Thesis for the Degree of Doctor of Philosophy

Topics on Harmonic analysis and Multilinear Algebra

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Abstract

The present thesis consists of six different papers. Indeed, they treat three different research areas: function spaces, singular integrals and multilinear algebra. In paper I, a characterization of continuity of the p-Λ-variation function is given and Helly's selection principle for $\Lambda BV^{(p)}$ functions is established. A characterization of the inclusion of Waterman-Shiba classes into classes of functions with given integral modulus of continuity is given. A useful estimate on the modulus of variation of functions of class $\Lambda BV^{(p)}$ is found. In paper II, a characterization of the inclusion of Waterman-Shiba classes into H_ω^q is given. This corrects and extends an earlier result of a paper from 2005. In paper III, the characterization of the inclusion of Waterman-Shiba spaces $\Lambda BV^{(p)}$ into generalized Wiener classes of functions $BV(q;\,\delta)$ is given. It uses a new and shorter proof and extends an earlier result of U. Goginava. In paper IV, we discuss the existence of an orthogonal basis consisting of decomposable vectors for all symmetry classes of tensors associated with Semi-dihedral groups SD_{8n}. In paper V, we discuss o-bases of symmetry classes of tensors associated with the irreducible Brauer characters of the Dicyclic and Semi-dihedral groups. As in the case of Dihedral groups [46], it is possible that $V_\phi(G)$ has no o-basis when ϕ is a linear Brauer character. Let $\vec{P} = (p_1, \ldots, p_m)$ with $1 < p_1, \ldots, p_m < \infty$, $1/p_1 + \cdots + 1/p_m = 1/p$ and $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$. In paper VI, we investigate the weighted bounds with dependence on aperture α for multilinear square functions $S_{\alpha,\psi}(\vec{f})$. We show that

$$\|S_{\alpha,\psi}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,\psi,\vec{P}}\ \alpha^{mn}[\vec{w}]_{A_{\vec{P}}}^{\max(\frac{1}{2},\frac{p_1'}{p},\ldots,\frac{p_m'}{p})} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}.$$

This result extends the result in the linear case which was obtained by Lerner in 2014. Our proof is based on the local mean oscillation technique presented firstly to find the sharp weighted bounds for Calderón–Zygmund operators. This method helps us avoiding intrinsic square functions in the proof of our main result.

Keywords: Generalized bounded variation, Helly's theorem, Modulus of variation, Generalized Wiener classes, Symmetry classes of tensors, Orthogonal basis, Brauer symmetry classes of tensors, Multilinear singular integrals, weighted norm inequalities, weighted bounds, local mean oscillation, Lerner's formula

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Mahdi Hormozi
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List of Papers

The thesis is based on the following papers.

I. **M. Hormozi**, A. A. Ledari and F. Prus-Wiśniowski, *On $p-\Lambda-$ bounded variation*, Bulletin of IMS. 37(4) (2011) 29–43

II. **M. Hormozi**, *Inclusion of $\Lambda BV^{(p)}$ spaces in the classes H_ω^q*, Journal of Mathematical Analysis and Applications 404(2) 195–200

III. **M. Hormozi**, F. Prus-Wiśniowski and H. Rosengren, *Inclusions of Waterman-Shiba spaces into generalized Wiener classes*, Journal of Mathematical Analysis and Applications 419(1) (2014) 428–432

IV. **M. Hormozi** and K. Rodtes, *Symmetry classes of tensors associated with the Semi-Dihedral groups SD_{8n}*, Colloquium Mathematicum (2013) 131(1) 59–67

V. **M. Hormozi** and K. Rodtes, *Orthogonal bases of Brauer symmetry classes of tensors for certain groups for Dicyclic and Semi-dihedral groups*, Submitted

VI. The Anh Bui, **M. Hormozi**, *Weighted bounds for multilinear square functions*, Submitted

Contents

1 Introduction

The present thesis consists of six different papers. Indeed, they treat three different research areas: function spaces, singular integrals and multilinear algebra. In the following, we will discuss these and give some aspects and summary of papers I-VI.

