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MASTER'S THESIS

Singular and de Rham cohomology for the Grassmannian

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Abstract

We compute the Poincaré polynomial for the complex Grassmannian using de Rham cohomology. We also construct a CW complex on the Grassmannian using Schubert cells, and then we use these cells to construct a basis for the singular cohomology. We give an algorithm for calculating the number of cells, and use this to compare the basis in singular cohomology with the Poincaré polynomial from de Rham cohomology.

We also explore Schubert calculus and the connection between singular cohomology on the complex Grassmannian and the possible triples of eigenvalues to Hermitian matrices $A + B = C$, and give a brief discussion on if and how cohomologies can be used in the case of real skew-symmetric matrices.

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Contents

1

Introduction

The idea behind cohomologies is to construct an algebraic structure (in fact a graded ring) based on a topological space such that the structure is invariant under homeomorphisms on the underlying space. If the space in question is a manifold, we also want the structure to be invariant under diffeomorphism. With such a structure on a space, one can analyze some of its geometrical, topological and analytical properties. Cohomology is one of the most – if not the most – important invariant for spaces.

The *Grassmannian* (sometimes referred to as the *Grassmann manifold*) of a vector space V is the space of all vector subspaces of a given dimension. This is very fundamental object in mathematics, used in a broad range of topics. In topology, the Grassmannian has in some contexts the role of a ”universal bundle”, i.e. a fiber bundle such that all other bundles are pullbacks of it (see [1] or [2]). It also naturally appears in enumerative geometry, a branch of algebraic geometry focusing on various counting problems; for example, how many lines in a three dimensional space intersects four given lines? In this context, we use the theory of cohomologies on the Grassmannian to arrive at the symbolic formalism known as *Schubert calculus* (see [3] or [4]).

In some areas, the appearance of the Grassmannian is more subtle. When multiplying elements in the cohomology ring of the Grassmannian, we obtain coefficients known as *Littlewood-Richardson numbers*. These numbers, originally defined using the combinatorial object *Young tableau*, also appears in a wide array of other subjects (see [5]). Perhaps most notably, these numbers were used in the proof of *Horn's conjecture*, which hypothesises about the relations between the eigenvalues of three Hermitian (or real symmetric) matrices such that one is the sum of the other two. We will go more into detail about Horn's conjecture later in this thesis.

In this thesis we approach cohomologies on the Grassmannian from two different directions. First from an analytical point of view with *de Rham* cohomologies, where we have to assume that the space we are working with is a differentiable manifold, and later from a topological point of view with *singular* cohomologies, which is more general. We then compare the two cohomologies and present a theorem that translate result from one context to the other. We will also explore some of the applications of cohomologies.

Left out from this thesis – which I would have preferred to include – is a deeper discussion about the ring structure of the cohomology of the Grassmannian. Most notably missing is the description of the cohomology ring of the Grassmannian of a complex vector field using Chern classes, which can be found in the last section of [2]. It would also be interesting to explore how the ring structure translates between de Rham and singular cohomology theory. The reason that the ring structure is important is that it gives yet another classifying structure to a space; two spaces with different ring structures on their cohomology rings can not be homeomorphic.

1.1 Outline of thesis

Chapter 2 is based on the de Rham cohomology of the Grassmannian. The first section of the chapter introduces differential forms and defines the de Rham cohomology for a manifold, and in the second section we demonstrate the theory by calculating the de Rham cohomology of the real projective space. In Sections 2.3 and 2.4 we introduce some more advanced theory, which we proceed to use in Section 2.5 by calculating the the de Rham cohomology of the complex projective space. The ring structure of the complex projective space is briefly described in the following section, 2.6.

In the last sections of Chapter 2, the cohomology of the Grassmannian of a complex vector space is described by calculating its Poincaré polynomial. This is done by first calculating this polynomial for the projectivization of complex vector spaces in Section 2.7, and then for projectivization of complex vector bundles in 2.8. In Section 2.9, we describe flag manifolds as a series of projectivization, which allow us to calculate the Poincaré polynomial for a flag bundle. Lastly, in section 2.10, the Grassmannian is defined and the previous results from the chapter is used to compute its Poincaré polynomial.

Chapter 3 is based on the geometry and the singular cohomology of the Grassmannian. In the first two sections we define what a CW complex is, and construct such a complex for the Grassmannian. In Section 3.3 we extend the cells of the complex introduced the the previous section to projective varieties. In Section 3.4 we introduce singular homology and cohomology, and relate these two with the Poincaré duality. In the last section, 3.5, we construct a basis to the singular cohomology of the Grassmannian using the varieties from Section 3.3, and give a brief discussion about the multiplication of the elements in the basis.

In Chapter 4, we give a concise description of how Horn's conjecture is related to cohomologies on the Grassmannian and speculate about the possibility of using these cohomologies on a similar problem.

In Chapter 5 we present a formula connecting de Rham cohomology with with singular cohomology. We also present an algorithm for calculating the number of Schubert cells on a given Grassmannian, and use this to compare the result from Chapter 2 and 3.

2

de Rham Cohomology

This chapter is mainly based on Bott & Tu's book *Differential forms in algebraic topology* [2]. This book is occasionally referred to as just *Bott*.

2.1 Differential forms

The goal of defining the de Rham cohomology is to categorize a manifold using a set of groups such that the categorization is invariant under diffeomorphism. This will allow us to examine geometrical and topological properties of the manifold by using algebra. This section will contain a very brief set of definitions of the most fundamental objects we will use in this chapter.

We begin by giving a purely algebraical definition of *differential forms* on $\mathbb{R}^n$. Let $x_1, \cdots, x_n$ be the linear coordinates on $\mathbb{R}^n$, and let Ω^* be the algebra over $\mathbb{R}$ generated by the symbols $dx_1, \cdots, dx_n$ such that its multiplication $\wedge$ satisfies the relations

$$\begin{cases} dx_i \wedge dx_i &= \quad 0 \\ dx_i \wedge dx_j &= \quad -dx_j \wedge dx_i, \ i \neq j. \end{cases}$$

Thus, as a vector space over $\mathbb{R}$, Ω^* has dimension 2^n with the basis

$$1, \quad dx_i, \quad dx_i \wedge dx_j, \quad dx_i \wedge dx_j \wedge dx_k, \quad \cdots, \quad dx_1 \wedge \cdots \wedge dx_n,$$

where the indexes are in increasing order.

We define a C^∞ differential form ω on R^n to be an element of

$$\Omega^*(\mathbb{R}^n) \quad = \quad \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*,$$

that is, ω is the unique expression $\sum f_{i_1 \cdots i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_q}$, where the coefficients $f_{i_1 \cdots i_q}$ are c^∞ functions on $\mathbb{R}^n$. For an easier notation, we will write $\omega = \sum f_I \ dx_I$, where $I = \{i_1, \cdots, i_q\}$ is an index set.

We extend the multiplication $\wedge$ to $\Omega^*(\mathbb{R}^n)$ by defining the *wedge product* $\omega \wedge \tau$ of two differential forms $\omega = \sum f_I \, dx_I$ and $\tau = \sum g_J \, dx_J$ as

$$\omega \wedge \tau \quad = \quad \sum f_I \, g_J \, dx_I \wedge dx_J.$$

We say that a differential from ω is a *q-form* if every nonempty term in ω has an index set with q elements. Let $\Omega^q(\mathbb{R}^n)$ be the subset of all q-forms. This endows a natural grading $\Omega^*(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n)$. We define a differential operator

$$d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$$

by first defining it on functions $f \in \Omega^0(\mathbb{R}^n)$ as

$$df \quad = \quad \sum \frac{\partial f}{\partial x_i} \, dx_i,$$

and then on general forms $\omega = \sum f_I \, dx_I$ as

$$d\omega \quad = \quad \sum df_I \, dx_I.$$

We call d the *exterior derivative.* Note that it is only really in the definition of d for functions that we use any analysis.

An important property of the exterior derivative d, which is easily proved by direct calculations, is that $d^2 = 0$. In fact, in a more general context, the property that the square of the operator is zero is what defines a differential operator. We say that a differential operator, along with its graded modules, is a *cochain complex* if the operator *increases* the grade by one. The specific complex $\Omega^*(\mathbb{R}^n)$ is called the *de Rham complex* on $\mathbb{R}^n$.

A q-form ω with the property that $d\omega = 0$ is called *closed*, and if there exist a $(q-1)$-form τ such that $\omega = d\tau$, then ω is called *exact.* Thus, the set of all closed forms is the kernel of d, while the set of all exact forms is the image of d. Since $d^2(\omega) = 0$, every exact form is closed. The object we are interested in is the quotient of the closed forms with the exact forms; let d_q be the exterior derivative mapping q-forms to $(q+1)$-forms, we define the q^{th} *de Rham cohomology* of $\mathbb{R}^n$ to be the vector space

$$H^q_{dR}(\mathbb{R}^n) \quad = \quad \ker d_q / \mathrm{im}\, d_{q-1}. \tag{2.1}$$

We will sometimes omit "dR" and just denote the de Rham cohomology with H^q when it is clear from the context what we refer to.

Since we want to use cohomologies to categorize all kind of manifolds, we need to extend the definition of H^q_{dR} to be usable on more than just $\mathbb{R}^n$. We will not go through the basics of manifold theory, all we really need here is that a *differentiable n-manifold* M is a Hausdorff space with a differentiable structure given by an *atlas*, that is, an open cover $\{U_\alpha\}_{\alpha \in A}$ of M where each U_α is homeomorphic to $\mathbb{R}^n$ through a homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$, and where each intersection $U_\alpha \cap U_\beta$ has a transition function

$$g_{\alpha\beta} \quad = \quad \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

which is a diffeomorphism of open subset of $\mathbb{R}^n$. For a more thoroughly review of manifolds and differential forms, consider [6].

Now, since a differentiable n-manifold M is basically a space that locally looks as $\mathbb{R}^n$, we can apply analysis on functions defined on the manifold. We say that a function $f : M \to \mathbb{R}^n$ is *differentiable* if $f \circ \phi_\alpha^{-1}$ is a differentiable function on $\mathbb{R}^n$ for every ϕ_α. Let $u_1, \cdots, u_n$ be the standard coordinates on $\mathbb{R}^n$, and let $x_1, \cdots, x_n$, where $x_i = u_i \circ \phi_\alpha$, be a coordinate system on U_α. The partial derivative $\partial f/\partial x_i$ at a point $p \in U_\alpha$ of a differentiable function f on M is defined as

$$\frac{\partial f}{\partial x_i}(p) \quad = \quad \frac{\partial(f \circ \phi_\alpha^{-1})}{\partial u_i}(\phi_\alpha(p)).$$

Since the definitions of the differential forms, the wedge product and the exterior derivative for $\mathbb{R}^n$ work just as good on open subsets of $\mathbb{R}^n$, we can now define a differential form ω_{U_α} along with a differentiable operator d on U_α completely analogous as for $\mathbb{R}^n$, using the definition of the partial derivative above.

Let $i_\alpha : U_\alpha \cap U_\beta \to U_\alpha$ be the inclusion map, and let $i_\alpha^* : \Omega^*(U_\alpha) \to \Omega^*(U_\alpha \cap U_\beta)$ be its induced *pullback* map, defined as

$$i^*\big(\sum f_I \, dx_I\big) \quad = \quad \sum f_I \circ i \, dx_I.$$

We define a *differential form* ω on M to be a collection of forms ω_{U_α} for each U_α in the atlas such that $i_\alpha^* \omega_{U_\alpha} = i_\beta^* \omega_{U_\beta}$ in $\Omega^*(U_\alpha \cap U_\beta)$. Extending the exterior derivative and wedge product to manifolds, we define $H^q_{dR}(M)$ analogous with (2.1).

We denote by T_pM the *tangent space* to M at point p. The tangent space is defined as the vector space over $\mathbb{R}$ spanned by the operators $\partial/\partial x_1(p), \cdots, \partial/\partial x_n(p)$. A smooth *vector field* on U_α is a linear combination $X_\alpha = \sum f_i \partial/\partial x_1$, where the f_i's are smooth functions on U_α, and a C^∞ vector field on M is a collection of vector fields X_α which agree on every intersection $U_\alpha \cap U_\beta$. In other words, a vector field associates every point p on a manifold M to an element in their tangent space T_pM.

Remark 2.1.1. *There is a natural pairing between differential forms and vector fields on a manifold, which allow us to define a dual between them. In fact, some literature (e.g. [6]) use this dual relationship in the very definition of differential forms. In this thesis however, we will only use this property in Section 2.2. Hence the purely algebraical definition of differential forms above.*

With the de Rham cohomology defined for a manifold M, we would like some basic results from it. We state the following theorems without proofs; for details, see Bott.

Theorem 2.1.2. *If two manifolds have the same homotopy type, then their de Rham cohomologies are isomorphic.*

Theorem 2.1.3. *The de Rham cohomology of $\mathbb{R}^n$ is*

$$H^q(\mathbb{R}^n) \quad = \quad H^q(point) \quad = \quad \begin{cases} \mathbb{R} & \text{for } q = 0, \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.1.4. *The de Rham cohomology of* $\boldsymbol{S}^n$ *is*

$$H^q(\boldsymbol{S}^n) = \begin{cases} \mathbb{R} & \text{for } q = 0,\, n, \\ 0 & \text{otherwise} \end{cases}$$

Remark 2.1.5. *In the following chapters, we will sometimes assume that our n-manifold is* orientable*. There are several equivalent definitions of this, and the details are not of much importance. It is enough to assume that this means that the manifold has a global nowhere vanishing n-form.*

2.2 Real projective spaces

This thesis will focus on the cohomologies for the *Grassmannian*, which we will define later in this chapter. In this section however, we will consider the perhaps most basic example of a Grassmannian: the *real projective space*.

