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**Optimal distribution of  
thermal insulation and ground  
heat losses**

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Building Research**



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OPTIMAL DISTRIBUTION OF THERMAL INSULATION AND GROUND HEAT LOSSES

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## 1 INTRODUCTION

The purpose of thermal insulations of buildings and other constructions, which maintain a temperature different from the surroundings, is to decrease the inevitable heat loss. A layer of insulation material is, wherever it is feasible, placed along the boundary of the construction.

Let us now assume that we have a given amount of the insulation material at our disposal in a particular case. The question arises how to distribute the insulation over the different parts of the boundary surface. It is clear that one should use more insulation on parts that are more exposed to the surrounding. The thickness of the insulation layer will then in general vary over the boundary surface. The insulation material is to be distributed over the boundary surface so that the total heat loss is minimized.

The optimization problem is often trivial. For example, the thickness of the insulation layer shall of course be constant for a wall of a building except for corner regions, where the situation is more complicated. The optimization problem arises, when the heat flow through the insulation is coupled to a multi-dimensional heat flow process in solid regions outside the insulation. Corner regions and other cases, for which the heat flow pattern is more complex, also pose an optimization problem.

An important case is the insulation of buildings and other structures along surfaces that border on the surrounding ground. The surfaces, over which the insulation is to be distributed in an optimal way, are the ground plate of the building or the walls and floor of the cellar. Other equally important cases are underground constructions such as district heating pipes, culverts and caverns.

A lot of systems for interseasonal storage of sensible heat in a large ground volume have been proposed and contemplated during the last years of energy crisis. A considerable thermal insulation

against heat losses to the ground surface and the surrounding ground is usually necessary. The amount of insulation material may become an economically important factor for the system. Then it is valuable to know how to distribute the insulation in an optimal way.

Let us directly in this introduction illustrate the theory and the results, which are presented in this paper, with an example. Consider a long house. The ground plate of the house lies directly on the earth. Between the plate and the ground surface there is an insulation layer. The thickness of the insulation layer is in the optimal case variable along the surface. Figure 1 shows a vertical cross-section of the house and the ground.

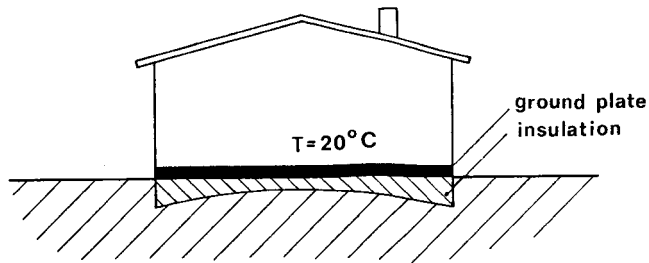


Figure 1 Introductory optimal insulation problem.

A given amount of insulation material is to be distributed along the ground plate so that the heat loss is minimized. The house is assumed to be quite long so that the heat flow process in the ground below the house is essentially two-dimensional in the plane of Figure 1.

The width of the house is 10 m ( $L=5$  m). The thermal conductivity of the insulation material is 0.05 of that of the ground below the



house ( $\lambda_i/\lambda_o = 0.05$ ). We have at our disposal a certain volume of insulation material, which corresponds to a thickness  $d_m$ , if the insulation were distributed uniformly along the ground plate.

There is a certain minimal value for the mean insulation thickness  $d_m$ . We have from formula (6.8) for a plate on the ground:

$$d_{m\min} = 0.05 \cdot 5 \left(1 - \frac{\pi}{4}\right) \approx 5.4 \text{ cm} \quad (1.1)$$

Let us first assume that the given  $d_m$  is precisely equal to this value:  $d_m = d_{m\min} = 5.4 \text{ cm}$ . The thickness  $\tilde{d}$  of the optimal insulation shall then from formula (6.7) vary along the boundary  $-5 < x < 5$  according to:

$$\tilde{d}(x) = 0.25 \cdot \left(1 - \sqrt{1 - \left(\frac{x}{5}\right)^2}\right) \quad -5 < x < 5$$

$$(d_m = 0.054 \text{ m}) \quad (1.2)$$

The insulation thickness is zero at the center  $x=0$ . It increases to a maximum value of 0.25 m at the edges  $x = \pm 5 \text{ m}$  of the ground plate. Curve II in Figure 2 shows this optimal insulation distribution.

Suppose next that we have an amount of insulation that corresponds to a higher  $d_m$  equal to say  $15+5.4 \text{ cm}$ . It is shown in the paper that the excess insulation above the critical amount  $d_{m\min}$  shall be distributed evenly along the insulation surface. The insulation thickness shall then in this case of optimal insulation vary as:

$$\tilde{d}(x) = 0.15 + 0.25 \left(1 - \sqrt{1 - \left(\frac{x}{5}\right)^2}\right) \quad -5 < x < 5$$

$$(d_m = 0.15 + 0.054 \text{ m}) \quad (1.3)$$

Suppose finally that the available insulation volume corresponds to a  $d_m$  below the critical value 5.4 cm. Then we must not put any insulation at all along a certain segment around the mid-point  $x=0$ . All insulation is to be placed along the remaining parts closer

to the edges of the ground plate. The precise form of the optimal thickness curve is not discussed in this study. This is deferred to a later study.

Figure 2 shows how the insulation thickness shall vary along the ground plate in the three discussed cases in order to obtain minimal total heat loss to the ground.

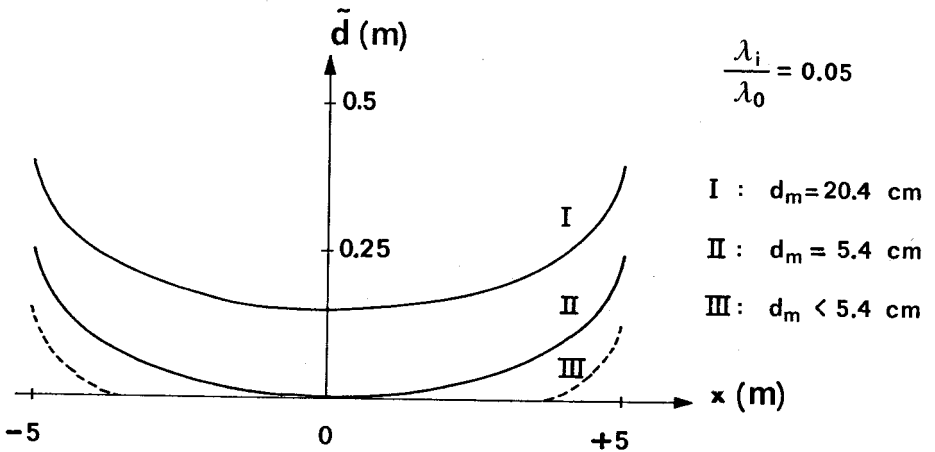


Figure 2 Optimal insulation thickness  $\tilde{d}$  for a ground plate for three different values of mean insulation thickness  $d_m$ .

The optimal insulation theory leads to quite simple formulas for the heat loss from the insulated area. This provides the second theme of this study. The concept of insulating soil thickness is introduced. The insulating capacity of the ground below the building depends on the solution of a multi-dimensional heat flow problem. The soil is from an insulating point of view replaced by a

single thickness. This is elaborated in sections 13 and 14.

Let us note that the insulating soil thickness for our considered case ( $L=5$  m) is from (13.4):

$$L \cdot u_m = L \cdot \frac{\pi}{4} = 5 \cdot \frac{\pi}{4} \approx 3.9 \text{ m} \quad (1.4)$$

The insulating capacity of the soil below the ground plate corresponds therefore to a soil layer with a thickness 3.9 m. The assumptions and precise meaning of this are discussed in the following.

There are some basic assumptions in this study. The heat flow problem is a steady-state one. This is not a severe restriction, since the net heat loss from a superimposed periodic heat flow process is zero. The finite thickness of the insulation layers are neglected. They are considered as an infinitely thin surface with a finite thermal resistance. This is a common and quite reasonable approximation. The thermal conductivity in the soil may have different values in different parts of the soil.

There are a lot of problems to which this optimal insulation theory may be applied. We will in this paper study cases, for which the whole heat loss surface is covered by insulation. Another study will be devoted to cases, when only a part of the heat loss surface is to be insulated. We also plan to treat cases, where the insulation boundary is chosen in an optimal way.



## 2 OPTIMIZATION PROBLEM

Let  $V$  denote the heat flow region of our optimization problem. The volume  $V$  may be the ground below a house. In general,  $V$  may be any region of solid material.

The boundary surface of  $V$  consists of two parts  $S_0$  and  $S_1$ . The surface  $S_1$  is the boundary area of the building or other structure which is to be insulated in an optimal way against heat losses out into the heat flow region  $V$ . The remaining part  $S_0$  of the boundary surface of  $V$  will normally border on the ambient air.

The thermal conductivity  $\lambda$  may vary in any way throughout  $V$ . The temperature at the boundary  $S_0$  is given as  $T=T_0$ . Along the surface  $S_1$  there is an insulation layer with varying thickness. Outside the insulation on the building side there is a prescribed temperature  $T=T_1$ . We assume that  $T_0$  and  $T_1$  are constants. The extension of the theory to cases of variable prescribed temperatures on  $S_0$  and outside the insulation at  $S_1$  is deferred to later studies. Other cases with more complicated boundary conditions on  $S_0$  and other possibilities are also left to later studies.

Figure 3 shows the considered type of heat flow process.

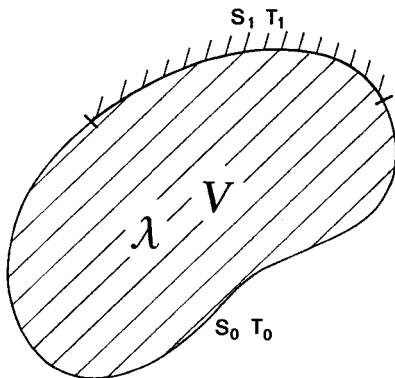


Figure 3 Considered type of heat flow process.  $S_1$  is covered by an insulation layer of variable thickness, which is to be chosen so that the heat loss over  $S_1$  into  $V$  is minimized.

In the example of Figure 1  $V$  is the ground below the house. The ground plate of the house represents the surface  $S_1$ , while the ground surface outside the house is  $S_0$ .

The temperature  $T(x,y,z)$  satisfies the steady-state heat conduction equation:

$$\nabla \cdot (\lambda \nabla T) = 0 \quad \text{in } V \quad (2.1)$$

Here  $\nabla = (\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z})$  denotes the gradient operator. On the boundary  $S_0$  we have:

$$T = T_0 \quad \text{on } S_0 \quad (2.2)$$

Let  $\lambda_i$  be the thermal conductivity of the insulation material. The varying thickness of the insulation layer is denoted  $d$ . The function  $d$  is defined over the surface  $S_1$ . We have the boundary condition:

$$\lambda \frac{\delta T}{\delta n} = \lambda_i \frac{T_1 - T}{d} \quad \text{on } S_1 \quad (2.3)$$

Here  $\frac{\delta T}{\delta n}$  is the outward normal derivative. The boundary condition (2.3) means that the finite thickness of the insulation layer is neglected. The layer is treated as an infinitely thin layer with a finite thermal resistance  $d/\lambda_i$ .

The rate of heat loss from the building over  $S_1$  into  $V$  is denoted  $Q_1$  (J/s). With a surface integral over  $S_1$  we have:

$$Q_1 = \iint_{S_1} \lambda \frac{\delta T}{\delta n} dS \quad (2.4)$$

Consider a certain insulation thickness distribution  $d$  over  $S_1$ . Let  $d_m$  denote the corresponding mean insulation thickness:

$$\iint_{S_1} d \cdot dS = d_m \cdot \iint_{S_1} dS \quad (2.5)$$

The integral on the right of (2.5) is the area of the surface  $S_1$ . The quantity  $d_m$  is a measure of the total amount of insulation material.

The heat loss  $Q_1$  decreases, if the insulation is increased throughout  $S_1$ . The optimization is always made for a fixed amount of insulation material, i.e. for given  $d_m$ .

We have the following optimization problem. The heat loss  $Q_1$  of the heat flow problem (2.1)-(2.3) is to be minimized for different choices of insulation thickness  $d$  over  $S_1$ . The only restriction on the function  $d$  on  $S_1$  is that its mean value  $d_m$  is prescribed - see (2.5). It should perhaps also be mentioned that  $d$  must be positive throughout  $S_1$ .





### 3 CRITERION FOR OPTIMAL INSULATION

In order to get the optimal distribution of the insulation material along  $S_1$  one has to compare different distributions and in particular study what happens to the heat loss, when the insulation material is redistributed.

Let  $d$  be any insulation distribution on  $S_1$ . The corresponding temperature  $T$  satisfies (2.1)-(2.3). Let  $\delta d$  be a small change of the insulation thickness:

$$d \rightarrow d + \delta d \quad \text{on } S_1 \quad (3.1)$$

The function  $\delta d$  is arbitrary on  $S_1$ . The heat loss  $Q_1$  changes to a new value, which depends on the choice of function  $\delta d$ . The change contains linear terms in  $\delta d$ , quadratic terms and so on. The first-order approximation means that only the linear contribution is retained. This is a reasonable approximation for a sufficiently small function  $\delta d$ . Let  $\delta Q_1$  denote the first-order variation or change of  $Q_1$  in the change (3.1).

It is shown in the appendix that

$$(T_1 - T_0) \delta Q_1 = - \iint_{S_1} \frac{\delta d}{\lambda_1} \left( \lambda \frac{\partial T}{\partial n} \right)^2 dS \quad (3.2)$$

This formula is remarkably simple. It provides the whole basis of the optimal insulation theory of this paper. The derivation of the formula involves variational calculus, vector analysis and a special thermodynamical concept, the thermality. The details are presented in the appendix.

Formula (3.2) involves only the given change  $\delta d$  and the boundary heat flow  $\lambda \frac{\partial T}{\partial n}$  of the original problem. The usefulness of the formula is due to the fact that the new temperature solution after the change (3.1) is not at all involved. We do not have to solve the new problem in order to get the new heat loss (in the first

order for small  $\delta d$ ).

The optimization is done with the subsidiary condition of a prescribed mean insulation thickness  $d_m$  according to (2.5). Let  $\tilde{d}$  be the optimal insulation distribution over  $S_1$ . The tilde sign  $\sim$  will be used to denote the optimal case. The corresponding temperature field is  $\tilde{T}$ . Formula (3.2) is of course applicable to the optimal case  $\tilde{T}$ , when the heat loss is minimal. The change  $\delta Q_1$  must in this case be non-negative for any permissible redistribution  $\delta d$  of the insulation over  $S_1$ . The mean thickness  $d_m$  is constant:

$$\iint_{S_1} \delta d \cdot dS = 0 \quad (3.3)$$

We must have the inequality

$$- \iint_{S_1} \frac{\delta d}{\lambda_i} \left( \lambda \frac{\partial \tilde{T}}{\partial n} \right)^2 dS \geq 0 \quad (3.4)$$

for any  $\delta d$  that satisfies (3.3). We will show that this implies that the heat flow  $\lambda \frac{\partial \tilde{T}}{\partial n}$  is constant over  $S_1$ . The integral (3.4) vanishes due to (3.3), if  $\lambda \frac{\partial \tilde{T}}{\partial n}$  is constant over  $S_1$ .

Suppose now that  $\lambda \frac{\partial \tilde{T}}{\partial n}$  is not constant over  $S_1$ . We single out two small areas on  $S_1$  such that  $\lambda \frac{\partial \tilde{T}}{\partial n}$  is greater throughout the first area than throughout the other one. Let  $\delta d$  be a variation which is positive on the first area, negative on the second, and zero elsewhere so that (3.3) is satisfied. Then from (3.2)  $\delta Q_1$  would become negative. The optimum assumption is violated. We may conclude that  $\lambda \frac{\partial \tilde{T}}{\partial n}$  is constant over  $S_1$ .

We have arrived at the important conclusion that the optimum thermal insulation requires that

$$\lambda \frac{\partial \tilde{T}}{\partial n} = q_1 \quad \text{on } S_1 \quad (\tilde{d} > 0) \quad (3.5)$$

Here  $q_1$  is a constant.

There is another restriction that has not been discussed yet. The insulation thickness  $d$  (and  $d+\delta d$ ) must be positive throughout  $S_1$ . Our conclusion (3.5) is only valid over the part of  $S_1$ , where  $\tilde{d}$  is strictly positive. In an area of  $A_1$ , where  $\tilde{d}$  is zero, we can only have positive variations  $\delta d > 0$ . Only half of our argument is so to speak valid. In areas, where  $\tilde{d}$  is zero we can merely conclude that  $\lambda \frac{\delta \tilde{T}}{\delta n}$  cannot exceed the constant value  $q_1$ :

$$\lambda \frac{\delta \tilde{T}}{\delta n} \leq q_1 \quad \text{on a part of } S_1 \text{ with } \tilde{d}=0 \quad (3.6)$$

Optimization cases, where  $\tilde{d}$  is zero on a part of  $S_1$ , will not be dealt with in this study. They will be the topic of another paper.

Condition (3.5) gives, except for the complication (3.6), a general criterion for optimal insulation. The insulation is to be distributed so that the heat flow is the same in all parts of the insulated area  $S_1$ .

This general criterion is intuitively reasonable. Suppose you have a higher heat loss at one point on  $S_1$  than at another one. It is then natural to remove some insulation from the latter point and use it to increase the insulation at the first point. But this redistribution will change the whole heat flow problem and in particular the heat flow through the rest of  $S_1$ . The advanced mathematical technique is required to really prove that the total heat loss will diminish for the considered redistribution of the insulation.

The criterion (3.5) provides an important guide-line in practical insulation problems. Consider an area  $S_1$  which is exposed to an indoor air with the temperature  $T_1$ . The heat flux  $\lambda \frac{\delta T}{\delta n}$  is proportional to the surface temperature. We can conclude that optimal insulation is equivalent to a constant surface temperature.

An insulation is not optimal if the surface temperature varies.



