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Stochastic Partial Differential Equations with Multiplicative Noise

Numerical simulations of strong and weak approximation errors

Thesis for the degree of Master of Science

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Abstract

A finite element Galerkin spatial discretization together with a backward Euler scheme is implemented to simulate strong error rates of the homogeneous stochastic heat equation with multiplicative trace class noise in one dimension. For the noise, two different operators displaying different degrees of regularity are considered, one of which is of Nemytskii type. The discretization scheme is extended to include discretization of the covariance operator of the Q-Wiener process driving the equation. The results agree with the theory. Furthermore, for exploratory reasons, weak error rates are also simulated using the classical Monte Carlo method and the multilevel Monte Carlo method.

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Contents

List of symbols

Symbol	Description
SPDE	Stochastic partial differential equation(s), page 13
FEM	Finite element method(s), page 16
$\mathbb{N}$	Set of all positive integers
$\mathbb{N}_0$	Set of all non-negative integers
$\mathbb{R}$	Set of all real numbers
T	A positive finite real number denoting the end of some time interval $[0, T]$
U, H	Real separable Hilbert spaces
$\langle \cdot, \cdot \rangle_H$	Inner product of a given Hilbert space H
H_0	Hilbert space $Q^{\frac{1}{2}}(H)$, page 11
B, B_1, B_2	Real Banach spaces
$dom(A)$	Domain of the operator A
$C(B_1; B_2)$	Space of all continuous mappings from B_1 to B_2
$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$	A filtered probability space with a normal filtration, page 4
$\mathbb{E}[X]$	Expectation of X, page 9
$\mathcal{B}(B)$	Borel σ-algebra of B
$\mathcal{P}_T$	σ-algebra of predictable stochastic processes, page 11
$\dot{H}^r$	$\dot{H}^r = dom(A^{\frac{r}{2}})$, page 7
$L^p(\Omega, B)$	Banach space of all p-fold integrable mappings from Ω to B
$L(B_1, B_2)$	Banach space of all bounded linear operators from B_1 to B_2
$L_{\mathrm{HS}}(U; H)$	Hilbert space of all Hilbert–Schmidt operators from U to H, page 5
$\|\| \cdot \|\|_B$	Norm of a given Banach space B
$\|\| \cdot \|\|_r$	Norm of the Hilbert space $\dot{H}^r$, page 7
I	Identity operator
1_D	Indicator function of a given measurable set D
I_h	Interpolation operator, page 33

R_h	Ritz projector, page 16
P_h	Orthogonal projector, page 16
$E(t)$	Semigroup generated by $-A$, page 7
$\mathrm{tr}(Q)$	Trace of Q, page 5
β	A real-valued Brownian motion
W	Q-Wiener process, page 10
$E_N[Y]$	Monte Carlo estimator, page 25
$E_L[Y_L]$	Multilevel Monte Carlo estimator, page 28

1 Introduction

In this thesis, we are concerned with the implementation of numerical approximation schemes of stochastic partial differential equations of evolutionary type, driven by multiplicative noise. These are partial differential equations where we have introduced a random noise term so that the solutions become stochastic processes taking values in some function space. Such equations are interesting for a number of reasons (see e.g. [13] or [16] for examples of applications) and we analyze them by considering the abstract problem of finding a solution $X : [0, T] \times \Omega \rightarrow H$ to

$$\begin{aligned} dX(t) + AX(t)dt &= G(X(t))dW(t), \text{ for } 0 \leq t \leq T \\ X(0) &= X_0. \end{aligned}$$

In the main part of this thesis, we will take $A = -\Delta$, where Δ is the Laplacian, $H = L^2([0,1], \mathbb{R})$ and W is a Q-Wiener process where Q is of finite trace. As the operator G controlling how the noise affects X also depend on X we say that this equation, which we refer to as as the *homogeneous stochastic heat equation*, has *multiplicative trace class noise*. It holds that under sufficient constraints on G and X_0 the equation admits a so called *mild solution*

$$X(t) = E(t)X_0 - \int_0^t E(t-s)G(X(s))dW(s)$$

which we want to approximate by some other process $\hat{X}_\ell$ that we can compute. Here E is the so called C_0-semigroup generate by A and the integrals are of Bochner and Itô type respectively. We are interested in the *strong error*

$$||X(T) - \hat{X}_\ell||_{L^2(\Omega;H)}$$

and the *weak error*

$$\left|\mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(\hat{X}_\ell)]\right|$$

where $\phi : H \rightarrow \mathbb{R}$ is some smooth functional.

When we want to implement this theory in a computer program, we have to be able to represent the solution $X(T)$ as an approximation $\hat{X}_\ell$ on finite partitions in space and time. For this we implement a spatio-temporal discretization scheme described in [11], which is the main source of the theory used in this thesis. The particular discretizations in this case are a Galerkin finite element method when discretizing with respect to H and a backward Euler-Maruyama scheme with respect to $[0, T]$. We are not aware

of previously published simulation results on SPDE with multiplicative noise using an implementation of this particular spatio-temporal scheme. The theory of Galerkin finite element methods is well-established and they do not require knowledge of the eigenvalues and eigenvectors of the operator A. We extend the discretization scheme slightly by proving results on a discretization of the covariance operator Q of the underlying Q-Wiener process, using an approach that is similar to the one used in [4].

The error estimate of this scheme given in [11] is in the form of strong errors. Since we are not aware of any papers providing results on the weak convergence rates when we consider SPDE with multiplicative noise with a discretization in both time and space we choose to simply note that we have weak convergence of this scheme since the weak error is bounded by the strong error:

$$\left|\mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(\hat{X}_\ell)]\right| \leq ||X(T) - \hat{X}_\ell||_{L^2(\Omega;H)}$$

for sufficiently smooth ϕ. However, many authors (see in particular [1] and [7] and the references therein for a setting similar to ours) have considered weak convergence in a semidiscrete setting (with respect to either space or time) and in anticipation of a combination of these we do an exploratory simulation on weak convergence rates of this spatio-temporal discretization scheme in the particular setting of the heat equation. There is a common rule of thumb [11, page 9] that in many situations the weak rate of convergence is almost twice the strong rate of convergence, and we wanted to see if we could find an indication of whether this was true in this case as well.

To actually simulate the expectations involved in the weak and strong errors above, we have to use estimators such as the classical Monte Carlo estimator and the so called *multilevel Monte Carlo estimator*. The multilevel Monte Carlo estimator can often reduce the computational work compared to the classical, or singlelevel, Monte Carlo estimator and its application to SPDE has been the subject of a number of recent papers (e.g. [3], [4]). We give proofs on the L^2-convergence of the obtained estimates to the true strong and weak error rates.

The computations involved in the simulations of these error rates are often very expensive. Fortunately, we were granted access to the cluster Glenn of the Chalmers Centre for Computational Science and Engineering (C3SE), and so we were able to also consider quite costly simulations.

In the end, our simulations of the strong error rates are consistent with the theory. The simulations of the weak error rates are also to some extent consistent with the beforementioned rule of thumb, which can be of interest for future research. Furthermore,

we are able to achieve similar results on the simulations of the weak error when using multilevel estimators to a smaller computational cost compared to singlelevel estimators.

1.1 Outline of the thesis

Chapter 2 is intended as an introduction to some notions needed from functional analysis that may be new to some readers. In particular we focus on *Hilbert-Schmidt operators*, *trace class operators* and *fractional operators* along with selected properties of them. We also give a short introduction to parts of stochastic calculus in infinite dimensions, especially the Q-Wiener process and the stochastic integral driven by it.

In Chapter 3 we present a semilinear SPDE of evolutionary type, different notions of solutions of it, as well as the main assumptions we make on the parameters involved. We also recapitulate an existence and uniqueness result on the mild solution.

Chapter 4 contains a brief summary of the spatio-temporal discretization scheme of [11] along with strong error estimates of this. In the last part of this chapter we prove a strong error estimate with respect to discretization of the covariance operator of the Q-Wiener process.

In Chapter 5 we describe the Monte Carlo method and prove results on its application to simulation of strong and weak rates of convergence. We also describe the multilevel Monte Carlo method and its application to estimating weak convergence rates.

Chapter 6 contains our implementation of the theory of the previous chapters. We estimate strong convergence rates and compare single- and multilevel Monte Carlo results on the estimation of the weak convergence rate.

1.2 On notation

In this thesis we mostly follow the notation of [11], with one notable exception in the form of Hilbert-Schmidt spaces. These we denote by $L_{\mathrm{HS}}(\cdot;\cdot)$. We also mention that we use *generic constants* denoted by C. These may vary from line to line in, for example, an equation and are always assumed not to depend on the spatial and temporal step sizes. Finally, we note that when we for real variables x, y write $x \simeq y$ we mean that there exists strictly positive constants C_1 and C_2 such that $C_1 x \leq y \leq C_2 x$.

2 Stochastic calculus in Hilbert spaces

The purpose of this chapter is to introduce some basic concepts needed for the definition of a stochastic partial differential equation (SPDE). It is assumed that the reader has some basic familiarity with measure theory and functional analysis. For an introduction to the material which presupposes less familiarity, [13] is an excellent resource. However, for details on the construction of the Q-Wiener process and the Itô integral, we follow the slightly different approach of [16], which is also a good introductory text.

In this whole chapter, let $0 < T < \infty$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space where $(\mathcal{F}_t)_{t\in[0,T]}$ is a so called *normal* filtration, i.e.

(i) $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$, and

(ii) $\mathcal{F}_t = \bigcap_{s>t}$.

Furthermore, let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ endowed with the Borel σ-algebra $\mathcal{B}(H)$. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis (ONB) of H.

2.1 Hilbert-Schmidt spaces and trace class operators

This subsection serves to introduce concepts from functional analysis that may be new to some readers. We start with the definition of compactness of operators.

Definition 2.1. Given two Banach spaces B_1 and B_2 with $G : X \to Y$ being linear, we say that G is *compact* if whenever a sequence $(x_i)_{i\in\mathbb{N}}$ is bounded in B_1 then $(Gx_i)_{i\in\mathbb{N}}$ has a convergent subsequence in B_2.

We now introduce so called Hilbert-Schmidt operators, which are a subset of linear bounded operators. They will play a very important role throughout the thesis.

Definition 2.2. Let U be another real separable Hilbert space with ONB $(f_i)_{i\in\mathbb{N}}$ and let $G \in L(U; H)$. Then we refer to G as an *Hilbert-Schmidt operator* if

$$\sum_{i=1}^{\infty} ||Gf_i||_H^2 < \infty.$$

The collection of all such operators is denoted by $L_{\mathrm{HS}}(U; H)$ or $L_{\mathrm{HS}}(H)$ if $U = H$. It holds that $L_{\mathrm{HS}}(U; H)$ is a separable Hilbert space when it is equipped with the inner

product

$$\left\langle G, \tilde{G} \right\rangle_{L_{\mathrm{HS}}(U;H)} := \sum_{i=1}^{\infty} \left\langle G f_i, \tilde{G} f_i \right\rangle_H .$$

Next, we prove that the norm defined by this inner product is an upper bound of the operator norm.

Lemma 2.3. *Let $A \in L_{\mathrm{HS}}(U;H)$. Then*

$$||A||_{L(U;H)} \leq ||A||_{L_{\mathrm{HS}}(U;H)}.$$

Proof. Take any $x \in U$ such that $||x||_U = 1$. Then, by the Cauchy–Schwarz inequality in the sequence space ℓ^2,

$$\begin{aligned} ||Ax||_H &= || \sum_{i \in \mathbb{N}} \langle x, f_i \rangle_U \, A f_i ||_H \\ &\leq (\sum_{i \in \mathbb{N}} \langle x, f_i \rangle_U^2)^{\frac{1}{2}} (\sum_{i \in \mathbb{N}} ||A f_i||_H^2)^{\frac{1}{2}} = ||A||_{L_{\mathrm{HS}}(U;H)} \end{aligned}$$

which implies the inequality by definition of the operator norm. □

Another important notion is that of the *trace* of an operator:

Definition 2.4. Let $Q \in L(U)$ be self-adjoint and positive definite. We define the *trace* of G by

$$\mathrm{tr}(Q) := \sum_{i \in \mathbb{N}} \langle Q f_i, f_i \rangle_U$$

Whenever this quantity exists it is independent of the choice of the orthonormal basis, see e.g. [9, page 18]. In this case we refer to Q as an operator of *finite trace* or a *trace class operator*.

Reasoning as in [11, page 12], from [6, Prop. C.3] it follows that such operators are compact. Therefore, by the Spectral Theorem [14, Theorem 4.24], Q diagonalizes with respect to an ONB $(f_i)_{i \in \mathbb{N}}$ of U, i.e.

$$Q f_i = \mu_i f_i \tag{1}$$

for all $i \in \mathbb{N}$ with $\mu_i \in \mathbb{R}^+$. We therefore have $\mathrm{tr}(Q) = \sum_{i=1}^{\infty} \mu_i$. Similarly, given the eigenbasis $(f_i)_{i \in \mathbb{N}}$, we can construct a trace class operator by choosing a positive real sequence $(\mu_i)_{i \in \mathbb{N}}$ of eigenvalues such that $\sum_{i=1}^{\infty} \mu_i < \infty$.

We will also use the following result.

Proposition 2.5. [11, Proposition 2.6] *Let $Q \in L(U)$ be positive definite and self-adjoint. Then there exists a unique self-adjoint and positive definite operator $Q^{\frac{1}{2}} \in L(U)$ such that $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$.*

Given the relation (1), it is easy to see that

$$Q^{\frac{1}{2}} f_i = \mu_i^{\frac{1}{2}} f_i \tag{2}$$

for the ONB $(f_i)_{i\in\mathbb{N}}$ of U. Hence, we have the following relationship between this operator and the trace of Q:

$$||Q^{\frac{1}{2}}||^2_{L_{\mathrm{HS}}(U)} = \mathrm{tr}(Q). \tag{3}$$

2.2 Semigroups and fractional powers of operators

In this section, we will consider densely defined, linear, self-adjoint positive definite operators A with compact inverse which are not necessarily bounded. An example is $-\Delta$ when $H = L^2([0,1];\mathbb{R})$. Here Δ is the Laplace operator, which will play a vital part in the SPDE considered in later parts of the thesis.

We first define the *semigroups* that such operators generate. They can be thought of as extensions of the exponential operator. The definitions come from [11, Appendix B.1].

Definition 2.6. Consider a Banach space B. A family $(E(t))_{t\in[0,\infty)}$ with $E(t) \in L(B)$ for all $t \in [0,\infty)$ is called a *strongly continuous semigroup* or a *C_0-semigroup* if

(i) $E(0) = I$, the identity operator,

(ii) $E(t+s) = E(t)E(s)$ for all $t, s \geq 0$ and

(iii) $\lim\limits_{t\searrow 0} E(t)b = b$ for all $b \in B$.

If in addition,

(iv) $||E(t)||_{L(B)} \leq 1$ for all $t > 0$,

then E is called a *semigroup of contractions*.

Definition 2.7. Let $(E(t))_{t\in[0,\infty)}$ and B be as in the previous definition. The linear operator $-A$ defined by

$$-Ab = \lim_{h\searrow 0} \frac{E(h)b - b}{h}$$

with domain

$$\mathrm{dom}(-A) = \left\{ b \in B : \lim_{h\searrow 0} \frac{E(h)b - b}{h} \text{ exists in } B \right\}$$

is called the *inifinitesimal generator of the semigroup* $(E(t))_{t\in[0,\infty)}$.

For our choice of A (again, think of the Laplace operator) the following two results on *fractional operators* hold. Here we take $B = H$, the Hilbert space considered in the beginning of this chapter.