The calculations in this section, unlike the rest of the thesis, is based on the differential geometry approach to differential forms, where the duality to vector fields is used. We will denote by $\omega_q(w_1, \cdots, w_k)$ the pairing (evaluation) of a k-form ω with k vector fields at a point q on the manifold. For a smooth map $f : M \to N$ between two manifolds, let $(f_*)_p : T_pM \to T_{f(p)}N$ denote its induced *pushforward* map at point p, which we define by generalizing the Jacobian matrix we have in the special case where $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ (see [6]). The pullback map $f^* : \Omega^*(N) \to \Omega^*(M)$ is now defined by how the image $f^*\omega$ of a k-form $\omega \in \Omega^*(N)$ is evaluated at a point p with vector fields $X_1, \cdots, X_k$:

$$(f^*\omega)_p\big(X_1, \cdots, X_k\big) = \omega_{f(p)}\big((f_*)_pX_1, \cdots, (f_*)_pX_k\big).$$

Let $\mathbf{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and i the antipodal map on $\mathbf{S}^n$:

$$i : (x_1, \cdots, x_{n+1}) \mapsto (-x_1, \cdots, -x_{n+1}).$$

The quotient we get by identifying $x \in \mathbf{S}^n$ with its image $i(x)$ is the *real projective space* $\mathbb{R}\mathbf{P}^n$.

Theorem 2.2.1. *The de Rham cohomology of* $\mathbb{R}\boldsymbol{P}^n$ *is*

$$H^q(\mathbb{R}\boldsymbol{P}^n) = \begin{cases} \mathbb{R} & \text{for } q = 0, \\ 0 & \text{for } 0 < q < n, \\ \mathbb{R} & \text{for } q = n \text{ odd}, \\ 0 & \text{for } q = n \text{ even}. \end{cases}$$

Proof. Let i be defined as above, and let $i^* : \Omega^*(\mathbf{S}^n) \to \Omega^*(\mathbf{S}^n)$ be the pullback of i. We say that a form ω is *invariant* if $i^*\omega = \omega$, and denote the set of all invariant forms on $\mathbf{S}^n$ with $\Omega^*(\mathbf{S}^n)^I$. Since $(i^*)^2 = I$, i splits $\Omega^*(\mathbf{S}^n)$ into the eigenspaces $\Omega^*(\mathbf{S}^n) = \Omega^*(\mathbf{S}^n)_+ \oplus \Omega^*(\mathbf{S}^n)_-$, where d of course respects the decomposition. But by construction, $\Omega^*(\mathbf{S}^n)^I = \Omega^*(\mathbf{S}^n)_+$, so $\Omega^*(\mathbf{S}^n)^I$ is in fact a differential complex. We first prove that $\Omega^*(\mathbf{S}^n)^I \cong \Omega^*(\mathbb{R}\mathbf{P}^n)$.

Let $\pi^* : \Omega^*(\mathbb{R}\mathbf{P}^n) \to \Omega^*(\mathbf{S}^n)$ be the pullback of the projection $\pi : \mathbf{S}^n \to \mathbb{R}\mathbf{P}^n$. From the equality $\pi \circ i = \pi$, we get that $i^* \circ \pi^* = \pi^*$, which shows that

$$\pi^*(\Omega^*(\mathbb{R}\mathbf{P}^n)) \subseteq \Omega^*(\mathbf{S}^n)^I.$$

Now, for $\eta \in \Omega^k(\mathbf{S}^n)$ and $\omega \in \Omega^k(\mathbb{R}\mathbf{P}^n)$, $0 \leq k \leq n$, we have that $\pi^*\omega = \eta$ if and only if $\omega_q(w_1, \cdots, w_k) = \eta_p(v_1, \cdots, v_k)$ for all $p \in \mathbf{S}^n$ such that $\pi(p) = q$ and for all $v_i \in \mathbf{S}^n_p$ such that $(\pi_*)_p v_i = w_i$, where $\mathbf{S}^n_p$ denotes the tangent space for $\mathbf{S}^n$ at p.

Let $\pi^*\omega = \eta$. We want to know which η have a uniquely defined ω. For every $q, w_1, \cdots, w_k$, let $\pi(p) = q$ and $\pi_{*p} v_i = w_i$. We now have that $\omega_q(w_1, \cdots, w_k) = \eta_p(v_1, \cdots, v_k)$, but we also have that $\omega_q(w_1, \cdots, w_k) = \eta_{i(p)}(i_* v_1, \cdots, i_* v_k)$. Hence η determines ω uniquely if and only if

$$\eta_p(v_1, \cdots, v_k) \quad = \quad \eta_{i(p)}(i_* v_1, \cdots, i_* v_k) \quad = \quad (i^*\eta)_p(v_1, \cdots, v_k),$$

that is, if and only if $\eta \in \Omega^k(\mathbf{S}^n)^I$. So π^* is the sought isomorphism.

Since π^* commutes with d, we also have that $H^*(\mathbf{S}^n)^I \cong H^*(\mathbb{R}\mathbf{P}^n)$. The next step of the proof is to prove that the the natural map $H^*(\mathbf{S}^n)^I \to H^*(\mathbf{S}^n)$, induced from the inclusion map $\Omega^*(\mathbf{S}^n)^I \to \Omega^*(\mathbf{S}^n)$, is injective.

Let ω be an invariant form and let $\omega = d\tau$ for some form τ on $(S)^n$, that is, let ω be mapped to zero in $H^*(\mathbf{S}^n)$. We want to prove that $\omega = d\eta$ for some invariant form η. Since $(i^*)^2 = I$, $\frac{\tau + i^*\tau}{2}$ is an invariant form, and since we also have that

$$d\frac{\tau + i^*\tau}{2} \quad = \quad \frac{d\tau + i^* d\tau}{2} \quad = \quad \frac{\omega + \omega}{2} \quad = \quad \omega,$$

we can take $\eta = \frac{\tau + i^*\tau}{2}$ and thus prove the injectivity.

Let $f : \mathbf{S}^n \to \mathbb{R}^{n+1}$ be the inclusion map. If ω is the standard volume form on $\mathbf{S}^n$, then there exists a form τ on $\mathbb{R}^{n+1}$ such that $\omega = f^*\tau$. This form is given by

$$\tau \quad = \quad \sum_{j=1}^{n+1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{n+1}$$

Since we can naturally extend the antipodal map to $\bar{i} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $f \circ i = \bar{i} \circ f$, we see that $i^*\omega = f^*\bar{i}^*\tau = (-1)^{n+1}\omega$ (i.e., the antipodal map is orientation-preserving for n odd and orientation-reversing for n even). Together with the isomorphism and injectivity proven above, the theorem now follows from the de Rham cohomology of $\mathbf{S}^n$. □

2.3 Fiber bundles

To compute the de Rham cohomology of $\mathbb{C}\mathbf{P}^n$, we will first introduce some more advanced theory. We begin by defining the notion of *fiber bundle*. The following definitions are taken from Bott.

Let G be a topological group acting on a space F on the left. A surjection $\pi : E \to B$ between topological spaces is a *fiber bundle* with *fiber* F and *structure group* G if B has an open cover $\{U_\alpha\}$ such that there are fiber-preserving homeomorphisms

$$\phi_\alpha : E\big|_{\pi^{-1}(U_\alpha)} \to U_\alpha \times F,$$

and the transitions functions are continuous functions with values in G:

$$g_{\alpha\beta}(x) \quad = \quad \phi_\alpha \phi_\beta^{-1}\big|_{\{x\}\times F} \in G.$$

If $x \in B$, the set $E_x = \pi^{-1}(x)$ is called the *fiber at x*. We will usually be a bit informal and refer to the *total space E* as the fiber bundle.

A *real vector bundle of rank n* is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $GL(n, \mathbb{R})$. Similarly, a *complex* vector bundle of rank n is a fiber bundle with fiber $\mathbb{C}^n$ and structure group $GL(n, \mathbb{C})$.

Let $f : N \to M$ be a map between manifolds M and N, and let $\pi : E \to M$ be a vector bundle over M. The *pullback* $f^{-1}E$ of E by f is a vector bundle on N induced by f. This bundle is defined to be the subset of $N \times E$ given by

$$\big\{(n, e) \,\big|\, f(n) = \pi(e)\big\}.$$

2.4 Čech-de Rham complexes

As we mentioned in the introduction, a differential complex is a general notion which includes more than just the de Rham complex. We will in this section define another differential operator, which we will denote by δ, acting on the set of differential forms defined on the *intersections* of open sets on a manifold.

Let M be a manifold, and let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in J}$ be a open cover of M, where the index set J is a countable ordered set. Denoting the intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ by $U_{\alpha_0\cdots\alpha_p}$, there is a sequence of inclusions of open sets

$$M \leftarrow \coprod U_{\alpha_0} \; \begin{smallmatrix} \xleftarrow{\partial_0} \\ \xleftarrow{\partial_1} \end{smallmatrix} \; \coprod_{\alpha_0<\alpha_1} U_{\alpha_0\alpha_1} \; \begin{smallmatrix} \xleftarrow{\partial_0} \\ \xleftarrow{\partial_1} \\ \xleftarrow{\partial_2} \end{smallmatrix} \; \coprod_{\alpha_0<\alpha_1<\alpha_2} U_{\alpha_0\alpha_1\alpha_2} \; \begin{smallmatrix} \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{smallmatrix} \; \cdots$$

where ∂_i is the inclusion which discards the index α_i from $U_{\alpha_0\cdots\alpha_i\cdots\alpha_p}$; for example,

$$\partial_0 : U_{\alpha_0\alpha_1\alpha_2} \hookrightarrow U_{\alpha_1\alpha_2}.$$

This sequence of inclusions of opens sets induces a sequence of restrictions of differential forms

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \begin{array}{c}\xrightarrow{\delta_0}\\ \xrightarrow{\delta_1}\end{array} \prod_{\alpha_0<\alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \begin{array}{c}\xrightarrow{\delta_0}\\ \xrightarrow{\delta_1}\\ \xrightarrow{\delta_2}\end{array} \prod_{\alpha_0<\alpha_1<\alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} \cdots$$

where r is the ordinary restriction and δ_i is the restriction induced from the inclusion (hat denoting omission of the index)

$$\partial_i : \coprod_{\alpha_i \in J} U_{\alpha_0\cdots\alpha_i\cdots\alpha_p} \to U_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_p},$$

i.e.

$$\delta_i : \Omega^*(U_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_p}) \to \prod_{\alpha_i \in J} \Omega^*(U_{\alpha_0\cdots\alpha_i\cdots\alpha_p}).$$

We define the difference operator $\delta : \prod \Omega^*(U_{\alpha_0\cdots\alpha_p}) \to \prod \Omega^*(U_{\alpha_0\cdots\alpha_{p+1}})$ to be the alternating difference

$$\delta \quad = \quad \sum (-1)^i \delta_i.$$

To make the definition clearer, we denote by $\omega_{\alpha_0\cdots\alpha_p} \in \Omega^q(U_{\alpha_0\cdots\alpha_p})$ the q-form component of $\omega \in \prod \Omega^q(U_{\alpha_0\cdots\alpha_p})$ on $U_{\alpha_0\cdots\alpha_p}$. We now have that

$$(\delta\omega)_{\alpha_0\cdots\alpha_{p+1}} \quad = \quad \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}},$$

where the forms on the right-hand side are restricted to $U_{\alpha_0\cdots\alpha_{p+1}}$.

We have that δ is indeed a differential operator, that is, $\delta^2 = 0$, and that the sequence

$$0 \longrightarrow \Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \cdots,$$

called the *generalized Mayer-Vietoris sequence*, is exact. For proofs of these facts, consult Bott.

We will now generalize the idea of differential complexes to *double complexes*. If a differential complex is represented by a chain of vector spaces with a differential operator d, the double complex is represented by a two-dimensional grid with two differential operators d and δ, where d acts vertically and δ acts horizontally. We will consider the case where d is our normal exterior derivative and where δ is the difference operator defined above for a given cover $\mathfrak{U}$ on a manifold M. We will denote this double complex $C^*(\mathfrak{U}, \Omega^*)$.