## 4 BASIC CONSTANT-FLOW SOLUTION

The optimal solution  $\tilde{T}$  shall have a constant heat inflow over  $S_1$ . The value of the constant is to be adjusted so that the mean insulation thickness becomes equal to the prescribed value  $d_m$ . We have to solve the following problem:

$$\begin{aligned} \nabla \cdot (\lambda \nabla \tilde{T}) &= 0 && \text{in } V \\ \lambda \frac{\delta \tilde{T}}{\delta n} &= q_1 && \text{on } S_1 \\ \tilde{T} &= T_0 && \text{on } S_0 \end{aligned} \quad (4.1)$$

The function  $(\tilde{T}-T_0)/q_1$  shall be zero on  $S_0$  and have unit heat flow on  $S_1$ . The essence of (4.1) is therefore to solve the heat flow problem for unit heat flow on  $S_1$  and zero temperature on  $S_0$ .

We will solve this problem in a dimensionless form. Let  $L$  be any length of the heat flow problem. We introduce dimensionless coordinates with  $x' = \frac{x}{L}$  and so on. The scaled volume is denoted  $V'$ . Let  $\lambda_0$  be a reference thermal conductivity. We use the dimensionless conductivity  $\lambda' = \frac{\lambda}{\lambda_0}$ .

We get the following heat flow problem:

$$\begin{aligned} \nabla' \cdot (\lambda' \nabla' u) &= 0 && \text{in } V' \\ \lambda' \frac{\delta u}{\delta n'} &= 1 && \text{on } S'_1 \\ u &= 0 && \text{on } S'_0 \end{aligned} \quad (4.2)$$

The solution of this problem will be called the fundamental constant-flow solution.

The optimal temperature  $\tilde{T}$  of (4.1) is from (4.2) given by:

$$\tilde{T}(x, y, z) = T_0 + \frac{q_1 L}{\lambda_0} \cdot u\left(\frac{x}{L}, \frac{y}{L}, \frac{z}{L}\right) \quad (4.3)$$

On the boundary  $S_1$  we must also satisfy the original boundary condition (2.3):

$$\tilde{T} + \frac{\tilde{d}}{\lambda_i} \lambda \frac{\delta \tilde{T}}{\delta n} = T_1 \quad \text{on } S_1 \quad (4.4)$$

Insertion of (4.3) and (3.5) gives:

$$T_0 + \frac{q_1 L}{\lambda_0} u + \frac{\tilde{d}}{\lambda_i} q_1 = T_1 \quad (4.5)$$

This formula shows how  $\tilde{d}$  is to be chosen in order to minimize the heat loss. The optimal distribution  $\tilde{d}$  is determined by the values of the basic constant-flow solution  $u$  on  $S_1$ . The constant  $q_1$  is determined by the prescribed mean insulation thickness  $d_m$ . Let  $u_m$  be the mean value of  $u$  over  $S_1$ :

$$u_m \cdot \iint_{S_1} dS = \iint_{S_1} u dS \quad (4.6)$$

Then we have, if we take the mean value of (4.5) over  $S_1$ :

$$T_0 + \frac{q_1 L}{\lambda_0} u_m + \frac{d_m}{\lambda_i} q_1 = T_1 \quad (4.7)$$

This formula determines  $q_1$ :

$$q_1 = \frac{T_1 - T_0}{\frac{d_m}{\lambda_i} + \frac{L u_m}{\lambda_0}} \quad (4.8)$$

From (4.8) and (4.5) we get the optimal insulation:

$$\tilde{d} = d_m + \frac{\lambda_i}{\lambda_o} L u_m - \frac{\lambda_i}{\lambda_o} L u \quad (4.9)$$

We finally have the restriction that  $d$  cannot be negative. The smallest value for  $\tilde{d}$  is obtained for the highest  $u$ . Let  $u_{\max}$  be the maximum value of  $u$  on  $S_1$ . Then we have the condition

$$d_m + \frac{\lambda_i}{\lambda_o} L(u_m - u_{\max}) \geq 0 \quad (4.10)$$

This is a requirement on the prescribed mean insulation thickness  $d_m$ .





## 5 BASIC FORMULAS

We will in this paragraph summarize the basic formulas and results.

Let  $L$  be a reference length of the heat flow problem. The coordinates are scaled with this length to a dimensionless formulation. Let  $\lambda_0$  be a constant reference thermal conductivity. The dimensionless thermal conductivity is then  $\lambda' = \frac{\lambda}{\lambda_0}$ .

The basic constant-flow solution  $u$  satisfies (4.2). The maximum and average values of  $u$  on  $S_1$  are denoted  $u_{\max}$  and  $u_m$  respectively.

The prescribed mean insulation thickness  $d_m$  must satisfy:

$$d_m \geq d_{\min} = \frac{\lambda_i}{\lambda_0} L \cdot (u_{\max} - u_m) \quad (5.1)$$

The results to be presented in the following are only valid, if inequality (5.1) is fulfilled. The optimization problem for smaller  $d_m$  is deferred to a later study. We will only mention that, in such a case, the optimal insulation is zero for some internal region of  $S_1$ . An example is indicated as III in Figure 2.

The optimal insulation distribution on  $S_1$  is from (5.1) and (4.9):

$$\begin{aligned} \tilde{d} &= d_m - d_{\min} + \frac{\lambda_i}{\lambda_0} L \cdot \tilde{u} \\ \tilde{u} &= u_{\max} - u \end{aligned} \quad (5.2)$$

The function  $\tilde{u}$  gives the functional variation of  $\tilde{d}$  over  $S_1$ . We will call  $\tilde{u}$  the optimal insulation function. This function is zero at the maximum point of  $u$  and positive elsewhere. The insulation thickness  $\tilde{d}$  is proportional to  $\tilde{u}$  in the limiting case, when  $d_m = d_{\min}$ . The proportionality factor contains the length  $L$  and the conductivity ratio  $\lambda_i/\lambda_0$ . The insulation thickness is in the limit-

ing case zero at  $u=u_{\max}$  and positive elsewhere.

It is important to note that any additional insulation  $d_m-d_{\min}$  is to be distributed evenly over  $S_1$ .

The heat loss  $q_1$  ( $J/m^2s$ ) through  $S_1$  is given by the simple formula

$$q_1 = \frac{T_1 - T_0}{\frac{d_m}{\lambda_i} + \frac{L \cdot u_m}{\lambda_o}} \quad (5.3)$$

This is an important formula. The heat flow  $q_1$  is equal to the temperature difference  $T_1 - T_0$  divided by a thermal resistance. The first term  $d_m/\lambda_i$  is the thermal resistance of an insulation layer with a thickness  $d_m$  equal to the mean insulation thickness. The second factor  $L \cdot u_m/\lambda_o$  may be interpreted as a mean thermal resistance of the soil. The equivalent thickness is  $L \cdot u_m$  and the thermal conductivity is  $\lambda_o$ . Figure 4 illustrates the heat loss formula (5.3).

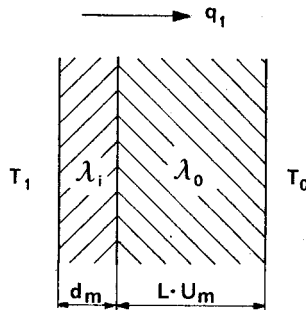


Figure 4 Heat loss formula (5.3). The soil may be regarded as a layer with the thickness  $L \cdot u_m$ .

The temperature distribution is:

$$\tilde{T}(x, y, z) = T_0 + \frac{L q_1}{\lambda_o} u\left(\frac{x}{L}, \frac{y}{L}, \frac{z}{L}\right) \quad (5.4)$$

## 6 ANALYTICAL SOLUTIONS

The optimal thermal insulation problem is solved, when the appropriate basic constant-flow solution  $u$  is determined. There are some cases, when it is possible to solve the problem analytically. The heat flow problem is more complicated than normally due to the mixed boundary conditions. The solution  $u$  is prescribed on one part of the boundary, while the normal derivative is prescribed on the remaining part. We will in this section present two important analytical solutions. The remaining part of this study will be devoted to numerically obtained results.

The two solutions that are presented here are derived in (\*). We will not repeat the derivations here.

The first solution concerns the two-dimensional case of a plate on the ground. The heat flow takes place in the vertical cross-section below a long building - see Figure 1. The ground plate has the width  $2L$ . The ground occupies the region  $-\infty < x < \infty$ ,  $z < 0$ . The plate lies along  $z=0$ ,  $-L < x < L$ . The thermal conductivity in the ground is  $\lambda_0$ .

The corresponding constant-flow solution  $u(x,z)$  shall satisfy:

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta z^2} = 0 \quad \begin{array}{l} -\infty < x < \infty \\ z < 0 \end{array} \quad (6.1)$$

The boundary conditions are:

$$u(x,0) = 0 \quad x > 1, x < -1 \quad (6.2)$$

$$\left. \frac{\delta u}{\delta z} \right|_{z=0} = 1 \quad -1 < x < 1 \quad (6.3)$$

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(\*) Johan Claesson: Theory of optimal thermal insulation. Mathematical Physics, Lund, Sweden. To appear.

The heat flow problem is illustrated in Figure 5.

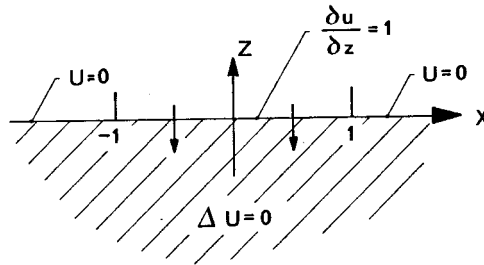


Figure 5 Basic constant-flow problem for a long plate on the ground.

The solution to problem (6.1)-(6.3) is:

$$u(x, z) = \sqrt{f^2 + x^2 z^2} + f + z \quad (6.4)$$

$$f = \frac{1+z^2-x^2}{2}$$

The temperature  $u(0, z)$  vertically downwards from the center of the plate is shown in Figure 7. The derivative  $\left. \frac{\partial u}{\partial z} \right|_{z=0}$  at the ground surface is shown in Figure 8.

The optimal insulation distribution is determined by the values of  $u$  along the ground plate. We have, when  $x$  is replaced by the dimensionless variable  $x/L$ :

$$u\left(\frac{x}{L}, 0\right) = \sqrt{1 - \frac{x^2}{L^2}} \quad -L < x < L \quad (6.5)$$

This function is shown in Figure 6.

In particular we have the maximum and mean values:

$$u_{\max} = 1 \quad u_m = \frac{\pi}{4} \quad (6.6)$$

The optimal thermal insulation distribution  $\tilde{d}$  is from (5.2) and (5.1):

$$\tilde{d} = d_m - d_{\min} + \frac{\lambda_i}{\lambda_o} L \cdot \left(1 - \sqrt{1 - \frac{x^2}{L^2}}\right) \quad (6.7)$$

Figure 2:I,II shows the character of the optimal insulation distribution. The formula is only valid, when the mean insulation thickness  $d_m$  exceeds the minimum value:

$$d_m \geq d_{\min} = \frac{\lambda_i}{\lambda_o} L \cdot \left(1 - \frac{\pi}{4}\right) \quad (6.8)$$

The heat loss  $q_1$  (J/m<sup>2</sup>s) is with (5.3):

$$q_1 = \frac{T_1 - T_o}{\frac{d_m}{\lambda_i} + \frac{L}{\lambda_o} \cdot \frac{\pi}{4}} \quad (6.9)$$

The other case, for which there is a relatively simple analytical solution, has cylindrical symmetry. Again, the ground plate lies directly on the ground surface. It has the shape of a circular disc. The radius of the disc is R.

The basic constant-flow solution is a function of the depth z and of the radial distance r from a vertical axis through the center of the disc:  $u=u(r,z)$ . The disc lies along  $z=0$ ,  $0 \leq r \leq R$ . The temperature distribution along the disc for the basic constant-flow solution becomes:

$$u(r,0) = \frac{2}{\pi} \sqrt{1 - \frac{r^2}{R^2}} \quad 0 \leq r \leq R \quad (6.10)$$

We have in this case:

$$u_{\max} = \frac{2}{\pi} \quad u_m = \frac{4}{3\pi} \quad (6.11)$$

The optimal thermal insulation distribution  $\tilde{d}$  is then:

$$\tilde{d} = d_m - d_{\min} + \frac{\lambda_i}{\lambda_o} R \frac{2}{\pi} \left(1 - \sqrt{1 - \frac{r^2}{R^2}}\right) \quad (6.12)$$

The formula is as usual only valid when the mean insulation thickness  $d_m$  exceeds the minimum value:

$$d_m \geq d_{\min} = \frac{\lambda_i}{\lambda_o} R \cdot \frac{2}{3\pi} \quad (6.13)$$

The heat loss  $q_1$  ( $\text{J}/\text{m}^2\text{s}$ ) becomes:

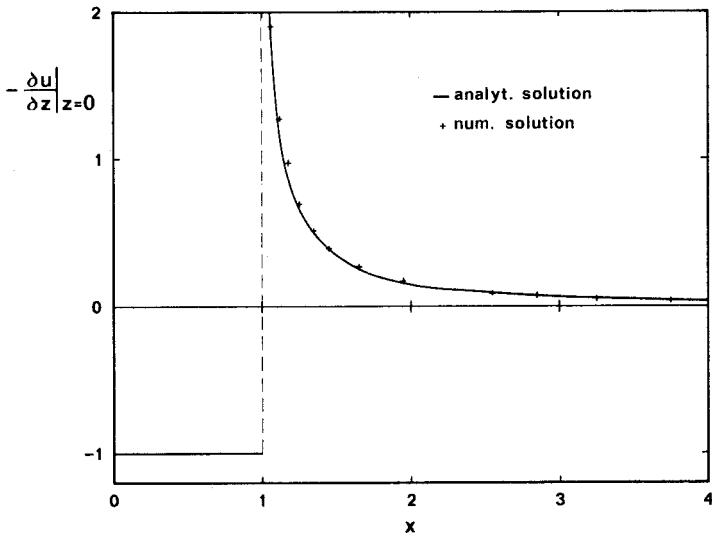
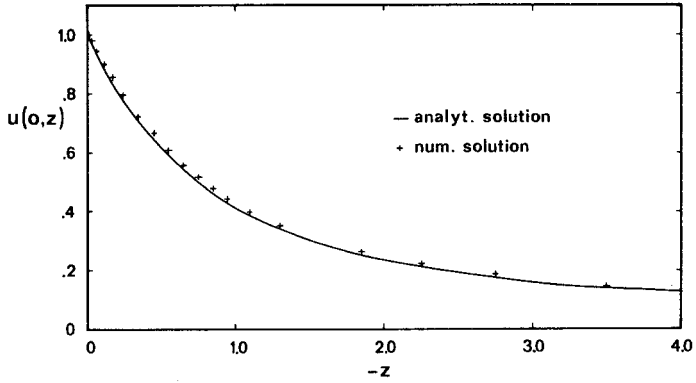
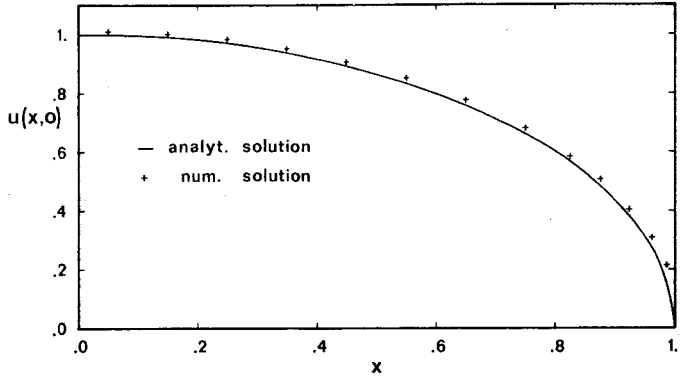
$$q_1 = \frac{T_1 - T_o}{\frac{d_m}{\lambda_i} + \frac{R}{\lambda_o} \cdot \frac{4}{3\pi}} \quad (6.14)$$

## 7 NUMERICAL PROCEDURE

The basic constant-flow problem will be solved numerically for a lot of different cases in the following. The steady-state temperature field is determined with finite differences. Overrelaxation is used.

The accuracy of the numerical method has been tested against some analytical solutions.

In the first test we have used the exact solution for the long plate on the ground - see Figures 1 and 5. The defining conditions are (6.1), (6.2) and (6.3). The analytical solution is given by (6.4). The ground region is divided into a rectangular mesh. The mesh distances are variable in the horizontal and in the vertical directions. Smaller distances are used near the edge of the plate. The mesh size is quite large far away from the plate. Figures 6,7, and 8 show a comparison between the analytical solution and the numerically computed one. Figure 6 shows the temperature distribution along the ground plate. This is the most important quantity, since it gives directly the optimal insulation distribution. The numerical accuracy is quite good. The largest errors occur at the edge ( $x=1$ ). The maximum absolute error is in this case about 0.03. The mesh consisted of  $39 \times 26$  points. Figure 7 shows the temperature distribution vertically downwards from the mid-point of the plate. Figure 8 shows the heat flow through the ground surface.



Figures 6,7,8 Numerical accuracy for the case of a plate on the ground.



The mesh consisted of thirty-nine points in the horizontal direction. Thirteen of these covered the length  $0 \leq x \leq 1$  of the plate. Coarser mesh divisions have also been tested. A particularly important quantity is  $u_m$ . We got the following relative errors for  $u_m$  for different meshes:

Total number of mesh points	39x26	19x13	17x12	15x11
Number of cells along the ground plate	13	6	5	4
Error for $u_m$	2.6%	6%	8%	16%

The finest mesh with an error of 2.6% is quite satisfactory. This mesh precision has been used in all two-dimensional cases (plane and cylindrical). A mesh precision that corresponds to 5 cells along the ground plate has been used in the three-dimensional cases. The error of 8% for  $u_m$  is judged to be acceptable.

It may be noted that the relative error at the edge of the plate was quite high for all mesh divisions. The influence from this error becomes smaller, when more cells are used. The reason for this edge error seems to be the complex character of the heat flow at the immediate vicinity of the edge of the plate.

