Encountering algebraic letters, expressions and equations:
A study of small group discussions in a Grade 6 classroom

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A study of small group discussions in a Grade 6 classroom

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Licentiate thesis in Mathematics Education at the Department of Pedagogical, Curricular, and Professional Studies, Faculty of Education, University of Gothenburg.

The licentiate thesis in full text can be downloaded from GUPEA – Gothenburg University Publikations – Elektronic Archive: http://hdl.handle.net/2077/40140

This licentiate thesis has been prepared within the framework of the graduate school in educational science at the Centre for Educational and Teacher Research, University of Gothenburg.

In 2004 the University of Gothenburg established the Centre for Educational Science and Teacher Research (CUL). CUL aims to promote and support research and third-cycle studies linked to the teaching profession and the teacher training programme. The graduate school is an interfaculty initiative carried out jointly by the Faculties involved in the teacher training programme at the University of Gothenburg and in cooperation with municipalities, school governing bodies and university colleges.

Abstract

Title: Encountering algebraic letters, expressions and equations: A study of small group discussions in a Grade 6 classroom
Author: Elisabeth Rystedt
Language: English with a Swedish summary
GUPEA: http://hdl.handle.net/2077/40140
Keywords: introductory algebra, algebraic letters, algebraic expressions, equations, small group discussions, manipulatives

Introductory algebra has a pivotal role for pupils' continued learning in algebra. The aim of this licentiate thesis is to contribute to knowledge about how pupils appropriate introductory algebra and the kind of challenges they encounter. In this thesis, the term introductory algebra is used to refer to the introduction of formal algebra at compulsory school level – algebraic letters, algebraic expressions and equations. The work is based on two research articles: *What's there in an n? Investigating contextual resources in small group discussions concerning an algebraic expression* and *Moving in and out of contexts in collaborative reasoning about equations*.

The studies are positioned in a socio-cultural tradition, which implies a focus on pupils' collaborative meaning making. A dialogical approach was applied when analyzing the pupils' communication. Two case studies have been conducted in classrooms, both consisting of video recorded small group discussions between 12-year-old pupils working with algebraic tasks.

The first study shows how pupils tried to interpret the algebraic letter *n* and provide an answer formulated as an algebraic expression. The results show that the pupils used a rich variety of contextual resources, both mathematical and non-mathematical, when trying to understand the role of the *n*. In addition, the meaning of the linguistic convention "expressed in *n*" was a barrier for the pupils. In the second study the pupils used their experiences of manipulatives (boxes and beans, which had been used during

prior lessons), as a resource when solving a task formulated as an equation expressed in a word problem. The study shows that the manipulatives supported the pupils in working out the equation, but did not help them to solve the task.

Three general conclusions can be drawn from the empirical studies. Firstly, the interpretations of an algebraic letter can be dynamic and the nature of the meaning making may shift quickly depending on the contextual resources invoked, indicating that an interpretation is not a static, acquired piece of knowledge, but more like a network of associations. Secondly, mathematical conventions may work as obstacles to the pupils' understanding. This indicates that learning mathematics is about learning a specific communicative genre in addition to learning about mathematical objects and relationships. Thirdly, the studies show that a critical part of appropriating introductory algebra is being aware of 'what is the example' and 'what is general' in different activities. A conclusion is that although pupils are able to mobilize resources that are helpful for managing specific cases, additional problems may arise when they try to comprehend fundamental algebraic principles.

Acknowledgements

It has been an exciting and educational journey to be a PhD student and to write this licentiate thesis. I have many people and groups of people to thank for support during this time. Above all I want to thank my three supervisors, Roger Säljö, Cecilia Kilhamn and Ola Helenius. They have helped me to see and be aware of things I would not have been able to discover by myself. They have continuously encouraged me and guided me during my work. Thanks to them I have had the opportunity to deepen and broaden my knowledge about research in general and about research on communication and mathematics education in particular. Cecilia and Ola are also co-authors of both my articles. This process of collaboration has really been, to me, a successful illustration of learning in and through interaction.

This work is about how pupils appropriate introductory algebra in small group discussions. My interest in pupils' interaction in small groups originates from my work as a teacher in the compulsory school over 20 years. I have often organized pupils into small groups and given them problems to discuss during math lessons. I have walked around in the classroom, like the teacher in my studies, listening to fragments of the pupils' discussions and interacting when I found reasons to intervene. However, I have never had the opportunity to follow the discussion from beginning to end in any of the groups. As a teacher you seldom have such an opportunity, because several groups work in parallel during group work and all require attention. My special interest in pupils' encounters with formal algebra primarily has two motives. One is the international studies and the national tests, which year after year reveal that many pupils do not succeed in algebra. Why are even tasks involving basic formal algebra problematic for many pupils and how might it be possible to support pupils so they are able to take further steps in their understanding of elementary formal algebra? The other motive relates to all the teachers I have met in connection with my work at the National Center for Mathematics Education (NCM). The NCM arranges different kinds of professional development and for several years my colleague Lena Trygg and I have performed two-day workshops with teachers in Sweden but also in Mexico and India. One of the activities carried out by the teachers during the workshops concerned how to introduce algebraic formulas to pupils. The aim of the activity was to give the pupils opportunities to make sense of a formula and to grasp why it is an effective and labor-saving tool to apply in mathematics. In all of our workshops we have encountered a really pronounced interest from the teachers in developing their teaching on the introduction of formal algebra.

When I became a PhD student, I got the opportunity to be a member of the VIDEOMAT project, which involves researchers in Sweden, Norway, Finland and the US (UCLA). I was fortunate, because my research interest coincided extremely well with the general aim of the VIDEOMAT project, which concerned analyzing teaching and learning of algebra. I have learned a lot from comments on my own text, from discussions during all of the videoconferences, from meetings in Gothenburg and from reading different kinds of texts produced in and circulated between the different countries. Thanks, all of you in the VIDEOMAT project!

I would also like to give special thanks to: the present manager Peter Nyström and all my workmates at NCM, always helping me whatever my questions were; the former manager at NCM, Bengt Johansson, for all his enthusiasm, knowledge and support over the years; Lena Trygg for genuine friendship and for our mutual exchange of knowledge and experiences when planning, performing and evaluating our work with teachers in in-service training; Görel Sterner for sharing difficulties as well as enjoyment and laughter in our time as PhD students; the director Jesper Boesen and my PhD student colleagues at the Centre for Educational Science and Teacher Research (CUL); LUM and FLUM for organizing valuable seminars and interesting discussions.

My thanks also go to my lovely family, to Johan & Dina and Anna & Paz for discussions about various designs of tables as well as the labor of writing academic texts, to my little grandchild Maxwell for happy smiles, to my husband Hans, for sharing so

many interests, from making trips with the four-wheeler in the forests in Stöpafors to dinners in the kitchen ending up with discussions related to current research.

Gothenburg, June 2015
Elisabeth Rystedt

Contents

PART ONE: DISSERTATION FRAME 13

CHAPTER 1 INTRODUCTION 15
1.1 Background 15
1.2 Aim of the thesis 20
1.3 Outline of the thesis 21

CHAPTER 2 INTRODUCTORY ALGEBRA AND LEARNING IN SMALL GROUP DISCUSSIONS 23
2.1 Introductory algebra 23
2.1.1 The history of algebra 23
2.1.2 Arithmetic thinking and algebraic thinking 26
2.1.3 What is algebra, then? 28
2.1.4 Research on introductory algebra 28
2.2 Small group discussions in the learning of mathematics 36

CHAPTER 3 LEARNING MATHEMATICS – A SOCIO-CULTURAL PERSPECTIVE 45
3.1 Appropriation, ZPD and intellectual tools 45
3.2 Dialogical approach in the analysis 48

CHAPTER 4 EMPIRICAL DATA AND METHODS 53
4.1 Design and data collection 53
4.2 Data analysis 56
4.3 Ethical considerations 57

CHAPTER 5 SUMMARY OF STUDIES 59
5.1 Study 1 59
5.2 Study 2 63

CHAPTER 6 DISCUSSION 69
6.1 Appropriating introductory algebra in small group discussions 69
6.2 Small group discussions as a zone of proximal development 73
6.3 Methodological reflections 76
6.4 Conclusions 79
6.5 Suggestions for further research 80

CHAPTER 7 SUMMARY IN SWEDISH 81

REFERENCES 91

PART TWO – RESEARCH ARTICLES 103

APPENDIX 105
Appendix 1. The original tasks in TIMSS 2007
Appendix 2. Forms for informed consent – parents
Appendix 3. Forms for informed consent – teachers

List of tables

Table 1. Stages in development of algebra 25
Table 2. The length of the video taped lessons 54
Table 3. The pupils' and the teacher's movements between contexts 65

List of figures

Figure 1. The pupils' written answer 66

Part One: Dissertation frame

Chapter 1 Introduction

Leo: *y* can be, be whatever […] that is the strange thing

Leo is 12 years old and is trying to solve an equation together with two classmates in a small group discussion during a mathematics lesson. This quote comes from one of my studies and gives an indication of what this licentiate thesis is about: how pupils appropriate introductory algebra in small group discussions. The general interest is to understand, as Dysthe (2003) suggests, a little bit more of what happens, how and why it happens or maybe does not happen and what supports or impedes learning.

1.1 Background

Introductory algebra has a pivotal position for pupils' continued learning in algebra. Many studies reveal the difficulties that students have at different school levels with respect to main concepts in introductory algebra: variables, algebraic expressions, equation solving and problem solving (Bednarz, Kieran & Lee, 1996).

International studies show that many pupils do not succeed in solving algebraic tasks (Mullis, Martin, Foy & Arora, 2012; OECD, 2013). In TIMSS 2011, four areas in mathematics were assessed: *Number, Algebra, Geometry* and *Data and Chance*. The international average achievement by students in Grade 8 was lowest in *Algebra* (Mullis et al., 2012). In TIMSS 2011, the Swedish pupils in Grade 8 performed 33 points lower on algebra tasks than the average in the EU/OECD (Swedish National Agency for Education, 2012). Comparing the Swedish pupils in TIMSS 2007 and in TIMSS 2011, there were no positive developments: the results had not changed. In the Nordic countries Finland had exactly the same score as the average in the EU/OECD, whereas Sweden and Norway were lower than average.

- Finland 492 p
- EU/OECD 492 p
- Sweden 459 p
- Norway 432 p

The results for Nordic countries[1] in Algebra and the average in the EU/OECD, Grade 8, TIMSS 2011 (Swedish National Agency for Education, 2012, p. 49).

The general status of algebra in school mathematic is that algebra is important, but difficult and it has been pinpointed as a gatekeeper for higher education (Cai & Knuth, 2011). As a result of its role in excluding pupils and an increasing concern about pupils' inadequate comprehension of algebra, policy makers and mathematics education researchers have drawn attention to algebra in the curriculum and teaching. One ambition is to develop algebraic thinking in earlier grades, with the aim of helping pupils to be better prepared for more formal studies of algebra in later grades. This is reflected in many policy documents all over the world (Cai & Knuth, 2011). In Common Core State Standards (NGA Center & CCSSO, 2010), *Algebraic thinking* is a part of the mathematical content even in kindergarten. In Sweden, *Algebra* was not listed as a main topic in the curriculum for the later grades in compulsory school until 1955 (Swedish National Agency for Education, 1997), but in the latest Swedish curriculum *Algebra* is taught as early as in grades 1-3 (Swedish National Agency for Education, 2011):

[1] Iceland did not participate in TIMSS 2007 and 2011. Denmark participated only with regard to pupils in Grade 4 in TIMSS 2011.

Algebra

In years 1–3[2]:

- Mathematical similarities and the importance of the equals sign.
- How simple patterns in number sequences and simple geometrical forms can be constructed, described and expressed.

In years 4–6:

- Unknown numbers and their properties and also situations where there is a need to represent an unknown number by a symbol.
- Simple algebraic expressions and equations in situations that are relevant for pupils.
- Methods of solving simple equations.
- How patterns in number sequences and geometrical patterns can be constructed, described and expressed.

In years 7–9:

- Meaning of the concept of variable and its use in algebraic expressions, formulae and equations.
- Algebraic expressions, formulae and equations in situations relevant to pupils.
- Methods for solving equations.

(Swedish National Agency for Education, 2011, pp. 60-63)

In mathematics education literature, terms like *pre-algebra, early algebra* and *introductory algebra* are often used, but the borders between them are not well defined. The difference between pre-algebra and algebra is not distinct and the concepts are interpreted differently depending on the country (Kilhamn & Röj-Lindberg, 2013). Similar mathematical content can have the heading "Pre-algebra" in American textbooks, while in Swedish textbooks the label can be "Algebra". Examples of *pre-algebra* in textbooks are when pupils are working with equations with a missing number without using letters to symbolize the missing number, working arithmetically with the

[2] In Sweden years 1–3 correspond to an age of 7–9, years 4–6 to an age of 10-12 and years 7–9 to an age of 13-15.

structural aspect of the equals sign, working with operations and inverse operations (doing/undoing) or numerical and geometrical patterns without expressing them algebraically (Kilhamn & Röj-Lindberg, 2013).

Discussions about *early algebra* emphasize that it is "not the same as algebra early" (Carraher, Schliemann & Schwartz, 2008, p. 235). To put it in short, there are three characteristics of early algebra: a) it is built on rich problem situations, b) formal notation is introduced gradually and c) it is tightly interwoven with existing topics in early mathematics curricula (Carraher et al., 2008). In the 1980s and 1990s, most of the research about algebra and young children had a pre-algebra approach, with the aim of bridging the gap between arithmetic and algebra (Kilhamn & Röj-Lindberg, 2013). After 2000, the early algebra approach has been dominant and regards algebraic thinking as an inherent feature of arithmetic (ibid.). In this thesis the concept *introductory algebra* will be used to signify the introduction to elementary formal algebra in the compulsory school. In the studies presented in this thesis, the topics chosen are algebraic letters, algebraic expressions and equations.

In my studies, I focus on algebra in small group discussions between pupils. There are three connected motives, without any hierarchical order, for selecting small group discussions. One motive is that communication has been a major focus in mathematics teaching reform for at least two decades. The importance of pupils interacting in small group discussions when learning mathematics is widely acknowledged (Nilsson & Ryve, 2010). Communication is described as a mathematical competence in frameworks for mathematics teaching and learning (NCTM, 2000; Niss & Højgaard-Jensen, 2002). It is argued that giving pupils opportunities to engage in speaking, writing, reading and listening has dual benefits: they communicate to learn mathematics and they learn to communicate mathematics (NCTM, 2000). However, the competence to communicate in mathematics is not the same as being able to communicate in general. The goal is to be able to communicate in, with, and about mathematics at different levels of theoretical or technical precision (Niss & Højgaard, 2011):

> This competency consists of, on the one hand, being able to study and interpret others' written, oral or visual mathematical expressions or "texts", and, on the other hand, being able to express oneself in different ways and with different levels of theoretical or technical precision about mathematical matters, either written, oral or visual, to different types of audiences.
>
> (Niss & Højgaard, 2011, p. 67)

In Common Core State Standards (NGA Center & CCSSO, 2010), it is argued that the students should be able to "justify their conclusions, communicate them to others, and respond to the argument of others" (pp. 6-7). In the Swedish curriculum it is expressed that through teaching, pupils should be given the opportunity to communicate about mathematics in daily life and mathematical contexts (Swedish National Agency for Education, 2011). In relation to reasoning, oral communication needs both a "speaker" and a "listener" and is about establishing shared meaning of the things talked about (Linell, 1998).

A second motive is that from a methodological perspective, communicative situations make it possible to analyze pupils' reasoning, because it is articulated. Sfard (2008) connects thinking and communication, and defines thinking as "the individualized form of (interpersonal) communication" (p. 91). She emphasizes the importance of studying communication because it helps us to learn more about mathematical learning (Sfard, 2001). A third motive is the socio-cultural tradition in which this thesis is positioned. When studying learning from this point of view, the object of analysis is not only the individual pupil, but the system of interacting individuals in a specific situation (Säljö, 2000). In this tradition communication is a tool for learning and therefore interesting to analyze.

