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Valuation of Interest Rate Swaps in the Presence of Counterparty Credit Risk

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Abstract

Insuring debt through credit default swaps (CDS) and collateralized debt obligations (CDO) has become increasingly more popular. Recent events such as the financial crisis of 2008 have shown that the credit models for these insurances have lacked severely in certain aspects. One commonly referred example of these ramifications that have ensued is the AIG, the largest insurance company in the United States, that were put into a serious liquidity crisis back in 2008 which prompted a large bailout by the U.S. government. The AIG incident made it evident that the instruments being used didn't properly address a part of the credit exposure that is known as counterparty risk. Counterparty risk means the risk that the counterparty (in this case the insurance company) fails to meet its contractual obligations.

Several models that account for this type of risk have been introduced during the past two decades. The purpose of this thesis is to explore this idea of accounting for counter party risk in financial derivatives and how it affects the pricing adjustment of interest rate swaps.

We estimate the value of IRS agreements in the presence of counterparty risk by adding a credit value adjustment that is estimated using an intensity based approach. The intensity is assumed to be piecewise constant and is calibrated against observed market CDS–quotes using the bootstrapping method.

We find that a 5–year IRS with a low–risk counterparty with 95.4% survival probability during this period yields a credit adjustment of about 40 basis points whereas a 30–year IRS with a high–risk counterparty with 13.5% survival probability yields a credit value adjustment of almost 1000 basis points.

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1 Introduction and overview

The presence of counterparty credit risk in the trades of financial instruments has caught the attention since the aftermath of the credit crisis of 2008. The derivatives market is huge reaching over US$ 600 trillion by the end of 2010. They are popular because they have opened up for a way to easily reallocate and more efficiently manage different types of risk. The three most common risk types to hedge against using derivatives are interest rate risk, credit risk and currency risk. Before the credit crisis of 2008 a lot of loans, mortgages and corporate bonds with poor credit quality were issued which created a demand for credit insurances which were issued through what is called credit default swap agreements (also known as CDS agreements). At the end of 2008 the AIG which is the largest insurance company in the U.S was on the brink of bankruptcy and the blame was put on its vast portfolio of CDS agreements. This made it evident that one big element of the credit risk was not accounted for in this situation; the counterparty default risk. The framework of management of counterparty credit risk extends beyond adding some extra premium on the exchange rates and the prices of financial instruments. It also affects the collateralization and decision making process of a bank. It is outlined both in the Basel II and the Basel III accords which are the regulations for how a bank should conduct their business in a safe and sound manner.

In this thesis we intend to look at the valuation of interest rate swaps in the presence of counterparty credit risk. In order to account for counterparty credit risk we need to understand credit risk and how we use CDS agreements to continuously quantify this risk in a given counterparty.

The rest of this paper is organized as follows; in Section 1 we take a tour on the swaps and derivatives market, explore how they are used to manage different kinds of credit risk. This section is finalized by discussing counterparty credit risk–which is the focus of this paper–and how it affects the valuation of financial derivatives. In Section 2 we establish a modeling framework for valuation of interest rate swaps with counterparty credit risk. The end of the section presents a valuation model of an interest rate swap that is adjusted to account for counterparty credit risk, we test this model under different risk scenarios and examine how these scenarios affect the counterparty adjustment.

1.1 Swaps and the Swap Market

A swap is an agreement that lets two entities swap their cash flows with each other. This is done without any initial monetary transactions which makes it more viable as an instrument as no transaction fees or limitations due to bound capital have to be dealt with.

Swaps can involve any kind of cash flows and the main idea is to let a floating cash flow where there is a risk that it can be either too high or too low be exchanged for a fixed cash flow or another floating cash flow which has a different risk profile.

When entering into such a contract it is set up so that both cash flows in the contract has the same expected net present value, i.e. the contract is set up so that it is fair to both parties. This basically means that the value of the contract is zero when being entered into but may change value over time depending on circumstances.

From a risk–neutral[1] perspective it is hard to see any reason for anyone to enter into a swap agreement since it is a fair game and there is no comparative advantage for either party to enter into such an agreement in an arbitrage free world. In the real world however, an institution or company may face limitations that can only be overcome by entering into such agreements. One party may have legally bound contracts that no longer match his risk profile due to changed circumstances or he may face any other type of institutional frictions. There may be tax differences between the two parties or information asymmetries may motivate a party to enter into a swap agreement with another. In other words the incentives behind entering into a swap agreement cannot be quantified by risk neutral measurements.

Swaps are rarely traded directly between parties, unless the parties are financial institutions. In the case of financial institutions, trades are more direct and they generally know exactly with whom they enter their swap agreements. Otherwise swaps are usually traded over the counter through financial intermediaries and it is generally not known with whom one swaps one's cash flows. These intermediaries serve the function of taking the opposite side of each transaction of the swaps and carry the responsibility of matching and covering for defaulting counterparties in the swap agreement. The spread inherent in the swap agreement is intended to cover for the default risk involved in the counterparties managed by the financial intermediary (Saunders, Cornett 2006, Bodie, Kane, Marcus 2008, Hull 2006). The financial intermediary may have a whole portfolio of entities that are in a swap agreement with each other and at his disposal he may have a set of tools for managing and mitigating the risk that any of the parties would default on his or her liabilities.

The most popular type of swaps involves *interest rate swaps* (IRS) where one party exchanges a floating rate loan for a fixed rate loan. The net present value of the fixed cash flows of an IRS is called the *fixed leg* and the expected net present value of the floating cash flows is called the *floating leg* (Lando 2004). If the fixed leg is paid and the floating leg is received we call the agreement a *payer* IRS whereas if the floating leg is received and the fixed leg is paid we call it a *receiver* IRS or receiver swap. A typical life time of a swap ranges from 2 to 15 years (Hull 2006). Each cash flow, be it annual, semiannual or quarterly can be represented by a forward contract that matures

[1] In a complete and arbitrage–free market there is a measure of valuation that is independent of investor's risk–aversion. An investor's individual choice with regards to his risk aversion is then explained by Fischer's separation principle (Copeland–Weston 2005).

at that date. Therefore during this lifetime the swap agreement can be seen (and therefore be valued) as a stream or portfolio of forward rate agreements (FRA) with different maturities up to the end of the swap period (Brigo, Mercurio 2008). The most popular floating interest rate is the LIBOR rate which is the London Interbank Offer Rate. It is estimated from the average rate at which banks lend and borrow unsecured funds in the London wholesale money market. It is widely accepted as a reference rate in the valuation of financial instruments such as interest rate swaps, foreign currency options and forward rate agreements. The LIBOR rate is estimated for 1, 3, 6 and 12 months maturities only (Bodie, Kane, Markus 2008). So if we are dealing with instruments that have a longer life–span than that, other sources of reference would be needed.

The second most popular type of swap is the *credit default swap* (CDS). In this case the swap acts as an insurance policy against default risk. We illustrate the roles of the parties within a CDS swap agreement in **Figure 1.1**. Let us say that one party A has a borrower, or reference asset C who may default on payments and therefore exposes the lending party A to credit risk. The lending party A may enter into a swap agreement with a protection seller B which generally is an investment bank or a financial institution. From this swap agreement, party A can exchange with B a fixed stream of cash flows (*premium leg*) which is the insurance premium, for a compensation (*default leg*) if the insured reference asset C would default on payments (Lando 2004).

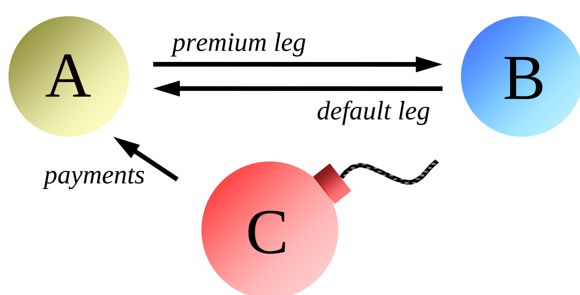

Figure 1.1: A CDS swap agreement that is established between a protection buyer A and a protection seller B where B insures against the credit risk inherent in a reference asset C (usually a borrower or a bond seller).

Source: Lando 2004

The protection seller has in this situation a better ability to diversify against this risk (over perhaps 1000 different clients) than the buyer of the insurance may have. So, in this situation it is easy to see the incentives for both parties to enter into such an agreement. As credit default swaps have become increasingly more popular over the past 10 years, they have become a useful tool in the assessment of the credit risk of a company. The premium rate of CDS contracts is denoted by their CDS spreads which are noted publicly for banks and larger companies. As shown in **Figure 1.2**, a notable event in the history of CDS trades is that the spread of many of them were considerably higher than normal about half a year before the credit crisis of 2008. The CDS–spreads of four large banks (Barclays Bank, BNP Paribas, Deutsche Bank and Royal Bank of Scotland) started to move turbulently after July 2007 and a peak around March 2008 came when Bear–Stearns was on the brink of bankruptcy and was offered to be acquired by J P Morgan (Roddy March 2008 Fortune). The acquisition by J P Morgan was finalized on May 30. We also see that the stock market crash at October 2008 gave rise to a spike, especially for the Royal Bank of Scotland. So the CDS market gave clear indications that something bad was about to happen and that the financial industry were aware of it before it happened.

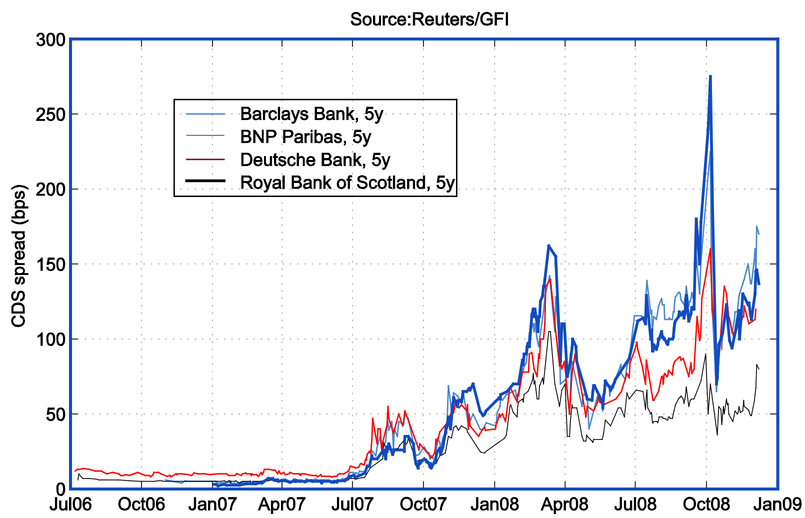

Figure 1.2: The CDS–spreads of four large banks illustrating the turbulence prior to the crisis of 2008.
Source: Reuters/GFI (Through Alexander Herbertsson's lecture notes)

There are other types of swap agreements such as *currency swaps* where two parties usually exchange fixed rate interest payments in one currency with the same interest rate payments in another currency. We also have *commodity swaps* which can be seen as a set of forward contracts on a certain commodity. They are most commonly used by airline companies on crude oil to hedge against sudden price increases. We also have *equity swaps* where a variable cash flow from equity is exchanged for cash flows from a debt with a fixed or floating interest rate.

The Bank for International Settlements reports statistical data on swap trades semiannually and the chart in **Figure 1.3** shows the total amount outstanding for the three most popular swaps.

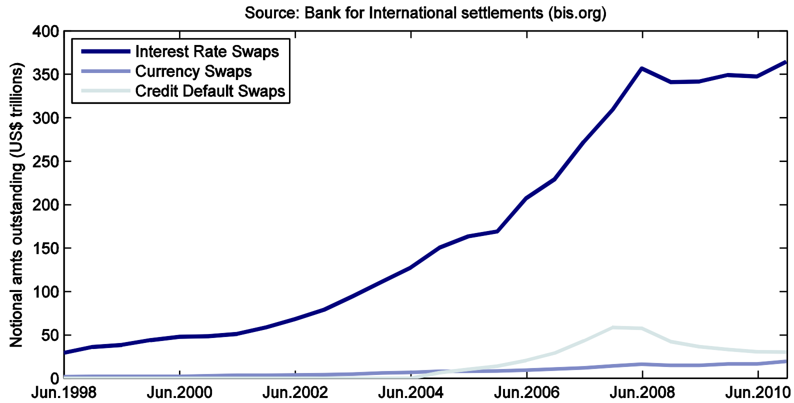

Figure 1.3: Interest rate swaps is the most traded type of financial instrument in the derivatives markets reaching almost US$ 400 trillion by December 2010. The chart also shows that the trading in credit default swaps declined after the financial crisis of 2008.

Source: Bank for international settlements (bis.org)

We see in **Figure 1.3** that the market for interest rate contracts and foreign exchange contracts (currency swaps) is expanding whereas the popularity for the credit default swaps has become a bit tainted since the financial crisis of 2008 and AIG's ensuing liquidity crisis stemming from its large portfolio of outstanding CDS agreements and collateralized debt obligations (CDO). Most swaps have historically been traded over the counter whereas some of them are also traded on public futures markets. In recent years however, regulations in the Basel III accords have forced swaps and particularly CDS swaps to be traded via so called central counterparty clearing houses or CCP's (bis.org 2010). So experience from the AIG incident shows that when trading in the OTC market, the credit quality of the counterparty is quite important.

1.2 Introduction to Credit Risk

Credit risk is defined as the risk that an obligor fails to meet obligations towards creditors. When this happens the obligor is said to default. A default happens for example when a company goes bankrupt, or fails to pay a coupon on one of its issued bonds in time or when a household fails to meet its amortization schedule. So credit risk and default risk are pretty much equivalent.

The credit worthiness or credit risk of an entity (be it an individual, a company, or even a whole country) is commonly assessed by a credit bureau or a rating agency which gives it a credit rating (Hull 2006). The agencies Moody's and Standards & Poor classify their ratings into brackets starting from AAA/Aaa (Standard & Poor's / Moody's) which is the highest rating proceeding with AA/Aa, A/A, BBB/Baa, BB/Ba, B/B and CCC/Caa. Each bracket is associated with a probability of default where a higher rating implies a lower probability of default (S&P Whitepaper 2009). From this rating a risk–premium is added to the interest rate of a loan or a bond that is issued by this entity. The scale of credit rating is rather coarse and the rating bureaus increase the granularity by dividing the lower brackets into subcategories (such as Aa1, Aa2, … or A+, A, A–, …). The challenge is to get a good estimate of the probability of default. Credit risk is not static. It varies over time and what is interesting to note is that for a bond with a high credit rating the default probability tend to increase with time whereas the default probability tend to decrease over time for a bond with a poor credit rating (Bodie, Kane, Marcus 2008). The reason for this is that for poor rating bonds the first couple of years may be critical whereas for high rating bonds there is the possibility that the financial health will decline over time. There are several ways to estimate the credit worthiness. The credit bureaus commonly look at financial history and the balance of assets and liabilities (Hull 2006). The downside of this is that these ratings are revised quite infrequently. So, financial institutions who deal with credit derivatives that require a more continuous assessment use more sophisticated statistical methods in their assessments.

If we ignore the influences from external market factors we have the following elements to consider involving credit risk and the modeling thereof (Schönbucher 2003):

- *Arrival risk* – This is also known as the probability of default within a given time–period.
- *Timing risk* – There is an uncertainty of the precise time of default, i.e. to know the timing of a default. In order to know the time of default one also needs to know about the arrival risk for all possible time horizons. This type of risk involves this type of uncertainty and one can say that timing risk is more detailed and specific than the arrival risk.
- *Recovery risk* – A default of an obligor doesn’t generally imply that the loss on the creditor’s behalf will be 100%. The amount the creditor can claim from a default or bankruptcy is called the recovery. The risk involved with the severity of a default given that it had occurred is therefore called recovery risk.
- *Default dependency risk* – This is the risk that several obligors default together. It is also known as default correlation risk and time has proven it to be one of the most crucial elements to consider when it comes to credit risk.