The boundaries far away from the plate region pose no difficulty. We assume zero heat flow at  $x=10$  and at  $z=-10$  ( $L=1$ ). The change of the solution, when these boundaries are moved to  $x=20$  and  $z=-20$ , is quite negligible.

The discussed comparison concerns a plane two-dimensional case. We have also compared the numerical results with the analytical solution for the circular disc on the ground. This is a problem with cylindrical symmetry. Using the fine mesh, the error was 4%.

Invariably, the numerical computations give values above the exact ones.

In order to get a truly three-dimensional test case we have used

a well-known analytical solution for the temperature distribution in the ground below a rectangular plate which is kept at constant temperature  $T_1$ , while the ground surface outside the plate is kept at a temperature  $T_0$ . The mesh consisted of five cells below the plate in the shortest direction. The numerically computed vertical flow at the mid-point of the rectangular plate deviated by 2% from the analytical value. The error in cell number four from the center was 8%.

From these comparisons between numerical and analytical values we estimate that we have the following accuracy. The errors in the two-dimensional cases (plane and cylindrical) are for  $u_{\max}$  and  $u_m$  less than 2% and 4% respectively. The error in  $u$  along  $S_1$  is less than 3% except for the immediate vicinity of edges and corners. The errors for  $u_{\max}$  and  $u_m$  are 2% and 10-15% respectively in the three-dimensional computations. The error for  $u$  is less than 10% except for the immediate vicinity of edges and corners.

The basic constant-flow solution has been determined for about 60 different cases. The results are given in the following paragraphs. A computer run of a two-dimensional case (including cylindrical cases) requires 0.5-5 minutes on a UNIVAC 1108. The mesh has in these cases consisted of from 39x26 to 50x80 points. A computer run of a three-dimensional case requires 2-7 minutes. The mesh has consisted of about 6000 points.

## 8 OPTIMAL INSULATION OF A GROUND PLATE

We will in this paragraph study the problem how to insulate a ground plate in an optimal way. The ground plate has a rectangular shape with the sides  $2L$  and  $2L_1$ , where  $L_1 \geq L$ . See Figure 9.

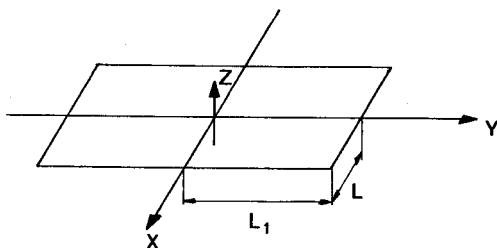


Figure 9 Rectangular ground plate with the sides  $2L$  and  $2L_1$ .

The ground fills the half-space  $z < 0$ . The problem is scaled with the length  $L$ . The fundamental constant-flow solution  $u(x,y,z)$  shall satisfy:

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = 0 \quad z < 0 \quad (8.1)$$

At the ground surface we have the boundary conditions:

$$\frac{\delta u}{\delta z} = 1 \quad z=0, \quad -1 < x < 1, \quad -\frac{L_1}{L} < y < \frac{L_1}{L} \quad (8.2)$$

$$u = 0 \quad z=0, \quad x \text{ and } y \text{ outside ground plate}$$

The solution depends on the shape parameter  $L_1/L$ . The problem has been solved numerically for  $L_1/L=1, 1.5, 2, 3, 5$ . The results are shown in Figure 10. The diagrams show the optimal insulation function  $\tilde{u}=u_{\max}-u$  as a function of  $x/L$  for different  $y/L$ . The accuracy of the values is distinctly lower near the edges of the plate. These results with lower accuracy are given with dashed lines.

Figure 11 shows the mean value  $u_m$  as a function of  $L_1/L$ , while Figure 12 shows  $u_{\max}-u_m$ .

Let  $d_m$  be the prescribed mean insulation thickness over the ground plate. Our results below are as usual only valid if

$$d_m \geq d_{\min} = \frac{\lambda_i}{\lambda_0} L \cdot (u_{\max} - u_m) \quad (8.3)$$

The factor  $u_{\max}-u_m$  is given in Figure 12.

The optimal insulation thickness over the ground plate is now

$$\tilde{d} = d_m - d_{\min} + \frac{\lambda_i}{\lambda_0} L \tilde{u} \quad (8.4)$$

The insulation thickness varies as  $\tilde{u}$ , which is given in Figure 10 for different  $L_1/L$ . The excess insulation above the minimum value,  $d_m-d_{\min}$ , is to be distributed evenly over the plate.

The limiting case  $L_1/L = \infty$  gives the previous two-dimensional case, which was solved analytically. The values of  $\tilde{u}$  for small  $y/L$  are rather close to the two-dimensional case, which is given by (6.5) and (6.6). We see from Figure 10 for  $L_1/L=3$  and 5 that the solution in the central region ( $y/L < 1.5$  and  $y/L < 2.5$  respectively) is not far from the two-dimensional one. The value of  $u_m$  for  $L_1/L=5$  is from Figure 11  $u_m=0.78$ . This value is however perhaps 10% too high, since the two-dimensional value is  $u_m = \frac{\pi}{4} = 0.785$ .

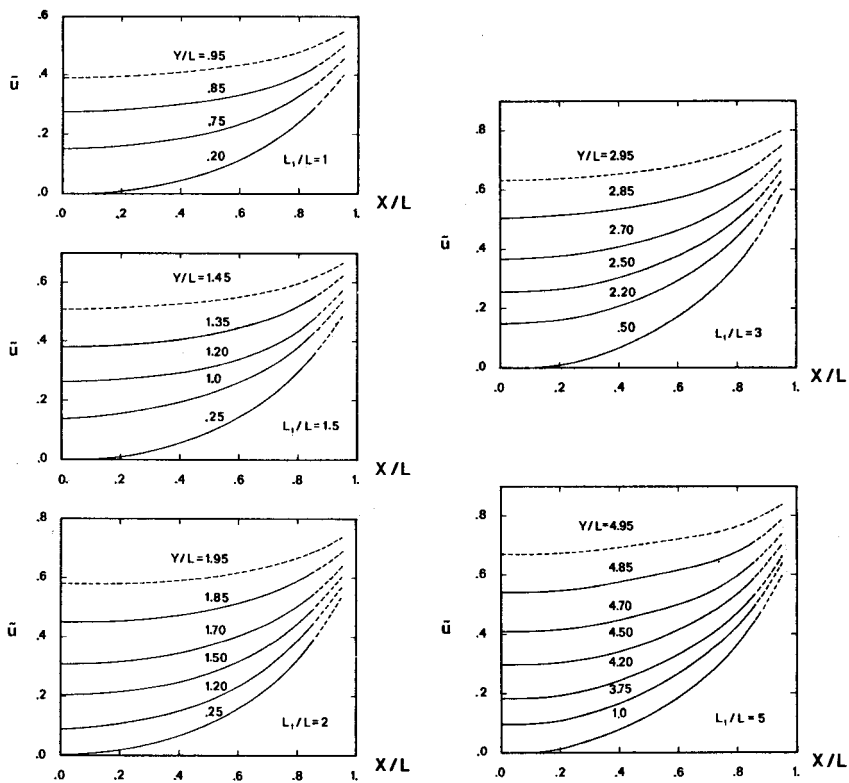


Figure 10 The optimal insulation function  $\tilde{u}$  for a rectangular plate on the ground. Each diagram refers to a certain shape  $L_1/L$ .

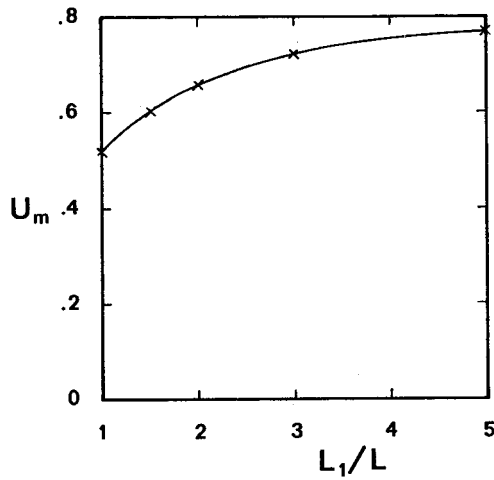


Figure 11 The mean value  $u_m$  as a function of the shape  $L_1/L$  of the rectangular plate.

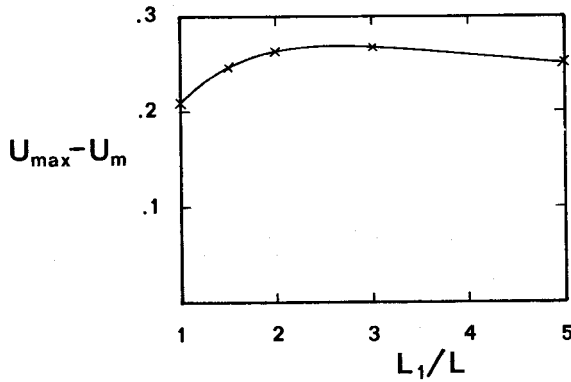


Figure 12 The quantity  $u_{\max} - u_m$  as a function of the shape  $L_1/L$  of the rectangular plate.

The heat loss ( $\text{J/m}^2\text{s}$ ) of the optimally insulated plate is given by formula (5.3):

$$q_1 = \frac{T_1 - T_0}{\frac{d_m}{\lambda_i} + \frac{L u_m}{\lambda_o}} \quad (8.5)$$

The function  $u_m$  is shown in Figure 11.





## 9 OPTIMAL INSULATION OF A CELLAR

The cellar or underground basement of a building has a floor and walls that border on the surrounding ground. The geometry is too complicated to allow any analytical determination of the corresponding basic constant-flow solution. We have to resort to numerical calculations.

A system for heat storage in the ground is another case with the same geometry. The storage region may be quite large. It is very important to insulate the storage thermally as well as possible. Heat storage systems is therefore an important application of the optimal insulation theory.

We will here consider three geometries for the cellar. The simplest case is a long house for which we study a vertical cross-section. We get a two-dimensional heat flow problem. The cross-section of the cellar is rectangular. We will use the denomination cellar cross-section for this case. The second case concerns a cellar of cylindrical shape. The cylindrical region may be used for heat storage. Finally we will study the most important case, when the cellar has the shape of a parallelepiped.

Figure 13 shows the first case of a cellar cross-section. The problem is two-dimensional in the vertical plane. The width of the house is  $2L$ . The height of the cellar is  $H$ . The problem is symmetrical relative to the dashed vertical line of Figure 13. Our task is to determine the optimal thermal insulation along the horizontal and vertical boundary of the cellar. Our problem is as usual scaled with the length  $L$ . The boundary curve is determined by a coordinate  $s$ , which runs from  $s=0$  to  $s=1+\frac{H}{L}$ . See Figure 13. The starting point  $s=0$  gives the mid-point of the floor of the cellar. The corner between the floor and the wall corresponds to  $s=1$ . The vertical wall of the cellar corresponds to  $1\leq s\leq 1+\frac{H}{L}$ . The coordinate  $s$  is also shown explicitly in Figure 13.

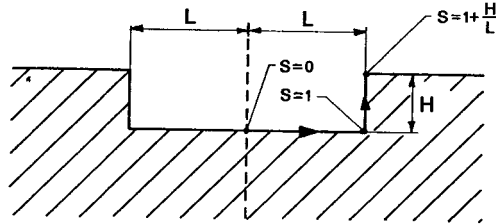


Figure 13 A cellar with a rectangular cross-section.  
Two-dimensional case.

The basic constant-flow solution  $u$  shall satisfy the Laplace equation  $\Delta u=0$  in the ground outside the cellar. The temperature shall be zero at the ground surface. At the vertical and horizontal boundary of the cellar the normal derivative is  $\frac{\partial u}{\partial n} = 1$ . The solution  $u$  depends on the shape parameter  $H/L$ . The problem has been solved for several values of  $H/L$ . The insulation is determined by the optimal insulation function  $\tilde{u} = u_{\max} - u$  along the boundary of the cellar. The results are shown in Figure 14. Each curve gives the optimal insulation function for a certain  $H/L$ . The curve  $H/L=0$  gives the previous case of a plate on the ground. The left part of the curves,  $0 \leq s \leq 1$ , gives the distribution on the floor from the mid-point to the corner. The right part,  $1 \leq s \leq 1+H/L$ , gives the values of  $\tilde{u}$  upwards along the vertical wall of the cellar.

Figures 15 and 16 show  $u_m$  and  $u_{\max} - u_m$  as a function of the shape  $H/L$ . The mean insulation thickness  $d_m$  must as usual exceed  $d_{\min} = L \lambda_i / \lambda_o (u_{\max} - u_m)$ .

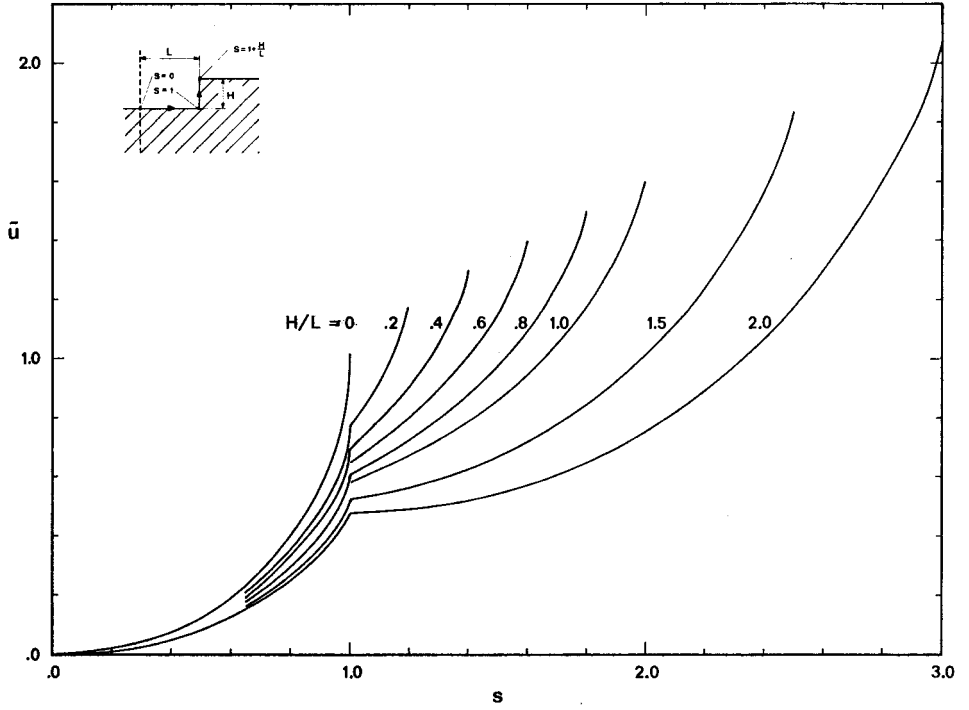


Figure 14 The optimal insulation function  $\tilde{u}$  for the cellular cross-section.

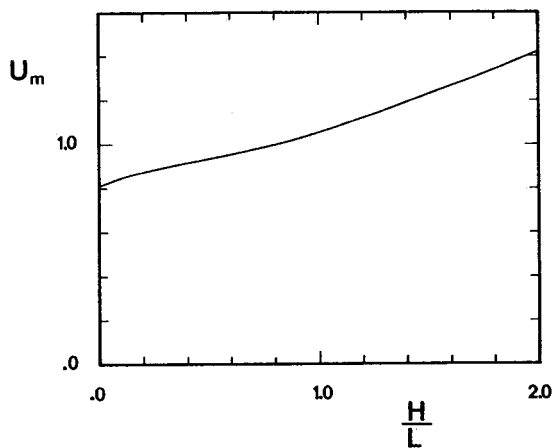


Figure 15 The mean value  $u_m$  for the cellar cross-section.

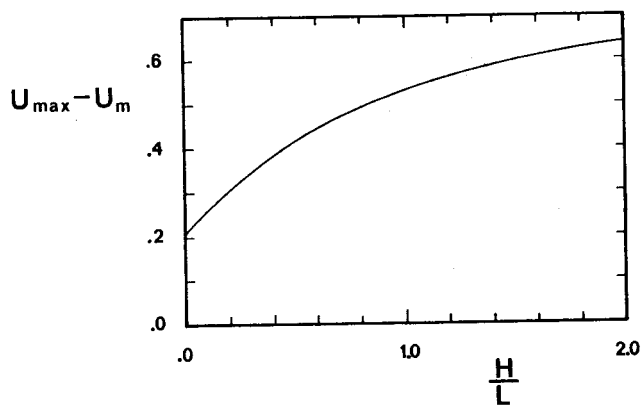


Figure 16 The quantity  $u_{\max} - u_m$  for the cellar cross-section.

The optimal insulation is from (5.2):

$$\tilde{d} = d_m - d_{\min} + \frac{\lambda_i}{\lambda_o} L \tilde{u} \quad (9.1)$$

The insulation thickness  $\tilde{d}$  is equal to a constant part  $d_m - d_{\min}$  plus  $\lambda_i/\lambda_o \cdot L \cdot \tilde{u}$ , where  $\tilde{u}$  is shown in Figure 14. The insulation  $\tilde{d}$  is directly proportional to the optimal insulation function  $\tilde{u}$  in the limiting case  $d_m = d_{\min}$ . Figure 17 shows the optimal insulation distribution in such a case ( $H/L=0.8$ ,  $d_m = d_{\min}$ ,  $L\lambda_i/\lambda_o=1$ ). The insulation thickness is zero at the center of the floor of the cellar ( $s=0$ ). The thickness increases to  $\tilde{d}=\tilde{u}=0.61$  at the corner between the floor and the wall. The thickness increases along the wall from  $\tilde{d}=\tilde{u}=0.61$  to  $\tilde{d}=\tilde{u}=1.5$  at the edge between the cellar and the ground surface. Figure 17 shows the shape of the optimal insulation distribution for  $d_m = d_{\min}$ . An additional amount of insulation  $d_m - d_{\min}$  is to be distributed evenly over the cellar surface.