Proposition 2.8. [11, Appendix B.2] *Let $A : dom(A) \subseteq H \to H$ be a densely defined, linear, self-adjoint and positive definite operator with compact inverse A^{-1}. Then A diagonalizes with respect to an eigenbasis of H $(e_i)_{i\in\mathbb{N}}$ in H with an increasing sequence of eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$. Furthermore, for $r \geq 0$, the fractional operators $A^{\frac{r}{2}} : dom(A^{\frac{r}{2}}) \subseteq H \to H$ are defined by*

$$A^{\frac{r}{2}} x := \sum_{n=1}^{\infty} \lambda_i^{\frac{r}{2}} \langle x, e_i \rangle_H e_i \text{ for } x \in dom(A^{\frac{r}{2}}).$$

It also holds that

$$\dot{H}^r := dom(A^{\frac{r}{2}}) = \left\{ x \in H : ||x||_r^2 := \sum_{i=1}^{\infty} \lambda_i^r \langle x, e_i \rangle_H^2 \right\}$$

are separable Hilbert spaces when equipped with the inner product

$$\langle \cdot, \cdot \rangle_r := \left\langle A^{\frac{r}{2}} \cdot, A^{\frac{r}{2}} \cdot \right\rangle_H .$$

The operator $-A$ generates a C_0-*semigroup* of contractions, which is explicitly expressed in the next corollary that finishes this section.

Corollary 2.9. [13, Lemma 3.21] *Let $A : dom(A) \subseteq H \to H$ be a densely defined, linear, self-adjoint and positive definite operator with compact inverse A^{-1}. Then, the*

family $(E(t))_{t\in[0,\infty)}$ *with* $E(t) \in L(H)$ *defined by*

$$E(t)h := \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle h, e_i \rangle_H e_i$$

is a C_0*-semigroup of contractions, generated by* $-A$*.*

2.3 Random variables

In this section, we generalize some common notions from real-valued probability theory to our setting. The solution X to the SPDE considered later will have to take values in a general Hilbert space, therefore the common definition of a real valued random variable must be extended to a more general notion.

Definition 2.10. Let $(B, ||\cdot||_B)$ be any Banach space. An $\mathcal{F} - \mathcal{B}(B)$ measurable function $X : \Omega \to B$ is called a *B-valued random variable*. If $B = \mathbb{R}$ then we refer to it as a *random variable*.

Definition 2.11. Let $(\mathcal{E}_i)_{i\in I}$ be a (possibly uncountable) family of sub-σ-algebras of $\mathcal{F}$. These are said to be *independent* if for any finite subset $J \subseteq I$ and every family $(E_j)_{j\in J}$ with $E_j \in \mathcal{E}_j$ we have

$$P\left(\bigcap_{j\in J} E_j\right) = \prod_{j\in J} P(E_j).$$

A family of B-valued random variables $(X_i)_{i\in I}$ is called independent if the corresponding family of generated σ-algebras $(\sigma(X_i))_{i\in I}$ is independent.

To define the expectation of X, one needs the so called *Bochner integral*, an extension of the Lebesgue integral to functions taking values in any Banach space. For the construction of it, we refer to [8, pages 156 and 179].

Definition 2.12. The *expectation* of a B-valued random variable X is given by

$$\mathbb{E}[X] := \int_{\omega\in\Omega} X(\omega) dP(\omega)$$

whenever $\mathbb{E}[||X||_B] < \infty$.

We note one important property of the Bochner integral:

$$||\mathbb{E}[X]||_B \leq \mathbb{E}[||X||_B]. \tag{4}$$

One can go on and define a covariance that takes values in Hilbert Spaces, but for our purposes we will incorporate this in the definition of a *Gaussian H-valued random variable*. There are several equivalent definitions of Gaussian law in Hilbert spaces, here we follow that of [16].

Definition 2.13. A probability measure μ on $(H, \mathcal{B}(H))$ is called *Gaussian* if for each $h \in H$, the bounded linear mapping $\langle \cdot, h \rangle_H$ has a *Gaussian law*, i.e. there exist real numbers m_h and $\sigma_h \geq 0$ such that if $\sigma_h > 0$

$$\mu(\{u \in H : \langle u, h \rangle_H \in D)\}) = \frac{1}{\sqrt{2\pi\sigma_h^2}} \int_A e^{-\frac{(x-m_h)^2}{2\sigma_h^2}} \, dx$$

for all $D \in \mathcal{B}(\mathbb{R})$, and if $\sigma_h = 0$,

$$\mu(\{u \in H : \langle u, h \rangle_H \in D)\}) = 1_D(m_h)$$

for all $D \in \mathcal{B}(\mathbb{R})$, where 1_D is the indicator function of D.

Theorem 2.14. [16, Theorem 2.1.2] *A probability measure μ on $(H, \mathcal{B}(H))$ is Gaussian if and only if its characteristic function*

$$\hat{\mu}(h) := \int_H e^{i\langle u,h \rangle_H} \mu(du) = e^{i\langle h,m \rangle_H - \frac{1}{2}\langle Qh,h \rangle_H} \tag{5}$$

for all $h \in H$ where $m \in H$ and $Q \in L(H)$ is of trace class.

Conversely, we have:

Theorem 2.15. [16, Corollary 2.1.7] *Let $Q \in L(H)$ be of trace class and let $m \in H$. Then there exists a Gaussian measure μ fulfilling (5).*

Definition 2.16. Let X be an H-valued random variable. X is called a *Gaussian H-valued random variable* if its image measure $P \circ X^{-1}$ is a Gaussian probability measure. In this case, Q in Theorem 2.14 is called the *covariance (operator)* of X, and we write $X \sim N(m, Q)$.

In connection to this, we also mention that by [16, Proposition 2.16]

$$\mathbb{E}[X] = m.$$

2.4 Q-Wiener Processes

In this section, we define an infinite-dimensional analogue to the Wiener process. First we need to introduce *stochastic processes*.

Definition 2.17. Given a Banach space B, a family of B-valued random variables $(X(t))_{t\in[0,T]}$ is called a *B-valued stochastic process*. It is said to be *adapted* if $X(t)$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$.

We can equally well think of a stochastic process as a function $X : [0,T] \times \Omega \to B$ and we will mostly use this notation. The next definition follows the lines of [16].

Definition 2.18. Let Q be a trace class operator $Q \in L(H)$. A stochastic process $W : [0,T] \times \Omega \to H$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ is called a *(standard) Q-Wiener process* if

- $W(0) = 0$,
- W has P-a.s. continuous trajectories,
- W has independent increments and
- for all $0 \le s < t \le T$ the increment $W(t) - W(s) \sim N(0, (t-s)Q)$.

If also the following holds,

- W is adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ and
- $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \le s < t \le T$,

then W is called a Q-Wiener process *with respect to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$.

When $H = \mathbb{R}$ we allow for $Q = I$. In this case we call W a *real-valued (standard) Wiener process* or a *Brownian motion* and we denote it by β. Using this process, we mention another representation of the general Q-Wiener process. This is called the *Karhunen–Loève expansion*.

Theorem 2.19. [13, Theorem 10.7] *Let Q be as above with eigenvectors $(e_i)_{i\in\mathbb{N}}$ and eigenvalues $(\mu_i)_{i\in\mathbb{N}}$. Then $W : [0,T] \times \Omega \to H$ is a Q-Wiener process if and only if*

$$W(t) = \sum_{i=1}^{\infty} \mu_i^{\frac{1}{2}} \beta_i(t) e_j \,, P\text{-a.s.}$$

where $(\beta_j)_{j=1}^{\infty}$ is a sequence of independent identically distributed real-valued Wiener processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$. The series converges in $L^2(\Omega, H)$ and even in $L^2(\Omega, C([0,T], H))$.

2.5 Stochastic integrals

Before defining the stochastic Itô integral which takes values in Hilbert spaces, it is useful to, as in [11], introduce the separable Hilbert space $H_0 := Q^{\frac{1}{2}}(H)$ together with the inner product

$$\langle \cdot, \cdot \rangle_{H_0} := \left\langle Q^{-\frac{1}{2}} \cdot, Q^{-\frac{1}{2}} \cdot \right\rangle_H \tag{6}$$

If Q is not one-to-one, $Q^{-\frac{1}{2}}$ denotes the pseudoinverse of $Q^{\frac{1}{2}}$.

Now note that if H is a Hilbert space, then so is $L(H)$ when equipped with the operator norm [14, Proposition 2.3]. This allows us to consider Bochner integrals with respect to $L(H)-$valued stochastic processes. We denote the *H-valued stochastic Itô integral* of a stochastic process $\Phi : [0,T] \times \Omega \to L(H)$ with respect to the Q-Wiener process W as

$$\int_0^T \Phi(s) dW(s).$$

As stated in [11, page 17], this is a well defined H-valued random variable if Φ is *integrable*, that is, if,

$$\Phi \in L^2([0,T] \times \Omega, \mathcal{P}_T, \mathrm{d}t \otimes P; L_{\mathrm{HS}}(H_0, H))$$

where $\mathcal{P}_T$ is the σ-algebra of *predictable stochastic processes*,

$$\mathcal{P}_T := \sigma(\{(s,t] \times F_s | 0 \leq s < t \leq T, F_s \in \mathcal{F}_s\} \cap \{\{0\} \times F_0 | F_0 \in \mathcal{F}_0\}).$$

We will not go into the construction of it here but refer to [16] for this. We will, however, mention two key properties of it, from [11, Chapter 2.2].

Theorem 2.20 (Itô isometry)**.** *For all integrable stochastic processes $\Phi : [0,T] \times \Omega \to L(H)$ the following holds:*

$$\mathbb{E}\left[\left|\left| \int_0^t \Phi(s)\, dW(s) \right|\right|^2 \right] = \int_0^t ||\Phi(s)||^2_{L_{HS}(H_0;H)}\, ds$$

for $t \in [0,T]$.

Theorem 2.21 (Burkholder-Davis-Gundy-type inequality)**.** *For any $p \in [2, \infty)$, $0 \leq t_1 < t_2 \leq T$ and for any predictable process $\Phi : [0,T] \times \Omega \to L_{\mathrm{HS}}(H_0; H)$ satisfying*

$$\mathbb{E}\left[\left(\int_{t_1}^{t_2} ||\Phi(s)||^2_{L_{\mathrm{HS}}(H_0;H)} ds\right)^{\frac{p}{2}}\right] < \infty,$$

there exists a constant $C > 0$ depending only on p such that

$$\mathbb{E}\left[\left|\left|\int_{t_1}^{t_2} \Phi(s) dW(s)\right|\right|^p\right] \leq C\mathbb{E}\left[\left(\int_{t_1}^{t_2} ||\Phi(s)||^2_{L_{\mathrm{HS}}(H_0;H)} ds\right)^{\frac{p}{2}}\right].$$

We end by proving the following upper bound on $||\Phi(s)||_{L_{\mathrm{HS}}(H_0;H)}$.

Lemma 2.22. *Let $\Phi : [0,T] \times \Omega \to L(H)$ be an integrable stochastic process. Then*

$$||\Phi(s)||_{L_{\mathrm{HS}}(H_0;H)} \leq tr(Q)^{\frac{1}{2}} ||\Phi(s)||_{L(H)} \tag{7}$$

Proof. By (6), we have that

$$||\Phi(s)||^2_{L_{\mathrm{HS}}(H_0;H)} = ||\Phi(s)Q^{\frac{1}{2}}||^2_{L_{\mathrm{HS}}(H)} = \sum_{i \in \mathbb{N}} ||\Phi(s)Q^{\frac{1}{2}} e_i||^2_H$$

Since $\Phi(s) \in L(H)$,

$$\begin{aligned}\sum_{i \in \mathbb{N}} ||\Phi(s)Q^{\frac{1}{2}} e_i||^2_H &\leq ||\Phi(s)||^2_{L(H)} \sum_{i \in \mathbb{N}} ||Q^{\frac{1}{2}} e_i||^2_H \\ &= ||\Phi(s)||^2_{L(H)} \operatorname{tr}(Q)\end{aligned}$$

where the equality follows by equation (3). □

3 Semilinear stochastic partial differential equations

In this chapter, we introduce the stochastic partial differential equation treated in the remainder of this thesis. We consider a simplified version of the setting used in [11], which will be outlined in the next section.

3.1 Setting and assumptions

From now on, we consider the separable Hilbert space $H = L^2([0,1];\mathbb{R})$. For the probability space, the same assumptions as in Chapter 2 apply.

We consider the equation

$$\begin{aligned} dX(t) + [AX(t) + F(X(t)]dt = G(X(t))dW(t), \text{ for } 0 \leq t \leq T \\ X(0) = X_0. \end{aligned} \tag{8}$$

This is to be understood as the integral equation

$$X(t) = X_0 - \int_0^t [AX(s) + F(X(s))]ds + \int_0^t G(X(S))dW(s),$$

where the left integral is of Bochner type while the second is an Itô integral, so that $X(t)$ is H-valued for all $t \in [0,T]$. We will return to what we mean by a solution to (8) in Section 3.2, but first we will describe our assumptions on the terms of the equation. We refer to r below as the *regularity parameter*.

Assumption 3.1.

(i) W is a Q-Wiener process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Given the ONB $(e_i)_{i\in\mathbb{N}}$ with $e_i = \sqrt{2}\sin(i\pi x)$, the trace class operator Q on H is defined through the relation $Qe_i = \mu_i e_i$ where $\mu_i = C_\mu i^{-\eta}$ for some constants $C_\mu > 0$ and $\eta > 1$.

(ii) The linear operator $-A : \mathrm{dom}(A) \to H$ is the Laplacian with zero boundary conditions.

(iii) Only the homogenous case is considered, i.e. $F = 0$.

(iv) Fix a parameter $r \in [0,1)$. The mapping $G : H \to L_{\mathrm{HS}}(H_0; H)$ satisfies for a constant $C > 0$

  (a) $G(h) \in L_{\mathrm{HS}}(H_0, \dot{H}^r)$ for all $h \in \dot{H}^r$,

(b) $||A^{\frac{r}{2}}G(h)||_{L_{\mathrm{HS}}(H_0;H)} \leq C(1+||h||_r)$ for all $h \in \dot{H}^r$,

(c) $||G(h_1)-G(h_2)||_{L_{\mathrm{HS}}(H_0;H)} \leq C||h_1-h_2||_H$ for all $h_1, h_2 \in H$ and

(d) $||G(h)e_i||_H \leq C||h||_H$ for all basis vectors e_i and $h \in H$.

(v) For $r \in [0,1)$ we assume that $X_0 \in \dot{H}^{1+r}$ is the deterministic initial value of the SPDE.

Regarding the choice of the linear operator A, we note that it is known (see e.g. [13, Example 1.90]) that Proposition 2.8 holds for A with the eigenbasis $\{e_i\}_{i\in\mathbb{N}}$ and eigenvalues $\lambda_i = i^2\pi^2$.

3.2 Strong and mild solutions

There are several notions of solutions to (8), two of which we will list here: the *strong* solution and the *mild* solution. In general, we expect that a strong solution is also mild but not vice versa [13, page 449] . The definition of the strong solution comes from [6].

Definition 3.2 (Strong solution)**.** A predictable H-valued process $X : [0,T]\times\Omega \to H$ is called a *strong solution* of (8) if for all $t \in [0,T]$:

(i) $X(t) \in \dot{H}^2 \quad \mathcal{P}_T$-almost surely,

(ii) $P\left(\int_0^T |X(s)| + |AX(s)|\, ds < \infty\right) = 1$,

(iii) $P\left(\int_0^T ||G(X(S))||^2_{L_{\mathrm{HS}}(H_0;H)}\, ds < \infty\right) = 1$ and

(iv) $X(t) = X_0 - \int_0^t [AX(s) + F(X(s))]ds + \int_0^t G(X(S))dW(s)$.

