UNIVERSITY OF GOTHENBURG
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Master Degree Project in Finance

Pricing Credit Default Index Swaptions

A numerical evaluation of pricing models

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Abstract

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This study examines the background and nature of the credit default index swaption (CDIS) and presents relevant methods for modelling credit risk. A CDIS is a credit derivative contract that gives the buyer right to enter into a credit default index swap (CDS index) contract at a given point in time. A CDS index, in turn, is a multi-name credit default swap (CDS). Within the field of research, this thesis identifies the CDIS pricing models presented by Jackson (2005), Rutkowski & Armstrong (2009) and Morini & Brigo (2011) as the most recognized and developed. These models are evaluated by reconstruction in a numerical software environment. Although the considered models are well-behaving under economic interpretation, they differ in constructional features regarding whether to model the so-called Armageddon event inside or outside the Black (1976) model. An Armadageddon event refers to a total default of the CDS index up to the expiry of the CDIS. Based on the criteria of required assumption boldness and calculation transparency, the model presented by Morini & Brigo (2011) have been evaluated in depth. The expected value of the front-end protection, i.e. the insurance against default events during the lifetime of the CDIS, is found to increase with pairwise correlation among reference names and the effect of the Armageddon scenario is only observable as the pairwise correlation approaches one. This implies that the choice of pricing model is found to be crucial during stressed economic climates and of less importance during calm economic climates.

Keywords: *Credit Default Index Swaptions, Options on CDS Indices, Credit Derivatives, Credit Default Swap, Credit Default Swaption, Credit Default Index Swap, Credit Risk, Credit Risk Modelling, Intensity-based Modelling, Black-Scholes*

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1. Introduction

Credit derivatives were growing dramatically for several years prior to the financial crisis starting in 2007. According to O'Kane (2008), there are plenty of institutions and researchers (e.g. The International Swaps and Derivatives Association) that have conducted surveys indicating the same exponential growth of the credit derivative market.

In the aftermath of the credit crunch, the majority of multi-name credit derivatives experienced a significant decline in liquidity. Morini & Brigo (2011), however, state that even if most of the credit derivatives decreased in liquidity, the credit default index swaptions (CDIS) did not. Moreover, Schmerken (2011) states that bid-ask spreads of this particular credit derivative have gotten tighter post-credit crunch due to higher liquidity. Schmerken (2011) also points out that major banks now provide these derivative securities in a larger variety with respect to strike spreads and maturities.

A credit default index swaption is an option to enter into a CDS index, where a CDS index fulfills the purpose of protection against credit defaults within a portfolio of reference names. The specific event of a total default of the CDS index up to the expiry of the CDIS is referred to as an Armageddon event. This event constitutes a part of the so-called front-end protection, which is an insurance against defaults prior to the expiry of the CDIS. The CDIS enables insurance against both idiosyncratic and macro-economic credit risks. According to Doctor & Goulden (2007) credit default index swaptions possess two particular advantageous features. Firstly, trading credit default index swaptions allows for expressing spread views. That is, an investor can express bullish or bearish views on the macro-economic climate by taking long or short positions on credit default swap indices, respectively. Secondly, it opens up for investors to trade implied volatility of the credit market without defining the direction of motion.

Credit default index swaptions come with a special feature in addition to ordinary swaptions. In a standard contract, the buyer is not protected against defaults up to the maturity of the swaption, while in the case of a CDIS, the buyer will receive protection even during the life of the swaption. As this fact creates an additional layer of complexity in the construction of the instrument, ordinary pricing models cannot be directly applied when valuing these kinds of derivatives. This complexity, together with the relatively young age of the derivative, has caused difficulties in reaching a consensus among researchers in how to price credit default index swaptions. Moreover, as of today CDIS contracts are traded over the counter (OTC) and therefore lack liquid market prices that could constitute a potential benchmark when evaluating pricing models. Although there exist several pricing models developed by eminent researchers, no previous study has implemented the models, compared their generated outcomes and examined their sensitivities with respect to input parameters.

This study aims at fully describing the nature of credit default index swaptions and clarifying several uncertainties surrounding them. The purpose of this thesis is therefore to examine the credit default index swaption and to evaluate its most recognized and developed pricing models. This objective will be fulfilled through answering a set of research questions.

· *Which are the most recognized and developed credit default index swaption pricing models?*

· *Can the models be reconstructed in a programmable environment?*

· *Are the models well-behaving in an economic interpretation?*

· *How do the models differ with respect to constructional features?*

· *Is there any superior model(s) worth evaluating further?*

· *What are the effects of front-end protection and Armageddon scenario?*

The thesis is organized as follows. In Section 2, central concepts necessary for the understanding of credit risk as well as related credit derivatives are presented. Section 3 outlines the main methodologies for modelling credit risk. Further, Section 3 also applies the methodologies for modelling credit risk to the credit derivative-specific features concerned in this study. In Section 4 an overview of the field of research is presented as it enables the study to pinpoint the most recognized and developed models. In this context, recognized refers to amount of citations and contribution to the field of research. Developed refers to the reputation of the model as well as the model being contemporary and accepted among researchers.

The pinpointed models will be examined mathematically in Section 4. This examination will provide the essential framework required for reconstruction in a numerical software environment. The models will be analyzed in Section 5 and evaluated with respect to sensitivities and interrelationships between input variables. To be able to compare the models, three aspects are taken into account. Firstly, whether they are well-behaving under economic interpretation, i.e. reacting reasonably to changes in the economic setting. Secondly, how well the their constructional features manage to capture the dynamics of the CDIS contract where constructional features refer to the mathematical design of the models. These two aspects are evaluated through conducting a set of sensitivity analyses (Subsection 5.2). The last aspect taken into account deals with the reliability of the models with respect to assumptions and transparency. In this context, assumptions are assessed on the level of boldness and to what extent calculations are model-specific. Transparency refers to clarity of calculations and if the researcher presents self-generated values. This aspect is discussed at the end of Subsection 5.2. If any superior model(s) worth evaluating further exists, this will be examined in an in-depth analysis (Subsection 5.3) which aims at examining how the model(s) reacts to simultaneous changes in the economic setting, i.e. how incremental changes in multiple input variables affect the modelled price of credit default index swaptions. In addition, the model(s) will be applied to real historical market data before, during and after the financial crisis of 2008.

Based on the models' behavior in the sensitivity analyses and in the real market data application, the effects of model-specific features will be discussed in Section 5. In Section 6 the study culminates into a set of conclusions in order to answer the research questions and fulfill the purpose.

2. Central concepts

This section aims at describing central concepts essential for conducting the thesis. In Subsection 2.1 a definition of credit risk is presented. Subsection 2.2 outlines the nature of the credit default swap. Further, in Subsection 2.3 a description of credit default swap index contracts is provided. Lastly, Subsection 2.4 aims at describing the nature of credit default index swaption contracts.

2.1 Credit risk

> *"Credit risk is most simply defined as the potential that a bank borrower or counterparty will fail to meet its obligations in accordance with agreed terms."*
>
> *(Basel Committee on Banking Supervision 2000, p. 1)*

Credit risk can, according to Schmid (2002) be categorized into two parts: default risk and spread risk. The default risk concerns the inability or unwillingness of an obligor (e.g. a company that has issued bonds) to fulfill payment obligations. The spread risk originates from changes in the credit quality which results in loss of market value. The scope of this thesis is restricted to only focus on default risk.

According to Moody's (2014) a default event can happen due to four scenarios: failure to pay an obligated cash flow on time, bankruptcy, restructuring of payments in order to avoid bankruptcy and changes in the payment terms of credit agreements. Hereafter the terms credit risk and default risk will be synonyms and refer to the definitions made by Moody's (2014).

2.2 Credit default swap (CDS)

This subsection outlines the nature of the credit default swap. These derivatives are frequently traded on liquid markets with great variations of companies and maturities. Figure 2.1 illustrates the timeline of the CDS contract.

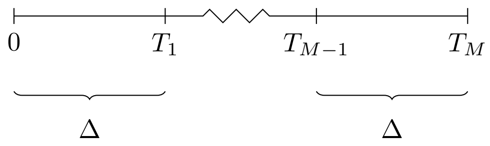


FIGURE 2.1: The timeline of CDS contracts

where T_M is the expiration time of the CDS contract, Δ is the interval between the cash flows and T_j is the time corresponding to the j^{th} CDS cash flow, where $j = 1, 2, ..., M$.

A credit default swap is a bilateral credit derivative insuring against third party default. The protection buyer **A** pays a quarterly fee to protection seller **B** in exchange for insurance against credit losses due to default of company **C** up to time T_M (see Figure 2.1). This insurance is, by the nature of the contract, set to compensate for the credit loss proportion $(1-\phi)$ of the outstanding notional amount N. The contract proceeds until the time of default τ for company **C** or until the CDS contract expires, whichever happens first (mathematically defined in Equation (3.2)). The CDS contract is graphically illustrated in Figure 2.2. Notations may differ among authors as buying protection is equivalent to selling risk.

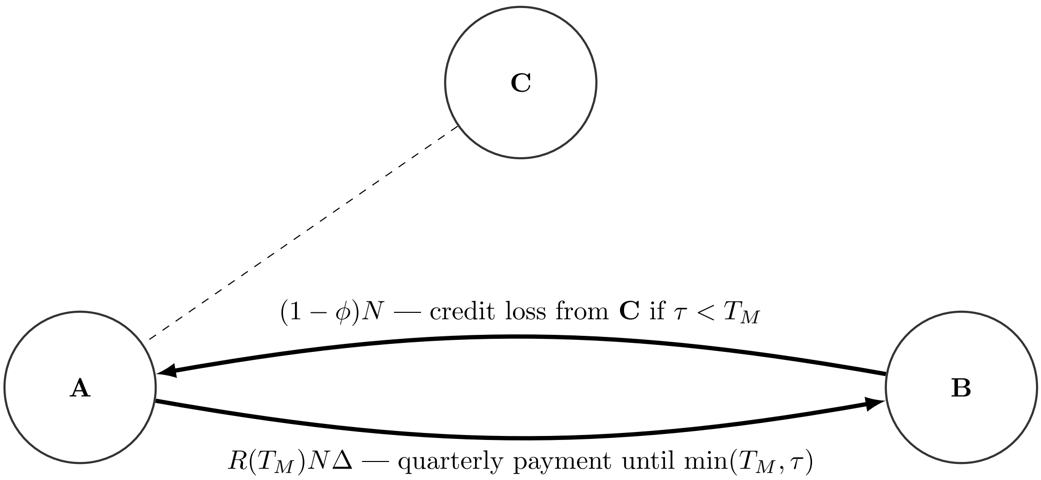


FIGURE 2.2: The structure of CDS contracts

where τ is the default time of company **C**, $R(T_M)$ is the T_M-year fair CDS spread calculated in Equation (2.3) and ϕ is the recovery rate, i.e. the proportion of the defaulted outstanding notional amount refunded without protection. The holder of protection against credit events (buyer of credit default swap) is therefore entitled to receive $1 - \phi$ in case of default of company **C**. For the purpose of this study, the recovery rate is assumed to be constant.

At the initiation of the contract, the fee is set so that the expected discounted cash flows between **A** and **B** are equal (the expectation is done under a risk-neutral measure). There are two ways to achieve this. Either the quarterly fee is set so that the expected discounted cash flows are identical, or, given a standardized quarterly fee (e.g. 100 bps), an upfront payment is paid/received by **A** at the start of the contract so that the value of the expected discounted cash flows are equal. There exist two types of contracted settlements in the event of a default of company **C**: physical settlement and cash settlement. The former refers to an actual delivery of the defaulted asset upon exercising while the latter refers to a netting transaction of money. (Herbertsson 2007) (Duffie & Singleton 2003)

Recall that τ is the default time for company **C**. Then the expected discounted cash flows from **B** to **A**, known as the default leg and denoted by Φ, and the value of receiving an annuity of one risky basis point during the life of the CDS, denoted by $DV01$, are mathematically expressed as follows

$$\Phi = \mathbb{E}[\mathbb{1}_{\{\tau \leq T_M\}} D(0,\tau)(1-\phi)] \tag{2.1}$$

and

$$DV01 = \sum_{j=1}^{M} \mathbb{E}[D(0,T_j)\Delta \mathbb{1}_{\{\tau > T_j\}} + D(0,\tau)(\tau - T_{j-1})\mathbb{1}_{\{T_{j-1} < \tau_i \leq T_j\}}] \tag{2.2}$$

where $\mathbb{1}_{\{\tau \leq T_M\}}$ is an indicator function taking the value of one if $\tau \leq T_M$ and zero otherwise, ϕ is the recovery rate, $D(0,T_j)$ is the discount factor from T_j to 0, j corresponds to the j^{th} cash flow of the swap contract, Δ is the interval between the cash flows (set to one quarter) and $\mathbb{1}_{\{\tau > T_j\}}$ is an indicator function taking the value of one if $\tau > T_j$ and zero otherwise. Further, $D(0,\tau)(\tau - T_{j-1})\mathbb{1}_{\{T_{j-1} < \tau_i \leq T_j\}}$ is the accrued premium term, where $D(0,\tau)$ is the discount factor from τ to 0 and $\mathbb{1}_{\{T_{j-1} < \tau_i \leq T_j\}}]$ is an indicator function taking the value of one if $T_{j-1} < \tau_i \leq T_j$ and zero otherwise.

The T_M-year CDS spread $R(T_M)$ is then calculated as the ratio between Equation (2.1) and Equation (2.2). In the case with no up-front premium, a CDS spread on company **C** is given by

$$R(T_M) = \frac{\mathbb{E}[\mathbb{1}_{\{\tau \leq T_M\}} D(0,\tau)(1-\phi)]}{\sum_{j=1}^{M} \mathbb{E}[D(0,T_j)\Delta \mathbb{1}_{\{\tau > T_j\}} + D(0,\tau)(\tau - T_{j-1})\mathbb{1}_{\{T_{j-1} < \tau_i \leq T_j\}}]}. \tag{2.3}$$

Note that the CDS spread formula is independent of the notional amount and priced under risk-neutral measure which is always used when pricing financial derivatives and requires an arbitrage-free market. The existence of such a measure is guaranteed under very weak assumptions as well as under the assumption of no arbitrage, and is further discussed in Bjork (2009).

Aligned with common practice, the following assumptions are defined. All Δ are assumed to be the same (one quarter). The accrued premium term (accumulated interest since last cash flow) in the CDS valuation formula is ignored. The loss is paid at the end of the quarter instead of immediately at the time of default τ for defaults occurring within a certain quarter during the lifetime of the CDS. By applying these assumptions, Equation (2.2) collapses to

$$DV01 = \sum_{j=1}^{M} \mathbb{E}[D(0, T_j)\Delta \mathbb{1}_{\{\tau > T_j\}}]. \tag{2.4}$$

Recalling Equation (2.3) where nominator in corresponds to the expected present value of an contingent claim, denoted by Φ, paying $1-\phi$ of the notional amount in case of default of company **C**. The denominator is the present value of receiving an annuity of one risky basis point ($DV01$) for company **C**. When rearranging terms in Equation (2.3) it becomes clear that the spread $R(T_M)$ is set so that the following equation holds

$$R(T_M)DV01 = \Phi. \tag{2.5}$$

That is, the expected present value of cash flows from **A** to **B** equals the expected present value of cash flows from **B** to **A**.

In this thesis a constant risk-free interest rate r is assumed, which implies that discount factors will be calculated as follows

$$D(t, T) = e^{-r(T-t)} \tag{2.6}$$

where t and T are arbitrary points in time and r is a constant (deterministic) interest rate.

2.3 Credit default swap index (CDS index)

This subsection provides a description of credit default swap index contracts.

A multi-name credit default swap is referred to as a credit default swap index, see e.g. O'Kane (2008). The main difference from an ordinary CDS contract is that in the case of default, the protection buyer of such contracts has the right to receive the notional amount times the defaulted proportion of the index, which is defined by a portfolio of m equally weighted reference names. Let $\tau_1, \tau_2, ..., \tau_m$ be the default times of the m reference names in the portfolio that constitutes the index. The number of defaults within the CDS index up to the arbitrary time t is calculated using the counting process N_t, given by

$$N_t = \sum_{i=1}^{m} \mathbb{1}_{\{\tau_i \leq t\}} \tag{2.7}$$

where $\mathbb{1}_{\{\tau_i \leq t\}}$ is an indicator function taking the value of one if $\tau_i \leq t$ and zero otherwise.