1.3 Credit derivatives and usage of swaps to manage credit risk

Like one can have insurance for one’s car or one’s house it is possible to insure a loan or a bond. The most popular way of insuring loans and bonds is through what is called credit default swaps (CDS). When one enters into a

CDS agreement one agrees to pay a stream of payments to the insurer for lending money to someone or buying obligations. Should the obligor default, the CDS agreement will terminate and the insurer will compensate for (the insured part of) whatever losses that will be incurred from the default. In derivatives terminology, as a buyer of a CDS it is said that one is *long* the premium leg (i.e. one pays the premium for the contract) and *short* the default leg (i.e. one receives payments from the contract in the event of a default) (Lando 2004). Another way to manage interest rate risk is through interest rate swaps where one can exchange a floating interest rate for a fixed rate. Managers of credit portfolios can also manage credit risk within a portfolio by buying *collateralized debt obligations* (CDO) which is a more complicated type of credit protection connected to a whole portfolio of bonds, loans or mortgages (Kane 2008). There are different types of CDOs but the most common is the synthetic CDO which is constructed synthetically from a portfolio of underlying CDS agreements. So the risk exposure in a synthetic CDO is taken on credit default swaps rather than directly on the bonds that the CDS agreements apply to. The obligations in the credit portfolio can be divided into tranches of asset classes and there are different CDOs that cover different tranches in a credit portfolio. All these insurances are classified as credit derivatives.

1.4 Counterparty Credit Risk and Credit Derivatives

In the transaction of credit derivatives (and other OTC traded derivatives) there is also a risk that the issuer of the derivatives defaults on his part of the contract and fails to honor the agreements within that contract. This type of risk is called *counterparty credit risk* (CCR) and recent events such as the financial crisis have marked their importance when dealing with credit derivatives. For years it has been a standard practice in the industry to mark portfolios of credit derivatives to market without taking this type of risk into account (Pykhtin, Zhu 2007). In the early days of the credit derivatives markets, only the financial institutions with the highest credit rating were dealing with them. They were and are offered over the counter, i.e. on the OTC market. Institutions with lower credit–worthiness were excluded entirely or had to meet additional trading requirements such as paying substantial premiums or were bound to rigid collateral terms. This was and is done by what is called margin agreements (Algoritmics Whitepapers 2011). A margin agreement limits the potential exposure by means of collateral requirements should the unsecured exposure exceed a pre–specified threshold. Whenever this threshold is exceeded, the other counterparty must supply additional collateral that is sufficient to cover this excess (Cesari, 2009). This is very much like the margin calls in a futures contract. In the early 2000's the trade with derivatives grew rapidly and the outstanding notional of derivatives transactions reached over $500 trillion by early 2008 and over $600 trillion by the end of that year. When it grows to such immensely huge amounts, it has turned out that counterparty credit risk plays a crucial role. By that time it was neglected because the financial institutions could act as clearing houses and limit the exposure by netting positive and negative positions and offset them with respect to a defaulting counterparty. While netting when used properly is a useful tool for mitigating risk, it complicates quantitative measurements significantly. Accounting for counterparty credit risk in OTC transactions is done by correcting by applying credit value adjustments (CVA). While CVA practices for adjusting with respect to counterparty credit risk are specified within the Basel II accords (Basel Committee on Banking Supervision July 2005) and in the IAS39 accounting standards, many institutions neglected this part before 2008 and therefore vastly underestimated the CCR since the exposure was with "too big to fail" counterparties. In fact, the American banks were still merely implementing the Basel I accords prior to the 2008 stock market crash. Events such as the bankruptcy of Lehman Brothers, the bailout of Bear Stearns and the AIG crisis have changed the attitude towards counterparty credit risk dramatically among investors according to a recent survey (Algorithmics Whitepapers 2011). It has been repeatedly called for an industry overhaul and stricter regulations have been pushed for in the Basel III accords since the swaps were blamed for being a chief contributor to the collapse of Lehman Brothers and the AIG. In the meantime an intensive research is done in this field and new models addressing this type of risk are being under heavy development.

As has been discussed in e.g. Canabarro, Duffie (2003) and Cesari (2009) there are the following definitions involved with counterparty credit risk:

- *Current Exposure* (CE) – This is the current value of counterparty credit exposure
- *Potential Future Exposure* (PFE) – This is a statistical measure of future exposure generated from a stochastic simulation. E.g. a 95% PFE with value 100 means that the future exposure of the forecasted horizon will not exceed 100 with 95% confidence.
- *Expected Exposure* (EE) – This is the expected value of the exposure up to the end of the forecasted period. It is in particular the expected positive exposure (EPE) that is looked at in the assessment of counterparty credit risk.

These are the tools to assess future exposure. There are several different models to estimate these exposures that are calibrated against measures such as trades agreements, legal entities, opinions, collateral holdings, limits etc, and they are commonly estimated through Monte Carlo simulations. The shape of the models is also different depending on the time horizons. It is for example generally required to add jump–diffusion processes for shorter time horizons (Das, Sundaram 1999) whereas jump diffusion becomes less important for longer time–horizons. There are mainly three ways to mitigate this type of exposure, some of which have already been mentioned in this paper:

- *Collateralization* – A margin account is set up with the counterparty and is generally managed in a fashion that is similar to margin accounts for futures positions in the public exchange market. The counterparty receives margin calls should the value of the positions go below a certain threshold and generally the overdraw should exceed a certain amount to mitigate the number of transactions to this account. So, if the overdraw limit is set to say 500 000, an overdraw by 100 000 will not generate a margin call but an overdraw of 1 000 000 will. The period that defines the frequency at which collateral is monitored and being called for is called the *call period* which is typically one day. The time interval necessary to close out the position with the counterparty and re–hedge its resulting market risk should the counterparty default is called the *cure period*. This is the period to cure the "wound" that is caused from the default of the counterparty. The total time interval from the last exchange until the defaulting counterparty is closed out is called *margin period of risk* which is the sum of the call period and the cure period. A collection of trades whose values should be added in order to determine the collateral to be posted or received with this collection is called a *margin node*.
- *Netting* – The exposure can be greatly reduced by what is called netting agreements. This agreement is a legally binding contract between two counterparties to aggregate the transactions between them in the event of default. This means that negative exposure in one contract will be offset by a positive exposure in another which can greatly reduce the overall counterparty exposure. A collection of trades within a position that can be netted is called a *netting node*.
- *Credit Value Adjustment* (CVA) – The premium or credit value is adjusted to cover the risks that are involved with the position. The adjustment can be either positive or negative depending on which party of the contract holds the highest risk. The CVA is generally calculated from PFE and/or EE estimations. The focus of this paper is on this adjustment.

Pykhtin, Zhu (2007) identifies three main components in the calculation of counterparty exposure:

- *Scenario Generation* – Future market scenarios are simulated for a given fixed set of dates using evolution models of the risk factors
- *Instrument Valuation* – For each simulation date and each realization of the underlying market risk factors, the holder's instruments are valued for each trade in the counterparty portfolio using the simulated scenarios. It should be noted that path dependent instruments such as American, Asian or Bermudan instruments require a different

approach than path independent instruments such as European instruments.

- *Portfolio Aggregation* – For each simulation date and for each realization of the underlying market risk factors, counterparty level exposure is obtained by aggregating the portfolio according to the holders' netting agreements

There is also another element of risk involved with counterparty exposure called right–way risk and wrong–way risk. This type of risk is involved with the credit quality of the counterparty. It is *wrong way* if the exposure towards the counterparty tends to increase when the credit quality of the counterparty worsens and it is called right way if the exposure tends to decrease with the credit quality of the counterparty. A typical wrong way risk scenario is when a bank enters into a swap contract with an oil producer where the bank receives a fixed rate whereas it pays the oil producer the floating crude oil price. Decreasing oil prices in this scenario will worsen the credit quality of the oil producer and increase the value of the swap to the bank. So the bank will be faced with a wrong way risk scenario as the swap goes the wrong way for the oil producer. A *right way* risk scenario will be faced by the bank if it instead takes the floating rate and pays the fixed rate to the oil producer. This will be beneficial to the oil producer (although his credit quality will worsen) as it goes the "right way" whereas it will be less beneficial for the bank. The emphasis with this type of risk is that in either way, the bank will be faced with a risk exposure within its swap position. There is no way for the bank to benefit from the increasing value of the swap if the oil producer defaults on his payments.

2 Modeling framework

Credit risk modeling is usually done by stochastic models that in one form or another use stochastic processes that capture the fluctuating nature of factors such as stock prices, interest rates etc. that influence the nature of the credit risk within a given entity. We discuss this further in Section 2.1.

The most common way of assessing credit risk is by using so called intensity models which we describe in Section 2.2. The results in intensity models often translate into a credit spread that can be put directly on top of a given interest rate in the valuation of financial instruments such as credit risk insurance policies of a given debt.

Interest rate swaps can be modeled in different ways depending on what factors that are to be taken into account. As we will see in Section 2.3, the idea is to identify the stream of payments within an interest rate agreement as a set or portfolio of more rudimentary forward rate agreements and value them according to what is a fair rate of the entire swap.

Section 2.4 discusses the how credit default risk is valued in general and Section 2.5 describes how an interest rate swap is valued in the presence of counterparty credit risk with the framework of Section 2.4 in mind. By assuming that the counterparty credit risk is stochastically independent from the underlying interest rate of an IRS agreement we can resort to a simplified framework that calculates the default probabilities and the expected cash flows from the IRS agreement separately using the basic Black–Scholes model for swaptions.

Section 2.6 introduces a few concepts to further enhance the valuation of the counterparty risk adjustment. We discuss an approach to accounting for when the interest rate and the default intensity follow a stochastic process and are correlated. We consider the case for when the interest rate follows a bivariate G2++ process and when the default intensity follows a CIR++ process. Calibration procedures for these models are also discussed and this Section is ended with a discussion about numerical methods for simulation which will see are required when there is a correlation between the interest rate and the default intensity.

2.1 Stochastic modeling

Credit risk is associated with a probability of default and we want to estimate this probability in such a way that we can find a premium to add on top of e.g. a lending rate. This is why so called intensity models have become popular. Using stochastic processes in the intensity models works very well but they add to the complexity of the calculations. The complexity of a credit risk model increases dramatically with the number of factors to be accounted for. The simpler models even have closed form expressions that one can use directly to calculate the credit risk. The more complex models don't have such solutions and therefore require simulations. Stochastic models such as the CIR model have become popular to use because they have properties that have been recognized in observations of e.g. interest rate movements. The problem however is that they are difficult to fit to real world data and are not as flexible as one would want to wish for.

Credit risk modeling in its traditional form generally assumes that there is a stock value representing the obligor that follows some form of stochastic process. The event of default is then defined as the point when this stock value dips below a certain threshold, usually representing the amount of debt held by the obligor.

An important and ground breaking credit risk model is the so called Merton model (Lando 2004) where it simply looks at debt as a European put option and the stock as a European call option. It is easy to use simply because it has closed–form solutions which are found through the Black–Scholes formula. The downside of this is that the Merton framework is simplified and doesn't account for all of the elements involved with credit risk, particularly default dependency risk. At the heart of this framework is the Wiener process which is a process where all increments are stochastically independent, i.e. they are not correlated. A way to make the Merton and the Black–Scholes model account for default dependency risk is to base it upon a process where the increments have some degree of correlation. This makes calculations more complicated and in most cases there won't be any closed form solutions. Also, some processes that in such calculations would replace the Wiener process in the Black–Scholes model don't have finite variance (i.e. Var(X_t) = ∞) or second order moment (E[X_t^2] = ∞) which would complicate things even further. Such processes with infinite variance are called leptokurtic because their increments have a fat–tailed distribution which is a consequence of the infinite variance. The fat–tail nature of these distributions makes them appealing to use to capture unforeseeable events such as stock market crashes or other observed phenomena that are not mathematically well–behaved.

2.2 Intensity Modeling

In this Section we study the assessment of credit risk using intensity models. We begin by explaining stopping–times which is the foundation of intensity models. We then show how an intensity translates into a probability and how it can be applied e.g. in the valuation of a risky bond. Intensities are commonly assumed to be piecewise constant and are usually calibrated against CDS–quotes that are observed on the market. The method of calibrating a piecewise constant intensity against market CDS–spreads is called *bootstrapping* which is explained further at the end of this section.

The time between the present and a given event of default is defined in stochastic calculus as the *stopping–time* (Klebaner 2005, Shreve 2008). Formally: A non–negative random variable τ given a filtration F_t is called a stopping time if for each t the event $\{\tau \leq t\} \in \mathsf{F}_t$. The term filtration is a theoretical concept in stochastic processes which in layman's terms means information, i.e. it is conditional on that a given series of events has happened up until time t.

This means that given the information in F_t it can be decided whether $\{\tau \leq t\}$ has occurred or not. So if the filtration is generated by a stochastic process $\{X_t\}$, then by observing it up to time t, from the generated values $X_0, X_1, \ldots, X_t$ we can decide whether the event $\{\tau \leq t\}$ has or has not occurred.

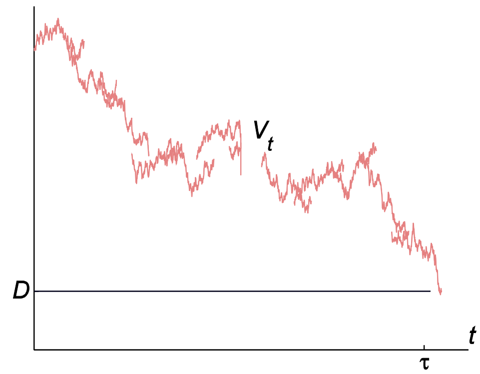

Figure 2.1: The stochastic variable τ is defined as the first time a given stochastic process V_t crosses the threshold D and is called a stopping time. A stopping time can be used to denote e.g. a default time which then is the time point when the value of a company goes below the value of its debt.

As can be seen in **Figure 2.1**, the stopping time τ is the time at which the stochastic process V_t hits the threshold D. If we look at a capital structure of a company, the stochastic process represents the market value of the company and the threshold is the amount of debt outstanding. In this framework the stochastic process is above the threshold and the default occurs at the point when the process intersects this threshold.

A popular way of modeling credit risk is through what is called the intensity based approach (Schönbucher 2003, Lando 2004, Brigo, Mercurio 2006). The intensity is defined as a probability of default within a given (infinitesimal) time period Δt, i.e. with respect to a given filtration $\mathcal{F}_t$ and a default time $\tau > t$. Given a stochastic process X_t and an mapping $\lambda_t(X_t)$ of that process where $\lambda(X_t) \in [0, \infty)$, the default probability within an infinitesimal time period dt is

$$\lim_{\Delta t \to 0} \mathrm{P}\left[\tau \in [t, t + \Delta t) \middle| \mathcal{F}_t \right] = \lambda(X_t)\, dt \text{, on } \tau > t \tag{2.1}$$

and the stopping time τ in the intensity modeling framework is defined as

$$\tau = \inf_{t \geq 0} \left\{ \int_0^t \lambda(X_s)\, ds \geq E \right\} \tag{2.2}$$

for an exponentially distributed random variable E that is independent of X_t. The mapping $\lambda_t(X_t)$ is also known as the intensity for the random variable τ. It should be noted that this intensity is a conditional parameter, i.e. it is a measure of default probability at time t conditional on no earlier default. So $\lambda(t)\Delta t$ is the probability of default between time t and $t + \Delta t$ conditional on no earlier default. In this framework it can be shown that the survival probability (the complement of the default probability) up to time *t* is

$$\mathrm{P}\left[\tau > t\right] = \mathrm{E}\left[\exp\left(-\int_0^t \lambda\left(X_s\right)ds\right)\right]. \tag{2.3}$$

The stochastic process X_s can be chosen arbitrarily and it can be designed to account for any type of risk involved with credit risk. Of course, the more factors that are involved, the more difficult it will be to make a sensible estimation. There is a lot more to say about this but that would reach beyond the scope of this paper. The beauty of this formula can be illustrated by assuming that the intensity is deterministic and constant with respect to time: Let $P\left(0,t\right)$ be the present value (at time 0) of a risk–free bond with a constant continuously compounded interest rate r that pays one unit of a given numeraire at a future time *t*. Assuming that the interest rate is constant a risky bond $\tilde{P}\left(0,t\right)$ with default probability $P\left[\tau < t\right]$ can then be valued as $\tilde{P}\left(0,t\right)$ = $e^{-(r+\lambda)t}$ since

$$\begin{aligned}\tilde{P}(0,t) &= E\left[P(0,t)\mathbf{1}_{\{\tau>t\}}\right] = E\left[P(0,t)\right]E\left[\mathbf{1}_{\{\tau>t\}}\right] \\ &= P(0,t)\,\mathrm{P}\left[\tau > t\right] = \exp(-rt)\,\mathrm{E}\left[\exp(-\lambda t)\right] = e^{-(r+\lambda)t}.\end{aligned} \tag{2.4}$$

In this case the intensity λ is a kind of a risk–premium over the risk–free rate that is estimated from credit risk assessments and P[τ > *t*] is the survival probability of the risky bond, i.e. the probability that it will *not* default prior to *t*. One very common implementation of λ is letting it follow a piecewise constant function as it is easier to calculate, computationally less intensive than stochastic functions and easier to properly fit with real world data than for example CIR models. The usage of piecewise constant default intensities is very common in the financial industry when calibrating a default probability distribution from market quotes of CDS–spreads.