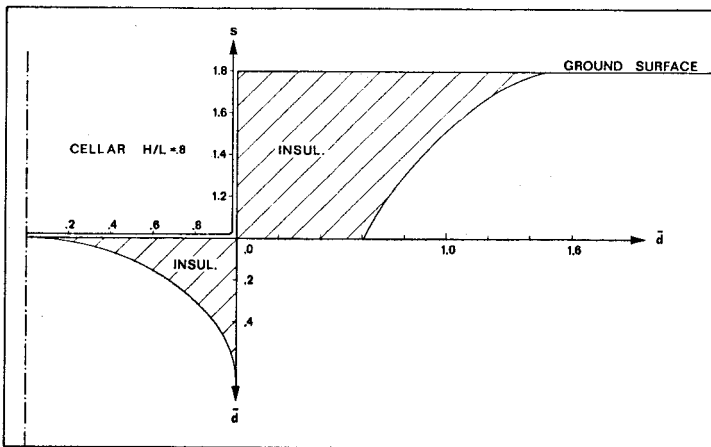


Figure 17 Optimal thermal insulation along a cellar cross-section for  $d_m = d_{\min}$ . ( $L\lambda_i/\lambda_o=1$ ).

Figure 18 shows the second case, when the cellar or heat storage region has a cylindrical shape. The radius of the cylinder is  $R$ , and the height is  $H$ . The problem is to insulate the bottom surface and the vertical envelope of the cylinder in an optimal way.

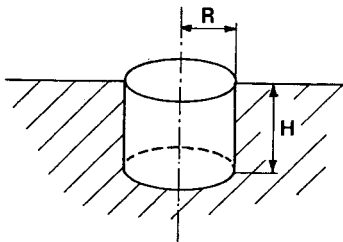


Figure 18 Cellar or heat storage region of a cylindrical shape with radius  $R$  and height  $H$ .

The problem is scaled with the length  $R$ . The basic constant-flow solution  $u$  depends on the shape factor  $H/R$ . The problem has been solved numerically for several values of  $H/R$ . The optimal insulation function  $\tilde{u}$ , which is defined on the cylinder surface, is shown in Figure 19. The independent variable  $s$  is defined by the small figure. The center of the bottom surface of the cylinder corresponds to  $s=0$ . The values of  $\tilde{u}$  for  $0 \leq s \leq 1$  show the radial increase along the bottom surface. The distribution upwards on the cylindrical envelope is given for  $1 \leq s \leq 1+H/R$ . The curve  $H/R=0$  gives the previous case of a circular disc on the ground.

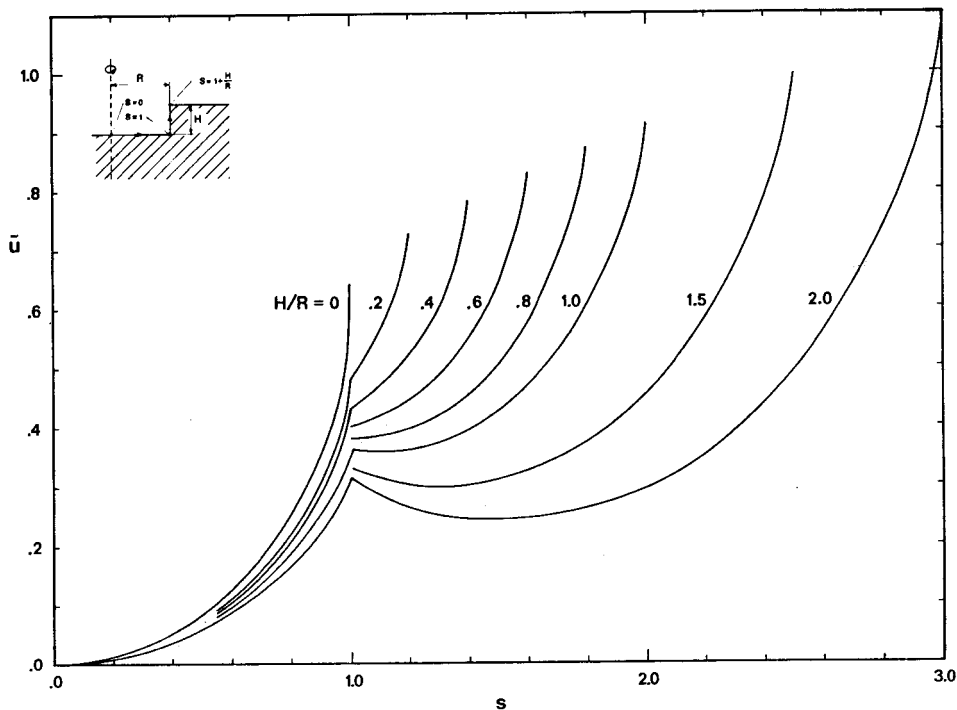


Figure 19 Optimal insulation function for a cylindrical cellar or heat storage.

The insulation distribution is not monotonously increasing for say  $H/R=2$ . There is a local maximum at the corner  $s=1$ . This is reasonable, since the protruding corner region is more exposed to the surrounding soil.

Figure 20 and Figure 21 show  $u_m$  and  $u_{\max} - u_m$  for different shapes  $H/R$ .

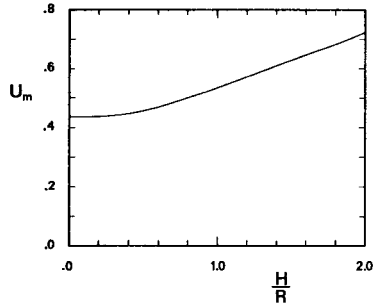


Figure 20 The mean value  $u_m$  for different shapes  $H/R$  of the cylinder.

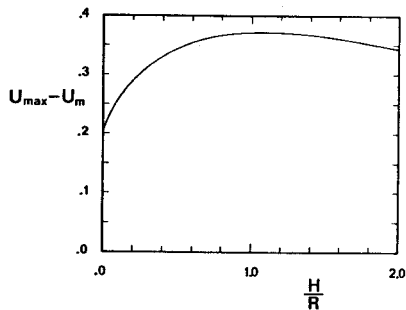


Figure 21 The quantity  $u_{\max} - u_m$  for different shapes  $H/R$  of the cylinder.

The optimal insulation distribution  $\tilde{d}$  is given by the basic formulas and Figures 19-21. The results are only valid, if  $d_m \geq d_{\min}$  according to (5.1) and Figure 21. The optimal insulation distribution is given by formula (5.2) and Figure 19. The heat loss is determined from formula (5.3) and Figure 20.

Figure 22 shows our third case, when the cellar has the shape of a parallelepiped.



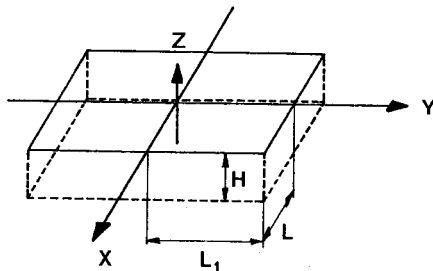


Figure 22 Cellar or heat storage with the shape of a parallelepiped.

The height of the cellar is  $H$ . The horizontal cross-section has a width  $2L$  and a length  $2L_1$ , where  $L_1 \geq L$ . There are two symmetry planes:  $x=0$  and  $y=0$  - see Figure 22. We need only to consider the problem for  $x \geq 0$ ,  $y \geq 0$ , and  $z \leq 0$ .

The basic constant-flow solution has been computed numerically for three different cases  $L_1/L=1, 2, 5$ . The height was  $H/L=0.4$  in all three cases.

We are interested in the optimal insulation function  $\tilde{u} = u_{\max} - u$  for the three rectangular surfaces. The floor rectangle of the cellar is defined by  $z = -H/L$ ,  $0 < x < 1$ , and  $0 < y < L_1/L$ . The two wall rectangles are given by  $y = L_1/L$ ,  $0 < x < 1$ ,  $-H/L < z < 0$  and  $x = 1$ ,  $0 < y < L_1/L$ ,  $-H/L < z < 0$  respectively - see Figure 22. The computed values for the three rectangles are shown in Table 1.

		y							
		.20	.50	.70	.85	.95			
x	.20	.0	.06	.14	.23	.32	.48	.53	.65
	.50	.06	.12	.19	.27	.35	.50	.54	.66
	.70	.14	.19	.24	.32	.39	.52	.56	.66
	.85	.23	.27	.32	.38	.44	.55	.59	.68
	.95	.32	.35	.39	.44	.49	.58	.61	.69
		.48	.50	.52	.55	.58	-.35		
		.53	.54	.56	.59	.61		-.25	
		.65	.66	.66	.68	.69			-.10

z

$L_1/L = 1 \quad H/L = 0.4$

		y									
		.25	.75	1.20	1.50	1.70	1.85	1.95			
x	.20	.00	.03	.11	.21	.30	.41	.51	.69	.75	.88
	.50	.09	.12	.19	.27	.36	.46	.54	.71	.77	.89
	.70	.19	.21	.27	.35	.42	.51	.59	.74	.78	.89
	.85	.31	.33	.38	.44	.50	.58	.64	.77	.81	.91
	.95	.42	.44	.48	.53	.58	.64	.70	.80	.84	.92
		.64	.65	.67	.70	.73	.77	.80	-.35		
		.72	.72	.74	.76	.78	.81	.84		-.25	
		.87	.87	.87	.88	.89	.91	.92			-.10

z

$L_1/L = 2 \quad H/L = 0.4$

		y											
		1.00	2.50	3.25	3.75	4.20	4.50	4.70	4.85	4.95			
x	.20	.00	.04	.09	.14	.23	.34	.44	.55	.65	.84	.91	1.04
	.50	.10	.14	.18	.23	.31	.41	.49	.60	.69	.86	.92	1.05
	.70	.22	.25	.29	.33	.40	.48	.56	.65	.73	.89	.94	1.05
	.85	.36	.38	.42	.46	.51	.58	.65	.72	.79	.92	.97	1.07
	.95	.48	.50	.53	.57	.62	.67	.73	.79	.85	.95	.99	1.08
		.74	.76	.77	.79	.82	.85	.88	.92	.95	-.35		
		.84	.85	.86	.87	.89	.91	.93	.96	.99		-.25	
		1.01	1.02	1.02	1.03	1.03	1.04	1.05	1.07	1.08			-.10

z

$L_1/L = 5 \quad H/L = 0.4$

Table 1 Optimal insulation function  $\tilde{u}$  for a cellar with the shape of a parallelepiped.

The coordinates  $x, y$ , and  $z$  of the mesh points are shown outside the three rectangles.

The values of  $u_m$  and  $u_{\max} - u_m$  for the three cases are given in Table 2.

$L_1/L$	1	2	5	
$u_m$	0.48	0.66	0.80	$H/L=0.4$
$u_{\max} - u_m$	0.39	0.45	0.46	

Table 2 The mean value  $u_m$  and the quantity  $u_{\max} - u_m$  for a parallelepipedic cellar.



## 10 OPTIMAL INSULATION OF A CULVERT

Heated pipes, tunnels, ducts, and culverts in the ground may have considerable heat losses due to their long extension. It is valuable to know how to insulate the heated region in the ground in an optimal way. The heat flow problem is two-dimensional in the vertical plane perpendicular to the duct in the ground.

We will here study the case, when the cross-section of the insulated region is rectangular. We will talk about a culvert in the ground. See Figure 23.

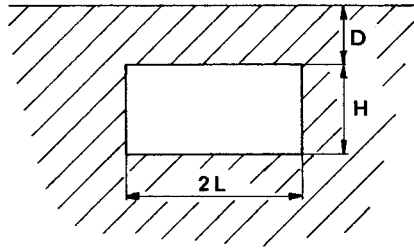


Figure 23 Culvert or tunnel in the ground with rectangular cross-section.

The height of the culvert is  $H$ , and the width is  $2L$ . The upper surface of the culvert lies at a depth  $D$  below the ground surface.

The problem is to determine the optimal insulation distribution along the culvert surface, i.e. along the rectangle. We need only to consider the right half of the culvert and the ground due to symmetry. See Figure 24. The problem is scaled with the length  $L$ . We introduce a coordinate  $s$  along the rectangular boundary curve of the culvert. The starting point  $s=0$  is the center of the lower boundary line. The lower corner is given by  $s=1$ . The vertical side

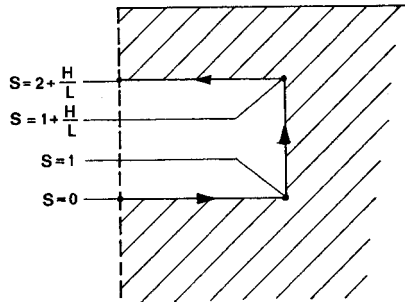


Figure 24 The right half of the culvert and the ground. The coordinate  $s$  gives the position along the rectangular boundary curve.

is represented by  $1 \leq s \leq 1 + \frac{H}{L}$ , where  $s = 1 + H/L$  gives the upper corner. The center of the upper surface is given by  $s = 2 + H/L$ .

The basic constant-flow solution  $u$  is zero at the ground surface. The normal derivative at the rectangular boundary of the culvert is equal to  $+1$ . The solution  $u$  depends on the two shape parameters  $H/L$  and  $D/L$ . The limit  $D/L = 0$  gives the previous case of a rectangular cellar.

The optimal insulation function  $\tilde{u}$  is defined along the curve  $0 \leq s \leq 2 + H/L$ . The numerically computed results are shown in Figures 25-28. The figures refer to  $H/L = 0.2, 0.5, 1.0,$  and  $2.0$  respectively. The four curves in each figure refer to different depths  $D/L$ . The dashed curve refers to the limit  $D/L = 0$ , which is the previously studied case of a rectangular cellar.

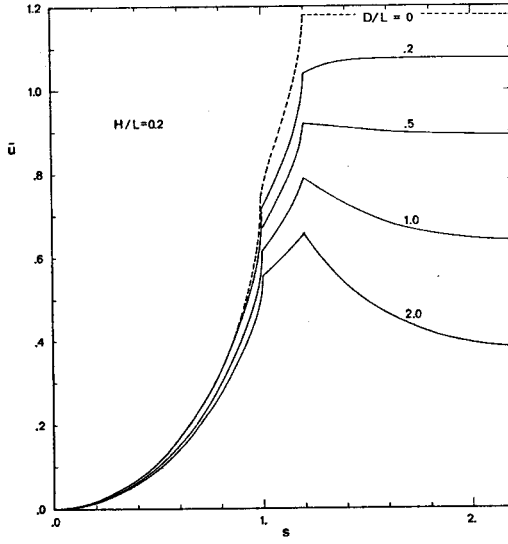


Figure 25 Optimal insulation function for a culvert.

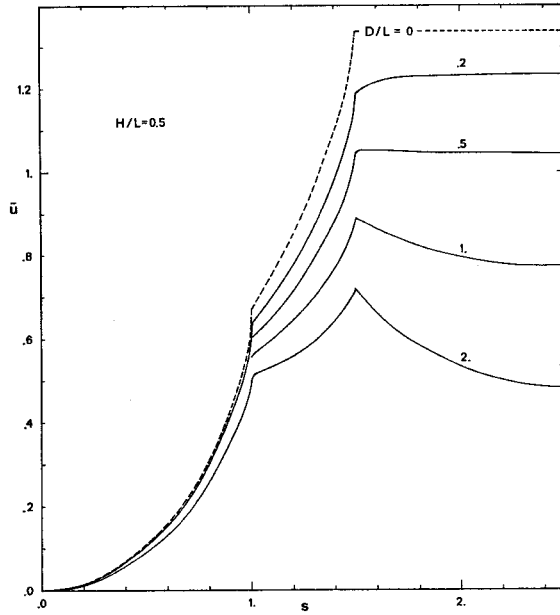


Figure 26 Optimal insulation function for a culvert.

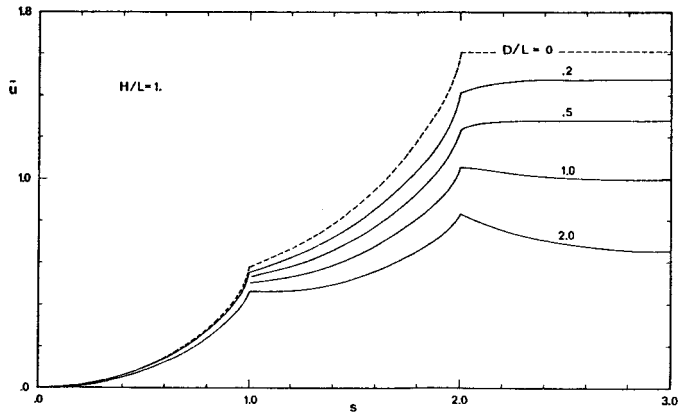


Figure 27 Optimal insulation function for a culvert.

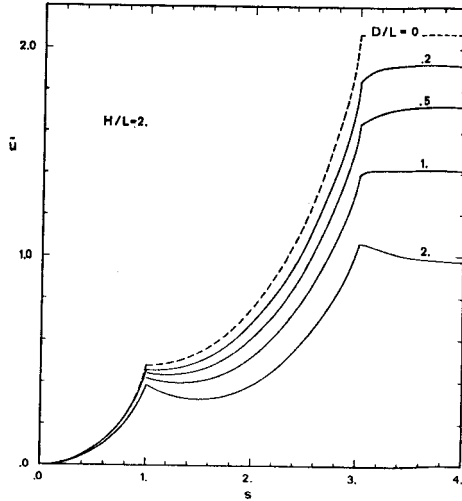


Figure 28 Optimal insulation function for a culvert.



The mean value  $u_m$  and the quantity  $u_{\max} - u_m$  are shown in Figures 29 and 30 as functions of the geometrical parameters  $H/L$  and  $D/L$ . The limiting case  $D/L=0$  corresponds to a rectangular cellar. The values of  $u_m$  differ due to the fact that we did not include the upper surface in the previous case with cellar. The values of  $u_m$  become consistent if, in the cellar case, the upper boundary is attributed the constant value  $\tilde{u}=u_{\max}$  and this is included in the mean value. This upper part corresponds to the horizontal part of the dashed lines in Figures 25-28.