Under Assumption 3.1(ii) $-A$ is the generator of the semigroup E of Corollary 2.9. Now, for the *mild solution*, we follow the definition in [11].

Definition 3.3 (Mild solution)**.** A predictable H-valued process $X : [0,T]\times\Omega \to H$ is called a *p-fold integrable mild solution* of (8) if

$$\sup_{t\in[0,T]} ||X(t)||_{L^p(\Omega;H)} < \infty$$

and for all $t \in [0,T]$ and $h \in H$, we have that P-a.s.

$$X(t) = E(t)X_0 - \int_0^t E(t-s)F(X(s))ds + \int_0^t E(t-s)G(X(s))dW(s). \tag{9}$$

This last definition is the one we will consider in this thesis, and in the next section, we cite an existence and uniqueness result.

3.3 Existence and uniqueness of the mild solution

Assumption 3.1 is stronger than Assumptions 2.13 to 2.17 of [11, Chapter 2] and hence we can use the corresponding result on existence and uniqueness of the mild solution.

Theorem 3.4. [11, Theorem 2.25] *Let Assumption 3.1 hold. Then there exists a unique (up to a modification) integrable mild solution* $X : [0,T] \times \Omega \to H$ *to (8) such that for every* $t \in [0,T]$ *and every* $s \in [0,1)$ *it holds that* $P(X(t) \in \dot{H}^s) = 1$ *with*

$$\sup_{t\in[0,T]} ||X(t)||_{L^2(\Omega;\dot{H}^s)} < \infty \tag{10}$$

Furthermore, for every $\delta \in (0, \frac{1}{2})$ *there exists a constant* $C > 0$ *such that*

$$||X(t_1) - X(t_2)||_{L^2(\Omega;H)} \leq C|t_1 - t_2|^\delta \tag{11}$$

for all $t_1, t_2 \in [0,T]$.

We also mention that due to the stronger assumptions made here, Assumption 2.19 and 2.20 of [11, Chapter 2] are also satisfied, and so the temporal regularity in Theorem 3.4 also holds for $\delta = \frac{1}{2}$, by Theorem 2.31 of [11]. Uniqueness is understood in the sense that if

4 Discretization methods for SPDE

In this chapter, we show how one can discretize the solution of (8) so that it can be simulated on a computer. From now on, we assume the conditions of Assumption 3.1 and consider approximations of the mild solution. Throughout the sections 4.1 and 4.2 we follow closely the approach of [11] but after that we leave this context and consider how the covariance operator Q can be discretized.

4.1 Galerkin finite element methods

In this section, we briefly describe the *Galerkin finite element method*, which is our first step in the discretization of (8). Here finite dimensional subspaces of H are considered, and so we speak of *spatial discretizations*.

Let $(V_h)_{h\in(0,1]}$ be a sequence of finite dimensional subspaces such that $V_h \subset \dot{H}^1 \subset H$. For these spaces, we follow the notation of [11] and consider two orthogonal projections: the usual $P_h : H \to V_h$ and the *Ritz* projection $R_h : \dot{H}^1 \to V_h$. These are defined by the relations

$$\langle P_h x, y_h\rangle_H = \langle x, y_h\rangle_H \text{ for all } x \in H, y_h \in V_h$$

and

$$\langle R_h x, y_h\rangle_1 = \langle x, y_h\rangle_1 \text{ for all } x \in \dot{H}^1, y_h \in V_h.$$

As in [11, Chapter 3.2], we make the following assumptions on these projections:

Assumption 4.1. For the given family of subspaces $(V_h)_{h\in(0,1]}$ and all $h \in (0, 1]$ there exists a constant C such that

(i) $||P_h x||_1 \leq C||x||_1$ for all $x \in \dot{H}^1$ and

(ii) $||R_h x - x||_1 \leq Ch^s||x||_1$ for all $x \in \dot{H}^s$ with $s \in \{1, 2\}$.

We will consider an explicit choice of $(V_h)_{h\in(0,1]}$ later on. Next we introduce the discrete version of the operator A, $A_h : V_h \to V_h$. For each $x_h \in V_h$ we define $A_h x_h$ to be the unique element of V_h such that

$$\langle Ax_h, y_h\rangle_H = \langle x_h, y_h\rangle_1 = \langle A_h x_h, y_h\rangle_H$$

for all $y_h \in V_h$. By using this relation with the properties of the inner product $\langle\cdot,\cdot\rangle_1$ one sees that A_h also is self-adjoint and positive definite on V_h. Therefore, as before, it is the generator of an analytic semigroup of contractions which we denote by $E_h(t)$ and

one can show (see e.g. [11, Section 3.4]) that there exists a unique stochastic process $X_h : [0,T] \times \Omega \to V_h$ which is the mild solution to the stochastic equation

$$dX_h(t) + A_h X_h(t)dt = P_h G(X_h(t))dW(t), \text{ for } 0 \leq t \leq T$$
$$X_h(0) = P_h X_0.$$

This is called the *semidiscrete approximation* of the solution to (8), but we will not consider it in detail in this thesis. The interested reader is referred to [11] for this. Instead, we will focus on the *fully discrete approximation* in which we also consider a discretization with respect to time.

4.2 The implicit Euler scheme

In this section, we mirror the approach of [11, Section 3.5] who in turn draws from [19, Chapter 7]. We refer to these sources for more details and generalizations of our informal introduction to the *implicit* (or *backward*) *Euler–Maruyama scheme*.

Let again $(V_h)_{h\in(0,1]}$ be a sequence of finite dimensional subspaces such that for all $h \in (0,1]$, $V_h \subset \dot{H}^1 \subset H$. Consider the homogenous equation

$$du(t) + A_h u(t)dt = 0$$

with inital value $u(0) = u_0 \in V_h$ for $t > 0$ and some fixed $h \in (0,1]$. One can then show (see e.g. [19]) that the solution to this is given by the semigroup generated by $-A_h$, namely $E_h(t)$. We may approximate this equation by defining the recursion

$$\hat{u}_j - \hat{u}_{j-1} + kA_h\hat{u}_j = 0, \qquad j \in \mathbb{N}$$

for some fixed time step $k \in (0,1]$ where $\hat{u}_j$ denotes the approximation of $u(t_j)$ with $t_j := jk$. A closed form of this is then given by

$$\hat{u}^j = (I + kA_h)^{-j}u_0, \qquad j \in \mathbb{N}_0$$

Now, following the notation of [11, Section 3.5] we write

$$E_{k,h}(t) := (I + kA_h)^{-j} \text{ if } t \in [t_{j-1}, t_j) \text{ for } j \in \mathbb{N}$$

and we call this operator the *rational approximation of the semigroup* $E(t)$ generated by $-A$. We end this section by citing the following smoothing property of the scheme, from [11, page 67]:

$$||A_h^\rho E_{k,h}(t)x_h|| \leq Ct_j^{-\rho}||x_h|| \tag{12}$$

which holds for any $t \in [t_{j-1}, t_j)$, $\rho \in [0,1]$ and $x_h \in V_h$.

4.3 A fully discrete strong approximation of the SPDE

In this section, we combine the Galerkin method with the linearly implicit Euler–Maruyama scheme and cite a convergence rate of the fully discrete approximation X_h^j of $X(t_j)$, where X is the mild solution of (8).

For this, consider the same sequence of subspaces $(V_h)_{h\in(0,1]}$ as before and let $T > 0$ be the fixed final time. Define a uniform timegrid with a time step $k \in (0, 1]$ by $t_j = jk$, $j = 0, 1, ..., N_k$ with $N_k k = T$. Denote the fully discrete approximation of $X(t_j)$, where X is the mild solution of (8), by X_h^j. The recursion scheme that approximates X is

$$\begin{aligned} X_h^j - X_h^{j-1} + k(A_h X_h^j) &= P_h G(X_h^{j-1})\Delta W^j \text{ for } j = 1, ..., N_k \\ X_h^0 &= P_h X_0 \end{aligned} \tag{13}$$

where ΔW^j are the Wiener increments $W(t_j) - W(t_{j-1})$.

In terms of the operator $E_{k,h}(t)$ one may equally well express this as

$$X_h^j = E_{k,h}(t_{j-1})P_h X_0 + \int_0^{t_j} E_{k,h}(t_j - s)P_h G_h(s)\, dW(s) \tag{14}$$

where

$$G_h(s) := \begin{cases} G(X_h^{j-1}) \text{ if } s \in (t_{j-1}, t_j], \\ G(P_h X_0) \text{ if } s = 0. \end{cases}$$

The following key theorem on convergence of the fully discrete approximation from [11] holds.

Theorem 4.2. [11, Theorem 3.14] *Under Assumptions 3.1 with $r \in [0, 1)$ and 4.1, for all $p \in [2, \infty)$ there exists a constant C independent of $k, h \in (0, 1]$ such that*

$$||X_h^j - X(t_j)||_{L^p(\Omega;H)} \leq C(h^{1+r} + k^{\frac{1}{2}}). \tag{15}$$

4.4 Noise approximation

An issue remaining when one wants to simulate a realisation of $X(t)$ for some $t \in [0, T]$ is how to simulate the Q-Wiener process W. We know that this can be expressed as an infinite sum of Brownian motions (see Theorem 2.19, the Karhunen–Loève expansion), but we cannot simulate an infinite number of Brownian motions on the computer.

Therefore, if one wants to use this expansion, one needs to truncate it at some point $\kappa \in \mathbb{N}$. We then end up with a new Q-Wiener process:

$$W^{\kappa}(t) = \sum_{j=1}^{\kappa} \mu_j^{\frac{1}{2}} \beta_j(t) e_j$$

with the corresponding covariance operator Q^{κ} defined by the relation

$$Q^{\kappa} e_j = 1_{\{j \leq \kappa\}} \mu_j e_j.$$

In the same way as before, we have a mild solution to (8) but now with truncated noise, and as in (14) it can be represented by

$$X_{\kappa,h}^j = E_{k,h}(t_{j-1}) P_h X_0 + \int_0^{t_j} E_{k,h}(t_j - s) P_h G_{\kappa,h}(s)\, dW^{\kappa}(s) \tag{16}$$

where

$$G_{\kappa,h}(s) := \begin{cases} G(X_{\kappa,h}^j) \text{ if } s \in (t_{j-1}, t_j] \\ G(P_h X_0) \text{ if } s = 0. \end{cases}$$

We also introduce

$$W^{c\kappa}(t) := W(t) - W^{\kappa}(t) = \sum_{j=\kappa+1}^{\infty} \mu_j^{\frac{1}{2}} \beta_j(t) e_j \tag{17}$$

which also is a Q-Wiener process with covariance operator $Q^{c\kappa} = Q - Q^{\kappa}$. It can be seen that for the stochastic integral it holds

$$\int_0^t \phi(s) dW(s) - \int_0^t \phi(s) dW^{\kappa}(s) = \int_0^t \phi(s) dW^{c\kappa}(s).$$

In the following proof, we use these notions to give an error bound for $X_{\kappa,h}^j$, c.f. Theorem 4.2, when $\kappa \in \mathbb{N}$ is chosen appropriately, to reflect the decay η, of the eigenvalues μ_j of Q, (see 3.1(i)). For this we take an approach that is very similar to the one found in [2].

Theorem 4.3. *Assume that Assumption 3.1 with $r \in [0,1)$ and Assumption 4.1 hold. Assume also that $\kappa \simeq h^{-\beta}$ for some $\beta > 0$. Furthermore, if $h^{1+r} \simeq k^{\frac{1}{2}}$ and $\beta(\eta - 1) = 2(1+r)$, for all $p \in [2, \infty)$ it holds that*

$$||X(t_j) - X^j_{\kappa,h}||_{L^p(\Omega;H)} \leq Ch^{1+r}.$$

for some constant $C > 0$.

Proof. Throughout this proof, we will use C to refer to any constant. First we split the error, by using Lemma A.2:

$$\begin{aligned} ||X(t_j) - X^j_{\kappa,h}||^2_{L^p(\Omega;H)} &\leq 2(||X(t_j) - X^j_h||^2_{L^p(\Omega;H)} + ||X^j_h - X^j_{\kappa,h}||^2_{L^p(\Omega;H)}) \\ &=: 2(\mathrm{I} + \mathrm{II}) \end{aligned}$$

For the first term, it holds that

$$\mathrm{I} \leq C(h^{1+r} + k^{\frac{1}{2}})^2 \simeq Ch^{2(1+r)} \tag{18}$$

by Theorem 4.2. By Lemma A.2 and the representation of the fully discrete approximation (14) and its truncated version (16) we have for II:

$$\begin{aligned} &\left|\left|\int_0^{t_j} E_{k,h}(t_j - s)P_h G_h(s)\, dW(s) - \int_0^{t_j} E_{k,h}(t_j - s)P_h G_{\kappa,h}(s)\, dW^k(s)\right|\right|^2_{L^p(\Omega;H)} \\ &\leq 2\left|\left|\int_0^{t_j} E_{k,h}(t_j - s)P_h(G_h(s) - G_{\kappa,h}(s))\, dW(s)\right|\right|^2_{L^p(\Omega;H)} \\ &\quad + 2\left|\left|\int_0^{t_j} E_{k,h}(t_j - s)P_h G_{\kappa,h}(s)\, dW(s)\right.\right. \\ &\qquad \left.\left. - \int_0^{t_j} E_{k,h}(t_j - s)P_h G_{\kappa,h}(s)\, dW^\kappa(s)\right|\right|^2_{L^p(\Omega;H)} =: 2\mathrm{II}_a + 2\mathrm{II}_b \end{aligned}$$