Hence, N_t counts the number of reference name defaults up to the arbitrary time t in CDS index. The defaulted proportion of the CDS index up to time t is therefore obtained by dividing this quantity by the total number of reference names within the CDS index m. If defaults occur within the CDS index, the premium payment is adjusted to the new number of underlying reference names within the index. That is, defaulted reference names are excluded from contract. The structure of a CDS index is graphically illustrated in Figure 2.3.

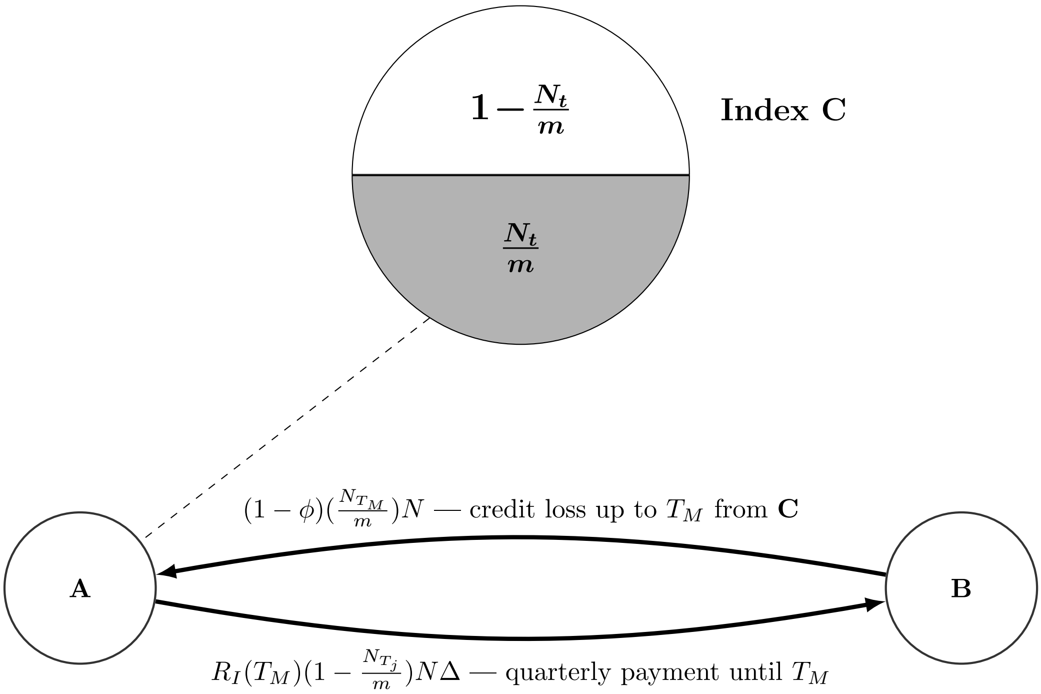


FIGURE 2.3: The structure of CDS index contracts

where N_{T_M} is a process counting the number of defaults up to time T_M, N_{T_j} is a process counting the number of defaults up to time T_j and $R_I(T_M)$ is the T_M-year fair CDS index spread.

In the same way as for a single-name CDS, see Figure 2.2, **A** and **B** is the buyer and the seller of protection, respectively. In contrast to an ordinary CDS, the protection buyer **A** now pays a quarterly fee to the protection seller **B** in exchange for insurance against all credit losses within the CDS index up to time T_M. Hence, the accumulated payment from **B** to **A** will be $(1-\phi)\frac{N_{T_M}}{m}N$. Unlike the ordinary CDS contract, the CDS index contract does not knock-out due to defaults unless all reference names defaults within the lifetime of the swap. Moreover, the fee $R_I(T_M)$ is set so that the expected discounted cash flows between **A** and **B** are equal at the time of inception.

The expected discounted cash flows from **B** to **A**, denoted by Φ, and the value of receiving an annuity of one risky basis point, denoted by $DV01$, can in accordance with e.g. Herbertsson &

Frey (2012), be mathematically expressed as follows

$$\Phi = D(0,T_M)\mathbb{E}[(1-\phi)(\frac{N_{T_M}}{m})N] + \int_0^{T_M} r_t D(0,t)\mathbb{E}\Big[(1-\phi)(\frac{N_t}{m})N\Big]dt \tag{2.8}$$

and

$$DV01 = \sum_{j=1}^{M} D(0,T_j)\left(1-\frac{\mathbb{E}[N_{T_j}]}{m}\right)\Delta \tag{2.9}$$

where Equation (2.8) follows from an integration by parts. (Herbertsson & Frey 2012)

The T_M-year fair CDS index spread $R_I(T_M)$ is then calculated as the ratio between Equation (2.8) and Equation (2.9)

$$R_I(T_M) = \frac{D(0,T_M)\mathbb{E}[(1-\phi)(\frac{N_{T_M}}{m})N] + \int_0^{T_M} r_t D(0,t)\mathbb{E}\Big[(1-\phi)(\frac{N_t}{m})N\Big]dt}{\sum_{j=1}^{M}\Big(1-\frac{1}{m}\mathbb{E}[N_{T_j}]\Big)\Delta}. \tag{2.10}$$

Prior to 2004 the market consisted of numerous competing CDS index products. These products, however, merged and resulted in two main indices: iTraxx (Europe and Asia) and CDX (North America). There exist a lot of variations of these indices but the most liquid ones, each containing 125 investment grade reference names, are European iTraxx and CDX.NA.IG (O'Kane 2008). Figure 2.4 illustrates the spreads of the 3-, 5-, 7- and 10-year European iTraxx indices before, during and after the financial crisis of 2008. Related to the default of Lehman Brothers during the early autumn of 2008, a steep increase in all the European iTraxx indices can be observed.

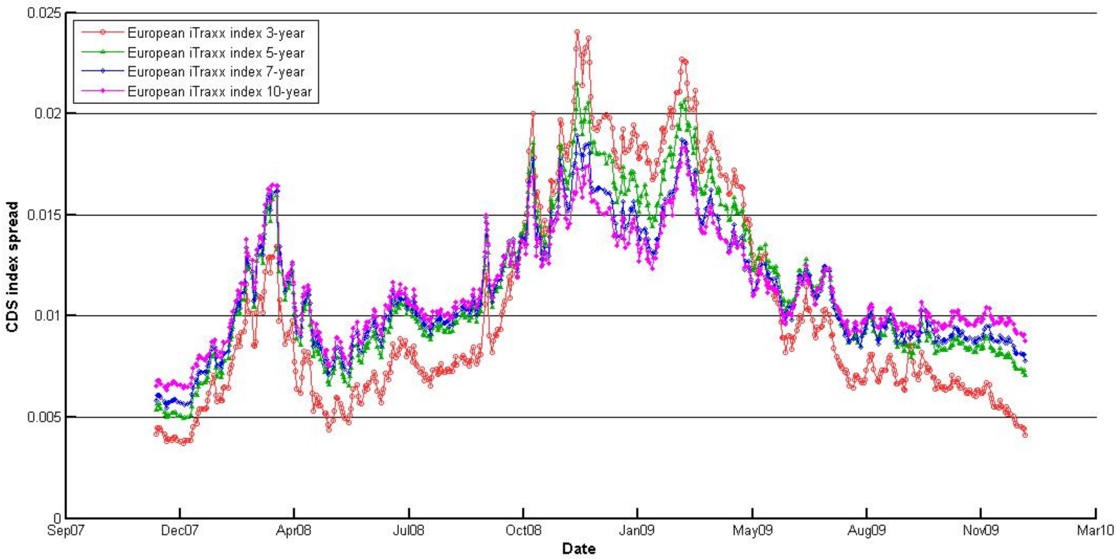


FIGURE 2.4: Historical spreads of European iTraxx indices
Source: Thomson Reuters

Every six month the composition of the references within CDS indices can be changed. This composition change is made to replace references due to two reasons; reduction of liquidity and downgrading of the credit rating below investment grade. The composition change is referred to as the index roll. (O'Kane 2008)

2.4 Credit default index swaption (CDIS)

This subsection aims at describing the nature of credit default index swaption contracts.

A credit default index swaption is a contract that gives the buyer right to enter into a CDS index contract at a given point in time. However, one technical feature differ the CDIS from ordinary swaptions; the former does not have a knock-out feature during the life of the swaption. This implies that a CDIS buyer is protected, in the same manner as in an ordinary CDS index contract, also before the maturity of the swaption. In the case of a default during the lifetime of the swaption, conditional on exercising the swaption, a payment (referred to as the front-end protection) is paid to the swaption holder at expiry of the contract. (O'Kane 2008) (Hull 2012)

The front-end protection is the present value of the payment a credit default index swaption buyer receives in case of defaults within the lifetime of the swaption. Recall that this feature does not exist in ordinary credit default swaptions due to the knock-out nature of single-name credit derivatives. As all reference names in a CDIS are equally weighted, the front-end protection corresponds to the defaulted proportion of the underlying index up to the expiry of the option, multiplied with the total notional amount. The special case of total default of the CDS index (all reference names) up to expiration of the swaption is referred to as an Armageddon scenario. Note that the Armageddon scenario constitutes in reality a part of the front-end protection but is often separated for the purpose of modelling.

Figure 2.5 illustrates the timeline of a credit default index swaption contract and the subsequent credit default swap index.

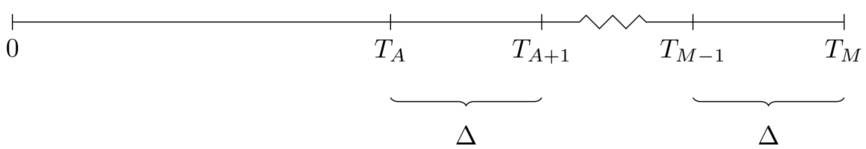


FIGURE 2.5: CDIS timeline

where T_A is the expiration time of swaption contracts, T_M is the expiration time of swap contracts, Δ is the interval between the cash flows in the swap contract and T_j is the time corresponding to the j^{th} cash flow in swap contracts where $j = A{+}1, A{+}2, ...M$. Note that the quarterly payments are consequently done at $T_{A+1}, T_{A+2}, ..., T_M$

O'Kane (2008) identifies several reasons why credit default index swaptions are attractive. For example, credit default index swaptions allows for low-cost positions (long and short) on broad credit indices. Further, the nature of swaptions (options) with the potential of non-linear payoffs might create an enhanced attractiveness to investors compared to just trading the underlying itself.

Recall that the discounted value of receiving an annuity paying one risky basis point during the life of the CDS contract starting at time T_A and maturing at time T_M, $DV01$, is given by

$$DV01 = \sum_{j=A+1}^{M} D(0, T_j)\left(1 - \frac{\mathbb{E}[N_{T_j}]}{m}\right) \Delta \tag{2.11}$$

where $D(0, T_j)$ is the discount factor from time T_j to time 0, m is the number of reference names within the CDS index and Δ is defined as the difference in time between the j^{th} and the $(j{-}1)^{\text{th}}$ cash flow (one quarter).

The expected value of the point process N_{T_j} counting the number of defaults up to time T_j is given by

$$\mathbb{E}[N_{T_j}] = \mathbb{E}\left[\sum_{i=1}^{m} \mathbb{1}_{\{\tau_i \leq T_j\}}\right] = \sum_{i=1}^{m} \mathbb{E}[\mathbb{1}_{\{\tau_i \leq T_j\}}] = \sum_{i=1}^{m} \mathbb{P}[\tau_i \leq T_j] = m\ \mathbb{P}[\tau \leq T_j] \tag{2.12}$$

where $\mathbb{1}_{\{\tau_i \leq T_j\}}$ is an indicator function taking the value of one if $\tau_i \leq T_j$ and zero otherwise, and $\mathbb{P}[\tau \leq T_j]$ is the individual probability of default up to time T_j

By inserting Equation (2.12) into Equation (2.11), $DV01$ collapses to

$$DV01 = \sum_{j=A+1}^{M} D(0, T_j)(1 - \mathbb{P}[\tau \leq T_j])\ \Delta. \tag{2.13}$$

Morini & Brigo (2011) as well as Herbertsson & Frey (2014) defines the payer CDIS payoff profile as follows

$$\Pi(T_A, T_M; \kappa) = \big(D(T_A, T_M)(R_{\mathcal{M}} - \kappa)\mathbb{1}_{\{N_{T_A} < m\}} + F_{T_A}\big)^{+} \tag{2.14}$$

where $D(T_A, T_M)$ is the discount factor from time T_M to time T_A, $R_{\mathcal{M}}$ is the T_M-year market CDS index spread observed at T_A, κ is the swaption strike spread, $\mathbb{1}_{\{N_{T_A} < m\}}$ is an indicator function taking the value of one if $N_{T_A} < m$ and zero otherwise, and F_{T_A} is the value of the front-end protection at time T_A. Note that the event $N_{T_A} < m$ means that the Armageddon scenario has not happened, i.e. there are still reference names alive in the index at time T_A.

This study will for the purpose of demarcation be restricted to solely focus on the perspective of a protection buyer, that is the perspective of agent **A**. Further, in addition to the assumptions stated in Subsection 2.2 and Subsection 2.3, the assumptions made by Black (1976), i.e. $R_I(T_M)$ being a Brownian motion, are applied when pricing credit default index swaptions.

3. Credit risk modelling

This section extends the discussion of credit risk as well as introduces different credit risk models and related concepts. First, in Subsection 3.1, the firm-value modelling approach is briefly discussed. Subsection 3.2 outlines the ideas of intensity-based models. Further, in Subsection 3.3, the intensity-based modelling approach is applied to the specific case of CDS pricing. Subsection 3.4 first presents a concise discussion about credit risk modelling without dependency in order to understand the subsequent, more rigorous, discussion of incorporating dependency when modelling default times within a credit portfolio. Lastly, in Subsection 3.5, previous subsections are applied to the specific case of modelling the Armageddon probabilities.

Credit risk modelling refers to the attempt of developing and managing models and sophisticated systems in order to describe and quantify credit risk. The outputs of these credit risk models are used by risk management units within banks and other financial institutions to measure performance, hedge portfolios, price risk and allocate securities. (Basel Committee on Banking Supervision 1999)

According to Bielecki & Rutkowski (2002) there exist two main methodologies for modelling credit risk: structural models and reduced-form models. Structural models require assumptions of a firm's capital structure and the dynamics of its assets. Such models link credit events to a firm's economic construction and are more commonly referred to as firm-value models. A default event is defined as the first time the value of the assets hits a specific threshold.

In reduced-form models, on the other hand, no assumptions are being made about the firm's capital structure. Instead an exogenous and stochastic model determines if and when a credit event will occur. Three reduced-form models are being presented by Bielecki & Rutkowski (2002): intensity-based approach, credit migrations and defaultable term structure.

3.1 Firm-value modelling

A firm-value model link credit events to a firm's capital structure. Recall that defaults events in a firm-value model is defined as the first time the value of the assets hits a specific threshold.

This method of modelling requires assumptions of a firm's capital structure and the dynamics of its assets.

One of the most recognized firm-value models is the Merton model. The basic behind the Merton model is to divide the firm's assets into two options, equity and liabilities. The idea relies on the assumption of limited liability among equity holders, i.e. equity holders always have the option of abandon the firm. Further, the firm's liabilities are assumed to take the form of zero-coupon bonds with a given maturity, allowing equity holders to keep the residual asset value after repaying their debt to the holder of liabilities at the time of maturity. Hence, holders of liabilities have a short put option contract on the firm's assets while equity holders have a long call option contract on the firm's assets, both with the debt level as strike price. The firm-value is determined at the time of maturity by the following relationship

$$\bar{B} = \begin{cases} K, & \text{if } V \geq K \\ V, & \text{if } V < K \end{cases} \tag{3.1}$$

where $\bar{B}$ is the price of a bond at maturity, K is the bond's face value and V is the value of the assets at the maturity of the bond.

A credit event therefore occurs when $V < K$ at the time of maturity. Note that the Merton model does not take in to account if $V < K$ happens before the time of maturity, that is, a credit event can only occur exactly at the time of maturity. (Lando 2004)

3.2 Intensity-based modelling

In this subsection a brief overview of intensity-based modelling is provided.