To conclude, the significant role of introductory algebra, the pupils' reported shortcomings in international studies, the role of algebra as a gatekeeper and the trend throughout the world of introducing algebra at an earlier stage of pupils' education, indicate that there is a need for further attention to be given to introductory algebra. This thesis concerns introductory algebra and how this content is enacted in small group discussions. Content and form are intertwined in the activities that pupils engage in when solving mathematical tasks. However, in my research they have been separated; both cannot be at the forefront of an analytical focus. Pupils' appropriation of introductory algebra is the figure and the small group discussions are the background.

1.2 Aim of the thesis

The overall aim of the thesis is to contribute knowledge on how pupils appropriate introductory algebra and the kind of challenges they encounter. Pupils' small group discussions are analyzed. In the first study, I investigate how a group of 12-year-old pupils try to make sense of the algebraic letter *n* when dealing with an algebraic expression. The analytical focus is how pupils use resources such as the surrounding physical situation, prior utterances in the discussion, and background knowledge. Together, these resources are called contextual resources (Linell, 1998). In the second study, I investigate how pupils use earlier experiences of manipulatives (boxes and beans) as a resource, when solving an equation expressed in a word problem. The specific research questions are:

- Study 1: What interpretations of the algebraic letter *n* emerge in the group, and how do these interpretations relate to the contextual resources made use of in the discussion?
- Study 2: How do the pupils contextualize the task given and how do they move between different contexts in their attempts to arrive at an answer? What support for and what obstacles to learning can be identified when pupils use manipulatives as a resource in the equation-solving process?

The ambition is to contribute to theoretical knowledge, and practical knowledge useful for teacher education and different kinds of in-service training. The vision is that an increasing numbers of pupils will get opportunities to succeed in learning introductory algebra.

1.3 Outline of the thesis

This licentiate thesis is based on two research articles: *What's there in an n? Investigating contextual resources in small group discussions concerning an algebraic expression* and *Moving in and out of contexts in collaborative reasoning about equations*. The thesis consists of two parts, where the first one will integrate and synthesize the two articles. In Chapter 1, the background and the aim of the thesis are described. Chapter 2 consists of a presentation of introductory algebra (the mathematical content) and small group discussions (the form of interaction) including research on these domains. Chapter 3 concerns learning mathematics from a socio-cultural perspective (theoretical framework) also involving a description of the dialogical approach (analytical framework). Chapter 4 explains how the empirical data on pupils' small group discussions has been collected and describes the methods that have been used to analyze the data. Chapter 5 presents summaries of the two studies. Chapter 6 discusses the results and general conclusions are drawn from the studies. Chapter 7 provides a summary of the thesis in Swedish. The second part of the thesis consists of the two research articles. Finally, the appendix includes the original tasks in TIMSS and forms for informed consent in the study.

Chapter 2 Introductory algebra and learning in small group discussions

This chapter has two main parts. The first addresses introductory algebra and the second concerns small group discussions.

2.1 Introductory algebra

First an overview is given of the history of algebra, followed by a description of similarities and differences between arithmetic and algebra, ending with an explanation of what is included in the concept of algebra in this work. Then there is a presentation of research on introductory algebra.

2.1.1 The history of algebra

The history of mathematic offers interesting information about the development of mathematical knowledge within a culture and across different cultures (Radford, 1997). It allows us to follow how insights have been gained and how they have changed over time. The history of algebra "can shed some light on the didactic problem of how to introduce algebra in school" (Radford, 2001, p. 34). However, it should not be over-interpreted: history should not be normative for teaching.

The word algebra comes from the Arabic *al-jabr*. It was used together with *al-muqabala* in a famous text concerning equation solving, written by al-Khwarizmi, who lived in Bagdad in the beginning of the 9th century AD (Kiselman & Mouwitz, 2008). The word *al-jabr* can be translated as "restoring" (Katz, 1993). It refers to the operation of moving one term with a minus sign from one side of an equation to the other side and changing the sign to a plus sign (Kiselman & Mouwitz, 2008). The word *al-muqabala* can be translated as "comparing" and denotes the operation involved when taking away equal amounts from both sides of an equation (Katz, 1993). Transforming the equation $3x + 2 = 4 – 2x$ into $5x + 2 = 4$, is an example of *al-jabr*, while converting the same equation to $5x = 2$ it is an example of *al-muqabala* (ibid.). It is notable, as Katz explains, that our word algebra, is a corrupted form of *al-jabr* and came into use when the work of al-Khwarizmi and other related treatises were translated into Latin. The word *al-jabr* was never translated and became the established general term for the entire science of algebra.

Thus, algebra has been developed as a human activity over a long time. The history of algebra is often divided into three periods: *rhetorical, syncopated* and *symbolic* (Radford, 1997). From the beginning all mathematical writing was *rhetorical*, expressed in common language and with words written out in full. In the *syncopated* period, algebraic thoughts were presented in a mixture of words and symbols. It was not until the 16th century that *symbolic* algebra emerged as a result of the work of the French mathematician Francois Viète (Sfard & Linchevski, 1994). Various stages in the development of algebra are summarized by Sfard and Linchevski (1994), as presented in the table below. Sfard (1991) divides both generalized and abstract algebra into two stages: operational and structural thinking. She considers that the same representation, for instance (n – 3), may sometimes be interpreted as a process (operational stage) and sometimes as an object (structural stage). Sfard (1991) finds that the computational operation in the process is the first step in the acquisition of new mathematical notions for most people.

Table 1. Stages in development of algebra (Sfard & Linchevski, 1994, p. 99)

Type	Stages	New focus on	Representation	Historical highlights
1. Generalized arithmetic	1.1. Operational	1.1.1. Numeric computations	Rhetoric	Rhind papyrus c. 1650 B.C.
			Syncopated	Diophantus c. 250 A.D.
	1.2. Structural	1.2.1.Numeric product of computations (algebra of a fixed value)	Symbolic (letter as an unknown)	16th century mainly Viète, (1540-1603)
		1.2.2. Numeric function (functional algebra)	Symbolic (letter as a variable)	Viète, Leibniz (1646-1716) Newton (1642-1727)
2. Abstract algebra	2.1. Operational	Processes on symbols (combinations of operations)	Symbolic (no meaning to a letter)	British formalist school, since 1830
	2.2. Structural	Abstract structures	Symbolic	19th and 20th century: theories of groups, rings, fields, etc., linear algebra

To sum up, algebra has developed from being rhetorical to using an increasingly sophisticated system of notation. Symbolic algebra, from a historical point of view, entered into mathematics comparatively late (Sfard, 1995). It is not surprising that what has taken time to develop from a historical perspective can also be quite difficult for the learner (Sfard, 1995).

Radford (1997) follows Sfard's use of history for epistemological reasons and argues that knowledge about historical conceptual developments may deepen our understanding of pupils' algebraic thinking. It may increase our capacity to enhance pupils' learning in algebra, Radford considers. However, he emphasizes that the historical development of algebra should not be seen as identical to the learning of algebra by individual pupils today. If follow the historical development of algebra in strictly chronological order, the pupils may only meet abstract mathematics late in their learning process. But today there are an increasing number of research studies showing that young pupils, even those with limited knowledge in arithmetic, are able to start learning algebraic concepts and work with algebraic notations (Cai & Knuth, 2011; Hewitt, 2012; Radford, 2012).

Radford (2001) describes how algebraic language has emerged as a tool over time. The pupils, however, often meet algebraic concepts and methods as well-established mathematical objects. Mathematical concepts and methods developed during the course of history need to be "unpacked", because it is not easy for the pupils to identify them from the outset (Furinghetti & Radford, 2008, p. 644).

2.1.2 Arithmetic thinking and algebraic thinking

Algebra is sometimes defined as "arithmetic with letters", a view Devlin (2011, Nov 20) and many others, oppose. Devlin views arithmetic and algebra as two different forms of thinking about numerical issues, stressing that the following distinction is all about school arithmetic and school algebra: When learning *arithmetic* the basic building blocks, numbers, emerge naturally around the pupil, when counting things, measuring things, buying things etc. Numbers may be abstract, but they are closely related to concrete things around the pupil. In *algebra*, the pupil needs to take a step away from the everyday world. Symbols, such as x and y, now denote numbers in general rather than specific numbers. According to Devlin, the human mind is not naturally suited to think at that level of abstraction. It requires a lot of training and effort. He summarizes the difference between arithmetic and algebra:

- First, algebra involves thinking logically rather than numerically.
- In arithmetic you reason (calculate) with numbers; in algebra you reason (logically) about numbers.
- Arithmetic involves quantitative reasoning with numbers; algebra involves qualitative reasoning about numbers.
- In arithmetic, you calculate a number by working with the numbers you are given; in algebra, you introduce a term for an unknown number and reason logically to determine its value.

(Devlin, 2011, Nov 20)

These distinctions, Devlin claims, make it clear that algebra is not the same as arithmetic with one or more letters. Kieran (2004) does not make the same strict dichotomous partition between arithmetic and algebra as Devlin. Their perspectives, however, can broadly be seen as consistent with each other, even if Kieran is more detailed in her comparison of algebra and arithmetic in the early grades:

> Algebra focuses on
> - relations – not merely on the calculation of a numerical answer
> - operations as well as their inverses, and on the related idea of doing/undoing
> - both representing and solving a problem – not merely solving it
> - both numbers and letters, which includes
>   - working with letters that may at times be unknowns, variables or parameters
>   - accepting unclosed literal expressions as responses
>   - comparing expressions for equivalence based on properties rather than on numerical evaluation
>   - the meaning of the equals sign as a statement of relationship, not an instruction to calculate
>
> (Kieran, 2004, pp. 140–141)

When algebra is introduced in school, pupils often try to solve the tasks by arithmetical thinking. Devlin explains that this approach usually works, because teachers initially choose “easy” problems. He also gives examples of how pupils who are strong in arithmetic are able to initially progress in algebra using arithmetical thinking.

The boundary between arithmetic and algebra “is not as distinct as often is believed to be the case” (Cai & Knuth, 2011, p. ix). Deep understanding of arithmetic requires, for instance mathematical generalizations that are algebraic in nature (Carraher & Schliemann, 2007). A central conclusion from this section is that even if it is theoretically possible to separate algebraic thinking from arithmetical thinking, there are many mathematical problems that can be treated with both forms of thinking.

2.1.3 What is algebra, then?

Algebra is not a static body of knowledge, which implies that the concept cannot be defined once and for all (Kaput, 2008). It is a substantial and extensive concept, generated over time, and may be further varied and elaborated on. How it is explained depends on who is explaining and may vary between e.g. teachers, researchers and mathematicians (ibid.). This thesis deals with school algebra and will use Bell's (1996) description of four approaches to school algebra: generalizing, problem solving, modeling and functions. In the first study, the pupils are working with a task that involves generalizing a relation between two people and the number of jackets that each has. In the second study the pupils are dealing with a task of a modeling character.

2.1.4 Research on introductory algebra

In relation to my two studies, the following parts of introductory algebra will be highlighted: letters in algebra, the equals sign, equations, algebraic expressions and conventions.

Letters in algebra

In algebra, letters can take on many roles, such as for instance labels, constants, unknowns, generalized numbers, varying quantities, parameters and abstract symbols (Philipp, 1992). In school mathematics, an algebraic letter (for instance *x* or *n*) is correctly used to represent a number in the following different categories of meaning (Kilhamn, 2014):

- a specific (unknown) number
- a generalized number representing several (or any) values
- a proper variable representing a range of values used to describe a relationship.

A person who is conversant in algebra has no difficulties recognizing and interpreting the different meanings of an algebraic letter, which is not the case when pupils initially encounter algebraic symbols (Häggström, 2008). In a study by Kilhamn (2014), two teachers' use of mathematical terminology and algebraic notations were analyzed, when introducing variables in Grade 6. All of the three categories above appeared:

$5 = 2 + x$	Equation with one unknown: x represented a specific but (still) unknown number, given that the equality is true.
$x + 2$	Algebraic expression: x represented a generalized number. One of the teachers explained the expression as "a number added with 2". It was not clear whether the number had a specific value (a specific number the teacher thought about) or whether it represented a general number.
$y = 2 + x$	Formula describing a relation between two or more variables: x represented a range of possible values.

These findings suggest that the different roles of algebraic letters need to be observed and discussed. The pupils need to meet a wide range of situations with algebraic letters and get the opportunity to become aware of the varying meanings.

The definition of the term variable differs. In some literature a clear distinction is made between unknown specific numbers and generalized numbers, on the one hand, and variables, on the other hand (Kilhamn, 2014; Küchemann, 1981). A more general definition of variable referred to by teachers and teacher educators in Sweden defines variable as a "quantity that may assume any value in a given set"[3] (Kiselman & Mouwitz, 2008, p. 21). Following this definition, a specific unknown, such as for example x in the equation $5 = 2 + x$, is also a variable since it represents any number in a stated set, for example the natural numbers, although only one of the numbers makes the equality true (Philipp, 1992). In the present thesis, I will make a distinction between a specific unknown number, a

[3] In the Swedish original language: "storhet som kan anta värden i en given mängd" (Kiselman & Mouwitz, 2008, p. 21). The translation to English (Kilhamn, 2014).

generalized number and a variable, in line with Kilhamn (2014) and Küchemann (1981).

Pupils' understanding, and misunderstanding, of algebraic letters has been the subject of research in numerous studies (see for example a literature review by Bush & Karp, 2013). In research on how pupils interpret algebraic letters, two classic studies are frequently referred to: Küchemann (1978, 1981) and MacGregor and Stacey (1997). These studies give an overview of how pupils treat tasks involving algebraic letters and the different ways pupils interpret variables. In the study by Küchemann (1978), 3000 pupils aged 13-15 years participated. The data came from a pencil and paper test involving 51 items. Küchemann (1978, 1981) found that different tasks invoked different interpretations of the letter. He organized the pupils' different interpretations in a hierarchical order of six levels starting with the least sophisticated. The percentage of correct answers from 14-year-old pupils are in parenthesis: *letter evaluated* (92%), *letter ignored* (97%), *letter as object* (68%), *letter as specific unknown* (41%), *letter as generalized number* (30%) and *letter as variable* (6%). As we can see, Küchemann's analysis showed that the greatest challenge was to understand a letter as a variable. In the test, this category concerned tasks such as "Which is larger, $2n$ or $n+2$? Explain." Solving such a task, Küchemann explains, requires the pupils to apprehend the dynamic relation between two expressions, depending on the value of the variable. Interpreting the letter as a generalized number was also difficult for most pupils. 30 percent managed this. However, in a later study by Knuth, Alibali, McNeil, Weinberg and Stephens (2011), a more positive result than those of Küchemann was observed: 50 percent of pupils in Grade 6 interpreted the algebraic letter as representing more than one value. This increased to 75 percent in Grade 8.

MacGregor and Stacey (1997) built on Küchemann's research (1978, 1981) when investigating pupils' understanding of algebraic notation. First, they analyzed the interpretations by 42 pupils, aged 11-12, who had not been taught any algebra. As opposed to ready-made algebraic symbols and expressions in Küchemann's tests, the pupils in MacGregor and Stacey's study were encouraged to construct an expression including a letter. One of the two tasks in the pre-test was: "Sue weighs 1 kg less than Chris. Chris weighs *y* kg. What can you write for Sue's weight?" This task is similar to the task the pupils struggled with in Study 1 in this thesis. MacGregor and Stacey also found six interpretations, though these were not exactly the same as in the ones in Küchemann's articles: *Letter ignored* (no letter at all in the answer), *numerical value* (a value related to the situation in the task), *abbreviated word* (w=weight), *alphabetical value* (*y*=25 because *y* was the 25th letter in the alphabet), *different letters for each unknown* (the pupils chose another letter for Sue's weight such as for instance: "o") and *unknown quantity* (*y* – 1: subtract 1 from number or quantity denoted by *y*). Two of these categories were not explicit in Küchemann's study: alphabetical value and different letters for each unknown. All of the six interpretations in the study by MacGregor and Stacey were also identified in the larger sample of pupils aged 13-15 who had been taught algebra. The task about Sue's weight was answered by nearly 1500 pupils but correctly answered by only 36 percent of the pupils in Grade 7, 46 percent in Grade 8, 60 percent in Grade 9 and 64 percent in Grade 10. It was surprising to the authors that the improvement between the grades was not higher. In Grade 10 there were 36 percent of the pupils, who still could not write Sue's weight as the expression (*y* – 1). In this thesis, MacGregor and Stacey's categories will be used in the analysis in Study 1.