Given a double complex, we can make it into a single complex by defining the differential operator

$$D \quad = \quad D' + D'' \quad = \quad \delta + (-1)^p d,$$

where p is the order of the column, starting from zero. The singly graded components of the double complex with respect to D are the diagonals. Applying this to $C^*(\mathfrak{U}, \Omega^*)$, we get the *Čech-de Rham complex.*

The closed forms in $\prod \Omega^0(U_{\alpha_0 \cdots \alpha_p})$ are the locally constant functions on $\coprod U_{\alpha_0 \cdots \alpha_p}$, i.e. arrays of real numbers representing the values of the functions on the disjoint set of intersections $U_{\alpha_0 \cdots \alpha_p}$, and we naturally denote this set $C^p(\mathfrak{U}, \mathbb{R})$. Together with δ, $C^p(\mathfrak{U}, \mathbb{R})$ is a differential complex itself, and its cohomology $H^p(\mathfrak{U}, \mathbb{R})$ is called the *Čech cohomology of the cover* $\mathfrak{U}$. In the same way the de Rham complex of M can be placed to the left of the double complex, mapped to the first column by the restriction r, the Čech complex can be placed on the bottom of the double complex, mapped to the first row by the inclusion i:

$$
\begin{array}{r|c|c|c}
0 \to \Omega^2(M) \xrightarrow{r} & \prod \Omega^2(U_{\alpha_0}) & \cdots & \cdots \\
0 \to \Omega^1(M) \xrightarrow{r} & \prod \Omega^1(U_{\alpha_0}) & \cdots & \cdots \\
0 \to \Omega^0(M) \xrightarrow{r} & \prod \Omega^0(U_{\alpha_0}) & \prod \Omega^0(U_{\alpha_0\alpha_1}) & \prod \Omega^0(U_{\alpha_0\alpha_1\alpha_2}) \\
\cline{2-4}
& i \uparrow & i \uparrow & i \uparrow \\
& C^0(\mathfrak{U}, \mathbb{R}) \longrightarrow & C^1(\mathfrak{U}, \mathbb{R}) \longrightarrow & C^2(\mathfrak{U}, \mathbb{R}) \longrightarrow \\
& \uparrow & \uparrow & \uparrow \\
& 0 & 0 & 0
\end{array}
$$

The cohomology $H_D(C^*(\mathfrak{U}, \Omega^*))$ of the Čech-de Rham complex is in fact isomorphic to $H^*_{dR}(M)$, which follows from the horizontal exactness of the double complex (i.e. the exactness of the Mayer-Vietoris sequence). Moreover, if the cover $\mathfrak{U}$ is *good*, we also have an isomorphism between the cohomology $H_D(C^*(\mathfrak{U}, \Omega^*))$ of the double complex and the Čech cohomology $H^p(\mathfrak{U}, \mathbb{R})$. "Good" here refers to the property that every nonempty finite intersection $U_{\alpha_0 \cdots \alpha_p} \in \mathfrak{U}$ should be diffeomorphic to $\mathbb{R}^n$, which means that the cohomology $H^q(U_{\alpha_0 \cdots \alpha_p})$ is trivial for $q \geq 1$, and thus that we also have *vertical* exactness in the double complex.

Combining the isomorphisms above, we get an isomorphism between the de Rham cohomology and the Čech cohomology for a good cover. Similarly, given a fiber bundle $\pi : E \to M$ and a good cover $\mathfrak{U}$ on M, then $\pi^{-1}\mathfrak{U}$ is a good cover of E and we can use the Čech-de Rham complex to prove the isomorphism

$$H^*_{dR}(E) \quad \cong \quad H^*_{dR}(M). \tag{2.2}$$

For formal proofs, again consult Bott.

Let K be a differential complex with differential operator D. A *subcomplex* K' of K is a subgroup such that $DK' \subset K'$. A sequence of subcomplexes

$$K \quad = \quad K_0 \supset K_1 \supset K_2 \supset K_3 \supset \cdots$$

is called a *filtration* on K. With a given a filtration, K is a *filtered complex*, with *associated graded complex*

$$GK \quad = \quad \bigoplus_{P=0}^{\infty} K_p / K_{p+1}$$

The only filtration we will consider is

$$K_p \quad = \quad \bigoplus_{i \geq p} \bigoplus_{q \geq 0} K^{i,\,q}$$

on the double complex $K = \bigoplus K^{i,\,q}$, where i stands for the order of the column and q for the order of the row. We can picture this as K_1 excludes the first column, K_2 excludes the two first columns etc.

A sequence of differential groups $\{E_r,\, d_r\}$ in which each E_r is the homology of its predecessor E_{r-1}, i.e.

$$\begin{aligned} E_n &= H(E_{n-1}) \text{ with differential } d_n, \\ E_{n+1} &= H(E_n) \text{ with differential } d_{n+1}, \\ E_{n+2} &= H(E_{n+1}), \text{ etc}, \end{aligned}$$

is called a *spectral sequence*. If E_r eventually becomes stationary, we denote the stationary value by E_∞, and if E_∞ is equal to the associated graded group of some filtered group H, then we say that the spectral sequence *converges* to H. A spectral sequence is said to *degenerate at the E_r term* if $d_r = d_{r+1} = \cdots = 0$, and for such a sequence we have that $E_r = E_{r+1} = \cdots = E_\infty$.

Using the filtration on the double complex defined above, we can make $B = \bigoplus K_p/K_{p+1}$ into a single complex with the differential operator D induced from K. Since the image of δ is zero on K_p/K_{p+1} for every p, the induced operator on B is in fact $(-1)^p d$, and thus we have

$$E_1 \quad = \quad H_D(B) \quad = \quad H_d(K).$$

Without going into the details, we can derive a new differential operator on E_1 and thus create a spectral sequence $\{E_r,\, d_r\}$. This sequence converge to the total cohomology $H_D(K)$, and each E_r has a bigrading on which the differential act as

$$d_r \;:\; E_r^{p,\,q} \to E_r^{p+r,\,q-r+1}.$$

For the first two elements of the sequence, we have that

$$\begin{aligned} E_1^{p,\,q} &= H_d^{p,\,q}(K), \\ E_2^{p,\,q} &= H_\delta^{p,\,q} H_d(K). \end{aligned}$$

We have already limited ourself by only considering double complexes, but we can in fact be even more specific and only consider the Čech-de Rham complex $K = C^*(\mathfrak{U},\, \Omega^*)$. Since K is a vector space, the associated graded complex of $H_D^*(K)$ with the above filtration is isomorphic to $H_D^*(K)$ itself. Moreover, for the spectral sequence $\{E_r,\, d_r\}$ associated with this filtration, we have

$$H_D^n(K) \quad \cong \quad GH_D^n(K) \quad \cong \quad \bigoplus_{p+q=n} E_\infty^{p,\,q}.$$

Let $\pi : E \to M$ be a fiber bundle with fiber F over a manifold M and let $\mathfrak{U}$ be a good cover of M. Since $\pi^{-1}\mathfrak{U}$ is a cover of E, we can form the double complex

$$K^{p,\,q} \quad = \quad C^p(\pi^{-1}\mathfrak{U}, \Omega^q) \quad = \quad \prod_{\alpha_0<\ldots<\alpha_p} \Omega^q(\pi^{-1}U_{\alpha_0<\ldots<\alpha_p}),$$

whose E_1 term in the spectral sequence $\{E_r, d_r\}$ is

$$E_1^{p,\,q} \quad = \quad H_d^{p,\,q} \quad = \quad \prod_{\alpha_0<\ldots<\alpha_p} H^q(\pi^{-1}U_{\alpha_0<\ldots<\alpha_p}) \quad = \quad C^p(\mathfrak{U}, \mathscr{H}^q),$$

where $\mathscr{H}^q$ is the *presheaf* $\mathscr{H}^q(U) = H^q(\pi^{-1}U)$ on M. We will not go into detail what a presheaf is, more than that it is a function on a topological space which assigns to every open set an abelian group (Bott gives a definition at pages 108-109).

From Section 13 in Bott, we have that since $\mathfrak{U}$ is a good cover, $\mathscr{H}^q$ is a locally constant presheaf with group $H^q(F)$. Moreover, if M is simply connected and $H^q(F)$ is finite-dimensional, then $\mathscr{H}^q$ is in fact the constant presheaf $\mathbb{R} \oplus \cdots \oplus \mathbb{R}$ on $\mathfrak{U}$, where the number of copies of $\mathbb{R}$ equals the dimension of $H^q(F)$. In this case, we have that the E_2 term is

$$\begin{aligned} E_2^{p,\,q} \quad &= \quad H_\delta^p(\mathfrak{U}, \mathscr{H}^q) \\ &= \quad H_\delta^p(\mathfrak{U}, \mathbb{R} \oplus \cdots \oplus \mathbb{R}) \\ &= \quad H_\delta^p(\mathfrak{U}, \mathbb{R}) \otimes H^q(F) \\ &= \quad H^p(M) \otimes H^q(F). \end{aligned}$$

By construction, the spectral sequence converges to $H_D^*(K)$. Since $\pi^{-1}\mathfrak{U}$ is a cover of E, we also have that $H_D^*(K) = H^*(E)$.

2.5 Complex projective spaces

Using the spectral sequence for the double complex of a fiber bundle defined in the previous section, we can now compute the de Rham cohomology of the complex projective space.

Consider the sphere

$$S^{2n+1} \quad = \quad \{(z_0, ..., z_n) \mid |z_0|^2 + ... + |z_n|^2 = 1\}$$

in $\mathbb{C}^{n+1}$. Let S^1 act on S^{2n+1} by

$$(z_0, ..., z_n) \mapsto (\lambda z_0, ..., \lambda z_n),$$

where $\lambda \in S^1$ is a complex number of absolute value 1. The quotient of S^{2n+1} by this action is the complex projective space $\mathbb{C}\mathbf{P}^n$. This gives S^{2n+1} the structure of a circle bundle over $\mathbb{C}\mathbf{P}^n$, represented by the short exact sequence

$$0 \to S^1 \to S^{2n+1} \to \mathbb{C}\mathbf{P}^n \to 0.$$

Theorem 2.5.1. *The de Rham cohomology of* $\mathbb{C}\boldsymbol{P}^n$ *is*

$$H^k(\mathbb{C}\boldsymbol{P}^n) = \begin{cases} \mathbb{R} & \textit{for } k = 0, 2, 4, ..., 2n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since $\mathbb{C}\mathbf{P}^n$ is simply connected, we have with the above circle bundle that

$$E_2^{p,\,q} = H^p(\mathbb{C}\mathbf{P}^n) \otimes H^q(S^1).$$

So E_2 has only two nonzero rows, $q = 0,\ 1$, both being equal to $H^*(\mathbb{C}\mathbf{P}^n)$. We also have that the columns $E_2^{p,\,*}$ is zero for $p \geq 2n+1$, because the dimension of $\mathbb{C}\mathbf{P}^n$ is $2n$, and that the first column $E_2^{1,\,*}$ is R, since $\mathbb{C}\mathbf{P}^n$ is connected. Summing up with a picture:

$$E_2 = \begin{array}{|c|c|c|c|c|c|c|} \mathbb{R} & X_1 & X_2 & ... & X_{2n} & 0 & ... \\ \mathbb{R} & X_1 & X_2 & ... & X_{2n} & 0 & ... \\ \hline \end{array}$$
$$\begin{array}{ccccccc} 0 & 1 & 2 & ... & 2n & 2n+1 & ... \end{array}$$

Since d_i maps down two or more rows for $i \geq 3$, we have that

$$d_3 = d_4 = \cdots = 0.$$

So the spectral sequence degenerates at the E_3 term and $E_3 = E_4 = \cdots = E_\infty = H^*(S^{2n+1})$. Since $E_3 = H_{d_2}(E_2)$, the columns $E_2^{p,\,*}$ is also zero for $p \geq 2n+1$, and together with the equality $H^k(S^{2n+1}) = \bigoplus_{p+q=k} E_3^{p,\,q}$ we get

$$E_3 = \begin{array}{|c|c|c|c|c|c|c|} 0 & 0 & 0 & ... & \mathbb{R} & 0 & ... \\ \mathbb{R} & 0 & 0 & ... & 0 & 0 & ... \\ \hline \end{array}$$
$$\begin{array}{ccccccc} 0 & 1 & 2 & ... & 2n & 2n+1 & ... \end{array}$$

The mapping $d_2 : E_2^{p,\,q} \to E_2^{p+2,\,q-1}$ is in fact an isomorphism when restricted to $E_2^{p,\,1}$ for $p = 0, 1, ..., 2n-1$, since we from the zeros in the E_3 table above get the exact sequences

$$0 \to E_2^{p,\,1} \to E_2^{p+2,\,0} \to 0.$$

This gives us that

$$\mathbb{R} = X_2 = X_4 = ... = X_{2n}.$$

But we also have the exact sequence $0 \to E_2^{1,\,0} \to 0$, which gives us that $0 = E_2^{1,\,0} = X_1$, so

$$0 = X_1 = X_3 = ... = X_{2n-1}$$

and we are done. ☐

Remark 2.5.2. *Comparing the cohomologies of* $\mathbb{R}\boldsymbol{P}^k$ *and* $\mathbb{C}\boldsymbol{P}^n$*, we see that we get non-trivial cohomologies on* $\mathbb{C}\boldsymbol{P}^n$ *other than the top and the bottom, unlike the case of* $\mathbb{R}\boldsymbol{P}^k$*. Notice the difference in the construction of these two spaces;* $\mathbb{C}\boldsymbol{P}^n$ *is the set of* complex *lines in* $\mathbb{C}^n$*. The space* $\mathbb{C}\boldsymbol{P}^n$ *is thus* not *a special case of* $\mathbb{R}\boldsymbol{P}^k$*, achieved by just doubling the dimensions.*

2.6 Ring structures

For the wedge product $\wedge$ on $\Omega^*(M)$, the Liebniz's rule holds

$$d(\omega \wedge \eta) \quad = \quad (d\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge (d\eta),$$

hence the wedge product makes the de Rham cohomology $H^*(M)$ into a graded algebra. Given a manifold M, it is thus interesting to investigate the ring structure of $H^*(M)$.

Using the wedge product, we can construct a more general product

$$\cdot \ : \ C^p(\mathfrak{U}, \Omega^q) \otimes C^r(\mathfrak{U}, \Omega^s) \to C^{p+r}(\mathfrak{U}, \Omega^{q+s})$$

on the whole double complex $C^*(\mathfrak{U}, \Omega^*)$. Let $\omega \in C^p(\mathfrak{U}, \Omega^q)$ and let $\eta \in C^r(\mathfrak{U}, \Omega^s)$, their product is defined to be

$$(\omega \cdot \eta)(U_{\alpha_0 \ldots \alpha_{p+r}}) \quad = \quad (-1)^{qr} \omega(U_{\alpha_0 \ldots \alpha_p}) \wedge \eta(U_{\alpha_p \ldots \alpha_{p+r}}), \tag{2.3}$$

where on the right-hand side both forms are restricted to $U_{\alpha_0 \ldots \alpha_{p+r}}$, with the convention that $\alpha_0 < \cdots < \alpha_{p+r}$. Let $\mathfrak{U}$ be a good cover, the isomorphism between the Čech-de Rham cohomology $H^*(\mathfrak{U}, \Omega^*)$ and both the de Rham cohomology $H^*(M)$ and the Čech cohomology $H^p(\mathfrak{U}, \mathbb{R})$ are then also *algebra* isomorphisms, where the product on the Čech cohomology of $\mathfrak{U}$ is naturally induced from (2.3).