The easiest way to arithmetically define a piecewise constant function is to formulate it using the indicator function[2]. So we have

$$\begin{aligned}\lambda(t) &= \sum_{j=1}^{N} \lambda_j \mathbf{1}_{\{\tilde{T}_{j-1} \le t < \tilde{T}_j\}} \\ &= \sum_{j=1}^{N} \lambda_j \cdot \left(\mathbf{1}_{\{t-\tilde{T}_{j-1}>0\}} - \mathbf{1}_{\{t-\tilde{T}_j>0\}}\right)\end{aligned} \tag{2.5}$$

[2] An indicator function $\mathbf{1}_X$ is a function that has the value 1 if X (e.g. x < 5 or A $\subseteq$ B) is satisfied and 0 otherwise. A notable result is that the expected value of an indicator function is the probability for X to happen, i.e. E[$\mathbf{1}_X$] = P[X].

for a set of N time intervals with N intensity values for each time–period. Let $m = m(t)$ be the largest interval where $t \notin [0, \tilde{T}_m]$, i.e. $m(t) = \max_j \{t \notin [0, \tilde{T}_j]\}$ for $t \in (0, \infty)$. The situation is illustrated in **Figure 2.2**.

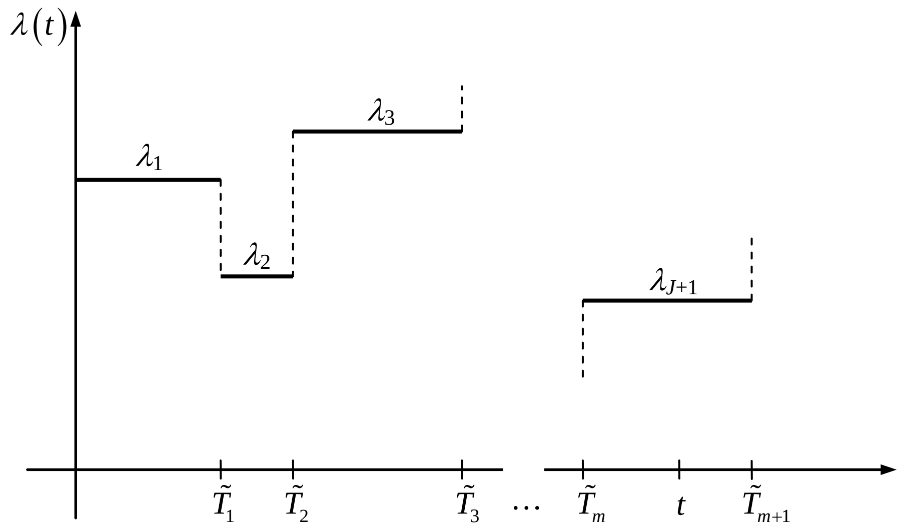

Figure 2.2 The intensity function $\lambda(t)$ as a piecewise constant function where $\tilde{T}_m$ encloses the largest interval $[0, \tilde{T}_m]$ that does not contain t.

From this we have the survival probability function

$$\mathrm{P}[\tau > t] = \exp\left[-\sum_{j=1}^{m(t)} \lambda_j \left(\tilde{T}_j - \tilde{T}_{j-1}\right) - \lambda_{J+1}\left(t - \tilde{T}_J\right)\right] \tag{2.6}$$

which is a result from the integration in Equation (2.6). The use of a piecewise constant intensity has the obvious drawback of being discontinuous. Using instead a piecewise linear intensity has proven to sometimes yield strange results when extrapolating up to 20 years (in some cases also with negative probabilities) (Brigo, Pallavicini 2008). What we can observe from real world data is the CDS spreads on CDS contracts for different maturities. If we assume that **a**) the accrued premium term is ignored and **b**) at a default τ in the period $\left[\frac{n-1}{4}, \frac{n}{4}\right]$, the loss is paid at time $t_n = t/4$, i.e. at the end quarter instead of immediately at τ, we have the following formula for calculating a CDS spread given the probability above using piecewise constant intensity

$$R(T) = \frac{(1-\phi)\sum_{n=1}^{4T} D(t_n)\left(F_\tau(t_n) - F_\tau(t_{n-1})\right)}{\frac{1}{4}\sum_{n=1}^{4T} D(t_n)\left(1 - F_\tau(t_n)\right)} \tag{2.7}$$

where ϕ is the recovery given default which is a percentage of the loss that can be recovered from the default should it happen, $D(t_n)$ is a discount factor and

$F_\tau(t_n)$ is the cumulative distribution function that represents the default probability which is $1 - \mathrm{P}[\tau > t]$, i.e. $F_\tau(t_n) = \mathrm{P}[\tau \le t]$. The cash–flows that are exchanged in a CDS–swap agreement is illustrated in **Figure 2.3**.

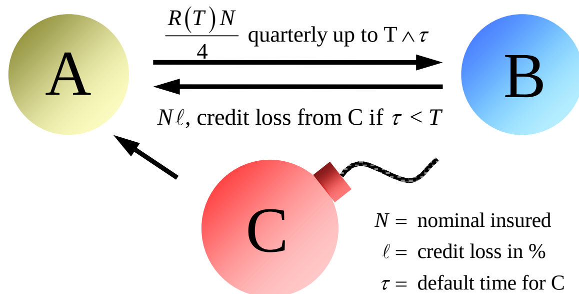

Figure 2.3: Cash–flows between parties in a CDS–agreement. Here, the protection buyer A pays B a quarterly fee given that C has not defaulted. If C defaults which happens when $\tau < T$, B pays A for the credit loss incurred by C which is $N\ell$.

Source: Alexander Herbertsson's Lecture notes

We can estimate the intensity parameters $\{\lambda_j\}_{j=1}^{N}$ by recursively finding intensity values that yield a CDS spread $R(T)$ that matches the real world data, starting at the first time interval, moving on to the next and so on. This way of estimation is called bootstrapping.

Bootstrapping is a method of calibrating the intensity towards observed market quotes of CDS spreads. The intensity is assumed to be deterministic and piecewise constant for different periods of time, see **Figure 2.2**. If we have observed CDS spreads for credit default swaps with J different maturities, the intensity is then given by

$$\lambda(t) = \begin{cases} \lambda_1 & \text{if } 0 \le t < \tilde{T}_1 \\ \lambda_2 & \text{if } \tilde{T}_1 \le t < \tilde{T}_2 \\ & \vdots \\ \lambda_{J-1} & \text{if } \tilde{T}_{J-2} \le t < \tilde{T}_{J-1} \\ \lambda_J & \text{if } \tilde{T}_{J-1} < t \end{cases} \tag{2.8}$$

for some term–structure $\mathrm{T} = \{\tilde{T}_1, \ldots, \tilde{T}_J\}$ and a given set of constant values $\{\lambda_1, \ldots, \lambda_J\}$. Starting with the lowest maturity we find a value λ_1 for the intensity such that $R(\tilde{T}_1) = R_M(\tilde{T}_1)$ as of Equation (2.7) for an observed CDS spread $R_M(\tilde{T}_1)$ of a CDS agreement with maturity $\tilde{T}_1$. In the next step we find a value λ_2 given the prior value λ_1 that yields the same value as the observed market CDS spread for a CDS contract with maturity $\tilde{T}_2$, i.e. satisfies $R(\tilde{T}_2)\big|_{\lambda_1} = R(\tilde{T}_2 \mid \lambda_1) = R_M(\tilde{T}_2)$. Then we proceed in a recursive manner to estimate the rest of the intensity values from the J observed CDS spreads given the prior estimations, i.e. in each iterative step we find a λ_j such that $R(\tilde{T}_j \mid \lambda_1, \lambda_2, ..., \lambda_{j-1}) = R_M(\tilde{T}_j)$ until $j = J$. It should be noted that a CDS contract $R_M(\tilde{T}_j)$ does not depend on $\{\lambda_{j+1}, ..., \lambda_J\}$ since it stops at maturity $\tilde{T}_j$.

2.3 IRS Valuation framework

A forward interest rate agreement (FRA) is legally binding agreement where the interest rate of a given debt is fixed so that it is constant until maturity. At the time of maturity one receives an interest that is accrued between a future start date (Brigo calls it date of expiry) and the date of maturity. So an FRA is effectively a swap agreement where a floating rate interest payment is swapped for a fixed rate payment which is settled at the time of maturity. The situation is illustrated in **Figure 2.4**.

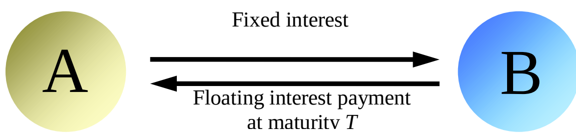

Figure 2.4: Party A enters an agreement with party B to pay a fixed interest rate payment in exchange for a floating payment at maturity T. This type of agreement is called forward rate agreement or FRA in short.

The difference between an interest rate swap and a forward interest rate agreement is that whereas a swap involves a stream of payments until maturity there is only one payment in a forward rate agreement which occurs at maturity. If we let T be the start date, S be the date of maturity, where $T < S$, $\tau(T, S)$ be a metric that measures the time between T and S in number of years (this metric is also known as year fraction) and let N be the nominal value of the contract, then the value of the contract is

$$N\tau(T,S)\big(K - L(T,S)\big)$$

where $L(T, S)$ is the spot rate resetting at time T and maturing at time S, and K is an agreed upon fixed rate. The value of the contract is thusly the difference between the two rates accrued over a given time period $\tau(T, S)$. When entered into at some time $t < T$, the fixed rate K is usually set so that the expected value of the contract is zero and the value is discounted by a factor $P(t, T)$. The function $P(t, T)$ represents the value of a bond at time $t < T$ that pays one unit of currency at time T. The floating rate $L(t, T)$ is the simply compounded spot interest rate at time t for the maturity T (i.e. this rate may have a term–structure) and is defined by the formula

$$L(t,T) = \frac{P(T,T) - P(t,T)}{\tau(t,T)P(t,T)} = \frac{1 - P(t,T)}{\tau(t,T)P(t,T)} \tag{2.9}$$

since $P(T, T) = 1$. This is the same formula as when calculating the return from a stock over time and converting it to the effective annual return by dividing it by a year–fraction term. The so called LIBOR rates are typically compounded this way which is why Brigo habitually denotes this rate by L (Brigo, Pallavicini 2008).

For the forward contract to be rendered fair, its value should be set to zero. Thus if we set the rate K equal to the forward rate F that makes the forward rate agreement zero, we have for a forward rate agreement FRA(T, S) between T and S that

$$\begin{aligned}\mathrm{FRA}(T,S) &= N\tau(T,S)\big(F - L(T,S)\big) = 0 \\ &= N\tau(T,S)\left[F - \frac{1-P(T,S)}{\tau(T,S)P(T,S)}\right] \\ &= N\left[\tau(T,S)F - \frac{1}{P(T,S)} + 1\right].\end{aligned} \tag{2.10}$$

The agreement is set up so that it is zero, both at the time of entry and at maturity. This is because the no–arbitrage argument implies that the payoff of the FRA is zero iff the current value is zero. If we consider the agreement in Equation (2.10) but at a future time point t where $t < T$, then we can define the discounted FRA(t,T,S) prevailing at t as

$$\mathrm{FRA}(t,T,S) = P(t,S)N\tau(T,S)F - P(t,T) + P(t,S). \tag{2. 11}$$

$$FRA(t,T,S) = P(t,S)FRA(T,S).$$

From the arbitrage argument (Brigo, Mercurio 2008) we can break up the discount factor $P(t, S)$ such that

$$P(t,S) = P(t,T)P(T,S)$$

which in (2. 11) yields

$$\mathrm{FRA}(t,T,S) = P(t,S)N\tau(T,S)F - P(t,T) + P(t,S). \tag{2.12}$$

By setting Equation (2.12) to zero just like Equation (2.9), we have the simply compounded forward interest rate $F(t, T, S)$ prevailing at time t with start date T, and maturity S defined as

$$F(t,T,S) = \frac{1}{\tau(T,S)}\left[\frac{P(t,T)}{P(t,S)} - 1\right]. \tag{2.13}$$

Mathematically, an interest rate swap can therefore be seen as a portfolio of forward rate agreements where the maturity of one FRA is the start date of another, i.e. we can formulate a payer IRS as

$$\text{PFS}\left(t,\left[T_\alpha,T_\beta\right],N,K\right)=\sum_{i=\alpha+1}^{\beta}P\left(t,T_i\right)N\tau\left(T_{i-1},T_i\right)\left(K-F\left(t,T_{i-1},T_i\right)\right) \qquad (2.14)$$

where the effective period of the IRS is between T_α and T_β. Hence, the fair rate at time t that sets the value of this swap agreement to zero is

$$S_{\alpha,\beta}\left(t\right)=S\left(t;T_a,T_b\right)=\frac{P\left(t,T_\alpha\right)-P\left(t,T_\beta\right)}{\sum_{i=\alpha+1}^{\beta}\tau\left(T_{i-1},T_i\right)P\left(t,T_i\right)}. \qquad (2.15)$$

This formula doesn't say anything at all about the value of the contract within the duration of this agreement, it merely states that the value is zero at the time of entry and at the time of maturity. Since we cannot be sure that both parties will honor their commitments to such a swap agreement we want to know the potential credit loss that would be incurred for one of the parties at the default of the other. The continuous valuation of an established swap agreement can be done by looking at a swap as a swap option or swaption.

A *swaption* is an option giving the buyer of it the right but not the obligation to enter into a swap agreement at a future time. We usually agree that it is European, i.e. that the option can only be exercised at the given date of maturity. Since there is no incentive for a swaption holder to exercise it if the net present value of the swap is negative, the payoff function is positive and we can treat it as a call option where we can value it using Black–Scholes formula. A payer swaption is then valued as

$$\text{PS}_{\alpha,\beta}\left(t,K,S_{\alpha,\beta}\left(t\right),\sigma_{\alpha,\beta}\right)=N\,\text{Bl}\left(K,S_{\alpha,\beta}\left(t\right),\sigma_{\alpha,\beta}\sqrt{T_\alpha},1\right)\sum_{i=\alpha+1}^{\beta}\tau_i P\left(t,T_i\right) \qquad (2.16)$$

where $\tau_i = \tau(T_{i\text{-}1}, T_i)$ and Bl represents Black–Scholes formula

$$\mathrm{Bl}\left(K, S_{\alpha,\beta}(t), \sigma_{\alpha,\beta}\sqrt{T_\alpha}, \omega\right) = S_{\alpha,\beta}(t)\,\omega\Phi\left(\omega d_1\left(K, S_{\alpha,\beta}(t), \sigma_{\alpha,\beta}\sqrt{T_a}\right)\right)$$
$$-K\omega\Phi\left(\omega d_2\left(K, S_{\alpha,\beta}(t), \sigma_{\alpha,\beta}\sqrt{T_a}\right)\right)$$
$$d_1\left(K, S_{\alpha,\beta}(t), \sigma_{\alpha,\beta}\sqrt{T_a}\right) = \frac{\ln\frac{S_{\alpha,\beta}(t)}{K} + \frac{\left(\sigma_{\alpha,\beta}\sqrt{T_a}\right)^2}{2}}{\sigma_{\alpha,\beta}\sqrt{T_\alpha}} \tag{2.17}$$
$$d_2\left(K, S_{\alpha,\beta}(t), \sigma_{\alpha,\beta}\sqrt{T_a}\right) = \frac{\ln\frac{S_{\alpha,\beta}(t)}{K} - \frac{\left(\sigma_{\alpha,\beta}\sqrt{T_a}\right)^2}{2}}{\sigma_{\alpha,\beta}\sqrt{T_\alpha}}$$

where Φ is a standard Gaussian cumulative distribution, K the strike price, and $\sigma_{\alpha,\beta}$ the volatility at the fair price $S_{\alpha,\beta}(t)$ as given in Equation (2.15). The extra coefficient ω is mainly used to mark the sign of the strike part of the payer swaption where ω is +1 for a payer swaption and –1 for a receiver swaption. For a more thorough treatment, see Brigo, Mercurio 2008.