The equals sign, equations and algebraic expressions

A well-documented difficulty is when pupils apprehend the equals sign as an instruction to calculate, not as a statement of relationship (e.g. Booth, 1984; Kieran, 1981). This means that the pupils may accept expressions such as "3 + 5 = 8 + 4 = 12", but reject "8 = 3 + 5" and 3 + 5 = 7 + 1" (Schliemann, Carraher & Brizuela, 2005, p. xi). When algebraic equations with variables on both sides of the equals sign are introduced (e.g. $3x = 5x - 14$), this will be problematic for pupils if they understand the equals sign as separating what they are supposed to calculate from the answer to the calculation (ibid.). The relation between pupils' understanding of the equals sign and their equation-solving performance has been examined by Knuth, Stephens, McNeil and Alibali (2006). In their study, 177 pupils in Grade 6–8 participated. The results show a strong positive relation in the sense that pupils who understand the equals sign as a relational symbol were more successful at solving equations than pupils who do not have this understanding. The finding holds when controlling for mathematical ability. It is notable that this finding suggests that even pupils who have not been taught formal algebra (Grade 6–7) have a better understanding of how to solve an equation when they conceived of the equals sign as a statement of relationship.

Several studies on algebra learning have highlighted pupils' mistakes when solving equations (Schliemann, Carraher & Brizuela, 2005). Difficulties concerning the interpretations of algebraic letters and the equals sign as a relational sign have already been mentioned. Additional difficulties are identified in a literature review by Bush and Karp (2013). One of these is a difficulty with establishing a meaning for the equation that they are solving. Another is a difficulty with checking answers by substituting solutions back into the equation. A third is a difficulty with combining, or not combining, similar terms. A fourth is with understanding the relationship between an equation and other representations such as tables and graphs.

When teaching pupils how to solve equations, there are two traditional approaches, Filloy and Rojano (1989) explain: One is grounded in the Viète model (transposition of terms from one side to the other), and the other is the Eulerian model (operating on both sides of the equation with additive and multiplicative inverses). However, as has been seen earlier in this thesis, these two traditional approaches have a long history. In the work of al-Khwarizmi from the 9th century, the Viète model was described as *al-jabr* and the Eulerian model as *al-muqabala* (Katz, 1993).

The Eulerian model gave rise to the balance metaphor (Filloy & Rojano, 1989), which has been widely applied when teachers introduce linear equations (Pirie & Martin, 1997). The balance metaphor involves "performing the same operations on both sides of the equation", with the purpose of giving an operative mental image of the manipulation associated with the principle of equality (Vlassis, 2002, p. 341). The balance metaphor is performed in various ways in different studies, such as illustrations of balance scales (see for example Vlassis, 2002), physical balance scales (see for example Suh & Moyer, 2007) or as virtual balance scales (see for example Kurz, 2013). The balance metaphor also appears in Study 2 in this thesis. With respect to this, the study by Vlassis (2002) is interesting because it investigated what happened when the pupils left the concrete drawings of the scale and tried to solve written equations in a more formal way.

The aim of Vlassis' (2002) study was to investigate learning of the formal way of solving an equation by performing the same operation on both sides of the equals sign. The analysis was based on classroom observations and examinations of pupils' written and/or drawn solutions. 40 pupils in Grade 8 were involved. In the first phase, the pupils were presented with drawings of balance scales illustrating different equations with unknowns on each side of the scale. The pupils did not have any serious difficulties understanding how to get the correct value for x. In the next phase, the pupils continued to solve equations, but without the drawings of the balance scales. The author found that all the pupils successfully applied the principle of performing the same operation on both sides, which had been demonstrated earlier with the scales. Most of the pupils started to remove the x's that it was necessary to remove in order to get a single x on one of the sides. However, three categories of errors appeared: a) Some pupils divided both sides by the coefficient of x before they cancelled out the constant. One example was when $3x + 4 = 19$ became $x + 4 = 6.333$. b) Mistakes of a syntactical nature, when the coefficient of x and the constant cancelled each other, as for instance when $4x + 4 + x$ became $x + x$. c) Many errors occurred when negative integers were part of the equation. $2 – 3x + 6 = 2x + 18$ became $2 – 1x + 6 = 18$ because the minus sign in front of $3x$ was not seen as attached to $3x$. Another issue was the order of subtracting, when cancelling out a negative expression. Vlassis concludes that the balance model can support pupils learning of how to solve an equation, because it gives meaning to the algebraic manipulations and it offers an "operative mental image that contains the principles to be applied" (Vlassis, 2002, p. 356). However, he argues, all equations are not compatible with the balance model. The model is not intended to be used for equations with negative numbers, Vlassis emphasizes.

An algebraic expression can be described as "a string of numbers, operations and algebraic letters without an equals sign" (Kilhamn, 2014, p. 87). An expression such as (n–3), can be treated both as a conceptual object in its own right as well as a process to be carried out when the variable is known (Hewitt, 2012; Gray & Tall, 1994). Sfard (1991) finds that the stage of computational operation often precedes the structural stage for most individuals, when learning mathematics. When dealing with an algebraic expression as something that should be calculated, it is difficult to accept an expression as a final solution, which is referred to as *acceptance of lack of closure* (Collis, 1975). Linchevski and Herscovics (1996) argue that the use of algebraic expressions requires a more advanced comprehension of algebraic letters than when solving equations. The reason is that in equations, the letter can be perceived as a placeholder or an unknown. The study by MacGregor and Stacey (1997), mentioned in the section "Letters in algebra", includes an analysis of how pupils aged 11-15 years dealt with algebraic expressions.

Conventions

In algebra, as well as in other areas of mathematics, conventions are important (Pierce & Stacey, 2007). Conventions are social constructs (Hewitt, 2012). There is no mathematical reason for writing $4 \cdot x$ as $4x$. Rather, people involved in mathematics began writing it in this way and then it became accepted within the mathematical community. Other alternatives could have been established (Hewitt, 2012).

Pupils need to be aware of different conventions. One example is when two symbols are placed together, which sometimes denotes addition ($5\frac{1}{3} = 5 + \frac{1}{3}$) and sometimes signifies multiplication ($5x$ = 5 multiplied by x) (Hewitt, 2012). Another example of a convention is the x, which means $1x$ (MacGregor & Stacey, 1997). In all other cases the coefficient is written in front of the x ($2x$, $3x$ and so on). For pupils who are used to writing and reading in a strict left-to-right order there can also be a conflict between the order of operations in expressions (Tall & Thomas, 1991). The expression $3x + 2$ is, for instance, both read and calculated from left to right. The expression $2 + 3x$ is also read from the left, but processed from right to the left, because of the convention of always doing multiplication before addition (Tall & Thomas, 1991).

No studies that focus purely on conventions in algebra, which would be relevant to this thesis, were found when searching in the database ERIC/EBSCO. Instead some individual observations relating to algebraic conventions from other studies will be presented. As part of a study (NCTM, 1981), the solution frequencies for two similar tasks, answered by 2 400 pupils aged 13 years, were compared. The first task 4 x □ = 24 was correctly solved by 91 percent of the pupils. However, when the same pupils were supposed find the value of m in $6m = 36$, the solving frequency was significantly lower, 65 percent. Mathematically the tasks are on the same level of difficulty: something multiplied by 4 is equal to 24; something multiplied by 6 is equal to 36. In the analysis the difference is generally explained by assuming that the format of the problem affects the performance. The pupils appeared to be more accustomed to using a box to represent the unknown value, the authors argue, than they were to using an algebraic letter. However, in the analysis the authors do not mention anything about conventions and how pupils deal with them.

In TIMSS 2011, one of the tasks in Grade 8 also includes algebraic conventions:

> What does $xy + 1$ mean?
> A. Add 1 to y, then multiply by x.
> B. Multiply x and y by 1
> C. Add x to y, then add 1.
> D. Multiply x by y, then add 1.
>
> (Foy, Arora & Stanco, 2013)

No calculation or manipulations with numbers or variables are necessary in this case. Instead the pupils need to be aware of the conventions of the invisible multiplication sign and the order of doing operations. The percentage of correct answers differed greatly between countries (Foy et al., 2013). Hong Kong was on the top, with 94 percent, and the international average was 65 percent. In the Nordic countries, only Finland achieved higher results than the international average: Finland, 72 percent, Sweden, 53 percent and Norway 36 percent. The results reveal that a great number of pupils in Grade 8 are not familiar with fundamental conventions in algebra.

To recap, the research uncovers several difficulties for many pupils when encountering introductory algebra. The pupils need to appropriate the principles of how to interpret algebraic letters in line with the expectations of contemporary school mathematics, to be aware of the equals sign as a relational sign, to become accustomed to formal equation solving, to accept algebraic expressions as final solutions and to make sense of hitherto unknown conventions. All these algebraic concepts and methods have developed over time, but as Radford (2001) points out, the pupils often meet them for the first time as well-established mathematical phenomena.

2.2 Small group discussions in the learning of mathematics

Pupils' interactions in groups can be labeled in many different ways, such as for instance cooperative learning, peer collaboration, peer interactions, group work, group discussions, collaborative reasoning or small group discussions. In this thesis, the last of these labels, small group discussions, will be applied and used in a broad sense to mean "students sitting together and working on a common mathematical task" (Jansen, 2012, p. 38). However, Jansen stresses that all small group work is not collaborative group work. In collaborative work, the pupils are dependent upon each other.

Teachers may have a range of intentions when organizing the pupils into small groups during mathematics lessons (Jansen, 2012). In addition to the directives in the curricula, the teachers may have several other aims, for instance for pupils to learn mathematics content through interaction or to give the pupils the opportunity to develop autonomy and/or be engaged in democratic processes of argumentation and knowledge sharing.

In the following, there is a focus on factors that may contribute to forming a culture of collaborative reasoning with the purpose of supporting pupils' progress "toward a more sophisticated understanding" of a specific problem and "a greater understanding of the power of algebraic generalization" (Koellner, Pittman & Frykholm, 2008, p. 310).

Certain conditions may create a culture of collaborative reasoning in a classroom and thus support the process of learning mathematics, Mueller (2009) argues. In such a culture, the pupils are used to presenting their ideas, to discussing the ideas of others and to jointly building on or challenging these ideas. In the study conducted by Mueller, the purpose was to investigate how mathematical reasoning developed over time among middle-school pupils, and how specific conditions influenced a culture of reasoning and contributed to mathematical reasoning. Participants were 24 pupils in Grade 6, who were video recorded during five sessions. There were four pupils in each group and they worked with open-ended problems concerning fractions. They were encouraged to develop and justify their solutions in the small group collaborations and then share their suggestions with the class.

Mueller's study has been given considerable attention in this section because the research settings are similar to those of my studies. Pupils of similar age collaborate and discuss mathematical tasks and their communication is in focus in the analysis. Mueller examined eight factors that are central when attempting to create a culture of reasoning in a mathematics classroom: *a collaborative environment, task design, representations, tools, inviting pupils to explain and justify their reasoning, emphasis on sense making, the teacher's role* and *mathematical discussion*. In the following each of the eight factors mentioned by Mueller will be commented on. Each factor is then expanded on, referring to additional studies that are not mentioned in her study.

A collaborative environment. In Mueller's study, the pupils sat in small groups, they were invited to collaborate and they had sufficient time to explore and consider the solutions of their fellow pupils. They listened to each other and incorporated ideas from peers into their own arguments. The pupils felt there was a supportive environment in the small group. Correcting each other and challenging conjectures by others were accepted. In a supportive environment, the pupils are comfortable with questioning their peers and asking for support, Mueller states. In a study by Webb and Mastergeorge (2003), pupils' support of each other has been analyzed in mathematics classrooms, Grade 7. The focus was to identify pupils' help-seeking and help-giving in relation to learning in mathematics. The authors found that effective help-seekers asked precise questions about what they did not understand, they continued to ask until they comprehended and they used the explanations they received. The effective help-givers provided detailed explanations, gave opportunities to the help-seekers to test the explanations by themselves and monitored the help-seekers' problem-solving attempts. In line with Mueller, Alrø and Skovsmose (2002) emphasize the need for a climate of mutual trust and confidence, and they argue that "dialogue cannot take place in any sort of fear or force" (p. 123). They discuss *running a risk* and note that what is going to happen in a classroom during a small group discussion is unpredictable both for the teacher and pupils. This can include risk-taking both in an epistemological and an emotional sense. When pupils engage in a discussion, they share both thoughts and feelings, they "invest part of themselves" (p. 122), which also makes them vulnerable. Their own assumptions and reasoning can lead them into blind alleys, which can make them feel uncomfortable. The element of risk should not be taken away, but it is necessary to establish a positive atmosphere where being uncertain is allowed. Risks are an integral part of dialogue and include both positive and negative possibilities (Alrø & Skovsmose, 2002).

Task design. Mueller's analysis showed that open-ended challenging tasks promote opportunities for multiple representations, multiple strategies and mathematical discussions. The relation between the tasks and what kind of learning they give rise to has been investigated in several other studies. In a randomized controlled study by Phelps and Damon (1989), 152 pupils in Grade 4 collaborated when solving mathematical tasks. The results of that study indicated that collaborative learning is more effective for conceptual learning than for rote learning. This was also confirmed in a literature review conducted by Cohen (1994). One conclusion of this fact is, as Phelps and Damon (1989) state, that it is necessary to consider when collaboration can make its most valuable contribution. The role of the task during peer interaction is also in focus in a study by Schwarz and Linchevski (2007). They found that even slight modification in the research setting, such as changing tasks, might lead to totally different results. In the study 60 pupils aged 15–16 solved proportional reasoning tasks. The pupils worked in pairs and firstly they were presented with tasks involving illustrations of a set of four blocks, where they were supposed to discuss the relative weight of two of the blocks. At the end of their discussion, the researcher introduced physical manipulatives in the form of a pan balance and concrete blocks with exactly the same configurations as the blocks in the illustrations. The pupils were encouraged to first formulate a hypothesis about the relations between the weights of the blocks, and then test their hypothesis on the pan balance. The tests showed that the pupils mainly learned from the interaction when they first formulated a hypothesis and then had the opportunity to test it. The authors conclude that it was the hypothesis testing that made the difference.

Representations and tools. In Mueller's explanation, the roles of representations and tools are similar. She gives examples of representations (e.g. words, models, drawings and symbols) and she uses corresponding examples when explaining tools (spoken and written language, physical models, drawings, diagrams and mathematical notations). The main point is, according to Mueller, that both representations and tools make the pupils' ideas public and possible to be communicated in the group. In her study, Cuisenaire rods are regarded as tools that were used by the pupils to create models of solutions. In the study of Koellner et al. (2008), the conversation between a group of four pupils in Grade 8 was analyzed while they tried to solve an algebraic task (a painted cube problem). The pupils were presented with a task about cubes built up of smaller cubes. They were asked to find the number of small cubes, which could be painted on only one face, two faces, three faces or be unpainted. The task required the pupils to find a formula to represent (and simplify) the inherent pattern in the problem. The authors found that the physical structure of the cubes helped the pupils to move from concrete ideas to more generalized solutions. The physical cube invited the pupils to discuss and to share ideas about the structure of the cube.

Inviting pupils to explain and justify their reasoning. In the study by Mueller, the pupils were continually asked to justify and defend their arguments. Also in other studies, didactical potential has been shown when pupils are encouraged to be explicit in their reasoning. This was shown in a video study by Weber, Maher, Powell and Lee (2008), which analyzed small group discussions between pupils about a statistical problem (which dice company produced the fairest dice) in Grade 7. When the object of the debate was mathematical principles, learning opportunities emerged from group discussions, as, for instance, when the pupils discussed how appropriate it was to draw conclusions from a relatively small sample size of data.

