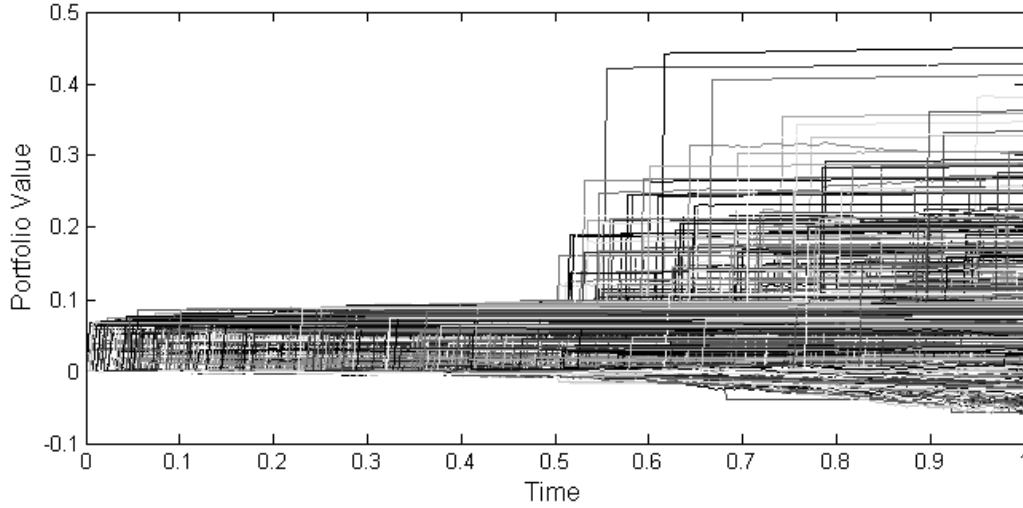


Figure A.1 – Development of hedging errors for a delta hedging using an option.

The figure shows that the majority of the jumps in the value of the hedging portfolio are positive and that no negative jumps occur prior to the rebalancing when the strike prices of the options becomes different.

A.6 Hedging error for delta hedging with an option for large movements in S

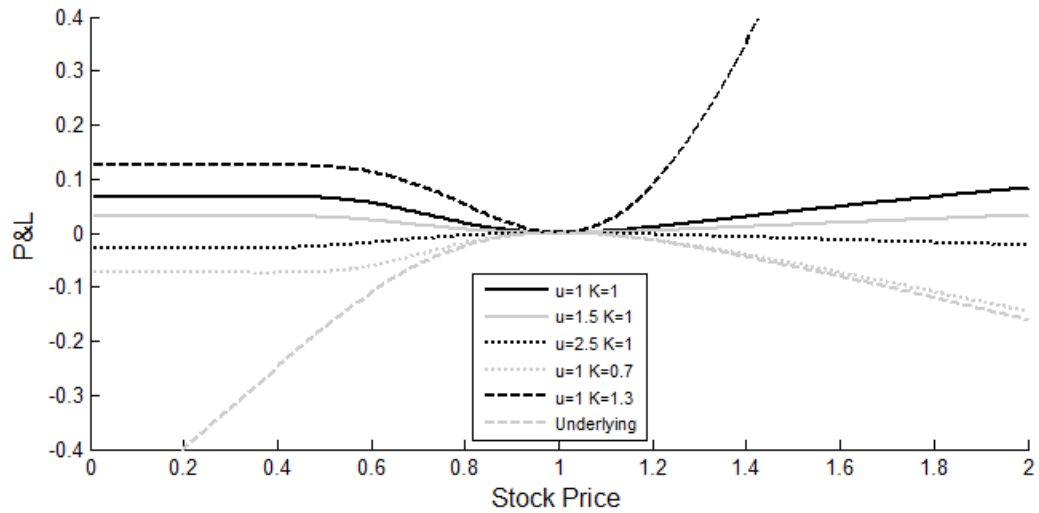


Figure A.2 - The hedging error arising from an instantaneous movement in the underlying asset. The initial stock price is set to 1, the strike of the target option is 1 and the time to maturity is set to 2 years. u is the time to maturity of the hedging option and K is the strike price.

From figure A.2 it is clear that the hedging error increases as the maturity of the hedging options becomes more different than that of the target option. The result is similar when altering the time to maturity of the hedging option.

A.7 Stock price distribution

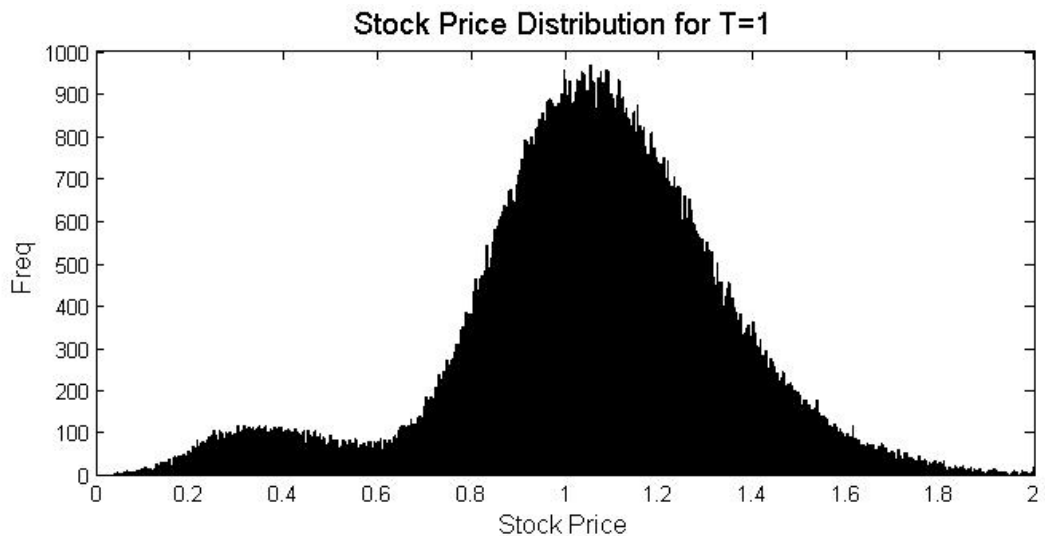


Figure A.3 – The stock price distribution at $T=1$ for $S_0 = 1$.

The figure A.3 shows the realized stock prices for 200,000 simulations at $T = 1$ and $S_0 = 1$. Because of the jumps there are two peaks in the distribution, one for the drift and one for the expected jump.

A.8 Range of Calibrated Strike Prices for the GHQ-strategy

Maturity Gap	<i>Number of options</i>				
	<i>3</i>	<i>5</i>	<i>10</i>	<i>15</i>	<i>20</i>
1	0.46 - 1.70	0.30 - 2.61	0.16 - 5.55	0.08 - 9.80	0.05 - 15.75
0.75	0.52 - 1.61	0.36 - 2.32	0.19 - 4.48	0.11 - 7.32	0.08 - 11.04
0.5	0.59 - 1.49	0.44 - 2.02	0.26 - 3.45	0.17 - 5.15	0.12 - 7.20
0.25	0.70 - 1.35	0.57 - 1.66	0.39 - 2.43	0.29 - 3.23	0.23 - 4.09

Table A.1 – Range of calibrated strike prices for different maturity gaps ($T - u$). $T = 2$ and the strike price of the target options equals 1. The parameters used for calibration are given in table 3.1.

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