Now, by Theorem 2.21:

$$\begin{aligned}
\mathrm{II}_a &= \mathbb{E}\left[\left|\left|\int_0^{t_j} E_{k,h}(t_j-s)P_h(G_h(s)-G_{\kappa,h}(s))\,dW(s)\right|\right|_H^p\right]^{\frac{2}{p}} \\
&\leq C\,\mathbb{E}\left[\left(\int_0^{t_j} ||E_{k,h}(t_j-s)P_h(G_h(s)-G_{\kappa,h}(s))||^2_{L_{\mathrm{HS}}(H_0;H)}\,ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} \\
&= C\,\mathbb{E}\left[\left(\int_0^{t_j} \sum_{i\in\mathbb{N}} ||E_{k,h}(t_j-s)P_h(G_h(s)-G_{\kappa,h}(s))Qe_i||_H^2\,ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} \\
&\leq C\,\mathbb{E}\left[\left(\int_0^{t_j} \sum_{i\in\mathbb{N}} ||(G_h(s)-G_{\kappa,h}(s))Qe_i||_H^2\,ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} \\
&= C\,\mathbb{E}\left[\left(\int_0^{t_j} ||(G_h(s)-G_{\kappa,h}(s))||^2_{L_{\mathrm{HS}}(H_0;H)}\,ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} \\
&\leq C\mathbb{E}\left[\left(k\sum_{n=1}^{j} ||X_h^{n-1}-X_{\kappa,h}^{n-1}||_H^2\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} \\
&\leq Ck\sum_{n=1}^{j} ||(||X_h^{n-1}-X_{\kappa,h}^{n-1}||_H^2)||_{L^{\frac{p}{2}}(\Omega;\mathbb{R})} = Ck\sum_{n=1}^{j} ||X_h^{n-1}-X_{\kappa,h}^{n-1}||^2_{L^p(\Omega;\mathbb{R})},
\end{aligned}$$

where the second inequality follows from the smoothing result (12) with $\rho = 0$, while the third follows from Assumption 3.1(iv)(c) and the fourth is the triangle inequality. For the other term, by the discussion preceeding this theorem and the representation (2) of $(Q^{c\kappa})^{\frac{1}{2}}$:

$$\begin{aligned}
\mathrm{II}_b &= \left|\left|\int_0^{t_j} E_{k,h}(t_j-s)P_hG_{\kappa,h}(s)\,dW^{c\kappa}(s)\right|\right|^2_{L^p(\Omega;H)} \\
&\leq C\,\mathbb{E}\left[\left(\int_0^{t_j} ||E_{k,h}(t_j-s)P_hG_{\kappa,h}(s)||^2_{L_{\mathrm{HS}}((Q^{c\kappa})^{\frac{1}{2}}[H];H)}\,ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} \\
&= C\,\mathbb{E}\left[\left(\int_0^{t_j} \sum_{i=\kappa+1}^{\infty} \mu_i||E_{k,h}(t_j-s)P_hG_{\kappa,h}(s)e_i||_H^2\,ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}
\end{aligned}$$

$$\leq C\,\mathbb{E}\left[\left(\int_0^{t_j}\sum_{i=\kappa+1}^{\infty}\mu_i||G_{\kappa,h}(s)e_i||_H^2\,ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}$$

where we have used Theorem 2.21 and (12) with $\rho = 0$ again. Now we note that, using Assumption 3.1(i) we have:

$$\sum_{i=\kappa+1}^{\infty}\mu_i = C_\mu\sum_{i=1}^{\infty}(i+\kappa)^{-\eta} \leq C_\mu\int_0^{\infty}(x+\kappa)^{-\eta}\,dx \leq Ch^{\beta(\eta-1)} \tag{19}$$

where we have used the fact that $\kappa \simeq h^{-\beta}$. We now use this observation along with the fact that due to Assumption 3.1(iv)(d) we have $||G_{\kappa,h}(s)e_i||_H \leq C||X^j_{\kappa,h}||_H$ if $s \in (t_{j-1}, t_j]$ to see that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^{t_j}\sum_{i=\kappa+1}^{\infty}\mu_i||G_{\kappa,h}(s)e_i||_H^2\,ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}\\
&\leq \mathbb{E}\left[\left((\sum_{i=\kappa+1}^{\infty}\mu_i)k\sum_{n=1}^{j}||X^{n-1}_{\kappa,h}||_H^2\,ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}\\
&\leq (\sum_{i=\kappa+1}^{\infty}\mu_i)k\sum_{n=1}^{j}||X^{n-1}_{\kappa,h}||^2_{L^p(\Omega;H)} \leq Ch^{\beta(\eta-1)}k\sum_{n=1}^{j}||X^{n-1}_{\kappa,h}||^2_{L^p(\Omega;H)}\\
&\leq Ch^{\beta(\eta-1)}k\sum_{n=1}^{j}\Big(||X^{n-1}_{\kappa,h}-X^{n-1}_h||^2_{L^p(\Omega;H)}\\
&\quad + ||X^{n-1}_h - X(t^k_{n-1})||^2_{L^p(\Omega;H)} + ||X(t^k_{n-1})||^2_{L^p(\Omega;H)}\Big)\\
&\leq Ch^{\beta(\eta-1)}(k\sum_{n=1}^{j}||X^{n-1}_{\kappa,h}-X^{n-1}_h||^2_{L^p(\Omega;H)} + (h^{1+r}+k^{\frac{1}{2}})^2+1)
\end{aligned}$$

where the second inequality is the triangle inequality for $L^{\frac{p}{2}}(\Omega;H)$, the third follows from (19) the fourth follows from Lemma A.2 and the fifth from Theorem 4.2 and (10), noting that $jk \leq T$.

Using the bounds on II_a and II_b, we get

$$\mathrm{II} \leq Ck(1+h^{\beta(\eta-1)})\sum_{n=1}^{j}||X^{n-1}_{\kappa,h}-X^{n-1}_h||^2_{L^p(\Omega;H)} + Ch^{\beta(\eta-1)}((h^{1+r}+k^{\frac{1}{2}})^2+1).$$

Now we can use the discrete Grönwall inequality, Theorem A.1, with $a_n = ||X^n_{\kappa,h} - X^n_h||^2_{L^p(\Omega;H)}$ to get:

$$\begin{aligned}
\text{II} &\leq Ch^{\beta(\eta-1)}((h^{1+r} + k^{\frac{1}{2}})^2 + 1)(1 + Ck(1 + h^{\beta(\eta-1)}))^j \\
&\leq Ch^{\beta(\eta-1)}((h^{1+r} + k^{\frac{1}{2}})^2 + 1)e^{Ckj(1+h^{\beta(\eta-1)})} \\
&\leq Ch^{\beta(\eta-1)}((h^{1+r} + k^{\frac{1}{2}})^2 + 1)e^{2CT} \\
&= Ch^{\beta(\eta-1)}((h^{1+r} + k^{\frac{1}{2}})^2 + 1) \leq Ch^{2(1+r)},
\end{aligned}$$

where we have used that $h^\alpha \leq 1$ for $\alpha > 0$ and also the assumption $\beta(\eta - 1) = 2(1 + r)$. Taken together with (18), we have the result. □

5 Monte Carlo methods

In this chapter, we will describe how one can estimate quantities involving $X(t)$. We start by defining the two types of errors we will analyse. Throughout this chapter, we use the notation $X^j_{\kappa,h}$ to refer to the truncated fully discrete approximation defined in (16).

5.1 Strong and weak errors

We refer to the error $||X(t_j) - X^j_{\kappa,h}||_{L^2(\Omega;H)}$ of Theorem 4.3 as the *strong error* of the truncated fully discrete approximation $X^j_{\kappa,h}$.

Often, one may not be interested in the paths of the solution to our SPDE (8) but rather the average value of some functional of its value at the final time T. Therefore, one is then interested in the *weak error*

$$|\mathbb{E}[\phi(X(T)] - \mathbb{E}[\phi(X^{N_k}_{h,\kappa})]| \tag{20}$$

where $\phi : H \to \mathbb{R}$ can be any sufficiently smooth test function.

In our case, we set $\Phi := ||\cdot||^2$ and refer to the expression

$$|\mathbb{E}[||X(T)||^2_H] - \mathbb{E}[||X^{N_k}_{h,\kappa}||^2_H]| \tag{21}$$

as *the weak error* of our truncated fully discrete approximation of $X(T)$.

Before we continue, we need to briefly mention the definition of a *Fréchet differentiable* operator.

Definition 5.1. Let B_1 and B_2 be Banach spaces and let $U \subseteq B_1$ be an open set. A function $\phi : U \to B_1$ is called *Fréchet differentiable* at $x \in U$ if there exists $\phi'(x) \in L(B_1; B_2)$ such that

$$\lim_{h\to 0} \frac{||\phi(x+h) - \phi(x) - \phi'(x)h||_{B_2}}{||h||_{B_1}} = 0.$$

Then $\phi'(x)$ is referred to as the *Fréchet derivative* of ϕ at $x \in U$.

The weak error is weaker than the strong error in the sense that (as is mentioned in e.g. [11, page 3]):

$$|\mathbb{E}[\phi(X(T)] - \mathbb{E}[\phi(X^{N_k}_{h,\kappa})]| \leq C||X(T) - X^{N_k}_{\kappa,h}||_{L^2(\Omega;H)}. \tag{22}$$

This holds true when ϕ is Fréchet differentiable and

$$||\phi'(x)||_{L(H)} \leq C(1 + ||x||_H^{p-1}).$$

Our choice of ϕ indeed fulfils this condition for all $p \geq 2$, as:

$$||\phi'(x)||_{L(H)} = ||\langle \cdot, x \rangle_H||_{L(H)} \leq ||x||_H$$

by the Cauchy–Schwarz inequality. Therefore, every strongly convergent approximation is also weakly convergent.

5.2 The Monte Carlo method

We first briefly review what the (ordinary) Monte Carlo method entails. Let $(\hat{Y}_i)_{i \in \mathbb{N}}$ be a sequence of independent, identically distributed (i.i.d.) U-valued random variables, where U may be any Hilbert space. Then, for large enough $N \in \mathbb{N}$, one could as in the real case expect to have

$$E_N(Y) := \frac{1}{N} \sum_{i=1}^{N} \hat{Y}_i \approx \mathbb{E}[Y].$$

That this is true is made clear by the following. We cite a simple form of the law of large numbers that holds true in general Hilbert spaces.

Lemma 5.2. [4, Lemma 4.1] *For $N \in \mathbb{N}$ and for $Y \in L^2(\Omega; U)$ it holds that*

$$||\mathbb{E}[Y] - E_N[Y]||_{L^2(\Omega;U)} \leq \frac{1}{\sqrt{N}} ||Y||_{L^2(\Omega;U)}.$$

Using this lemma, we can estimate the additional error when estimating the strong (L^2-)error.

Proposition 5.3. *Let the assumptions of Theorem 4.3 be fulfilled. Then, the Monte Carlo estimator with $N \in \mathbb{N}$ of $||X(t_j) - X^j_{\kappa,h}||_{L^2(\Omega;H)}$ satisfies*

$$\begin{aligned} &\left|\left| E_N\left[||X(t_j) - X^j_{\kappa,h}||_H^2 \right]^{\frac{1}{2}} - ||X(t_j) - X^j_{\kappa,h}||_{L^2(\Omega;H)} \right|\right|_{L^2(\Omega;\mathbb{R})} \\ &\leq \frac{1}{N^{\frac{1}{4}}} ||X(t_j) - X^j_{\kappa,h}||_{L^4(\Omega;H)}. \end{aligned}$$

Proof. We have that

$$\begin{aligned}
&\left|\left|E_N\left[||X(t_j)-X^j_{\kappa,h}||^2_H\right]^{\frac{1}{2}}-||X(t_j)-X^j_{\kappa,h}||_{L^2(\Omega;H)}\right|\right|^2_{L^2(\Omega;\mathbb{R})}\\
&=\left|\left|E_N\left[||X(t_j)-X^j_{\kappa,h}||^2_H\right]^{\frac{1}{2}}-\mathbb{E}\left[||X(t_j)-X^j_{\kappa,h}||^2_H\right]^{\frac{1}{2}}\right|\right|^2_{L^2(\Omega;\mathbb{R})}\\
&\leq\left|\left|\left|E_N\left[||X(t_j)-X^j_{\kappa,h}||^2_H\right]-\mathbb{E}\left[||X(t_j)-X^j_{\kappa,h}||^2_H\right]\right|^{\frac{1}{2}}\right|\right|^2_{L^2(\Omega;\mathbb{R})}\\
&=\left|\left|E_N\left[||X(t_j)-X^j_{\kappa,h}||^2_H\right]-\mathbb{E}\left[||X(t_j)-X^j_{\kappa,h}||^2_H\right]\right|\right|_{L^1(\Omega;\mathbb{R})}\\
&\leq\left|\left|E_N\left[||X(t_j)-X^j_{\kappa,h}||^2_H\right]-\mathbb{E}\left[||X(t_j)-X^j_{\kappa,h}||^2_H\right]\right|\right|_{L^2(\Omega;\mathbb{R})}\\
&\leq\frac{1}{\sqrt{N}}\mathbb{E}\left[||X(t_j)-X^j_{\kappa,h}||^4_H\right]^{\frac{1}{2}}=\frac{1}{\sqrt{N}}||X(t_j)-X^j_{\kappa,h}||^2_{L^4(\Omega;\mathbb{R})},
\end{aligned}$$

where the first inequality follows from the fact that $|\sqrt{a}-\sqrt{b}|\leq\sqrt{|a-b|}$ for $a,b\geq 0$. The second inequality is the Hölder inequality while the third follows from Lemma 5.2. □

When it comes to the weak error, there are (at least) two ways of approximating it with a Monte Carlo method, namely

$$\left|\mathbb{E}\left[||X(T)||^2_H\right]-E_N\left[||X^{N_k}_{h,\kappa}||^2_H\right]\right| \tag{23}$$

and

$$\left|E_N\left[||X(T)||^2_H-||X^{N_k}_{h,\kappa}||^2_H\right]\right|. \tag{24}$$

In practice, neither $\mathbb{E}[||X(T)||^2_H]$ nor $||X(T)||^2_H$ will be known exactly, so one has to estimate them. However, there is an important distinction. The quantity $\mathbb{E}[||X(T)||^2_H]$ is a real number that can be estimated independently of $E_N\left[||X^{N_k}_{h,\kappa}||^2_H\right]$ while $||X(T)||^2_H$ is a real-valued random variable that must be simulated using the same realisation of the Q-Wiener process as $||X^{N_k}_{h,\kappa}||^2_H$. We will return to this in more detail in later sections. For now, we prove the following result, analogously to Proposition 5.3.

Proposition 5.4. *Let the assumptions of Theorem 4.3 be fulfilled. Then, the Monte Carlo estimators (23) and (24) with $N \in \mathbb{N}$ of $|\mathbb{E}[||X(T)||_H^2] - \mathbb{E}[||X_{h,\kappa}^{N_k}||_H^2]|$ satisfy:*

$$\begin{aligned} &\left|\left||\mathbb{E}\left[||X(T)||_H^2\right] - E_N\left[||X_{h,\kappa}^{N_k}||_H^2\right]| - |\mathbb{E}\left[||X(T)||_H^2 - ||X_{h,\kappa}^{N_k}||_H^2\right]|\right|\right|_{L^2(\Omega;\mathbb{R})} \\ &\leq \frac{C}{\sqrt{N}} \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\left|\left||E_N\left[||X(T)||_H^2 - ||X_{h,\kappa}^{N_k}||_H^2\right]| - |\mathbb{E}\left[||X(T)||_H^2 - ||X_{h,\kappa}^{N_k}||_H^2\right]|\right|\right|_{L^2(\Omega;\mathbb{R})} \\ &\leq \frac{C}{\sqrt{N}} \left|\left|X(T) - X_{\kappa,h}^{N_k}\right|\right|_{L^2(\Omega;H)}. \end{aligned} \tag{26}$$

Proof. By the reverse triangle inequality, the left hand side of (25) is bounded by $||\mathbb{E}\left[||X_{h,\kappa}^{N_k}||_H^2\right] - E_N\left[||X_{h,\kappa}^{N_k}||_H^2\right]||_{L^2(\Omega;\mathbb{R})}$. The inequality now follows from Lemma 5.2 and the fact that $||X_{h,\kappa}^{N_k}||_{L^2(\Omega;H)} \leq C < \infty$ for some $C > 0$, which in turn is a consequence of Theorem 4.3 and (10).