1.1 Function spaces

In paper I, II and III, we are concerned with functions of generalized bounded variation. Jordan [51] introduced the functions of bounded variation (BV) in 1881. Since that time, BV has had an important place in Fourier analysis going back to the Dirichlet-Jordan theorem [51]. After the first generalization of the space BV by Wiener [82], many people extended the convergence results of the Dirichlet-Jordan theorem by generalizing the concept of bounded variation to larger classes of functions. In this area one may cite the papers of L.C. Young [85], Salem [70], Musielak and Orlicz [67], Garsia and Sawyer [37], Waterman [79] and Z. A. Chanturiya [16].

We begin by defining concepts and giving the past results which are pertinent to paper I.

Suppose $f : [a,\, b] \to \mathbb{R}$, for $I = [x,\, y] \subset [a,\, b]$ we write $f(I) = f(y) - f(x)$. Let $\{I_i\}$ be a collection of nonoverlapping intervals with $I_n = [a_n,\, b_n]$, where by nonoverlapping we mean that any two such intervals intersect in at most one point. A function $f : [a,\, b] \to \mathbb{R}$ is said to be of **bounded variation** on $[a,\, b]$ if there exists a constant M such that

$$\sum |f(I_i)| \; < \; M,$$

whenever $\{I_i\}$ is a collection of nonoverlapping intervals in $[a, b]$. Suppose $f(x+)$ and $f(x-)$ are the right and left limits respectively of f at x. By *regulated*, we mean a function with only simple discontinuities such that for each x we have $f(x) = \frac{1}{2}(f(x+) + f(x-))$. The number of discontinuities of a regulated function is at most countable.
We will now consider some results connecting BV and Fourier series.

We first state the Dirichlet-Jordan theorem which gives a condition on f which leads to the convergence of its Fourier series $S(f)$.

Theorem 1.1. *(Dirichlet-Jordan) If f is regulated, periodic and of bounded variation on $[0, 2\pi]$, then*
(i) The Fourier series $S(f, x)$ converges to f everywhere.
(ii) If f is continuous at every point of a closed interval P, then $S(f, x)$ converges uniformly on P.

In 1972, Waterman introduced functions of Λ-bounded variation.

Definition 1.2. Let $\Lambda = \{\lambda_i\}$ be a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_i} = +\infty$. Let f be a real function on $[a,\, b]$ and $\{I_i\}$ a collection of nonoverlapping intervals in $[a, b]$. If for each such collection we have

$$\left(\sum_{i=1}^{\infty} \frac{|f(I_i)|}{\lambda_i} \right) \; < \; +\infty,$$

then f is said to be of Λ-**bounded variation** on $[a, b]$.

It is clear that $BV \subset \Lambda BV$. The class ΛBV with $\lambda_k = k$ is referred to as HBV, the functions of **harmonic bounded variation.**

Let $V_\Lambda(f) \;:=\; \sup \left\{ \sum \frac{|f(I_i)|}{\lambda_i} \right\}$, where the supremum is over all collections $I_i \subseteq [a, b]$. It can be seen [79] that $f \in \Lambda BV$ if and only if $V_\Lambda(f) < \infty$. ΛBV equipped with the norm $\|f\|_\Lambda := |f(a)| + V_\Lambda(f)$ is a Banach space. The discontinuities of a ΛBV function are simple and ΛBV functions are bounded [79].
Since $BV \subset HBV$, the theorem below includes the Dirichlet-Jordan theorem as a special case.