Intensity based modelling does not require any assumptions on the dynamics of a firm's assets. Instead it uses the default intensity λ, also known as arrival intensity, which cannot be observed in real life but is the modelled pace at which an obligor approaches its default time τ. The default time τ is defined as

$$\tau = \inf \left\{ t \geq 0 : \int_0^t \lambda(X_s)\ ds \geq E_1 \right\} \tag{3.2}$$

where t is a point in time, $\lambda(X_s)$ is the value of λ driven by the stochastic process X_s and E_1 is a random threshold representing the default level.

From Equation (3.2) one can see that the default intensity can be interpreted as the speed by which the integral approaches the random threshold E_1. Apparently, a higher λ will increase the probability of default for any arbitrary time interval.

Furthermore, by using Equation (3.2) one can show that the individual default probability $\mathbb{P}[\tau \leq t]$ up to the arbitrary time t, is given by

$$\mathbb{P}[\tau \leq t] = 1 - \mathbb{E}\left[e^{-\int_0^t \lambda(X_s)\ ds}\right]. \tag{3.3}$$

For a rigorous proof of Equation (3.3), see e.g. Lando (2004), Bielecki & Rutkowski (2002) or Herbertsson (2014).

Intensity-based modelling will be applied throughout this thesis as it is common practice when pricing credit derivatives as well as it, unlike firm-value modelling, does not require any assumptions of the capital structure of the firm.

3.3 Pricing CDS with intensity-based modelling

Within the intensity-based modelling framework there exist several ways of modelling the default intensity λ. Two common approaches are the process developed by Vasicek (1977) (referred to as the Vasicek-process) as well as through the further developed process presented by Cox et al. (1985) (referred to as the CIR-process).

The Vasicek-process of λ is defined as

$$d\lambda_t = \alpha(\mu - \lambda_t)dt + \sigma dW_t \tag{3.4}$$

where μ is the long-term default intensity mean, α corresponds to the speed of mean reversion, σ is the volatility and W_t is a Brownian motion process, see e.g. Vasicek (1977) and Bjork (2009).

Furthermore the CIR-process of λ is defined as

$$d\lambda_t = \alpha(\mu - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t \tag{3.5}$$

where μ is the long-term default intensity mean, α corresponds to the speed of mean reversion, σ is the volatility and W_t is a Brownian motion process, see e.g. Cox et al. (1985).

Note that the only difference between Equation (3.4) and Equation (3.5) is the latter incorporates the term $\sqrt{\lambda_t}$ in the diffusion part of the equation. This additional feature adapts the process to the magnitude of the intensity variable λ_t and prevents the process to take negative values.

If desired, default intensities can be modelled deterministically. For example, piecewise constant default intensities allow for a deterministic default intensities without setting λ to one specific level exclusively. Such models are conducted by defining a set of time points $\mathbf{T} = \{T_1, T_2, \ldots, T_J\}$ and assigning each interval in between one given level of default intensity λ_t, i.e.

$$\lambda_t = \begin{cases} \lambda_1, & \text{if } 0 \leq t < \tilde{T}_1 \\ \lambda_2, & \text{if } \tilde{T}_1 \leq t < \tilde{T}_2 \\ \vdots & \\ \lambda_J, & \text{if } \tilde{T}_{J-1} \leq t < \tilde{T}_J. \end{cases} \tag{3.6}$$

Recalling Equation (3.3), the probability of default under piecewise constant default intensities is given by the following relationship

$$\mathbb{P}[\tau \leq t] = \begin{cases} 1 - e^{-\lambda_1 t}, & \text{if } 0 \leq t < \tilde{T}_1 \\ 1 - e^{-\lambda_1 \tilde{T}_1 - \lambda_2 (t - \tilde{T}_1)}, & \text{if } \tilde{T}_1 \leq t < \tilde{T}_2 \\ \vdots & \\ 1 - e^{-\sum_{j=1}^{J-1} \lambda_j (\tilde{T}_j - \tilde{T}_{j-1}) - \lambda_J (t - \tilde{T}_{J-1})}, & \text{if } \tilde{T}_{J-1} \leq t < \tilde{T}_J. \end{cases} \tag{3.7}$$

For the purpose of this thesis, when pricing CDS using intensity-based modelling, λ is, however, assumed to be constant (neither piecewise constant nor dependent on any stochastic process). In addition, as this study assumes constant recovery rates, λ can be implied from observed market spot spreads, $R_{\mathcal{M}}$. That is, from the market spot spread of a CDS, the implied default intensity λ can be obtained under the assumption of constant default intensities and constant recovery rates as follows

$$R_{\mathcal{M}} = 4(1 - \phi)(e^{\frac{\lambda}{4}} - 1) \tag{3.8}$$

where $R_{\mathcal{M}}$ is the observed market spot spread of the CDS and ϕ is the recovery rate. By approximating using the Taylor expansion and rearranging terms in Equation (3.8), the expression collapses to the following

$$\lambda = \frac{R_{\mathcal{M}}}{1 - \phi}. \tag{3.9}$$

Equation (3.9), commonly known as the credit triangle, holds also for CDS indices under the assumption of a homogeneous portfolio. For a proof of Equation (3.8) and Equation (3.9) see Herbertsson (2014).

Furthermore, from Equation (3.3) it follows that, under the bold assumption of constant default intensities, the individual default probability $\mathbb{P}[\tau \leq t]$ up to the arbitrary time t is given by

$$\mathbb{P}[\tau \leq t] = 1 - e^{-\lambda t}. \tag{3.10}$$

3.4 Dependency modelling in credit risk

This subsection firstly presents a brief discussion about credit risk modelling without dependency in order to understand the subsequent, more rigorous, discussion of incorporating dependency when modelling default times within a credit portfolio.

The most simplistic way of modelling default times within a credit portfolio is to assume independency between reference names, that is, intensity-based modelling without dependency. In this thesis, the Binomial distribution is presented to fulfill this purpose.

The binomial probability distribution is a sum of Bernoulli experiments, that is, only two mutually exclusive and collectively exhaustive outcomes can be generated in each experiment. In the setting of credit derivatives, these events refer to the default or survival of an underlying reference name up until the arbitrary time t. The exogenous probabilities of such events are $\mathbb{P}[\tau \leq t]$ for default and $1 - \mathbb{P}[\tau \leq t]$ for survival until the arbitrary time t, respectively. In a world of multiple reference names, e.g. index-based credit derivatives, the binomial model requires the assumptions of a homogeneous portfolio as well as independent and identically distributed (i.i.d.) reference names. The binomial distribution probability of having $k \in [0, m]$ default events up to the arbitrary time t out of m reference names is given by

$$\mathbb{P}[N_t = k] = \binom{m}{k} \mathbb{P}[\tau \leq t]^k (1 - \mathbb{P}[\tau \leq t])^{(m-k)}. \tag{3.11}$$

(Newbold et al. 2013)

The probability of default before the arbitrary time t, $\mathbb{P}[\tau \leq t]$, is in this thesis modelled using an intensity-based approach (Subsection 3.2). This probability can, however, in general be modelled using other credit risk models, e.g. through a firm-value approach presented in Subsection 3.1. Throughout this thesis the binomial probability distribution will be used to model non-Armageddon default probabilities.

Although independence between reference names allows for a simplistic way of modelling, it is a bold and somewhat unrealistic assumption to make. From here, this subsection will therefore incorporate dependency when modelling default times within a credit portfolio. Throughout this thesis, correlated default times among reference names within a credit index will be modelled using the so-called Gaussian copula.

Copulas are multivariate distribution functions, constructed from known marginal distributions of random variables with a uniform distribution $\in [0, 1]$. The framework of the Gaussian copula consist of a portfolio of m reference names, with default times $\tau_1, \tau_2, \ldots, \tau_m$ and the probability of default $\mathbb{P}[\tau_i \leq t]$ up to the arbitrary time t, calibrated from market CDS spreads $R_{\mathcal{M}}$.

According to Herbertsson (2014), this way of modelling interdependence between reference names has been the standard approach up to the financial crisis of 2008. In the early stages of the crisis the Gaussian copula had a monopolistic position in modelling interdependence between default times. Jones (2009) describes the extent to which the Gaussian copula was being used as follows

> *"The development of the model had, ironically, changed the nature of the reality it was modelling."*
>
> *(Jones 2009)*

Jones (2009) further states that banks began to incur huge losses when the defaults of debts (in particular sub-prime mortgages) started to increase. This in turn created uncertainties about the solvency of financial institutions, which decreased the willingness to lend them money. As a result of the decline in financial activities, the whole economy started to stagnate. According to Jones (2009), the Gaussian copula manages to properly capture binary outcomes, that is default or non-default events, but often fail to reproduce more intricate interrelationships and abstract outcomes in the economic environment. (Jones 2009)

In order to derive the Gaussian copula, in accordance with Herbertsson (2014), a sequence of random variables X_i is defined as follows

$$X_i = \sqrt{\rho}Z + \sqrt{1-\rho}Y_i \tag{3.12}$$

where ρ is the pairwise correlation parameter being a constant $\in [0, 1]$, Y_i is a sequence of standard normal distributed random variables, and Z is a standard normal random variable, also referred to as the background factor, independent of the sequence Y_i.

Next, defining a sequence of thresholds $D_i(t)$, one for each reference name i as follows

$$D_i(t) = \mathbf{N}^{-1}\Big(\mathbb{P}[\tau_i \leq t]\Big). \tag{3.13}$$

Modifying the definition of τ from Equation (3.2) to incorporate m number of reference names and to be a function of X_i and $D_i(t)$ as follows

$$\tau_i = \inf\Big\{t \geq 0 : X_i \leq D_i(t)\Big\}. \tag{3.14}$$

That is, the default time of reference name i is the first point in time fulfilling the condition of the random variable X_i being smaller or equal to the threshold $D_i(t)$. From Equation (3.12) and Equation (3.14) it is implied that $\tau_i \leq t$ if and only if $X_i \leq D_i(t)$, which in turn renders

$$\mathbb{P}[\tau_i \leq t] = \mathbb{P}[X_i \leq D_i(t)] = \mathbb{P}[\sqrt{\rho}Z + \sqrt{1-\rho}Y_i \leq D_i(t)]. \tag{3.15}$$

Recall that, conditional on the random variable Z, the default times of the reference names τ_i are independent. By rearranging terms in Equation (3.16) and make them conditional on Z, the following expression is obtained

$$\mathbb{P}[\tau_i \leq t|Z] = \mathbb{P}\left[Y_i \leq \frac{D_i(t) - \sqrt{\rho}Z}{\sqrt{1-\rho}}|Z\right] = \mathbf{N}\left(\frac{D_i(t) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right). \tag{3.16}$$

From here, the last term of Equation (3.16), i.e. the probability of default for reference name i up to the arbitrary time t conditional on the variable Z, will be defined as $p_t(Z)$. By plugging in the results from Equation (3.16) into Equation (3.11), the final Gaussian copula formula for calculating the number of defaulted reference names N_t up to the arbitrary time t is given by

$$\mathbb{P}[N_t = k|Z] = \binom{m}{k} p_t(Z)^k (1 - p_t(Z))^{(m-k)} \tag{3.17}$$

where the term $p_t(Z)$ is defined as follows

$$p_t(Z) = \mathbf{N}\left(\frac{D_i(t) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right). \tag{3.18}$$

Hence unconditional on Z, the random variable N_t is binomially distributed with probability $p_t(Z)$, mathematically expressed as follows

$$\mathbb{P}[N_t = k] = \int_{-\infty}^{\infty} \binom{m}{k} p_t(z)^k (1 - p_t(z))^{(m-k)} \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}} dz. \tag{3.19}$$

Equation (3.19) is, however, incapable of rendering well-defined outcomes in numerical software due to the great integers generated by the combination formula $\binom{m}{k}$ for certain values of k as the number of reference names m in the portfolio becomes large. Another numerical problem occurs as probability calculated in Equation (3.18) is raised by a relatively large number k which will produce values approximated to zero.

The probability of having k number of defaults, can also be calculated as the difference between the probability of having less or equal to k defaults and the probability of having less or equal to $k-1$ defaults, i.e.

$$\mathbb{P}[N_t = k] = \mathbb{P}[N_t \leq k] - \mathbb{P}[N_t \leq k-1]. \tag{3.20}$$

If the number of reference names m within the portfolio is sufficiently large, the terms in Equation (3.20) can be calculated, using the conditional law of large numbers in accordance with Lando (2004) and Herbertsson (2014), with a so-called large portfolio approximation (LPA) as follows

$$\mathbb{P}\Big[\frac{N_t}{m} \leq \frac{k}{m}\Big] \to \mathbb{P}[p_t(Z) \leq \frac{k}{m}] \quad \text{as} \quad m \to \infty. \tag{3.21}$$

Equation (3.21), in turn, can be rewritten as follows

$$\begin{aligned}\mathbb{P}[p_t(Z) \leq \frac{k}{m}] &= \mathbb{P}\Big[\mathbf{N}\Big(\frac{\mathbf{N}^{-1}(\mathbb{P}[\tau_i \leq t]) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\Big) \leq \frac{k}{m}\Big] \\ &= \mathbf{N}\left(\frac{1}{\sqrt{\rho}}\Big(\sqrt{1-\rho}\,\mathbf{N}^{-1}\Big(\frac{k}{m}\Big) - \mathbf{N}^{-1}\big(\mathbb{P}[\tau_i \leq t]\big)\Big)\right)\end{aligned} \tag{3.22}$$

where ρ is the homogeneous pairwise correlation between reference names, k is the number of defaults, m is the number of reference names in the underlying index, and $\mathbb{P}[\tau_i \leq t]$ is one reference name's individual default probability up to the arbitrary time t.

Under the assumption of a homogeneous credit portfolio, the probability of having k number of defaults up to the arbitrary time t, $\mathbb{P}[N_t = k]$, can be approximated using the following formula

$$\begin{aligned}\mathbb{P}[N_t = k] \approx \mathbf{N}&\left(\frac{1}{\sqrt{\rho}}\Big(\sqrt{1-\rho}\,\mathbf{N}^{-1}\Big(\frac{k}{m}\Big) - \mathbf{N}^{-1}\big(\mathbb{P}[\tau \leq t]\big)\Big)\right) \\ &- \mathbf{N}\left(\frac{1}{\sqrt{\rho}}\Big(\sqrt{1-\rho}\,\mathbf{N}^{-1}\Big(\frac{k-1}{m}\Big) - \mathbf{N}^{-1}\big(\mathbb{P}[\tau \leq t]\big)\Big)\right).\end{aligned} \tag{3.23}$$

This method might, however, also be unable to render well-defined outcomes when k and m are large. The reason for this is that Equation (3.23) is defined as a difference between two probabilities dependent on k and $k-1$ respectively. The only distinction between the terms is found in the nominators where k enters. This implies that the difference between the terms approaches zero as k increases, which in turn causes problems for programming software to distinguish the terms from each other. The outcome of the equation may therefore be approximated to zero.

To be able to handle this problem and still incorporating dependence into the pricing models concerned in this thesis, one could for example use the central limit theorem applied to the counting process N_t, stated by Frey et al. (2008). Furthermore, through a modification presented by Herbertsson (2012), the probability of having k number of defaults up to the arbitrary time t is given by

$$\mathbb{P}[N_t = k] = \int_{-\infty}^{\infty} \mathbf{N}\left(\frac{k + 0.5 - mp(z)}{\sqrt{mp(z)\big(1 - p(z)\big)}}\right) f_Z(z)\ dz - \int_{-\infty}^{\infty} \mathbf{N}\left(\frac{(k-1) + 0.5 - mp(z)}{\sqrt{mp(z)\big(1 - p(z)\big)}}\right) f_Z(z)\ dz \quad (3.24)$$

where $f_Z(z)$ is the standard normal distribution density function and $p(z)$ is defined as

$$p(z) = \mathbf{N}\left(\frac{\mathbf{N}^{-1}\big(\mathbb{P}[\tau \le t]\big) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \quad (3.25)$$

where ρ is the homogeneous pairwise correlation between reference names and z is the variable integrated upon in Equation (3.24).

The Gaussian copula will be used in this study to model default probabilities under Armageddon environment. Even if Gaussian copulas have drawbacks as stated by Jones (2009), it is a well-recognized, commonly used, intuitively clear and a simplistic approach capturing the phenomenon of correlation among reference names.