Using the product $\cdot$ defined above and the spectral sequences introduced in the previous section, we can compute the ring structure of $H^*(\mathbb{C}\mathbf{P}^n)$. Note that the differential operator d_r in the spectral sequence $\{E_r, d_r\}$ follows the Liebniz's rule relative to the double complex product.

Theorem 2.6.1. *The ring structure of* $H^*(\mathbb{C}\boldsymbol{P}^n)$ *is*

$$H^*(\mathbb{C}\boldsymbol{P}^n) \quad = \quad \mathbb{R}[x]/(x^{n+1}),$$

where x *is a two-form.*

Proof. Just as in the proof of (2.5.1), we apply the spectral sequence of the fiber bundle

$$0 \to S^1 \to S^{2n+1} \to \mathbb{C}\mathbf{P}^n \to 0$$

and get that the differential operator d_2 is an isomorphism when restricted to $E_2^{p,\,1}$ for $p = 0, 1, ..., 2n-1$. Let a be a generator of $E_2^{0,\,1} = \mathbb{R}$, then $x := d_2 a$ is a generator of

$$E_2^{2,\,0} \quad = \quad H^2(\mathbb{C}\mathbf{P}^n) \quad = \quad \mathbb{R}$$

and $x{\cdot}a$ is a generator of $E_2^{2,\,1}$. Applying d_2 again, we get a generator of $E_2^{4,\,0} = H^4(\mathbb{C}\mathbf{P}^n)$:

$$d_2(x \cdot a) \quad = \quad x \cdot d_2 a \quad = \quad x^2.$$

Continuing this procedure of mapping with d_2 and multiplying with a, we see that all non-trivial cohomologies of $\mathbb{C}\mathbf{P}^n$ will be generated by a power of x, i.e.,

$$H^*(\mathbb{C}\mathbf{P}^n) \quad = \quad \mathbb{R}[x]/(x^{n+1}).$$

□

2.7 Projectivization of complex vector spaces

A *complex line bundle* is a fiber bundle with fiber $\mathbb{C}^n$ and structure group $GL(1,\,\mathbb{C})$, i.e., a complex vector bundle of rank 1. Being a complex vector bundle, the structure group of the complex line bundle can be reduced to $U(1)$. Since $U(1)$ and $SO(2)$ are isomorphic, there is a one-to-one correspondence between the complex line bundles and the oriented rank 2 real bundles.

We define the *first Chern class* $c_1(L)$ of a complex line bundle L over a manifold M to be the *Euler class* of its corresponding real bundle $L_{\mathbb{R}}$, $c_1(L) := e(L_{\mathbb{R}}) \in H^2(M)$. For a definition of Euler class, see pages 116-118 in Bott. Intuitively, we can view the Euler class of an oriented real vector bundle of rank k over a manifold M to be a cohomology class in $H^k(M)$ that in some sense measure how twisted the bundle is.

Let V be a complex vector space of dimension n, and let $P(V)$ be its projectivization:

$$P(V) \quad = \quad \{\text{one-dimensional subspaces of } V\}.$$

Earlier in this chapter, we worked with the special cases $P(\mathbb{R}^n) = \mathbb{R}\mathbf{P}^n$ and $P(\mathbb{C}^n) = \mathbb{C}\mathbf{P}^n$.

On $P(V)$ there are three naturally induced vector bundles: the *product bundle* $\hat{V} = P(V) \times V$, the *universal subbundle* $S = \{(\ell,\, v) \in \hat{V} \mid v \in \ell\}$ (the fiber of ℓ is ℓ itself) and the *universal quotient bundle* Q, defined by the *tautological exact sequence*

$$0 \to S \to \hat{V} \to Q \to 0.$$

Tautological here refers to the property of being induced from the already established constructions. We will sometimes refer to these three bundles as the *tautological bundles*.

Let σ denote the composition of the following inclusion and projection:

$$\sigma : S \hookrightarrow P(V) \times V \to V.$$

The inverse image of σ at a point $v \in V$ is

$$\sigma^{-1}(v) \quad = \quad \{(\ell,\, v) \mid v \in \ell\},$$

so if $v \neq 0$, then $\sigma^{-1}(v) = (\ell,\ v)$ where ℓ is the line trough the origin and v. However, $\sigma^{-1}(0)$ consists of many points, and is in fact isomorphic to the whole $P(V)$ since every one-dimensional subspace of V goes through the origin.

To compute the cohomology of $P(V)$, we first need to endow V with a Hermitian metric. Let E be the unit sphere bundle of the universal subbundle S related to this metric,

$$E \quad = \quad \{(\ell,\ v) \mid v \in \ell,\ \|v\| = 1\}.$$

Note that $\sigma^{-1}(0)$ is the zero section of S, and thus $S \setminus \sigma^{-1}(0)$ is naturally diffeomorphic to $V \setminus \{0\}$. Therefore, E is diffeomorphic to the sphere S^{2n-1} in V and the map $\pi : E \to P(V)$ gives a fibering

$$0 \to S^1 \to s^{2n-1} \to P(V) \to 0.$$

Notice the similarities with the fiber bundle over $\mathbb{C}\mathbf{P}^n$ we used to compute the cohomology and its ring structure. In fact, with computations similar to those used for the complex projective space, we can compute the ring structure for $H^*(P(V))$. In this case, the generator is the first Chern class $c_1(S)$ of the universal subbundle S. Actually, it is customary to choose $-c_1(S)$ as generator instead, since this is the Chern class of the dual to S: the so called *hyperplane bundle* S^*. With $x = -c_1(S)$, we have:

$$H^*\big(P(V)\big) \quad = \quad \mathbb{R}[x]/(x^n), \qquad \text{where } n = \dim_{\mathbb{C}} V. \tag{2.4}$$

We define the *Poincaré series* of a manifold M to be

$$P_t(M) \quad = \quad \sum_{i=0}^{\infty} \dim H^i(M)\, t^i. \tag{2.5}$$

By (2.4), the Poincaré series of the projective space $P(V)$ is

$$P_t\big(P(V)\big) \quad = \quad 1 + t^2 + \cdots + t^{2(n-1)} = \frac{1 - t^{2n}}{1 - t^2}.$$

2.8 Projectivization of complex vector bundles

Let $\rho : E \to M$ be a complex vector bundle with structure group $GL(n,\mathbb{C})$ and let E_p denote the fiber over p. We define the *projectivization* of E, $\pi : P(E) \to M$, to be the fiber bundle whose fiber at a point p in M is the projective space $P(E_p)$ and whose structure group is the projective general linear group $PGL(n,\ \mathbb{C}) = GL(n,\ \mathbb{C})/\{\text{scalar matrices}\}$. Thus a point of $P(E)$ is a line ℓ_p in the fiber E_p.

Similar to $P(V)$, we have three naturally induced vector bundles: the pullback bundle $\pi^{-1}E$, the universal subbundle S and the universal quotient bundle Q. The pullback bundle $\pi^{-1}E$ is the vector bundle over $P(E)$ whose fiber at ℓ_p is E_p. Since $\rho : E_p \to \{p\}$ is a trivial bundle, we also get a trivial bundle when restricting the pullback bundle to the fiber $\pi^{-1}(p) = P(E_p)$, i.e.

$$\pi^{-1}E\,\big|_{P(E_p)} \quad = \quad P(E_p) \times E_p.$$

The fiber at ℓ_p of the universal subbundle S over $P(E)$ is defined to be ℓ_p itself,

$$S \quad = \quad \{(\ell,\, v) \in \pi^{-1}E \mid v \in \ell\},$$

and the universal quotient bundle is defined by the corresponding tautological exact sequence, just as in the case $P(V)$.

Set $x = -c_1(S)$, similar as before but here x is a cohomology class in $H^2(P(E))$ instead of $H^2(P(V))$. However, since the restriction of $\pi^{-1}E$ to a fiber $P(E_p)$ is trivial, we get that the universal subbundle S restricted to $P(E_p)$ is the universal subbundle $\tilde{S}$ of the projective space $P(E_p)$, and since the Euler class is functorial, it follows that $c_1(\tilde{S})$ is the restriction of $c_1(S)$ to $P(E_p)$. Hence the global cohomology classes $1,\ x,\ \cdots,\ x^{n-1}$ on $P(E)$ restricted to each fiber $P(E_p)$ freely generate the cohomology of the fiber.

By the Leray-Hirsch theorem (see [2, p. 50]), the cohomology $H^*(P(E))$ is a free module over $H^*(M)$ with basis $\{1,\ x,\ \cdots,\ x^{n-1}\}$. In particular, x^n can be written as a linear combination of $1,\ x,\ \cdots,\ x^{n-1}$ with coefficients in $H^*(M)$. We use these coefficients to extend the notion of first Chern class to *Chern classes* of the complex vector bundle E, and thus giving a second, but equivalent, definition of the first Chern class. We define the i^{th} *Chern class* $c_i(E)$ of E to be the unique element in $H^{2i}(M)$ such that

$$x^n + c_1(E)x^{n-1} + \cdots + c_n(E) \quad = \quad 0,$$

and the *total Chern class* $c(E) \in H^*(M)$ to be

$$c(E) \quad = \quad 1 + c_1(E) + \cdots + c_n(E).$$

With this definition of the Chern classes, we get from the above discussion that the ring structure of the cohomology of $P(E)$ is given by

$$H^*\big(P(E)\big) \quad = \quad H^*\big(M\big)/\big(x^n + c_1(E)x^{n-1} + \cdots + c_n(E)\big),$$

where n is the rank of E. Disregarding the ring structure of the cohomology and viewing it purely as a vector space, we have

$$H^*\big(P(E)\big) \quad = \quad H^*(M) \otimes H^*(\mathbb{C}\mathbf{P}^{n-1}),$$

since every fiber space $P(E_p)$ has the cohomology of $\mathbb{C}\mathbf{P}^{n-1}$. Thus, the Poincaré series of $P(E)$ is, by definition (2.5),

$$P_t\big(P(E)\big) \quad = \quad P_t(M)\frac{1-t^{2n}}{1-t^2}. \tag{2.6}$$

2.9 Flag Manifolds

Let V be a complex vector space of dimension n. We define a *flag* in V to be a sequence of subspaces $A_1 \subset A_2 \subset \cdots \subset A_n = V$, where $\dim_{\mathbb{C}} A_i = i$, and denote the collection of all flags in V with $F(V)$. Given an arbitrary flag $A \in F(V)$, we can transform A to any other flag in $F(V)$ by letting an element M in the general linear group $GL(n, \mathbb{C})$ act on all the subspaces A_i. The stabilizer of A, i.e. all elements $M \in GL(n, \mathbb{C})$ such that $MA = A$, is the subgroup H of the upper triangular matrices, which is easy to see if we first change the basis of V to $\{a_1, a_2, \cdots, a_n\}$, where a_1 spans A_1, $\{a_1, a_2\}$ spans

A_2, etc. So as a set $F(V)$ is isomorphic to the coset space $GL(n, \mathbb{C})/H$. The quotient of a Lie group by a closed subgroup is also a manifold (see [7, p. 120]), so $F(V)$ can be made into a manifold. We call $F(V)$ the *flag manifold* of V.

Similar to the case of projectivization of vector spaces, we can extend the notion of flag manifolds to vector bundles. Given a vector bundle E, the *associated flag bundle* $F(E)$ is the bundle obtained from E by replacing each fiber E_p by the flag manifold $F(E_p)$. Thus the fiber space becomes $F(\mathbb{C}^n)$, and we can take the transition functions of $F(E)$ to be those of E. The flag bundle $F(E)$ does not, however, become a vector bundle over E.

The concept of associated flag bundles is closely related to the projectivization of vector bundles; in the case of projectivization we consider one dimensional subspaces of the fiber, and in the case of flag bundles we consider subspaces of *all* positive dimensions of the fiber. As it turns out, we can actually construct the associated flag bundle by a series of projectivizations.