Theorem 1.3. *(Dirichlet-Jordan-Waterman) Let f be a 2π-periodic function of harmonic bounded variation on the interval $[-\pi, \pi]$, then*
(i) at every point x, $S(f, x) \to \frac{1}{2}(f(x+) + f(x-))$; in particular, if f is continuous at x then $S(f, x)$ converges to f.
(ii) if f is continuous at every point of a closed interval P, then $S(f, x)$ converges uniformly on P.
(iii) if $\Lambda BV \setminus HBV \neq \emptyset$ then there is a continuous f in ΛBV whose Fourier series diverges at a point.

Let p be a number greater than or equal to 1. A function $f : [a,\, b] \to \mathbb{R}$ is said to be of bounded p-Λ-variation on a not necessarily closed subinterval $P \subset [a,\, b]$ if

$$V(f) \;:=\; \sup \left(\sum_{i=1}^{n} \frac{|f(I_i)|^p}{\lambda_i} \right)^{\frac{1}{p}} \; < \; +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping subintervals of P. The symbol $\Lambda BV^{(p)}$ denotes the linear space of all functions of bounded p-Λ-variation with domain $[0,\, 1]$. The Waterman-Shiba class $\Lambda BV^{(p)}$ was introduced in 1980 by M. Shiba in [74] and it clearly is a generalization of the Waterman class ΛBV. Some of the basic properties of functions of class $\Lambda BV^{(p)}$ were discussed by R.G. Vyas in [76]. More results concerned with the Waterman-Shiba classes and their applications can be found in [11], [12], [54], [71], [72], [75] and [77]. $\Lambda BV^{(p)}$ equipped with the norm $\|f\|_{\Lambda,p} := |f(0)| + V(f)$ is a Banach space.

Functions in a Waterman-Shiba class $\Lambda BV^{(p)}$ are integrable [76, Thm. 2], and thus it makes sense to consider their integral modulus of continuity

$$\omega_1(\delta,\, f) \;:=\; \sup_{0\le h\le \delta} \int_0^{1-h} |f(t+h) - f(t)|\, dt,$$

for $0 \le \delta \le 1$. However, if f is defined on $\mathbb{R}$ instead of on $[0,\, 1]$ and if f is 1-periodic, it is convenient to modify the definition and put

$$\omega_1(\delta,\, f) \;:=\; \sup_{0\le h\le \delta} \int_0^{1} |f(t+h) - f(t)|\, dt,$$

since the difference between the two definitions is then nonessential in all applications of the concept. We will use the second definition.

A function $\omega :\, [0,\, 1] \to \mathbb{R}$ is said to be a modulus of continuity if it is nondecreasing, continuous and $\omega(0) = 0$. If ω is a modulus of continuity, then H_1^ω denotes the class of functions $f \in L^1_{[0,\,1]}$ for which $\omega_1(\delta,\, f) = \mathrm{O}(\omega(\delta))$ as $\delta \to 0+$. We may now state our main results.

Theorem 1.4. *If (f_j) is a sequence in $\Lambda BV^{(p)}$ with $\|f_j\|_{\Lambda,p} \le M$, then there exists a subsequence (f_{j_k}) converging pointwise to a function f in $\Lambda BV^{(p)}$ with $\|f\|_{\Lambda,p} \le M$.*

Theorem 1.5. *If $f \in \Lambda BV^{(p)}$, then*

$$|\hat{f}(n)| \;\le\; \frac{V(f)}{2\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}}.$$

The main result of paper I provides a characterization of the embedding of Waterman-Shiba classes into classes of functions with given integral modulus of continuity.

Theorem 1.6. *The inclusion $\Lambda BV^{(p)} \subset H_1^\omega$ holds if and only if*

$$\liminf_{n\to\infty}\, \omega^p(\tfrac{1}{n}) \sum_{i=1}^{n} \frac{1}{\lambda_i} \;>\; 0. \tag{1.1}$$

In 2005, Goginava [38] tried to find the sufficient and necessary condition for inclusion $\Lambda BV \subset H_q^\omega$ for $q \in [1,\, +\infty)$. Unfortunately, although the result given is correct, there are serious mistakes in the proof. However, the result of paper II provides a characterization of the embedding $\Lambda BV \subset H_q^\omega$.