The optimal insulation distribution  $\tilde{d}$  and the heat loss  $q_1$  are given by the basic formulas (5.2) and (5.3) provided that (5.1) is valid.

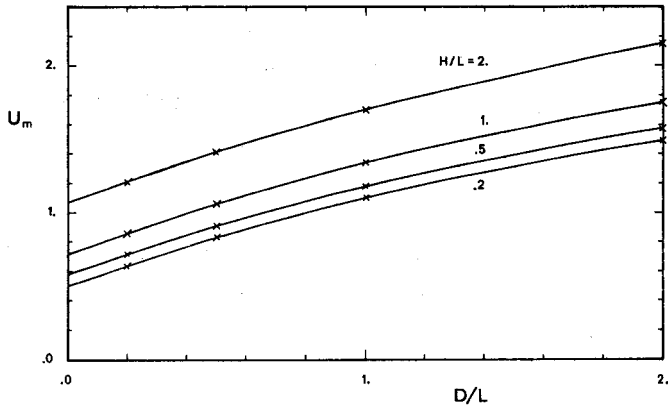


Figure 29 The mean value  $u_m$  for a culvert.

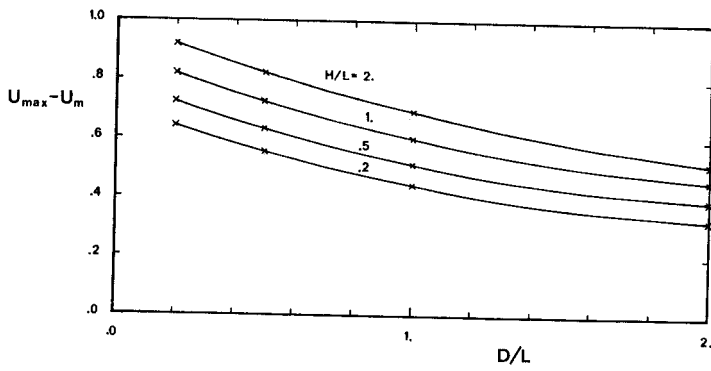


Figure 30 The quantity  $u_{max} - u_m$  for a culvert.

## 11 TWO REGIONS OF SOIL

The thermal conductivity in the ground has been constant in the cases that we have discussed so far. But the optimal insulation theory is valid for a variable thermal conductivity through the heat flow region. It is quite common that the soil consists of different strata with different thermal conductivities. The difference in conductivity is considerable between soil and rock.

We will here only consider one particular case. The ground consists of granite rock except for a relatively thin covering soil layer of for example moraine. This case is quite common in Sweden. The moraine has the thermal conductivity  $\lambda_0$ , and the underlying granite the higher conductivity  $3\lambda_0$ .

We will study the optimal insulation of a cellar with the height  $H$ . The cellar is built down to the granite. The thickness of the moraine stratum is therefore also  $H$ .

The first numerical computation concerns a rectangular cellar. We have previously studied this case for constant thermal conductivity in the ground - see Figure 13. The width of the cellar is again  $2L$ . The situation is shown in Figure 31.

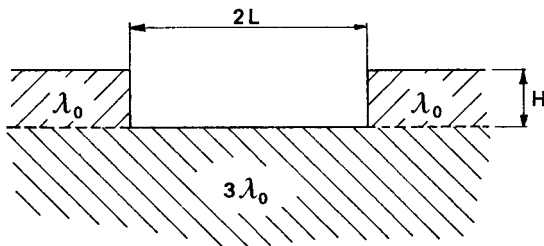


Figure 31 Rectangular cellar in a ground of granite ( $3\lambda_0$ ), which is covered by a moraine layer ( $\lambda_0$ ).

The problem is scaled with the length  $L$ . The dimensionless thermal conductivity  $\lambda' = \lambda/\lambda_0$  is equal to +1 in the moraine and +3 in the granite. The basic constant-flow solution satisfies (4.2). The normal derivative of  $u$  is now equal to +1 on the vertical boundary against the moraine, and  $+\frac{1}{3}$  on the bottom of the cellar that borders on the granite.

Figure 32 shows the computed optimal insulation function for  $H/L=1$ . We have in this case:

$$u_m = 0.59 \quad u_{\max} - u_m = 0.12 \quad (11.1)$$

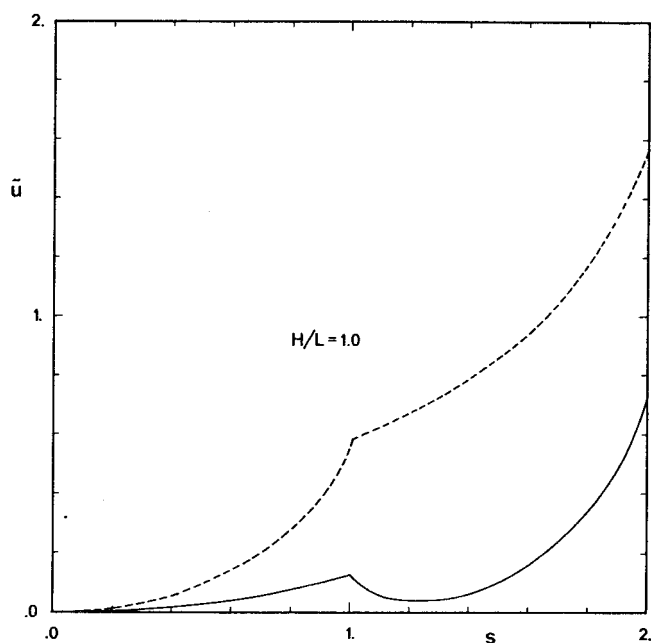


Figure 32 Optimal insulation function for a rectangular cellar on a granite ground with a layer of moraine. The dashed curve refers to the homogeneous ground.

The dashed line shows the same case for a homogeneous ground - see Figure 14. The higher thermal conductivity in the granite increases considerably the heat loss through the floor of the cellar. The relative amount of insulation to be put on the floor increases therefore. This is shown by the two curves.

Our second numerical study concerns a cylindrical cellar. The radius is  $R$ . The situation is essentially shown in Figure 18. The upper layer of the soil is again moraine with a conductivity  $\lambda_0$ . Downwards from the depth  $H$  the ground consists of granite with the thermal conductivity  $3\lambda_0$ . The circular bottom of the cellar borders on the granite, while the vertical envelope borders on the moraine.

Figure 33 shows the computed optimal insulation function for  $H/R=0.4$  and  $1.0$ . We have in this case:

H/R	0.4	1
$u_m$	.22	.33
$u_{\max} - u_m$	.079	.026

(11.2)

The dashed lines in Figure 33 show the same case for a homogeneous ground from Figure 19.

The shape of the curves (dashed and full lines) changes drastically between the two cases. The difference is most pronounced for the deeper cellar  $H/R=1.0$ . The optimal insulation thickness  $\tilde{d}$  is relatively constant over the bottom surface of the cylinder. There is a local maximum at the corner. Then the insulation  $\tilde{u}$  decreases upwards to a minimum for  $s \approx 1.35$ , i.e. at the depth  $0.65H$  below the ground surface.

This example illustrates that there may be a drastic change of the optimal insulation, when the soil contains different parts with highly different thermal conductivities.

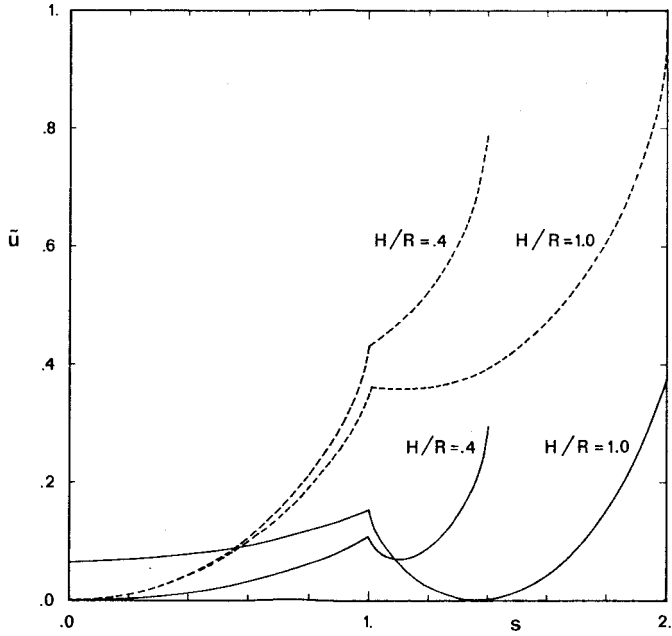


Figure 33 Optimal insulation function for a cylindrical cellar in a ground with moraine and granite. The dashed curves refer to the homogeneous ground.

## 12 EFFECT OF GROUND WATER

Moving ground water, which is not too far below the insulated structure, will change the heat flow problem. The effect depends strongly on the velocity of the ground water flow.

We will here only consider one simple situation. We take the two-dimensional case with a rectangular cellar. The water table lies at a depth  $D$  below the floor of the cellar. We will consider the extreme case of a strong ground water flow. The velocity of the water is sufficiently high to keep the temperature equal to the ambient level  $T_0$  in the ground water. We get a lower boundary at the horizontal water table, which lies at the depth  $D+H$  below the ground surface. The temperature on this boundary is the same as at the ground surface. This horizontal line is in the previous general description to be included in the boundary surface  $S_0$ . The situation is shown in Figure 34.

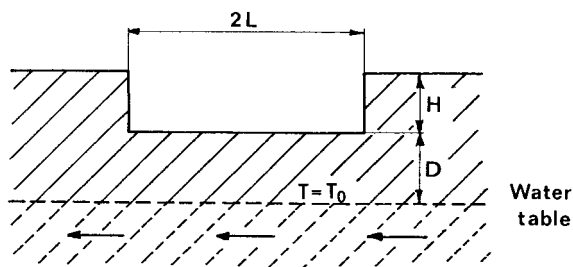


Figure 34 Rectangular cellar on a ground with a strong ground water movement.

The basic constant-flow solution  $u$  is zero at the water table. Figure 35 shows the computed optimal insulation function for  $H/L = 0.4$  for three different depths  $D$ . The dashed line shows the case without the ground water disturbance from Figure 14.

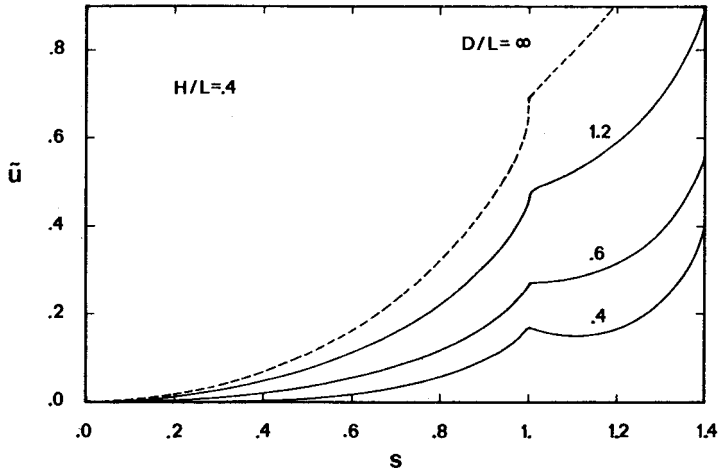


Figure 35 Optimal insulation function for a rectangular cellar, when there is a strong ground water movement.

We also get:

D/L	0.4	0.6	1.2	$\infty$
$u_m$	.31	.42	.63	.91
$u_{\max} - u_m$	.080	.14	.26	.39

(12.1)



## 13 INSULATING SOIL THICKNESS

Formula (5.3) for the heat flow  $q_1$  through the optimally insulated surface  $S_1$  is an important one:

$$q_1 = \frac{T_1 - T_0}{\frac{d_m}{\lambda_i} + \frac{L \cdot u_m}{\lambda_0}} \quad (13.1)$$

The quantity  $q_1$  is the heat flow per unit time and unit area. The total heat loss through the insulated area  $S_1$  per unit time,  $Q_1$ , is:

$$Q_1 = A_1 \cdot q_1 \quad (13.2)$$

Here  $A_1$  denotes the area of the insulated surface  $S_1$ .

The expression for  $q_1$  is a temperature difference  $T_1 - T_0$  divided by a thermal resistance.

Consider now the following one-dimensional steady-state heat flow process. We have a slab of the soil material with a thickness  $L \cdot u_m$ . The thermal conductivity is  $\lambda_0$ . The thermal resistance is then  $L u_m / \lambda_0$ . To this soil we add a slab of the insulation material. The thickness is  $d_m$ , and the thermal conductivity is  $\lambda_i$ . The thermal resistance of the insulation slab is  $d_m / \lambda_i$ . We have obtained a composite slab - see Figure 4.

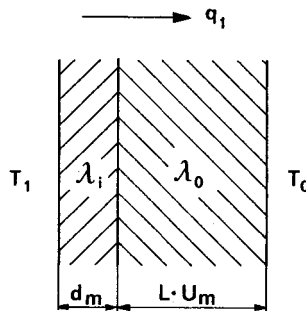


Figure 4 Heat loss formula (13.1). The soil may be regarded as a layer with the thickness  $L u_m$ .

The total thermal resistance of the composite slab is  $Lu_m/\lambda_0 + d_m/\lambda_1$ . The temperature difference over the slab is  $T_1 - T_0$ . The heat flow in steady-state is equal to the temperature difference over the slab divided by the thermal resistance of the composite slab. We get precisely the expression of formula (13.1).

The soil is, for an optimally insulated surface  $S_1$ , equivalent to a slab with the thickness  $L \cdot u_m$  from a heat loss point of view. Here  $L$  is the length that is used to obtain a dimensionless formulation. We can say that  $L \cdot u_m$  is the equivalent mean insulating soil thickness. We will call

$$L \cdot u_m \tag{13.3}$$

the insulating soil thickness.

It must be emphasized that the surface  $S_1$  is insulated in an optimal way  $\tilde{d}$  according to formula (5.2). A soil layer of the insulating soil thickness together with an insulation layer of constant thickness, equal to the given mean thickness  $d_m$ , gives the same heat loss  $q_1$ .

The introduced concept provides a simple and tangible way to assess the thermal insulation capacity of the ground.

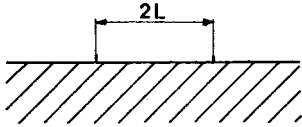
Another way to express the thermal insulating capacity is to use the so-called "k"-value. We note that the equivalent k-value of the soil is

$$\frac{\lambda_0}{Lu_m}$$

We have in the previous paragraphs computed the insulating soil thickness  $L \cdot u_m$  in several important cases. These are a plate on the ground (two-dimensional), a disc on the ground, a rectangular ground plate, cellar cross-section, a cylindrical cellar, a cellar with a parallelepipedic shape, a culvert, and some cases with va-

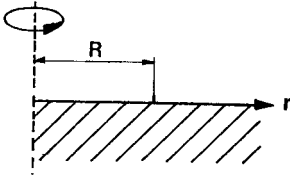
riable conductivity and with ground water effects. The results concerning the insulating soil thickness  $L \cdot u_m$  are summarized below.

Plate on the ground  
(two-dimensional)



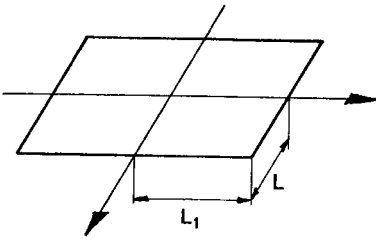
$$u_m = \frac{\pi}{4} \approx 0.785 \quad (13.4)$$

Circular disc on the ground



$$u_m = \frac{4}{3\pi} \approx 0.424 \quad (13.5)$$

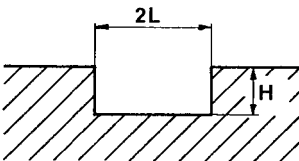
Rectangular ground plate



$L_1/L$	1	1.5	2	3	5
$u_m$	.52	.61	.66	.72	.78

(13.6)

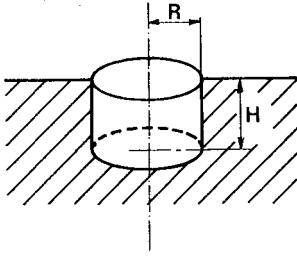
Cellar cross-section



$H/L$	.0	.05	.1	.2	.4	.6	.8	1.0
$u_m$	.81	.84	.85	.87	.91	.95	1.0	1.1
$H/L$	1.5	1.8	2.0					
$u_m$	1.2	1.3	1.4					

(13.7)

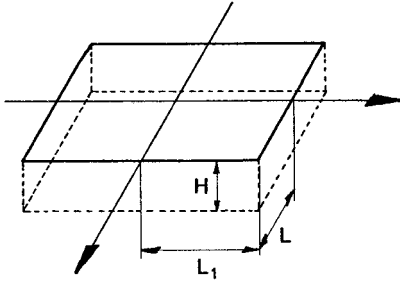
## Cylindrical cellar or heat storage



H/R	0	.2	.4	.6	.8	1.0
$u_m$	.42	.44	.45	.47	.50	.54
H/R	1.5	1.8	2.0			
$u_m$	.63	.69	.72			

(13.8)

## Parallelepipedic cellar

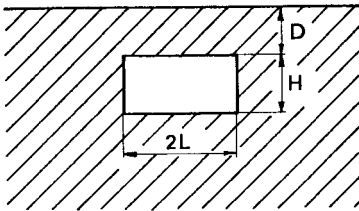


$$H/L=0.4$$

$L_1/L$	1	2	5
$u_m$	.48	.66	.80

(13.9)

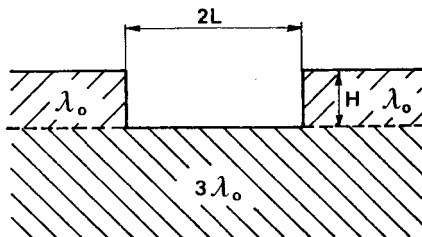
## Culvert



	D/L				
	.2	.5	1.0	2.0	
H/L	.2	.63	.84	1.1	1.5
	.5	.71	.91	1.2	1.6
	1.0	.86	1.1	1.3	1.8
	2.0	1.2	1.4	1.7	2.2

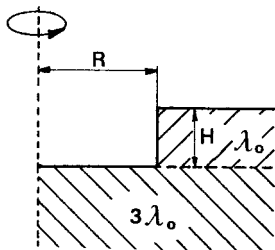
(13.10)

Cellar cross-section,  
granite and moraine



$$\frac{H}{L}=1: u_m=0.59 \quad (13.11)$$

Cylindrical cellar,  
granite and moraine

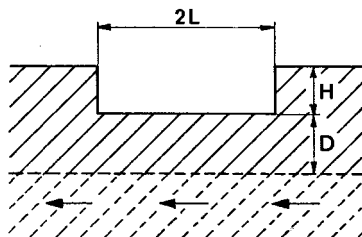


$$H/R=0.4 : u_m=0.22$$

$$H/R=1 : u_m=0.33$$

(13.12)

Cellar cross-section,  
strong ground water flow



$$H/L=0.4$$

D/L	0.4	0.6	1.2	∞
$u_m$	0.31	0.42	0.63	0.91

(13.13)

Graphs of  $u_m$  have been given previously for the rectangular ground plate (Figure 11), the cellar cross-section (Figure 15), the cylindrical cellar (Figure 20) and the culvert (Figure 29).