3.5 Computing the Armageddon probabilities

One uncertainty in pricing of credit default index swaptions is how to compute the probability of an Armageddon scenario. The definition of an Armageddon scenario in a CDIS contract is having m number of defaults up to time T_A, i.e. $\mathbb{P}[N_{T_A} = m]$, where T_A is the expiry time of the swaption.

Equation (3.20) will be reintroduced in the particular case of computing the Armageddon probability, where the number k will be replaced by the number of reference names m in the underlying CDS index. Hence, for the specific case when calculating the Armageddon probability, Equation (3.20) becomes

$$\mathbb{P}[N_{T_A} = m] = \mathbb{P}[N_{T_A} \le m] - \mathbb{P}[N_{T_A} \le m-1]. \quad (3.26)$$

Recall that Equation (3.23) prevents the rendering of well-defined probabilities as k is large. As m is a relatively large number (most commonly 125 as stated in Subsection 2.3), the probability of having m defaults within the swaption lifetime ($N_{T_A} = m$) can be calculated using Equation (3.24).

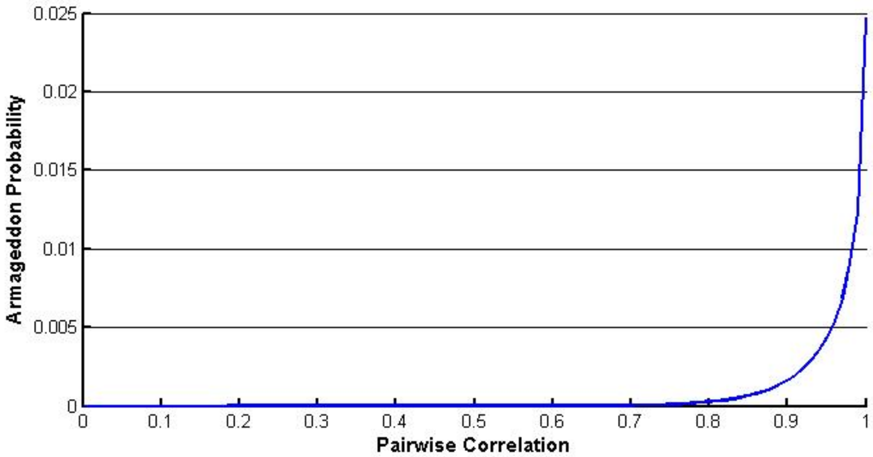


FIGURE 3.1: The Armageddon probability as a function of pairwise correlation

Figure 3.1 illustrates the probability of an Armageddon event as a function of the pairwise correlation ρ generated from Equation (3.24). Note that the values of the input parameters can be found in Table 5.1. Two remarks can be made from the graph. Firstly, the Armageddon probability is only observable as the pairwise correlation is relatively large. Secondly, under the assumption of a homogeneous portfolio, as the pairwise correlation approaches unity, the probability of an Armageddon scenario converges to the individual default probability of a single reference name.

4. Pricing credit default index swaptions

The purpose of this section is firstly to provide a relatively non-technical overview of models presented by influential contributors to the field of pricing CDIS (Subsection 4.1). Further, in subsection 4.2 the most recognized and developed models are identified and their fundamental mathematical frameworks are presented. These frameworks will form the basis of reconstruction in a numerical software environment.

4.1 Field of research overview

One of the first to enter the research field of CDIS pricing was Pedersen (2003). He argues that the standard way of valuing options on single name CDS's using the formula developed by Black (1976) is not applicable on options with CDS indices as underlying. From ordinary option theory, it is intuitively clear that the value of a standard out-of-the-money call option should approach zero when the strike price approaches infinity. Pedersen (2003) states that by applying Black (1976) formula to options on CDS indices the value of the option will not go towards zero even if it is extremely out of the money. In other words, due to the inclusion of the front-end protection in the Black (1976) formula, the price of the option will converge towards the value of this protection instead of zero. This is, however, according to Pedersen (2003) counter-intuitive as the front-end protection is worthless unless the option is exercised, which is very unlikely in the extremely out of the money scenario.

To handle this problem Pedersen (2003) suggests that the front-end protection should be incorporated directly in the CDS spread and then use the so called default-adjusted portfolio spread when pricing the CDIS. In the presented model Pedersen (2003) assumes the default-adjusted portfolio spread to follow a standard Brownian motion process. Even if it is not that hard to imagine that jumps could take place in the spread, Pedersen (2003) argues that the adoption of log-normality does not impair the model at the same magnitude as in the model with a single entity. Usually the spread is already high before an entity defaults, which probably will make a jump to have less impact and therefore not affect the portfolio spread to any great extent.

Further Pedersen (2003) states that the CDS indices are traded at lower spreads compared to combining the corresponding single name CDS's, i.e. the sum of spreads SoS, even if they technically should be the same by definition. This is due to lack of liquidity in the single name CDS market. When performing the valuation of credit default index swaptions, one have to adjust the single name CDS spreads according to the market spread of the CDS index. In other words the intrinsic CDS index spread needs to be the same as the observed market spread.

For the pricing of credit default index swaptions, Pedersen (2003) presents a model that differs significantly from the one presented by Black (1976). This methodology ensures that the pricing is done under consistency of the individual single name credit curves.

Jackson (2005) presents a new method for valuation credit default index swaptions. He combines the valuation technique of single name CDS with the survival measure presented by Schönbucher (2000) (see Appendix A.1). This combination allows for expressing swaption prices using the Black (1976) formula. Although the starting point is single name credit default swaptions, Jackson (2005) emphasizes the importance of treating it differently from credit default index swaptions.

By conditioning upon the loss variable at the expiry of the swaption, i.e. all possible scenarios of number of defaults, Jackson (2005) derives a pricing formula as a sum of Black (1976) formulas, weighted by their respective probabilities. The Armageddon scenario is valued separately as it does not require any Black (1976) formulas to be undertaken. However, these results are dependent on certain assumptions connected to the annuity of cash flows contained in the nature of the underlying swap contract. Jackson (2005) also outlines how his method captures the significant negative consequences on the underlying CDS index, caused by just a small number of defaults.

Rutkowski & Armstrong (2009) present a fairly general model which can be applied to several credit derivatives. Even though the model introduces several specific features, its main tool is the presentation of an appropriate choice of information filtration. To be able to absorb reference name defaults, the Rutkowski & Armstrong (2009) model is built upon an expected value of losses which is used to adjust the strike level in order to capture the value of front-end protection. Hence, in contrast to Pedersen (2003) and Jackson (2005), the model presented by Rutkowski & Armstrong (2009) uses the fair (not default-adjusted) spread and is not conditional on each default event.

Brigo & Morini (2009) argues that the probability of Armageddon scenarios are underestimated (or even excluded) in previous research when pricing of credit default index swaptions. Brigo & Morini (2009) shows that different probabilities of Armageddon events are obtained before and after the start of the credit crisis of 2008. Therefore, the approach suggested by the authors deals to a higher extent with the extreme events of high correlation among defaults. Further, Brigo

& Morini (2009) criticize previous research and suggest that the front-end protection needs to be incorporated directly within the Black (1976) model. The reason for this is that putting the front-end protection outside the formula ignores the fact that the front-end protection payment can be received only upon exercising of the swaption. Brigo & Morini (2009) introduce an arbitrage-free model that, opposed to previous research, brings only the Armageddon scenario outside of the Black (1976) formula.

A comparison between the arbitrage-free model presented by Brigo & Morini (2009) and previous research shows no difference in the spread output before credit crisis of 2008. Although, the same comparison on data from 2008 shows significant differences in the obtained CDIS spreads. The increase of the Armageddon default probabilities during the credit crisis shows that the arbitrage-free model includes crucial features when valuing these kinds of instruments. (Brigo & Morini 2009)

Moreover, in the further developed article written by Morini & Brigo (2011), three main problems of previous models are stated. All these problems are associated with the value of receiving an annuity paying one risky basis point during the life of the CDS index contract, $DV01$, not being strictly positive. That is, in the case of an Armageddon scenario there is no reference names still alive and therefore the value of receiving an annuity payment, $DV01$, is zero.

Morini & Brigo (2011) solve these problems by modelling the information through an appropriate subfiltration. In contrast to previous researchers, the subfiltration introduced by Morini & Brigo (2011) excludes the scenario of $DV01$ being equal to zero (explained in Appendix A.2). Morini & Brigo (2011) build a rigorous theoretical framework and present an arbitrage-free swaption formula challenging previous research.

Martin (2012) presents a method for pricing credit default index swaptions which handles the problems of valuing accrued payouts from defaults during the life of the swaption (front-end protection) and taking Armageddon events into account. Even if there exists a strong criticism against the assumption of credit spreads following a Brownian motion process, the author still uses a modified version of Black (1976) model as it is the most common and established model used on trading desks. Martin (2012) states that this approach is not revolutionary but it treats the payout more carefully and intuitively.

Furthermore Martin (2012) shows that when the strike spread is significantly high, the exercise payoff does not converge towards zero as stated by Pedersen (2003) or towards the expected value of the Armageddon scenario as stated by Brigo & Morini (2009). Instead Martin (2012) suggests that the payoff of the CDIS as $\kappa \to \infty$ will converge to $1-\phi$. These results, however, appear counter-intuitive as the front-end protection is conditional upon exercising the swaption and for a strike spread approaching infinity there exists only one case when exercising of the swaption is economically justified — the Armageddon scenario. The expected present value of a CDIS as the strike spread goes towards infinity is rather the discounted front-end protection conditional on Armageddon, multiplied by the probability of the Armageddon event to happen, which is consistent with the findings of Brigo & Morini (2009).

4.2 Considered models

Based on the criteria of recognition and development (defined in Section 1), Jackson (2005), Rutkowski & Armstrong (2009) and Morini & Brigo (2011) has been identified as the most recognized and developed models and the thesis will thus here from solely focus on these three models.

In the following subsections, models are presented independently of specific probability distributions. However, for the purpose of modelling in the numerical software environment, the Armageddon- and non-Armageddon probability distributions are modelled using the Gaussian copula and the Binomial probability distribution (both presented in Subsection 3.4), respectively. Furthermore, all models are preceded by the calculations presented in Section 2 and Section 3, which are generic and model-independent.

Recall that, in order to facilitate the modelling of credit default index swaptions, the following set of assumptions are defined. Firstly, the recovery rates are assumed to be constant and pre-defined. Secondly, interest rates are assumed to be constant over time (and thus consequently deterministic). Thirdly, in addition to identical default intensities, they are also assumed to be constant over time. Lastly, the time of valuation is set to 0 in all pricing models.

4.2.1 The Jackson (2005) model

Jackson (2005) examines the payoff of the instrument as a construction of two parts, the forward spread process of the underlying CDS index and the loss-at-expiry variable ℓ_{T_A}. He introduces the index survival measure (see Appendix A.1) which is used to simplify the price expression for the credit default index swaption. Jackson (2005) suggest a Gaussian copula approach (described in Subsection 3.4) for modelling the joint distribution of default times.

By conditioning upon the loss variable at the expiry of the swaption ℓ_{T_A}, Jackson (2005) allows for constructing a consistent model that takes into account both the process of the spread and losses during the life of the swaption. This is modelled through stating i different exogenous defined loss distributions, where $i = 0, 1, ..., m-1$ and m is the number of reference names in the underlying CDS index.

Jackson (2005) uses a weighted sum of Black (1976) model, conditioning on the number of defaults i up to time T_A. The Armageddon scenario $i = m$ is, however, separated from the other defaults scenarios and valuated outside the Black (1976) model. Assuming homogeneous recovery rates ϕ, the loss variable $\ell_{T_A,i}$ is defined as

$$\ell_{T_A,i} = \frac{i(1-\phi)}{m} \tag{4.1}$$

that is, the loss given default for i number of defaults. For the purpose of modelling, the function defined by Jackson (2005) used to combine the portfolio losses along with the cash payment $f(\ell_{T_A,i})$ is assumed to be equivalent to the loss variable $\ell_{T_A,i}$. The reason for this is that Jackson (2005) outlines two equivalent methods in taking the loss up to T_A into account. Either a cash payment corresponding to the defaulted proportion of the nominal is paid out or a strike adjustment is made in order to compensate for these losses.

Jackson (2005) uses the approach of adjusting the strike spread κ in order to account for a given number, i, of defaulted reference names within the index at time T_A. The adjusted strike spread $\bar{\kappa}_i$ is defined as

$$\bar{\kappa}_i = \kappa - \frac{f(\ell_{T_A,i})}{DV01 + A\ell_{T_A,i}}. \tag{4.2}$$

The denominator is derived from the assumption that the stochastic process $DV01_{T_A}$ (the value at time T_A of receiving an annuity paying one risky basis point during the life of the CDS index contract) is a linear function of the losses incurred up to T_A, given by

$$DV01_{T_A}|(L_{T_A} = \ell_{T_A,i}) = \begin{cases} DV01 + A\ell_{T_A,i}, & \text{if } i \neq m \\ 0, & \text{if } i = m \end{cases} \tag{4.3}$$

where L_{T_A} is a stochastic process counting the actual accumulated loss due to defaults up to time T_A, still under the assumption of homogeneous recovery rates ϕ, that is $L_{T_A} = \sum_{i=1}^{m}(1-\phi)\mathbb{1}_{\{\tau_i \leq T_A\}}$ and A is a constant given by

$$A = \frac{\sum_{i=1}^{m-1} q_0^i f(\ell_{T_A,i}) DV01(1 - D(0,T_A))}{\sum_{i=1}^{m-1} q_0^i D(0,T_A)\ell_{T_A,i} f(\ell_{T_A,i})} \tag{4.4}$$

where q_0^i is an arbitrary measure, not exclusively determined by Jackson (2005), calculating the probability of each default scenario i. The probability measure q_0^i is for the purpose of modelling set to follow a Binomial probability distribution.

Jackson (2005) constructs a model consisting of three parts, the present value of receiving an annuity paying one risky basis point, $DV01$, multiplied with the weighted sum of Black (1976) models, plus the expected value of an Armageddon scenario. Jackson (2005) distinguishes the probability measure of the Armageddon scenario from all other scenarios. The intuition behind this is that the probability of m defaults is likely to be greater than the probability of $m-1$

defaults. For modelling purpose the Armageddon event is assigned a probability q_0^m rendered from a Gaussian copula. The final valuation formula is given by

$$C_0^{J05} = DV01 \cdot \mathbb{1}_{\{\tilde{\tau}>0\}} \sum_{i=1}^{m-1} q_0^i [R_{\mathcal{M}} \mathbf{F}(d_{1i}) - \bar{\kappa}_i \mathbf{F}(d_{2i})] + q_0^m D(0, T_A) f(\ell_{T_A,m}) \tag{4.5}$$

where $\tilde{\tau}$ is the default time of the last-to-default reference name in the CDS index, that is $\tilde{\tau} = \max(\tau_1, \tau_2, ..., \tau_m)$, $R_{\mathcal{M}}$ is the observed market spot spread of the CDS index, $\mathbf{F}(x)$ is an arbitrary cumulative distribution function (in line with Black (1976) set to be a normal cumulative distribution $\mathbf{N}(x)$ for the purpose of modelling), $f(\ell_{T_A,m})$ is the loss given default of all reference names within the CDS index up to T_A and d_{1i} and d_{2i} are defined as follows

$$d_{1i} = \frac{\ln(R_{\mathcal{M}}/\bar{\kappa}_i) + \sigma^2 \frac{T_A}{2}}{\sigma\sqrt{T_A}}, \qquad d_{2i} = d_{1i} - \sigma\sqrt{T_A}. \tag{4.6}$$

If desired, the volatility σ of the underlying CDS index can be modified to be a function dependent on i number of defaults. This will, however, be excluded from the scope of this thesis.

4.2.2 The Rutkowski & Armstrong (2009) model

To be able to absorb reference name defaults, the Rutkowski & Armstrong (2009) model is built upon an expected value of losses which is used to adjust the strike spread level in order to capture the value of front-end protection prior to the swaption maturity. The model uses the fair CDS index spread, not conditional on each default event.