Theorem 1.7. *For $p, q \in [1, \infty)$, the inclusion $\Lambda BV^{(p)} \subset H_q^\omega$ holds if and only if*

$$\limsup_{n\to\infty} \frac{1}{\omega(1/n)n^{\frac{1}{q}}} \max_{1\le k\le n} \frac{k^{\frac{1}{q}}}{(\sum_{i=1}^{k} \frac{1}{\lambda_i})^{\frac{1}{p}}} < +\infty.$$

H. Kita and K. Yoneda introduced a new function space which is a *generalization of Wiener classes* [53] (see also [52] and [1]). The concept was further extended by T. Akhobadze in [2] who studied many properties of the generalized Wiener classes $BV(q,\, \delta)$ thoroughly (see [3], [4], [5], [6], [7]).

Definition 1.8. Let $q = (q(n))_{n=1}^{\infty}$ be an increasing positive sequence and let $\delta = (\delta(n))_{n=1}^{\infty}$ be an increasing and unbounded positive sequence. We say that a function $f : [0,\, 1] \to \mathbb{R}$ belongs to the class $BV(q; \delta)$ if

$$V(f, q;\, \delta) := \sup_{n\ge 1} \sup_{\{I_k\}} \left\{ \left(\sum_{k=1}^{s} |f(I_k)|^{q(n)} \right)^{\frac{1}{q(n)}} : \inf_k |I_k| \ge \frac{1}{\delta(n)} \right\} < \infty,$$

where $\{I_k\}_{k=1}^{s}$ are non-overlapping subintervals of $[0, 1]$.

The main result of Paper III formulated below extends the main theorem of [39] essentially and furnishes a new and much shorter proof.

Theorem 1.9. *For $p \in [1, \infty)$ and q and δ sequences satisfying the conditions in Definition* 1.8*, the inclusion $\Lambda BV^{(p)} \subset BV(q;\, \delta)$ holds if and only if*

$$\limsup_{n\to\infty} \left\{ \max_{1\le k\le \delta(n)} \frac{k^{\frac{1}{q(n)}}}{(\sum_{i=1}^{k} \frac{1}{\lambda_i})^{\frac{1}{p}}} \right\} < +\infty. \tag{1.2}$$

1.2 Multilinear Algebra

In this subsection, we begin by defining concepts and results which are pertinent to papers IV and V.

1.2.1 Orthogonal bases of symmetry classes of tensors

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of an n-dimensional complex inner product space V and G be a permutation group on m elements. Let χ be any irreducible character of G. For any $\sigma \in G$, define the operator

$$P_\sigma : \bigotimes_1^m V \to \bigotimes_1^m V$$

by

$$P_\sigma(v_1 \otimes ... \otimes v_m) = (v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(m)}). \tag{1.3}$$

The symmetry classes of tensors associated with G and χ is the image of the symmetry operator

$$T(G,\chi) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)P_\sigma, \tag{1.4}$$

and it is denoted by $V_\chi^n(G)$. We say that the tensor $T(G,\chi)(v_1\otimes\cdots\otimes v_m)$ is a decomposable symmetrized tensor, and we denote it by $v_1 * \cdots * v_m$. We call $V_\chi^n(G)$ the symmetry class of tensors associated with G and χ. Let Γ_n^m be the set of all sequences $\alpha = (\alpha_1, ..., \alpha_m)$, with $1 \leq \alpha_i \leq n$. Define the action of G on Γ_n^m by

$$\sigma.\alpha = (\alpha_{\sigma^{-1}(1)}, ..., \alpha_{\sigma^{-1}(m)}).$$

Now let us denote by e_α^* the tensor $e_{\alpha_1} * \cdots * e_{\alpha_m}$. A basis which consists of decomposable symmetrized tensors e_α^* is called an orthogonal $*$-basis. If χ is not linear, it is possible that $V_\chi(G)$ has no orthogonal $*$-basis.

