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Aggregate information, common knowledge, and agreeing not to bet*

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Abstract

I consider a gamble where the sum of the distributed payoffs is proportionate to the number of participants. I show that no subset of the population can agree to participate in the bet, if the size of the group is commonly known. Repeated announcements of the number of the participants leads the population to agree not to bet.

1. Introduction

Aumann (1976) was the first one to formalize the concept of common knowledge. In his seminal paper he showed that if two people have the same prior, and their posterior probabilities about an event are commonly known, then these posteriors are identical. Geanakoplos and Polemarchakis (1982) proved that if the individuals have a common prior, and they update their beliefs through communication, they will eventually agree on a common posterior, while McKelvey and Page (1986) generalized the previous results,

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by showing that common knowledge of an aggregate statistic suffices for consensus to be achieved.

Sebenius and Geanakoplos (1983) extended Aumann's result to expectations, by proving that if the expectations of two people about a random variable are commonly known, then they must necessarily be equal. A number of generalizations appeared in the literature ever since (Nielsen, *et.al.*, 1990; Nielsen, 1995; Hanson, 1998; Hanson, 2002). A direct application of this proposition is the famous no-bet theorem, which states that two rational, risk-averse individuals with a common prior will never agree to participate in a gamble, if their willingness to bet is commonly known. Milgrom and Stokey (1980) had already addressed this problem, by showing that common knowledge precludes trading among rational, risk-averse agents in an uncertain environment.

The present paper extends the no-bet theorem to cases where the gamble can take place even if a subset of the population rejects participation. In this case the total payoff to be distributed among the participants depends on the number of people who have accepted the bet. In other words, rejection of the gamble by a group of people does not cancel it, but the prize becomes smaller due to limited participation.

I show that no group of individuals can agree to bet, if the number of participants is commonly known. Notice that this result is different to the standard no-bet theorem, in the sense that it does not require common knowledge of which, but how many, individuals participate. The number of participants is an aggregate statistic of the population's willingness to bet, whilst the identity of the participants explicitly reveals the full information about the composition of the group of the potential participants. Nielsen, *et.al.* (1990), and Nielsen (1995) showed that if an aggregate statistic of every individual's expected payoff is commonly known, the population will agree not to bet.

There are some central features that distinguish their result from my conclusions. First, they aggregate what everybody believes about one's expected payoff. They show that if all individuals' aggregate expected payoffs are commonly known, then they are necessarily identical. On the other hand, I consider a unique measure which aggregates everybody's willingness to participate into a single statistic, *ie.* the number of individuals who accept participation. Second, I do not assume that the bet is canceled, once an individual has rejected participation. In this case a side bet takes place with a reduced prize, depending on the number of participants.

Finally, I show that repeated announcements of the number of participants eventually induce a no-bet agreement.

2. Agreeing not to bet with aggregate information

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a finite population $N = \{1, ..., n\}$. The measure $\mathbb{P}$ determines the (common) prior beliefs of the individuals in the population about every event $E \in \mathcal{F}$. Every individual is endowed with a finite, non-delusional, information partition $\mathcal{I}_i \subset \mathcal{F}$. Let $\mathcal{J} = \vee_{i=1}^{n} \mathcal{I}_i$, and $\mathcal{M} = \wedge_{i=1}^{n} \mathcal{I}_i$ denote the join (coarsest common refinement), and the meet (finest common coarsening) of the information partitions respectively. We assume that $\mathbb{P}[J] > 0$ for every $J \in \mathcal{J}$. We define knowledge as usual, *ie.* we say that i knows some $E \in \mathcal{F}$ at ω, and we write $\omega \in K_i(E)$, whenever $I_i(\omega) \subseteq E$, where $I_i(\omega)$ denotes the member of $\mathcal{I}_i$ that contains ω. The event E is commonly known if $M(\omega) \subseteq E$, where $M(\omega)$ denotes the member of the meet that contains ω.