2.4 General valuation of counterparty risk

The general procedure for valuing a cash flow in the presence of a counterparty default risk involves adding a premium term that represents this type of risk. We let the filtration F_t denote all observable market quantities but the default event up to time t. Moreover we let $\mathsf{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$ be the right–continuous filtration generated by the default event. From this we set $\mathsf{G}_t := \mathsf{F}_t \vee \mathsf{H}_t$ (so F_t is a sub–filtration of G_t, i.e. $\mathsf{F}_t \subseteq \mathsf{G}_t$) and $\mathrm{E}_t[.] := \mathrm{E}[.|\mathsf{G}_t]$. If we let $\Pi^D(t, T)$ (which we also abbreviate to $\Pi^D(t)$) be a discounted payoff function of a generic defaultable claim and $\mathrm{CF}(t, T)$ be the cash flows between time t and T from a contingent claim without counterparty risk, then the net present value of these cash flows at the default time τ is defined as $\mathrm{NPV}(\tau) = \mathrm{E}_\tau[\mathrm{CF}(\tau)]$ for the filtration G_τ at τ, and

$$\Pi^D(t) = 1_{\{\tau \geq T\}} CF(t,T) + 1_{\{t \leq \tau \leq T\}} \left[CF(t,\tau) + D(t,\tau)\left(\phi(NPV(\tau))^+ - \left(-NPV(\tau)^+\right)\right)\right] \tag{2.18}$$

where $D(u, v)$ is a stochastic discount factor at time u with maturity v, $\mathbf{1}_A$ is an indicator function for the event A and ϕ is the recovery given default. The function x^+ is the maximum of x and 0, i.e. $x^+ = \max(0, x)$ and is commonly used to describe the payoff of a put or a call option. In this setting we assume that the counterparty default risk is unilateral, i.e. we assume that only one counterparty may default whereas the other counterparty is assumed to be risk–free. The rationale of this is that if the counterparty doesn't default, then the cash flows will go as usual until maturity T. Should a default happen at a time prior to maturity, i.e. $\tau < T$, then the cash flows will stop at that point of time and if the net present value of the contract is positive, only a certain percentage of what is left of the counterparty will be recovered. If the net present value is negative, the entire value has to be paid to the counterparty if it defaults.

In this framework it can be shown that the expected payoff of a defaultable claim is given by the following formula (Brigo, Mercurio 2008):

$$\mathrm{E}_t\left[\Pi^D(t)\right] = \mathrm{E}_t[\Pi(t)] - \underbrace{L_{GD}\, \mathrm{E}_t\left[1_{\{t<\tau\leq T\}} D(t,\tau)(NPV(\tau))^+\right]}_{\text{Positive counterparty risk adjustment}} \tag{2.19}$$

where L_{GD} is loss given default which is the same as $(1 - \phi)$ which is assumed to be deterministic. This is a general model and it is not specified in the model how the default risk probability is constructed. In this thesis we construct the counterparty default risk probability from default intensity models. The focus in this thesis is on this counterparty risk adjustment term.

2.5 Counterparty Risk and IRS Valuation

Let the value of a risk–free interest rate swap agreement be IRS(t). In the general framework of Equation (2.19) we can then express the value of a defaultable swap agreement $\mathrm{IRS}^{\mathrm{D}}(t)$ as

$$\mathrm{IRS}^{D}(t) = \mathrm{IRS}(t) - L_{GD} \cdot \mathrm{DP}(t). \tag{2.20}$$

where the adjustment term DP(t) is defined as (Brigo, Masetti 2005)

$$\begin{aligned} \mathrm{DP}(t) &= \mathrm{E}_t\left[1_{\{t<\tau\le T\}} D(t,\tau)(NPV(\tau))^{+}\right] \\ &= \int_{T_a}^{T_b} \mathrm{PS}_{s,T_b}\left(t, K, S(t;s,T_b), \sigma_{s,T_b}\right) d_s\mathbf{Q}(\tau \le s) \end{aligned} \tag{2.21}$$

where PS is the value of the payer swap as a swaption and K is the forward swap rate rendering the contract fair at t, i.e. $K = S(t;\, T_a,\, T_b)$ as in Equation (2.15). Since we are going to express the values in terms of swap rates, the DP(t) adjustment will be in terms of a spread and not a net present value in the regular sense as in Equation (2.19). If we assume that the intensity and the cash–flows are independent, the calculations become straightforward. We can simplify further without notable loss of accuracy by assuming that defaults only occur at the points of payment T_i. In this setting we can either assume that the default occurs before the last payment (the payoff is then said to be postponed) or after the last default (Brigo denotes this as anticipated default). Under these assumptions we have for the postponed payoff (P) and the anticipated default (A) respectively that

$$\begin{aligned} \mathrm{DP}^{P}(t) &= \sum_{i=a+1}^{b-1} \mathbf{Q}_t\{\tau \in (T_{i-1}, T_i]\}\, \mathrm{PS}_{i,b}\left(t; K, S_{i,b}(t), \sigma_{i,b}\right) \\ &= \sum_{i=a+1}^{b-1} \left(\mathbf{Q}_t(\tau > T_{i-1}) - \mathbf{Q}_t(\tau > T_i)\right) \mathrm{PS}_{i,b}\left(t; K, S_{i,b}(t), \sigma_{i,b}\right) \end{aligned} \tag{2.22}$$

where $\mathrm{PS}_{i,b}(t;K,S_{i,b}(t),\sigma_{i,b})$ is defined as in (2.16) and

$$\begin{aligned} \mathrm{DP}^{A}(t) &= \sum_{i=a+1}^{b} \mathbf{Q}_t\{\tau \in (T_{i-1}, T_i]\}\, \mathrm{PS}_{i-1,b}\left(t; K, S_{i-1,b}(t), \sigma_{i-1,b}\right) \\ &= \sum_{i=a+1}^{b} \left(\mathbf{Q}_t(\tau > T_{i-1}) - \mathbf{Q}_t(\tau > T_i)\right) \mathrm{PS}_{i-1,b}\left(t; K, S_{i-1,b}(t), \sigma_{i-1,b}\right). \end{aligned} \tag{2.23}$$

Here $\mathbf{Q}_t$ is the conditional risk–neutral probability measure for the default probabilities of the counterparty, that is $\mathbf{Q}_t(.)=\mathbf{Q}_t(. |\mathcal{F}_t)$. These probabilities are computed from the bootstrapped piecewise constant intensity function and the payer swap is calculated using Black–Scholes formula for a European option with a payer swap payoff. For more details on the derivation of (2.22) and (2.23) see Brigo Masetti 2005.

2.6 Enhancing counterparty risk valuation

The methods presented previously for valuing counterparty risk are simplified and rather crude. In this section we will discuss methods for enhancing the valuation of this adjustment. One important enhancement is to account for stochastic dependence between the interest rate and default intensity. Accounting for this dependence tends to lead to complex calculations and a limited set of options if we seek to yield mathematically feasible expressions. We conduct a further discussion about this in Subsection 2.6.1. In [Brigo Pallavicini 2006] a valuation of the counterparty risk in the presence of correlation is assessed. There it is suggested to let the interest rate follow a G2++ process and the default intensity follow a CIR++ process. We consider these cases in Subsection 2.6.2 and 2.6.3 and discuss how they can get correlated in Subsection 2.6.4. Correlated processes do not generally lead to closed form solutions so we need to conduct numerical simulations to be able to assess the counterparty risk adjustment under these settings. In Subsection 2.6.5 at the end of this section we therefore introduce some numerical methods such as the Euler–Maruyama method and the Milstein method for numerical simulation.

2.6.1 Risk valuation under stochastic dependence

We can enhance the counterparty risk adjustment term further in the IRS valuation by assuming that there is some degree of stochastic dependence between interest rates and the counterparty risk. This can be done by assuming that the stochastic process behind the interest rate and the stochastic process behind the counterparty risk intensity have some degree of correlation. In [Lando 2004] and [Brigo, Mercurio 2006] a modeling of a joint behaviour between interest rates and default intensities are suggested by incorporating correlated Brownian motions as drivers behind these processes. Two approaches are suggested to introduce this correlation. The first approach is to tie the correlation with the noise term. If we assume that the interest rate r_t and the default intensity λ_t follow a mean–reversing process then the joint behaviour can be modeled as

$$dr_t = \kappa_r (r_t - \theta_r) dt + \sigma_r dW_t^1$$
$$d\lambda_t = \kappa_\lambda (\lambda_t - \theta_\lambda) dt + \sigma_\lambda \left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2 \right)$$

where the correlation is introduced through the correlation term ρ and the independent Brownian motion processes W_t^1 and W_t^2. For each of these two processes, the mean–reversion parameter κ then determines how quickly the impact from the fluctuations of the diffusion term dissipates, the parameter θ determines the mean level or equilibrium value the process reverts towards and the parameter σ determines the sensitivity to the fluctuations or the Brownian motion processes above. The other suggested approach is to model the correlation through an affine dependence between the default intensity and the interest rate, i.e. as

$$dr_t = \kappa_r (r_t - \theta_r) dt + \sigma_r dW_t^1$$
$$\lambda(t) = \alpha + \beta r_t$$

When using such models one has to account for negative interest rates and intensities. Whereas a negative interest rate is consistent with the no–arbitrage/risk–neutral assumption, a negative default intensity doesn't make any sense at all. Lando suggests using an analytical approximation where the intensity is set to zero whenever the intensity process becomes negative. Another approach is to let the intensity follow a quadratic Gaussian process or use CIR–like specifications as CIR processes are nonnegative.

If we revisit the credit value adjustment term in Equation (2.19) we have

$$\begin{aligned} L_{GD} E_t \left[1_{\{t<\tau\le T\}} D(t,\tau) (NPV(\tau))^+ \right] &= L_{GD} E_t \left[\{\mathbf{I}\} \{\mathbf{II}\} \{\mathbf{III}\} \right] \\ = L_{GD} E_t \Big[&\{ \textit{the condition that default time } \tau \textit{ has occurred at } (t,T] \} \\ &\{ \textit{discount from } t \textit{ to } \tau \} \{ NPV^+ \textit{ at } \tau \} \Big] \end{aligned} \qquad (2.24)$$

where the default time in **I** is commonly modeled through a default intensity and is strongly linked to the discount factor **II**. The net present value **III** of the underlying asset given default is the present value of the interest rate agreement from the default time until final maturity of the agreement. As we have stated before, the value of the agreement only has an impact on the adjustment term when it is positive, otherwise it is zero since there is no loss to recover in the counterparty risk adjustment. The net present value of an interest rate swap agreement is modeled as a so called swaption, i.e. a European option that gives the holder the right but not the obligation to enter the current swap agreement at time τ. The no–arbitrage assumption allows us to discount using zero–coupon bond valuations. The price $P(t, T)$ at time t of a zero–coupon bond maturing at T with a unit face value and an interest rate following the G2++ process above can then be evaluated through

$$P(t,T) = E\left\{\exp\left(-\int_t^T r(s)\,ds\right)\middle|\mathcal{F}_t\right\} \tag{2.25}$$

see [Brigo, Mercurio 2006; Section 4.2.2] for details. Note that this formula will be needed in the swaption formula below. If we consider a European swaption with strike rate X, maturity T and a nominal value N (that is $S_{\alpha,\beta}(t)|_{t=0}$ in (2.16)) which at $t_0 = T$ gives the holder the right to enter an interest rate swap agreement with payment times $\mathsf{T} = \{t_1, \ldots, t_n\}$, $t_1 > T$, where he in exchange for a fixed rate gets a floating rate, then it can be shown that the arbitrage–free price at time $t = 0$ is given by the following formula for a European swaption

$$\mathrm{ES}(t,T,\mathsf{T},N,X,\omega) = NP(t,T)E^T\left[\left\{\omega\left[1-\sum_{i=1}^{n(T)} c_i(X)P(t,t_i)\right]\right\}^+\right] \tag{2.26}$$

where ω is +1 for a payer swaption and –1 for a receiver swaption. In many concrete situations, a useful numeraire is the zero–coupon bond whose maturity T coincides with that of the derivative we want to price. In such a case, in fact $S_T = P(T,T) = 1$ which we took advantage of in Equation (2.9), so that pricing the derivative can be achieved by calculating an expectation of its payoff (divided by one). The measure associated with the bond maturing at time T is referred to as T–forward risk–adjusted measure, or more briefly as T –forward measure, and is commonly denoted by Q^T. The related expectation is denoted by E^T as we see in Equation (2.26).The pricing function $\mathrm{P}(t, t_i)$ is given in (2.25). The coupons c_i constitutes the remaining payments within the interest rate agreement, i.e.

$$\begin{aligned} c_i(X) &= \tilde{\tau}_i\left(F(t;t_{i-1},t_i) - X\right) \\ &= \tilde{\tau}_i\left(\tilde{\tau}_i^{-1}\int_{t_{i-1}}^{t_i} r(s)\,ds - X\right) \end{aligned}$$

where $\tilde{\tau}_i$ is the year fraction of the coupon payments and F is the forward rate for each time interval, which is the interest rate over that interval. Note that this formula is model independent, i.e. it doesn't make any assumptions on the behaviour of the underlying interest rate.

In the case where the default intensity and the interest rate are independent we assumed that the underlying asset followed a geometric Brownian motion and could therefore apply Black–Scholes formula on a replicating portfolio of forward agreements that constitutes the remainder of the swap payments until final maturity. So we ended up with a sum of Black–Scholes calculations where the payoff function is modified to represent the payoffs until the final maturity. If we assume that the underlying asset behaves in some other way than the geometric Brownian motion, we may still treat it as a swaption but we can no longer use the Black–Scholes framework to evaluate it. In such case we have to find another way to evaluate the swaption. Even though we may have stochastic dependence between the default intensity interest rate it may be possible to make some simplifying assumptions so as to find a closed form solution to the swaption without considerable loss of precision in the valuation. One such assumption may be that the value of the swaption given default is independent of the default intensity. If we cannot find a closed expression we will have to resort to simulations using either Milstein or Euler methods. As we can see can see in Equation (2.24), the two drivers behind the adjustment are the default intensity and the interest rate so we shall give them a separate treatment given that they are correlated in the following sections.

2.6.2 Using bivariate interest rate

In [Brigo, Pallavicini 2006] it is suggested to use the G2++ model for the interest rates. In this subsection we will explore how incorporating this process with the interest rates will affect the valuation of the credit value adjustment while still assuming that the interest rate and the default intensity are stochastically independent. The G2 process is a bivariate Gaussian process. The "++" means that it is calibrated through a deterministic function $\varphi(t;\alpha)$ against observations in the market which in this case is the term–structure of interest rates. This function is well–defined over a time–interval $[0, T^*]$ or time–horizon T^* which could be 10, 30 or 50 years. We will apply this process in the CVA evaluation by incorporating it with a European swap. We will also construct a pricing function $P(t, T)$ from this process and show how it is used in the calibration process against observed market prices by the end of this subsection.