Marcus and Chollet (1986) asserted that orthogonal bases consisting of decomposable symmetrized tensors never existed if the degree of the character was greater than one [64]. Wang and Gong (1991) showed that this statement was false by exhibiting an orthogonal basis in the case of the degree two character of the dihedral group of order eight [78]. The paper Holmes and Tam [45] generalized that result. In paper IV, we find a new counterexample in the case of the degree two character of the semi-dihedral group. The reader can find further information about the symmetry classes of tensors in e.g. [10], [24],[26],[35], [44], [45] and [65].

In paper IV, we discuss the existence of an orthogonal basis consisting of decomposable vectors for all symmetry classes of tensors associated with semi-dihedral groups

$$SD_{8n} = \langle a, b \mid a^{4n} = b^2 = 1, bab = a^{2n-1}\rangle,$$

where the embedding of SD_{8n} into the symmetric group S_{4n} is given by $T(a)(t) := \overline{t+1}$ and $T(b)(t) := \overline{(2n-1)t}$, where $\overline{m}$ is the remainder of m divided by $4n$.

Definition 1.10. Define $C_{even}^\dagger := \{2, 4, ..., 2n-2\}$, $C_{odd}^\dagger = \{1, 3, 5, \ldots, 2[n/2]-1, 2n+1, 2n+3, \ldots, 2[3n/2]-1\}$, $C^* := \{1, 3, 5, ..., n, 2n+1, 2n+3, 2n+5, ..., 3n\}$.

We define two-dimensional representations, for each natural number h and $\omega = e^{\frac{i\pi}{2n}}$;

$$\rho^h(a) = \begin{pmatrix} \omega^h & 0 \\ 0 & \omega^{(2n-1)h} \end{pmatrix} \text{ and } \rho^h(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Denote $\chi_h = Tr(\rho^h)$. According to paper IV, the non-linear irreducible complex characters of SD_{8n} are the characters are the characters χ_h where $h \in C^{\dagger}_{even}$ or $h \in C^{\dagger}_{odd}$.

If χ is a linear character of G then the symmetry class of tensors associated with G and χ has an orthogonal basis [36]. Therefore, the existence of orthogonal bases for symmetry classes of tensors associated with SD_{8n} and non-linear irreducible complex characters is of interest. The following Theorems of paper IV answer the question.

Theorem 1.11. *Let $G = SD_{8n}$ be a subgroup of S_{4n}, denote $\chi = \chi_h$ for $h \in C^{\dagger}_{even}$, and assume $d = \dim V \geq 2$. Then $V_\chi(G)$ has an orthogonal $*$-basis if and only $\nu_2(\frac{h}{2n}) < 0$.*

In Theorem 1.11, ν_2 denotes 2-adic valuation, that is $\nu_2(\frac{2^k m}{n}) = k$ for m and n odd. Then, the condition $\nu_2(\frac{h}{2n}) < 0$ means that every power of 2 that divides h also divides n. In particular, this condition is never satisfied when n is odd, so in that case $V_\chi(G)$ never has an orthogonal $*$-basis.

Theorem 1.12. *Let $G = SD_{8n}$, be a subgroup of S_{4n}, denote $\chi = \chi_h$ for $h \in C^{\dagger}_{odd}$ (even n) or $h \in C^* \setminus \{n, 3n\}$(odd n), and assume $d = \dim V \geq 2$. Then $V_\chi(G)$ does not have orthogonal $*$-basis.*

1.2.2 Orthogonal bases of Brauer symmetry classes of tensors

Let G be a subgroup of the full symmetric group S_m and p be a fixed prime number. An element of G is *p-regular* if its order is not divisible by p. Denote by $\hat{G}$ the set of all p-regular elements of G. A Brauer character is a certain function from $\hat{G}$ to $\mathbf{C}$ associated with an FG-module where F is a suitably chosen field of characteristic p. The Brauer character is an irreducible if the associated module is simple. If the order of G is not divisible by p then $\hat{G} = G$ and each irreducible Brauer character is irreducible (ordinary) character. Replacing G by $\hat{G}$ and χ by a Brauer character ϕ in (1.4), we can define *Brauer symmetry class of tensors* and o-bases exactly as in §1.2.1.