Emphasis on sense making and the teacher's role. When the pupils worked to convince their classmates of something, Mueller suggests, they needed to think about whether their ideas, models and/or arguments made sense. It was the pupils themselves who determined what made sense, not the teacher. However, the teacher's role was important. She asked verification questions with the intention of making the pupils' arguments public, but she did not correct the comments or give the answer. By working in this manner, Mueller claims, pupils are encouraged to use each other as resources and be engaged in the collaborative argumentation. In the study by Koellner et al. (2008), the teacher's role is also accentuated. The teacher did not give the solutions that would "unlock the problem", but helped the pupils to make sense of what the question was about, for instance when attempting to organize the information in a table (Koellner et al., p. 309).

Mathematical discussion. When analyzing the form of mathematical argumentation in the pupils' discussions, Mueller found three main categories: *building on each other's ideas*, *questioning each other* and *correcting each other*. The first category, building on each other's ideas, was then divided into three sub-categories: *reiterating, redefining* and *expanding*.

The first, *reiterating*, means that the pupils repeated the statement or restated it in their own words. The second category, *redefining*, implies that the original statement was complemented by, for example, relevant concepts. The third category, *expanding*, implies that, for example, alternative justifications were created based on the ideas presented by their peers. Two types of co-action were identified in the pupils' constructions of arguments: *co-construction* and *integration*. Evidence of co-construction occurred when pupils built their argumentation collaboratively "from the ground up" (Mueller, 2009, p. 148) and the idea grew explicitly out of their discussion. There was a constant back-and-forth dialogue involving a cycle of reiterating, redefining and expanding of ideas. These types of arguments, Mueller argues, might not have appeared without the opportunity to collaborate. When pupils integrated the ideas of others into their solutions, the collaboration was classified as integration. This happened when some pupils individually tried to solve the task and come up with arguments, but in discussions with peers they then modified and developed their argumentation. When comparing the first and last session, Mueller concludes that pupils' mathematical reasoning deepened.

Finally, the study by Mueller showed that each of these eight factors contributed to pupils' co-constructions of arguments and integration of ideas in their collaborative reasoning. In a subsequent study by Mueller, Yankelewitz and Maher (2012), the number of collaborative actions carried out by pupils was compared over time. The analysis showed that the pupils became more active in communicating about mathematics. The number of co-constructions of ideas increased between the first and the fifth (last) session.

However, when pupils are organized in small groups and expected to work together, it cannot be taken for granted that learning in mathematics happens (Gillies, 2008; Radford, 2011a; Sfard & Kieran, 2001; Weber et al. 2008). In a study by Sfard and Kieran (2001), this is illustrated by the conversation between two 13-year-old boys learning algebra. Both the mathematical communication and the interaction were analyzed. The results showed that one of the boys was so occupied with trying to interpret his classmate's self-communication that he gave up on his own mathematical thinking. The authors point to the difficulty of taking part in an ongoing conversation and at the same time trying to be creative in solving problems. Their detailed analysis of the boys' collaboration shows, they argue, that understanding needs effort: "The road to mutual understanding is so winding and full of pitfalls that success in communication looks like a miracle. And if effective communication is generally difficult to attain, in mathematics it is a real uphill struggle" (p. 70). However, at the end of their analysis Sfard and Kieran (2001) state that they "believe in the didactic potential of talking mathematics", but "the art of communication has to be taught" (p. 71).

To summarize, many conditions may contribute to fostering a culture of reasoning that supports mathematical discussions in small groups. A fundamental condition is that the environment is supportive and that pupils feel comfortable (Mueller, 2009). Dialogues cannot take place "in any sort of fear or force" (Alrø & Skovsmose, 2002, p. 123). If participants do not want to share their thoughts in the group, there will not be any discussions. The design of the task needs to allow the possibility of multiple representations, multiple strategies and mathematical discussion. (Mueller, 2009). Even slight modification of tasks may lead to completely different results (Schwarz & Linchevski, 2007). Representations and tools make ideas public (Mueller, 2009; Koellner et al., 2008). To contribute to learning in mathematics through group discussions, the pupils need to be challenged and encouraged to justify and defend their reasoning (Mueller, 2009; Weber et al., 2008). One central role of the teacher is to encourage the pupils to engage in making sense of their own and their peers' arguments, instead of delivering or confirming a correct answer (Mueller, 2009; Koellner et al., 2008). Two categories of co-action were found in pupils' mathematical lines of argumentation: co-construction and integration (Mueller, 2009). When pupils co-constructed arguments and ideas grew out of the collaborative discussions, alternative ways of reasoning followed, and when the pupils integrated peers' ideas into their own solutions, their arguments became stronger. A later analysis of the same study showed that the pupils were more active in their mathematical argumentation in the fifth session in comparison to the first one. However, even if instruction is organized in small groups, it will not ensure that learning in mathematics takes place (Gillies, 2008; Radford, 2011a; Sfard & Kieran, 2001; Weber et al., 2008). In this thesis, some factors presented by Mueller (2009) will be further discussed in relation to my empirical results.

Chapter 3 Learning mathematics – a socio-cultural perspective

The focus of this thesis is how pupils collaboratively co-construct meanings of concepts and methods related to introductory algebra, a subject matter that they have not yet mastered. The analytical focus is on how the pupils interact with each other and the intellectual and material resources they draw on to make sense of the problems they encounter and the results they achieve. From this point of departure, learning mathematics will be analyzed from a socio-cultural perspective, implying an interest in how pupils develop ways of dealing with problems related to introductory algebra, the conditions for how this is enabled and the role of mental and material tools in this process. First, some central concepts in socio-cultural theories that are relevant to this thesis are highlighted: *appropriation* as a metaphor for learning, *the zone of proximal development* and *intellectual tools*. Secondly, the analytical focus will be further developed by presenting concepts that are central to a dialogical approach, in this case the concepts of *contextual resources* and *contextualization*.

3.1 Appropriation, ZPD and intellectual tools

Why does this thesis take an interest in pupils' discussions? In what sense does an exploration of their reasoning say something about how they learn? To understand such issues, it is important to outline the theoretical foundations for how learning is understood from a socio-cultural perspective. This view on learning implies that knowledge does not exist in a vacuum: the social and cultural contexts are always present (Radford, 1997). The term *social* denotes that we are all anchored in a community and the ways that we think and act are related to this anchoring (Dysthe, 2003). This implies that understanding what impedes and supports pupils' learning presupposes that the specific situation is taken into account. How we learn is an issue that has to do with how we appropriate the intellectual and physical tools that are part of our culture and our community (Säljö, 2000). Knowledge and competence of this kind do not originate from our brain as biological phenomena. Even if the processes in the brain are important prerequisites for our abilities to "analyze concepts, solve equations and write poetry" (p. 21), these concepts, equations and poems are not located in the brain as such. Instead, Säljö states that cognitive processes have to do with meaning making and meaning is a communicative, not a biological, phenomenon. To understand the role of mathematical symbols and concepts, it is important to note that knowledge and skills, from a socio-cultural view, originate from insights that people have developed throughout history. This means that the human being – with brain, mind and body – becomes a socio-cultural being using already existing intellectual, linguistic and physical tools in the community (Säljö, 2015). Every new generation is, so to speak, born into the use of these tools and may then develop them. The introduction to formal algebra serves as an exemplary case of how the use of socio-historically developed tools is appropriated by a new generation.

The overall aim of this thesis is to understand how pupils appropriate introductory algebra and what kinds of challenges they encounter. The notion of *appropriation* is therefore an important concept. It is seen as a metaphor for learning and it implies that in each situation we have the chance to appropriate knowledge or insights in interactions (Säljö, 2000). Learning is the ability to see something new as an example of, or as a variety of, something already known. In line with Säljö (2000), Radford (2000) describes learning algebra as "the appropriation of a new and specific mathematical way of acting and thinking", which is "interwoven with a novel use and production of signs" (p. 241). In this thesis, for instance, this connects to how an algebraic symbol, n, should be interpreted, something that is far from obvious for novices. It is thus of central interest how this develops in small group discussions. The notion of appropriation implies that learning is a gradual process, in which individuals learn to use tools in different activities and gradually learn to practice them on their own in a more productive way. Appropriation of knowledge or skills is not necessarily ever completed. The borders between understanding and not understanding are seldom definite. Complex concepts, knowledge and skills may always be refined and cultivated. For these reasons, processes of appropriation are of central interest, regardless of whether they lead to a correct understanding of algebra from a disciplinary point of view or not.

The notion of *zone of proximal development, ZPD*, provides a frame for understanding how pupils' meaning making develops in small group discussions. It offers a dynamic perspective on learning that originates from the works of Vygotsky (1934/2012). To put it briefly, it refers to the zone between what a child can manage on her own and what she can do with the assistance of an adult or some more competent partner. With assistance, every child can do more than she can do on her own. From a socio-cultural perspective it is not plausible to assume that pupils can discover abstract knowledge on their own (Säljö, 2000). Knowledge does not exist in the objects in themselves, instead it lies in our descriptions and analyses of them. This means that abstract concepts need to be unpacked in collaboration (Furinghetti & Radford, 2008). For this thesis ZPD relates to an interest in how the understanding of algebra develops in the interplay between pupils in small group discussions.

In the present thesis, examples of *intellectual tools* are variables, algebraic expressions and equations. Intellectual tools, as well as physical tools, are key concepts in the socio-cultural perspective, because they are parts of individuals' learning, thinking, working and living (Säljö, 2015). Tools are the resources we have access to and which we use when we understand our surrounding world and act upon it. Physical tools are used when we travel, dig a ditch, hammer a nail, cook food or write a text. Intellectual tools, such as percent, kilometer, and triangle are applied when we think and communicate about the world in specific ways that have developed in our culture over time. This implies that tools include experiences and insights of earlier generations, which we utilize when making use of these resources. The historical development is present in the existing tools (Säljö, 2000; 2015). The most powerful and important intellectual tool of all is language, which we can use to understand and think about the world on our own and which enables the mediation of our understanding to others. In language, important parts of our knowledge are available. That is why development and learning are issues relating to understanding linguistic distinctions about new and hitherto unknown phenomena – such as football, cell biology or probability calculus – so they can be understood in a more differentiated and varied manner (Säljö, 2000). In schools, language is the most important tool for mediating knowledge. It is through listening, reading, writing and discussing that most learning takes place. From this point of departure, it follows that the pupils' communication when dealing with algebraic problems is key to understanding how they learn. In the present thesis a *dialogical approach* is applied as a framework for conceptualizing how the pupils in small group discussions make meaning of tasks related to introductory algebra.

3.2 Dialogical approach in the analysis

The dialogical approach (Linell, 1998) provides an opportunity to analyze how meaning emerges in dialogue, and why the pupils end up with particular solutions when working with algebraic tasks. The concept of the dialogical approach (analytical framework) and its origin, dialogism (theoretical perspective), are used in many ways by different scholars and cannot be seen as coherent and well defined (Ryve, 2008). Dialogism, to put it briefly, can be seen as the opposite to monologism, which accounts for communication as transfer of information, cognition as individual processes and contexts as fixed environments (Linell, 2011). Dialogism is an epistemology for understanding cognition and communication as different aspects of the same phenomenon (Linell, 1998). Meaning gradually arises in a dialogue, as utterances are successively constructed in collaboration with the other actors. Individual utterances are not well-formed intentions, ready to be transferred to other individuals. The human mind is instead understood as a flow of thoughts, where some things are made explicit and others are vaguely present. This conceptualization of mind leads to an interest in how actors collaboratively interact and co-construct meaning. A fundamental way of sharing meanings in a communicative interaction involves at least three steps: Person A wants to communicate meaning to person B and makes an utterance a_1. Person B indicates her understanding of this by another utterance b_1. A shows the reaction to B's response by yet another utterances, a_2.

Two concepts from the dialogical approach are used as analytical tools in the thesis. *Contextual resources* are central in the first study, whilst *contextualization* is central in the second. In the following section there will be a presentation of how they are applied in my empirical studies. Linell (1998) uses the concepts *context* and *resource* together, because he claims that context is nothing in and of itself; it cannot be seen "objectively" or in the singular. Context is a multifaceted concept, which should rather be described in terms of different kinds of contexts (Linell, 1998). There are different traditions to explain contexts. From a socio-cultural perspective, contexts are seen "as embedded within and emergent from" the activities themselves (Linell, 2011, p. 93). Utterances actualize contexts and actualized contexts initiate new utterances in a mutual process between utterances and contexts. In other traditions, context is often regarded as more or less "stable outside environments" (ibid.), which implies that contexts already exist, before the sense making activity. A dialogical approach implies that contextual resources are defined as potential contexts made relevant through the dialogue (Linell, 1998). From this point of view, the gap between the two aspects of contexts can be closed, by treating contexts both as resources for and products of the interlocutors' activity. Linell (1998) distinguishes between three different dimensions of contextual resources:

1. Surrounding physical situation
   in which the interaction takes place, the "here-and-now" environment with its people, objects and artifacts
2. Co-text
   what has been said on the same topic before the utterance or episode in focus
3. Background knowledge
   such as knowledge and assumptions about specific topics, the world in general, people involved, the specific activity type or communicative genre (for instance a family dinner-table conversation or a theatre performance) etc.

A contextual resource, Linell (1998) explains, is thus not something given and inherent in things or processes. A resource is a resource for somebody, for a purpose in a situation. Some of the contextual resources can be relatively stable and constant over time, e.g. general background knowledge. Although considered stable, the resources need to be invoked and made appropriate in the actual talk. Other resources can be local and temporary, emerging from the dialogue itself and readily dropped (Linell, 1998).

Concerning *contextualization*, pupils may make different contextualizations of the same task, depending on how the pupils in collaboration define the on-going activity (Ryve, 2006). This in turn is intertwined with the resources they rely on in making sense of the task (Linell, 1998). This implies that even if pupils are working with the same mathematical task during the same mathematics lesson, they may encounter various mathematical problems, depending on how they contextualize the task (Ryve, 2006).

In the second study the concept of contextualization is applied (Linell, 1998), a concept that has been further developed in mathematics education by Nilsson (2009). Describing the process of pupils' meaning making through the construct of contextualization is beneficial, Nilsson argues, because it is based on the assumption that understanding does not occur through adding small isolated elements to already existing elements in a linear and hierarchical order. Instead, he continues, understanding is seen as being in the form of a "comprised system of linked, interrelated and coordinated knowledge elements and bits of information" (Nilsson, 2009, p. 65). A fundamental idea of contextualization is that learning mathematics implies that the pupils develop contextualizations, "networks of interpretations", where a mathematical treatment appears relevant and meaningful (p. 66).

To summarize, in this thesis the aim is to understand how pupils make meaning of algebraic letters, algebraic expressions and equations. From a socio-cultural perspective, these concepts are examples of intellectual tools that have been developed through history and now are in the process of being appropriated by a new generation – the pupils in these studies. Analyzing their communication gives us the opportunity to increase our knowledge about how pupils learn basic formal algebra. The zone of proximal development, ZDP, is a frame for understanding how pupils' meaning making develops in the interplay between them in small group discussions. The dialogical approach gives us an opportunity to look at pupils' mathematical activity from a broader perspective – not only *what* they say, but also *how* meaning emerges in the small group discussions. The analytical concepts of contextual resources and contextualization offer tools to track the reason *why* the pupils end up with various mathematical interpretations and mathematical solutions.

Chapter 4 Empirical data and methods

In this chapter there is a description of the design of the research studies, how the empirical data from the pupils' small group discussions was collected and a presentation of the methods used in the data analysis. Finally, there is a section about ethical considerations.

4.1 Design and data collection

The studies are part of a research project, entitled VIDEOMAT[4], which consists of video studies of algebra learning in Sweden, Finland, Norway and the US (Kilhamn & Röj-Lindberg, 2013). Within the VIDEOMAT project, five consecutive lessons were recorded in five different Grade 6 and 7 classrooms in four schools in Sweden. The teachers were recruited through the school authorities or through personal contacts. The teachers were informed that they would be video taped during their first four lessons on introductory algebra, following the ordinary curriculum. The fifth lesson was guided by the project. It consisted of small group discussions around three tasks adapted from Grade 8 in TIMSS 2007 (Foy & Olson, 2009). The first task concerned an equation, the second the variable n and an algebraic expression, the third a matchstick problem. This thesis is about the first and the second task. In TIMSS the tasks were multiple-choice questions (see Appendix 1), but for the VIDEOMAT project they were changed to

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[4] The VIDEOMAT-project is fund by the Joint Committee for Nordic Research Councils for the Humanities and the Social Sciences (NOS-HS) through a grant to professor Roger Säljö, director of the Linnaeus Centre for Research on Learning, Interaction and Mediated Communication in Contemporary Society (LinCS), the Swedish Research Council (dnr 349-2006-146).

open questions in order to generate discussions. The tasks were given to the teachers after the fourth lesson. The aim of the fifth lesson was to study pupils' communication when working with algebra tasks they had not been specifically prepared for. All the pupils in the classes were informed that the tasks were intended for Grade 8.