Let $\rho : E \to M$ be a vector bundle of rank n, and let $\pi : P(E) \to M$ be its projectivization. The universal quotient bundle Q previously defined is in fact a vector bundle of rank $n-1$ over $P(E)$, since the universal subbundle S is a vector bundle of rank 1 and the tautological exact sequence

$$0 \to S \to \pi^{-1}E \to Q \to 0$$

gives an isomorphism between $\pi^{-1}E$ and $S \oplus Q$ as vector bundles. We can thus apply the projectivization again, but this time on Q by changing the role of $P(E)$ to M and Q to E. Repeating this process, we get a sequence of bundles

$$M \leftarrow P(E) \leftarrow P(Q_1) \leftarrow \cdots \leftarrow P(Q_{n-1}),$$

where Q_i is the universal quotient bundle of $P(Q_{i-1})$. A point in $P(E)$ can be represented as a point p in M and a line ℓ_1 in E_p, a point in $P(Q_1)$ can be represented as a point in $P(E)$ and a line ℓ_2 in E_p/ℓ_1, etc. Thus, a point in $P(Q_{n-1})$ can be represented as the $n+1$-tuple $(p, \ell_1, \ell_2, \cdots, \ell_n)$, where $\bigoplus_i \ell_i = E_p$. But if we rewrite this representation as

$$(p,\ \ell_1 \subset \{\ell_1, \ell_2\} \subset \cdots \subset \{\ell_1, \cdots, \ell_n\} = E_p),$$

we get exactly a point in $F(E)$, the associated flag bundle of E. So we have proven that

$$F(E) \quad = \quad P(Q_{n-1}).$$

The representation of $F(E)$ as a series of projectivization makes it easy to prove the following theorem:

Theorem 2.9.1. *The Poincaré series of the flag bundle $F(E)$ is*

$$P_t\big(F(E)\big) \quad = \quad P_t(M)\frac{(1-t^2)(1-t^4)\cdots(1-t^{2n})}{(1-t^2)(1-t^2)\cdots(1-t^2)} \quad = \quad P_t(M)\frac{(1-t^4)\cdots(1-t^{2n})}{(1-t^2)^{n-1}}$$

Proof. From (2.6) we have that a projectivization of a complex vector bundle of rank n over M changes the Poincaré series by a factor $\frac{1-t^{2n}}{1-t^2}$. By the discussion above, $F(E)$ can be viewed as a repeated projectivization where the rank of the vector bundle is reduced by one each step. The theorem follows. □

Remark 2.9.2. *We can easily use (2.9.1) to also compute the Poincaré series of a flag manifold $F(V)$, since we can take $F(V)$ to be the flag bundle of a vector bundle over a single point with fiber V. The Poincaré series of a point is just 1, so we get*

$$P_t\big(F(V)\big) \quad = \quad \frac{(1-t^2)(1-t^4)\cdots(1-t^{2n})}{(1-t^2)(1-t^2)\cdots(1-t^2)} \quad = \quad \frac{(1-t^4)\cdots(1-t^{2n})}{(1-t^2)^{n-1}}$$

2.10 Grassmannians

As we have previously seen, the projective space $P(V)$ of a complex vector space V consists of the 1-dimensional subspaces of V. We will now generalize this by considering the k-dimensional subspaces of V instead. Actually, in this section, we will consider the k-*co*dimensional subspaces instead, that is, if V has complex dimension n, we will consider the subspaces of complex dimension $n-k$, which we will refer to as $(n-k)$-planes.

Let $G^k(V)$ denote the set of all $(n-k)$-planes in a n dimensional complex vector space V. This will be referred to as the *complex Grassmannian* or just *Grassmannian.*

Remark 2.10.1. *It is only in this section that the Grassmannian will refer to the set of k-*co*dimensional subspaces. In Chapters 3, 4 and 5, the Grassmannian will refer to the set of k-dimensional subspaces instead. To make this distinction clear, $G^k(V)$ will always denote the k-codimensional case, and $G_k(V)$ – with a subscript instead of superscript – will always denote the k-dimensional case.*

Endowing V with a Hermitian inner product, we define the unitary group $U(n)$ to be the group of all endomorphisms in V that preserves the inner product. The action of $U(n)$ is clearly transitive on $G^k(V)$, and the stabilizer of a fixed $(n-k)$-plane is the matrices in $U(n)$ that sends the plane to itself. Because of the preservation of the inner product, the elements in the stabilizer will also send the complementary orthogonal k-plane to itself, and so the stabilizer of the $(n-k)$-plane in V is $U(n-k)\times U(k)$. Thus the Grassmannian can be represented as the coset space

$$G^k(V) \quad = \quad \frac{U(n)}{U(n-k)\times U(k)},$$

and since $U(n)$ is a Lie group and the stabilizer a closed subgroup, $G^k(V)$ is a differentiable manifold. Note that with this notation standard, $G^{n-1}(V)$ is the projective space $P(V)$.

The tautological bundles of $P(V)$ has counterparts over the Grassmannian $G^k(V)$: the *product bundle* $\hat{V} = G^k(V)\times V$, the *universal subbundle* S whose fiber at each point

Λ of $G^k(V)$ is the $(n-k)$-plane Λ itself and the *universal quotient bundle* Q defined by the tautological exact sequence

$$0 \to S \to \hat{V} \to Q \to 0.$$

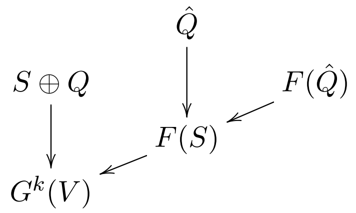
Over $G^k(V)$ the universal subbundle S is a vector bundle of rank $n-k$ and the universal quotient bundle Q is a vector bundle of rank k.

In the previous section we used the projectivization of a vector bundle to compute the Poincaré series of the flag bundle $F(E)$. Since the Grassmannian is just a generalized projective space, it is not surprising there is also a connection between the flag construction and the Grassmannian. In fact, we will use a series of flag constructions together with the known Poincaré polynomial of $F(E)$ to prove the following theorem:

Theorem 2.10.2. *The Poincaré series of the complex Grassmannian $G^k(V)$ is*

$$P_t(G^k(V)) \quad = \quad \frac{(1-t^2)\cdots(1-t^{2n})}{(1-t^2)\cdots(1-t^{2k})(1-t^2)\cdots(1-t^{2(n-k)})}$$

Proof. Let S be the universal subbundle of $G^k(V)$ and let $F(S)$ be its flag bundle. We can define a vector bundle $\hat{Q}$ over $F(S)$ by taking the pullback of the universal quotient bundle Q over $G^k(V)$, and from this vector bundle we get the flag bundle $F(\hat{Q})$ over $F(S)$.



Since we can view Q as the "complement dimensions" of S in V, and since $F(\hat{Q})$ encodes information from a flag structure on both S and Q, $F(\hat{Q})$ has enough information to describe the whole flag manifold $F(V)$. In fact, we will show that $F(\hat{Q}) = F(V)$.

A point of $F(S)$ is a pair $(\Lambda,\ L)$ consisting of an $(n-k)$-plane Λ in V and a flag $L : L_1 \subset \cdots \subset L_{n-k-1} \subset \Lambda$ in Λ, and a point in $F(\hat{Q})$ consists of a point in $F(S)$ together with a flag $L' : L'_1 \subset \cdots \subset L'_{k-1} \subset V/\Lambda$ in V/Λ. By taking the direct sum of the subspace Λ with every subspaces in the flag L', we get a sequence of inclusions that is a continuation of the flag L all the way to V, and thus the pair $(L,\ L')$ represents a single flag in V. We now have that every point in $F(\hat{Q})$ represents a flag in V, and that every flag in V can be divided into a pair $(L,\ L')$ which represents a point in $F(\hat{Q})$. But this gives exactly that $F(\hat{Q}) = F(V)$.

Since $F(\hat{Q})$ was obtained from $G^k(V)$ by two consecutive flag constructions, applying (2.9.1) two times enables us to express the Poincaré series of $F(V)$ in terms of $P_t(G^k(V))$

$$P_t(F(V)) \quad = \quad P_t(G^k(V))\frac{(1-t^2)\cdots(1-t^{2k})(1-t^2)\cdots(1-t^{2(n-k)})}{(1-t^2)\cdots(1-t^2)(1-t^2)\cdots(1-t^2)}.$$

But (2.9.2) gives us the Poincaré series of the left-hand side, so by eliminating the second factor from the right-hand side we get

$$P_t(G^k(V)) \quad = \quad \frac{(1-t^2)\cdots(1-t^{2n})}{(1-t^2)\cdots(1-t^{2k})(1-t^2)\cdots(1-t^{2(n-k)})}.$$

□

3

Singular Cohomology

This chapter will be more oriented towards geometry. We will see how we can compute the cohomology with integer coefficients on the Grassmannian by first constructing it as a CW complex. The cells of this complex will be interpreted geometrically by taking the closure in Zariski topology, and the resulting varieties will be studied with homology which in turn can be related by Poincaré duality to the cohomology of the Grassmannian.

This chapter is mainly based on Bott [2] and Hatcher's *Algebraic Topology* [8], occasionally referred to as *Hatcher*.

3.1 CW complexes

We will begin with by defining a *CW complex*. Intuitively, a CW complex can be described as a topological space resulting from successively gluing together cells of increasing dimension.

Let X^0 denote a discrete set of points, referred to as *0-cells*, and let D^n_α denote an n-dimensional closed disk index by α. We construct the *n-skeleton* X^n from X^{n-1} by attaching *n-cells* e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$, i.e. we define X^n to be the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ under the identification $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Since we will only study finite dimensional CW complexes, the whole complex X will be the same as the last skeleton $X = X^n$.

We define the *characteristic map* $\Phi_\alpha : D^n_\alpha \to X$ to be an extension of the attaching map ϕ_α, and the n-cell e^n_α to be the image of the interior of D^n_α under this map. More precisely, we define Φ_α to be the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \to X^n \hookrightarrow X$$

where the middle map is the quotient map defining X^n. With these definition, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$ as a set, where the n-cells e^n_α are perceived as open disks.

3.2 CW structure on Grassmannians

To be able to construct the Grassmannian as a CW complex, we first need a convenient description of the cells. We will here only consider the complex Grassmannian, $G_k(\mathbb{C}^n)$, but the construction of $G_k(\mathbb{R}^n)$ as a CW complex is almost completely analogous. This section is based on Hatcher.

For every element $V \in G_k(\mathbb{C}^n)$ we can find a k-tuple of complex n-vectors that spans V. Perceiving these vectors as row-vectors, we can construct a $k \times n$ matrix. From linear algebra we know that we can find a row reduced echelon form by a finite sequence of elementary row operations, where we want the lower left corner to consist of zeroes. Here we instead want a echelon form with zeros in the upper right corner, with every row ending with an 1 with zeros below and to the right of it. For example, a row reduced matrix from a plane in $G_3(\mathbb{C}^7)$ might look like this

$$\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & 0 & * & 1 & 0 \end{bmatrix}$$

where the asterisks denote arbitrary elements in $\mathbb{C}$. We can encode the shape of a reduced matrix A with the *Schubert symbol* $\lambda(A)$, which is a k-tuple of integers where the i^{th} integer represents the column coordinate of the special entry "1" in the i^{th} row vector of the matrix. For example, the Schubert symbol of the matrix above is $(\lambda_1,\ \lambda_2,\ \lambda_3) = (3,\ 4,\ 6)$. By construction we have that $\lambda_1 < \cdots < \lambda_k$.

The Schubert symbol does not depend on the particular reduction of the matrix, which in turn gives us that the echelon itself only depends on the k-plane V we started with. To see the first claim, we first note that the row vectors of our reduced matrix still span V, since the row operations does not alter the plane spanned by the rows. So if we project the plane V onto its last $n-i$ coordinates in $\mathbb{C}^n$, we see from the reduced matrix that the dimension of this projection will be reduced by one exactly when i hits one of the elements λ_j in the Schubert symbol. With this geometrical interpretation of the Schubert symbol, we see that it depends only on the plane V. Using this, we can define a projection from the λ_i coordinate of the i^{th} vector spanning V, which will then be a bijection between V and $\mathbb{C}^k$ since multiplication of a row in the echelon matrix again does not change the plane the row vector spans. But this bijection shows that it can only be one k-tuple of vectors with 1's in the coordinates corresponding to the Schubert symbol, which proves the second claim.

Given a Schubert symbol λ, we define the *Schubert cell* to be the subset $\Omega_\lambda^\circ \subset G_k(\mathbb{C}^n)$ of all k-planes in $\mathbb{C}^n$ having λ as their Schubert symbol. Since every k-plane has a unique echelon matrix, the dimension of Ω_λ° will be the number of arbitrary elements in a reduced matrix with Schubert symbol λ. Using the above matrix as an example again, we see that the (complex) dimension of $\Omega_\lambda^\circ = e(3{,}4{,}6)$ is the number of asterisks, i.e. 7. More generally, we get that the dimension of $\Omega_\lambda^\circ = e(\lambda_1, \cdots, \lambda_k)$ is $(\lambda_1 - 1) + \cdots + (\lambda_k - k)$, so Ω_λ° is homeomorphic to a an open disk of this dimension.

Theorem 3.2.1. *The Schubert cells Ω_λ° are the cells of a CW structure on $G_k(\mathbb{C}^n)$.*

Proof. We need to find a characteristic map for every λ which maps the interior of a closed disk to a cell Ω_λ° of equal dimension. To do this, we will consider a different echelon form which also characterizes the k-planes in Ω_λ°; the *orthonormal echelon form.* We get this form from the previous echelon form by first allowing the 1's to be any *positive* real number and the zeros below to be any element in $\mathbb{C}$, and then imposing the condition that all row vectors should be orthonormal. This echelon form will also be unique for the k-plane V it represents, since if we let V_i denote the subspace of V spanned by the first i rows of the standard echelon form, then there is a unique unit vector in V_i orthogonal to V_{i-1} and with positive real λ_i^{th} coordinate.

Ignoring the last zeros after the λ_i^{th} coordinate in the i^{th} row vector, we see that this vector is an element in a closed "hemisphere" H_i of the complex unit sphere $S^{\lambda_i - 1} \subset \mathbb{C}^{\lambda_i} \subset \mathbb{C}^n$. More precisely, H_i is defined to be the closed subset of S^{λ_i-1} with *non-negative* real λ_i^{th} coordinates. Since we do not allow any imaginary part in the last coordinate, the *real* dimension of H_i is $2\lambda_i - 2$.

Let E_λ be the set of all k-tuples $(v_1, \cdots, v_k) \in (S^{\lambda_n - 1})^k$ such that $v_1, \cdots, v_k$ are orthogonal and $v_i \in H_i$ for each i. Note that the set of row vectors of the orthonormal echelon forms representing the k-planes in Ω_λ° is exactly the interior of E_λ, so we have a natural bijection between the interior of E_λ and Ω_λ°. This natural map is in fact a homeomorphism, since we can interpret the topology on $G_k(\mathbb{C}^n)$ as the quotient topology from the space of k-tuples of orthonormal vectors in $\mathbb{C}^n$.