The given data provide a lot of information on the insulating capacity of the ground. The insulating soil thickness for a plate on the ground is  $L \cdot 0.785$ . A circular disc has the distinctly smaller value  $R \cdot 0.424$ . This illustrates the difference between two- and three-dimensional situations. A quadratic ground plate has  $Lu_m = L \cdot 0.52$  ( $L_1/L = 1$ ).

The difference between two and three dimensions is also illustrated, when we compare  $u_m$  for the rectangular and cylindrical cellars. The insulating thickness is roughly twice as big in the plane case.

The values for the rectangular ground plate and the parallelepipedic cellar ( $H/L=0.4$ ) are rather close to each other.

The influence of granite below the moraine is rather strong in the considered example. The insulating thickness is diminished from  $1.1 L$  to  $0.59 L$ , when the moraine below the rectangular cellar is replaced by granite. The effect is similar for a cylindrical cellar.

It is of interest to compare the magnitude of the two thermal resistance terms in the denominator of (13.1). Let  $d_{eq}$  be the thickness of the insulating material that corresponds to the thermal resistance  $Lu_m/\lambda_0$  of the soil:

$$d_{eq} = \frac{\lambda_i L}{\lambda_0} u_m \quad (13.14)$$

The insulating soil thickness  $Lu_m$  is to be multiplied by the thermal conductivity ratio  $\lambda_i/\lambda_0$  in order to give the insulating capacity of the soil expressed as a thickness  $d_{eq}$  of the insulating material.

## 14 FIRST-ORDER GROUND HEAT LOSS FORMULA

We have hitherto studied the optimal insulation problem. Let us now instead assume that we have an arbitrary insulation distribution  $d$  over the insulation surface  $S_1$ . The most important particular case is when  $d$  is constant over  $S_1$ . Let  $d_m$  be the mean insulation thickness for the distribution  $d$ . The quantities  $d$  and  $d_m$  are of course equal, when  $d$  is constant over  $S_1$ .

Let  $\tilde{d}$  as usual be the optimal insulation distribution corresponding to the same mean insulation thickness  $d_m$ . The function  $\tilde{d}$  is given by the basic formula (5.2). The two insulation distributions  $d$  and  $\tilde{d}$  have the same mean value over  $S_1$ . We may regard  $d$  as obtained from  $\tilde{d}$  in a variation:

$$d = \tilde{d} + (d - \tilde{d}) = \tilde{d} + \delta d \quad \text{on } S_1 \quad (14.1)$$

The insulation thickness is changed by an amount  $\delta d = d - \tilde{d}$  at each point on  $S_1$ .

The heat loss  $Q_1$  increases from the minimal value for the optimal insulation to a higher one, when the insulation is changed from  $\tilde{d}$  to  $d$ . Formula (3.2) gives the first-order contribution to this change as an integral of  $q_1^2 \cdot \delta d$  over  $S_1$ . The heat flow  $q_1$  is constant over  $S_1$  in the original optimal insulation case. The first-order change of the heat loss is thus proportional to the integral of  $\delta d = d - \tilde{d}$  over  $S_1$ . But this integral vanishes, since we have the same mean insulation thickness in the two cases. We have arrived at the important conclusion that the first-order change of the heat loss is zero. We have the following theorem.

Let  $d$  be an arbitrary insulation distribution over  $S_1$  with the mean thickness  $d_m$ . Let  $\tilde{d}$  be the optimal insulation distribution with the same mean thickness  $d_m$ . The heat loss  $Q_1$  through the surface  $S_1$  is then, up to the first order in the change of the insulation thickness from  $\tilde{d}$  to  $d$ , the same in the two cases:

$$Q_1 \approx A_1 q_1 = A_1 \frac{T_1 - T_0}{\frac{d}{\lambda_i} + \frac{L u_m}{\lambda_0}} \quad (14.2)$$

(arbitrary insulation)

Here  $A_1$  is the area of the insulated surface  $S_1$ . The total temperature difference is  $T_1 - T_0$ . The thermal conductivity of the insulation is  $\lambda_i$ . The corresponding constant-flow solution is scaled with  $L$ , and it has  $\lambda_0$  as reference thermal conductivity. The mean value of the constant-flow solution  $u$  over  $S_1$  is  $u_m$ .

It must be pointed out that formula (14.2) is only a first-order approximation. It may be completely useless, if the variation  $d - \tilde{d}$  is large.

The case of constant insulation thickness  $d$  over  $S_1$  is of particular interest. The first-order approximation of the heat loss is then with formula (14.2):

$$Q_1 \approx A_1 \lambda_i \frac{T_1 - T_0}{d + \frac{\lambda_i L u_m}{\lambda_0}} \quad (14.3)$$

(constant insulation thickness)

The simplicity of this formula is noteworthy. The heat loss  $Q_1$  is equal to the area  $A_1$  times the thermal conductivity  $\lambda_i$  of the insulation times the quotient of the temperature difference  $T_1 - T_0$  and a length. The length in the denominator is equal to  $d$  plus a constant. The constant is equal to the insulating soil thickness  $L u_m$  times the conductivity correction  $\lambda_i / \lambda_0$  from soil to insulation material. The insulating effect of the ground is accounted for by the addition of the length  $\lambda_i L u_m / \lambda_0$  to the insulation thickness  $d$ .

It is not possible to give any general rule for the accuracy of formula (14.2) and in particular of (14.3). The accuracy increases, when the relative variation  $(d - \tilde{d}) / \tilde{d}$  decreases. The accuracy of formula (14.3) increases, when the thickness  $d$  increases.



The accuracy of (14.3) has been tested for some cases. The heat loss  $Q_1$  has been computed numerically for constant insulation thickness over  $S_1$ . We will denote this loss  $Q_{1,\text{const}}$ . The corresponding heat loss for an optimal insulation is denoted  $Q_{1,\text{op}}$ . We have in the previous section given a lot of data for  $Q_{1,\text{op}}$ . Formula (14.3) means that  $Q_{1,\text{const}}$  is approximated with  $Q_{1,\text{op}}$  (for  $d=d_m$ ).

The two heat losses  $Q_{1,\text{op}}$  and  $Q_{1,\text{const}}$  are compared for

$$\frac{\lambda_o(T_1 - T_o)}{L} = 1 \quad L=1 \quad (14.4)$$

This is not any restriction, since other cases are obtained by a simple scaling.

We have the following results in different cases. The values  $Q_{1,\text{const}}$  are computed numerically, while the values  $Q_{1,\text{op}}$  are obtained from the previous paragraphs.

Our first comparison concerns the plate on the ground (two-dimensional). The insulation thickness is a multiple of the minimum value  $d_{\text{min}}$ . We got the following results:

$u_m = 0.806$	$d_m/d_{\text{min}}$	$Q_{1,\text{op}}$	$Q_{1,\text{const}}$	Increase %
$d_{\text{min}} = 0.2081 \cdot \frac{\lambda_i \cdot L}{\lambda_o}$	1	0.986	1.107	12
	2	0.818	0.870	6.4
	3	0.699	0.727	4.0
	5	0.542	0.553	2.0
	10	0.346	0.349	0.9

The values of  $Q$  refer to the right half of the plate. The values for  $u_m$  and  $d_m$  are not the analytical ones. We have instead used the values that we got in a numerical simulation of the optimal case. We think that it is more reasonable to base a comparison completely on numerical values, since these tend to give similar numerical errors.

We see from the table above that the error in formula (14.3) is

12%, when the insulation thickness is equal to the special value  $d_{\min}$ . A three-fold increase of the insulation diminishes the error to 4%. The error is only 0.9% for an insulation thickness of  $10 \cdot d_{\min}$ .

For a circular disc we got in the same way:

$$u_m = 0.440$$

$$d_{\min} = 0.2049 \cdot \frac{\lambda_i R}{\lambda_o}$$

$d_m/d_{\min}$	$Q_{i,op}$	$Q_{l,const}$	Increase %
1	4.871	5.318	9.2
2	3.697	3.856	4.3
3	2.979	3.054	2.5
5	2.146	2.171	1.2
10	1.262	1.267	0.4

For a cylindrical cellar with  $H/R=0.4$  we got:

$$u_m = 0.447$$

$$d_{\min} = 0.3347 \cdot \frac{\lambda_i L}{\lambda_o}$$

$d_m/d_{\min}$	$Q_{l,op}$	$Q_{l,const}$	Increase %
1	7.231	7.985	10.4
2	5.065	5.291	4.5
3	3.897	3.994	2.5
5	2.666	2.696	1.1
10	1.490	1.495	0.3

For a quadratic plate on the ground we got ( $L_1=L$ ):

$$u_m = 0.516$$

$$d_{\min} = 0.2065 \cdot \frac{\lambda_i L}{\lambda_o}$$

$d_m/d_{\min}$	$Q_{l,op}$	$Q_{l,const}$	Increase %
1	1.384	1.489	7.7
2	1.076	1.118	3.9
3	0.881	0.902	2.4
5	0.646	0.653	1.1
10	0.387	0.389	0.5

A rectangular plate on the ground with  $L_1/L=2$  gives:

$u_m=0.660$	$d_m/d_{min}$	$Q_{1,op}$	$Q_{1,const}$	Increase %
$d_{min}=0.2645 \cdot \frac{\lambda_i L}{\lambda_o}$	1	2.163	2.339	8.1
	2	1.682	1.751	4.1
	3	1.376	1.410	2.5
	5	1.009	1.021	1.2
	10	0.605	0.608	0.5

The heat losses for these cases of quadratic and rectangular plates refer to one fourth of the plate (with a total length  $2L_1$  and a total width  $2L$ ).

We have in the special situation, when  $d_m=d_{min}$ , studied some additional cases.

For a cellar cross-section we got:

$d_m=d_{min}$	H/L	$\frac{d_{min} \lambda_o}{\lambda_i L}$	$Q_{1,op}$	$Q_{1,const}$	Increase %
	0	0.2081	0.986	1.107	12
	0.2	0.307	1.020	1.221	20
	0.4	0.386	1.084	1.280	18

The given heat losses refer to the right half of the cellar.

For a cylindrical cellar we got:

$d_m=d_{min}$	H/R	$\frac{d_{min} \lambda_o}{\lambda_i L}$	$Q_{1,op}$	$Q_{1,const}$	Increase %
	0	0.2049	4.871	5.318	9
	0.2	0.2900	6.053	6.835	13
	0.4	0.3350	7.231	7.985	10

For a culvert with  $D/L=0.2$  we got:

$d_m=d_{\min}$ $D/L=0.2$	$H/L$	$\frac{d_{\min} \lambda_o}{L \lambda_i}$	$Q_{1,op}$	$Q_{1,const}$	Increase %
	0.5	0.723	1.743	1.982	14
	1.0	0.815	1.791	2.039	14

The given heat losses refer to the right half of the culvert.

All given values of heat losses are computed under assumption (14.4). The heat losses for other cases are obtained in the following way. The values are multiplied by the scale factor  $\lambda_o(T_1-T_o)/L$  and by an area scaling factor. In the three-dimensional cases the scale factor for the area is  $L^2$ . In the plane two-dimensional cases we have to multiply by the factor  $L$ , instead of  $L^2$ .

We are now able to give some guide-lines on the accuracy of formula (14.3), where the heat loss for constant insulation thickness is approximated by the corresponding optimal insulation heat loss. The insulation thickness is  $d$ . The mean insulation thickness  $d_m$  is equal to this value  $d$ . Let  $d_{\min}$  be the limiting case for the optimal insulation. We know that an interior part of  $S_1$  is to be left uninsulated, when  $d$  is less than  $d_{\min}$ .

We can say that formula (14.3) underestimates the heat loss with roughly 10%, when the constant insulation thickness  $d$  is near  $d_{\min}$ . This underestimation falls to roughly 6, 4, 2, and below 1%, when the insulation  $d$  is near  $2 \cdot d_{\min}$ ,  $3 \cdot d_{\min}$ ,  $5 \cdot d_{\min}$ , and  $10 \cdot d_{\min}$  respectively. The value of  $d_{\min}$  has been given for different cases in the foregoing.

The comparisons of this section are of great interest from another point of view. It is in many applications customary to use a constant insulation thickness over the insulation surface  $S_1$ . A most pertinent question is how much one gains, when the insulation is distributed optimally instead. This question is readily answered.

The gain is roughly 10% when the insulation thickness is near the limiting case  $d_{\min}$ . The gain drops to roughly 6, 4, and 2%, when the insulation thickness is near  $2 \cdot d_{\min}$ ,  $3 \cdot d_{\min}$ , and  $5 \cdot d_{\min}$  respectively. The gain is therefore negligible for an insulation thickness above say  $3 \cdot d_{\min}$ .

It should be remembered that the gain may become considerably higher, when the thickness lies below  $d_{\min}$ , i.e. when parts of the surface are to be uninsulated. The optimal insulation theory is more important in these cases. But this is deferred to a later study.



## 15 EXAMPLES

All examples and illustrations of the optimal insulation theory have been put off to this section. They are quite important for a proper appreciation of the presented theory.

SI-units are used consistently.

The thermal conductivity of soil and rock material ranges from  $\lambda_0 = 0.8$  to  $\lambda_0 = 3.5$  J/ms<sup>0</sup>C. The conductivity of the insulation material lies in the region  $\lambda_i = 0.1$  to 0.03. The thermal conductivity ratio  $\lambda_i/\lambda_0$  will lie between 0.01 and 0.1.

The examples will follow the different headings of the preceding sections.

15A Plate on the ground

Example A1. Two-dimensional. Optimal insulation distribution.

The two-dimensional case of a plate on the ground is shown in Figure 1. The problem is solved analytically in section 6. Let us take:

$$\begin{aligned} \lambda_0 &= 2 \text{ J/ms}^0\text{C} & \lambda_i &= 0.05 \text{ J/ms}^0\text{C} \\ 2L &= 10 \text{ m} & d_m &= 0.1 \text{ m} & T_1 - T_0 &= 10 \text{ }^0\text{C} \end{aligned}$$

Then we have:

$$(6.8): \quad d_{\min} = \frac{0.05}{2} \cdot 5 \cdot \left(1 - \frac{\pi}{4}\right) \approx 0.027 \text{ m}$$

$$(6.7): \quad \tilde{d} = 0.073 + 0.125 \cdot \left(1 - \sqrt{1 - \left(\frac{\lambda_i}{\lambda_0}\right)^2}\right) \text{ m}$$

The second part of  $\tilde{d}$  is given in Figure 14 for  $H/L=0$ . The optimal insulation function  $\tilde{u}$  is to be multiplied by the scale factor 0.125.

$$(6.9): \quad Q_1 = 10 \cdot \frac{10}{\frac{0.10}{0.05} + \frac{5}{2} \cdot \frac{\pi}{4}} \approx 25 \text{ J/ms}$$

The minimum insulation thickness  $d_{\min}$  is well below the given  $d_m$ , so all of the plate is to be covered by insulation. There is a constant part  $d_m - d_{\min} = 0.073$  m. On top of this there is an variable insulation, which is zero at the center  $x=0$ . This part increases to 0.125 m at the edges  $x = \pm 5$  m. The optimal insulation thickness is 0.073 m at the center. It increases to 0.198 m at the edges.

Example A2. Two-dimensional. Minimum insulation criterion.

Let us illustrate criterion (6.8)

$$d_m \geq d_{\min} = \frac{\lambda_i}{\lambda_o} L \cdot \left(1 - \frac{\pi}{4}\right)$$

A certain part around the center of the plate is to be left un-insulated, if this inequality is not fulfilled.

$$A. \quad \frac{\lambda_i}{\lambda_o} = 0.1$$

$$d_m \geq 0.021 \cdot L$$

$$2L=10 \text{ m}: \quad d_m \geq 0.11 \text{ m}$$

$$B. \quad \frac{\lambda_i}{\lambda_o} = 0.03$$

$$d_m \geq 0.0064 \cdot L$$

$$2L=10 \text{ m}: \quad d_m \geq 0.032 \text{ m}$$

$$C. \quad \frac{\lambda_i}{\lambda_o} = 0.01$$

$$d_m \geq 0.0021 \cdot L$$

$$2L=10 \text{ m}: \quad d_m \geq 0.011 \text{ m}$$



The minimum insulation  $d_{\min}$  increases with  $L$  and with the quotient  $\lambda_i/\lambda_o$ .

Example A3. Circular plate.