In my case the process of video recording was already completed when I entered the project. I have viewed the video data from a school in the west of Sweden, consisting of 5 lessons in each of 2 classes of 12-year-old pupils. The method of collecting data by video tape recording gives an opportunity to catch, at least partly, the complexity in a small group discussion with its manifold interactional phenomena (Derry et al., 2010). It provides the possibility of observing eye gazes, gestures, movements, facial expressions etc., which can deepen the verbal representations and contribute to making sense of the situations. Another advantage is that video sequences can be slowly replayed, seen repeatedly and be analyzed and reanalyzed together with other researchers (Derry et al., 2010).

Table 2. The length of the video taped lessons

	Class 6a	Class 6b
Lesson 1	42 min	40 min
Lesson 2	42 min	39 min
Lesson 3	36 min	39 min
Lesson 4	29 min	41 min
Lesson 5	Group 1: 31 min Group 2: 31 min	Group 1: 37 min Group 2: 37 min
Total	3 h 31 min	3 h 53 min

The video data has been analyzed through an inductive approach (Derry et al., 2010), where the actual video corpus was initially investigated with broad questions in mind. Two different group discussions in the fifth lesson in Class 6b were selected. The reason for selecting the first group was that it showed a discussion that seemed very productive, while the resulting written answer was simple and incorrect. Was the apparent productivity of the discussion only a surface feature or was the incorrect answer a poor representation of the learning that took place? The reason for selecting the second group was that when the pupils solved an equation, they referred to manipulatives (boxes and beans) that had been employed during prior lessons and used these as a resource. The specific character of this small group discussion was that they applied this resource on their own initiative – without any requests in the written task, suggestion from the teacher or hints from classmates. This provided opportunities to investigate the pros and cons of using manipulatives in mathematics.

Key video passages from the discussion in the two groups were discussed by the members of the Swedish part of the project. There was consensus in the project that analysis of these discussions could contribute to our understanding of how pupils appropriate introductory algebra, and what challenges pupils may meet when they deal with algebraic letters, algebraic expressions and equations. In the next phase, the formulation of specific research questions began. As a consequence of this procedure of selecting data, the data in both studies comes from the same class and the same lesson with the same teacher, but from two different groups working with two separate tasks. In the first study, data is based on a 15-minute discussion in a group of three 12-year-old pupils. In the second one, data comes from a 26-minute discussion in another group of three 12-year-old pupils. In total, there were 15 pupils in the class during this lesson, divided into 5 groups. The teacher was certified to teach Grades 4-6, and had 22 years of teaching experience. The tasks presented were not to be regarded as a regular test. However, it was suggested to the teacher that the lesson could be seen as an opportunity for formative assessment in a diagnostic tradition. During the lesson the teacher refrained from whole-class teaching and instead moved around the room observing the pupils, only interacting when they asked for help.

The pupils' written solutions and the teacher's plans were collected from all five lessons. In the data collection, three stationary cameras were used. During the fifth lesson, one camera was at the back of the classroom, operated manually to follow the teacher and getting close-ups when she worked with pupils or at the front of the classroom. Each of the other two cameras focused on one group of pupils. Wireless microphones were placed in the center of each group and one was worn by the teacher. The discussions in the two groups were transcribed verbatim, including non-verbal events that seemed relevant to the analysis.

4.2 Data analysis

Data analyses have been guided by the research questions. The process of turning raw data into researchable and presentable units followed an analytical model introduced by Powell, Francisco and Maher (2003). The model consists of seven interacting, non-linear phases:

1. Viewing the video data attentively
2. Describing the video data
3. Identifying critical events (an event is critical in its relation to the research question)
4. Transcribing
5. Coding
6. Constructing a storyline (the result of making sense of the data based on the identified codes)
7. Composing a narrative.

The process has been cyclic, involving multiple movements back and forth in the video data (Powell et al., 2003; Derry et al., 2010). As suggested by Derry et al. (2010), the transcripts have been iteratively revised throughout the process of analysis. The translation of the excerpts from Swedish to English has provided a further step in the analysis and has been a way of deepening the interpretations of the pupils' communication. In the excerpts, the original Swedish language and the English translations have been present in parallel during the analysis and have been repeatedly adjusted. In the articles, the pupils' utterances in Swedish were taken away at the end of the analytic process because of the limitation of space in research journals.

In the first study, the coding and analysis of what contextual resources the pupils make use of were built on the work of Linell

(1998). Categories suggested by Linell were then modified by the empirical data in an iterative process. The coding and analysis of different interpretations of *n* used a priori categories identified by MacGregor and Stacey (1997). In the second study, the data analysis focused on how pupils contextualize the mathematical task and how they move between different contexts. The construct of contextualization for analyzing data (Linell, 1998; Nilsson, 2009) was used. In mathematics education, the construct of contextualization is an analytical tool used with the aim of investigating "how and why a certain way of reasoning takes form and what it contains in terms of mathematical potential" (Nilsson, 2009, p. 64). In design and analysis of mathematical learning activities, four categories are central: 1. The mathematical potential of the problems; 2. Issues of familiarity; 3. Variation in contextualizations; 4. Reflection on viability. In the second study the focus was on the connections between contextualizations, which is included in the third category.

Three fundamental principles in the dialogical approach have guided the analysis throughout: *sequentiality, joint construction* and *act-activity interdependence* (Linell, 1998). The first principle, *sequentiality*, implies that an utterance is always a part of a sequence and cannot be understood if it is taken out of its context. A dialogue is a *joint construction*, because no part is regarded as being completely the product of one single individual. The third principle, *act-activity interdependence*, means that acts and activities are intrinsically related and are affected by each other.

4.3 Ethical considerations

An increasing number of educational researchers are using video recordings as a method for data collection (Derry et al., 2010). The use of video implies that specific ethical issues need to be addressed to secure the participants' integrity.

My studies are parts of the VIDEOMAT research project. The Swedish part of the project is approved by the regional ethical board of Västra Götaland.

The project has been conducted in line with the general ethical requirements formulated by the Swedish Research Council (2011) – concerning *consent, information, usage* and *confidentiality*. Informed *consent* has been obtained in writing from the parents and teachers and orally from the pupils. In signing the forms (see Appendix 2 and 3), the participants confirmed that they had received *information* of the main purpose of the project, that they were aware of the fact that participation was voluntarily, and that it was possible to withdraw whenever they wished. If they agreed to participate, they could choose between two options. One option allowed *usage* of the video recordings only for research purposes. The other alternative allowed use of the video recordings both for research purposes and for instruction in teacher education. In my studies all participants agreed that the video recordings could be used for both purposes. To protect the participants' *confidentiality,* the pupils have been given fictitious names. In all publications from the VIDEOMAT project, images have been stylized so that the participants cannot be identified. In Study 1, for instance there is a photo, which I have converted to a sketched image.

In following the Personal Data Act (Ministry of Justice, 2006) and its stipulated rules, I and my research colleagues in the VIDEOMAT project, have ensured that personal data is treated in a lawful manner that will protect the pupils' and the teachers' personal integrity. The video material, is stored in a locked safe at a video lab at the University of Gothenburg.

Chapter 5 Summary of studies

The two studies, as mentioned earlier, were conducted in the same classroom during the same lesson, but concerned two different groups of pupils working with two different algebraic tasks. The studies are connected by their focus on the contextual resources the pupils invoke when trying to solve tasks involving introductory algebra and the consequences in mathematical terms of applying these resources. In the first study, different contextual resources used by pupils have been identified, when they try to make sense of the algebraic letter *n* and create an algebraic expression using *n*. The second study presents an analysis of the process when the pupils use one specific contextual resource to solve an equation expressed in a word problem. In this case, the contextual resource consists of physical manipulatives (boxes and beans) used during prior instruction.

I have written the two articles together with two of my supervisors. I am the first author on both articles. My responsibility has been the selection of the research focus and theoretical framework, the initial analysis, the transcriptions and the literature reviews. The analysis was carried out and the final texts were written in collaboration with my co-authors.

5.1 Study 1

What's there in an n? Investigating contextual resources in small group discussions concerning an algebraic expression
(Accepted for publication in *Nordic Studies in Mathematics Education*.)

The study analyzes the process when three 12-year-old pupils in a video recorded small group discussion struggle to make meaning of the algebraic letter *n* and to form the expected algebraic expression ($n - 3$). Which contextual resources they make use of, and how these resources lead to different interpretations of the *n*, are in focus. Data for the study comes from a 15-minute dialogue, when pupils worked with the following task:

> Hasse has 3 jackets more than Anna. If *n* is the number of jackets Hasse has, how many jackets does Anna have expressed in *n*? Write an expression to describe how many jackets Anna has expressed in *n*.

MacGregor and Stacey's (1997) categories are used in the analysis of the pupils' different interpretations of *n*. Five of the six categories in MacGregor and Stacey's study appeared in the pupils' reasoning: *letter ignored* (no letter at all in the answer); *numerical value* (a value related to the situation in the task); *alphabetical value* (n=14; i=9 because of the order of the letters in the alphabet); *different letters for each unknown* (the pupils chose another letter, in this case "i"); *unknown quantity* (n–3). The only category not found was the interpretation of the letter as an abbreviated word.

The analysis of which contextual resources the pupils made use of was built on the work of Linell (1998) and was then adjusted to the empirical data. The dialogical approach (Linell, 1998) made it possible to identify eight categories of *contextual resources* the pupils utilized in their discussion (marked with italics below). It is important to notice that the presented result is not a complete record of all potential resources, but the ones which were found in the analysis to be relevant to the pupils' solving process. The pupils' different interpretations of *n* are presented in relation to the contextual resources.

1. *The surrounding physical situation*

   This means the situation in which the interaction takes place, the "here-and-now" with its people, objects and artifacts. The pupils made use of the competence in their group by asking each other questions and by explaining to each other. They read the text of the task several times, they overheard comments from pupils in another group and listened to the teacher, when she gave examples of how the value of *n* can vary and suggested writing a formula. This contextual resource gave rise to the following interpretations of the algebraic letter *n*: alphabetical value; numerical value; different letters for each unknown; unknown quantity.

2. *Co-text*

   The concept co-text signifies connections to what has been said earlier on the same topic. Initially one of the pupils tried to give the algebraic letter *n* the same value as the order in the alphabet, *n*=14. Later in the discussion another pupil picked up this strategy and suggested that they should check the order of the letter "i" in the alphabet, which resulted in i=9. However, the letter "i" that appeared in the task was not an algebraic letter, it was a preposition in the Swedish language. The "i" (in) was a part of the Swedish task, which asked the pupils to answer how many jackets Anna has "expressed in *n*" (in Swedish: "uttryckt i *n*"). If *n* =14 → maybe i = 9. This resource gave rise to the following interpretation of *n*: alphabetical value.

3. *Background knowledge*

   3.1. Prior knowledge in relation to mathematics

   When the pupils deal with the known relation in the task ("3 jackets more") in arithmetic terms, it appeared easy for them to give correct examples of the mathematical generalization: the number of Hasse's jackets = the number of Anna's jackets plus 3 → the number of Anna's jackets = the number of Hasse's jackets minus 3. Surprisingly, it was not until the end of the discussion that they applied their prior knowledge about algebraic letters when saying that "*n* is like *x* and *y*". This resource gave rise to the following interpretations of *n*: numerical value; unknown quantity.

   3.2. Prior knowledge in relation to situations outside mathematics

   They saw the letter *n* in the task and they were familiar with the alphabet. Attempts were made to use the alphabet as a resource to give the letter *n* a numerical value. (See also 2.) This resource gave rise to the following interpretation of *n*: alphabetical value.

   3.3. Knowledge and assumptions about the real world

   The term "real world" is here used in contrast to the "mathematical world" with its mathematical signs and symbols. The very first answer in the group was "2" [jackets], which could be seen as a realistic number of jackets. This resource gave rise to the following interpretation of *n*: letter ignored. When the pupils interpreted n=14, they giggled and said, "there are not many people who have 14 jackets".

3.4. Assumption about the communicative project (in this case to solve the task)

The pupils did not only want to solve the task, they also posed meta-questions such as: "How do you figure that out?" and "How are you supposed to think?". This resource gave rise to the following interpretation of n: unknown quantity.

3.5. Assumptions about the actual topic, in terms of socio-mathematical norms

Several socio-mathematical norms were visible in the discussion: the pupils took for granted that the answer was a number; that tasks and answers do not need to be realistic in math; that in a math problem you are supposed to work with the given numbers; and that the task has a meaning and that it is possible to solve it. This resource gave rise to the following interpretation of n: numerical value.

3.6. Knowledge and assumptions about each other

There were few misunderstandings between the pupils; they seemed to understand each other's humor and they laughed at the same things. An example from the data is when they investigated the alphabet track, and one of the pupils finally suggested that they could try the Spanish alphabet instead "to see if it will work better", at which they all laughed.

The results of this study may be read as a demonstration of pupils' engagement in meaning making when attempting to understand the role of n and their initial attempts to appropriate significant elements of what an algebraic expression implies. First, the study shows that pupils use a rich variety of contextual resources, both mathematical and non-mathematical, when making meaning of the algebraic letter. Second, the group displayed most of the interpretations of an algebraic letter identified by MacGregor & Stacey (1997) in a very short period of time in a group discussion. This is in contrast to previous research where each student were assigned to one particular interpretation only. The conclusion is that interpretations of an algebraic letter can be dynamic and that the nature of the meaning making may change rapidly depending on the contextual resources evoked. This indicating that an interpretation is not a static, acquired piece of knowledge, but more like a network of associations. Third, the study shows that the more advanced interpretation of the algebraic letter emerges as a result of the invoking of contextual resources of a mathematical nature. Although such resources are potentially available, they might not emerge unless the more basic interpretations are found to be invalid. Pupils' collaborative struggle to make sense of a phenomenon can be a fruitful way to approximate more sophisticated interpretations. Fourth, it is also noteworthy that what the pupils struggled with was not only a problem that could be solved by means of mathematics or logic, but also the linguistic convention of the term "expressed in *n*". An important conclusion is that learning mathematics is not only learning about mathematical objects and relationships, it is also learning about a specific communicative genre. In summary, we interpret this as indicating that the pupils appropriated important elements of algebraic representations and that the discussion served as a tool for this learning.

5.2 Study 2

Moving in and out of contexts in collaborative reasoning about equations
(Submitted to an international mathematics education journal.)

This study investigates how another group of 12-year-old pupils contextualize a task formulated as an equation expressed in a word problem. A special focus is put on which support for and what obstacles to learning can be identified, when pupils use manipulatives as a resource in an equation-solving process. In this study the pupils applied their experiences of manipulatives (boxes and beans used in earlier instruction) as a resource to solve the task. They were not explicitly instructed to do so and they did not have access to the physical objects themselves.