We now prove that E_λ is homeomorphic to a closed disc. We begin by noting that H_i is homeomorphic to a closed disk of complex dimension $\lambda_i - 1$. Let $\pi : E_\lambda \to H_1$ be the projection $\pi(v_1, \cdots, v_k) \mapsto v_1$. We can identify $\pi^{-1}(v_0)$ with $E(\lambda')$, where $v_0 = (0, \cdots, 0, 1) \in \mathbb{C}^{\lambda_1}$ and $\lambda' = (\lambda_2 - 1, \cdots, \lambda_k - 1)$; the -1 appearing in λ' since all vectors in $\pi^{-1}(v_0)$ needs to be orthogonal to v_0.

To construct the homeomorphism, we only need to find another projection $p : E_\lambda \to \pi^{-1}(v_0)$ which is a homeomorphism on the fibers of π, since then the map $\pi \times p : E_\lambda \to H_1 \times \pi^{-1}(v_0)$ is a homeomorphism, and thus we can with the identification $\pi^{-1}(v_0) = E(\lambda')$ inductively construct a homeomorphism

$$E_\lambda \to H_1 \times H_1' \times \cdots \times H_1^{(k-1)},$$

where $H_1^{(i)}$ is a hemisphere in $S^{\lambda_1^{(i)}-1}$ and $\lambda^{(i)} = (\lambda_{i+1} - i, \cdots, \lambda_k - i)$. But we can easily construct the map p by defining its restriction $p\,\big|_{\pi^{-1}(v)} : \pi^{-1}(v) \to \pi^{-1}(v_0)$ for every $v \in H_1$ as the map obtained by applying the rotation $\rho_v \in SU(n)$ to every vector in $(v, v_1, \cdots, v_{k-1}) \in \pi^{-1}(v)$, where ρ_v is uniquely defined to be the transformation that maps v to v_0 and fixes the orthogonal $(n-2)$-complex dimensional subspace.

Let D_λ be a closed disk of dimension $(\lambda_1 - 1) + \cdots + (\lambda_k - k)$. For a given cell Ω_λ°, we can now define the characteristic map $\Phi_\lambda : D_\lambda \to G_k(\mathbb{C}^n)$ to be the composition

$$D_\lambda \to H_1 \times H_1' \times \cdots \times H_1^{(k-1)} \to E_\lambda \to G_k(\mathbb{C}^n)$$

where the last map takes the orthonormal vectors in an element of E_λ to its span. By the above discussion, Φ_λ restricted to the interior of D_λ is homeomorphic to Ω_λ°. The boundary of D_λ is mapped to cells of lower dimensions obtained from λ by decreasing some λ_i's, since the boundary of E_λ consists of the k-tuples where at least one v_i is in $\partial H_i = S^{i-2}$.

We can now create the CW complex on $G_k(\mathbb{C}^n)$ with induction. Let X^i be the union of the cells in $G_k(\mathbb{C}^n)$ having dimension at most i. Assume that X^i is a CW complex, and construct the space Y by attaching every $(i+1)$-cell $\Omega_\lambda^\circ \in X^{i+1}$ via $\Phi_\lambda\big|_{\partial D_\lambda} : \partial D_\lambda \to X^i$. Y is then a CW complex with a natural continuous bijection $Y \to X^{i+1}$, which in fact is a homeomorphism since Y consists only of a finite number of cells and is thus compact. So X^{i+1} is a CW complex and we are done. □

3.3 Schubert varieties

While the Schubert cells Ω_λ° had the topological property of being homeomorphic to the interior of a closed disk, which allowed us to create the CW complex for $G_k(\mathbb{C}^n)$ in the previous section, it is less useful from a geometrical point of view. It turns out, however, that we can extend the Schubert cells to become *projective varieties*.

A projective variety (over $\mathbb{C}$) is a subset of $\mathbb{C}\mathbf{P}^n$ cut out by the zeroes of some finite family of *homogeneous polynomials* in $\mathbb{C}[X_1, \cdots, X_{n+1}]$, i.e. polynomials where every term has the same degree. We also require a variety to be *irreducible*, which means that it can not be written as a nontrivial union of two subsets also cut out by homogeneous polynomials.

Theorem 3.3.1. *The Grassmannian $G_k(\mathbb{C}^n)$ is a projective variety in $\mathbb{C}\boldsymbol{P}^{\binom{n}{k}-1}$.*

Proof. Let $\bigwedge V$ be the exterior algebra over a n-dimensional complex vector space V, and let $\bigwedge^k V$ be the subset of all k-vectors in $\bigwedge V$. A given k-plane V in $G_k(\mathbb{C}^n)$ represents a line in $\bigwedge^k \mathbb{C}^n$, since if we let $(v_1, \cdots, v_k)$ and $(u_1, \cdots, u_k)$ be two arbitrary k-tuples of vectors $v_i, u_j \in \mathbb{C}^n$ both spanning V, and let C be the $k \times k$-matrix expressing the coordinates of one k-tuple in terms of the other, the alternating property of the wedge product $\wedge$ then gives us that one of the k-vectors $v_1 \wedge \cdots \wedge v_k$ or $u_1 \wedge \cdots \wedge u_k$ is just a factor ($\det C$) times the other.

Thus we get a well defined map $\pi : G_k(\mathbb{C}^n) \to \mathbf{P}(\bigwedge^k \mathbb{C}^n)$ taking k-planes V to k-vectors $v_1 \wedge \cdots \wedge v_k$, where $v_1, \cdots v_k$ are any vectors spanning V. This map is called the *Plücker embedding*.

The Plücker embedding has its name motivated by the fact that it is injective. To see this, we define the map $\phi : \bigwedge^k \mathbb{C}^n \to G_k(\mathbb{C}^n)$ where the image of a k-vector w is defined as

$$\phi(w) \quad = \quad \{v \in \mathbb{C}^n \mid v \wedge w = 0\} \in G_k(\mathbb{C}^n).$$

It is clear from the definition of ϕ that all multiples of w are mapped to the same k-plane V, and it is also clear that $\phi(\pi(V)) = V$, which proves the injectivity of π.

The image of π in $\mathbf{P}(\bigwedge^k \mathbb{C}^n)$ is the subset of *simple* k-vectors, i.e. the k-vectors that can be written as $v_1 \wedge \cdots \wedge v_k$, which excludes those k-vectors which can only be written as a nontrivial *sum* of k-vectors. The simple k-vectors are cut out by a family of homogeneous polynomials, called the *Plücker relations.* For a proof of this, consider [9] or [10]. A slightly different proof based on the algebra of multivectors is given in [11].

Since the dimension of $\bigwedge^k \mathbb{C}^n$ is $\binom{n}{k}$, the Plücker embedding together with the Plücker relations proves the theorem. □

We have seen that the Schubert cells represents the k-planes in $G_k(\mathbb{C}^n)$ whose dimension, when projected onto its last $n-i$ coordinates in $\mathbb{C}^n$, is reduced by one exactly when i equals one of the $\lambda_j \in \lambda$. We can reformulate this as

$$\Omega_\lambda^\circ \quad = \quad \left\{ V \in G_k(\mathbb{C}^n) \mid \forall t \in \{1, \cdots, k\} : \begin{array}{l} \dim\,(V \cap F_{\lambda_t}) = t \text{ and} \\ \dim\,(V \cap F_i) < t \text{ for all } i < \lambda_t \end{array} \right\}.$$

where $F : F_0 = 0 \subset \cdots \subset F_n = \mathbb{C}^n$ denote the standard flag in $\mathbb{C}^n$, i.e. F_j is spanned by the standard basis elements $e_1, \cdots, e_j$. This definition looks a little bit awkward, and indeed its image under the Plücker embedding can not be defined as a zero set of homogeneous functions in $\mathbb{CP}^{\binom{n}{k}-1}$. Instead, we define the *Schubert variety*

$$\Omega_\lambda \quad = \quad \{V \in G_k(\mathbb{C}^n) \mid \forall t \in \{1, \cdots, k\} : \dim\,(V \cap F_{\lambda_t}) \geq t\}.$$

We demonstrate the difference between the Schubert cell and Schubert variety with an example. Let $\lambda = (2,4)$ be the Schubert symbol for the Schubert cell Ω_λ°. This cell is represented by all 2-planes in $G_2(\mathbb{C}^4)$ whose reduced matrix is on the form

$$\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix}.$$

The Schubert variety Ω_λ on the other hand, includes the 2-planes with the above reduced matrix, but also includes the 2-planes with Schubert symbol $\lambda' = (\lambda'_1,\lambda'_2)$ such that $\lambda'_1 \leq 2$ and $\lambda'_2 \leq 4$. For example, Ω_λ includes 2-planes with matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix}$$

but not 2-planes with matrix

$$\begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}.$$

We can also use the above example to demonstrate the Plücker relations introduced in the proof of (3.3.1). We saw that the Plücker embedding π was well defined, i.e. independent of vectors spanning $V \in G_k(\mathbb{C}^n)$, so we can take $\pi(V)$ to be the k-vector $v_1 \wedge \cdots \wedge v_k$ where v_i is the i^{th} row vector in the reduced matrix representing V. Notice that the largest Schubert variety by definition is the whole $G_k(\mathbb{C}^n)$, so in our example above we have that $\Omega_{(3,4)} = G_2(\mathbb{C}^4)$, and we can thus get an explicit description of the Plücker relations for $G_2(\mathbb{C}^4)$ by applying the Plücker embedding on $\Omega_{(3,4)}$.

Let $V \in G_2(\mathbb{C}^4)$ have the reduced matrix

$$A = \begin{bmatrix} a_1 & a_2 & 1 & 0 \\ a_3 & a_4 & 0 & 1 \end{bmatrix}.$$

If we let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $\mathbb{C}^4$ and $\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$ the standard basis for $\bigwedge^2 \mathbb{C}^4$, the Plücker embedding for V becomes

$$\begin{aligned} (a_1e_1 + a_2e_2 + e_3) \wedge (a_3e_1 + a_4e_2 + e_4) &= \\ (a_1a_4 - a_2a_3)e_{12} - a_3e_{13} + a_1e_{14} - a_4e_{23} + a_2e_{24} + e_{34}, \end{aligned} \tag{3.1}$$

where we have used the alternating property of the wedge product. Notice that the coefficient of e_{ij} is the sub-determinant from column i and j in A. This is no coincidence; the calculation above is just a reformulation of calculations with sub-determinants. We call these coefficients *Plücker coordinates* and denote them as p_{ij}. From (3.1), we can formulate two relations between the coordinates p_{ij}

$$\begin{aligned} p_{12} &= -p_{14}p_{23} + p_{24}p_{13} \\ p_{34} &= 1. \end{aligned}$$

Homogenising the first equation with respect to p_{34}, we get one *homogeneous* equation

$$p_{12}p_{34} - p_{24}p_{13} + p_{14}p_{23} = 0, \tag{3.2}$$

which is the Plücker relation for $G_2(\mathbb{C}^4)$. We can easily see that this equation holds for *all* 2-planes in $\Omega_{(3,4)}$, and not just for the 2-planes in the Schubert cell $\Omega^\circ_{(3,4)}$. Take for example a 2-plane with Schubert symbol (2,4), then

$$\begin{aligned} p_{13} &= p_{14}p_{23} \\ p_{34} &= 0 \\ p_{24} &= 1, \end{aligned}$$

which gives us two homogeneous equations

$$\begin{aligned} p_{13}p_{24} - p_{14}p_{23} &= 0 \\ p_{34} &= 0. \end{aligned}$$

But these two equations are just the special case of (3.2) when p_{34} vanish.

This pattern also holds in the general case $G_k(\mathbb{C}^n)$, i.e. the relation we obtain from the Schubert cell with maximal Schubert symbol λ will be the Plücker relation, and decreasing some λ_j will give us a Schubert variety defined by a special case of the Plücker relation where some $p_{i_1 i_2 \cdots i_k} = 0$. Thus, with a slight modification of the construction of the Plücker relations, we can prove that every Schubert variety is in fact a projective variety. It can also be shown that the Schubert variety is the smallest projective variety containing a given Schubert cell, that is, Ω_λ is the *Zariski closure* of Ω°_λ.

3.4 Singular homology

In the previous chapter we studied the de Rham cohomology. Later in this chapter we will study another cohomology – *singular cohomology* – which has a dual relationship with *singular homology* we define in this section. This section is based mainly on Bott and Hatcher; Hatcher gives an extensive description of singular homology, while Bott is more brief. Since we will not need to do any direct calculations with homology, we leave out the definition of the more intuitive but less theoretically useful *simplicial homology.*

Let e_i be the standard basis in $\mathbb{R}^\infty$, and let e_0 be the origin. We define the *standard q-simplex* Δ_q to be the set

$$\Delta_q = \left\{ \sum_{j=0}^{q} t_j e_j \mid \sum_{j=0}^{q} t_j = 1, t_j \geq 0 \right\}.$$

Let X be a topological (Hausdorff) space, a *singular q-simplex* in X is a continuous map $s : \Delta_q \to X$. The word "singular" refers to the fact that the image of s might have singularities where it does not look like a simplex at all; we only require s to be continuous. A *finite* linear combination with *integer* coefficients of singular q-simplices is called a *singular q-chain*, and the set $S_q(X)$ of all such chains is an Abelian group. We define a boundary map $\partial_q : S_q(X) \to S_{q-1}(X)$ by first defining the i^{th} *face map* $\partial_q^i : \Delta_{q-1} \to \Delta_q$ between standard simplices to be the function given by

$$\partial_q^i \left(\sum_{j=0}^{q-1} t_j e_j \right) = \sum_{j=0}^{i-1} t_j e_j + \sum_{j=i+1}^{q} t_{j-1} e_j.$$

Given a singular q-chain s, the boundary map can now be defined as

$$\partial_q s = \sum_{i=0}^{q} (-1)^i s \circ \partial_q^i,$$

where the i^{th} term can be seen as the restriction of s to the $(q-1)$-simplex where the basis element e_i is excluded. We have that $\partial_q \partial_{q+1} = 0$, since the term in the composition excluding e_i and e_j and the term excluding the same elements but in revers order will be equal except for sign. Thus, the complex $S_*(X) = \bigoplus S_q(X)$ together with the boundary map ∂_q gives us a homology $H_q(X) = \text{Ker } \partial_q / \text{Im } \partial_{q+1}$, the *singular homology.* The q-chains in Ker ∂_q are called q-cycles, and the q-chains in Im ∂_{q-1} are called q-boundaries.