Next, we again use the reverse triangle inequality to see that the left hand side of (26) is bounded by

$$\begin{aligned} &\left|\left|E_N\left[||X(T)||_H^2 - ||X_{h,\kappa}^{N_k}||_H^2\right] - \mathbb{E}\left[||X(T)||_H^2 - ||X_{h,\kappa}^{N_k}||_H^2\right]\right|\right|_{L^2(\Omega;\mathbb{R})} \\ &\leq \frac{1}{\sqrt{N}} \left|\left|||X(T)||_H^2 - ||X_{h,\kappa}^{N_k}||_H^2\right|\right|_{L^2(\Omega;\mathbb{R})} \\ &= \frac{1}{\sqrt{N}} \left|\left|\left\langle X(T) + X_{h,\kappa}^{N_k}, X(T) - X_{h,\kappa}^{N_k}\right\rangle_H\right|\right|_{L^2(\Omega;\mathbb{R})} \\ &\leq \frac{C}{\sqrt{N}} \left|\left|X(T) - X_{h,\kappa}^{N_k}\right|\right|_{L^2(\Omega;H)} \end{aligned}$$

Here, the first inequality follows from Lemma 5.2, while the second follows from the Cauchy–Schwarz inequality along with the fact that $||X_{h,\kappa}^{N_k}||_{L^2(\Omega;H)} \leq C < \infty$ and $||X(T)||_{L^2(\Omega;H)} \leq C < \infty$ for some $C > 0$. □

5.3 The Multilevel Monte Carlo method

One problem with the Monte Carlo estimator of the strong and weak errors described in the previous section is the large number of samples needed for a good estimate. This is not a problem when the approximation in time and space is rough, since it is computationally relatively cheap. However, this estimator has a high bias. A compromise is not to solve all samples on the same discretization level - we want to generate a large number of samples on a coarse grid and fewer samples on a fine grid and then add them together to get an estimator that has both low variance and bias and is computationally cheaper than the estimator $E_N(Y)$, which we from now on will refer to as the *singlelevel Monte Carlo estimator*. In this section, we introduce the *multilevel Monte Carlo* in a general framework, similar to that of [3].

Assume that $(Y_\ell)_{\ell\in\mathbb{N}_0}$ is a sequence of approximations of the U-valued random variable Y, where we have denoted the set of non-negative integers by $\mathbb{N}_0$ as opposed by the set of positive integers which we denote by $\mathbb{N}$. For any $L \in \mathbb{N}$ it holds that

$$Y_L = Y_0 + \sum_{\ell=1}^{L} (Y_\ell - Y_{\ell-1})$$

and by the linearity of the expectation operator

$$\mathbb{E}\left[Y_L\right] = \mathbb{E}\left[Y_0\right] + \sum_{l=1}^{L} \mathbb{E}\left[(Y_\ell - Y_{\ell-1})\right]. \tag{27}$$

This motivates the *multilevel Monte Carlo* estimator

$$E^L[Y_L] := E_{N_0}[Y_0] + \sum_{\ell=1}^{L} E_{N_\ell}(Y_\ell - Y_{\ell-1})$$

where $E_{N_\ell}(Y_\ell - Y_{\ell-1})$ is the singlelevel Monte Carlo estimator with a number of independent samples N_ℓ depending on the level ℓ.

We prove the following slightly modified version of [3, Lemma 2.2].

Lemma 5.5. *Let $(Y_\ell)_{l\in\mathbb{N}_0}$ be a sequence of approximations to $Y \in L^2(\Omega,\mathbb{R})$ and assume further that $Y_\ell \in L^2(\Omega;\mathbb{R})$ for all $\ell \in \mathbb{N}_0$. Then it holds that*

$$\begin{aligned} &||\mathbb{E}\left[Y\right] - E^L[Y_L]||_{L^2(\Omega,\mathbb{R})} \\ &\qquad \leq |\mathbb{E}\left[Y - Y_L\right]| + \left(\frac{1}{N_0}||Y_0||^2_{L^2(\Omega;\mathbb{R})} + \sum_{l=1}^{L} \frac{1}{N_\ell}||Y_\ell - Y_{\ell-1}||^2_{L^2(\Omega;\mathbb{R})}\right)^{\frac{1}{2}} \end{aligned}$$

Proof. By the triangle inequality

$$\begin{aligned}||\mathbb{E}\left[Y\right] - E^L[Y]||_{L^2(\Omega,\mathbb{R})} &\leq ||\mathbb{E}\left[Y\right] - \mathbb{E}\left[Y_L\right]||_{L^2(\Omega,\mathbb{R})} + ||\mathbb{E}\left[Y_L\right] - E^L[Y_L]||_{L^2(\Omega,\mathbb{R})} \\ &= |\mathbb{E}\left[Y - Y_L\right]| + ||\mathbb{E}\left[Y_L\right] - E^L[Y_L]||_{L^2(\Omega,\mathbb{R})}.\end{aligned}$$

For the second term, we have using the telescoping sum at the right hand side of (27),

$$\begin{aligned}&||\mathbb{E}\left[Y_L\right] - E^L[Y_L]||^2_{L^2(\Omega,\mathbb{R})} \\ &= ||\mathbb{E}\left[Y_0\right] - E_{N_0}[Y_0] + \sum_{l=1}^{L}(\mathbb{E}\left[Y_\ell - Y_{\ell-1}\right] - E_{N_\ell}[Y_\ell - Y_{\ell-1}])||^2_{L^2(\Omega,\mathbb{R})} \\ &= ||\mathbb{E}\left[Y_0\right] - E_{N_0}[Y_0]||^2_{L^2(\Omega,\mathbb{R})} + \sum_{l=1}^{L}||(\mathbb{E}\left[Y_\ell - Y_{\ell-1}\right] - E_{N_\ell}[Y_\ell - Y_{\ell-1}])||^2_{L^2(\Omega,\mathbb{R})},\end{aligned}$$

where the last equality follows from the linearity of the $||\cdot||^2_{L^2(\Omega;\mathbb{R})}$-operator for real-valued independent zero-mean random variables. The result now follows from Lemma 5.2. □

We now make an explicit application of this multilevel Monte Carlo estimator. Recalling the notation and setting of Theorem 4.3, we set $h_\ell = h_0 2^{-\ell}$ for $\ell \in \mathbb{N}_0$ with some real $h_0 > 0$. Let $\hat{X}_\ell := X^{N_{k_\ell}}_{\kappa_\ell,h_\ell}$ where $k_\ell = h_\ell^2$ and $\kappa_\ell = h_\ell^{\frac{2}{\eta-1}}$. Then we get a series of H-valued random variables such that $\hat{X}_\ell \in L^2(\Omega, H)$, and by Theorem 4.3, for all $p \geq 2$ there exists a constant $C > 0$ independent of h_ℓ, κ_ℓ and k_ℓ such that:

$$||\hat{X}_\ell - X(T)||_{L^2(\Omega;H)} \leq Ch_\ell = Ch_0 2^{-\ell}. \tag{28}$$

We now apply Lemma 5.5 to this sequence of H-valued random variables.

Proposition 5.6. *Let $\hat{X}_\ell := X^{N_{k_\ell}}_{\kappa_\ell,h_\ell}$ where for some real $h_0 > 0$, $h_\ell = h_0 2^{-\ell}$, $k_\ell = h_\ell^2$ and $\kappa_\ell = h_\ell^{\frac{2}{\eta-1}}$. Let $\delta > 0$ and for $\ell \leq L$ with $L, \ell \in \mathbb{N}_0$ let N_ℓ of the multilevel estimator E^L fulfil $N_\ell \simeq h_0^{2(1-\alpha)}\ell^{1+\delta}2^{2(\alpha L-\ell)}$ for $\ell \geq 1$ and $N_0 \simeq h_0^{-2\alpha}2^{2\alpha L}$. Assuming that there exists constants $C_1, \alpha > 0$ such that for all $\ell \in \mathbb{N}_0$ the weak error satisfies*

$$|\mathbb{E}\left[||X(T)||^2_H - ||\hat{X}_\ell||^2_H\right]| \leq C_1 h_0^\alpha 2^{-\alpha\ell},$$

then there exists a constant C_2 not depending on L such that

$$||\mathbb{E}\left[||X(T)||^2_H\right] - E^L[||\hat{X}_L||^2_H]||_{L^2(\Omega,\mathbb{R})} \leq C_2 h_0^\alpha 2^{-\alpha L}.$$

Proof. By Lemma 5.5 we have

$$\begin{aligned}
&||\mathbb{E}\left[||X(T)||_H^2\right] - E^L[||\hat{X}_L||_H^2]||_{L^2(\Omega,\mathbb{R})} \\
&\leq |\mathbb{E}\left[||X(T)||_H^2 - ||\hat{X}_L||_H^2\right]| \\
&\quad + C\left(\frac{1}{N_0}||\hat{X}_0||_{L^4(\Omega;\mathbb{R})}^4 + \sum_{l=1}^{L}\frac{1}{N_\ell}\left|\left|||\hat{X}_\ell||_H^2 - ||\hat{X}_{\ell-1}||_H^2\right|\right|_{L^2(\Omega;\mathbb{R})}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\left|\left|||\hat{X}_\ell||_H^2 - ||\hat{X}_{\ell-1}||_H^2\right|\right|_{L^2(\Omega;\mathbb{R})}^2 = \left|\left|\left\langle \hat{X}_\ell + \hat{X}_{\ell-1}, \hat{X}_\ell - \hat{X}_{\ell-1}\right\rangle_H\right|\right|_{L^2(\Omega;\mathbb{R})}^2 \\
&\leq C\left|\left|\hat{X}_\ell - \hat{X}_{\ell-1}\right|\right|_{L^2(\Omega;H)}^2 \leq 2C\left(\left|\left|X(T) - \hat{X}_\ell\right|\right|_{L^2(\Omega;H)}^2 + \left|\left|X(T) - \hat{X}_{\ell-1}\right|\right|_{L^2(\Omega;H)}^2\right) \\
&\leq Ch_0^2(2^{-2\ell} + 2^{-2(\ell-1)}) = Ch_0^2(2^{-2\ell} + 4\cdot 2^{-2\ell}) \leq Ch_0^2 2^{-2\ell},
\end{aligned}$$

where the first inequality follows from the Cauchy–Schwarz inequality and the fact that the truncated fully approximations are bounded (c.f. the proof of Proposition 5.4). The second inequality is Lemma A.2 while the third is the convergence rate of the strong error, as outlined above. Therefore, using the fact that the truncated fully discrete approximations are bounded once again:

$$\begin{aligned}
&\frac{1}{N_0}||\hat{X}_0||_{L^4(\Omega;\mathbb{R})}^4 + \sum_{\ell=1}^{L}\frac{1}{N_\ell}\left|\left|||\hat{X}_\ell||_H^2 - ||\hat{X}_{\ell-1}||_H^2\right|\right|_{L^2(\Omega;\mathbb{R})}^2 \\
&\leq C\left(\frac{1}{N_0} + \sum_{\ell=1}^{L}\frac{1}{N_\ell}h_0^2 2^{-2\ell}\right) \leq Ch_0^{2\alpha}\left(2^{-2\alpha L} + 2^{-2\alpha L}\sum_{\ell=1}^{L}\ell^{-(1+\delta)}\right) \\
&\leq Ch_0^{2\alpha}(1+\zeta(1+\delta))2^{-2\alpha L} \leq Ch_0^{2\alpha}2^{-2\alpha L},
\end{aligned}$$

where ζ denotes the Riemann zeta function. Summing up, using the assumption on the weak convergence, we get our result

$$\begin{aligned}
&||\mathbb{E}\left[||X(T)||_H^2\right] - E^L[||\hat{X}_L||_H^2]||_{L^2(\Omega,\mathbb{R})} \\
&\leq C(h_0^\alpha 2^{-\alpha L} + (h_0^{2\alpha}2^{-2\alpha L})^{\frac{1}{2}}) \leq Ch_0^\alpha 2^{-\alpha L}.
\end{aligned}$$

□

We remark that the assumption on the weak convergence rate of Proposition 5.6 holds true at least for $\alpha = 1$ since the weak error is bounded by the strong error.

6 Simulations

In this chapter, some numerical experiments in connection to the results of the previous chapters will be described. We will focus on the simulation of the strong and weak approximation errors.

However, we start by defining two noise operators (the term G in (8)), G_1 and G_2, which, we recall, are functions on $H = L^2([0,1];\mathbb{R})$ taking values in $L_{\mathrm{HS}}(H_0;H)$.

6.1 Geometric Brownian motion in infinite dimensions

The first operator G_1 is taken from [11, Section 6.4]. For $h \in H$ and $h_0 \in H_0$ it is defined by

$$G_1(h)h_0 := \sum_{j=1}^{\infty} \langle h, e_j \rangle_H \langle h_0, e_j \rangle_H e_j.$$

We have to check the conditions of Assumption 3.1(iv) to see that they hold. To see that for all $h \in H$, $G_1(h) \in L_{\mathrm{HS}}(H_0;\dot{H}^r)$, note that

$$||G_1(h)||_{L_{\mathrm{HS}}(H_0;\dot{H}^r)} = ||A^{\frac{r}{2}} G_1(h)||_{L_{\mathrm{HS}}(H_0;H)} \leq \mathrm{tr}(Q)^{\frac{1}{2}} ||A^{\frac{r}{2}} G_1(h)||_{L(H)}$$

by Lemma 2.22 and that by Lemma 2.3

$$\begin{aligned} ||A^{\frac{r}{2}} G_1(h)||^2_{L(H)} \leq ||A^{\frac{r}{2}} G_1(h)||^2_{L_{\mathrm{HS}}(H)} &= \sum_{i=1}^{\infty} ||A^{\frac{r}{2}} \langle h, e_i \rangle_H e_i||^2_H \\ &= \sum_{i=1}^{\infty} ||A^{\frac{r}{2}} \langle h, e_i \rangle_H e_i||^2_H = \sum_{i=1}^{\infty} ||\lambda_i^{\frac{r}{2}} \langle h, e_i \rangle_H e_i||^2_H \\ &= \sum_{i=1}^{\infty} \lambda_i^r \langle h, e_i \rangle^2_H = ||h||^2_r \end{aligned}$$

using Parseval's identity and Proposition 2.8. Therefore for all $r \geq 0$ we have

$$||G_1(h)||_{L_{\mathrm{HS}}(H_0;\dot{H}^r)} \leq \mathrm{tr}(Q)||h||_r,$$

which shows the first two assumptions of Assumption 3.1(iv) - the third also follows from this since for $h_1, h_2 \in H$:

$$\begin{aligned} ||G(h_1) - G(h_2)||_{L_{\mathrm{HS}}(H_0;H)} &= ||G(h_1 - h_2)||_{L_{\mathrm{HS}}(H_0;H)} \\ &\leq \mathrm{tr}(Q)||h_1 - h_2||_H. \end{aligned}$$

Finally, the fourth assumption follows from the fact that the ONB of H (see Assumption 3.1(i)) is uniformly bounded.

Now, this choice of G admits an analytical solution of (8), as is shown in [11, Section 6.4] - for $t \in [0, T]$ we get

$$X(t) = \sum_{i=1}^{\infty} \langle X_0, e_i \rangle_H \exp\left(-(\lambda_i + \frac{\mu_i}{2})t + \mu_i^{\frac{1}{2}} \beta_i(t)\right) e_i. \tag{29}$$

So on each basis function of H, the process follows a geometric Brownian motion, which is the reason for the name of this section and this process.

Since $\beta_i(T) \sim \mathrm{N}(0, T)$, it is easy to show that $\mathbb{E}\left[\exp\left(2\mu_i^{\frac{1}{2}} \beta_i(T)\right)\right] = \exp\left(2\mu_i T\right)$ and so by Parseval's identity we have

$$\mathbb{E}\left[||X(T)||_H^2\right] = \sum_{i=1}^{\infty} \langle X_0, e_i \rangle_H^2 \exp\left(-(2\lambda_i + \mu_i)T\right). \tag{30}$$

6.2 The heat equation with multiplicative Nemytskii-type noise

The next operator G_2 is analysed in detail in [10]. For this, we let $\gamma : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function and we then define $G_2 : H \to L_{\mathrm{HS}}(H_0; H)$ for $h \in H$, $h_0 \in H_0$ and $x \in [0, 1]$ by

$$(G_2(h)h_0)[x] := \gamma(h(x))h_0(x).$$

As it is noted in [11, Example 2.23], the analysis of [10, Section 4] shows that G_2 is globally Lipschitz and that there furthermore exists a constant $C > 0$ such that for $h \in H$

$$||G_2(h)||_{L_{\mathrm{HS}}(H_0;H)} \leq C \operatorname{tr}(Q)\left(1 + ||h||_H\right)$$

and under the additional assumption

$$\sum_{i=1}^{\infty} \mu_i \sup_{x \in [0,1]} |e_i'(x)|^2 < \infty \tag{31}$$

there exists a constant $C > 0$ such that

$$||A^{\frac{r}{2}} G(h)||_{L_{\mathrm{HS}}(H_0;H)} \leq C(1 + ||h||_r)$$

for all $r \in [0, \frac{1}{2})$ and $h \in \dot{H}^r$. Since $e_i'(x) = \sqrt{2}i\pi\cos(i\pi x)$, (31) holds if $\eta > 3$. This shows the first three assumptions of Assumption 3.1(iv) for any $r \in [0, \frac{1}{2})$ while the last one again follows from the uniform bound of the ONB of H. In the remainder of the thesis we take $\gamma(x)$ to be $\sin(x)$.