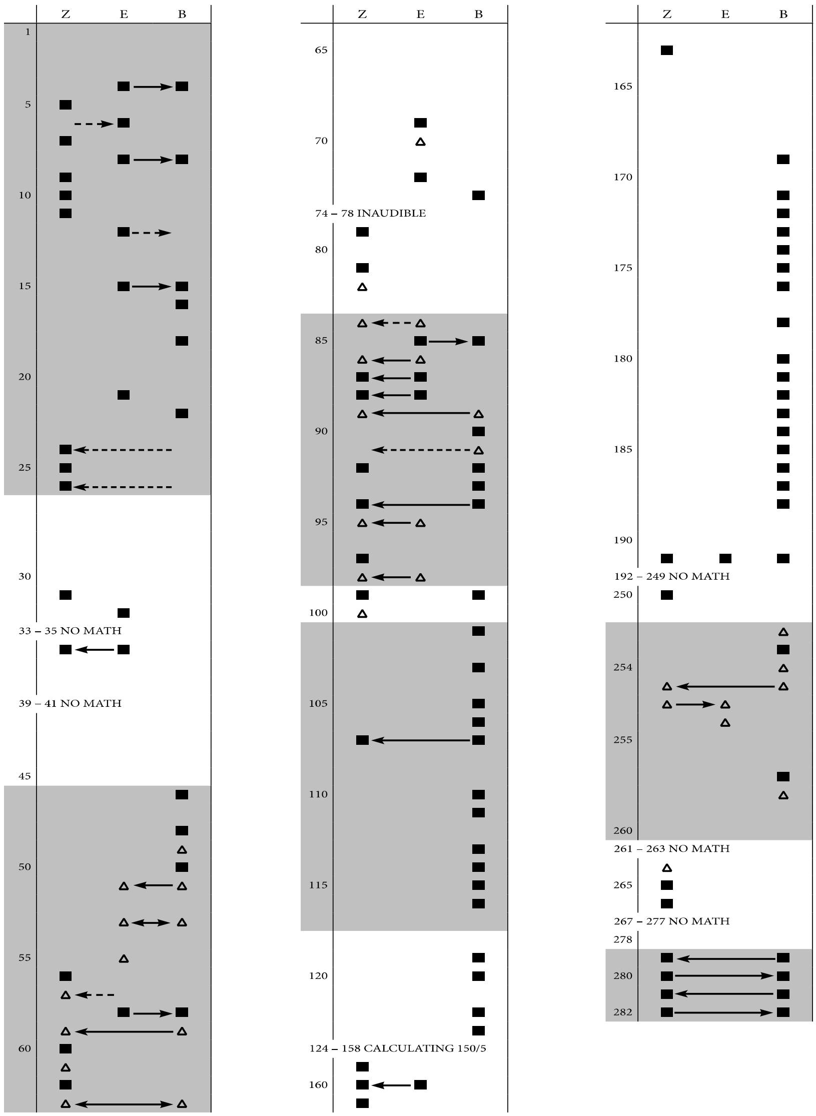
The pupils' small group discussion was video taped and their reasoning was analyzed using the construct of contextualization (Linell, 1998; Nilsson, 2009). In mathematics education, this is an analytical tool that is used with the aim of investigating "how and why a certain way of reasoning takes form and what it contains in terms of mathematical potential" (Nilsson, 2009, p. 64). Data comes from a 26-minute discussion, when the pupils worked with the following task:

> In Zedland, the cost of shipping a parcel is calculated using the following equation: $y = 4x + 30$, where x is the weight in grams and y is the cost in zed dollars. A parcel that costs 150 zed dollars to ship can be written using the following equation: $150 = 4x + 30$. How many grams does that parcel weigh?

Early in the analysis, we observed three forms of contextualizations in the pupils' discussions: *the Zedland context* (connected to the weight and cost of a parcel in Zedland); *the Equation context* (reasoning mainly involving algebraic symbols); *the Boxes-and-beans context* (pupils referred to the use of manipulatives used in prior lessons). At the beginning of the discussion one of the pupils, Leo, takes his point of departure from the equation ($y = 4x + 30$) and immediately makes associations to boxes and beans. The x in the equation is connected to boxes and the constant 30 to 30 spare (beans). In terms of contextualization, Leo first contextualizes the task so as to concern equations and then recontextualizes the task in terms of boxes and beans.

The flowchart below (Table 3), inspired by Sfard and Kieran (2001), visualizes how the task is contextualized in the pupils' continuing reasoning and how the whole discussion moves between different contexts. Each utterance is represented in the flowchart, except for breaks when they discuss the calculation of 150/5, when the utterances are inaudible or when they lack a relation to mathematics. Although each utterance comes from an individual, the analysis focuses on the joint construction (Linell, 1998) of the discussion.

Table 3. The pupils' and the teacher's movements between different contexts in chronological order in the whole discussion



Z= the Zedland context; E= the Equation context; B= the Boxes-and-beans context. Every fifth utterance is numbered. The shaded areas illustrate excerpts presented in the article. Pupils' utterances are represented by ■ and the teacher's utterances by △. Arrows indicate movements between contexts. A solid arrow indicates an explicit movement within one utterance. A dotted arrow indicates an implicit movement initiated in an earlier utterance.

In brief, the flowchart demonstrates how the discussion initially oscillated between all three contexts, with the utterances evenly distributed (U01-26). In large sections of the discussion, the pupils contextualized the task by means of boxes and beans (U101-123 and U169-188). The flowchart also shows the problem in the pupils' solving process, namely that they move from the Equation to the Boxes-and-beans context (U04; U08; U15; U58; U85), but never in the opposite direction. This indicates that they did not apprehend the connections between the two contexts as a translation in a two-way process (Filloy & Rojano, 1989). When they find out the correct numerical value 30 by referring to the manipulatives, their answer 30 is in terms of boxes and beans, "30 beans in each box", not as "x = 30". When they leave the Boxes-and-beans context, they bypass the Equation context, moving straight from the Boxes and beans to the Zedland context (U94; U107; U279; U281).

The pupils' final written answer illustrates that the pupils ended up in the Boxes-and-beans context, having great difficulties in recontextualizing their answer in terms of cost and weight of a parcel.

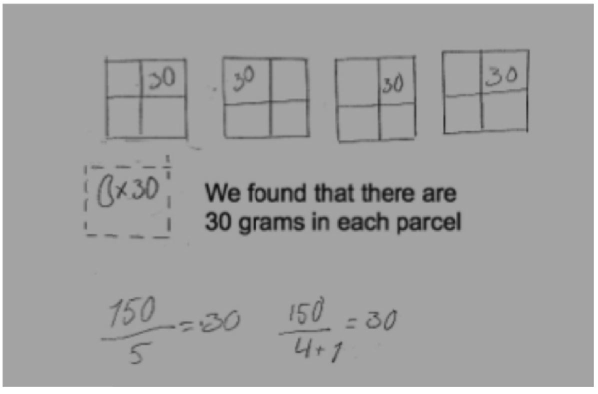


Figure 1. The pupils' written answer. (In Swedish: Vi kom fram till att det är 30 gram i varje paket)

The study shows that the use of concrete material supported the pupils in finding the correct numeric value of x in the equation. However, it did not help them to solve the task. The pivotal problem is that the pupils do not translate their results from the concrete manipulative context into the symbolic equation context. Hence, they never conclude that it is the value of x that is 30 and so it became problematic to connect the numerical value of 30 to other contexts. Learning about an abstract principle through the introduction of a concrete manifestation requires that students can see the general through the particular (Mason, 2008). Pupils need to be exposed to several tasks, handling the equation formulated in different ways, with qualitative discussions about 'what is the example' and 'what is general' in the activity. The results highlight the importance of giving pupils opportunities to comprehend the particular position of symbolic mathematical representations, when dealing with mathematical concepts. While a symbolic mathematical representation describes something general, concrete representations and specific real world examples always describe something particular. No one particular example incorporates the rich meaning of a symbolic mathematical concept. Teachers need to include this central distinction in their teaching practices.

Chapter 6 Discussion

This chapter involves an overall discussion about introductory algebra and is divided into three parts. It starts with a discussion of the conditions for pupils' appropriation of introductory algebra in small group discussions. The second part addresses issues on the zone of proximal development and in the last part I will present some methodological reflections.

6.1 Appropriating introductory algebra in small group discussions

The studies are positioned in a socio-cultural tradition, which implies a focus on how pupils interact to make sense of algebraic letters, algebraic expressions and equations. These tools have been developed over a long period of time (Säljö, 2000). The pupils in my studies are in the process of appropriating these tools, into which earlier generations' insights have been built. However, the pupils meet them as ready-made concepts (Radford, 2001). Nevertheless, some historical voices can be regarded as embedded in the voices of the pupils (Radford, 2011b). When the teacher asked Leo what x is, Leo answered "x are things" (Study 2, Excerpt 6). In a work of al-Khwarizmi one of the equations was written: "A square, which is equal to forty *things*[5] minus four squares." (van der Waerden, 1985, p. 4). Today we may write the same equation as $x^2 = 40x - 4x^2$ (ibid.). It is notable that the word "things" is used to denote the unknown number (the symbol x) both by one of the pupils in my study and historically by al-Khwarizmi in Bagdad.

[5] The word *things* has been italicized by me.

When comparing the development of algebra, as summarized by Sfard and Linchevski (1994) in Table 1 (p. 25), and the pupils' discussion, it is possible to link some of the interpretations to specific stages in the historical development. Firstly, using a *letter as an unknown* appeared during the 16th century. In Study 1 this is demonstrated when the pupils first gave the n a fixed value of 7 (Study 1, Excerpt 4, 6 and the written answer). Secondly, when the pupils described n as "like x and y" (Study 1, Excerpt 6) and explain that "y can be, be whatever" (Study 2, Excerpt 1), their utterances are related to the interpretation of a *letter as a variable*, which appeared in the next stage of the historical development of algebra. Thirdly, when the pupils made a computational operation and got the answer $n - 3 = 4$ (*operational stage*), it indicates that they did not see the relation between n and "three less" as an object (*structural stage*). This finding is in line with Sfard (1991), who states that operational thinking is the first step both in generalized and abstract algebra.

Even if there are parallels between historical and individual development, history should not be understood as a predestined trajectory for planning and assessing pupils' learning in algebra. This is pointed out in recent research that demonstrates that even young pupils are able to develop algebraic ideas (Cai & Knuth, 2011; Hewitt, 2012; Radford, 2012). What we can learn from history, as Sfard (1995) ascertains, is that what has taken time in history to develop can also be complicated for learners today. The magnitude of the challenges pupils encounter can be compared to the fact that it has taken about 4 000 years to develop algebra and it was not until the 16th century that the algebraic symbols were introduced (Sfard & Linchevski, 1994).

Similarly, many mathematical conventions have developed over time. As shown in Study 1, the pupils were unaware of the convention of using n in the sense required in the task and of the linguistic convention "expressed in n". Conventions are social constructs (Hewitt, 2012) that do not need to be difficult as such. There is nothing to understand mathematically; they need only be accepted and adopted. Problems occur, as illustrated in Study 1, when the pupils do not master the meaning of specific conventions that they did not have any prior opportunity to become familiarized with and accustomed to. When describing an ideal way of teaching mathematics, Lampert (1990) mentions conventions: the pupils "are introduced to the tools and conventions used in the discipline, which have been refined over the centuries to enable the solution of theoretical and practical problems" (Lampert, 1990, p. 42). Study 1 shows that the meanings of the mathematical conventions constituted obstacles. A conclusion is that learning mathematics is learning about a specific communicative genre as well as learning about mathematical objects and relationships.

Both studies illustrate that the appropriation of introductory algebra can involve the use of a wide range of contextual resources (cf. Linell, 1998). In the first study, the pupils relied on both mathematical and non-mathematical resources in their efforts to make sense of the algebraic letter *n*. In the second study, the pupils, on their own initiative, applied manipulatives (boxes and beans) used during previous mathematics lessons, as a contextual resource for solving the task. Apparently they all agreed that the boxes and beans were the only resources to use for solving an equation. What is remarkable is that they do not have access to these resources on this particular occasion; instead they make use of their previously gained familiarity with them.

In the first study it may seem surprising that the pupils initially failed to make use of their prior knowledge in mathematics, "*n* is like *x* and *y*". In the beginning the situation did not seem to invoke this knowledge, even if it appeared later. This indicates that even if the pupils have prior knowledge of a specific mathematical content, it should not be taken for granted that they will immediately apply this when encountering new problems. Furthermore, the results of Study 1 support the claim that socio-mathematical norms (Yackel & Cobb, 1996) have a strong impact on how pupils handle a task. Some of the socio-mathematical norms supported the pupils' reasoning (the task has a meaning and is possible to solve), but there were also norms that were less helpful (the answer is a number; operate with the given number). Socio-mathematical norms are often tacit, but it seems to be beneficial to pupils' in problem solving if explicit attention is paid to these norms.

A critical part of appropriating introductory algebra, which is also demonstrated in the empirical results, is to "see through the particular to the general" (Mason, 2008, p. 63). In Study 1, the pupils asked the teacher about the n and the teacher gave examples of how the value of n can vary, "n can be any number of jackets, it could be that Hasse has 7 jackets, it could be that Hasse has 15 jackets". However, the discussion indicates that the pupils interpreted the teacher's input as an invitation to choose "any number", for instance 7, and then decide to fix the value of n at exactly 7. According to Mason (1996), the pupils do not seem to apprehend that the teacher tried to illustrate the generality of how the value of n can vary; instead, they saw the examples as complete in themselves. Another example of the pupils' difficulties with seeing the general in the specific instance is demonstrated in the second study, in which the pupils talked about the equation $150 = 4x + 30$ as "4 boxes and 30 spare [beans] should be 150 beans then" (Excerpt 2, Study 2). The pupils were able to manage the specific case of solving an equation with boxes and beans, but were not able to use the strategy on a general level. The conclusion is that the pupils are able to appropriate intellectual tools (the algebraic letter n and equations) and material resources such as manipulatives for specific cases, but problems arise when they try to apprehend fundamental algebraic principles. It is notable that in their discussion about what could be in the boxes, they switched between specific concrete objects (chocolates, plastic icons, pizza and coffee beans) and the general statement "it doesn't matter what we have into them". This shows that the pupils were aware that x could represent something unknown, even if all their suggestions were some kind of concrete objects. They saw the specific cases and had apprehended some part of the basic algebraic principle.

6.2 Small group discussions as a zone of proximal development

In none of the studies were the pupils' final written answers completely correct. Nevertheless, we can see that they are en route to dealing with the algebraic tasks in accordance with conventional school algebra. If one puts the pupils' interaction in the frame of ZPD (Vygotsky, 1934/2012), it is possible to argue that the pupils, through their own interaction in the groups, were really close to understanding the meaning of significant elements in basic introductory algebra.

It is also important to remember that the pupils in my studies are 12-year-olds and that the tasks that they were given are really challenging even for older pupils. In both studies the pupils are working with tasks adapted from TIMSS 2007, which makes it possible to look at the frequency of correct solutions by 15-year-old pupils in that substantial and international study. It is important to keep in mind that in TIMSS, the tasks were formulated as multiple-choice tasks with four alternative answers. In the studies presented here, the pupils did not have access to possible answers. The result from TIMSS 2007 shows that the task about the algebraic expression (in Study 1) was correctly solved by 61 percent of the Swedish pupils in Grade 8 (Swedish National Agency for Education, 2010). The pupils in Hong Kong were not substantially better, 68 percent, but the result in Taiwan was higher: 78 percent. The Zedland item (Study 2) was correctly solved by only 23 percent of the Swedish pupils in Grade 8. This result was no better than a random guess, since the pupils could choose between four alternatives. In Hong Kong 71 percent of the pupils solved the item correctly and in Taiwan 75 percent. These statistics demonstrate that the actual tasks are also problematic for many pupils in Grade 8, not only for pupils who are in Grade 6, as in these studies. Many pupils in Grade 8 still struggle to understand and solve tasks involving algebraic letters, algebraic expressions and equations, even if they have been taught introductory algebra.

Considering that the pupils in my studies appropriated some elements of introductory algebra, and the result of TIMSS showed that many pupils in Grade 8 do not succeed in solving similar tasks, one might ask what conditions are critical for small group discussions to be productive. Mueller (2009) points to eight factors that are central for fostering a culture of collaborative reasoning in a mathematics classroom. In this section some of these factors are discussed in relation to the empirical studies. Mueller's description of a supportive *collaborative environment* is in line with what characterizes the interactions between the teacher and the pupils and among the pupils in my empirical material. The pupils listened to each other, were comfortable asking their peers for support and made considerable efforts to provide explanations to anyone who asked. No noticeable signs of "fear" could be observed in the participants' interactions (Alrø & Skovsmose, 2002).

Mueller (2009) argues that the *task design* is crucial and in her study she found that tasks of an open-ended nature promoted mathematical discussions. In both of the present studies, each of the tasks has only one correct answer and thus was not open-ended. However, collaborative engagement in mathematical reasoning can also be enhanced through problem-solving tasks where pupils do not have a given solution strategy (Schoenfeld, 1985). This seems to be the situation in both cases studied here.