Remark 3.4.1. *To be more specific, the homology defined above is singular homology* with integer coefficients*. We can in fact choose the coefficients to be in any fixed abelian group, but the set of integers is the most general case. In Chapter 5, we will see that singular homology with* real *coefficients is equivalent to de Rham cohomology.*

With the singular homology defined, we can now define *singular cohomology* (with integer coefficients) by considering the dual elements to the q-chains defined above. To be more precise, we define a *singular q-cochain* on a topological space X as a linear functional on the $\mathbb{Z}$-module $S_q(X)$ of singular q-chains. Thus, if we denote the group of all singular q-cochains by $S^q(X)$, we have $S^q(X) = \text{Hom}\ (S_q(X),\mathbb{Z})$. From the group of all singular cochains $S^*(X) = \bigoplus S^q(X)$ we get a differential complex by defining the coboundary operator $d_q : S^q(X) \to S^{q+1}(X)$ as the operator such that

$$(d_q\omega)(s) \quad = \quad \omega(\partial_{q+1}s)$$

for all $(q+1)$-chains s and all q-cochains ω. The (singular) cohomology with integer coefficients is defined as the homology $H^q(X) = \text{Ker}\ d_q/\text{Im}\ d_{q-1}$ of this complex. To avoid cumbersome notations, we will sometimes omit the subscript for d and ∂.

There is a natural product on the complex $S^*(X)$, the *cup product*

$$\smile : \ S^p \otimes S^q \to S^{p+q},$$

given by

$$(\omega \smile \eta)(s) \quad = \quad \omega(s\,\big|_{[e_0,\cdots,e_p]})\eta(s\,\big|_{[e_p,\cdots,e_{p+q}]})$$

where ω is a p-cochain, η a q-cochain and s a $(p+q)$-simplex. By linearity $\omega \smile \eta$ is a $(p+q)$-cochain, i.e. maps from the whole of $S_q(X)$. By explicitly calculating $(d\omega \smile \eta)(s)$ and $(-1)^{\deg\omega}(\omega \smile d\eta)(s)$, we get the Liebniz's rule

$$d(\omega \smile \eta) \quad = \quad (d\omega) \smile \eta + (-1)^{\deg\omega}\omega \smile (d\eta),$$

and thus the cup product of cochains also induces a cup product of cohomology classes.

Remark 3.4.2. *The cup product is the analog of the wedge product for the de Rham complex. In fact, the analogy can be taken even further; the Mayer-Vietoris sequence for singular cochains is exact, and so if we create the double complex $C^*(\mathfrak{U}, S^*)$, we have an isomorphism between the cohomology of the double complex and the singular cohomology. This isomorphism is also an algebra isomorphism if we define a product on $C^*(\mathfrak{U}, S^*)$ as we did in (2.3), replacing the wedge product with the cup product.*

Remark 3.4.3. *Let X be a topological space and $\mathfrak{U}$ a good cover on X. As in the de Rham case, we can also use the double complex $C^*(\mathfrak{U}, S^*)$ to construct an isomorphism between the singular cohomology $H^*(X)$ and the Čech cohomology $H^*(\mathfrak{U}, \mathbb{Z})$ for the cover $\mathfrak{U}$ with coefficients in the constant presheaf $\mathbb{Z}$. Note that the singular 0-simplices are just points on X, so $S^0(X) = Hom\ (S_0(X),\mathbb{Z})$ is just the set of $\mathbb{Z}$-valued functions on X. A function $\omega \in S^0(X)$ is a 0-cocycles if and only if $\omega(\partial c) = 0$ for all paths $c \in S_1(X)$, hence ω needs to be constant on each path component of X. This gives us that the cocycles of $\prod S^0(U_{\alpha_0\cdots\alpha_p})$, where $U_{\alpha_0\cdots\alpha_p} \in \mathfrak{U}$, are exactly the elements of $C^p(\mathfrak{U}, \mathbb{Z})$.*

While the cup product maps two cochains to a third on a given space X, we can define a similar product, the *cap product*, between a *chain* and a cochain

$$\frown : \ S^q \otimes S_p \to S_{p-q}$$

for $p \geq q$. Take $\sigma \in S_p$ and $\phi \in S^q$, their cap product is defined to be

$$\phi \frown \sigma \quad = \quad \phi(\sigma\big|_{[e_0,\cdots,e_q]})\sigma\big|_{[e_q,\cdots,e_p]}.$$

For a differential form The cap product makes the singular chains S_* a left module over the singular cochains S^*. Notice however that the cap product *reduces* the degree of the p-chain σ, while the cup product increases the degree.

As for the cup product, the cap product has the equivalent of the Liebniz's rule

$$\partial(\phi \frown \sigma) \quad = \quad (-1)^q(\phi \frown \partial\sigma - \delta\phi \frown \sigma),$$

which is easily checked, and thus the cap product induces a product between the homology H_* and cohomology H^*,

$$\frown: \ H^q \otimes H_p \to H_{p-q}.$$

The cap product $\frown$ can be seen as the dual to the cap product $\smile$ by defining the duality

$$< \cdot, \cdot >: \ S^p \times S_p \to \mathbb{Z}$$

as the evaluation

$$< \phi, \sigma > \quad = \quad \phi(\sigma)$$

for $\phi \in S^p$ and $\sigma \in S_p$. Take $\alpha \in S_{p+q}$, $\omega \in S^p$ and $\eta \in S^q$, directly from the definitions we get the duality relation

$$< \omega \smile \eta, \alpha > \quad = \quad < \omega, \eta \frown \alpha > .$$

The above equality makes it clear that the analog of cap product in multivector algebra is the *interior product* on $\bigwedge V$, see [11].

Remark 3.4.4. *The cap product also bears resemblance to the interior product ι_X : $\Omega^q(M) \to \Omega^{q-1}(M)$ (for a vector field X and manifold M) found in differential geometry. Let $< \cdot \, , \, \cdot >$ denote the duality between differential forms and vector fields and let ω be a q-form, ι_X is defined by*

$$< \iota_X\omega, \ (X_1, \cdots, X_{q-1}) > \quad = \quad < \omega, \ (X, X_1, \cdots, X_{q-1}) > .$$

Note that this analogy with the cap product is a little bit awkward, since it reduces the degree of differential forms whose differential operator increases the degree, while the cap product reduces the degree of singular chains whose differential operator also reduces the degree.

Up to this point, we have studied homology and cohomology of a general topological (Hausdorff) space X. Restricting our attention to manifolds as in previous sections, we find a deeper connection between the homology and cohomology, which in our case will manifest itself as the *Poincaré duality for compact orientable manifolds.* "Orientable" here refers to the standard notion of orientability on differentiable manifolds, which is also called $\mathbb{Z}$-*orientation*.

For a compact connected orientable real n-manifold (i.e. of dimension n), the top homology $H_n(M)$ is isomorphic to $\mathbb{Z}$, and the homologies $H_i(M)$ above the top homology, $i > n$, are zero. For proofs, see Hatcher. An element of $H_n(M)$ generating the whole $\mathbb{Z}$ is called a *fundamental class* for M. We can now formulate the Poincaré duality. For a proof, we again refer to Hatcher.

Theorem 3.4.5 (Poincaré duality). *If M is a compact orientable n-manifold without boundary and with fundamental class $[M] \in H_n(M)$, then the map $D : H^k(M) \to H_{n-k}(M)$, defined by*

$$D(\alpha) \quad = \quad \alpha \frown [M],$$

is an isomorphism for all k.

The Poincaré duality allows us to identify the homology groups with the cohomology groups, which will come to use later. The analog of the Poincaré duality in multivector algebra is the *Hodge dual*.

3.5 Schubert calculus

The point of extending the Schubert cells Ω_λ° to Schubert varieties Ω_λ, is that projective varieties are complex manifolds, so that we in this section can apply the homology results on manifolds from the previous section.

From section 3.3, we have that Ω_λ is a projective complex variety of complex dimension $p = \sum \lambda_i - i$, and thus also a *real* $2p$-manifold. In fact, Ω_λ is a *subvariety* of $G_k(\mathbb{C}^n)$, so we can also view it as a submanifold. From Section 3.4, the top homology $H_{2p}(\Omega_\lambda)$ is isomorphic to $\mathbb{Z}$, and we can define a fundamental class $[\Omega_\lambda]$ generating $\mathbb{Z}$.

Embedding the submanifold Ω_λ in $G_k(\mathbb{C}^n)$, the top homology in Ω_λ maps to the $2p$ homology in $G_k(\mathbb{C}^n)$, and by applying the Poincaré duality on the Grassmannian, we see that each Schubert variety Ω_λ determines a singular cohomology class in $H^{2k(n-k)-2p}(G_k(\mathbb{C}^n))$. The notation $[\Omega_\lambda]$ will now refer to this cohomology class, instead of the dual homology class as above.

Theorem 3.5.1. *The integer cohomology classes $[\Omega_\lambda]$ form a basis for $H^*(G_k(\mathbb{C}^n))$ over $\mathbb{Z}$.*

To prove this, we will use the following lemma

Lemma 3.5.2. *Let X be a projective nonsingular variety with a filtration $X = X_s \supset X_{s-1} \supset \cdots \supset X_0 = \emptyset$ of algebraic subsets such that $X_i \setminus X_{i-1}$ is a disjoint union of sets $U_{i,\,j}$, each isomorphic to an affine space $\mathbb{C}^{n(i,\,j)}$. Then the cohomology classes $[\bar{U}_{i,\,j}]$, where the bar denotes closure in the Zariski topology, give an additive basis for $H^*(X)$ over $\mathbb{Z}$.*

For a proof of the lemma, see the appendix B in [9].

Proof of theorem 3.5.1. Give the Grassmannian $G_k(\mathbb{C}^n)$ the filtration

$$G_k(\mathbb{C}^n) \quad = \quad X_{k(n-k)} \supset X_{k(n-k)-1} \supset \cdots \supset X_0 \quad = \quad \emptyset$$

where X_i is the union of all Ω_λ of dimension i or lower. From Section 3.3 we have that the Schubert varieties Ω_λ are subvarieties of $G_k(\mathbb{C}^n)$, and thus the unions X_i are algebraic subsets. From the same section we also have that the Schubert cells Ω_λ° are isomorphic to $\mathbb{C}^p$, $p = \sum \lambda_j - j$, and that $\Omega_\lambda \setminus \Omega_\lambda^\circ$ is the union of all Schubert varieties $\Omega_{\lambda'}$ where λ' is given from λ by decreasing one λ_j. Since every two Schubert cells are disjoint, $X_i \setminus X_{i-1}$ is a disjoint union of the Schubert cells Ω_λ° with dimension i. Lemma 3.5.2 now gives the result. □

Remark 3.5.3. *There is an alternative way to relate the Schubert cells to the integer cohomology on the Grassmannian using* cellular homology *(see Hatcher). This method uses the fact that $G_k(\mathbb{C}^n)$ is a CW complex with no odd real-dimensional cell, which gives that $H_i(G_k(\mathbb{C}^n))$ is free abelian with basis in one-to-one correspondence with the i-cells of $G_k(\mathbb{C}^n)$. Using Poincaré duality, we thus get a one-to-one relation between the Schubert cells Ω_λ° and the basis for the integer cohomology $H^*(G_k(\mathbb{C}^n))$.*

The problem with this method is that it is not clear that the cohomology classes $[\Omega_\lambda]$ constructed above is the actual basis, i.e., that they are linearly independent.

Using the cup product defined in the previous section, by theorem 3.5.1 there is a unique expression

$$[\Omega_\alpha] \smile [\Omega_\beta] \quad = \quad \sum_\gamma d_{\alpha\,\beta}^{\,\gamma} [\Omega_\gamma] \tag{3.3}$$

for integers $d_{\alpha\,\beta}^{\,\gamma}$. These coefficients are referred to as *Littlewood-Richardson numbers*, and methods for calculating them is referred to as *Schubert calculus.*

The Littlewood-Richardson numbers encode a lot of geometrical information, but this is not clear from the rather abstract definition (3.3). Indeed, the cup product is just a natural product appearing in the context of cochains, and even though we manage to tie the geometrical Schubert varieties to the cochains, it is still unclear how the cup product interacts with the geometry of the varieties.

There is, however, a connection to be made; the cup product between cocycles represents intersection of the corresponding cycles. In the context of Schubert varieties, this means that, with some reformulations, the cup product represents intersection of the Schubert varieties. For exact formulation and proof, consider [12, p. 366-372]. Bott also have this result, but in the de Rham cohomology context.

4

Applications of cohomologies

4.1 Horn's conjecture

It turns out that the Littlewood-Richardson numbers $d_{\alpha\,\beta}^{\,\gamma}$ defined in (3.3) appears in many different areas, from algebraic geometry and algebraic topology to representation theory and combinatorics.