The Dicyclic group T_{4n} is defined as follows:

$$T_{4n} = \langle r, s | r^{2n} = e, r^n = s^2, s^{-1}rs = r^{-1} \rangle.$$

The group T_{4n} can be embedded in S_{4n} and has $n+3$ conjugacy classes. We refer the reader to Paper V to see the complete list of irreducible Brauer characters of T_{4n}—for different cases of n—consisting of linear irreducible Brauer characters $\hat{\chi}_h, \hat{\chi'}_h$ and nonlinear ones $\hat{\psi}_j, \hat{\psi'}_j$.

Theorem 1.13. *Let $G = T_{4n}$, $0 \leq h < \epsilon$ where $\epsilon = 4$ if $p \neq 2$ and $\epsilon = 1$ if $p = 2$, and put $\phi = \hat{\chi}_h$ or $\hat{\chi'}_h$. The space $V_\phi(G)$ has an o-basis if and only if at least one of the following holds:*

(i) $\dim V = 1$
(ii) $p = 2$,
(iii) $2n$ *is not divisible by* p.

Theorem 1.14. *Let* $G = T_{4n}$, $1 \le j < \frac{l}{2}$, $2n = lp^t$ *such that* $p \nmid l$ *and put* $\phi = \hat{\psi}_j, \hat{\psi'}_j$. *If the space* $V_\phi(G)$ *has an o-basis, then* p *is odd and* $\nu_2(\frac{j}{n}) < 0$.

In case of SD_{8n} for $n \geq 2$, we write $4n = lp^t$ with prime p and integer l not divisible by p. The reader can consult Paper V to see the complete list of irreducible Brauer characters of SD_{8n}—for different cases of p and n—consisting of linear irreducible Brauer characters $\hat{\chi}_j, \hat{\chi'}_j$ and nonlinear ones $\hat{\psi}_h, \hat{\psi'}_h$.

Theorem 1.15. *Let* $\dim V > 2$, $G = SD_{8n}$, $0 \le j \le \epsilon$, *and put* $\phi = \hat{\chi}_j$ *or* $\hat{\chi'}_j$. *Then, space* $V_\phi(G)$ *has an o-basis if and only if* $p = 2$ *or* $4n$ *is not divisible by* p.

Remark 1.16. Theorem 1.15 shows that if $\dim V > 2$, unlike the case for an irreducible character, it is possible that $V_\phi(G)$ has no o-basis when ϕ is a linear Brauer character. This holds when $\dim V = 2$ as well by some simple calculations. □

Theorem 1.17. *Let* $\dim V > 2$, $G = SD_{8n}$ *and write* $4n = lp^t$ *such that* $p \nmid l$. *Let* $h \in \Pi$ *where* Π *is defined in Paper V. Put* $\phi = \hat{\psi}_h$ *or* $\hat{\psi'}_h$. *If the space* $V_\phi(G)$ *has an o-basis, then* p *is odd and* $\nu_2(\frac{h}{n}) \le 0$.

1.3 Singular Integrals

In this subsection, we begin by defining concepts and results which are pertinent to paper VI.

For a general account on multiple weights and related results we refer the interested reader to [60].