So, in the G2++ framework, the interest rate is modeled as a bivariate process under the risk neutral measure **Q** and given by

$$r(t) = x(t) + y(t) + \varphi(t;\alpha), \quad r(0) = r_0 \tag{2.27}$$

where α is a defined set of parameters and the stochastic processes x and y are adapted to the filtration F_t and satisfying the following SDE's

$$\begin{aligned} dx(t) &= -ax(t)\,dt + \sigma dZ_1(t), \quad x(0) = 0 \\ dy(t) &= -by(t)\,dt + \eta dZ_2(t), \quad y(0) = 0 \end{aligned} \tag{2.28}$$

where (Z_1, Z_2) is a two–dimensional Brownian motion with an instantaneous correlation $\rho_{1,2} \in [-1, 1]$ (i.e. the correlation of the Brownian increments) given as

$$dZ_1(t)\,dZ_2(t) = \rho_{1,2}\,dt$$

where r_0, a, b, σ, and η are positive constants which together with the correlation $\rho_{1,2}$ constitute the parameter set α as defined above. The function $\varphi(t;\alpha)$ can be set to a value that automatically calibrates against the initial zero coupon curve observed in the market where $\varphi(0;\alpha) = r_0$ [Brigo Mercurio 2006]. By integrating (2.28) from 0 to t and inserting the results into (2.27) we get

$$r(t)=\sigma\int_0^t e^{-a(t-u)}dZ_1(u)+\eta\int_0^t e^{-b(t-u)}dZ_2(u)+\varphi(t;\alpha).$$

For each $s < t$, the above equation becomes

$$\begin{aligned} r(t) &= x(s)e^{-a(t-s)}+y(s)e^{-b(t-s)} \\ &\quad +\sigma\int_s^t e^{-a(t-u)}dZ_1(u)+\eta\int_s^t e^{-b(t-u)}dZ_2(u)+\varphi(t;\alpha). \end{aligned}$$

The dynamics of the processes x and y can also be expressed in terms of two independent Brownian motions W_1 and W_2 such that

$$\begin{aligned} dx(t) &= -ax(t)dt+\sigma dW_1(t) \\ dy(t) &= -by(t)dt+\eta\rho dW_1(t)+\eta\sqrt{1-\rho^2}dW_2(t) \end{aligned}$$

where

$$\begin{aligned} dZ_1(t) &= dW_1(t) \\ dZ_2(t) &= \rho dW_1(t)+\sqrt{1-\rho^2}dW_2(t). \end{aligned}$$

We have seen that the credit value adjustment as is outlined in Equation (2.24) is evaluated as a payer swaption in Equation (2.21). As we shall see, the swaption formula can be extended to interest rates following G2++ processes. If we assume that the interest rate is independent of the default intensity just as in Subsection 2.3 and that the interest rate follows a G2++ process, we may evaluate Equation (2.26) by numerically computing the following expression (Brigo Mercurio 2006)

$$\begin{aligned} &\mathrm{ES}^{G2++}(0,T,\mathsf{T},N,X,\omega)= \\ &\frac{N\omega P(0,T)}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{x-\mu_x}{\sigma_x}\right]^2}\left[\Phi(-\omega h_1(x))-\sum_{i=1}^{n}\lambda_i(x)e^{\kappa_i(x)}\Phi(-\omega h_2(x))\right]dx \end{aligned}$$

where we set $\omega = +1$ since we consider a payer swaption, i.e

$$\mathrm{PS}^{G2++}(0,T,\mathsf{T},N,X)=\mathrm{ES}^{G2++}(0,T,\mathsf{T},N,X,1)$$

(as in Equation (2.17)) and

$$h_1(x)=\frac{1}{\sqrt{1-\rho_{xy}^2}}\left[\frac{\bar{y}-\mu_y}{\sigma_y}-\frac{\rho_{xy}(x-\mu_x)}{\sigma_x}\right]$$
$$h_2(x)=h_1(x)+B(b,T,t_i)\sigma_y\sqrt{1-\rho_{xy}^2}$$
$$\lambda_i(x)=c_iA(T,t_i)e^{-B(a,T,t_i)x}$$
$$\kappa_i(x)=-B(b,T,t_i)\left[\mu_y-\frac{1}{2}(1-\rho_{xy}^2)\sigma_y^2B(b,T,t_i)+\rho_{xy}\sigma_y\frac{x-\mu_x}{\sigma_x}\right]$$

$\bar{y}=\bar{y}(x)$ is the unique solution to

$$\sum_{i=1}^{n}c_iA(T,t_i)e^{-B(a,T,t_i)x-B(b,T,t_i)\bar{y}}=1$$
$$\mu_x=-M_x^T(0,T)$$
$$\mu_y=-M_y^T(0,T)$$
$$\sigma_x=\sigma\sqrt{\frac{1-e^{-2aT}}{2a}}$$
$$\sigma_y=\eta\sqrt{\frac{1-e^{-2bT}}{2b}}$$
$$\rho_{xy}=\frac{\rho\sigma\eta}{(a+b)\sigma_x\sigma_y}\left[1-e^{-(a+b)T}\right]$$

respectively. The functions $A(T,t_i)$ and $B(b,T,t_i)$ constitutes the volatility and correlation structures in two–factor models in the Heath–Jarrow–Morton (1992) framework. The mean functions $M_x^T(0,T)$ and $M_y^T(0,T)$ will be detailed by the end of this subsection through Lemma 4.2.1 below which presents the mean and variance functions of a markovian process over a time interval $[t, T]$. We will also find an explicit formula for the pricing function $P(t, T)$ that we will use in the calibration process against observed market prices $P^M(t, T)$. The deterministic function $\varphi(t;\alpha)$ in (2.27) is presented in Corollary 4.2.1 below which also outlines the proper calibration procedures for this function.

Lemma 4.2.1 [Brigo Mercurio 2006]: *For each t, T the random variable*

$$I(t,T)=\int_t^T[x(u)+y(u)]du$$

conditional to the filtration $\mathcal{F}_t$ (which is also known as a sigma–field) is normally distributed mean M(t, T) and variance V(t, T) respectively given by

$$M(t,T)=\frac{1-e^{-a(T-t)}}{a}x(t)+\frac{1-e^{-b(T-t)}}{b}y(t)$$

and

$$V(t,T)=\frac{\sigma^2}{a^2}\Xi(a)+\frac{\eta^2}{b^2}\Xi(b)+$$
$$+2\rho\frac{\sigma\eta}{ab}\left[T-t+\frac{e^{-a(T-t)}-1}{a}+\frac{e^{-b(T-t)}-1}{b}-\frac{e^{-(a+b)(T-t)}-1}{a+b}\right]$$

where

$$\Xi(\xi)=T-t+\frac{2}{\xi}e^{-\xi(T-t)}-\frac{1}{2\xi}e^{-2\xi(T-t)}-\frac{3}{2\xi}.$$

□

If we assume that the term–structure of discount factors that is observed on the market follows a sufficiently smooth function $T \mapsto P^M(0,T)$ and denote the instantaneous forward rate at time 0 for a maturity T by

$$f^M(0,T)=-\frac{\partial P^M(0,T)}{\partial T}$$

then the following corollary applies in the calibration process for the deterministic function $\varphi(t;\alpha)$ in Equation (2.27):

Corollary 4.2.1 [Brigo, Mercurio 2006]: *The model in Equation (2.28) fits the currently observed term structure of discount factors if and only if for each time of maturity T*

$$\varphi(T;\alpha)=f^M(0,T)+\frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2+\frac{\eta^2}{2b^2}\left(1-e^{-bT}\right)$$
$$+\rho\frac{\sigma\eta}{ab}\left(1-e^{-aT}\right)\left(1-e^{-bT}\right)$$

i.e. if and only if

$$\exp\left\{-\int_t^T\varphi(u)\,du\right\}=\frac{P^M(0,T)}{P^M(0,t)}\exp\left\{-\frac{1}{2}\left[V(0,T)-V(0,t)\right]\right\} \qquad (2.29)$$

such that the corresponding zero–coupon bond prices at time t are given by

$$P(t,T)=\frac{P^M(0,T)}{P^M(0,t)}\exp\{\mathsf{A}(t,T)\}$$

$$\mathsf{A}(t,T)=\frac{1}{2}[V(t,T)-V(0,T)+V(0,t)] -\frac{1-e^{-a(T-t)}}{a}x(t)-\frac{1-e^{-b(T-t)}}{b}y(t).$$

□

One important remark about the calibration is that we don't need to derive the whole φ curve. What matters is the integral between two given instants which is computed in Equation (2.29). From this expression we see that the only curve that is needed is the observed market discount curve which don't even need to be differentiated (which you do when estimating the instantaneous forward curve $f^M(0, T)$), and we only need this at times corresponding to the maturities of the market discount rates that are observed through bonds or other desired interest rates which considerably limits the need for interpolation.

Another important remark is that it can be shown that the risk–neutral probability of negative interest rates is negligible in the G2++ framework. A proof is found in [Brigo Mercurio 2006].

The functions $A(T,t_i)$ and $B(b,T,t_i)$ are then defined as

$$A(t,T)=\frac{P^M(0,T)}{P^M(0,t)}\exp\left\{\frac{1}{2}[V(t,T)-V(0,T)+V(0,t)]\right\}$$

$$B(z,t,T)=\frac{1-e^{-z(T-t)}}{z}$$

so that we can write $P(t, T)$ in (2.25) as

$$P(t,T)=A(t,T)\exp\{-B(a,t,T)x(t)-B(b,t,T)y(t)\}.$$

This can be used to understand which market–volatility structures can be fitted by the model. A more thorough treatment is conducted in [Brigo, Mercurio 2006]. Finally by using Lemma 4.2.1 we can prove that $M_x^T(s,t)$ and $M_y^T(s,t)$ are given by

$$M_x^T(s,t) = \left[\frac{\sigma^2}{a^2} + \rho\frac{\sigma\eta}{ab}\right]\left(1-e^{-a(t-s)}\right) - \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - e^{-a(T+t-2s)}\right) - \frac{\rho\sigma\eta}{b(a+b)}\left(e^{-b(T-t)} - e^{-bT-at+(a+b)s}\right)$$

$$M_y^T(s,t) = \left[\frac{\eta^2}{b^2} + \rho\frac{\sigma\eta}{ab}\right]\left(1-e^{-b(t-s)}\right) - \frac{\eta^2}{2b^2}\left(e^{-b(T-t)} - e^{-b(T+t-2s)}\right) - \frac{\rho\sigma\eta}{a(a+b)}\left(e^{-a(T-t)} - e^{-aT-bt+(a+b)s}\right)$$

We have now treated the credit value adjustment we stated in (2.24) as a payer swaption where the interest rate follows the bivariate G2++ process which is calibrated against observed market quotes via Lemma 4.2.1. This closed formula assumes that the payer swaption to be evaluated is independent of (and thusly not correlated to) the default intensity.

2.6.3 Using CIR++ based default intensity

The CIR++ process which is a CIR process that is calibrated against the credit default spreads through a deterministic function just like the G2++ is calibrated against the term–structure of zero–coupon bonds. A CIR model is chosen here because it is nonnegative. So we set

$$\lambda_t = y_t + \psi(t;\beta), \quad t \geq 0, \tag{2.30}$$

where $\psi(t,\beta)$ is a deterministic function that depends on a parameter vector β and is integrable over some closed interval $[0, T^{**}]$. The initial condition for this deterministic function is set such that it is congruent with the relation in Equation (2.30), i.e.

$$\psi(0;\beta) = \lambda_0 - y_0 .$$

The term y_t is set to follow a Cox–Ingersoll–Ross (CIR) process, i.e.

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}\,dZ_3(t) \tag{2.31}$$

where the parameter vector is $\beta = (\kappa, \mu, \nu, y_0)$ and κ, μ, ν, y_0 are positive deterministic constants such that $2\kappa\mu > \nu^2$ which ensures that the origin is inaccessible to y_t so that we won't have to worry about negative default intensities and the process remains positive. Here $Z_3(t)$ is a one–dimensional Brownian motion under the risk neutral measure just like the Brownian motion processes $Z_1(t)$ and $Z_2(t)$ in the G2++ model. The deterministic function ψ is then calibrated against observed market CDS spreads. The calibration procedure that is introduced in [Brigo Pallavicini 2008] assumes that the intensity is piecewise constant and is calibrated through a bootstrapping method similar to what is discussed in Section 2.2 that is applied to closed form expressions. If we integrate the terms in Equation (2.30) over [0, t] we get

$$\int_0^t \lambda_s ds = \int_0^t y_s ds + \int_0^t \psi(s;\beta) ds$$

or that

$$\Lambda(t) = Y(t) + \Psi(t;\beta)$$

if the functions $\Lambda(t)$, $Y(t)$ and $\Psi(t;\beta)$ are primitives of the terms λ_t, y_t and $\psi(t;\beta)$ respectively. If we assume that the default intensity is independent of the short rate r, it can be shown that the CDS valuation becomes model independent and is given by

$$\begin{aligned} CDS_{a,b}(0,R) &= R\sum_{i=a+1}^{b} P(0,T_i)\mathbf{Q}(\tau \geq T_i) - L_{GD}\sum_{i=a+1}^{b} P(0,T_i)\mathbf{Q}(\tau \in [T_{i-1},T_i]) \\ &= \sum_{i=a+1}^{b} P(0,T_i)[R\mathbf{Q}(\tau \geq T_i) - L_{GD}\mathbf{Q}(\tau \in [T_{i-1},T_i])] \end{aligned}$$

for a given protection period [T_a, T_b] where R is a periodic premium rate that is paid to the protection seller until final maturity or the first T_i following default. Here L_{GD} is the loss given default that is received by the protection buyer at the first T_i following default. It should be noted that the quoted market spread R_M is such that $\text{CDS}_{a,b}(0,R_M) = 0$ while $\text{CDS}_{a,b}(t,R_M) \neq 0$ for $0 < t < T_b$. This model allows us to extract the default probabilities inherent in the observed CDS market quotes and so we need only to make sure that these probabilities are correctly reproduced by the CIR++ model. Since the survival probabilities in the CIR++ model are given by

$$Q(\tau > t)_{model} = E\left[e^{-\Lambda(t)}\right] = E\left[e^{-\Psi(t;\beta)-Y(t)}\right],$$

the CIR++ model is calibrated by setting $\Psi(t;\beta)$ such that

$$\begin{aligned} E\left[e^{-\Psi(t;\beta)-Y(t)}\right] &= E\left[e^{-\Psi(t;\beta)}e^{-Y(t)}\right] \\ &= \exp(-\Psi(t;\beta))E\left[e^{-Y(t)}\right] = \mathbf{Q}(\tau > t)_{market} \end{aligned}$$

i.e.

$$\Psi(t;\beta) = \log\left[\frac{E\left[e^{-Y(y)}\right]}{\mathbf{Q}(\tau > t)_{market}}\right] = \log\left[\frac{P^{CIR}(0,t,y_0;\beta)}{\mathbf{Q}(\tau > t)_{market}}\right]$$

where the parameters β are chosen such that ψ is a positive function and P^{CIR} is the closed form expression for bond prices in the time–homogeneous CIR model, i.e.

$$P^{CIR}(0,t,y_0;\beta) = \exp\left(\bar{A}(t;\beta) - \bar{B}(t;\beta)\,y_0\right)$$

where

$$\bar{A}(s;\beta) = \frac{2\kappa\mu}{\nu^2}\log\left[\frac{2\gamma e^{(\gamma+\kappa)s/2}}{(\gamma+\kappa)\left(e^{\gamma s}-1\right)+2\gamma}\right]$$

$$\bar{B}(s;\beta) = \frac{2\left(e^{\gamma s}-1\right)}{(\gamma+\kappa)\left(e^{\gamma s}-1\right)+2\gamma}$$

and

$$\gamma = \sqrt{\kappa^2 + 2\nu^2}\,.$$

In [Brigo Pallavicini 2006] the market survival probability is simplified to

$$\mathbf{Q}(\tau > t)_{market} = \exp(-\zeta t)$$

for a constant deterministic value of ζ. This assumption doesn't yield precise CDS valuations but it is argued to suffice in qualitative analyses of the model. In a quantitative study it is desired to conduct a proper calibration as described above using a piecewise constant function like in (2.8). For more details see Part VII Credit in [Brigo Mercurio 2006].