In Mueller's study, the pupils used Cuisenaire rods which helped them to make their ideas public and possible to discuss in the group. In my studies the pupils did not have immediate access to any physical manipulatives. What is remarkable is that physical manipulatives, in the form of boxes and beans, at all events, played a central role in the activities carried out in Study 2. Throughout the discussion the pupils referred to the boxes and beans used during prior lessons.

Inviting pupils to explain and justify their reasoning is an important part of forming a culture of collaborative reasoning according to Mueller (2009). In the video data as a whole, there are several occasions on which the teacher encouraged the pupils to explain their way of reasoning to each other. One obvious example is when the teacher said "listen to Tom now, then let us hear", and then looked at Tom and said to him, "and explain to them" (Excerpt 7, Study 2). Also in the interactions between the pupils, the pupils ask each other to provide explanations, for instance: "How did you figure that out" (Excerpt 1, Study 1) and "How do you know that?" (Excerpt 3, Study 2).

Another point brought forward by Mueller (2009) is the *emphasis on sense making*. It is likely that there is a difference between the groups' ambitions in my studies. The pupils' interaction in the first study gives the impression that the pupils really try to understand, not only to find an answer. One boy, Max, expressed it like this: "I still don't understand. Do you understand, Moa? How are you supposed to think?" (Excerpt 6, Study 1). In contrast, one of the pupils in the second study is more occupied with completing the task quickly by suggesting at an early stage in the discussion, "Then this must totally be 120 … then we have the answer" (Excerpt 3, Study 2).

The *teacher's role* and how she/he acts during small group discussions is pivotal, according to Mueller (2009) and Koellner et al. (2008). They both agree that the teacher should not provide the correct answer; instead, teachers should ask verification question (Mueller, 2009) and/or support the pupils in making sense of the tasks (Koellner et al., 2008). The teacher in my studies followed the principle that the teacher should not provide the answer or explicitly confirm whether the answer was correct or not. One example is (Excerpt 5, Study 1) when Moa suggested the accurate answer "n minus 3 … or…?" and the teacher implicitly supported it by saying: "I think that sounds good". Nevertheless, the pupils did not assess the algebraic expression as an accurate answer, instead they continued the discussion and finally ended up in the written answer "$n - 3 = 4$". On the one hand, this can be interpreted as meaning that the pupils did not understand that their own suggestion was the correct answer so there was no point in writing it down as an answer. On the other hand, if the teacher had distinctly confirmed their answer, it could have provided the pupils with an opportunity to observe and learn what an algebraic expression looks like. The pupils' answer, "$n - 3 = 4$" indicates that they wanted a numerical answer and were reluctant to accept a lack of closure (cf. Collis, 1975; Kieran, 1981).

Mueller (2009) found that co-action resulted in stronger arguments and alternative ways of reasoning. In my studies, the groups tackled the tasks in different ways, although the conditions seemed on the surface to be identical: the same teacher and all pupils sitting in groups working with algebraic tasks during the same lesson. Of course there are many variations but this can also be understood in terms of act-activity interdependence (Linell, 1998). In the first study, the pupils seemed to define the activity (the task) as giving opportunities to act in many ways (using different contextual resources to try to make meaning of the *n*), which culminated in several alternative interpretations of the *n*. The pupils conjectured, tested and sometimes abandoned an insufficient interpretation such as the alphabetical value of the *n*. The analysis showed that the pupils progressed from non-algebraic meanings to more advanced interpretations of the *n* during the discussion. In the second study the pupils seemed to frame the activity (the equation in the task) as giving signals to act in one specific way (to use boxes and beans when solving the equation). This act in turn, led to the pupils becoming stuck in a frame of solving the task with help of the manipulatives. No other strategies were tried out or discussed. The pupils were fixed in the context of boxes and beans and were not able, by themselves, to recontextualize the task so as to relate it to the equation. Since the groups of pupils contextualized the two tasks differently, they also acted in divergent ways, which had decisive implications for their chances of solving the task (Linell, 1998; Ryve, 2006).

6.3 Methodological reflections

In these studies the focus is on the processes involved, when pupils tried to interpret the algebraic letter *n*, write an algebraic expression and solve an equation. As argued earlier, this makes the discussion a fruitful object of study. If the choice instead had been to study written answers, pupils' interpretations and solution processes would not have been visible (Sfard, 2008). This was the case in the classic studies by Küchemann (1978) and MacGregor and Stacey

(1997). They used pencil and paper tests[6] and obtained access to a huge number of written answers. Their research has contributed to well-grounded and valuable knowledge about how pupils interpret algebraic letters. However, the processes behind the written answers are not known. MacGregor and Stacey (1997) have for instance added "likely explanations" (p. 5) and "assumed reasoning" (p. 6), when they present the pupils' answers to the tasks. In my studies, in contrast, I had access to pupils' reasoning in detail and by applying a dialogical approach it was possible to analyze collaborative and communicative aspects in the mathematics classrooms. Such aspects might be decisive for what happens or does not happen during a lesson in mathematics.

The video recording has many benefits for analyzing discussions, mainly because it gives the opportunity to capture the complexity in the interactions between the pupils and their environment. Another advantage is the opportunity to see the video clips repeatedly and analyze the raw data together with other researchers. Of course, the method might have disadvantages. The presence of camera and researchers in the classroom may have an impact on pupils and teachers. An example of this was a comment from a Finnish pupil to the teacher in an earlier part of the VIDEOMAT project: "you do not usually talk with such a soft voice" (Kilhamn & Röj-Lindberg, 2013, p. 320). The two studies in this thesis concern the fifth and last video recorded lesson and there were no comments at all from the pupils about variations between lessons that were filmed and lessons that were not filmed. When I entered the project the video data had already been collected, so I did not participate in the video recording. In this project, however, the researchers had the role of "complete observers" (Cohen, Manion, & Morrison, 2011, p. 457) when they filmed the lessons, so this should not make any difference to the analysis of the video data. However, one of the researchers (Kilhamn) who collected the presented data has also participated in the data analyses.

[6] In the study by MacGregor and Stacey (1997) the entire investigation includes approximately 2000 pupils, but 14 pupils were also interviwed while working with selected items.

The technique is of course a critical factor when using video tape recording. When analyzing data from the small group discussions there were some smaller sections that were inaudible. It appeared when someone talked very quietly or when all the pupils in the group talked at the same time. It would also have been favorable if there had been more close-ups when the pupils were pointing to their papers.

The verification process in the analysis has some features in common with Kvale's (1996) notion of communicative validity. The latter implies that knowledge claims are proved in rational discussions, where it is always possible "to argue for or against an interpretation, to confront interpretations and to arbitrate between them" (Kvale, 1996, p. 245). In my case, research questions have been discussed continuously between my three supervisors and me. Two of my supervisors are also co-authors of both the articles, which means that we have had a mutual ambition to scrutinize all arguments for and against different aspects of the research coming up in the discussion. Nevertheless, there are also aspects that are taken for granted and will not come to light and be discussed. Therefore it has been valuable to present my research, show video-clips and discuss the analysis both within the VIDEOMAT project and also in several presentations and seminars outside the project: the Centre for Educational Science and Teacher Research at Gothenburg University (CUL), the National Center for Mathematics Education (NCM) and The Swedish Mathematics Education Research Seminar (MADIF 9). It has been an advantage to receive comments from people who work in the field of mathematical educational research, but who are not involved in the research project. The discussions during these meetings made me aware of issues that had not been visible before. In addition, reviewers have given input to the first submitted article.

This thesis is composed of two small case studies, each involving a group of three pupils. A common question in such qualitative studies is about the possibility of generalizing the results. In both my studies, there are thick descriptions of the cases, evidence is supported by excerpts, figures and tables, and the arguments are made explicit. This makes it possible for the reader to judge "the extent to which the findings" from these studies "can be used as a guide to what might occur in another situation" (Kvale, 1996, p. 233). But it is also a limitation that the two groups involved only six pupils in all. One limitation of small case studies, mine as well as others, is the restricted numbers of participants and the restricted opportunities to find rich variation in pupils' reasoning and to identify more general patterns.

6.4 Conclusions

Three general conclusions about how pupils appropriate introductory algebra and the kind of challenges they encounter can be drawn from the empirical studies. Firstly, the results show how the pupils made use of a rich variety of contextual resources (Linell, 1998), both mathematical and non-mathematical ones, when trying to make sense of the algebraic letter *n*. A general conclusion is that interpretations of an algebraic letter can be dynamic and the nature of the meaning making may shift quickly depending on the contextual resources invoked, indicating that an interpretation is not a static, acquired piece of knowledge, but more like a network of associations.

Secondly, it was found that mathematical conventions might work as obstacles to pupils' understanding. The first study reveals how an understanding of the convention of using *n* in the sense required in the task is necessary for solving the task as well as comprehending the linguistic convention of the term "expressed in *n*". A general conclusion is that learning mathematics is about learning a specific communicative genre in addition to learning about mathematical objects and relationships.

Thirdly, empirical studies show that a critical part of appropriating introductory algebra is to "see through the particular to the general" (Mason, 2008, p. 63). In the first study the teacher gave examples of how the value of *n* can vary, while the discussion indicates that the pupils interpreted the teacher's input as an invitation to choose "any number" and then decided to fix the value of *n* at that exact number. In line with Mason (1996), the pupils did not apprehend that the teacher illustrated the generality of how the value of n can vary; instead they understood the examples as complete in themselves. In the second study the pupils find the correct numeric value (30) by contextualizing the equation to a contexts of boxes and beans. This can work well as long as such reasoning is carried out within the same context, but it also shows that this does not ensure that pupils grasp the generality in the Boxes-and-beans activity. The number value 30 was seen in terms of 30 beans in each box, not as $x=30$. A general conclusion is that even if pupils are able to mobilize resources that are helpful for managing specific cases, additional problems may arise when they try to comprehend fundamental algebraic principles.

6.5 Suggestions for further research

The two studies indicate a need for further research. The mathematical conventions seem to be challenging to pupils. One suggestion is to analyze pupils' discussions when they are co-operating on a sequence of tasks involving mathematical conventions. Which conventions constitute challenges to pupils? How do pupils try to make sense of the content in the conventions? What characterizes the challenges the pupils encounter when they try together to understand the content of conventions?

The results also demonstrate that when equations are contextualized in terms of manipulatives, such as the boxes and beans, care has to be taken. Even if such contextualization can help individuals to solve equations it will not necessarily follow that they gain competence in using equations in other circumstances. In our case, pupils could calculate answers to equations but could not translate their answer to the situation that had generated the equation. The research presented here however concerns only one case. Further and more systematic research is suggested to examine in greater depth these phenomena and similar processes in other circumstances.

Chapter 7 Summary in Swedish

Elevers möte med algebraiska bokstäver, uttryck och ekvationer – en studie av smågruppsdiskussioner i åk 6

Sökord: inledande algebra, algebraiska bokstäver, algebraiska uttryck, ekvationer, smågruppsdiskussioner, laborativt material

Det övergripande syftet med licentiatavhandlingen är att få kunskap om hur elever approprierar inledande algebra och vilka utmaningar de möter. Inledande algebra omfattar i denna avhandling introduktion till formell algebra i grundskolan och berör specifikt algebraiska bokstäver, algebraiska uttryck samt ekvationer. Avhandlingen är en sammanläggning av två forskningsstudier: *What's there in an n? Investigating contextual resources in small group discussions concerning an algebraic expression* och *Moving in and out of contexts in collaborative reasoning about equations.*

I skolans matematikundervisning har algebra utpekats som en grindvakt, vilken kan bidra till att elever exkluderas från högre utbildning (Cai & Knuth, 2011). Algebraiskt tänkande har därför under senare tid förts in som ett matematikinnehåll i tidigare skolår i kursplaner och policydokument världen över, i syfte att elever ska vara bättre förberedda inför algebra i senare årskurser (Cai & Knuth, 2011). I Sverige blev algebra ett huvudmoment för senare årskurser i grundskolan först 1955 (Skolverket, 1997), men idag finns algebra som ett matematikinnehåll redan i åk 1-3 (Skolverket, 2011). Tidigare studier visar att inledande algebra har en central roll för elevers fortsatta lärande i algebra (Bednarz, Kieran & Lee, 1996) samtidigt som internationella undersökningar redovisar elevers svårigheter i algebra (Mullis, Martin, Foy & Arora, 2012; OECD, 2013). I TIMSS 2011 presterade svenska elever i åk 8 under genomsnittet i algebra inom EU/OECD (Skolverket, 2012). Vid jämförelse mellan svenska elevers resultat på algebrauppgifter i TIMSS 2007 och TIMSS 2011 fanns ingen positiv utveckling. Resultatet hade inte förändrats mellan de två mätningarna (Skolverket, 2012). Sammanfattningsvis, finns således behov av att få ytterligare kunskap om hur elever tillägnar sig inledande algebra.

Mina studier är positionerade inom den socio-kulturella traditionen och fokuserar på elever som i mindre grupper diskuterar algebrauppgifter. Det finns tre sammanvävda motiv till varför smågruppsdiskussioner valts. Ett motiv är att kommunikation är en viktig kompetens som elever ska få möjlighet att utveckla inom matematikundervisningen. Ett annat är att i kommunikativa situationer blir elevernas tolkningar explicita och därigenom möjliga att analysera. Ett tredje är att den socio-kulturella traditionen innebär ett intresse för lärande ur ett kommunikativt perspektiv. När lärande studeras från denna utgångspunkt är det inte bara den enskilde individen utan interaktionen i gruppen i en specifik situation, som är i fokus för analysen.

Det matematiska innehåll som eleverna möter – algebraiska bokstäver, algebraiska uttryck samt ekvationer – är i en socio-kulturell tradition intellektuella redskap som utvecklats under lång tid i historien (Säljö, 2000, 2015). En intressant fråga är då hur dessa redskap kan approprieras av en ny generation. I analyserna av kommunikation har en dialogisk ansats tillämpats (Linell, 1998). Det innebär att analyserna är riktade mot hur eleverna interagerar med varandra och med de intellektuella och materiella resurser som de använder för att försöka lösa de algebraiska uppgifterna. Den dialogiska ansatsen gör det möjligt att se på elevernas matematiska resonemang utifrån ett bredare perspektiv – inte bara *vad* de säger, utan också *hur* de försöker förstå. Två centrala begrepp i avhandlingen är *kontextuella resurser* och *kontextualisering*. Dessa analytiska begrepp utgör verktyg för att undersöka *varför* elever gör de olika tolkningarna och varför de kommer fram till mer eller mindre korrekta matematiska lösningar. Med kontextuell resurs avses en resurs för någon i ett speciellt syfte i en viss situation, inte något som ”finns” inneboende i objekt eller processer. Linell skiljer mellan tre dimensioner av kontextuella resurser: den omgivande fysiska situationen; vad som tidigare sagts i diskussionen samt bakgrundskunskap (Linell, 1998). I båda studierna har den dialogiska ansatsen kombinerats med analyser utifrån matematikdidaktisk forskning. I den första studien har analysen av hur eleverna tolkar den algebraiska bokstaven *n*, utgått från MacGregors och Staceys (1997) tidigare identifierade kategorier. I den andra studien har Nilssons (2009) ramverk för hur elever kontextualiserar en matematikuppgift använts för att undersöka varför och hur olika resonemang tar form och vad det innebär i termer av matematisk förståelse.