The insulation forms a circular disc on the ground. The radius of the disc is  $R$ . The analytical solution of this case is given in section 6. Let us take:

$$\begin{array}{lll} \lambda_o = 1.1 \text{ J/ms}^\circ\text{C} & \lambda_i = 0.04 \text{ J/ms}^\circ\text{C} & \\ R = 6.77 \text{ m} & d_m = 0.1 \text{ m} & T_1 - T_o = 15^\circ\text{C} \end{array}$$

The minimum insulation thickness is from (6.13):

$$d_{\min} = \frac{0.04}{1.1} \cdot 6.77 \cdot \frac{2}{3\pi} \approx 0.052 \text{ m}$$

Our theory is applicable, since  $d_m > d_{\min}$ . The optimal insulation distribution is from (6.12):

$$\tilde{d} = 0.048 + 0.157 \cdot (1 - \sqrt{1 - (\frac{r}{6.77})^2}) \text{ m}$$

The insulation thickness ranges from 0.048 m at the center of the disc to 0.205 m at the circumference. The heat loss through the disc is given by (6.14):

$$Q_1 = \pi \cdot (6.77)^2 \cdot \frac{15}{\frac{0.10}{0.04} + \frac{6.77}{1.1} \cdot \frac{4}{3\pi}} = 423 \text{ J/s}$$

Example A4. Quadratic plate.

This case is shown by Figure 9 with  $L_1=L$ . Let us take:

$$\begin{array}{lll} \lambda_o = 1.1 \text{ J/ms}^\circ\text{C} & \lambda_i = 0.04 \text{ J/ms}^\circ\text{C} & T_1 - T_o = 15^\circ\text{C} \\ 2L = 12 \text{ m} & 2L_1 = 12 \text{ m} & d_m = 0.1 \text{ m} \end{array}$$

The minimum insulation thickness is from (5.1) and Figure 12:

$$d_{\min} = \frac{0.04}{1.1} \cdot 6 \cdot 0.21 = 0.046 \text{ m}$$

The insulation distribution  $\tilde{d}$  is given by (5.2):

$$\tilde{d} = 0.054 + 0.218 \cdot \tilde{u}$$

The two-dimensional distribution  $\tilde{u}(\frac{x}{L}, \frac{y}{L})$  is given by Figure 10. The heat flow per unit time through the insulated surface is from Figure 11:

$$Q_1 = 144 \cdot \frac{15}{\frac{0.10}{0.04} + \frac{6}{1.1} \cdot 0.52} = 405 \text{ J/s}$$

Example A5. Rectangular plate.

The case is shown in Figure 9. Let us take:

$$\begin{array}{lll} \lambda_0 = 1.1 \text{ J/ms}^0\text{C} & \lambda_1 = 0.04 \text{ J/ms}^0\text{C} & T_1 - T_0 = 15 \text{ }^0\text{C} \\ 2L = 8.50 \text{ m} & 2L_1 = 17 \text{ m} & d_m = 0.1 \text{ m} \end{array}$$

The minimum insulation thickness is from (5.1) and Figure 12:

$$d_{\min} = \frac{0.04}{1.1} \cdot 4.25 \cdot 0.26 = 0.040 \text{ m}$$

The insulation distribution  $\tilde{d}$  is given by (5.2)

$$\tilde{d} = 0.060 + 0.155 \cdot \tilde{u}$$

Here  $\tilde{u}$  is given by Figure 10 for  $L_1/L=2$ . The heat flow per unit time through the insulated surface is from Figure 11

$$Q_1 = 144 \cdot \frac{15}{\frac{0.10}{0.04} + \frac{4.25}{1.1} \cdot 0.66} = 428 \text{ J/s}$$

Commentary.

Examples A3, A4, and A5 treat the problem of a plate on the ground. The plate area is the same in all cases, but the geometry differs. The heat losses from the plates are 423 J/s (circular plate), 405 J/s (quadratic plate) and 428 J/s (rectangular plate,  $L_1/L=2$ ).

It is obvious that the loss from the circular plate should be the smallest one. However, the circular case is solved analytically, while the other two cases are based on a three-dimensional numerical calculation.

According to the considerations in chapter 7, there is an estimated error in  $u_m$  of about 10-15% in the three-dimensional calculations. The calculated values of  $u_m$  are always too high. This means that, in the three-dimensional cases,  $Q_1$  is attributed a value that is about 5-8% too low. A correct value for the square plate should probably be something like 425-440 J/s. The conclusion is that the difference in heat loss between a circular plate and a quadratic one is just a few per cent.

In the comparison of the quadratic plate and the rectangular one, we may compare the results 405 J/s and 428 J/s directly, since both values contain approximately the same error. The difference in heat loss between the two geometries is about 6%.

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 Example A6. Long rectangular plate.

This case could be a building without cellar (see Figure 9). Let us take:

$$\begin{array}{lll} \lambda_o = 1.1 \text{ J/ms}^\circ\text{C} & \lambda_i = 0.04 \text{ J/ms}^\circ\text{C} & T_1 - T_o = 15 \text{ }^\circ\text{C} \\ 2L = 10 \text{ m} & 2L_1 = 50 \text{ m (} L_1/L = 5 \text{)} & d_m = 0.13 \text{ m} \end{array}$$

The minimum insulation thickness is from (5.1) and Figure 12 ( $L_1/L=5$ )

$$d_{\min} = \frac{0.04}{1.1} \cdot 5 \cdot 0.25 = 0.045 \text{ m}$$

The insulation distribution  $\tilde{d}$  is given by (5.2):

$$\tilde{d} = 0.085 + 0.182 \cdot \tilde{u}$$

Here  $\tilde{u}$  is given by Figure 10 ( $L_1/L=5$ ). The heat flow per unit time through the insulated surface is from (13.2) and Figure 11:

$$Q_1 = 10 \cdot 50 \cdot \frac{15}{\frac{0.13}{0.04} + \frac{5}{1.1} \cdot 0.78} = 1100 \text{ J/s}$$

### 15B. Cellars

Example B1. Cellar cross-section.

The cellar cross-section has a rectangular shape. See Figure 13. The results refer to the two-dimensional cross-section for a long house. The results will not be valid for the end regions of the long house, where three-dimensional effects must be considered. Let us take:

$$\begin{array}{lll} \lambda_0 = 1.1 \text{ J/ms}^\circ\text{C} & \lambda_1 = 0.04 \text{ J/ms}^\circ\text{C} & T_1 - T_0 = 15 \text{ }^\circ\text{C} \\ 2L = 12 \text{ m} & H = 2.4 \text{ m (H/L=0.4)} & d_m = 0.12 \text{ m} \end{array}$$

The minimum insulation thickness  $d_{\min}$  is given by (5.1) and Figure 16:

$$d_{\min} = \frac{0.04}{1.1} \cdot 6 \cdot 0.39 = 0.085 \text{ m}$$

The insulation distribution  $\tilde{d}$  is given by (5.2):

$$\tilde{d} = 0.035 + 0.218 \cdot \tilde{u} \text{ m}$$

Here  $\tilde{u}$  is given by Figure 14 with  $H/L=0.4$ .

The insulation thickness at the edge of the cellar is:

$$0.035 + 0.218 \cdot 1.3 = 0.32 \text{ m}$$

Formulas (13.2) and (13.7) give the heat flow per unit time through the insulated area:

$$Q_1 = (2L+2H) \cdot \frac{T_1 - T_0}{\frac{d_m}{\lambda_i} + \frac{L u_m}{\lambda_o}} = (12+4.8) \cdot \frac{15}{\frac{0.12}{0.04} + \frac{6 \cdot 0.91}{1.1}} = 32 \text{ J/sm}$$

Example B2. Cellar cross-section used for heat storage. Two regions of soil.

The situation is shown in Figure 31. The granite bedrock is covered by a layer of moraine with a thickness  $H$ . The cellar is built down to the bedrock. Let us assume that the cellar shape is used for heat storage. The temperature difference  $T_1 - T_0$  is relatively high. Let us take:

$$\begin{array}{lll} \lambda_o = 1.1 \text{ J/ms}^\circ\text{C} & \lambda_i = 0.04 \text{ J/ms}^\circ\text{C} & T_1 - T_0 = 50 \text{ }^\circ\text{C} \\ 2L = 10 \text{ m} & H = 5 \text{ m (H/L=1.0)} & d_m = 0.20 \text{ m} \end{array}$$

The minimum insulation  $d_{\min}$  is given by (5.1) and (11.1):

$$d_{\min} = \frac{0.04}{1.1} \cdot 5 \cdot 0.12 = 0.022 \text{ m}$$

The optimal insulation distribution is from (5.2):

$$\tilde{d} = 0.178 + 0.182 \cdot \tilde{u}$$

The optimal insulation function  $\tilde{u}$  is shown in Figure 32. The insulation thickness at the edge of the storage is

$$0.178 + 0.182 \cdot 0.72 = 0.31 \text{ m}$$

Formulas (13.2) and (11.1) give the heat flow per unit time through the insulated area

$$Q_1 = (10+10) \cdot \frac{50}{\frac{0.20}{0.04} + \frac{5 \cdot 0.59}{1.1}} = 130 \text{ J/sm}$$

Example B3. Cellar cross-section. Strong ground water effect.

We take example B1 with the ground water complication. There is a strong ground water flow below the cellar and the temperature of the water is the same as that of undisturbed ground ( $T=T_0$ ). See Figure 34. The depth to the ground water table is  $D=2.4 \text{ m}$  ( $D/L=0.4$ )

The data are the same as in example B1.

Formulas (5.1) and (12.1) give

$$d_{\min} = \frac{0.04 \cdot 6}{1.1} \cdot 0.08 = 0.017 \text{ m}$$

The optimal insulation distribution  $\tilde{d}$  is:

$$\tilde{d} = 0.103 + 0.218 \cdot \tilde{u} \text{ m}$$

The optimal insulation function is given in Figure 35 with  $D/L=0.4$ . The insulation thickness at the mid-point of the cellar floor ( $s=0$ ), at the corner in the ground ( $s=1$ ), and at the corner at the ground surface ( $s=1.4$ ) are:

$$\tilde{d} = 0.103 \text{ m}$$

$$\tilde{d} = 0.103 + 0.218 \cdot 0.17 = 0.140 \text{ m}$$

$$\tilde{d} = 0.103 + 0.218 \cdot 0.40 = 0.190 \text{ m}$$

respectively.

The heat loss is from (13.2) and (12.1):

$$Q_1 = (12 + 4.8) \cdot \frac{15}{\frac{0.12}{0.04} + \frac{6 \cdot 0.31}{1.1}} = 54 \text{ J/sm}$$

Example B4. Cylindrical cellar.

This case could be an example of a heat storage. See Figure 18.  
Let us take:

$$\begin{array}{lll} \lambda_0 = 1.1 \text{ J/ms}^\circ\text{C} & \lambda_1 = 0.04 \text{ J/ms}^\circ\text{C} & T_1 - T_0 = 20 \text{ }^\circ\text{C} \\ R = 6 \text{ m} & H = 2.4 \text{ m (H/R=0.4)} & d_m = 0.15 \text{ m} \end{array}$$

The minimum insulation thickness is given by (5.1) and Figure 21:

$$d_{\min} = \frac{0.04}{1.1} \cdot 6 \cdot 0.34 = 0.074 \text{ m}$$

The optimal insulation distribution  $\tilde{d}$  is given by (5.2):

$$\tilde{d} = 0.076 + 0.218 \tilde{u} \quad \text{m}$$

Here  $\tilde{u}$  is given by Figure 19 with  $H/R=0.4$ . The insulation thickness at the edge of the cellar is

$$0.076 + 0.218 \cdot 0.78 = 0.25 \text{ m}$$

The heat flow per unit time through the insulated surface is given by (13.2) and Figure 20:

$$Q_1 = (\pi \cdot 6^2 + 2\pi \cdot 6 \cdot 2.4) \cdot \frac{20}{\frac{0.15}{0.04} + \frac{6 \cdot 0.45}{1.1}} = 660 \text{ J/s}$$

Example B5. Cylindrical cellar. Two regions of soil.

The figure at formula (13.12) shows the considered case. The thermal conductivity of the granite bedrock is  $3\lambda_0$ .

Let us take essentially the same values as in the preceding example B4:

$$\begin{array}{lll} \lambda_0 = 1.1 \text{ J/ms}^\circ\text{C} & \lambda_{\text{granite}} = 3.3 \text{ J/ms}^\circ\text{C} & T_1 - T_0 = 20 \text{ }^\circ\text{C} \\ R = 6 \text{ m} & H = 2.4 \text{ m (H/R=0.4)} & d_m = 0.15 \text{ m} \\ \lambda_i = 0.04 \text{ J/ms}^\circ\text{C} & & \end{array}$$

The minimum insulation thickness is given by (5.1) and (11.2) with  $H/R=0.4$ :

$$d_{\min} = \frac{0.04}{1.1} \cdot 6 \cdot 0.079 = 0.017 \text{ m}$$

The insulation distribution  $\tilde{d}$  is given by (5.2):

$$\tilde{d} = 0.133 + 0.218 \tilde{u} \text{ m}$$

Here  $\tilde{u}$  is given by Figure 33 ( $H/R=0.4$ ). The insulation thickness at the edge of the cellar ( $s=1.4$ ) is

$$0.133 + 0.218 \cdot 0.30 = 0.20 \text{ m}$$

The heat flow per unit time through the insulated surface is given by (13.2) and (11.2):

$$Q_1 = (\pi \cdot 6^2 + 2\pi \cdot 6 \cdot 2.4) \cdot \frac{20}{\frac{0.15}{0.04} + \frac{6 \cdot 0.22}{1.1}} = 820 \text{ J/s}$$

Example B6. Cylindrical heat storage.

This case is shown by Figure 18. Let us take:

$$\begin{array}{lll} \lambda_0 = 1.1 \text{ J/ms}^\circ\text{C} & \lambda_i = 0.04 \text{ J/ms}^\circ\text{C} & T_1 - T_0 = 55 \text{ }^\circ\text{C} \\ R = 30 \text{ m} & H = 7.5 \text{ m (H/R=0.4)} & d_m = 0.3 \text{ m} \end{array}$$

The minimum insulation thickness is given by (5.1) and Figure 21,



$H/R=0.4$ :

$$d_{\min} = \frac{0.04}{1.1} \cdot 30 \cdot 0.34 = 0.37 \text{ m}$$

Since  $d_m > d_{\min}$ , the condition for the presented theory of optimal insulation is not fulfilled. We just know that a certain central part of the bottom must not be insulated. The condition (5.1) is fulfilled for soils with a higher thermal conductivity. For example, for a granite ground with  $\lambda_0=3.5 \text{ J/ms}^\circ\text{C}$  we get:

$$d_{\min} = \frac{0.04}{3.5} \cdot 30 \cdot 0.34 = 0.12 \text{ m} \quad (< d_m = 0.30 \text{ m})$$

Example B7. Parallelepipedic cellar.

This case is a cellar with a square bottom. The bottom area is the same as in example B4. The height is about the same. Let us take:

$\lambda_0=1.1 \text{ J/ms}^\circ\text{C}$	$\lambda_i=0.04 \text{ J/ms}^\circ\text{C}$	$T_1-T_0=20 \text{ }^\circ\text{C}$
$2L=10.63 \text{ m}$	$2L_1=10.63 \text{ m}$	$L_1/L=1$
$H=2.13 \text{ m}$	$H/L=0.4$	$d_m=0.15 \text{ m}$

The minimum insulation thickness is given by (5.1) and Table 2 ( $L_1/L=1$ )

$$d_{\min} = \frac{0.04}{1.1} \cdot 5.32 \cdot 0.39 \text{ m} = 0.075 \text{ m}$$

The insulation distribution  $\tilde{d}$  is given by (5.2):

$$\tilde{d} = 0.075 + 0.193 \tilde{u} \text{ m}$$

Here  $\tilde{u}$  is given by Table 1 ( $L_1/L=1$ ).

The heat flow per unit time through the cellar is from (13.2) and

Table 2 ( $L_1/L=1$ ).

$$Q_1 = (10.63^2 + 4 \cdot 10.63 \cdot 2.13) \cdot \frac{20}{\frac{0.15}{0.04} + \frac{5.32}{1.1} \cdot 0.48} = 670 \text{ J/s}$$

Example B8. Parallelepipedic cellar.

This case is a cellar with a rectangular bottom. The size of the bottom is identical to the plate in example A6. We have:

$\lambda_0 = 1.1 \text{ J/ms}^\circ\text{C}$	$\lambda_1 = 0.04 \text{ J/ms}^\circ\text{C}$	$T_1 - T_0 = 15 \text{ }^\circ\text{C}$
$2L = 10 \text{ m}$	$2L_1 = 50 \text{ m}$	$L_1/L = 5$
$H = 2 \text{ m}$	$H/L = 0.4$	$d_m = 0.13 \text{ m}$

The minimum insulation thickness is from (5.1) and Table 2 ( $L_1/L=5$ ,  $H/L=0.4$ ):

$$d_{\min} = \frac{0.04}{1.1} \cdot 5 \cdot 0.46 = 0.084 \text{ m}$$

The insulation distribution  $\tilde{d}$  is given by (5.2):

$$\tilde{d} = 0.046 + 0.182 \tilde{u} \text{ m}$$

Here  $\tilde{u}$  is given by Table 1 ( $L_1/L=5$ ,  $H/L=0.4$ ). The heat flow per unit time through the cellar is from (13.2) and Table 2:

$$Q_1 = (10 \cdot 50 + 2 \cdot (10+50) \cdot 2) \cdot \frac{15}{\frac{0.13}{0.04} + \frac{5}{1.1} \cdot 0.80} = 1600 \text{ J/s}$$

### 15C. Culverts

Example C.