One specific problem related to these numbers is *Horn's conjecture*, which was proved to be true quite recently in the 1990s. Horn's conjecture concerns triples $(\alpha,\ \beta,\ \gamma)$ of eigenvalues to Hermitian (or real symmetric) n by n matrices A, B and C such that $C = A + B$. More specifically, it conjectures that a given triple occurs as eigenvalues if and only if $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$ and a set of inequalities of the form

$$\sum_{k \in K} \gamma_k \quad \leq \quad \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \tag{4.1}$$

holds for certain subsets I, J and K of $\{1,\ \cdots,\ n\}$. A good overview of the conjecture, its proof and how it relates to problems in different areas is given by [5].

The subsets defining the inequalities (4.1) in Horn's conjecture can be related to Schubert calculus by the following theorem, which is part of the more general theorem 17 in [5]. To make the notation easier, the theorem is stated with the cohomology classes $[\Omega_\lambda] \in H^*(G_k(\mathbb{C}^n))$ denoted by σ_α, where α is a weakly decreasing sequence such that $\alpha_i = n - k + i - \lambda_i$.

Theorem 4.1.1. *Let α, β and γ be weakly decreasing sequences of real numbers such that $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$. Order the elements of the subsets I, J and K of $\{1,\ \cdots,\ n\}$ such that $I = \{i_1 < \cdots < i_r\}$, and let f be a function converting subsets I indexed as above to weakly decreasing sequences,*

$$f(I) \quad = \quad (i_r - r,\ i_{r-1} - (r-1),\ \ldots,\ i_1 - 1).$$

Then α, β and γ are eigenvalues of Hermitian matrices $A + B = C$ if and only if

$$\sum_{k\in K} \gamma_k \quad \leq \quad \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j$$

for all subsets I, J and K of same cardinality r such that $\sigma_{f(K)}$ occurs in $\sigma_{f(I)} \smile \sigma_{f(J)}$ in $H^*(G_r(\mathbb{C}^n))$, and all $r < n$.

With this theorem, the problem of characterizing eigenvalue triples for Hermitian matrices $A+B = C$ is basically the same as finding out when the Littlewood-Richardson numbers $d_{\alpha\,\beta}^{\,\gamma}$ are positive.

4.2 Eigenvalue triples for skew symmetric matrices

While Horn's conjecture characterize eigenvalue triples for sums of Hermitian and real symmetric matrices, the problem of characterizing the eigenvalues for real skew-symmetric matrices is still open.

A triple $(\alpha,\ \beta,\ \gamma)$ of eigenvalues for skew symmetric matrices $A + B = C$ is, by ignoring the factor i, also a triple for the Hermitian matrices $Ai + Bi = Ci$, so the inequalities (4.1) must also be satisfied for $(\alpha,\ \beta,\ \gamma)$. The problem is that in the skew-symmetric case, the inequalities (4.1) are not enough, and the extra conditions in the lower dimensional cases can not be describe by only using inequalities on a specific form. Because of this, one can not just mimic the approach in the proof of Horn's conjecture by reformulating the problem with Schubert calculus.

Nevertheless, if one would be able to conjecture a categorization of the extra constraints on the triple $(\alpha,\ \beta,\ \gamma)$, it is possible that it can be related to the cohomologies of the Grassmannian similar to Horn's conjecture. Because of the need of extra conditions tough, one would need a larger ring structure than $H^*(G_k(\mathbb{C}^n))$.

A comprehensive text about the problem is given by [13], which includes some ideas on how to solve it.

5

Comparing the cohomologies

5.1 de Rham's theorem

We have defined two kinds of cohomologies from two different branches of mathematics. The de Rham cohomologies arose from the definition of differential forms and the differential operator, which restricts us to work with smooth manifolds. The singular cohomology on the other hand, was defined from the construction of homology, which allows us to work with a much larger class of topological spaces.

Since this thesis only treats the Grassmannian, which is a smooth manifold, both cohomologies are defined. In fact, in this context, the both cohomologies are basically the same, which one can show using Poincaré duality (defined in Section 3.4) and two other theorems, *the universal coefficient theorem for homology* and the *de Rham's theorem*.

The universal coefficient theorem gives a relation between singular homologies with different coefficients. In Chapter 3, we only concerned ourselves with the most fundamental case when the coefficients are the integers, but we can easily generalize this to take the coefficients from any given abelian group G. Let $H_i(X; G)$ denote the i^{th} homology with coefficients in an abelian group G of a space X (if G is not specified, we assume the standard case of $\mathbb{Z}$). The universal coefficient theorem for homology states the existence of the split exact sequence

$$0 \to H_i(X) \otimes_{\mathbb{Z}} G \to H_i(X;\ G) \to \mathrm{Tor}(H_{i-1}(X),\ G) \to 0$$

which implies that

$$H_i(X;\ G) \quad \cong \quad H_i(X) \otimes_{\mathbb{Z}} G \oplus \mathrm{Tor}(H_{i-1}(X),\ G).$$

We will not go into detail how to define the Tor functor; all we really need is that $\mathrm{Tor}(A,\ B)$ vanishes if A or B is torsionfree, since we are only interested in the case $G = \mathbb{R}$. I refer to Hatcher for a proof and a more in depth description of the universal coefficient theorem.

In most applications of singular homology, the choice of $\mathbb{Z}$ as the coefficients group is the most convenient one. However, to connect the singular cohomologies to the de Rham cohomologies, we need the singular cohomologies to have *real* coefficients. More specifically, we have the de Rham's theorem, which gives us the following isomorphism between de Rham and singular cohomology of a smooth manifold M

$$H^i_{dR}(M) \quad \cong \quad H^i(M;\, \mathbb{R}).$$

For a proof of de Rham's theorem, see [7, p. 206].

Assuming that we have a compact orientable n-manifold, we can use the Poincaré duality to identify the singular cohomologies, used in the de Rham's theorem, with the singular homologies, used in the universal coefficient theorem. Summing up, we get the following theorem

Theorem 5.1.1. *If M is a compact orientable n-manifold without boundary, we have the following isomorphism*

$$H^i_{dR}(M) \quad \cong \quad H_{n-i}(M) \otimes_{\mathbb{Z}} \mathbb{R}$$

5.2 Comparing cohomologies on Grassmannians

Since the Grassmannian is a compact oriented finite dimensional manifold, theorem 5.1.1 can be used to connect the main results from Chapter 2 and 3. In Chapter 2, we calculated the Poincaré polynomial for the Grassmannian $G_k(V)$ over a complex vector space V of dimension n, where the coefficient for t^i in the polynomial corresponded to the dimension of the i^{th} de Rham cohomology $H^i_{dR}(G_k(V))$. In Chapter 5.1.1 on the other hand, we proved that the Schubert cells in $G_k(\mathbb{C}^n)$ corresponds to a basis for the singular integer homology. So, to compare the results from Chapter 2 and 3, we first need an algorithm to compute the number of Schubert cells of a given dimension.

In Section 3.2 we saw that the Schubert cells in $G_k(\mathbb{C}^n)$ corresponds to k by n matrices on a specific reduced form, where the (complex) dimension of a Schubert cells is the number of arbitrary complex numbers (denoted by asterisks) in the matrix that represent it. So the calculation of the number of Schubert cells of dimension i is simply a combinatorial task of calculating the number of different matrices that have exactly i number of asterisks. We demonstrate these calculations on $G_3(\mathbb{C}^6)$, where the largest cell has the form

$$\begin{bmatrix} * & * & * & 1 & 0 & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & * & 0 & 0 & 1 \end{bmatrix}$$

which has 9 asterisks, and thus the Schubert cell it represents has dimension 9. In Section 3.5, we saw that this cell generates a homology in $H_{18}(G_3(\mathbb{C}^6))$. Since the above matrix is obviously the only one we can construct with 9 asterisks, the corresponding Schubert cell generates the whole of $H_{18}(G_3(\mathbb{C}^6))$. To get matrices representing Schubert cells of lower dimension, we can simply start the above matrix and move the ones to the left such that the resulting matrix still has the columns with ones in a strictly increasing order. For example, the only possible way to move the ones only one step to the left, is to move the one in the top row. The resulting matrix is

$$\begin{bmatrix} * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 \\ * & * & 0 & * & 0 & 1 \end{bmatrix}$$

which has 8 asterisks. It is easy to see that every time we move a one one step to the left, we loose exactly one asterisk. Thus, we have only one matrix with 8 asterisks, and the corresponding Schubert cell generates the whole of $H_{16}(G_3(\mathbb{C}^6))$. If we move a total of two steps to the left, we can do this in two ways; either we move the top row one two steps to the left, or we move both the top one and the middle one one step to the left. This gives us that $H_{14}(G_3(\mathbb{C}^6))$ is generated by exactly two cells.

With this algorithm, we see in the general case that the numbers of i dimensional cells in $G_k(\mathbb{C}^n)$ is the same as the number of weakly decreasing partitions λ of $k(n-k)-i$, i.e., the number of weakly decreasing sequences $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\sum \lambda_j = k(n-k)-i$ and $\lambda_j \leq n$. Since we can construct a completely analogues algorithm where we instead start from the matrix representing the zero dimensional cell and move the ones to the right, we get that the number of i cells is the same as the number of $k(n-k)-i$ cells.

Finishing the calculations for the number of cells in $G_3(\mathbb{C}^6)$, we get

$$H_i(G_3(\mathbb{C}^6)) \quad = \quad \begin{cases} \mathbb{Z} & \text{for } i = 18,\, 16,\, 2,\, 0, \\ \mathbb{Z}^2 & \text{for } i = 14,\, 4, \\ \mathbb{Z}^3 & \text{for } i = 12,\, 10,\, 8,\, 6, \end{cases}$$

Since we have no torsion for the singular integer cohomology on $G_k(\mathbb{C}^n)$, and since

$$\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R} \quad = \quad \mathbb{R}^n,$$

we can use theorem 5.1.1 to calculate $H^i_{dR}(G_k(\mathbb{C}^n))$ by simply replacing $\mathbb{Z}$ by $\mathbb{R}$ in $H_{n-i}(G_k(\mathbb{C}^n))$. In fact, it is even easier than that; because of the symmetry described above, $H^i_{dR}(G_k(\mathbb{C}^n))$ is actually equal to $H_i(G_k(\mathbb{C}^n))$, with $\mathbb{Z}$ replaced by $\mathbb{R}$.

Now, computing the de Rham cohomology of $G_3(\mathbb{C}^6)$ through theorem 5.1.1, how does the result compare with Poincaré series from Chapter 2? From theorem 2.10.2, we get that

$$P_t(G_3(\mathbb{C}^6)) \quad = \quad \frac{(1-t^2)(1-t^4)(1-t^6)(1-t^8)(1-t^{10})(1-t^{12})}{(1-t^2)(1-t^4)(1-t^6)(1-t^2)(1-t^4)(1-t^6)}$$

whose series representation is

$$t + t^2 + 2t^4 + 3t^6 + 3t^8 + 3t^{10} + 3t^{12} + 2t^{14} + t^{16} + t^{18}$$

whose coefficients, unsurprisingly, completely agrees with the dimensions of $H^i_{dR}(G_k(\mathbb{C}^n))$.

5.3 Concluding remarks

We have seen that the two different ways of approaching cohomologies on the complex Grassmannian basically give the same result. The methods used, however, were very different.

In Chapter 2, we generalized the de Rham complex to a double complex and applied the theory of spectral sequences to compute the cohomology of $\mathbb{C}\mathbf{P}^n$. By generalizing the wedge product to the double complex, we could also use spectral sequences to compute the ring structure of $H^*(\mathbb{C}\mathbf{P}^n)$. In 2.7, we treated the cohomologies of the projectivization of any complex vector space V, which we later used to compute the cohomologies for the projectivization of a vector bundle. In Section 2.9, we constructed the flag bundle as repeated projectivizations and could thus compute its cohomology, which we later used to compute the cohomology of the Grassmannian in Section 2.10.

In Chapter 3, we started out by defining the Schubert cells and constructing a CW complex on the complex Grassmannian. We then gave a geometrical meaning to the cells by extending them to varieties. In Section 3.4, we defined the singular homology and cohomology, and a product on singular cohomology analogous to the wedge product. We also related the homology to the cohomology trough the Poincaré duality. In Section 3.5, we showed that the Schubert cells (varieties) generated a basis for both the homology and the cohomology of the complex Grassmannian, and that multiplication of these basis had a geometrical interpretation that was useful in applications.

While the two different methods both compute the cohomology, the method in Chapter 2 gives a framework in which we can easily compute the ring structure $H^*_{dR}(G_k(V))$. When it comes to the ring structure of $H^*(G_k(\mathbb{C}^n))$ in the context of Chapter 3, we need to turn to Schubert calculus instead. It is probably possible that, with careful calculations, one could connect the ring structure of $H^*(G_k(\mathbb{C}^n))$ with $H^*_{dR}(G_k(V))$, similarly to the isomorphism 5.1.1. With such a connection, one could use the theory of Schubert calculus to the ring structure of the de Rham cohomology and vice versa.

A problem with the method in Chapter 2 is that it breaks down when trying to generalize it to real Grassmannian by replacing the Chern classes with the *Pontrjagin* classes. The method in Chapter 3 also breaks down for the real case, since the CW structure on the real Grassmannian does not skip the odd dimensional cells, but this is easier to overcome. In fact, there is an article [14] that conjectures that the ring structure of $H^*(G_k(\mathbb{R}^n), \mathbb{R})$ is *almost* as simple as the ring structure on the complex Grassmannian with the Chern classes replaced by the Pontrjagin classes; the key to prove this is perhaps to work in the context of singular cohomologies rather than de Rham cohomologies.

Describing the ring structure of the cohomology of the *oriented* Grassmannian $\tilde{G}_k(\mathbb{R}^n)$ in the case where $k = 2$ or $n \leq 8$ is done in another article [15].

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