Studierna ingår i forskningsprojektet VIDEOMAT, som består av videostudier av hur elever tillägnar sig inledande algebra. Projektet genomförs i samarbete med forskargrupper i Sverige, Finland, Norge och USA (Kilhamn & Röj-Lindberg, 2013). Inom VIDEOMAT-projektet har fem konsekutiva lektioner spelats in i fem olika klasser (åk 6–7) i fyra skolor i Sverige. De fyra första lektionerna om inledande algebra har läraren genomfört utifrån ordinarie kursplan och egen planering. Den femte lektionen i serien har varit styrd av projektet och utgörs av elevers smågrupps-diskussioner kring tre uppgifter som ursprungligen kommer från TIMSS 2007, avsedda för elever i åk 8 (Foy & Olson, 2009). Två av uppgifterna ingår i mina studier. I de ursprungliga TIMSS-uppgifterna fanns fyra alternativa svar som eleverna kunde välja mellan (Appendix 1), men inom VIDEOMAT projektet har de omarbetats till frågor utan svarsalternativ i syfte att öppna upp för diskussion i grupperna. Läraren fick tillgång till elevuppgifterna först efter att hon genomfört sina egna fyra lektioner. VIDEOMAT:s syfte med den femte lektionen är att studera elevers kommunikation när de arbetar med algebrauppgifter som de inte specifikt är förberedda inför. Läraren var medveten om att hon kunde betrakta den femte lektionen som ett tillfälle för formativ bedömning och att hon själv kunde avgöra i hur hög grad hon ville intervenera i grupperna. Alla elever var informerade om att uppgifterna var avsedda för åk 8. Informerat samtycke hade inhämtats skriftligt från föräldrar och lärare samt muntligt från eleverna (Appendix 2 och 3).

Båda studiernas videoinspelade data kommer från samma klass under den femte lektionen inom VIDEOMAT-projektet. I studierna analyseras hur två olika elevgrupper tar sig an två olika algebraiska uppgifter. I Studie 1 undersöks hur en grupp av tre 12-åriga elever strävar med att tolka symbolen *n* och hur de försöker formulera ett förväntat algebraiskt uttryck (*n* – 3). Forskningsintresset är riktat mot vilka kontextuella resurser som eleverna använder sig av och hur dessa resurser leder fram till olika tolkningar av *n*. Data utgörs av elevernas smågruppsdiskussion, när de arbetade med följande uppgift under 15 minuter:

> Hasse har 3 jackor fler än Anna.
> Om *n* är antalet jackor som Hasse har, hur många jackor har Anna uttryckt i *n*?
> Skriv ett uttryck som beskriver hur många jackor Anna har uttryckt i *n*.

Analysen av hur eleverna tolkar bokstaven *n* utgår från MacGregor och Stacey (1997). Fem av sex kategorier i MacGregors and Staceys studie (1997) återkom i elevernas resonemang: *bokstaven ignorerad* (ingen bokstav alls i svaret); *numeriskt värde* (ett värde som passar i sammanhanget); *alfabetiskt värde* (*n*=14 beroende på bokstavens ordningstal i alfabetet); *olika bokstäver för varje okänt tal* (eleverna använde en annan bokstav, i detta fall “i”); *okänd kvantitet* (*n*–3). Den enda kategori som inte återfanns i vår studie var tolkningen av en algebraisk bokstav som en förkortning av ett ord.

I sammanfattningen nedan, presenteras åtta kontextuella resurser som identifierades (markerade med kursiverad stil) samt vilka olika tolkningar av *n* de gav upphov till. Viktigt att betona är att detta inte är en sammanställning av samtliga potentiella resurser, däremot de resurser som i analysen visade sig vara relevanta i elevernas problemlösningsprocess.

1. *Den omgivande fysiska situationen*

   Denna resurs avser den aktuella ”här-och-nu situationen” med dess personer, objekt och artefakter: Eleverna tog hjälp av kompetensen i den egna gruppen genom att fråga och förklara för varandra. De läste frågan ett flertal gånger, de uppmärksammade resonemanget i en annan grupp samt lyssnade på läraren (som gav exempel på hur värdet av *n* kan variera samt förslog eleverna att skriva en formel). Denna kontextuella resurs gav upphov till följande tolkningar av bokstaven *n*: alfabetiskt värde; numeriskt värde; olika bokstäver för varje okänt tal; okänd kvantitet.

2. Co-text

Resursen syftar på vad som tidigare sagts inom ”samma” ämne: En av eleverna gav inledningsvis bokstaven *n* ett värde som var identiskt med dess ordningstal i alfabetet, *n*=14. Senare i diskussionen återgick en annan elev till denna strategi och föreslog att de skulle räkna till bokstaven ”i” i alfabetet, vilket gav resultatet i=9. Om *n* =14 → kanske i = 9. Bokstaven ”i” var dock inte en algebraisk bokstav utan prepositionen ”i” i uppmaningen att de skulle skriva svaret ”uttryckt i *n*”. Denna resurs gav upphov till följande tolkning av *n*: alfabetiskt värde.

3. Bakgrundskunskap

3.1. Tidigare kunskap relaterad till matematik

När eleverna använde det kända sambandet i uppgiften (”3 jackor fler”) i aritmetiska termer, tycktes det vara enkelt för eleverna att ge korrekta exempel på den matematiska generaliseringen: Hasses antal jackor = Annas antal jackor plus 3 → Annas antal jackor = Hasses antal jackor minus 3. Överraskande var att eleverna inte förrän i slutet av diskussionen använde sin tidigare kunskap om algebraiska symboler, då de förklarade att ”*n* är som *x* och *y*”. Denna resurs gav upphov till följande tolkningar av *n*: numeriskt värde; okänd kvantitet.

3.2. Tidigare kunskap i relation till situationer utanför matematik

Eleverna såg bokstaven *n* i uppgiften och de var välbekanta med alfabetet. Försök gjordes med att använda alfabetet som en resurs för att ge *n* ett numeriskt värde. (Se även punkt 2). Denna resurs gav upphov till följande tolkning av *n*: alfabetiskt värde.

3.3. Kunskap och antaganden om "the real world"

Termen "real world" används här i motsats till "the mathematical world" med matematiska tecken och symboler. Det allra första svar som gavs i gruppen var ”2” [jackor], vilket skulle kunna ses som ett tänkbart antal jackor. Denna resurs gav upphov till följande tolkning av *n*: bokstaven ignorerad. När eleverna fick fram värdet *n*=14, fnissade de och sa ”det är inte många personer som har 14 jackor”.

3.4. Antaganden om det kommunikativa projektet (i detta fall att lösa uppgiften)

Eleverna sökte inte endast svaret på uppgiften, utan ställde även metafrågor för att försöka förstå på vilket sätt som de skulle resonera: ”Hur kom du på det?” och ”Hur ska man tänka?”. Denna resurs gav upphov till följande tolkning av *n*: okänd kvantitet.

3.5. Antaganden om det aktuella ämnet, i termer av socio-kulturella normer

Flera sociomatematiska normer kom till uttryck i deras diskussion. Eleverna tog för givet att svaret var ett tal; att frågor och svar inte behöver vara realistiska i matematik; att givna tal i uppgiften ska användas och att uppgiften har en mening och är möjlig att lösa. Denna resurs gav upphov till följande tolkning av *n*: numeriskt värde.

3.6. Kunskap och antaganden om varandra

Det var få missförstånd mellan eleverna, de tycktes förstå varandras humor och skrattade åt samma saker. Ett exempel var när de undersökte alfabetsspåret och en av eleverna till sist frågade om de skulle testa det spanska alfabetet för att ”se ifall det går bättre” och alla skrattade tillsammans åt förslaget.

Resultatet av studien illustrerar hur elever försöker förstå en algebraisk bokstav och deras inledande försök att appropriera viktiga element av vad ett algebraiskt uttryck innebär. För det första visar studien att eleverna använder en stor variation av kontextuella resurser, såväl matematiska som icke-matematiska, när de försöker tolka bokstaven *n*. För det andra framgår att merparten av MacGregors's and Stacey's (1997) identifierade tolkningar av en algebraisk bokstav kan uppstå under en förhållandevis kort tidsperiod i en och samma gruppdiskussion. Detta resultat står i kontrast till tidigare forskning, där elever tillskrivits enskilda tolkningar. Slutsatsen är att elevers tolkning av en algebraisk bokstav kan vara dynamisk och meningsskapandets karaktär kan hastigt förändras beroende på vilka kontextuella resurser som tas i anspråk. Detta visar att en tolkning inte är ett statiskt, förvärvat kunskapsobjekt, utan snarare ett nätverk av potentiella associationer. För det tredje visar studien att de mer avancerade tolkningarna är resultat av kontextuella resurser av matematisk natur. Men även om sådana resurser är potentiellt tillgängliga, så tycks inte eleverna använda dem förrän de övergett sina mer basala tolkningar. Elevernas gemensamma strävan att försöka förstå, kan därigenom vara en fruktbar väg för dem att uppnå en mer sofistikerad tolkning. För det fjärde kan noteras, att det som eleverna strävade med var inte bara ett problem som kunde lösas med hjälp av matematik eller logik, utan krävde även förståelse av den lingvistiska konventionen "uttryckt i n". Det leder fram till slutsatsen att lärande i matematik är att lära om en specifik kommunikativ genre, likaväl som att lära om matematiska objekt och relationer.

I Studie 2 analyseras hur en annan grupp av tre 12-åriga elever kontextualiserar en uppgift formulerad som en ekvation även uttryckt som en situation beskriven i ord. Särskild uppmärksamhet riktas mot vilket stöd och vilka hinder som kan identifieras, när elever använder erfarenheter av laborativt material som resurs i en ekvationslösningsprocess. I denna studie tar eleverna, på eget initiativ, hjälp av tidigare använt laborativt material (askar och bönor) men utan att ha tillgång till de fysiska objekten i sig under tiden de arbetar med uppgiften.

Elevernas diskussion videofilmades och deras resonemang analyserades utifrån ett dialogiskt perspektiv med fokus på hur eleverna kontextualiserade uppgiften (Linell, 1998; Nilsson, 2009). Data kommer från en 26-minuter lång diskussion, där eleverna arbetar med följande uppgift:

> I Zedland beräknar man hur mycket det kostar att skicka ett paket med
> formeln $y = 4x + 30$, där x är vikten i gram och y är kostnaden i zed-dollar.
> Ett paket som kostar 150 zed-dollar att skicka kan skrivas på ekvationen:
> $150 = 4x+30$
> Hur många gram väger det paketet?

I analysen identifierades tre former av hur eleverna kontextualiserade uppgiften: Zedlandkontext (resonemang om vikt och kostnad av ett paket i Zedland); Ekvationskontext (resonemang som huvudsakligen involverar algebraiska symboler); Ask-och-bönkontexten (eleverna refererar till laborativt material som de använt under tidigare matematiklektioner).

I inledningen av diskussionen tog en av eleverna utgångspunkt i ekvationen "$y = 4x + 30$" och relaterade den omedelbart till askar och bönor. Variabeln x i ekvationen kopplades till askar med bönor inuti och konstanten 30 till 30 lösa (bönor). I termer av kontextualisering, innebär det att eleven först kontextualiserar uppgiften till att handla om ekvationer och därefter rekontextualiserar uppgiften till att handla om askar och bönor.

I elevernas fortsatta lösningsprocess pendlade diskussionen inledningsvis mellan alla tre kontexter. I stora delar av diskussionen kontextualiserade de dock uppgiften till att endast vara en fråga om askar och bönor. De förflyttade sig från Ekvationskontexten till Ask-och-bönkontexten, men aldrig i motsatt riktning. Det indikerar att de inte uppfattat sambandet mellan kontexterna som en översättning i en två-vägsprocess (Filloy & Rojano, 1989). Vid de tillfällen som de lämnade Ask-och-bönkontexten förflyttade de sig direkt till Zedlandkontexten.

Elevernas slutliga svar bestod av uppritade kvadrater med talet 30 inskrivet i varje, samt texten ”Vi kom fram till att det är 30 gram i varje paket.” Deras tecknade bild och skriftliga svar kan ses som en illustration av att de avslutade inom Ask-och-bönkontexten och att de hade stora svårigheter att rekontextualisera sitt svar i termer av kostnad och vikt av ett paket.

Studien visar att användningen av laborativt material kan vara ett stöd för elever när de arbetar med ekvationens lösning. Men, det hjälpte inte eleverna att få fram det efterfrågade svaret på uppgiften. I analysen framgick att svaret 30 relaterar till ”30 bönor i varje ask”, inte till ”x = 30”. Det centrala problemet är att eleverna aldrig förflyttar sig från Ask-och-bönkontexten till Ekvationskontexten. De drar inte den generella slutsatsen att det är värdet av x som är 30, vilket gör det problematiskt för dem att använda resultatet från Ask-och-bönkontexten i andra sammanhang. Att lära en abstrakt princip genom en konkret aktivitet kräver att eleverna kan uppfatta det generella i det specifika exemplet (Mason, 2008). Elever behöver få möta åtskilliga ekvationer formulerade på olika sätt och få möjlighet att diskutera ‘vad som är exempel’ och ‘vad som är generellt’ i aktiviteten. Resultaten visar på betydelsen av att elever får möjlighet att komma till insikt om den särställning som symboliska matematiska representationer har. Medan den symboliska matematiska representationen beskriver något generellt, så relaterar en konkret representation och ”a real world example” alltid till något specifikt. Inget enskilt exempel kan införliva ett matematiskt begrepps rika innebörd. Lärare behöver uppmärksamma denna olikhet i sin undervisning.

Sammanfattningsvis kan tre generella slutsatser dras från de empiriska studierna om hur elever approprierar inledande algebra:

- Elevers tolkningar av algebraiska bokstäver kan vara dynamiska och meningsskapandets karaktär kan skifta snabbt beroende på vilka kontextuella resurser som används. Det tyder på att en tolkning inte är ett statiskt, förvärvat kunskaps-objekt, utan att den mer kan liknas vid ett nätverk av associationer.
- Matematiska konventioner kan innebära ett hinder i elevernas förståelse. Lärande i matematik är därför att lära om en specifik kommunikativ genre, likaväl som att lära om matematiska objekt och relationer.
- En kritisk del i elevernas appropriering av inledande algebra är att uppfatta vad som är exempel och vad som utgör den generella principen. Även om eleverna kan ta hjälp av resurser som stöd för att hantera specifika fall, så kan andra svårigheter tillkomma när de ska försöka förstå grundläggande algebraiska principer.

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Part two – Research articles

Article 1: What's there in an n? Investigating contextual resources in small group discussions concerning an algebraic expression

Article 2: Moving in and out of contexts in collaborative reasoning about equations

Appendix

The original tasks in TIMSS 2007

Study 1

The number of jackets that Haley has is 3 more than the number Anna has.
If n is the number of jackets Haley has, how many jackets does Anna have in terms of n?

A. $n - 3$
B. $n + 3$
C. $3 - n$
D. $3n$

Study 2

In Zedland, total shipping charges to ship an item are given by the equation $y = 4x + 30$, where x is the weight in grams and y is the cost in zeds. If you have 150 zeds, how many grams can you ship?

A. 630
B. 150
C. 120
D. 30

Appendix 2

Samtycke till elevs deltagande i forskningsstudien VIDEOMAT

Vänligen underteckna formuläret och återlämna så snart som möjligt till er matematiklärare. Vid frågor som läraren inte kan besvara vänd er till studiens kontaktpersoner, i första hand xxx.
Tel: xxx Mail: xxx

Jag och mitt barn har tagit del av information om forskningsprojektet VIDEOMAT. Vi är informerande om att deltagandet i videoinspelningarna är frivilligt och att mitt barns medverkande när som helst kan avbrytas.

☐ Ja, mitt barn har tillstånd att delta i studien. Inspelningarna får användas i studien samt i universitets forskning och lärarutbildning.

☐ Ja, mitt barn har tillstånd att delta i studien. Inspelningarna får användas i universitets forskning.

☐ Nej, mitt barn får inte medverka i studien.

Datum____

Elevens namn:____________________________________

Underskrift vårdnadshavare 1:

____

Underskrift vårdnadshavare 2:

____

Samtycke till lärares deltagande i forskningsstudien VIDEOMAT

Vänligen underteckna formuläret och lämna till xxx. Vid ytterligare frågor kontakta studiens kontaktpersoner, i första hand xxx.
Tel: xxx Mail: xxx

Jag har tagit del av information om forskningsprojektet VIDEOMAT. Jag är informerad om att deltagandet i studien är frivilligt och att medverkan när som helst kan avbrytas.

☐ Ja, jag deltar i studien. Inspelningarna får användas i studien samt i universitets forskning och utbildningar med fokus på lärande och undervisning.

☐ Ja, jag deltar i studien. Inspelningarna får endast användas i universitets forskning.

Datum:____

Namn:_____________________________________

Underskrift:____