Consider m weights $w_1, \ldots, w_m$ and denote $\overrightarrow{w} = (w_1, \ldots, w_m)$. Also let $1 < p_1, \ldots, p_m < \infty$ and p be numbers such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and denote $\overrightarrow{P} = (p_1, \ldots, p_m)$. Set

$$\nu_{\vec{w}} := \prod_{i=1}^{m} w_i^{\frac{p}{p_i}}.$$

We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \Big(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \Big) \prod_{j=1}^{m} \Big(\frac{1}{|Q|} \int_Q w_j^{1-p'_j} \Big)^{p/p'_j} < \infty. \tag{1.5}$$

This condition, introduced in [60], was shown to characterize the classes of weights for which the *multilinear maximal function* $\mathcal{M}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ into $L^p(\nu_{\vec{w}})$ (see [60, Thm. 3.7]). In the linear case—i.e. m is one—we call (1.5) the A_p condition.

1.3.1 History of sharp weighted estimates

The problem of the optimal quantitative estimates for the $L^p(w)$ norm of a given operator T in terms of the A_p constant of the weight w has been very challenging and interesting in the last decades.

First, the problem for the Hardy–Littlewood maximal operator was solved by S. Buckley [13] who proved

$$\|M\|_{L^p(w)} \leq C_p\, [w]_{A_p}^{\frac{1}{p-1}}, \tag{1.6}$$

where C_p is a dimensional constant. We say that (1.6) is a sharp estimate since the exponent $1/(p-1)$ cannot be replaced by a smaller one.

However, for singular integral operators the question was much more complicated. In 2012, T. Hytönen [48] proved the so-called A_2 theorem, which asserted that the sharp dependence of the $L^2(w)$ norm of a Calderón–Zygmund operator on the A_2 constant of the weight w was linear. More precisely,

$$\|T\|_{L^p(w)} \leq C_{T,n,p}[w]_{A_p}^{\max\left(1, \frac{1}{p-1}\right)}, \ 1 < p < \infty. \tag{1.7}$$

Shortly after that, A.K. Lerner gave a much simpler proof [57] of the A_2 theorem proving that every Calderón–Zygmund operator is bounded from above by a supremum of *sparse operators*. Interested readers can consult [49] for a survey on the history of the proof.

The versatility of Lerner's techniques is reflected in the extension of the A_2 theorem to multilinear Calderón–Zygmund operators in [23],

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,\vec{P},T}[\vec{w}]_{A_{\vec{P}}}^{\max(1, \frac{p_1'}{p}, \ldots, \frac{p_m'}{p})} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}. \tag{1.8}$$

See for example [22; 61]. For further details on the theory of multilinear Calderón–Zygmund operators, we refer to [40; 41] and the references therein.

1.3.2 Square functions

Let $S_{\alpha,\psi}$ be the square function defined by means of the cone Γ_α in $\mathbb{R}^{n+1}_+$ of aperture α, and a standard kernel ψ as follow

$$S_{\alpha,\psi}(f)(x) = \Big(\int_{\Gamma_\alpha(x)} |f \star \psi_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \Big)^{1/2},$$

where $\alpha > 1$ and $\psi_t(x) = t^{-n}\psi(x/t)$. In [59], Lerner by applying *intrinsic square function*, introduced in [83], proved sharp weighted norm inequalities for $S_{\alpha,\psi}(f)$. Later on, Lerner himself improved the result— in the sense of determination of sharp dependence on α — in [58] by using the local mean oscillation formula. More precisely,

$$\|S_{\alpha,\psi}\|_{L^p(w)} \lesssim \alpha^n [w]_{A_p}^{\max\left(\frac{1}{2},\frac{1}{p-1}\right)}, \ 1 < p < \infty. \tag{1.9}$$

Motivated by these works, the main aim of Paper VI is to investigate the weighted bounds for certain multilinear square functions. Let us recall definition of multilinear square functions considered in this paper.