6.3 Simulation setting

In the next few sections, we will use the single level Monte Carlo method to simulate both strong and weak error rates and use the multilevel Monte Carlo method to simulate weak error rates of the equation (8). The computations will be done in MATLAB™, in part on a desktop computer and in part on a computer cluster.

We now consider equidistant partitions in space on the domain $[0,1]$ with $h = \frac{1}{N_h}$, $x_j := jh$, $j = 0, 1, 2, ..., N_h$. For each such partition we let V_h be given by the set of all continuous functions on $[0,1]$ that are piecewise linear on the intervals $[x_j, x_{j+1}]$ for $j = 0, 1, 2, ...N_h - 1$ and zero at the boundary of the domain. From [11, Example 3.6] we see that this choice of V_h fulfils Assumption 4.1. The sequence of *hat functions* $(\Phi_j)_{j=1}^{N_h-1}$ defined by their nodal values

$$\Phi_j(x_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

forms a basis for V_h.

We will compute the numerical approximations $X_{\kappa,h}^{N_k}$ by recursively solving the numerical equation

$$\begin{aligned} X_{\kappa,h}^{j} - X_{\kappa,h}^{j-1} + k(A_h X_{\kappa,h}^{j}) &= G_i^h(X_{\kappa,h}^{j-1})\Delta W^{\kappa,j} \text{ for } j = 1, ..., N_k \\ X_{\kappa,h}^{0} &= I_h X_0. \end{aligned} \tag{32}$$

where $\Delta W^{\kappa,j}$ are the Wiener increments $W^\kappa(t_j) - W^\kappa(t_{j-1})$ and the *interpolation operator* $I_h : H \to V_h$ is defined by

$$I_h f(x) = \sum_{j=1}^{N_h - 1} f(x_j)\Phi_j(x)$$

and for $G_i^h : V_h \to L_{\text{HS}}(H_0; V_h)$, $i \in \{1, 2\}$ we set

$$G_1^h(v)[h_0] := \sum_{i=1}^{\kappa} \langle v, I_h e_i \rangle_H \langle I_h h_0, I_h e_i \rangle_H I_h e_i$$

and

$$G_2^h(v)[h_0] := I_h G_2(v) I_h h_0.$$

We do not exactly know how or if these changes of projections from P_h to different constellations of interpolation operators I_h affect the order of convergence. However, this kind of replacement of projectors seems to be quite common in practice and they make the computation easier. We leave this choice unjustified, subject to future research.

We now fix $\eta = 5$, $T = 1$ and for $\ell \in \mathbb{N}_0$ we set $h_\ell = 2^{-\ell}$, $k_\ell = h_\ell^2$ and $\kappa_\ell = h_\ell^{-1}$ to get a series of solutions to (32) which we denote by $\hat{X}_\ell := X_{\kappa_\ell, h_\ell}^{N_{k_\ell}}$. We note that by our choice of η it would have been sufficient to set $\kappa_\ell = h_\ell^{-1/2}$ and still have, by Theorem 4.3 with $r = 0$,

$$||X(T) - \hat{X}_\ell||_{L^2(\Omega;H)} \leq 2^{-\ell} \tag{33}$$

but for computational reasons we abstain from this and note that still (33) holds with this choice. We also set $X_0(x) := x - x^2$ and note that this satisfies Assumption 3.1(v).

We end this section by remarking that in practice, given a finite element space V_h, we will estimate the norm in H by $||f||_H \approx \sqrt{\sum_{j=1}^{N_h - 1} |f(x_j)|^2}$.

6.4 Results: Strong convergence rates

We start by using the singlelevel Monte Carlo estimator to try to estimate the strong error rate. When we say that the strong error converges with a *rate* of α, we mean in this context that there exists a constant $C > 0$ such that

$$||X(T) - \hat{X}_\ell||_{L^2(\Omega;H)} \leq C h_l^\alpha = C 2^{-\alpha l}. \tag{34}$$

In general, when simulating the quantity on the left hand side of this expression, we do not have access to the exact value of $X(T)$ for a given realisation of X. This is true even when, for the case of $G = G_1$, we can use the expression (29) for $t = T$, since we cannot generate an infinite number of Brownian motions. Instead, we use a so called *reference solution*, that is, we replace the expression $||X(T) - \hat{X}_\ell||_{L^2(\Omega;H)}$ by $||\tilde{X}_L - \hat{X}_\ell||_{L^2(\Omega;H)}$ where we let $\tilde{X}_L = \hat{X}_L$ for $L > \ell$ when we consider $G = G_2$ while we let $\tilde{X}_L$ be given by the expression (29) truncated at $\kappa_L = 2^L$ when we consider the case $G = G_1$.

Now we investigate the strong rate of convergence for different choices of C_μ, which influence the overall level of variance in the realisations of X (recall that the eigenvalues

of Q are $\mu_i = C_\mu i^{-5}$). For $M = 2 \cdot 10^4$, $\ell = 1, 2, 3, 4, 5$ and $L = 6$ we compute the quantities

$$E_M \left[||\tilde{X}_L - \hat{X}_\ell||_H^2 \right]^{\frac{1}{2}}$$

for G_1 and G_2. Informed by Proposition 5.3 we hope that these will approximate the strong error well.

In Figure 1 we plot these values against 2^ℓ for $\ell = 1, 2, 3, 4, 5$ where we use a logarithmic scale for both axes: a so called *log-log plot*. By [18] it holds that in this graph a convergence of order α corresponds to a line with slope $-\alpha$.

It should be noted that given a fixed C_μ all observations of $\tilde{X}_L$ and $\hat{X}_\ell$ are computed on the same M instances of Q-Wiener processes W^{κ_L}. Therefore, the variance will be much lower than if we for each level ℓ had generated $\hat{X}_\ell$ independently of the other levels. Also, the same set of Q-Wiener processes was used for the strong error corresponding to G_1 and the strong error corresponding to G_2, so in Figure 1 the paths are independent of one another with respect to different values of C_μ, but for the same value of C_μ a path in the lower picture is not independent of the corresponding path in the upper picture.

The theoretical results of Chapters 3 and 4 make us expect a convergence rate of order 1 as noted in (34). Furthermore, when $C_\mu = 0$, the equation reduces to the deterministic case, and from [12, Chapter 10] we expect a convergence of order 2 in this case. The results of Figure 1 seem to be more or less in line with this. For small values of C_μ we appear to get a rate of order 2 but asymptotically it seems reasonable to expect a convergence of order 1 in this case. When $C_\mu > 10$ the variance is so great that the curves appear very erratic, despite the use of a single set of Q-Wiener processes for all levels ℓ.

Next, we choose to look more closely at how the strong convergence behaves when $C_\mu = 10$, so we take levels ℓ up to $\ell = 7$ with $L = 8$ and set $M = 3 \cdot 10^3$. Despite the relatively low number of samples, this computation takes a very long time if we were to run it on a typical desktop computer.

Fortunately, we have access to the Glenn cluster at Chalmers Centre for Computational Science and Engineering (C3SE). The system consists of 379 compute nodes with a total of 6080 cores [17]. With ℓ and M as above, the total computation for all levels takes 10 hours and 32 seconds using 8 computing nodes with 128 cores. The result is shown in Figure 2. From this we can see that the trend of a convergence of order 1 seems to hold, even with this somewhat low number of samples.

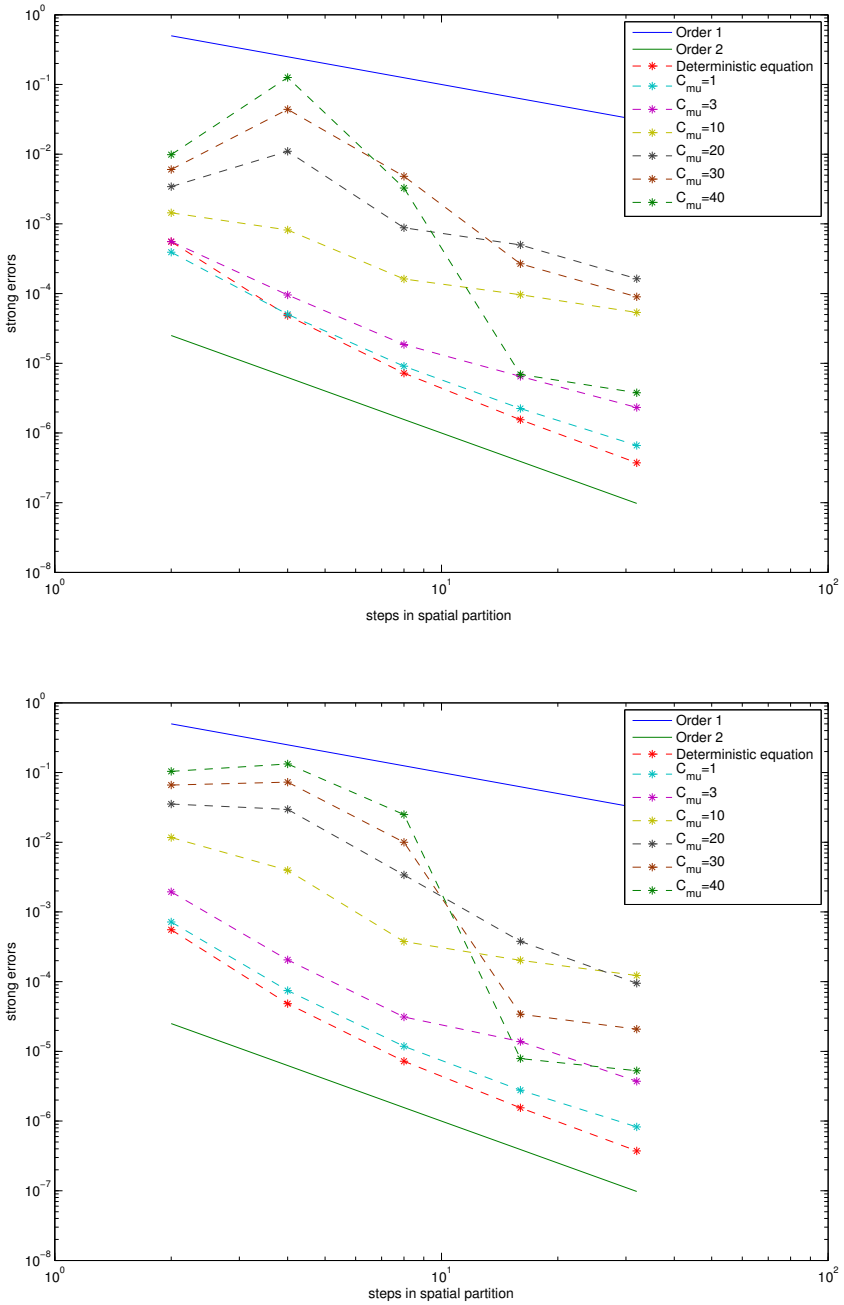

Figure 1: Approximations of the strong error rate for $\ell = 1, 2, 3, 4, 5$ and different values of C_μ. The upper image shows the case $G = G_1$ and the lower image $G = G_2$. For each C_μ, all levels of error approximations have been computed using the same set of $M = 2 \cdot 10^4$ realisations of the Q-Wiener processes.

To utilize the resources of the cluster in an optimal way, it is crucial to make sure that the code can run in parallel. In our case, this essentially amounts to changing the for loop of our Monte Carlo computation to a so called parfor loop, which is a feature of the MATLAB Distributed Computing Server™, installed on the cluster. In the user manual of the Parallel Computing Toolbox™ [15] we can read:

> A parfor-loop is useful in situations where you need many loop iterations of a simple calculation, such as a Monte Carlo simulation. parfor divides the loop iterations into groups so that each worker executes some portion of the total number of iterations. parfor-loops are also useful when you have loop iterations that take a long time to execute, because the workers can execute iterations simultaneously.

From this manual we also note that each worker is assigned a unique random number stream so we can be sure that we are getting M independent samples of the Q-Wiener process.

The code used to produce the results of Figure 1 and 2 can be found in Section B.

6.5 Results: Weak convergence rates

Next, we compare the strong convergence rates to the weak rates. These are defined completely analogously to how we defined the strong rate in (34). To our knowledge, no simulations of the weak convergence rates of fully discrete approximations (using the FEM) of SPDE with multiplicative noise have been published, so this could be of interest for future research. A common "rule of thumb" (see e.g. the introduction of [11]) within this field is that the weak rate of convergence is twice that of the strong rate. Therefore, in particular we investigate whether one can achieve a rate of order 2 in the same context as the previous simulation, that is, when we consider ℓ from 1 to 7, $L = 8$ and we take $M = 3 \cdot 10^3$.

Now, as we recall from Section 5 there are (at least) two ways of approximating it with a Monte Carlo method, namely

$$\left| \mathbb{E}\left[||\tilde{X}_L||_H^2 \right] - E_M\left[||\hat{X}_\ell||_H^2 \right] \right|$$

and

$$\left| E_M\left[||\tilde{X}_L||_H^2 - ||\hat{X}_\ell||_H^2 \right] \right|.$$

which we refer to as the weak error rate of *type I* and *type II* respectively.

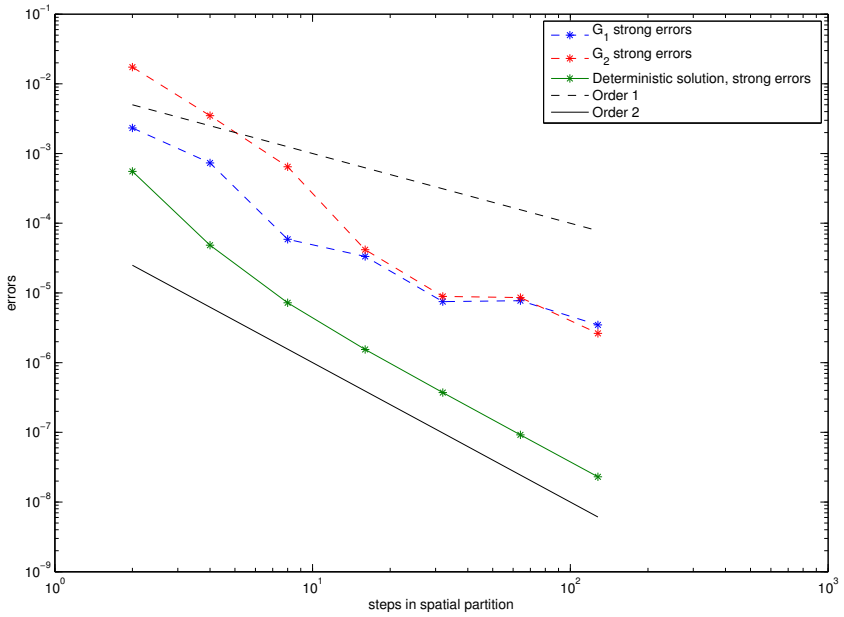

Figure 2: Approximations of the strong error rate for $\ell = 1, 2, 3, 4, 5, 6, 7$ for $C_\mu = 10$. All levels of error approximations has been computed using the same set of $M = 2 \cdot 10^4$ realizations of the Q-Wiener processes with a reference solution at level $L = 8$.