This CDS calibration procedure assumes that the default risk and interest rates are stochastically independent and is therefore not applicable by principle on models that assumes a correlation between the default risk and interest rate movements. However, it has been shown in [Brigo Alfonsi 2006] that the impact of this correlation is negligible on CDS spreads which suggests that we can apply this calibration procedure even though the interest rates and the default risk are stochastically dependent.

2.6.4 Introducing cross–correlation

The stochastic dependence between the interest rate and the default intensity is introduced in the simulation of the noise terms if the interest rate process and the default intensity process by letting the Brownian motions Z_1, Z_2, Z_3 that we have defined in Sections 2.6.2 and 2.6.3 (Equations (2.27) and (2.30)) be instantaneously correlated according to

$$\begin{aligned} dZ_1(t)\,dZ_3(t) &= \rho_{1,3}dt \\ dZ_2(t)\,dZ_3(t) &= \rho_{2,3}dt \end{aligned}$$

which can be rewritten as

$$\begin{aligned} dZ_1(t) &= \rho_{1,3}dW_1 + \sqrt{1-\rho_{1,3}^2}\,dW_3 \\ dZ_2(t) &= \rho_{2,3}dW_2 + \sqrt{1-\rho_{2,3}^2}\,dW_3 \\ dZ_3(t) &= dW_3 \end{aligned} \tag{2.32}$$

which is equivalent to performing a Cholesky decomposition on the variance–covariance matrices of the pairs ($W_1(t)$, $W_3(t)$) and ($W_2(t)$,$W_3(t)$). The correlation between instantaneous interest–rate and credit–spread can then be obtained by

$$\bar{\rho} = \frac{\sigma\rho_{1,3} + \eta\rho_{2,3}}{\sqrt{\sigma^2 + \eta^2 + 2\sigma\eta\rho_{1,2}}}$$

2.6.5 Numerical simulation procedures

When we do not have closed form solutions to a stochastic differential equation such as the one in Equation (2.28) or (2.31) that we use when evaluating the credit value adjustment in Equation (2.24) via Equation (2.25) or (2.26), we have to resort to numerical methods to solve them. The simplest approach is to use the Euler method that basically breaks up the continuous time into a fine mesh of discrete time–steps upon which the solution is iterated [Kloeden Platen 1995]. The Euler method is rather crude and prone to errors although we have a convergence in the solution. One can increase the convergence rate by looking at higher order terms in a so called Taylor–Itô expansion which is what is done when finding a numerical solution to an SDE using the Milstein method. The problem formulation is as follows; consider a stochastic differential equation of the form

$$\begin{aligned} dX(t) &= \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t), \quad 0 \le t \le T \\ X(0) &= X_0 \end{aligned} \tag{2.33}$$

where $\mu(x,t)$ and $\sigma(x,t)$ are given and well–defined deterministic functions, $B(t)$ is a Brownian motion process and $X(t)$ is the unknown process to be found. If the integrals

$$\int_0^t \mu(X(s),s)\,ds \text{ and } \int_0^t \sigma(X(s),s)\,dB(s)$$

exist for all $t > 0$ then a process $X(t)$ as defined by the following equation

$$X(t) = X(0) + \int_0^t \mu(X(s),s)\,ds + \int_0^t \sigma(X(s),s)\,dB(s) \qquad (2.34)$$

is a so called strong solution to the SDE in Equation (2.33). The second integral is a so called Itô–integral which is also known as a Stieltjes integral [Evans 2011]. The Stieltjes integral is a generalization of the ordinary Riemann integral to make it applicable on stochastic processes. The Stieltjes Integral of f with respect to a monotone function g over an interval $(a, b]$ is defined as

$$\int_a^b f dg = \int_a^b f(t)\,dg(t) = \lim_{\delta\to 0}\sum_{i=1}^{n} f\left(\xi_i^n\right)\left(g\left(t_i^n\right) - g\left(t_{i-1}^n\right)\right).$$

where $\xi_i^n \in \left(t_{i-1}^n, t_i^n\right]$ for all i and n (which follows of Cauchy's mean value theorem on integrals) [Klebaner 2005]. If we set $g(t) = t$ we see that we arrive at the ordinary Riemann integral. The Itô–integral is then defined using the Stieltjes framework as

$$\int_0^t \sigma(X(s),s)\,dB(s) = \lim_{\delta\to 0}\sum_{i=1}^{n} \sigma\left(X\left(\xi_i^n\right),\xi_i^n\right)\left(B\left(t_i^n\right) - B\left(t_{i-1}^n\right)\right).$$

Sometimes it may simplify calculations to use the Stratonovich integral instead which is defined as follows [Evans 2011]

$$\begin{aligned}\int_0^t \sigma(X(s),s)\,\partial B(s) &= \int_0^t \sigma(X(s),s)\circ dB(s) \\ &= \lim_{\delta\to 0}\sum_{i=1}^{n} \frac{\sigma\left(X\left(t_i^n\right),t_i^n\right) + \sigma\left(X\left(t_{i-1}^n\right),t_{i-1}^n\right)}{2}\left(B\left(t_i^n\right) - B\left(t_{i-1}^n\right)\right)\end{aligned}.$$

To use the Milstein method we also need to use Itô's formula which is a result of the following theorem [Klebaner 2005]: Let $f(t)$ be a twice continuously differentiable function (i.e. a C^2 function) and the stochastic process $X(t)$ have a stochastic differential for $0 \le t \le T$ where

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dB(t)$$

then the stochastic differential of the process $Y(t) = f(X(t))$ exists and is given by

$$
\begin{aligned}
df(X(t)) &= f'(X(t))\,dX(t) + \frac{1}{2} f''(X(t))\,d[X,X](t) \\
&= f'(X(t))\,dX(t) + \frac{1}{2} f''(X(t))\sigma^2(t)\,dt \\
&= \left[f'(X(t))\mu(t) + \frac{1}{2} f''(X(t))\sigma^2(t) \right] dt + f'(X(t))\sigma(t)\,dB(t)
\end{aligned}
\tag{2.35}
$$

where $d[X,X](t)$ is the quadratic covariation which is defined as

$$
[f,g](t) = \lim_{\delta \to 0} \sum_{i=0}^{n-1} \left(f\left(t_{i+1}^n\right) - f\left(t_i^n\right) \right) \left(g\left(t_{i+1}^n\right) - g\left(t_i^n\right) \right)
$$

when the limit is taken over partitions $\{t_i^n\}$ of $[0, t]$ with $\delta_n = \max_i \left(t_{i+1}^n - t_i^n \right)$. This implies that we have the following integral representation for the process $Y(t)$ (7.5)

$$
f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\,dX(s) + \frac{1}{2} \int_0^t f''(X(s))\sigma^2\,ds \tag{2.36}
$$

which will be useful when defining the Milstein method. If we define a mesh where

$$
0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T
$$

and let $h_n = t_{n+1} - t_n$ and $h = \max h_n \leq 1$, then we have that the strong solution $X(t)$ satisfies [Kloeden Platen 1995]

$$
X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} \mu(X(s),s)\,ds + \int_{t_n}^{t_{n+1}} \sigma(X(s),s)\,dB(s). \tag{2.37}
$$

For a mesh where N is large enough we can find an approximation $Y(t_n)$ where $Y(0) \approx X(0)$ and

$$
\begin{aligned}
Y(t_n) &= Y(t_n) + \mu(Y(t_n),t_n) \int_{t_n}^{t_{n+1}} ds + \sigma(Y(t_n),t_n) \int_{t_n}^{t_{n+1}} dB(s) \\
&= Y(t_n) + \mu(Y(t_n),t_n)\,h_n + \sigma(Y(t_n),t_n)\,\Delta B(t_n)
\end{aligned}
\tag{2.38}
$$

where $\Delta B_n = B(t_{n+1}) - B(t_n)$. Equation (2.38) defines the so called Euler–Maruyama method which is a more general variation of the ordinary Euler method. It can be evaluated simply by simulating the Brownian motion $B(t)$. We can without loss of generality make the deterministic functions μ and σ independent of t, i.e.

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t) \tag{2.39}$$

so we have the representation

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} \mu(X(s))ds + \int_{t_n}^{t_{n+1}} \sigma(X(s))dB(s). \tag{2.40}$$

If we integrate (2.35) from t_0 to t we get

$$\begin{aligned} f(X(t)) - f(X(t_0)) &= \int_{t_0}^{t} \left[f'(X(t))\mu(t) + \frac{1}{2} f''(X(t))\sigma^2(t) \right] dt \\ &\quad + \int_{t_0}^{t} f'(X(t))\sigma(t)dB(t) \\ &= \int_{t_0}^{t} L_0 f(X(s))ds + \int_{t_0}^{t} L_1 f(X(s))dB(s) \end{aligned} \tag{2.41}$$

where L_0 and L_1 are the differential operators

$$\begin{aligned} L_0 &= \mu \frac{d}{dx} + \frac{1}{2}\sigma^2 \frac{d^2}{dx^2} \\ L_1 &= \sigma \frac{d}{dx} \end{aligned}.$$

If we set $f(x) = x$ we get Equation (2.40) where $L_0 f = \mu$ and $L_1 f = \sigma$. We can set $f(X(t)) = \mu(X(t))$ and $f(X(t)) = \sigma(X(t))$ to get an expression for each of the deterministic functions μ and σ in Equation (2.39). Making this ansatz and inserting the results in Equation (2.41) yields

$$\begin{aligned} X(t) &= X(t_0) + \int_{t_0}^{t} \left(\mu(X(t_0)) + \int_{t_0}^{s} L_0\mu(X(u))du + \int_{t_0}^{s} L_1\mu(X(u))dB(u) \right) ds \\ &\quad + \int_{t_0}^{t} \left(\sigma(X(t_0)) + \int_{t_0}^{s} L_0\sigma(X(u))du + \int_{t_0}^{s} L_1\sigma(X(u))du \right) dB(s) \\ &= X(t_0) + \mu(X(t_0)) \int_{t_0}^{t} ds + \sigma \int_{t_0}^{t} dB(s) + R_1(t, t_0) \end{aligned}$$

where $R_1(t, t_0)$ is

$$R_1(t,t_0)=\int_{t_0}^{t}\int_{t_0}^{s}L_0\mu(X(u))\,du\,ds+\int_{t_0}^{t}\int_{t_0}^{s}L_1\mu(X(u))\,dB(u)\,ds$$
$$+\int_{t_0}^{t}\int_{t_0}^{s}L_0\sigma(X(u))\,du\,ds+\int_{t_0}^{t}\int_{t_0}^{s}L_1\sigma(X(u))\,dB(u)\,ds.$$
$$+\int_{t_0}^{t}\int_{t_0}^{s}L_1\sigma(X(u))\,dB(u)\,dB(s)$$

This is the motivation for Eulers method where we neglect the higher order terms in $R_1(t, t_0)$. Now, if we look at the last term of $R_1(t, t_0)$, we can find an expression for it by making the ansatz $f(x) = L_1\sigma$ in Itô's formula and insert the results in this term. From this we get

$$X(t)=X(t_0)+\mu(X(t_0))\int_{t_0}^{t}ds+\sigma\int_{t_0}^{t}dB(s)$$
$$+\sigma(X(t_0))\sigma'(X(t_0))\int_{t_0}^{t}\int_{t_0}^{s}dB(u)\,dB(s)+R_2(t,t_0)$$

where $R_2(t, t_0)$ involves some higher order terms that is not relevant here. This is the motivation of Milstein's method where we also account for the double Itô integral. To evaluate this integral we can apply Itô's formula for a Brownian motion process such that

$$f(B(t))=f(0)+\int_0^t f'(B(s))\,dB(s)+\frac{1}{2}\int_0^t f''(B(s))\,ds$$

and take $f(x) = x^m$, $m \geq 2$ which yields

$$B^m(t)=m\int_0^t B^{m-1}(s)\,dB(s)+\frac{m(m-1)}{2}\int_0^t B^{m-2}(s)\,ds.$$

In particular, with $m = 2$ we have

$$B^2(t)=2\int_0^t B(s)\,dB(s)+t$$

which can be rearranged to

$$\int_0^t B(s)\,dB(s)=\frac{1}{2}B^2(t)-\frac{1}{2}t.$$

With this result we yield for the double Brownian integral

$$\int_{t_0}^{t}\int_{t_0}^{s} dB(u)dB(s) = \int_{t_0}^{t}(B(s)-B(t_0))dB(s)$$
$$= \frac{1}{2}\left((B(t)-B(t_0))^2-(t-t_0)\right)$$

Thus we extend the formula in the Euler–Maruyama method with this integral such that

$$Y(t_{n+1}) = Y(t_n) + \mu(Y(t_n))h_n + \sigma(Y(t_n))\Delta B_n$$
$$+\frac{1}{2}\sigma(Y(t_n))\sigma'(Y(t_n))\left((\Delta B_n)^2 - h_n\right)$$

where once again

$$h_n = t_{n+1} - t_n, \quad \Delta B_n = B(t_{n+1}) - B(t_n).$$

This idea can be further extended to derive methods of even higher order but the coefficients will yield many times iterated integrals which are difficult to evaluate [Kloeden Platen 1995].

The CIR++ process must be evaluated using numerical methods such as the Milstein or Euler–Maruyama method if the background process is correlated [Brigo, Alfonsi 2005]. If we use the same parameters as before in the Euler–Maruyama scheme but with the correlated process $Z(t) = \rho W_t + \sqrt{1-\rho^2 W_t'}$ where W_t and W_t' are two stochastically independent Wiener processes, the Euler–Maruyama equation for simulating this process is then

$$\tilde{y}(t_{i+1}) = \tilde{y}(t_i) + \kappa(\mu - \tilde{y}(t_i))h_i + \nu\sqrt{\tilde{y}(t_i)}\Delta Z_i$$

and the Milstein equation for this process becomes

$$\tilde{z}(t_{i+1}) = \tilde{z}(t_i) + \kappa(\mu - \tilde{z}(t_i))h_i + \nu\sqrt{\tilde{z}(t_i)}\Delta Z_i + \frac{1}{4}\nu^2\left[(\Delta Z_i)^2 - h_i\right]$$

where

$$h_i = t_{i+1} - t_i, \quad \Delta Z_i = B(t_{i+1}) - B(t_i)$$

One major drawback with these numerical schemes is that they don't ensure the positivity of $\tilde{y}(t)$ and $\tilde{z}(t)$. This can be overcome by simulating a so called Brownian bridge on each interval $[t_i, t_{i+1}]$ with a small enough time step to retrieve the positivity that is ensured by the relation $2\kappa\mu > \nu^2$ for the continuous process (Brigo Alfonsi 2005). So whenever the value becomes negative, one simulates a Brownian bridge for that interval. Another drawback that is related to the aforementioned implementation is that the positivity preserving property does not always apply when running the simulation above. This property is important since taking a positive initial condition normally ensures that the process stays positive during the simulation. A further discussion is covered in [Brigo Alfonsi 2005].

3 Results and Computations

Brigo, Masetti 2005 construct three risk scenarios; Low Risk, Medium Risk and High Risk for the credit quality of a given credit party. Each of these scenarios has a set of intensities and survival probabilities, for a given set of dates. It is assumed that the intensities follow a piecewise constant step function as defined in Equation (2.5). The values for these three risk profiles are given in **Table 3.1**.

	Low Risk		Medium Risk		High Risk	
Date	Intensity	Survival	Intensity	Survival	Intensity	Survival
10-mar-04	0.0036	100.00%	0.0202	100.00%	0.0534	100.00%
12-mar-05	0.0036	99.64%	0.0202	97.96%	0.0534	94.70%
12-mar-07	0.0065	98.34%	0.0231	93.48%	0.0564	84.47%
12-mar-09	0.0099	96.38%	0.0266	88.57%	0.0600	74.78%
12-mar-11	0.0111	94.24%	0.0278	83.71%	0.0614	66.03%
12-mar-14	0.0177	89.31%	0.0349	75.27%	0.0696	53.42%
12-mar-19	0.0177	81.64%	0.0349	63.05%	0.0696	37.53%
12-mar-24	0.0177	74.63%	0.0349	52.80%	0.0696	26.36%
12-mar-29	0.0177	68.22%	0.0349	44.23%	0.0696	18.51%
12-mar-34	0.0177	62.36%	0.0349	37.05%	0.0696	13.01%

Table 3.1: An estimation of intensities and survival probabilities $\mathbf{Q}[\tau > T]$ at different dates (called *intensities nodes*) for the risk scenarios Low Risk, Medium Risk and High Risk for the credit quality of the counterparty.