The culvert geometry is shown in Figure 23. Let us take:

$$\begin{array}{lll} \lambda_0 = 1.2 \text{ J/ms}^\circ\text{C} & \lambda_1 = 0.04 \text{ J/ms}^\circ\text{C} & T_1 - T_0 = 13 \text{ }^\circ\text{C} \\ 2L = 8 \text{ m} & H = 4 \text{ m} & D = 4 \text{ m} \\ d_m = 0.10 \text{ m} & & \end{array}$$

We have  $H/L=1$  and  $D/L=1$ . The minimum insulation thickness  $d_{\min}$  is from (5.1) and Figure 30:

$$d_{\min} = \frac{0.04}{1.2} \cdot 4 \cdot 0.61 = 0.081 \text{ m}$$

The optimal insulation distribution becomes:

$$\tilde{d} = 0.019 + 0.133 \tilde{u} \text{ m}$$

Here  $\tilde{u}$  is given by Figure 27 ( $D/L=1$ ,  $H/L=1$ ). The insulation thickness at the center of the bottom surface ( $s=0$ ), at the lower corner ( $s=1$ ), at the upper corner ( $s=2$ ), and at the center of the top surface ( $s=3$ ) becomes:

$$\begin{array}{l} \tilde{d} = 0.019 \text{ m} \\ \tilde{d} = 0.019 + 0.133 \cdot 0.51 = 0.087 \text{ m} \\ \tilde{d} = 0.019 + 0.133 \cdot 1.06 = 0.160 \text{ m} \\ \tilde{d} = 0.019 + 0.133 \cdot 0.99 = 0.151 \text{ m} \end{array}$$

respectively.

The heat loss per unit time from the insulated culvert is from (13.2) and Figure 29:

$$Q_1 = (2 \cdot 8 + 2 \cdot 4) \cdot \frac{13}{\frac{0.10}{0.04} + \frac{4}{1.2} \cdot 1.34} = 45 \text{ J/sm}$$

### 15D. Insulating soil thickness

The insulating soil thickness  $Lu_m$  is a fundamental quantity. It represents the insulating capacity of the soil. We will in this section give explicit values for this length in various cases. All values are taken from section 13.

We note that the heat loss  $Q_1$  is immediately obtained from  $Lu_m$ . We have from (13.1) and (13.2):

$$Q_1 = A_1 \cdot \frac{T_1 - T_0}{\frac{d_m}{\lambda_i} + \frac{Lu_m}{\lambda_o}}$$

The quantity

$$\frac{\lambda_o}{Lu_m}$$

represents an equivalent "k"-value of the soil.

Ground plate  
(Two-dimensional)

$$2L=10 \text{ m:} \quad Lu_m=3.9 \text{ m}$$

Circular disc

$$R=5 \text{ m:} \quad Ru_m=2.1 \text{ m}$$

Rectangular ground plate

2 L = 10 m	2L <sub>1</sub> (m)	10	20	50
	Lu <sub>m</sub> (m)	2.6	3.3	3.9

Cellar cross-section

$$2L=10 \text{ m} \quad H=0 \quad : \quad Lu_m=3.9 \text{ m}$$

2L=10 m	H=2 m	:	$Lu_m=4.6$ m
2L=10 m	H=5 m	:	$Lu_m=5.5$ m
2L=10 m	H=10 m	:	$Lu_m=7.0$ m

#### Cylindrical cellar or heat storage

R=5 m	H=0 m	:	$Ru_m=2.1$ m
R=5 m	H=2 m	:	$Ru_m=2.3$ m
R=5 m	H=5 m	:	$Ru_m=2.7$ m
R=5 m	H=10 m	:	$Ru_m=3.6$ m

We note that the insulating soil thickness is roughly twice as big in the plane case of a cellar cross-section as in the corresponding ( $R=L$ ) cylindrical case.

#### Parallelepipedic cellar

2L=10 m	$2L_1=10$ m	:	$Lu_m=2.4$ m
H=2 m	$2L_1=20$ m	:	$Lu_m=3.3$ m
	$2L_1=50$ m	:	$Lu_m=4.0$ m

We note that the difference between the parallelepipedic cellar and the corresponding ground plate is quite small.

#### Cellar cross-section, granite and moraine.

$$(\lambda_{\text{granite}} = 3 \cdot \lambda_{\text{moraine}} = 3\lambda_0)$$

2L=10 m	H=5 m	:	$Lu_m=3.0$ m
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We note that the introduction of the granite bedrock diminishes  $Lu_m$  from 5.5 m to 3.0 m.

#### Cylindrical cellar, granite and moraine

R=5 m	H=2 m	:	$Ru_m=1.1$ m
	H=5 m	:	$Ru_m=1.7$ m

## Cellar cross-section, strong ground water effect

2L=10 m	D=2 m	:	$Lu_m=1.6$ m
H=2 m	D=3 m	:	$Lu_m=2.1$ m
	D=6 m	:	$Lu_m=3.2$ m
	D= $\infty$ m	:	$Lu_m=4.6$ m

## Culverts

2L=10 m	H=1 m	:	$Lu_m=3.2$ m
D=1 m	H=2.5 m	:	$Lu_m=3.6$ m
	H=5 m	:	$Lu_m=4.3$ m
2L=10 m	H=1 m	:	$Lu_m=4.2$ m
D=2.5 m	H=2.5 m	:	$Lu_m=4.6$ m
	H=5 m	:	$Lu_m=5.5$ m

Let us note that the insulating soil thickness is directly proportional to the scale L (or R). Other values than 2L=10 m are immediately obtained by proportionality.

15E. Approximate heat loss

We will with some examples illustrate the approximate first-order heat loss formula and the results of section 14. We have now instead a constant insulation thickness  $d$  over the insulation surface  $S_1$ .

The heat loss over  $S_1$  is approximated with formulas (14.2) and (14.3):

$$Q_1 = A_1 \cdot \frac{T_1 - T_0}{\frac{d}{\lambda_i} + \frac{Lu_m}{\lambda_o}}$$

Here  $Lu_m$  is the insulating soil thickness. Numerous examples were given in 15D. We will use the following values in the examples be-

low:

$$\lambda_0 = 2 \text{ J/ms}^\circ\text{C}$$

$$\lambda_1 = 0.04 \text{ J/ms}^\circ\text{C}$$

$$T_1 - T_0 = 15 \text{ }^\circ\text{C}$$

Example E1. Plate on the ground. Two-dimensional.

Let us take:

$$2L = 10 \text{ m}$$

$$d = 0.10 \text{ m}$$

The heat loss is approximately with (14.2):

$$Q_1 \approx 10 \cdot \frac{15}{\frac{0.10}{0.04} + \frac{5}{2} \cdot \frac{\pi}{4}} = 34 \text{ J/ms}$$

The corresponding  $d_{\min}$  is from (5.1) and (6.8):

$$d_{\min} = \frac{0.04}{2} \cdot 5 \cdot \left(1 - \frac{\pi}{4}\right) \approx 0.021 \text{ m}$$

The given  $d$  is about  $5 \cdot d_{\min}$ . The value  $Q_1 \approx 34$  is according to section 14 an underestimation with about 2%.

Example E2. Rectangular ground plate.

Let us take:

$$2L = 10 \text{ m}$$

$$2L_1 = 20 \text{ m}$$

$$d = 0.05 \text{ m}$$

The heat loss is approximately with (14.2) and (13.6):

$$Q_1 \approx 10 \cdot 20 \cdot \frac{15}{\frac{0.05}{0.04} + \frac{5 \cdot 0.66}{2}} = 1030 \text{ J/s}$$

The corresponding  $d_{\min}$  is from (5.1) and Figure 12:

$$d_{\min} \approx \frac{0.04}{2} \cdot 5 \cdot 0.265 = 0.027 \text{ m}$$

The thickness  $d$  is about twice  $d_{\min}$ , so the error for the heat loss is about 6%.

Example E3. Cylindrical heat storage.

Let us take:

$$R=20 \text{ m} \quad H=4 \text{ m} \quad d=0.30 \text{ m}$$

The heat loss is approximately with (14.2) and (13.8):

$$Q_1 \approx (\pi \cdot 20^2 + 2\pi \cdot 20 \cdot 4) \cdot \frac{15}{\frac{0.30}{0.04} + \frac{20 \cdot 0.44}{2}} \approx 2200 \text{ J/s}$$

The corresponding  $d_{\min}$  is from (5.1) and Figure 21:

$$d_{\min} = \frac{0.04}{2} \cdot 20 \cdot 0.29 = 0.12 \text{ m}$$

The error in  $Q_1$  is about 5%.

Example E4. Culvert.

Let us take:

$$2L=4 \text{ m} \quad H=2 \text{ m} \quad D=2 \text{ m} \quad d=0.50 \text{ m}$$

The heat loss is approximately with (14.2) and (13.10):

$$Q_1 \approx (4+2+4+2) \cdot \frac{15}{\frac{0.50}{0.04} + \frac{2 \cdot 1.3}{2}} = 13 \text{ J/ms}$$



The corresponding  $d_{\min}$  is from (5.1) and Figure 30:

$$d_{\min} = \frac{0.04}{2} \cdot 2 \cdot 0.60 = 0.024$$

The thickness  $d$  is more than ten times  $d_{\min}$ , so the error for  $Q_1$  is less than 1%.



## 16 CONCLUDING REMARKS

The heat losses for the optimal insulation and a corresponding constant-thickness distribution are compared in section 14 for several cases. The gain in the optimal case is 10%, or less for the considered cases. This moderate gain applies to smaller building structures which are well insulated. The gain is small if the heat flux across the non-optimal insulation is relatively constant.

The insulating soil thickness is a very useful concept for these smaller structures. The simple heat loss formula in section 14 is quite accurate.

The optimal insulation theory is more important for larger structures. An inner protected part of the boundary surface is to be left uninsulated. These situations are not dealt with in the present paper. So there remains a large number of basic cases to be investigated. The current ideas to construct large heat storage systems in the ground will provide another set of important applications of the optimal insulation theory.

The insulation layers do not always lie along the boundary between the structure and the ground. They may for example be put directly on the ground surface immediately outside the structure. These cases provide an almost infinite variety of other configurations for the insulation sheets. All such cases may be analysed with the aid of the present theory.

There is a whole field of pertinent cases that require an analysis with the aid of this optimal insulation theory. Some generalizations of the theory will be necessary.

Appendix. First-order variation of heat loss.

We will in this appendix derive the fundamental formula (3.2), which is the basis for the optimal insulation criterion. The derivation is done for a somewhat more general heat flow problem than that of (2.1)-(2.3).

We have an original heat flow problem in a volume  $V$  with the boundary surface  $S$ . The thermal conductivity  $\lambda$  may be variable through  $V$ . The temperature solution  $T$  satisfies the heat conduction equation:

$$\nabla \cdot (\lambda \nabla T) = 0 \quad \text{in } V \quad (\text{A1})$$

Along the boundary  $S$  there is an insulation layer with the thickness  $d$  and thermal conductivity  $\lambda_i$ . Outside the insulation layer there is a prescribed temperature  $f$ . The boundary condition on  $S$  is then:

$$T + \frac{d}{\lambda_i} \cdot \lambda \frac{\delta T}{\delta n} = f \quad \text{on } S \quad (\text{A2})$$

The prescribed temperature  $f$  may be any function over  $S$ . The thickness  $d$  may also be variable over  $S$ . In particular we have a prescribed boundary temperature  $f$  at points, where  $d$  is zero.

Consider now the following new heat flow problem. The thickness of the insulation layer is at each point on  $S$  changed from  $d$  to  $d + \delta d$ :

$$d \rightarrow d + \delta d \quad \text{on } S \quad (\text{A3})$$

The function  $\delta d$  is arbitrary over  $S$ , except for the condition that  $d + \delta d$  is non-negative. We have a new solution to this changed heat flow problem:

$$T \rightarrow T + \delta T \quad \text{in } V \quad (\text{A4})$$

The temperature change satisfies:

$$\nabla \cdot (\lambda \nabla \delta T) = 0 \quad \text{in } V \quad (A5)$$

The new temperature  $T + \delta T$  satisfies (A2), when  $d$  is replaced by  $d + \delta d$ . The difference between the boundary condition in the two cases is:

$$\delta T + \frac{d}{\lambda_i} \lambda \frac{\delta(\delta T)}{\delta n} + \frac{\delta d}{\lambda_i} \lambda \frac{\delta T}{\delta n} + \frac{\delta d}{\lambda_i} \lambda \frac{\delta(\delta T)}{\delta n} = 0 \quad \text{on } S \quad (A6)$$

The further analysis is based on a special thermodynamical concept; namely the so-called thermality. This is developed in (\*). The thermality transfer is equal to the temperature in centigrades times the heat transfer. It represents the first-order term of the entropy transfer and is an expansion of the centigrade temperature divided by the absolute temperature level.

The thermality consumption in our original heat flow process is by definition given by:

$$\Gamma = \iint_S f \lambda \frac{\delta T}{\delta n} d S \quad (A7)$$

The corresponding thermality consumption in the new heat flow problem is:

$$\Gamma' = \iint_S f \lambda \left( \frac{\delta T}{\delta n} + \frac{\delta(\delta T)}{\delta n} \right) d S \quad (A8)$$

We will first derive a formula for the first-order variation of the

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(\*) Johan Claesson: Thermodynamics of sensible heat storage systems. Thermality concept. August 1979. Department of Mathematical Physics, Lund, Sweden.

thermality in the change (A3). From this we will get the desired formula (3.2) for the first-order variation of the heat loss.

The change of thermality consumption in the heat flow process, when the insulation thickness is changed according to (A3) is from (A7) and (A8):

$$\Gamma' - \Gamma = \iint_S \left( T + \frac{d}{\lambda_i} \lambda \frac{\delta T}{\delta n} \right) \lambda \frac{\delta(\delta T)}{\delta n} dS \quad (A9)$$

Here (A2) has been inserted in the integral.

We need the following identity:

$$\begin{aligned} \iint_S T \lambda \frac{\delta(\delta T)}{\delta n} dS &= \iiint_V \nabla \cdot (T \lambda \nabla(\delta T)) dV = \\ &= \iiint_V \nabla \cdot (\delta T \lambda \nabla T) dV = \iint_S \delta T \lambda \frac{\delta T}{\delta n} dS \end{aligned} \quad (A10)$$

Gauss' formula and the identity

$$\nabla \cdot (T \lambda \nabla(\delta T)) = \lambda \nabla T \cdot \nabla \delta T = \nabla \cdot (\delta T \nabla T) \quad (A11)$$

have been used in (A10). Formulas (A1) and (A5) are used in (A11).

The integrand of (A9) becomes with (A10):

$$\lambda \frac{\delta T}{\delta n} \left( \delta T + \frac{d}{\lambda_i} \lambda \frac{\delta(\delta T)}{\delta n} \right) \quad (A12)$$

The second factor of (A12) coincides with the first two terms of (A6).

The other two terms of (A6) are therefore of major interest to us. We aim to derive an expression for the first-order variation of  $\Gamma$  in the change (A3). This means that only first-order terms in  $\delta d$

are to be retained. The change of temperature  $\delta T$  is of the first order. The fourth term of (A6) is therefore of the second order. Hence it is neglected.

The first-order variation  $\delta \Gamma$  is then from (A6), (A12) and (A9):

$$\delta \Gamma = - \iint_S \frac{\delta d}{\lambda_i} \left( \lambda \frac{\delta T}{\delta n} \right)^2 d S \quad (\text{A13})$$

This is a fundamental formula. Its simplicity is note-worthy. The change of thermality is determined by the given change  $\delta d$  of insulation thickness and by the heat flow  $\lambda \frac{\delta T}{\delta n}$  of the original problem. The important thing is that the temperature  $\delta T$  is not involved. We need not solve the new problem in order to determine the first-order thermality change.

Let us now consider the more special case, when  $S$  consists of two parts  $S_0$  and  $S_1$ . The prescribed temperature  $f$  is  $T_1$  on  $S_1$  and  $T_0$  on  $S_0$ . Then we have:

$$\Gamma = \iint_{S_1} T_1 \lambda \frac{\delta T}{\delta n} d S + \iint_{S_0} T_0 \lambda \frac{\delta T}{\delta n} d S = (T_1 - T_0) Q_1 \quad (\text{A14})$$

We have used that the surface integrals over  $S_1$  and  $S_0$  of  $\lambda \frac{\delta T}{\delta n}$  gives  $Q_1$  and  $-Q_1$  respectively. Here  $Q_1$  is the heat flow through  $S_1$  as defined by (2.4).

The variation of  $\Gamma$  is therefore given by:

$$\delta \Gamma = (T_1 - T_0) \delta Q_1 \quad (\text{A15})$$

We are in this study only considering cases, when the insulation is changed on  $S_1$ , i.e.  $\delta d=0$  on  $S_0$ . We finally have from (A15) and (A13):

$$(T_1 - T_0) \delta Q_1 = - \iint_{S_1} \frac{\delta d}{\lambda_i} \left( \lambda \frac{\delta T}{\delta n} \right)^2 d S \quad (\text{A16})$$





## SUMMARY

The thermal insulation of a building structure against the surrounding ground poses an optimization problem. How is a given amount of insulation material to be distributed along the boundary surface between building and ground in order to minimize the heat losses? A mathematical theory for the optimization problem is presented. The basic criterion is that the heat flux across the insulation is to be constant along the insulation surface.

The optimal insulation distribution has been computed explicitly in several cases: Rectangular plate on the ground for different shapes, circular disc on the ground, cellar (parallelepipedic shape), rectangular cellar cross-section (two-dimensional case), and cylindrical cellar or heat storage. The optimal distribution is also given for a culvert of rectangular cross-section for different heights, widths, and depths below the ground surface. A few cases, for which the ground consists of two layers (granite under a top layer of moraine), have been analysed. Finally, some cases with a cooling ground water stream below the insulated cellar are dealt with.

The ground below and around the building structure gives a thermal insulation. The magnitude of the insulation capacity of the ground is obtained from the solution of a multi-dimensional heat flow problem. The presented theory provides an equivalent mean insulation soil thickness. This insulating soil thickness is given for the discussed cases (plate on the ground, cellar, culvert).

The insulating soil thickness makes it possible to give a simple approximate formula for the heat loss to the ground for any insulation distribution.

The presented formulas and results are illustrated by numerous examples.



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