For the weak error of type I we choose to estimate the quantity $\mathbb{E}\left[||\tilde{X}_L||_H^2\right]$ for the case $G = G_1$ by the deterministic quantity

$$\sum_{i=1}^{10^6} \langle X_0, e_i\rangle_H^2 \exp\left(-(2\lambda_i + \mu_i)\right) \tag{35}$$

since we have access to the analytical solution as expressed in (30). For the case of $G = G_2$ we do not have this, so we instead estimate $\mathbb{E}\left[||\tilde{X}_L||_H^2\right]$ by $E_M\left[||\hat{X}_L||_H\right]$. In this case we will estimate $\mathbb{E}\left[||\tilde{X}_L||_H^2\right]$ on a different set of Q-Wiener processes than that used to generate $E_M\left[||\hat{X}_\ell||_H^2\right]$.

For the weak error of type II we generate both $||\tilde{X}_L||_H^2$ and $||\hat{X}_\ell||_H^2$ on the same set of Q-Wiener processes where we again make use of (29) truncated at $\kappa = h_L^{-1}$ and h_ℓ^{-1} respectively to generate these in the case of $G = G_1$. The resulting simulation is shown in Figure 3. In this case we see that the error is much smaller than when we simulated the strong error in Figure 2 which is what we expect. However, it is hard to gauge any particular rate of convergence since the variance seems to dominate the weak error rate. We also note that we appear to have no rate of convergence at all when we consider the case $G = G_1$ and an independent (in this case deterministic) reference solution, an issue that we will return to shortly. For $G = G_2$, the addition of an independent estimate of the reference solution seems to have little to no influence on the behaviour of the simulation of the weak rate of convergence. The behaviour of the last two points could be due to the fact that we consider a reference solution at the next (L=8) level instead of taking the exact solution. The computation to create this picture takes 19 hours, 59 minutes and 32 seconds using 8 computing nodes with 128 cores.

In Figure 4 we repeat these simulations for $C_\mu = 5$. We note that we get clearer indications of a rate of convergence which is somewhere between a rate of 1 and 2. We now also include the case of the type I error when all of the approximations of $E_M\left[||\hat{X}_\ell||_H^2\right]$ are independent of one another. In this case we have increased M to 10^5, but despite this, the noise is so great that it is impossible to say anything about the order of convergence. The computation time for these is presented in Table 1.

Remark 6.1. *The complete lack of convergence in the case of a deterministic reference solution in Figure 3 needs to be addressed. To do this purpose we can use the fact that we have an analytical expression of $X(T)$ to investigate the behaviour of $E_M\left[||\hat{X}_L||_H^2\right]$*

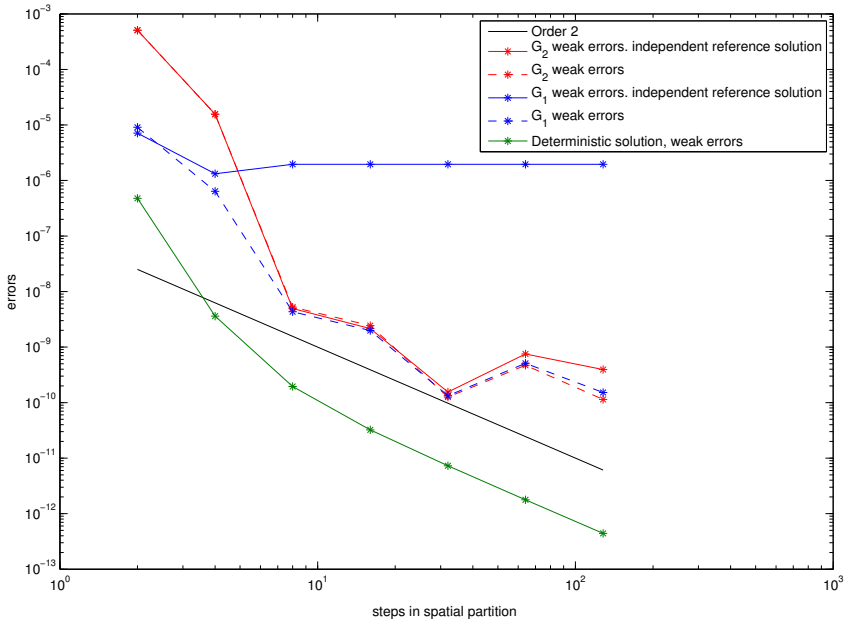

Figure 3: Approximations of the weak error rate for $\ell = 1, 2, 3, 4, 5, 6, 7$ and $C_\mu = 10$. In one set of cases the reference solution has been generated on an independent set of Q-Wiener processes and in one set it has been generated on the same set as the observations for the levels $\ell = 1, 2, 3, 4, 5, 6, 7$.

Level ℓ	Time for $G = G_1$	Time for $G = G_2$
2	00:01:28	00:00:49
3	00:01:09	00:00:38
4	00:00:51	00:00:51
5	00:02:31	00:02:14
6	00:21:09	00:16:28
7	07:13:20	04:57:38

Table 1: Time needed to compute the independent weak error estimates in Figure 4 using 8 computing nodes with 128 cores. The time for $\ell = 1$ was not recorded. 10^5 samples were taken at each level.

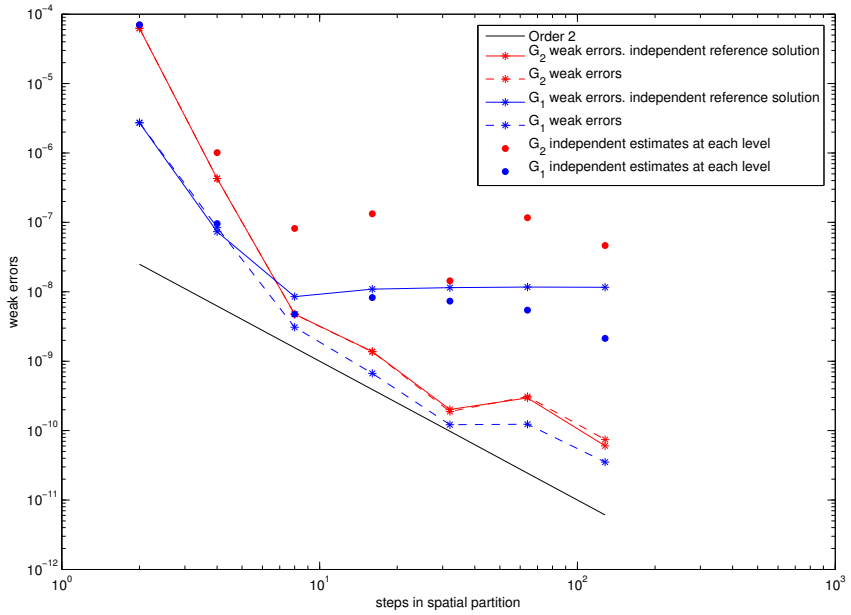

Figure 4: Approximations of the weak error rate for $\ell = 1, 2, 3, 4, 5, 6, 7$ and $C_\mu = 5$. For the lines: in one set of cases the reference solution has been generated on an independent set of Q-Wiener processes and in one set it has been generated on the same set as the observations for the levels $\ell = 1, 2, 3, 4, 5, 6, 7$. For the dots: all weak error estimates have been generated on independent sets of 10^5 Q-Wiener processes with a reference solution generated at $L = 8$ using 10^4 Q-Wiener processes.

when M is large and $L = 8$. Using (29) and the fact that for all $i \in \mathbb{N}$, $\beta_i(T) \sim N(0,1)$, we estimate $||\hat{X}_L||_H^2$ by

$$|| \sum_{i=1}^{N_{h_L}} \langle X_0, e_i \rangle_H \exp\left(-(\lambda_i + \frac{\mu_i}{2}) + \mu_i^{\frac{1}{2}} Z_i \right) I_{h_L} e_i ||_H^2$$

where $Z_i \sim N(0,1)$ i.i.d. and we use this to compute $E_M\left[||\hat{X}_L||_H^2\right]$ which we plot against M ranging from 1 to 10^6 in Figure 5 (note that for an increase in M we just add another observation of $||\hat{X}_L||_H^2$ as opposed to generating another $M+1$ number of observations). We see that for the final value of M the value of $E_M\left[||\hat{X}_L||_H^2\right]$ is almost entirely attributable to one single observation of $||\hat{X}_L||_H^2$. This indicates that the distribution is highly skewed, so that a large number of observations will be very close to zero but the sample mean will be much larger due to the presence of very large outliers.

Another (not entirely rigorous) way to think about this is to realize that since $\langle X_0, e_i \rangle_H^2$ decreases rapidly with increasing values of i and $\lambda_i + \frac{\mu_i}{2}$ is increasing in i, we should often have

$$\begin{aligned} &|| \sum_{i=1}^{N_{h_L}} \langle X_0, e_i \rangle_H \exp\left(-(\lambda_i + \frac{\mu_i}{2}) + \mu_i^{\frac{1}{2}} Z_i \right) I_{h_L} e_i ||_H^2 \\ \approx &|| \langle X_0, e_1 \rangle_H \exp\left(-(\lambda_i + \frac{\mu_1}{2}) + \mu_1^{\frac{1}{2}} Z_1 \right) I_{h_L} e_1 ||_H^2 \end{aligned}$$

which has a log-normal distribution. A well-known fact of the log-normal distribution is that it is highly skewed to the right when the variance of the underlying normal distributed variable (in this case $\mu_1 Z_1$) is big, so we should expect the presence of a small number of very large outliers which in turn means that in the majority of cases, for reasonable values of M, $E_M\left[||\hat{X}_L||_H^2\right]$ will be relatively far from the true mean.

6.6 Results: Multilevel Monte Carlo estimations

We end our numerical exploration by implementing the multilevel Monte Carlo estimator for the weak error. As in Figure 4 we let $C_\mu = 5$ and for $\ell = 0, 1, 2, ...$ we set $h_\ell = h_0 2^{-\ell}$, $k_\ell = h_\ell^2$ and $\kappa_\ell = h_\ell^{-1}$ to get a series of solutions to (32) which we denote by $\hat{X}_\ell := X_{\kappa_\ell, h_\ell}^{N_{k_\ell}}$. For computational reasons we let $h_0 = 2^{-1}$.

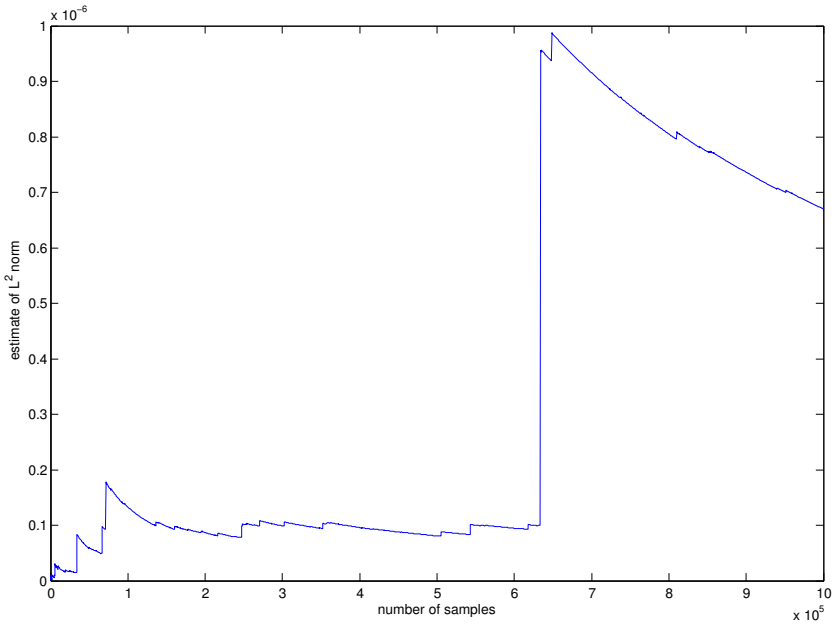

Figure 5: A plot of $E_M[||\hat{X}_8||_H^2]$ for M ranging from 1 to 10^6.

Let us make the bold assumption that the weak error rate actually is twice that of the strong error rate. Then we have

$$|\mathbb{E}\left[||X(T)||_H^2 - ||\hat{X}_\ell||_H^2\right]| \leq C_1 h_0^2 2^{-2\ell}$$

and so, from Proposition 5.6 we get

$$||\mathbb{E}\left[||X(T)||_H^2\right] - E^L[||\hat{X}_L||_H^2]||_{L^2(\Omega,\mathbb{R})} \leq C_2 h_0^2 2^{-2L}.$$

Let us now test this for $L = 1, 2, 3, 4, 5$. As in Section 6.5, we replace $\mathbb{E}\left[||X(T)||_H^2\right]$ by (35) in the case of $G = G_1$ and in the case of $G = G_2$ we replace it by a reference solution $E_M[||\hat{X}_7||_H^2]$ where we let $M = 10^4$. We generate all quantities independently of one another, but when computing a single multilevel estimate $E^L[||\hat{X}_L||_H^2$ it is vital to let the differences $||\hat{X}_\ell||_H^2 - ||\hat{X}_{\ell-1}||_H^2$ be computed on the same Q-Wiener process. In the case of $G = G_1$, the total computation time for all levels was 10 minutes and 39 seconds and in the case of $G = G_2$, the total computation time was 9 minutes. The computation time is thus reduced by more than a factor of two when compared to taking $M = 10^5$ in the singlelevel estimator.

The errors $\left|\mathbb{E}\left[||X(T)||_H^2\right] - E^L[||\hat{X}_L||_H^2]\right|$ are shown in the upper part of Figure 6. For comparison purposes, we have also included the independent weak errors from Figure 4. We note that for both noise operators, we are able to achieve similar results to a smaller computational cost using the multilevel estimator. In the lower part of Figure 6 we take the L^2-average of 100 realizations of the multilevel algorithm, that is we plot

$$E_{100}\left[\left|\mathbb{E}\left[||X(T)||_H^2\right] - E^L[||\hat{X}_L||_H^2]\right|^2\right]^{\frac{1}{2}}$$

and we see that we get an order of convergence which is very close to 2.

6.7 Concluding discussion

In the numerical experiments outlined above, we first tried to simulate the strong rate of convergence predicted by the theory of the previous chapters of this thesis. The results seem to be consistent with this theory, although the noise makes it hard to say anything for certain about this.