Source: Brigo, Masetti 2005 or Brigo, Mercurio 2007

It is common practice to use the LIBOR rate as a discount rate of financial instruments such as interest rate swaps. A problem with these rates is that they are only specified for maturities of up to one year. It is possible to use different long–term government bonds (such as Treasury bonds or Treasury Inflation–Protected Securities [TIPS] which offer maturities up to 30 years) as a risk free discount rate. The collected rates for usage as risk–free swap rates are presented in **Table 3.2**.

Maturity (years)	Risk–free swap rate
5	3.249%
10	4.074%
15	4.463%
20	4.675%
25	4.775%
30	4.811%

Table 3.2: Risk–free implied swap rate for different maturities T_j to be used when discounting the cash flows in the interest rate swap agreements.

Source: Brigo, Masetti 2005

The discount factors can be estimated recursively by solving for each maturity T_j starting at the lowest and working one's way upwards using

$$P(0,T_j) = \frac{1}{1+\beta K_j}\left[1-\beta K_j \sum_{i=1}^{j-1} P(0,T_j)\right] \tag{3.1}$$

where K_j are the swap rates in **Table 3.2** and β is the year–fraction (which is 5 years). From this formula we could estimate the discount rate r_j by using

$$r_j = -\frac{\ln P(t,T)}{T-t} = -\frac{\ln P(0,T_j)}{T_j}. \tag{3.2}$$

This rate varies over time and we could either let it be piecewise constant or piecewise linear with respect to t. We derived the following formula that automatically generates a piecewise linear function with respect to time, given the values r_j:

$$y(t) = y(t_0) + \sum_{i=1}^{N} \frac{y(t_i)-y(t_{i-1})}{t_i - t_{i-1}} \left(\max\left(\min\left(t,t_i\right),t_{i-1}\right)-t_{i-1}\right) \tag{3.3}$$

where $y(t_i)$ are given interest rates at maturities t_i (where we set $r_i = r(t_i) = y(t_i)$). We see a comparison between the piecewise constant interest rate and the piecewise linear function in **Figure 3.1**. We can see that for maturities below 5 years, the piecewise linear function tends to underestimate the real interest rate as it starts at 0% at time zero and linearly transitions to the rate at the lowest maturity at five years. For higher maturities the function tends to overestimate the real interest rate.

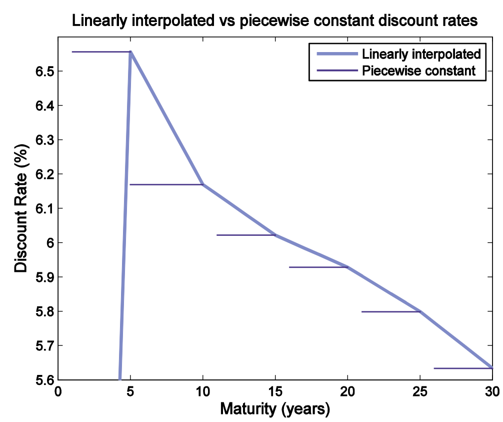

Figure 3.1: A comparison between letting discount rate follow a piecewise constant function and a piecewise linear function. The piecewise linear function tends to underestimate the interest rate for maturities below 5 years and overestimate it at higher maturities, but overall seems to be a more sensible option than the piecewise constant function.

It is fully possible to tweak the formula and let it start at a positive interest rate at time zero but overall we find it a reasonable estimate as the short term underestimation counterweighs the long term overestimation. Now, we have all the background data required to estimate the counterparty credit adjustments as in Equations (2.22) and (2.23).

	Low Risk		Medium Risk		High Risk	
Mty	Antic.	Postp.	Antic.	Postp.	Antic.	Postp.
5	41.846	30.543	183.816	147.936	435.908	362.184
10	67.758	52.943	248.775	209.076	557.582	482.079
15	115.572	96.474	340.856	299.484	701.723	631.995
20	170.248	149.574	434.015	394.046	829.879	768.778
25	218.666	197.501	509.891	472.070	923.037	869.753
30	257.691	236.552	566.757	531.082	985.443	938.411

Table 3.3: Resulting calculations from Equation (2.22) and (2.23) for the counterparty default adjustments assuming anticipated default (DP^A) and postponed payoff given default (DP^P) for the maturities ranging from 5 to 10 years.

Our calculations from the three distinct risk scenarios are in **Table 3.3**. We see overall that the anticipated default assumption yields higher adjustments than the postponed payoff given default assumption. **Figure 3.2** illustrates the difference between these two assumptions.

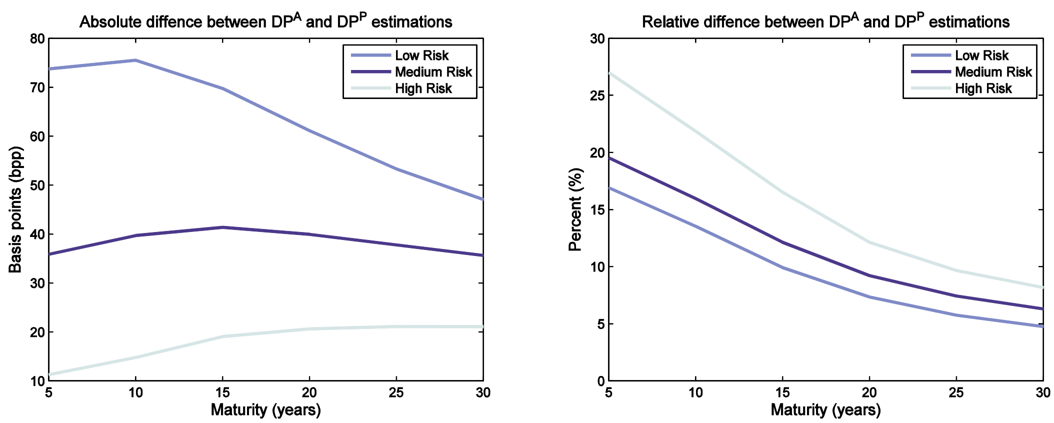

Figure 3.2: The absolute and relative difference between the anticipated default assumption and the assumption of postponed payoff given default is bigger for lower maturities and for higher risk scenarios whereas it is lower for lower risk scenarios and at longer maturities.

It is rather sensible that the difference is higher at lower maturities than at higher maturities and tends to zero at infinity. It also makes sense that the added risk in a high risk scenario also add to the difference between the two assumptions. A visualization of the credit valuation adjustment with respect to risk and maturity assuming anticipated default is given in **Figure 3.3**.

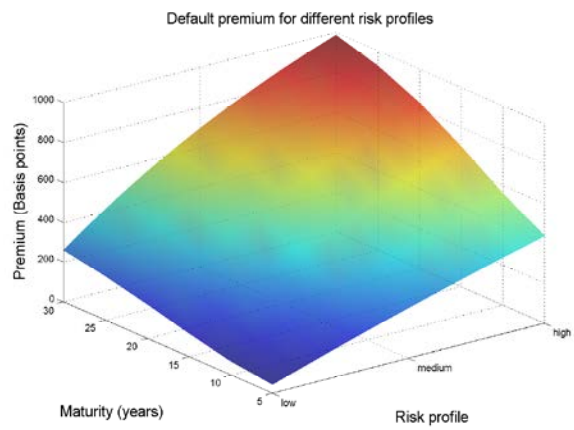

Figure 3.3: A 3D visualization of the credit value adjustment added to the spread of an IRS for different maturities and risk profiles linearly transitioned between the Low Risk, Medium Risk and the High Risk scenarios.

In **Figure 3.4** we have the following cross section plots from the meshplot in **Figure 3.3**.

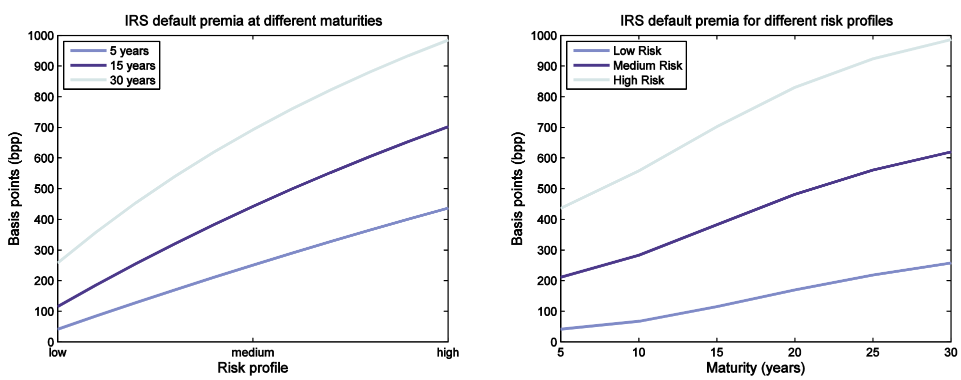

Figure 3.4 Comparative cross–sections of the credit value adjustment to be added to the spread of an IRS for different risk profiles at different maturities (left) and for different maturities at different risk profiles (right).

4 Conclusion

We see that accounting for counterparty risk adds to the risk premium in the valuation of the interest rate swap. At one extreme a survival probability of 13.5% at 30 years maturity demands a risk–adjustment by almost 1000 basis points. At the other extreme a survival probability of 96.38% at 5 years maturity demands a risk–adjustment of around 40 basis points. The model yields a risk premium that is larger for high risk scenarios and longer maturities whereas it is rather small in lower risk scenarios and lower maturities which is in line with what we would expect from real world data.

5 Final Thoughts

There are several factors that are not accounted for in the model. The most important factor is the correlation between counterparty credit risk and interest rates. This requires that we assume that the default risk and the interest rates have some degree of stochastic dependence. In this scenario we need to find a stochastic model for the interest rates that is (partly) correlated with a stochastic model for the underlying intensities requiring a more complex valuation scheme for the interest rate swap agreement. So we have to ditch the assumption of stepwise constant intensity and model it with a stochastic process. A further treatment regarding this can be found in Section 2.6. Brigo, Pallavicini 2008 use what they call the CIR++ model. The major advantage of the CIR process in the intensity framework is that it is always positive so we don't risk getting negative default probabilities in our estimations. The downside is that the CIR model is a lot more inflexible and complex than the piecewise constant deterministic function and it is more difficult to find a good fit with real world data. They overcome this limitation by calibrating it implicitly through an arbitrary deterministic function and call it the CIR++ process. The CIR++ process is still able to yield a satisfactory fit with the real world data, especially for the shorter term estimations which is why they add a jump diffusion process to the stochastic intensity. Their stochastic model for intensity values that involves jump diffusion is called the JCIR++ process. The interest rates are modeled using a stochastic process which they call the G2++ process. It is a two factor Gaussian process that they calibrate against real–world interest rates using an arbitrary deterministic function in much the same way as they calibrate the CIR++ and the JCIR++ processes against the real–world CDS market data. The JCIR++ process is correlated with the G2++ process and the counterparty credit adjustment is calculated using Equation (2.21) through Monte–Carlo simulations since we no longer have closed form expressions. Also, we can no longer use the Black–Scholes model to value the swaption but use a more complex expression which is found in Brigo, Mercurio 2008 (which is shown in Theorem 4.2.3 in section 4.2.4).

Appendix

Matlab code of the calculations performed in Section 3. The file `dpcalcfun.m` is the function that contains the core algorithms for the computations and it is used to run different scenarios in `dprun1.m` that also summarized the computations in a 3D mesh plot. The file `dprun2.m` analyzes the results further by producing additional comparative 2D plots.