For any $t \in (0,\infty)$, let $\psi(x,\vec{y}) := K_t(x,y_1,\dots,y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \ldots = y_m$ in $\mathbb{R}^{n\times(m+1)}$. We assume that $\psi(x,\vec{y})$ is *standard kernel*. For $\vec{f} = (f_1,\dots,f_m) \in \mathcal{S}(\mathbb{R}^n)\times\cdots\times\mathcal{S}(\mathbb{R}^n)$ and $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$ we define

$$\psi_t(\vec{f})(x) = \frac{1}{t^{mn}} \int_{(\mathbb{R}^n)^m} \psi\Big(\frac{x}{t},\frac{y_1}{t},\dots,\frac{y_m}{t}\Big) \prod_{j=1}^m f_j(y_j)dy_j.$$

For $\lambda > 2m, \alpha > 0$, the multilinear square functions $g^*_{\lambda,\psi}$ and $S_{\psi,\alpha}$ associated to $\psi(x,\vec{y})$ are defined by

$$g^*_{\lambda,\psi}(\vec{f})(x) = \Big(\int_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} |\psi_t(\vec{f})(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}$$

and

$$S_{\alpha,\psi}(\vec{f})(x) = \Big(\int_{\Gamma_\alpha(x)} |\psi_t(\vec{f})(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \alpha t\}$.

These two mutilinear square functions were introduced and investigated in [18; 73; 84]. The study on the multilinear square functions has important applications in PDEs and other fields. For further details on the theory of multilinear square functions and their applications, we refer to [19; 20; 21; 25; 32; 33; 34; 43; 17; 84; 18] and the references therein.

In paper VI, we assume that there exist some $1 \le p_1,\dots,p_m \le \infty$ and some $0 < p < \infty$ with $\frac{1}{p} = \frac{1}{p_1}+\cdots+\frac{1}{p_m}$, such that $g^*_{\lambda,\psi}$ maps continuously $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. The next theorems give the sharp weighted bounds with sharp dependence on α for multilinear square functions $S_{\alpha,\psi}(\vec{f})$ and $g^*_{\lambda,\psi}(\vec{f})$.

Theorem 1.18. *Let $\alpha \ge 1$, $\vec{P} = (p_1,\dots,p_m)$ with $1 < p_1,\dots,p_m < \infty$ and $1/p_1+\cdots+1/p_m = 1/p$. If $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{P}}$, then*

$$\|S_{\alpha,\psi}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C_{n,m,\psi,\vec{P}} \alpha^{mn} [\vec{w}]_{A_{\vec{P}}}^{\max(\frac{1}{2},\frac{p_1'}{p},\dots,\frac{p_m'}{p})} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}. \tag{1.10}$$

Theorem 1.19. *Let $\lambda > 2m$, $\vec{P} = (p_1, \ldots, p_m)$ with $1 < p_1, \ldots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$. If $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$, then*

$$\|g^*_{\lambda,\psi}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,\psi,\vec{P}}[\vec{w}]_{A_{\vec{P}}}^{\max(\frac{1}{2},\frac{p_1'}{p},\ldots,\frac{p_m'}{p})} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}. \tag{1.11}$$

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Corrections to paper I

- Throughout the proof of Lemma 2.1, $\delta/2$ should be replaced by $(\delta/2)^p$ in all inequalities of the form $a_k < \delta/2$ or $a_k \geq \delta/2$.
- In the definition of S in case 1, the term a_{k+2}/λ_{k+2} should be a_{k+1}/λ_{k+2}.
- On page 47, V^p is not subadditive on intervals. The easiest solution is to write $\|f\| \leq \sum_{k=1}^{\infty} \|f_k\|$. Making the same estimates, $2^p \sum_{i=1}^{\infty} \frac{1}{n_{i-1}^{2p}}$ will be replaced by $2 \sum_{i=1}^{\infty} \frac{1}{n_{i-1}^{2}}$, which is still convergent.
- At the bottom of page 47, two occasions of 4 should be 2 and the outer exponent $-\frac{1}{p}$ should be $-\frac{1}{2}$.
- In reference [14], the article number should be 23.