Rutkowski & Armstrong (2009) further defines the expected accumulated loss up to the arbitrary time t as

$$\mathbb{E}[\ell_t] = \frac{(1-\phi)\mathbb{E}[N_t]}{m}. \tag{4.7}$$

Rutkowski & Armstrong (2009) defines κ_t^n as the non-Armageddon fair forward CDIS spread at the arbitrary time t. Setting $t = 0$, however, renders

$$\kappa_0^n = \frac{\Phi}{DV01}. \tag{4.8}$$

The strike spread adjustment, used to absorb reference name defaults through calculating the expected value of losses, $\bar{\ell}_0$, is defined by Rutkowski & Armstrong (2009) as

$$\bar{\ell}_0 = \frac{\mathbb{E}[\ell_{T_A}]}{DV01}. \tag{4.9}$$

The final formula by Rutkowski & Armstrong (2009) is calculated as follows

$$C_0^{RA09} = \mathbb{1}_{\{\tilde{\tau}>0\}} DV01 \cdot \left(\kappa_0^n \, \mathbf{N}(d_1) - (\kappa - \bar{\ell}_0)\mathbf{N}(d_2)\right) \tag{4.10}$$

where κ is the strike spread, and the terms d_1 and d_2 are defined as follows

$$d_1 = \frac{\ln\left(\frac{\kappa_0^n}{\kappa-\ell_0}\right) + \frac{1}{2}\int\limits_0^{T_A}\sigma^2(u)du}{\left(\int\limits_0^{T_A}\sigma^2(u)du\right)^{\frac{1}{2}}}, \qquad d_2 = \frac{\ln\left(\frac{\kappa_0^n}{\kappa-\ell_0}\right) - \frac{1}{2}\int\limits_0^{T_A}\sigma^2(u)du}{\left(\int\limits_0^{T_A}\sigma^2(u)du\right)^{\frac{1}{2}}}. \tag{4.11}$$

Lastly, under the assumption of constant volatility, d_1 and d_2 collapses to

$$d_1 = \frac{\ln\left(\frac{\kappa_0^n}{\kappa-\ell_0}\right) + \sigma^2\frac{T_A}{2}}{\sigma\sqrt{T_A}}, \qquad d_2 = \frac{\ln\left(\frac{\kappa_0^n}{\kappa-\ell_0}\right) - \sigma^2\frac{T_A}{2}}{\sigma\sqrt{T_A}}. \tag{4.12}$$

4.2.3 The Morini & Brigo (2011) model

Morini & Brigo (2011) identify three main problems associated with previous models. Firstly, the forward CDS spread is not globally defined as the denominator in the spread formula is not strictly positive. That is, when the value of receiving one risky basis point $DV01$ is zero (i.e. in the case of an Armageddon event) the CDS spread becomes undefined. Secondly, the problem of a zero value of $DV01$ will further cause an undefined valuation formula of the CDIS. Although in practise the contracts do have explicit and positive values even under such conditions due to the front-end protection. Thirdly, $DV01$ is an inappropriate choice of numerarie, again as it can take the value of zero. Morini & Brigo (2011) describe the situation as follows:

> *"To the best of our knowledge, the current literature does not solve these problems. The only partial exception is Jackson (2005), that deals with the second problem while not considering the first and the last one (in particular he uses a numerarie which is not strictly positive)."*
>
> *(Morini & Brigo 2011, p. 579).*

Morini & Brigo (2011) solve these three problems by modelling the information through an appropriate subfiltration. In contrast to previous researchers, Morini & Brigo (2011) introduces a new subfiltration that excludes the scenario of $DV01$ being equal to zero (see Appendix A.2).

In consistency with Jackson (2005), Morini & Brigo (2011) isolate the Armageddon scenario from all other scenarios. The Armageddon probability measure $\mathbb{Q}$ is distinguished by Morini & Brigo (2011) from the non-Armageddon probability measure. For the purpose of modelling throughout this thesis, these probability measures are set to be the Gaussian copula and the Binomial probability distribution, respectively. Further, the value of front-end protection in case of (but not conditional on) an Armageddon scenario is defined as

$$(1-\phi)D(0,T_A)\mathbb{Q}[\tilde{\tau} \leq T_A]. \tag{4.13}$$

The modification of the Black (1976) formula made by Morini & Brigo (2011) uses an adapted spot spread $\hat{S_0}$, which consists of the spread $\tilde{S_0}$ less an adjustment for the Armageddon scenario. $\hat{S_0}$ is thus calculated as follows

$$\hat{S_0} = \tilde{S_0} - \frac{(1-\phi)D(0,T_A)\mathbb{Q}[\tilde{\tau} \leq T_A]}{DV01} \tag{4.14}$$

where $\tilde{S_0}$ is the spread which, when multiplied with $DV01$, equals the present value of the default leg plus the present value of the front-end protection. That is

$$\tilde{S_0} = \frac{\mathbb{E}[\Phi] + \mathbb{E}[F_0]}{\mathbb{E}[DV01]} \tag{4.15}$$

where Φ is the default leg and F_0 is the value of the front-end protection at time 0. The final valuation formula is given by

$$C_0^{MB11} = DV01\left(\hat{S_0}\mathbf{N}(d_1) - \kappa\mathbf{N}(d_2)\right) + (1-\phi)D(0,T_A)\mathbb{Q}[\tilde{\tau} \leq T_A] \tag{4.16}$$

where κ is the strike of the swaption and d_1 and d_2 are defined as follows

$$d_1 = \frac{\ln(\hat{S_0}/\kappa) + \sigma^2\frac{T_A}{2}}{\sigma\sqrt{T_A}}, \qquad d_2 = d_1 - \sigma\sqrt{T_A}. \tag{4.17}$$

5. Numerical studies and results

This section aims to present the sensitivities and interrelationships between input parameters and swaption spreads with respect to the different models in order to examine whether the models are well-behaving under economic interpretation. Subsection 5.1 outlines how the analyses are conducted. In Subsection 5.2 the models presented by Jackson (2005), Rutkowski & Armstrong (2009) and Morini & Brigo (2011) are compared and evaluated with respect to the conducted analyses. Further, as Morini & Brigo (2011) are identified as superior in demanding the least bold assumptions and having the highest level of transparency, an in-depth analysis is made in Subsection 5.3. This subsection will also present an application on real historical market data before, during and after the financial crisis of 2008.

5.1 Specification of analysis

In order to fulfill the purpose and answering the research questions of this thesis, a mapping of the considered models' behavior with respect to changes in input parameters needs to be undertaken. The behavior of the models, in turn, will be evaluated through conducting a set of sensitivity analyses. This method intends to find interrelationships between input variables and swaption spreads in order to examine whether the models are well-behaving under economic interpretation.

Firstly, all models fulfilling the criteria of recognized and developed (Jackson (2005), Rutkowski & Armstrong (2009) and Morini & Brigo (2011)) will be analyzed over separate changes in input variables. This approach, known as one-factor-at-a-time (OFAT), allows for evaluating the dynamics and mutually comparing the models. For illustrative purpose, these sensitivity analyses will be presented in two-dimensional graphs.

Regarding the in-depth analysis (Subsection 5.3), sensitivities will be divided into two parts, CDIS prices modelled over sets of input pairs as well as for relevant ad-hoc scenarios. The first part aims at showing what forces that drives the CDIS prices in multi-variable environments and how they are interrelated with each other. In Subsection 5.3 index spot spread will reappear as an input variable in all analyses. This modelling setup relies mainly on two reasons, the large impact index spot spread has on CDIS prices as well as the clear intuition behind the effect of changes in the index spot spread. However, swaption strike spread has an almost equally intuitive interpretation and as its impact on the CDIS price is practically a mirror of the index spot spread. Hence, for the sake of consistency the parameter index spot spread will reappear as a variable on one of the axes in all analyses. The second part consists of ad-hoc scenarios concerning derivative-specific features composed by, but not limited to, the isolated effect of front-end protection and Armageddon as well as the CDIS value when the swaption strike spread approaches infinity ($\kappa \to \infty$). The latter ad-hoc scenario is of interest due to the existence of, in accordance with Subsection 4.1, a clear inconsistency in the interpretation among researchers.

When conducting the aforesaid analyses, the relationships will be graphically illustrated using three-dimensional diagrams. The in-depth sensitivities will later constitute the fundamentals of the discussion when evaluating the dynamics of the model(s) considered. Table 5.1 outlines the exogenous inputs and their assigned values for the purpose of modelling. That is, a certain variable which is not varied in all subsequent analyses will be assigned the value given in Table 5.1.

TABLE 5.1: Input parameters of numerical studies

Input	**Value**
Index Spot Spread $R_{\mathcal{M}}$	0.0300
Swaption Strike Spread κ	0.0300
Swaption Maturity T_A	0.5
Swap Lifetime T_M	5
Recovery Rate ϕ	0.4
Number of Reference Names m	125
Index Volatility σ	0.3
Risk-free Interest Rate r	0.01
Interval between Swap Cash Flows Δ	0.25
Pairwise Correlation ρ	0.3

5.2 Comparison of Jackson (2005), Rutkowski & Armstrong (2009) and Morini & Brigo (2011)

Out of the main contributors of this field of study, Jackson (2005), Rutkowski & Armstrong (2009) and Morini & Brigo (2011) are identified as the most recognized and developed models in pricing CDIS. It is, however, of importance to illuminate that there exist other models, excluded from this thesis, worth to evaluate if the demarcations would have been defined differently. The reason why models in this thesis has been restricted to be both recognized and developed is that the former criteria by itself does not necessarily reveal anything about the quality of the models. That is, authors can be cited due to criticism as well as acceptance. Hence, the criterion of development captures only the accepted models. This criterion by itself, on the other hand, does not say anything about the contribution to the field of research. Although the demarcations affect the outcome of the study, the literature review (Subsection 4.1) as well as the discussions about derivative-specific features are independent of how the considered models are selected.

Figure 5.1 presents CDIS values generated from the three models considered over a set of index spot spreads $R_{\mathcal{M}}$. As can be seen from the zoomed circle, Jackson (2005) produces slightly lower values than Rutkowski & Armstrong (2009). Over the generous interval of $R_{\mathcal{M}}$, the two models, however, produce similar values while Morini & Brigo (2011) render significantly higher values all over the defined interval.

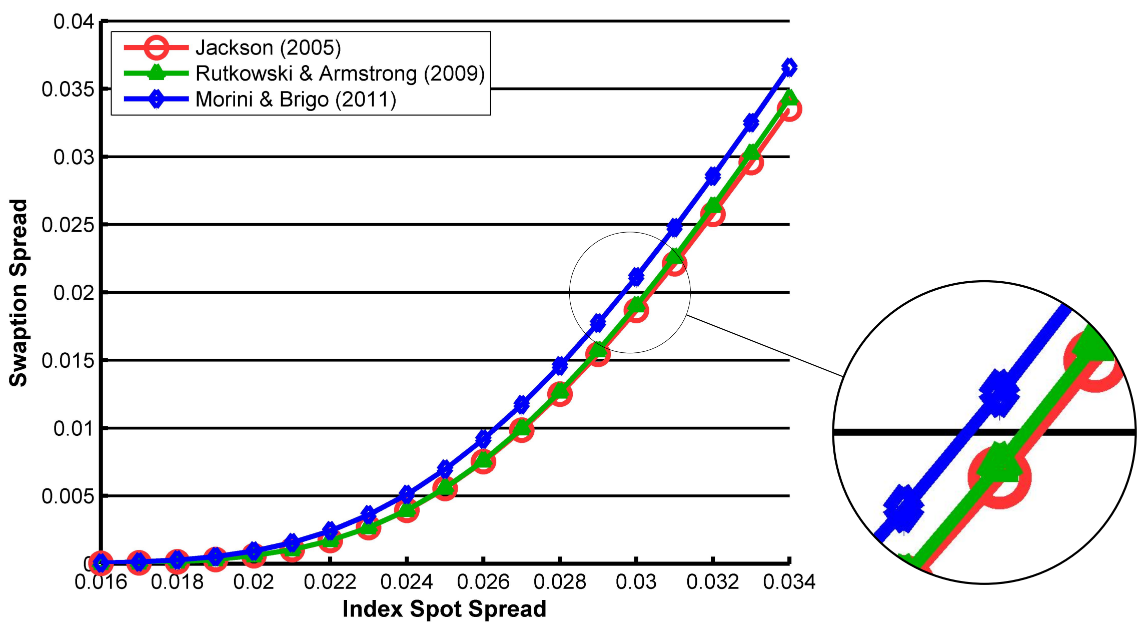


Figure 5.1: CDIS spread as a function of index spot spread

The reason why Jackson (2005) and Rutkowski & Armstrong (2009) generate values close to each other can be derived from the constructional features of the models. From Equation (4.2) and Equation (4.9) it is notable that Jackson (2005) and Rutkowski & Armstrong (2009) uses similar methods when adjusting the swaption strike spread κ in their models. Morini & Brigo (2011), on the other hand, uses an adjustment of the spot spread in Black (1976) model, as seen in Equation (4.14). Even though these are important explanatory factors, they of course do not constitute the entire difference between the models.

Figure 5.2 illustrates CDIS spreads as a function of swaption strike spread κ. Intuitively, the graph almost illustrates a mirrored relationship from that in Figure 5.1. The probability of the swaption ending up in-the-money decreases as swaption strike spread increases.

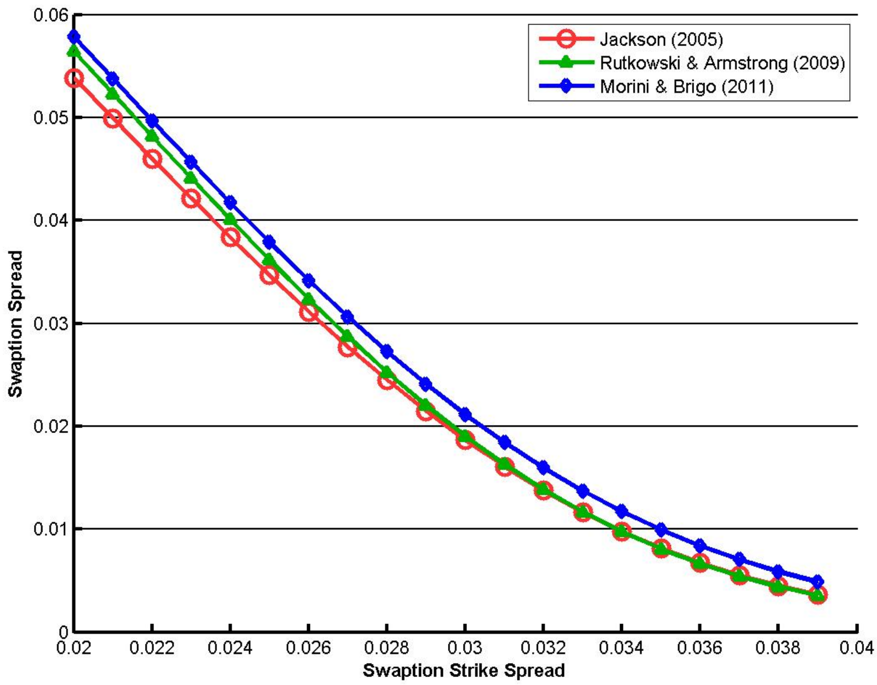


FIGURE 5.2: CDIS spread as a function of swaption strike spread

In the extreme case when the swaption strike spread approaches infinity ($\kappa \to \infty$), the value of the swaption approaches the expected value of an Armageddon scenario in the models presented by Jackson (2005) and Morini & Brigo (2011). However, in the case of Rutkowski & Armstrong (2009), the value of the CDIS goes towards zero. The reason for this is that Jackson (2005) and Morini & Brigo (2011) model the Armageddon scenario outside the Black (1976) formula, while Rutkowski & Armstrong (2009) incorporates the Armageddon scenario directly inside the Black (1976) formula. In the case of an Armageddon event there is no reason for not exercising the swaption as it is independent of the relationship between the index spot spread and the swaption strike spread. Remarkably, this is not taken into account by Rutkowski & Armstrong (2009).

Figure 5.3 graphs the relationship between CDIS value and the level of index volatility σ. Recall that the rest of the parameters are given as in Table 5.1. All three models are strictly increasing in index volatility and therefore consistent with ordinary option theory. Notable is that the Morini & Brigo (2011) model is more sensitive (greater first-order derivative) with respect to index volatility, when the volatility is relatively high.