- dpcalcfun.m -

```
%
% Counterparty risk and contingent CDS-valuation
%
% R, Axelsson
% Handelshögskolan, Göteborg
%
%

% +-------------------------------------------------------------------
----
%
######################################################################
% ##################################################################
% ////////////////////////////////////////////////////
|
% /////////////////////////
-+-
% ///////
|
%
%            Definitions
%

% Additional information:
%
%  t = 0 since PS(t) is not defined at other time values

function DPP = dpcalcfun(DEFPAR, MKTSWP, sigmaBS)

% The inparameters for this function are:
%
%  DEFPAR.lambda  -  a vector of default intensities
%  DEFPAR.prob    -  a vector of default probabilities
%  DEFPAR.dT      -  a vector of interval lenghts for eact default
%                    intensity/probability
%
%  sigmaBS        -  a sigma parameter for the B-S swaption valuation
```

```
%
%  MKTSWP.R        -  a vector of observed swap spreads
%  MKTSWP.Mt       -  a vector of maturities for the observed swap
spreads
%
%  It is assumed that the vectors have the same lenght in the DEFPAR
% parameter and in the MKTSWP parameter.

% I want to find a way to construct a detection algorithm that matches
the
% elements in LR, MR, and HR with the proper year.Perhaps find(pT ==
Mt(i))
% to return those indices and then use them on LR, MR and HR where

pT = cumsum(DEFPAR.dT);

% The function for a fair strike price of a CDS agreement is
% (WARNING: For valid S_ab the parameters a,b must be a < b)

S_ab = @(P_0T,a,b, beta)((P_0T(a)-P_0T(b))./(beta*sum(P_0T(a+1:b))));

% This comes directly from equation (4.2).

% So we need to use the Black-Scholes model for swaptions as
implemented by
% Brigo (eq 1.26 Brigo-Pallavicini)

d_1= @(K, F, v)((log(F./K)+v.^2/2)./v);
d_2= @(K, F, v)((log(F./K)-v.^2/2)./v);

Bl = @(K, F, v, w)(F.*w.*normcdf(w.*d_1(K,F,v)) - ...
                                           K.*w.*normcdf(w.*d_2(K,F,v))
);

% The cumulative density function for a piecewise constant intensity is
%

F_tau = @(t, lambda)(1-exp(-lambda(:,1)'*(((t>lambda(:,3)).* ...
                    (lambda(:,3)-lambda(:,2)))+((t<=lambda(:,3))- ...
                    (t<=lambda(:,2)))*(t-
lambda(:,2)'*((t<=lambda(:,3))-...
                    (t<=lambda(:,2)))))));

% where
% lambda = a matrix containing the parameters of the intensity which is
%           assumed to be a step-function with respect to time. The
first
%           column of this matrix contains the intensity values, the
second
%           column contains the time points where each intensity value
```

```
%           starts and the third column contains the time points where
%           each of them stop. So the second and third column defines
the
%           start and the end of each time step such that the entire
%           intensity function can be defined as a sum of Heaviside step
%           functions H(t), i.e.
%
%           lambda(t) = lambda(:,1)'*(H(t-lambda(:,2))-H(t-lambda(:,3)))
%                     = lambda(:,1)'*((t > lambda(:,2))-(t >
lambda(:,3)))
%                     = lambda(:,1)'*((t <= lambda(:,3))-(t <=
lambda(:,2)))
%
%          so the sums are expressed here as scalar products
%
% So in this case: lambda = [#R(:,1) pT [pT(2:end);100] ] where # is L,
M
% or H and Q(\tau > t) =  1 - F_tau(t, [#R(:,1) pT [pT(2:end);100])

% The risk-free discount functions are only defined on discrete points
in
% time. If we want to find out the values between these points we need
to
% interpolate:

% This defines a discount function that follows a piecewise constant
% interest rate

P0c_t0 = @(t, R0T, Mt)(exp(-[R0T(1);R0T]'*[0 diff(t <= Mt)]'.*t));
P0c_t = @(t, R0T, Mt)(exp(-[R0T(1);R0T]'*...
         [0*(1:length(t))' diff(meshgrid(t,Mt) <=
meshgrid(Mt,t)')']'.*t));

% This defines a discount function that uses an interest rate that
follows
% a linearly interpolated continuous function. It is also adapted to
accept
% time as a vector.

P0f_t = @(t, R0T, Mt)(exp(-(diff([0;R0T])'./diff(Mt))* ...
                     (max(min(meshgrid(t, Mt(2:end))' , ...
                     meshgrid(Mt(2:end),t)),meshgrid(Mt(1:end-1),t))-
...
                     meshgrid(Mt(1:end-1),t))'.*t));

% This is the interest rate function (for verification). I assume that
this
% is the interest rate with respect to maturity, i.e. that it is
already
```

```
% integrated and I can apply it directly as a "fixed" rate with respect
to
% the given time-period.

R0f_t = @(t, R0T, Mt)((diff([0;R0T])'./diff(Mt))*(max(min(meshgrid(t,
...
                      Mt(2:end))' , meshgrid(Mt(2:end),t)), ...
                      meshgrid(Mt(1:end-1),t))-meshgrid(Mt(1:end-
1),t))');

% We also need to redefine the S_ab with respect to these
interpolations

S_abf = @(t, R0T, Mt)((P0f_t(t(1), R0T, Mt)-P0f_t(t(end), R0T, Mt))./
...
                       (mean(diff(t))*sum(P0f_t(t(2:end), R0T, Mt))));

% We also redefine S_ab for use with a piecewise constant interest rate
% function

S_abf2 = @(t, R0T, Mt)((P0c_t(t(1), R0T, Mt)-P0c_t(t(end), R0T, Mt))./
...
                        (mean(diff(t))*sum(P0c_t(t(2:end), R0T, Mt))));

% Note that the mean(diff) is put there because I assume that the time
% points are 'monospaced'. If the time intervals are heterogeneous, all
we
% have to do is to remove the mean() and the sum() and change the
% expression denominator to a scalar product, i.e. to
% 'diff(t)'*P0x_t(t(2:end))'.

% The following is for calculations straight off the table values and
the
% table time-grid (5 years) using piecewise constant interest rates

ADPP.antpure = zeros(length(MKTSWP.Mt)-1, 1);

% We also want to try the same evaluations for the interpolated values

ADPP.antinterp  = zeros(length(MKTSWP.Mt)-1, 1);
ADPP.postinterp = ADPP.antinterp;

% Let's also try the same evaluations for the piecewise constant
interest
% rate but with a finer time-grid

ADPP.antstep = zeros(length(MKTSWP.Mt)-1, 1);
ADPP.poststep = ADPP.antstep;
```

```
% +------------------------------------------------------------------
----
%
######################################################################
#
% ##########################################################
% /////////////////////////////////////////////////////        |  \ /
/
% //////////////////////////                                  -+-   X
/
% ///////                                                      |  / \
/
%
%         Computations
%

% Computing the P(0, T_i) from the table values

P0T = zeros(1,length(MKTSWP.R))';
for j = 1:length(MKTSWP.R)
    P0T(j) = (1-5*MKTSWP.R(j)*sum(P0T))./(1+5*MKTSWP.R(j));
end

R0T = -log(P0T(2:end))./MKTSWP.Mt(2:end)';
ADPP.R0T = R0T;

% Here begins the big foor-loop that does all the necessary
calculations.
% The variable k is for the different maturities of the swaps where k =
2 3
% 4 5 6 7 (length(Mt)) which is from the table values. The for-loops
below
% basically use equations (1.28) and (21.41) in Brigo-Pallavicini.

for k = 2:length(MKTSWP.Mt)

    % Calculations of anticipated values done pretty much straigh from
the
    % table values

    for i = 2:k
        SabC = S_ab(P0T(i-1:k), 1, k-i+2, 5);
```

```
    % SabC(find(isnan(SabC))) = 0;  % This is in case SabC returns NaN
    % (when inp arg a = b)
        ADPP.antpure(k-1, :) = ADPP.antpure(k-1, :) + ...
            (DEFPAR.prob(find(MKTSWP.Mt(i-1)==pT),:) ...
           - DEFPAR.prob(find(MKTSWP.Mt(i)==pT),:)).* ...
           Bl(MKTSWP.R(k),SabC,sigmaBS.* ...
           MKTSWP.Mt(i-1).^.5,1).*5.*sum(P0T(1:k));
    end
    for i = 1:MKTSWP.Mt(k)
    % We now calculate the anticipated adjustments for the interpolated
    % functions using annual time-grid

        ADPP.antinterp(k-1) = ADPP.antinterp(k-1) + ...
           (F_tau(i, [DEFPAR.lambda pT [pT(2:end);100]]) - ...
           F_tau(i-1, [DEFPAR.lambda pT [pT(2:end);100]])).* ...
           Bl(MKTSWP.R(k), S_abf(i-1:MKTSWP.Mt(k), R0T, MKTSWP.Mt), ...
           sigmaBS.*(i-1)^.5, 1).* ...
           sum(P0f_t(i-1:MKTSWP.Mt(k), R0T, MKTSWP.Mt));

        % We now calculate the anticipated adjustments for the stepwise
        % constant interest rate functions but with an annual time-grid
        % instead of the "quintoannual" (5 years) time-grid that was
used
        % for the table-based calculations above.

        ADPP.antstep(k-1) = ADPP.antstep(k-1) + ...
            (F_tau(i, [DEFPAR.lambda pT [pT(2:end);100]]) - ...
            F_tau(i-1, [DEFPAR.lambda pT [pT(2:end);100]])).* ...
            Bl(MKTSWP.R(k), S_abf2(i-1:MKTSWP.Mt(k), R0T, ...
            MKTSWP.Mt), sigmaBS.*(i-1)^.5, 1).* ...
            sum(P0f_t(i-1:MKTSWP.Mt(k), R0T, MKTSWP.Mt));
        if i < MKTSWP.Mt(k)
            % The postponed adjustments for the interpolated functions
are
            % calculated here

            ADPP.postinterp(k-1) = ADPP.postinterp(k-1) + ...
            (F_tau(i, [DEFPAR.lambda pT [pT(2:end);100]]) ...
            - F_tau(i-1, [DEFPAR.lambda pT [pT(2:end);100]])).* ...
            Bl(MKTSWP.R(k), S_abf(i:MKTSWP.Mt(k), R0T, MKTSWP.Mt), ...
            sigmaBS.*(i)^.5, 1).* ...
            sum(P0f_t(i:MKTSWP.Mt(k), R0T, MKTSWP.Mt));
            % The same adjustments are calculated but for the stepwise
            % constant interest rate function

            ADPP.poststep(k-1) = ADPP.poststep(k-1) + (F_tau(i, ...
               [DEFPAR.lambda pT [pT(2:end);100]]) - ...
               F_tau(i-1, [DEFPAR.lambda pT [pT(2:end);100]])).* ...
               Bl(MKTSWP.R(k), S_abf2(i:MKTSWP.Mt(k), R0T, MKTSWP.Mt),
...
```

```
               sigmaBS.*(i)^.5, 1).* ...
               sum(P0f_t(i:MKTSWP.Mt(k), R0T, MKTSWP.Mt));
        end
    end
end

DPP = ADPP;
```

- dprun1.m -

```
%
% Counterparty risk and contingent CDS-valuation
%
% R, Axelsson
% Handelshögskolan, Göteborg
%
%

% Maturities and rates for observed market swaps

MKTSWP.Mt   = [0 5     10    15    20    25    30    ];
MKTSWP.R = [0 3.249 4.074 4.463 4.675 4.775 4.811]*.01;

% Risk profiles with their intensities and probabilities

% In this risk profile we assume that the first period starts at t = 0.
This
% is a qualified assumption since the survival probability is 1 at that
time.
% Alternatively we can assume a later time which then would be a matter
of
% adding a discount factor. The year differences are specified in the
column
% vector 'dT'. The first column in LR (Low Risk), MR (Medium Risk) and
HR
% (Human Resources) contains the intensities and the second column
contains
% the probabilities.

% Since the probabilities already are computed we can use them directly
% without further computations.

LR.lambda = [  .0036 .0036 .0065 .0099 .0111 .0177 .0177 .0177 .0177
.0177]';
LR.prob   = [ 1      .9964 .9834 .9638 .9424 .8931 .8164 .7463 .6822
.6236]';
LR.dT     = [ 0     1     2     2     2     3     5     5     5     5
]';
```

```
MR.lambda = [  .0202 .0202 .0231 .0266 .0278 .0349 .0349 .0349 .0349
.0349]';
MR.prob   = [ 1      .9796 .9348 .8857 .8371 .7527 .6305 .5280 .4423
.3705]';
MR.dT     = [ 0     1     2     2     2     3     5     5     5     5
]';

HR.lambda = [  .0534 .0534 .0564 .0600 .0614 .0696 .0696 .0696 .0696
.0696]';
HR.prob   = [ 1      .9470 .8447 .7478 .6603 .5342 .3753 .2636 .1851
.1301]';
HR.dT     = [ 0     1     2     2     2     3     5     5     5     5
]';

sigmaBS = .17;


DPPLR = dpcalcfun(LR, MKTSWP, sigmaBS);
DPPMR = dpcalcfun(MR, MKTSWP, sigmaBS);
DPPHR = dpcalcfun(HR, MKTSWP, sigmaBS);

fprintf('My calculations of a payer swap using the Low, Medium and High
risk\nscenarios\n\n')
fprintf('Calculations of anticipated payer swaps using pure table
values\n(columns: Maturities, LR, MR, HR):\n')

[MKTSWP.Mt(2:end)' DPPLR.antpure DPPMR.antpure DPPHR.antpure]

fprintf('Calculations of anticipated payer swaps using fully
interpolated interest\nrates and manually calculated default
probabilities:\n')

[MKTSWP.Mt(2:end)' DPPLR.antinterp DPPMR.antinterp DPPHR.antinterp]

fprintf('The same calculations but on postponed accruals:\n')

[MKTSWP.Mt(2:end)' DPPLR.postinterp DPPMR.postinterp DPPHR.postinterp]

fprintf('Calculations of anticipated payer swaps assuming piecewise
constant interest\nrates and manually calculated default
probabilities:\n')

[MKTSWP.Mt(2:end)' DPPLR.antstep DPPMR.antstep DPPHR.antstep]

fprintf('The same calculations but on postponed accruals:\n')

[MKTSWP.Mt(2:end)' DPPLR.poststep DPPMR.poststep DPPHR.poststep]

x_i = 0:.1:1;
DR.prob = LR.prob;
```

```
DR.dT = LR.dT;
DPMESH = zeros(length(MKTSWP.Mt)-1,length(x_i));
for i = 1:length(x_i)
    DR.lambda = (HR.lambda - LR.lambda)*x_i(i) + LR.lambda;
    DPPDR = dpcalcfun(DR, MKTSWP, sigmaBS);
    DPMESH(:,i) = DPPDR.antinterp;
end


% Just a mockup of how the mesh plotter works:
% mesh(1:3, MKTSWP.Mt(2:end), [DPPLR.antinterp DPPMR.antinterp
% DPPHR.antinterp ]*10000);

FIGS.fig1 = figure('Color',[1 1 1]);
surf(x_i, MKTSWP.Mt(2:end), DPMESH*1e4);
surface(x_i, MKTSWP.Mt(2:end), DPMESH*1e4);
alpha .5
shading interp
material metal
lighting phong
camlight('right')

% Note that the big font size is chosen so that the image can be scaled
down
% so as to compensate for the resolution limitations in the pixel based
% z-buffer plot (only z-buffer can render 3D images with lighting
effects
% and it cannot make vector based images, at least not in Matlab
2010a).

zlabel('\fontsize{16}Premium (Basis points)')
ylabel('\fontsize{16}Maturity(years)')
xlabel('\fontsize{16}Risk profile')
title('\fontsize{16}Default premium for different risk profiles')

set(gca,'XTick',[0 .5 1])
set(gca,'XTickLabel',{'low', 'medium', 'high'});

% (1) A comparative graph to show how postponed vs anticipated affect
the CVA
% and (2) a comparative graph to show interpolated vs stepwise interest
rate
% would ba a good thing to add.
% (3) The three risk profiles for anticipated and postponed would be
good
% to add
%
```

- dprun2.m -

```
%
% Counterparty risk and contingent CDS-valuation
%
% R, Axelsson
% Handelshögskolan, Göteborg
%
%

fprintf('Note that this should be run AFTER dprun1.m\n\n')

FIGS.LineColors = [[0, 0, 123]; [75, 56, 138]; [127, 138, 199]; ...
   [214, 229, 230]; [115, 115, 115]; [165, 169, 176]; [183, 202,
211]]/255;

FIGS.fig2 = figure('Color',[1 1 1]);
plot(MKTSWP.Mt(2:end),100*(DPPHR.antinterp-DPPHR.postinterp)./ ...
         DPPHR.antinterp, 'LineWidth', 2, 'Color', FIGS.LineColors(3,
:))
hold on
plot(MKTSWP.Mt(2:end),100*(DPPMR.antinterp-DPPMR.postinterp)./ ...
         DPPMR.antinterp, 'LineWidth', 2, 'Color', FIGS.LineColors(2,
:))
plot(MKTSWP.Mt(2:end),100*(DPPLR.antinterp-DPPLR.postinterp)./ ...
         DPPLR.antinterp, 'LineWidth', 2, 'Color', FIGS.LineColors(4,
:))
title('\fontsize{12}Relative diffence between DP^A and DP^P
estimations')
xlabel('\fontsize{12}Maturity (years)')
ylabel('\fontsize{12}Percent (%)')
legend('Low Risk', 'Medium Risk', 'High Risk')


FIGS.fig3 = figure('Color',[1 1 1]);
plot(MKTSWP.Mt(2:end),10000*(DPPHR.antinterp-DPPHR.postinterp), ...
                        'LineWidth', 2, 'Color', FIGS.LineColors(3, :))
hold on
plot(MKTSWP.Mt(2:end),10000*(DPPMR.antinterp-DPPMR.postinterp), ...
                         'LineWidth', 2, 'Color', FIGS.LineColors(2,
:))
plot(MKTSWP.Mt(2:end),10000*(DPPLR.antinterp-DPPLR.postinterp), ...
                         'LineWidth', 2, 'Color', FIGS.LineColors(4,
:))
title('\fontsize{12}Absolute diffence between DP^A and DP^P
estimations')
xlabel('\fontsize{12}Maturity (years)')
ylabel('\fontsize{12}Basis points (bpp)')
legend('Low Risk', 'Medium Risk', 'High Risk')
```

```
R0f_t = @(t, R0T, Mt)((diff([0;R0T])'./diff(Mt))*(max(min( ...
                      meshgrid(t, Mt(2:end))' , meshgrid(Mt(2:end),t)),
...
                      meshgrid(Mt(1:end-1),t))-meshgrid(Mt(1:end-
1),t))');
R0c_t = @(t, R0T, Mt)([R0T(1);R0T]'*[0*(1:length(t))' ...
                      diff(meshgrid(t,Mt) <= meshgrid(Mt,t)')']');
P0f_t = @(t, R0T, Mt)(exp(-(diff([0;R0T])'./diff(Mt))*(max(min( ...
                      meshgrid(t, Mt(2:end))' , meshgrid(Mt(2:end),t)),
...
                      meshgrid(Mt(1:end-1),t))- ...
                      meshgrid(Mt(1:end-1),t))'.*t));
P0c_t = @(t, R0T, Mt)(exp(-[R0T(1);R0T]'*[0*(1:length(t))' ...
                      diff(meshgrid(t,Mt) <= meshgrid(Mt,t)')']'.*t));

FIGS.fig4 = figure('Color',[1 1 1]);
plot(100*R0f_t(1:max(MKTSWP.Mt), DPPLR.R0T, MKTSWP.Mt), ...
                            'LineWidth', 2, 'Color', FIGS.LineColors(3,
:))
hold on
CVx = 1:30;
CVx(6)=5;CV(11)=10;CV(16)=15;CV(21)=20;CV(26)=25;
CVy = 100*R0c_t(1:max(MKTSWP.Mt), DPPLR.R0T, MKTSWP.Mt);
for i = 1:5:30;
    plot(CVx(i:(i+4)), CVy(i:(i+4)),'LineWidth', 1, ...
                                    'Color', FIGS.LineColors(2, :))
end
ylim([5.6 6.6]);
legend('Linearly interpolated', 'Piecewise constant')
xlabel('\fontsize{12}Maturity (years)')
ylabel('\fontsize{12}Discount Rate (%)')
title('\fontsize{12}Linearly interpolated vs piecewise constant
discount rates')
```

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