In the case of weak convergence, we noted that when we tried to estimate the weak errors using the same set of Q-Wiener processes there were indications that the rule of

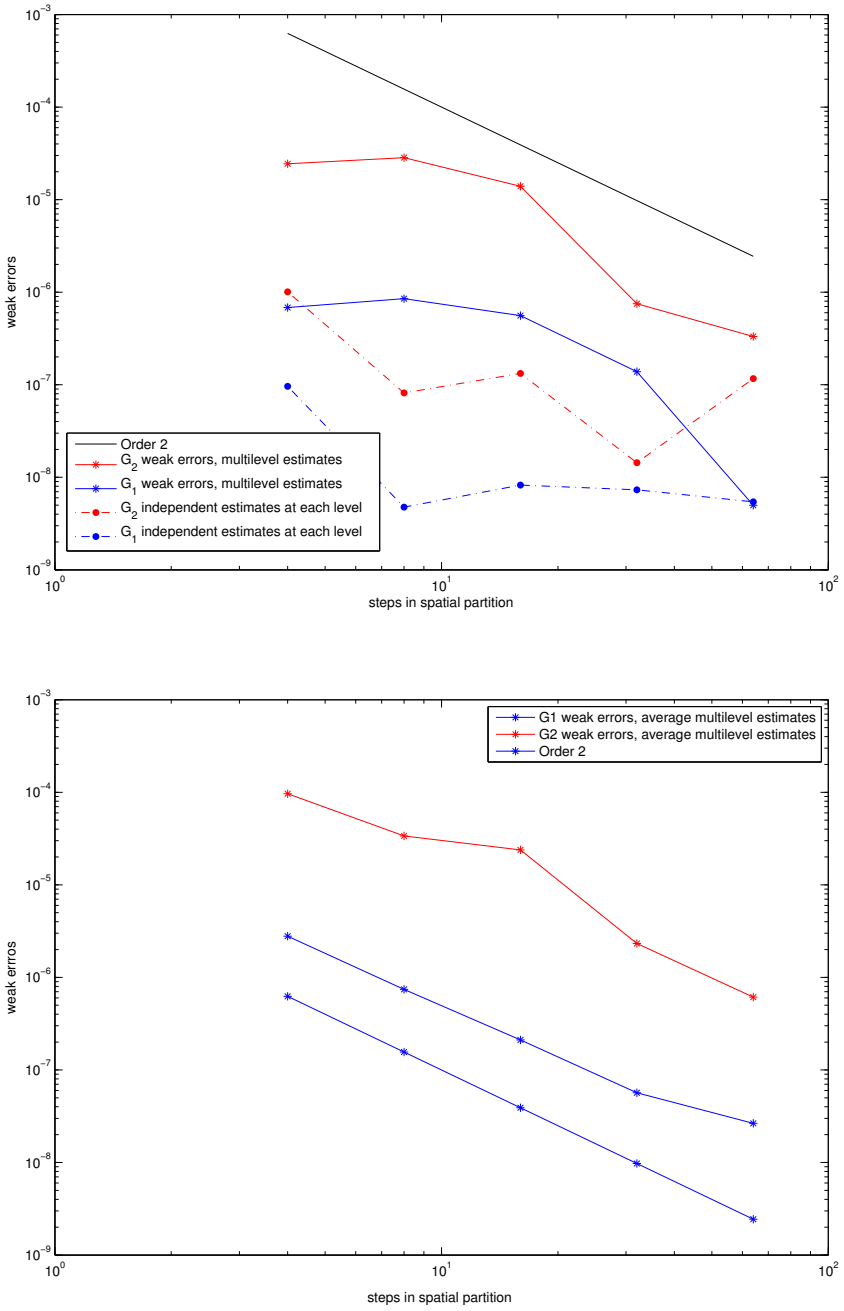

Figure 6: Upper picture: A comparison between multilevel estimates computed at $\ell = 1, 2, 3, 4, 5$ and the corresponding (single level) weak errors where each level is simulated independently of one another of Figure 4. Lower picture: Two L^2 averages of 100 realisations of the multilevel schemes of the upper picture.

thumb stating that the weak rate often is twice the strong rate seemed to hold. However, especially in the analytical case, when comparing the estimates to a reference solution which had been computed independently of these, the pattern was not very clear. When we finally had all the estimates computed independently of one another there were close to no indications of convergence, despite the relatively expensive computations. This shows how it, even in these relatively simple cases, can be hard to actually estimate quantities of the solution in practice due to the variance of these, and perhaps due to properties of their distribution as well.

However, in the case of the multilevel estimates, we were able to get a rate of convergence that seems to be close to 2 despite having each level estimate be independent of one another. The pattern was much clearer, though, when an average of multiple runs of the multilevel algorithm was taken. We stress, though, that the application of the multilevel algorithm assumes that the weak rate of convergence actually is two, something that we have not provided theory for. The result nevertheless shows the practical value of the multilevel algorithm and together with the results on the singlelevel weak error rates, may indicate some interesting directions for future research.

A Appendix

This appendix contains two useful inequalities that do not fit in well elsewhere. We start with the following *discrete Grönwall inequality* from [5].

Lemma A.1. [5, page 280] *Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be non-negative sequences such that*

$$a_n \leq b_n + \sum_{k=0}^{n-1} c_k a_k$$

for $n \geq 0$. Then

$$a_n \leq \max_{\theta\in\{0,1,\ldots,n\}} b_\theta \prod_{k=0}^{n-1}(1+c_k)$$

The second result is an arithmetic inequality which we use in several places throughout the thesis and prove below.

Lemma A.2. *Let $n \in N$ and let a_i, $i = 1, 2, ...n$ be non-negative numbers. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^p \leq n^{p-1} \sum_{i=1}^{n} a_i^p$$

Proof. Consider the measure space $(\tilde{\Omega}, \mathcal{A}, \mu_c)$ with $\tilde{\Omega} = \{1, 2, ..., n\}$,
$\mathcal{A} = \mathcal{P}(\tilde{\Omega})$, the power set of $\tilde{\Omega}$, and μ_c being the counting measure. Let $f, g : \tilde{\Omega} \to \mathbb{R}$ be defined by $f(k) = 1$ and $g(k) = a_k$. Then, by Hölder's inequality:

$$\sum_{i=1}^{n} a_i = ||fg||_1 \leq ||f||_q ||g||_p = n^{\frac{1}{q}} \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}$$

where $p^{-1} + q^{-1} = 1$. The result now follows by raising each side of the inequality to the p-th power. □

B Source code

B.1 HSHE_strong_errors

This function is used to produce the results of Figure 1 and 2.

```
function [strong_errors_G1, ...
    strong_errors_G2]=HSHE_strong_errors(maximum_ell, M, eigenconstant, ...
    eigendecay)
% Produces the strong error approximations for the homogeneous  of Figure X.
% Uses M realizations of the $Q$-Wiener process with
% Qe_j=eigenconstant*j^(-eigendecay) to compute approximations for h
% ranging from 2^-1 to 2^-maximum_ell.
% A reference level of maximum_ell+1 is used to compute the reference
% solution which is exact for G_1 and approximated for G_2.

%M samples of the error in L2-norm squared
error_samples_G1=zeros(M,maximum_ell);
error_samples_G2=zeros(M,maximum_ell);

ref_N_h=2^(maximum_ell+1); % Inverse space step size for reference level

rng('shuffle');
parfor m = 1:M
    % Generate a Wiener process - each row is a realisation of \beta_j
    W=cumsum([ zeros(ref_N_h-1,1), randn(ref_N_h-1,ref_N_h^2)],2)./ref_N_h;
    % Generate the 'exact' reference solution for G=G_1
    ref_X_G1 = zeros(ref_N_h+1,1);
    start = ((-1).^((1:ref_N_h-1)+1)+1)'.*(1:ref_N_h-1).^(-3)'; % Initial ...
        value is x-x^2
    start = sqrt(8)/pi^3*start;
    xgrid = linspace(0,1,ref_N_h+1);
    E = sqrt(2)*sin(pi*xgrid(2:end-1)'*(1:(ref_N_h-1))); % E_ij=e_j(x_i), ...
        symmetric
    ref_X_G1(2:end-1) = E*(start.*exp(-pi^2*(1:(ref_N_h-1))' .^ ...
        2-eigenconstant/2 * (1:(ref_N_h-1))' .^ ...
        (-eigendecay)+(sqrt(eigenconstant) * (1:(ref_N_h-1))' .^ ...
        (-eigendecay/2)) .* W(:,end)));
    % Generate the reference solution for G=G_2
    [~,ref_X_G2] = solve_HSHE(ref_N_h, ref_N_h^2, eigenconstant, ...
        eigendecay, W, ref_N_h^2);
    % For each level, calculate the fully discrete approximation and record
    % the sample of the errors
    for level=1:maximum_ell
```

```
        [Xpath_G1, Xpath_G2] = solve_HSHE(2^level, 2^(2*level), ...
            eigenconstant, eigendecay, W, ref_N_h^2);
        error_samples_G1(m,level) = sum( (interp1( ...
            linspace(0,1,2^level+1), Xpath_G1(:,end), ...
            linspace(0,1,ref_N_h+1))'-ref_X_G1(:)).^2 ) / ref_N_h;
        error_samples_G2(m,level) = sum( ( ...
            interp1(linspace(0,1,2^level+1), Xpath_G2(:,end), ...
            linspace(0,1,ref_N_h+1))'-ref_X_G2(:,end)).^2 ) / ref_N_h;
    end
end
% Take the square of the mean value of the errors to get the final estimate
strong_errors_G1 = sqrt(mean(error_samples_G1));
strong_errors_G2 = sqrt(mean(error_samples_G2));
end

function [Xpath_G1, Xpath_G2] = solve_HSHE(spacesteps, coarsetimesteps, ...
    eigenconstant, eigendecay, Wfine, finetimesteps)
% Given a realisation of W, solves HSHE for G1 and G2 by implementing the
% backwards Euler scheme.

h = 1/coarsetimesteps; % time step size
xgrid = linspace(0,1,spacesteps+1);
% Initialize memory for output variable
Xpath_G1 = zeros(spacesteps+1, coarsetimesteps+1);
Xpath_G2 = zeros(spacesteps+1, coarsetimesteps+1);

% Translate the brownian motions to coarsetimesteps, truncate the ...
    Karhunen-Loeve expansion

Wcoarse=zeros(spacesteps-1,coarsetimesteps+1);
for j=1:(spacesteps-1)
    Wcoarse(j,:)=interp1(linspace(0,1,finetimesteps+1), Wfine(j,:), ...
        linspace(0,1,coarsetimesteps+1));
end

% Generate matrices
A = spacesteps*(2*diag(ones(spacesteps-1,1)) - ...
    diag(ones(spacesteps-2,1),1) - diag(ones(spacesteps-2,1),-1)); % ...
    stiffness
M = (6*spacesteps)^-1*(4*diag(ones(spacesteps-1,1)) + ...
    diag(ones(spacesteps-2,1),1) + diag(ones(spacesteps-2,1),-1)); % mass
E = sqrt(2)*sin(pi*xgrid(2:end-1)'*(1:(spacesteps-1))); % E_ij=e_j(x_i), ...
    symmetric
D = sqrt(eigenconstant)*diag((1:(spacesteps-1)).^(-eigendecay/2)); % ...
    eigenvalues of Q
P = E*D; O = E*M; % for efficiency
```

```
% Evaluate initial condition
Xpath_G1(:,1) = initial(xgrid)';
Xpath_G2(:,1) = initial(xgrid)';

% Compute exact path implementing the backward Euler scheme
for j = 1:coarsetimesteps
    Xpath_G1(2:end-1,j+1) = (M + h*A) \ (M*(Xpath_G1(2:end-1,j) + ...
        P*diag(Wcoarse(:,j+1)-Wcoarse(:,j))*O*Xpath_G1(2:end-1,j)));
    Xpath_G2(2:end-1,j+1) = (M + h*A) \ (M*(Xpath_G2(2:end-1,j) + ...
        sin(Xpath_G2(2:end-1,j)) .* ...
        (P*diag(Wcoarse(:,j+1)-Wcoarse(:,j))*ones(spacesteps-1,1))));
end

end

function init = initial(xgrid)
% Returns X(0) on a given xgrid
init = xgrid-xgrid.^2;
end
```

B.2 multilevel_estimate_G2

This function computes the multilevel estimate used in 6.6 in the case of $G = G_2$. The code for the case $G = G_1$ is entirely analogous and is therefore not provided here.

```
function [MLMC]=multilevel_estimate_G2(L, eigenconstant, eigendecay)

delta = 0;
h0=1/2; %
space_list = (h0)^(-1)*2.^((0:L)); % List of space steps, h_0=2^{-1}
time_list = space_list.^2; % List of time steps
M_list = round([space_list(end)^4, ...
    space_list(2:end).^(-2).*space_list(end)^4.*(1:L).^(1*(1+delta))]); % ...
    Monte Carlo iterations
MC = 0; % MC estimate at each level
tic;
% X_0 :
parfor m = 1:M_list(1)
        % Generate a Wiener process - each row is a realisation of \beta_j
        W=cumsum([ zeros(space_list(1)-1,1), ...
            randn(space_list(1)-1,time_list(1))],2)./space_list(1);
```

```
        X = solve_HSHE(space_list(1),time_list(1), eigenconstant, ...
            eigendecay,W);
        L2norm = sum( X(:,end).^2 )/(space_list(1));
        MC = MC + L2norm/M_list(1);
end
MLMC = MC;
% X_1 to X_L
for k = 2:length(space_list)
    MC = 0;
    finetimesteps=time_list(k);
    coarsetimesteps=time_list(k-1);
    finespacesteps=space_list(k);
    coarsespacesteps=space_list(k-1);
    M=M_list(k);
    parfor m = 1:M
        % Generate a Wiener process - each row is a realisation of \beta_j
        fineW=cumsum([ zeros(finespacesteps-1,1), ...
            randn(finespacesteps-1,finetimesteps)],2)./finespacesteps;
        % Interpolate Wiener process to the coarser time grid
        coarseW=zeros(coarsespacesteps-1,coarsetimesteps+1);
        for j=1:(coarsespacesteps-1)
             coarseW(j,:)=interp1(linspace(0,1,finetimesteps+1), ...
                 fineW(j,:),linspace(0,1,coarsetimesteps+1));
        end
        X = solve_HSHE(finespacesteps,finetimesteps, eigenconstant, ...
            eigendecay,fineW);
        Y = solve_HSHE(coarsespacesteps,coarsetimesteps, eigenconstant, ...
            eigendecay,coarseW);
        L2normX = sum( X(:,end).^2 )/finespacesteps;
        L2normY = sum( Y(:,end).^2 )/coarsespacesteps;
        L2norm =  L2normX-L2normY;
        MC = MC +L2norm/M; % Singlelevel estimate of difference
    end
    %toc
    MLMC = MLMC+MC; % Update multilevel estimate
end
toc
end

function [Xpath] = solve_HSHE(spacesteps, timesteps, eigenconstant, ...
    eigendecay, W)
% Given a realisation of W, solves HSHE for G1 and G2 by implementing the
% backwards Euler scheme.

h = 1/timesteps; % time step size
xgrid = linspace(0,1,spacesteps+1);
```

```
% Initialize memory for output variable
Xpath = zeros(spacesteps+1, timesteps+1);

% Generate matrices
A = spacesteps*(2*diag(ones(spacesteps-1,1)) - ...
    diag(ones(spacesteps-2,1),1) - diag(ones(spacesteps-2,1),-1)); % ...
    stiffness
M = (6*spacesteps)^-1*(4*diag(ones(spacesteps-1,1)) + ...
    diag(ones(spacesteps-2,1),1) + diag(ones(spacesteps-2,1),-1)); % mass
E = sqrt(2)*sin(pi*xgrid(2:end-1)'*(1:(spacesteps-1))); % E_ij=e_j(x_i), ...
    symmetric
D = sqrt(eigenconstant)*diag((1:(spacesteps-1)).^(-eigendecay/2)); % ...
    eigenvalues of Q
P = E*D;

% Evaluate initial condition
Xpath(:,1) = initial(xgrid)';

% Compute exact path implementing the backward Euler scheme
for j = 1:timesteps
    Xpath(2:end-1,j+1) = (M + h*A) \ (M*(Xpath(2:end-1,j) + ...
        sin(Xpath(2:end-1,j)) .* ...
        (P*diag(W(:,j+1)-W(:,j))*ones(spacesteps-1,1))));
end

end
```

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