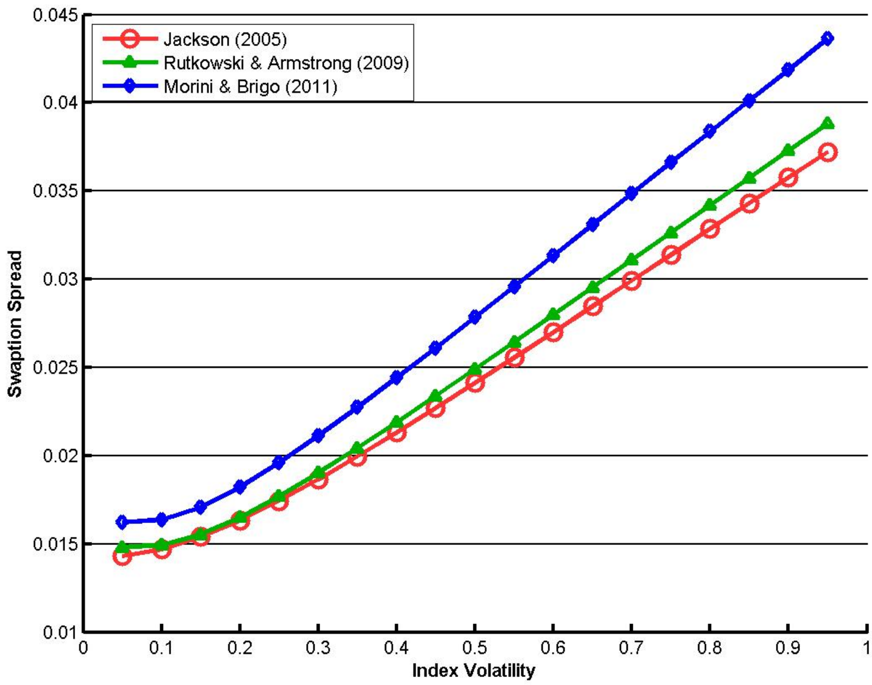


FIGURE 5.3: CDIS spread as a function of index volatility

Furthermore, the CDIS value over a range of all possible values of pairwise correlation ρ is shown in Figure 5.4. Again, recall that the rest of the parameters are given as in Table 5.1. For the purpose of this study, the pairwise correlation parameter enters only in the valuation of the Armageddon scenario through the Gaussian copula presented in Subsection 3.4. That is the reason why the graphs dramatically increase when the pairwise correlation is high, i.e. close to one. If pairwise correlation were to be incorporated in all loss scenarios, the graphs would still begin and end up in the same coordinates in the diagram, but their shapes would likely be more evenly increasing.

As shown by Preis et al. (2012) the pairwise correlation among reference names ρ is in reality a changing process over time, taking higher values during stressed economic climates. Hence, the Armageddon event is important to include when valuing the CDIS even if its initial value is negligible. Therefore, if desired the pairwise correlation parameter ρ can be modified to be a function of the economic environment. This will be further discussed in Subsection 5.3.2.

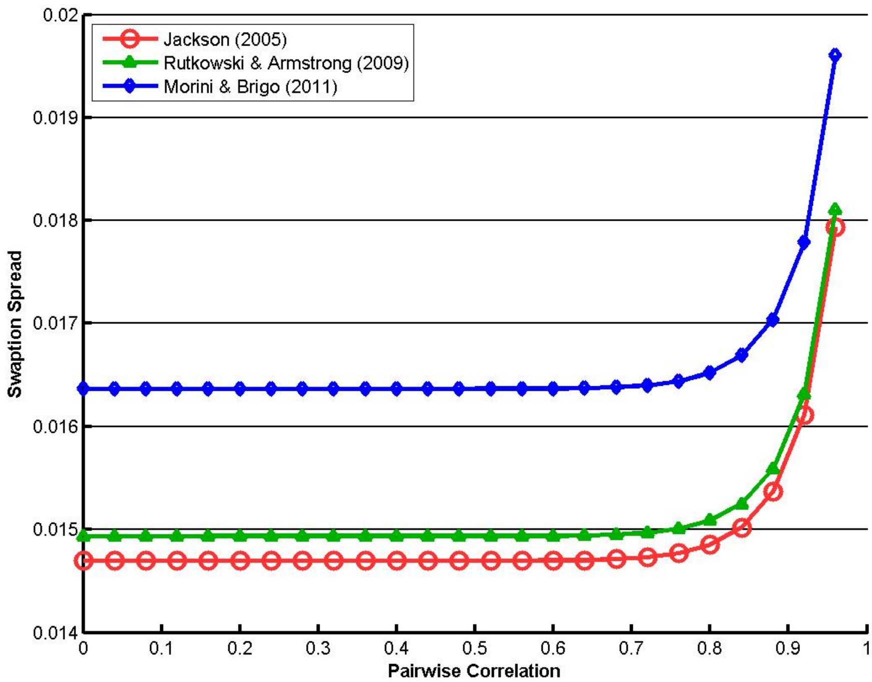


FIGURE 5.4: CDIS spread as a function of pairwise correlation

The three considered models all make sense under economic interpretation and are well-behaving with respect to changes in all input parameters. In this context, well-behaving only refers to the direction and relative movements of output changes due to changes in input parameters, and does not necessarily reveal anything about the absolute magnitude of movement. As shown in Subsection 5.3.3, the CDIS price is to a great extent derived from the Black (1976) model. In a non-stressed economic environment the Black (1976) model might even generate a reasonable approximation of the CDIS price. This implies that when the economic climate is calm, the derivative-specific features are negligible due to a low probability of default within the swaption lifetime. In economic turmoil, on the other hand, the probability of default during the swaption lifetime increases drastically and causes the derivative-specific features to constitute a larger proportion of the CDIS price.

Further, the models diverge the most in the derivative-specific features which causes the difference in output between the models to escalate as the economic climate worsens. This implies that the way of modelling the derivative-specific features, and therefore the choice of model, are crucial during economic turmoil and of less importance during non-stressed economic climates.

Further, Jackson (2005) and Rutkowski & Armstrong (2009) demand bold assumptions in order to reconstruct their models and produce CDIS values, due to lack of transparency in defining certain calculation procedures. Moreover, the interpretation of the CDIS value as the swaption strike spread approaches infinity made by Rutkowski & Armstrong (2009) is inconsistent with the nature of the contract. Finally, neither Jackson (2005) nor Rutkowski & Armstrong (2009) concretize their models by presenting generated CDIS values.

All aforesaid shortcomings affect the reliability of the models presented by Jackson (2005) and Rutkowski & Armstrong (2009). With this awareness, even though the researchers are eminent and their models are robust, the study will from here focus solely on an in-depth analysis of the model presented by Morini & Brigo (2011) as it is superior in demanding the least bold assumptions and having the highest level of transparency.

5.3 In-depth analysis of Morini & Brigo (2011)

5.3.1 Sensitivity analysis

This subsection provides in-depth sensitivity analyses as well as an application on real historical market data before, during and after the financial crisis of 2008 on the model presented by Morini & Brigo (2011).

The sensitivity analyses undertaken in this subsection aims to examine how incremental changes in multiple input parameters affect the prices of credit default index swaptions rendered from the model presented by Morini & Brigo (2011). Recall that unless explicitly stated, all parameters are given as in Table 5.1.

Firstly, CDIS spread as a function of index spot spreads is presented in Figure 5.5. Similarly to an ordinary call option, it can be roughly divided into three intervals. When the swaption is deeply out-of-the-money, the value of the CDIS is close to zero and the graph appears flat. For the deep-in-the-money interval, the value of the CDIS increases almost proportional to the index spot spread $R_{\mathcal{M}}$. The interval in between is characterized by a convex curvature affected to a great extant by the level of index volatility σ and the time to swaption maturity T_A in relation to the absolute distance between the index spot spread $R_{\mathcal{M}}$ and the swaption strike spread κ.

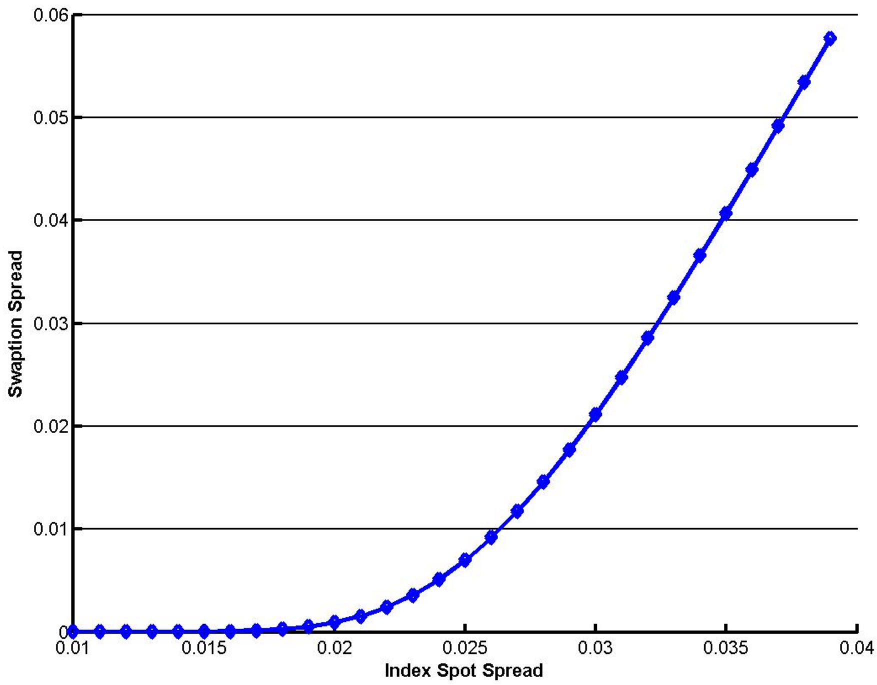


FIGURE 5.5: Morini & Brigo (2011) CDIS spread as a function of index spot spread

Figure 5.6 illustrates a surface of CDIS spreads as a function of index spot spread and swaption strike spread. The flat area of the surface, when rendering a CDIS spread close to zero, occurs due to the swaption being way out-of-the-money. At first sight the output may seem linear in identical changes of index spot spread and swaption strike spread (for any given absolute difference between the two inputs, the output will be the same). However, a constant volatility causes a growth in absolute movements as index spot spread increases. That is, the surface is in fact asymmetric. Note that Morini & Brigo (2011) do not incorporate the index spot spread $R_{\mathcal{M}}$ directly into the Black (1976) model, instead it enters the formula through determining the individual default probability λ (see Subsection 4.2).

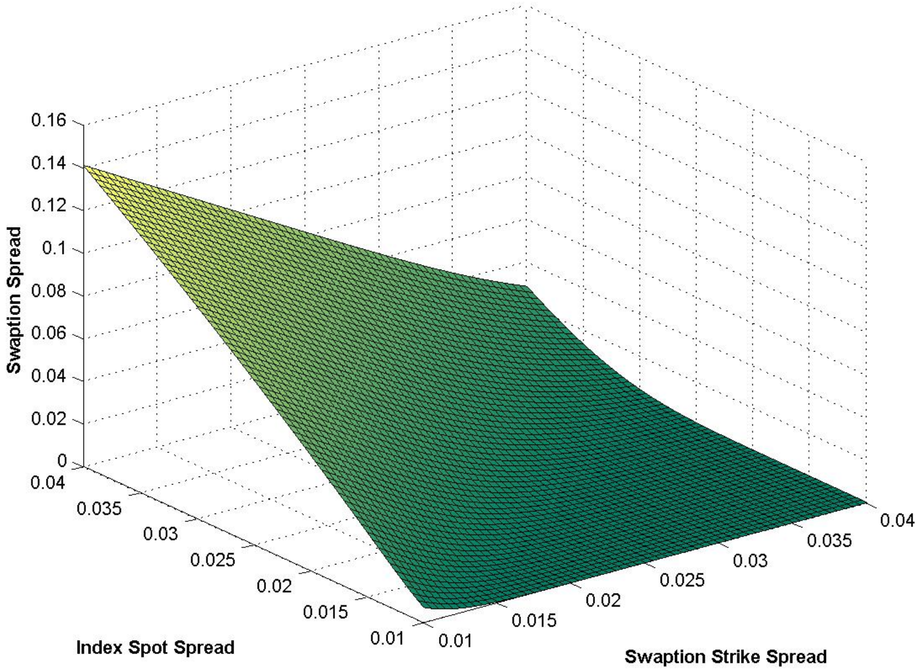


FIGURE 5.6: Morini & Brigo (2011) CDIS spread as a function of index spot spread and swaption strike spread

Figure 5.7 differs from Figure 5.6 in the sense that swaption strike spread κ is replaced by pairwise correlation ρ on the axis. The surface is almost uniform for all levels of pairwise correlation ρ except when approaching one. This is related to the fact that the expected value of an Armageddon scenario is not negligible when pairwise correlation is high, due to being assigned relatively high probability. This is discussed further in Subsection 5.3.3.

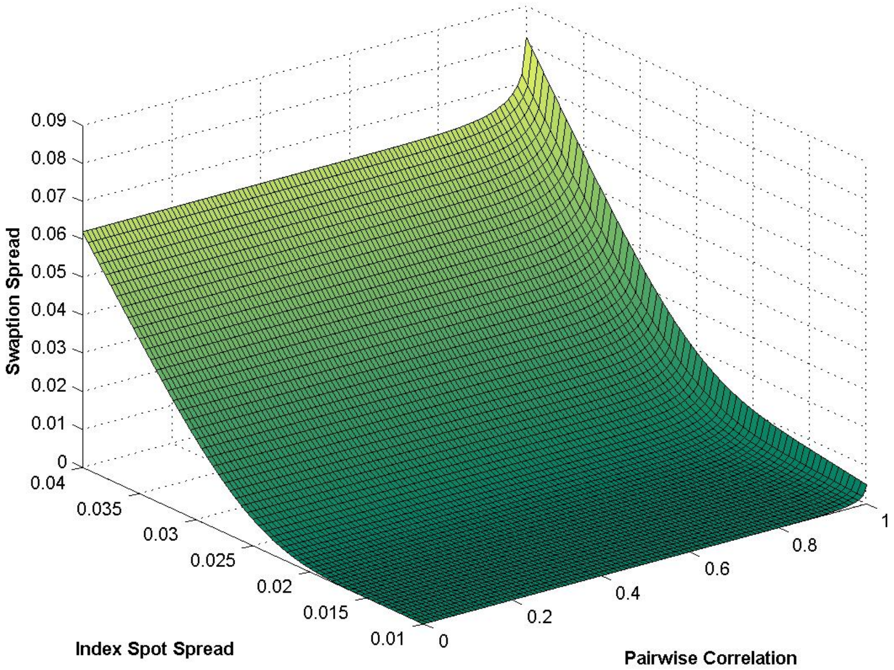


FIGURE 5.7: Morini & Brigo (2011) CDIS spread as a function of index spot spread and pairwise correlation

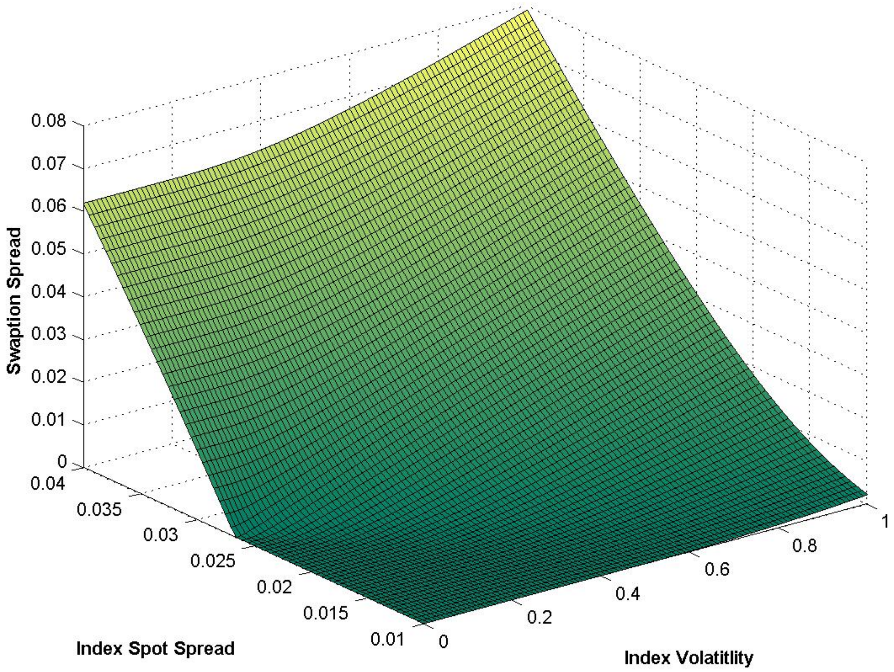


FIGURE 5.8: Morini & Brigo (2011) CDIS spread as a function of index spot spread and index volatility

Figure 5.8 illustrates a surface constructed from a set of index spot spreads $R_{\mathcal{M}}$ and index volatility σ, respectively. Intuitively, CDIS spreads are strictly increasing in index volatility for all levels of spot spreads. Furthermore, when uncertainty of index spot spread $R_{\mathcal{M}}$ is removed from the model (i.e. σ=0), apart from front-end protection the swaption price converges to the known-for-sure return of the swaption. The graph therefore takes the form of two straight lines. Finally, high index volatility might cause CDIS spreads not to be negligible even for deep-out-of-the-money swaptions.

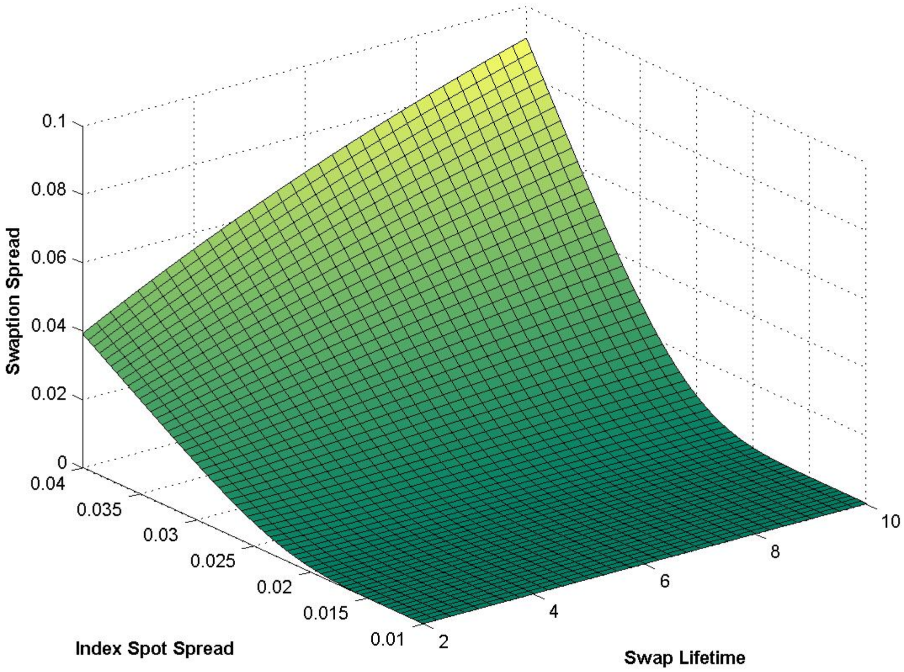


Figure 5.9: Morini & Brigo (2011) CDIS spread as a function of index spot spread and swap lifetime

The effect of changing the swap lifetime T_M on CDIS spread is shown in Figure 5.9 over different levels of index spot spreads. This effect is, intuitively, almost negligible when the swaption is deeply out-of-the-money. On the other hand, as the swaption is deep-in-the-money the CDIS spread increases concavely over the swap lifetime. This can be explained by the diminishing marginal value of an incremental change in swap lifetime due to discount factors and the nature of the swap contract.

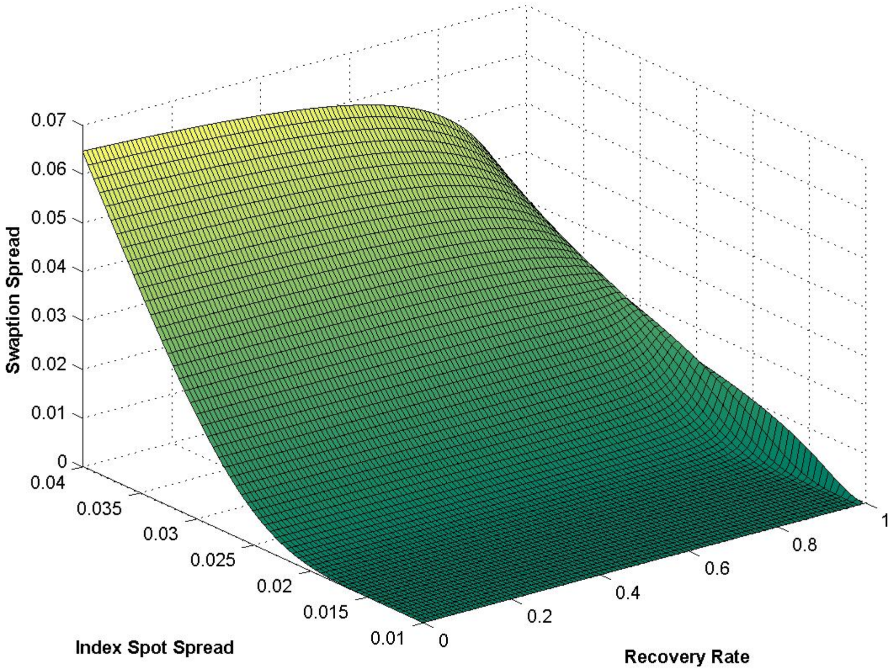


FIGURE 5.10: Morini & Brigo (2011) CDIS spread as a function of index spot spread and recovery rate

Figure 5.10 illustrates how the CDIS spread behaves in relation to index spot spreads and recovery rates. As one minus the recovery rate $(1-\phi)$ determines the proportion paid back to the swaption holder in case of default, the value of CDIS approaches zero as the recovery rate approaches one. However, it seems somewhat counter-intuitive that the value of the CDIS increases in Figure 5.10 for low values of index spot spread as the recovery rate increases. To explain this phenomenon, two separate forces have to be taken into account. These forces are more easily shown in Figure 5.11, which has the same properties as Figure 5.10 but with a tighter interval of both index spot spreads and recovery rates.

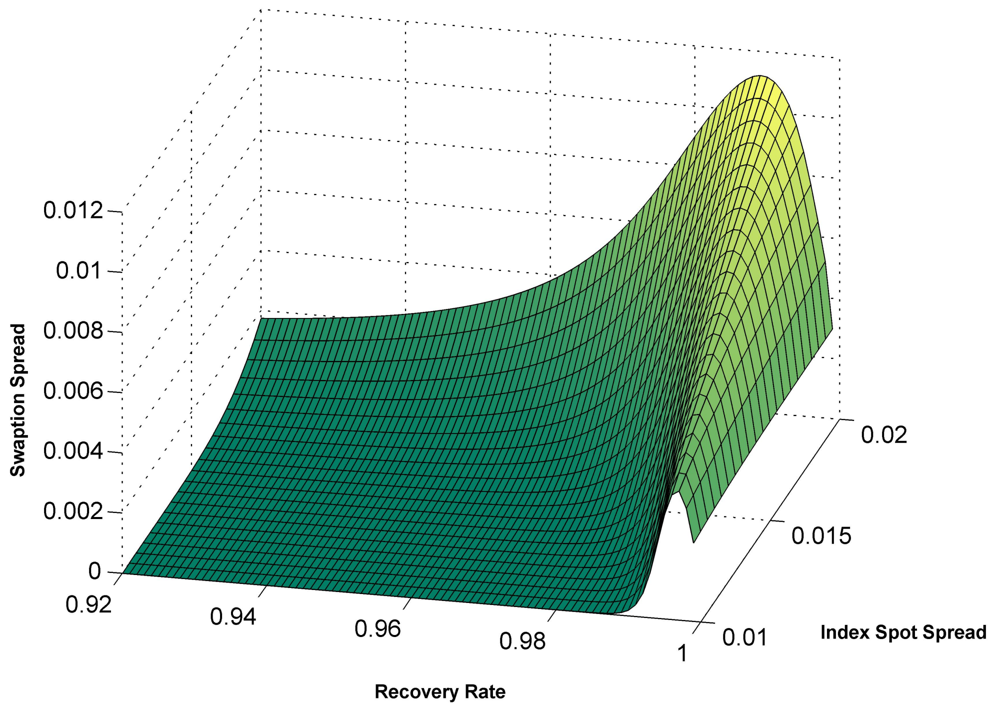


FIGURE 5.11: Morini & Brigo (2011) CDIS spread as a function of index spot spread and high values of recovery rate

The first force is driven by Equation (3.9) which calculates the individual reference name default probability λ. The default intensity λ approaches infinity as ϕ approaches one, which causes the CDIS spread to increase drastically as the probability of default goes towards one. Consequently, the swaption will be exercised almost for sure.

However, as recovery rate ϕ increases further the second force, which is the diminishing proportion the swaption holder is paid back in case of defaults, becomes dominant. This is illustrated by the sudden drop in value in Figure 5.11. The second force can be explained by the fact that the CDIS contract becomes worthless as the proportion paid back in case of default disappears. The phenomenon shown in Figure 5.11 is, due to the front-end protection, unique for the credit default index swaptions.

5.3.2 Application to historical market data

In this subsection, the CDIS pricing model presented by Morini & Brigo (2011) is applied to real historical market data of the European iTraxx 5-year index before, during and after the financial crisis of 2008. Again, recall that unless explicitly stated, all parameters are given as in Table 5.1. Moreover, as CDIS contracts lack liquid market prices that could constitute a potential benchmark when evaluating pricing models, the volatilities will in this subsection be calculated from historical data of CDS indices. If liquid CDIS prices could be observed, their implied volatility could be backed out in a similar way as for ordinary liquid options.

Figure 5.12 illustrates real historical market CDS index spot spreads of the European iTraxx 5-year index. As can be seen in the graph, the spread of the CDS index dramatically increased during the financial crisis of 2008, while being on a lower level both before and after. Further, the historical volatility of the European iTraxx 5-year index, shown in Figure 5.13, is calculated from a rolling six months interval with daily observations and presented annually. That is, each CDS index volatility observation corresponds to the respective previous six months interval. For example, the volatility corresponding to 1 July 2008 is calculated from a range of daily observation between 1 January 2008 and 30 June 2008 and thereafter annualized.

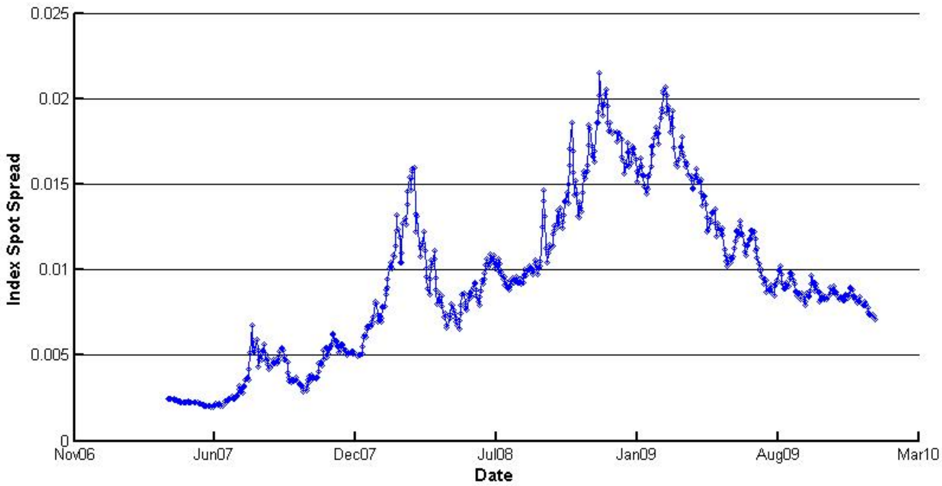


FIGURE 5.12: Historical spread of European iTraxx 5-year index

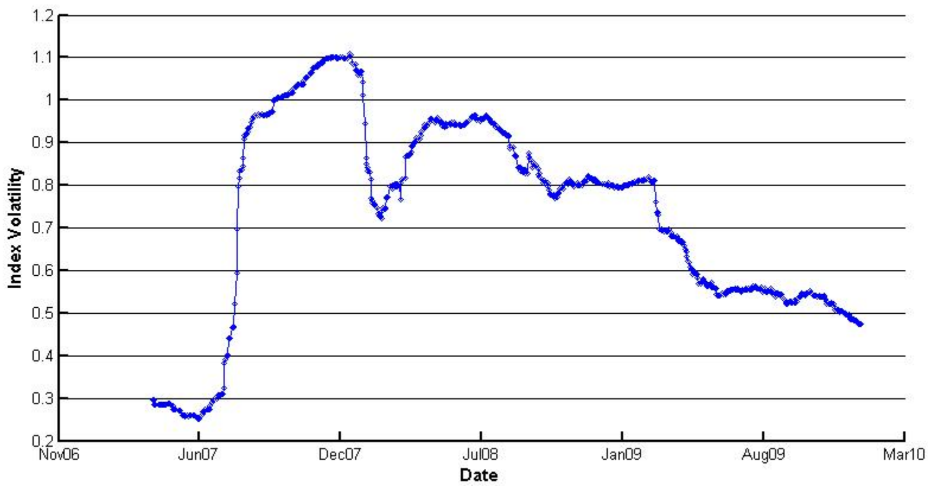


FIGURE 5.13: Historical volatility of European iTraxx 5-year index

Figure 5.14 shows the results of applying the Morini & Brigo (2011) model to the data presented in Figure 5.12 and Figure 5.13. Two remarks can be made from this. Firstly, during the economically calm periods before and after the credit crunch, the CDIS spread is relatively small in comparison to the spread of the CDS index. The reason for this fact has to with the low probability of the swaption ending up in-the-money, due to the high strike spread ($\kappa = 0.0300$) in relation to the spot spreads of the CDS index. Further, although the volatility of the CDS index drastically increases during the late summer of 2007, the spot spread of the CDS index remains on a low level causing the absolute fluctuations to be small. This, in turn, results in a low CDIS spreads due to the unlikeliness of the swaption ending up in-the-money.

The second remark is that the combination of high CDS index spreads and relatively high volatilities in the end of 2008/beginning of 2009 renders high CDIS spreads. The reason for this is that, as opposed to the previous paragraph and even though the volatility has diminished slightly, the absolute fluctuations now are higher due to the greater levels of CDS index spot spreads. These fluctuations, in addition to the reduced absolute distance between the CDS index spot spread and the CDIS strike spread κ, cause the probability of ending up in-the-money now to be relatively large.

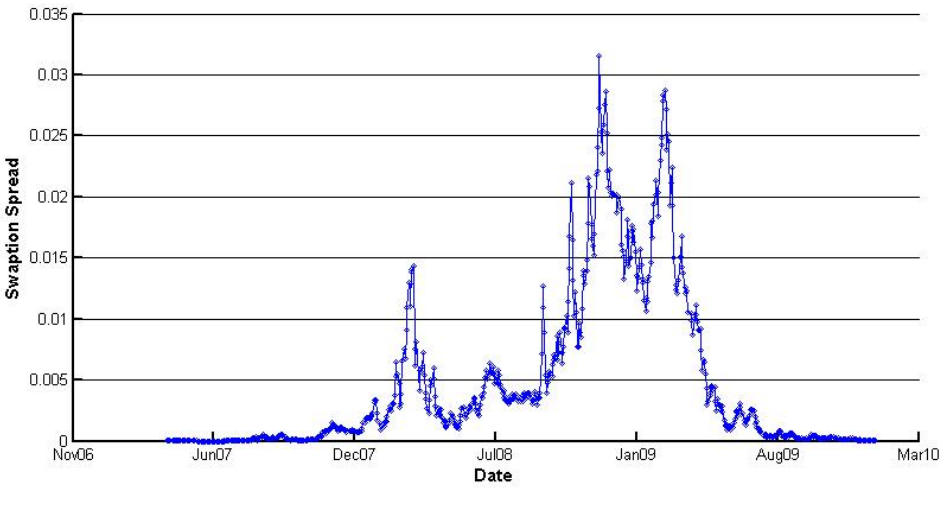


FIGURE 5.14: CDIS spreads generated from Morini & Brigo (2011) using historical data of European iTraxx 5-year index

After the peak of the credit crunch, in the end of the time interval in Figure 5.12 and 5.13, low levels of both spot spreads and volatility of the CDS index can be observed. These facts cause, similar to the period preceding the crisis of 2008, the CDIS spreads to be low due to the low probability of the CDS index ending up in-the-money at the expiry of the swaption.

In addition to the relation between the CDS index spot spread and the swaption strike spread κ discussed in previous paragraphs, the CDIS contract comes with the feature of front-end protection. The value of this feature increases with high levels of CDS index spot spreads due to the default intensity λ being calculated using the credit triangle in Equation (3.9). In other words, the CDIS spread is determined by the relation between the CDS index spot spread and the swaption strike spread as well as by the value of the front-end protection. The impact of the latter, which also includes the Armageddon event, will be further discussed in Subsection 5.3.3.

The previous graph (Figure 5.14) is based on the bold assumption of constant pairwise correlation. In reality, however, one could expect the pairwise correlation among reference names ρ to be a changing process over time, taking higher values during stressed economic climates. For the purpose of illustration, the CDS index spot spread will in this subsection be assumed to constitute a measure of the economic climate, where low spot spreads correspond to calm economic periods and high spot spreads correspond to economic turmoil. In Figure 5.15 this is taken into account by defining the pairwise correlation ρ as a linear function of the historical spot spread during the defined interval, taking the value of zero at the minimum spot spread level and the value of one at the maximum spot spread level.

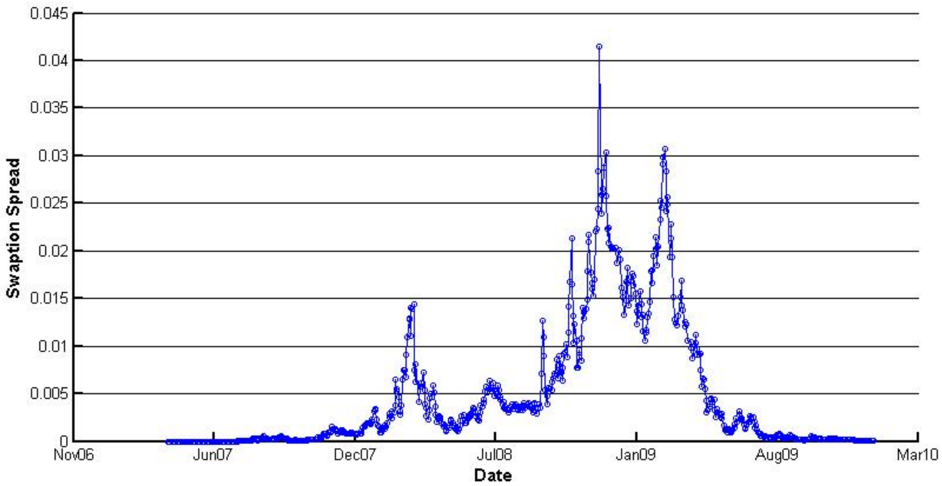


FIGURE 5.15: CDIS spreads generated from Morini & Brigo (2011) using historical data of European iTraxx 5-year index with pairwise correlation as linear function of the spot spread

The effect of letting the pairwise correlation be a function of the CDS index spot spread is almost negligible for low levels of ρ, while having a major impact on the CDIS value as the CDS index spot spread takes on high values. In comparison to Figure 5.14, the maximum CDIS spread in Figure 5.15 is now approximately 100 basis points higher. This demonstrates the importance of including the macro-economic climate as a dynamic variable when pricing CDIS.

5.3.3 The effect of front-end protection and Armageddon events

This subsection aims to isolate the effects of front-end protection and the inclusion of an Armageddon scenario when valuing credit default index swaptions. The graph below (Figure 5.16) illustrate CDIS spreads rendered from a modified version of Morini & Brigo (2011), excluding the front-end protection (therefore also the Armageddon scenario) for a set of index spot spreads and pairwise correlation, respectively.

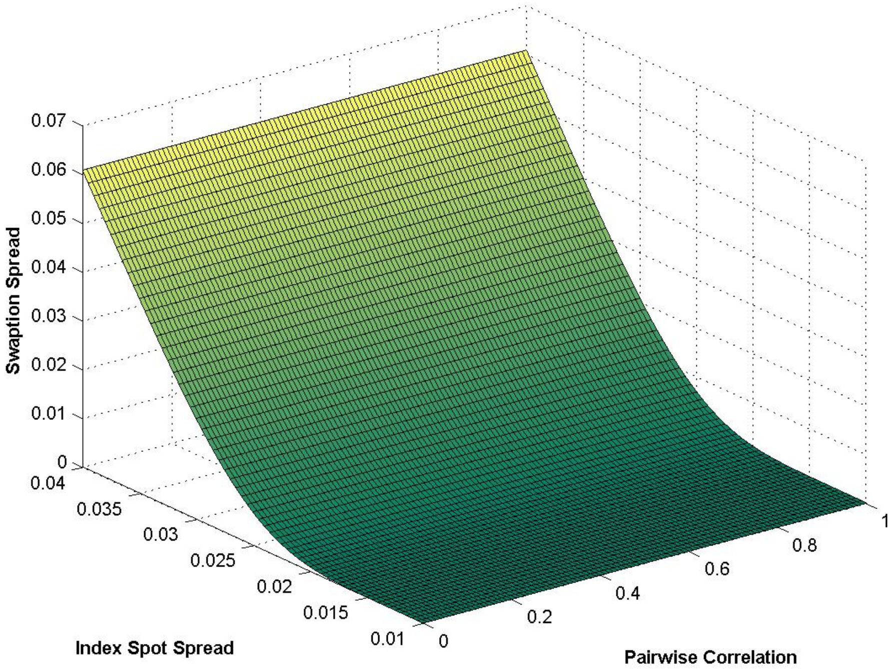


FIGURE 5.16: Morini & Brigo (2011) CDIS spread, without front-end protection and Armageddon scenario, as a function of index spot spread and pairwise correlation

Recall Figure 5.7, in which the front-end protection and Armageddon scenario is taken into account when valuing the CDIS spread. The curvature of the graph for high values of pairwise correlation is the effect of the Armageddon scenario in the model. Note that the Armageddon scenario is the only case for which the pairwise correlation parameter enters the model. However, if the pairwise correlation would be taken into account in all default scenarios, another shape of the surface would be observed.

As shown in Figure 5.16, excluding the front-end protection and the Armageddon scenario decreases the value of the CDIS. As the value of the Armageddon scenario increases with high values of pairwise correlation, the effect of excluding the Armageddon scenario is only observable as the pairwise correlation approaches one. For clarification, note that the surface is uniform over all levels of pairwise correlation as it only enters the model in case of an Armageddon scenario.

Figure 5.17 illustrates the probability of an Armageddon scenario to happen over an interval before, during and after the credit crunch of 2008 for different levels of pairwise correlation. The probability of an Armageddon event is, aligned with previous findings, almost negligible for low levels of pairwise correlation. As pairwise correlations approaches unity, the probability of an Armageddon event increases exponentially.

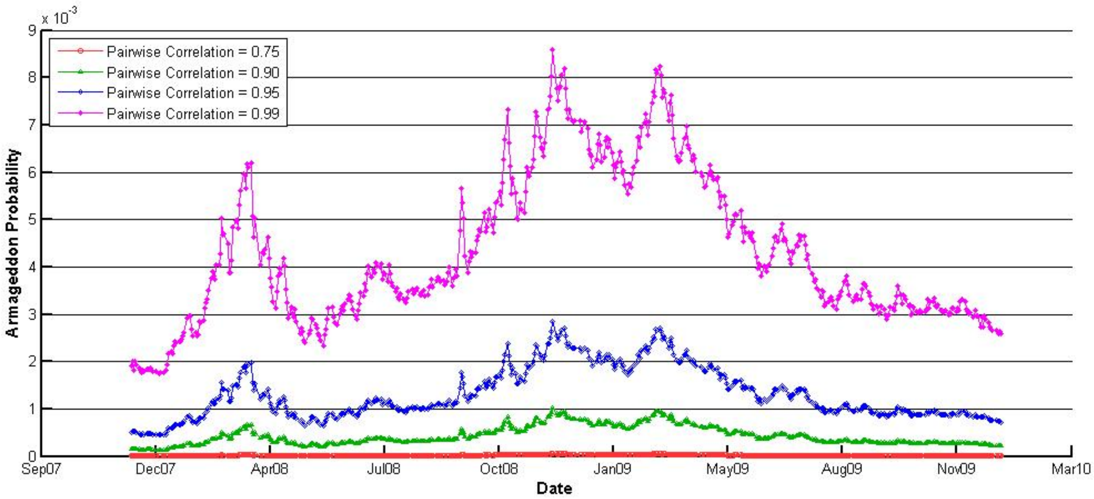


Figure 5.17: The Armageddon probability for different levels of pairwise correlation using historical data of European iTraxx 5-year index

6. Conclusions

The models presented by Jackson (2005), Rutkowski & Armstrong (2009) and Morini & Brigo (2011) are all reconstructable in a programmable environment. The two former models, however, lack transparency and therefore demand more bold assumptions compared to Morini & Brigo (2011). Further, all the models make sense under economic interpretation and are well-behaving with respect to changes in the input parameters.

As the swaption strike spread converges to infinity ($\kappa \to \infty$), the value of the CDIS approaches the expected value of an Armageddon scenario in the models presented by Jackson (2005) and Morini & Brigo (2011), while according to Rutkowski & Armstrong (2009) the same value goes towards zero. In the case of an Armageddon event there is no reason for not exercising the swaption as it is independent of the relationship between the index spot spread and the swaption strike spread. Remarkably, this is not taken into account by Rutkowski & Armstrong (2009).

The pairwise correlation among reference names ρ is in reality a changing process over time, taking higher values during stressed economic climates. The expected value of the front-end protection, including the Armageddon scenario, increases with this correlation. Although the effect the Armageddon scenario is only observable as the pairwise correlation approaches one, the dynamic features of ρ causes in reality the inclusion of an Armageddon scenario be crucial when valuing the CDIS to, even if its value at the time of observation is negligible.

Although, the lack of market information and liquid prices has prevented this study from identifying any superior model in pricing credit default index swaptions, Morini & Brigo (2011) have been evaluated further as their model is superior in demanding the least bold assumptions and having the highest level of transparency. Moreover, Morini & Brigo (2011), unlike Jackson (2005) and Rutkowski & Armstrong (2009), also concretize their model by presenting self-generated CDIS values.

In a non-stressed economic environment the ordinary Black (1976) model generates a reasonable approximation of the CDIS price. This implies that, during economically calm climates, the derivative-specific features are negligible due to low probabilities of default within the swaption lifetime. In economic turmoil, on the other hand, the probabilities of default during the swaption lifetime increase drastically and causes the derivative-specific features to constitute a larger proportion of the CDIS price. This in turn implies that the way of modelling the derivative-specific features, and therefore the choice of model, is crucial during economic turmoil and of less importance during non-stressed economic climates.

A. Appendix: Measures and filtrations

A.1 Survival measure presented by Schönbucher (2000)

Schönbucher (2000) introduces a survival measure $\bar{P}_k$ such that it is equal to the P_k forward measure conditional on survival up to time T_k. Schönbucher (2000) shows this with an example considering an event $A \in \mathcal{F}_{T_k}$ where he calculates its expected value given $\tau > T_k$, that is

$$\mathbb{E}^{P_k}[\mathbb{1}_{\{A\}} | \tau > T_k] = \mathbb{E}^{\bar{P}_k}[\mathbb{1}_{\{A\}}] \tag{A.1}$$

. As seen above, the survival probability $\bar{P}_k$ of event A is equivalent to the probability P_k of event A conditional on survival up to T_k.

A.2 Subfiltrations presented by Morini & Brigo (2011)

Morini & Brigo (2011) subfiltrate the information contained in $\mathcal{F}_t$ into one additional filtration $\mathcal{H}_t$, which excludes the information of an Armageddon event. Information containing the Armageddon event is filtrated into $\mathcal{J}_t$. The relationship is defined as follows

$$\mathcal{F}_t = \hat{\mathcal{J}}_t \vee \hat{\mathcal{H}}_t. \tag{A.2}$$

This implies that probability of an Armageddon event occurring after time t is strictly positive when modelling through filtration space $\hat{\mathcal{H}}_t$, i.e.

$$\mathbb{Q}[\hat{\tau} > t \mid \hat{\mathcal{H}}_t] > 0. \tag{A.3}$$

(Morini & Brigo 2011)

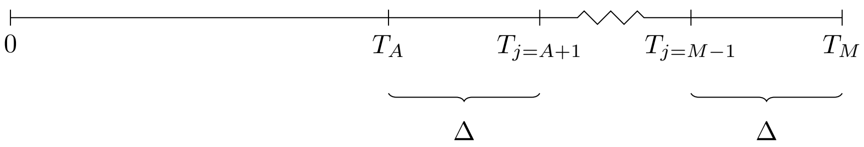
B. Appendix: Definitions

$\boldsymbol{T_A}$ is the expiration time of swaption contracts.

$\boldsymbol{T_M}$ is the expiration time of swap contracts.

$\boldsymbol{T_j}$ is the time corresponding to the j^{th} cash flow in swap contracts where $j = A+1, A+2, ..., M$.

$\boldsymbol{\Delta}$ is the interval between the cash flows in the swap contract.



$\boldsymbol{F_t}$ (*Front-end protection*) is the present value at the arbitrary time t of the payment a credit default index swaption buyer receives in case of defaults within the lifetime of the swaption. This feature does not exist in ordinary credit default swaptions due to the knock-out nature of single-name credit derivatives. As all reference names in a CDIS are equally weighted, the front-end protection corresponds to the defaulted proportion of the underlying index up to the expiry of the option, multiplied with the outstanding nominal amount.

$\boldsymbol{\phi}$ (*Recovery rate*) is the proportion of the defaulted outstanding notional amount refunded without protection. The protection holder against credit events (buyer of credit default swap) is therefore entitled to receive $1 - \phi$ in case of default. For the purpose of this study, the recovery rate is assumed to be constant both over reference names (homogeneous portfolio) and over the entire lifetime of the contracts (during the swaption as well as the swap lifetime).

$\boldsymbol{\lambda}$ (*Default intensity*) also known as arrival intensity, cannot be observed in real life but is the modelled pace at which an obligor approaches its default time τ. For the purpose of this study λ is assumed to be constant. In addition, as this study assumes constant recovery rates, λ can be implied from observed market spot spreads of CDS's or CDS indices.

$\boldsymbol{\tau}$ (*Time of default*) is defined as

$$\tau = \inf\left\{t \geq 0 : \int_0^t \lambda(X_s)\ ds \geq E_1\right\} \tag{B.1}$$

where t is a point in time, $\lambda(X_s)$ is the value of λ driven by the stochastic process X_s and E_1 is a random threshold representing the default level.

The default intensity can be interpreted as the speed by which the integral approaches the random threshold E_1. Apparently, a higher λ will increase the probability of default for any arbitrary time interval.

Armageddon refers to the scenario of total default of the index (all reference names) up to expiration of the swaption. The Armageddon scenario constitutes in reality a part of the front-end protection but is often separated for the purpose of modelling.

$\boldsymbol{\Phi}$ (*Default leg*) is expected present value of a contingent claim paying $1-\phi$ of the notional amount in case of default of the contracted company or index.

$\boldsymbol{DV01}$ (*Premiuim leg*) is the present value of receiving an annuity of one risky basis point during the life of the swap.

$\boldsymbol{\kappa}$ (*Swaption strike spread*) is the strike spread in swaption contracts.

$\boldsymbol{R_{\mathcal{M}}}$ (*Index spot spread*) is the observed market CDS index spread.

$\boldsymbol{r}$ is the constant (deterministic) risk-free interest rate.

$\boldsymbol{m}$ is the number of reference names contained in a certain index.

$\boldsymbol{N_t}$ is a discrete stochastic process counting the number of defaults up to the arbitrary point in time t. That is, $N_t = \sum_{i=1}^{m} \mathbb{1}_{\{\tau_i \leq t\}}$.

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