We define a gamble as an $\mathcal{F}$-measurable partition $\mathcal{G} = \{G_1, ..., G_n\}$ of Ω. Participating in the gamble has a fixed cost, which for simplicity we assume is equal to 1 unit. If some $\omega \in G_i$ occurs, the i-th individual receives the n units. Sebenius and Geanakoplos (1983) showed that if the individual willingness to participate in $\mathcal{G}$ is commonly known, they cannot agree to bet. Their argument is based on the fact that, if $\mathbb{E}[U_i|I_j(\omega)]$ is commonly known for every $j \in N$, then all individuals must hold the same conditional expectation. Nielsen, *et.al.* (1990) extended this result by showing that if $F(\mathbb{E}[U_i|I_1(\omega)], ..., \mathbb{E}[U_i|I_n(\omega)])$ is commonly known, for every $i \in N$, then the conditional $\mathbb{E}[U_i|I_i(\omega)] = \mathbb{E}[U_j|I_j(\omega)]$, for every $i, j \in N$ and therefore any agreement to bet is precluded. Nielsen, *et.al.* (1990) discuss the properties of the function F.

Let $S \subseteq N$ denote the subset of the population that has agreed to participate in $\mathcal{G}$. If now some $\omega \in G_i$ occurs, the i-th individual receives the amount that has been bet, *ie.* s units, where s denotes the cardinality of S. If the winner has chosen not to participate, then nobody wins. Consider for instance a lottery. Each player buys one ticket for a fixed cost, and the winner takes all.

Notice that the random payoff received by the winner, depends on the number of participants, and can be explicitly formulated as follows,

$$U_i^s(\omega) = (s1_{\{\omega\in G_i\}} - 1)1_{\{i\in S\}}, \tag{1}$$

where $1_{\{\cdot\}}$ denotes the indicator function. An individual i agrees to participate in $\mathcal{G}$, given the number of participants, whenever the expected payoff is positive, given the private information $I_i(\omega)$. Clearly if $S = \{i\}$, *ie.* if i is the only participant, then $U_i^s(\omega) \leq 0$, for every $\omega \in \Omega$, implying that there is no point playing alone. The expected payoff of the gamble with s participants for the i-th individual, while being at ω, is equal to

$$\mathbb{E}[U_i^s|I_i(\omega)] = s\mathbb{P}[G_i|I_i(\omega)] - 1, \tag{2}$$

if i participates, and equal to 0 otherwise.

It is commonly known that s individuals participate, if for every $\omega' \in M(\omega)$ there are exactly s individuals such that $\omega' \in R_i^s$, where

$$R_i^s = \{\omega \in \Omega : \mathbb{E}[U_i^s|I_i(\omega)] > 0\}. \tag{3}$$

If it was commonly known who, instead of how many people, participated, then the same s individuals (and only them) would be willing to accept $\mathcal{G}$ at every $\omega' \in M(\omega)$. On the other hand, in this model we do not require the same people to participate at every $\omega' \in M(\omega)$, as long as at every state contained in the meet exactly s individuals accept the bet.

Proposition 1. *Consider a finite population N, and a gamble $\mathcal{G}$. Then no group $S \subseteq N$ can agree to bet, if the cardinality of S is commonly known.*

Proof. If $s = 1$, the proof is straightforward, since $U_i^1(\omega) \leq 0$, for every $\omega \in \Omega$, and therefore i will not participate either. Suppose now that $s \geq 2$. By definition, $s\mathbb{P}[G_i|I_i(\omega)] > 1$, for every $\omega \in R_i^s$. Since, $\mathbb{P}[J] > 0$ for every $J \in \mathcal{J}$, it follows that $\mathbb{P}[I_i|M(\omega)] > 0$ for every $I_i \in \mathcal{I}_i$. Hence, $s\mathbb{P}[G_i|I_i(\omega)]\mathbb{P}[I_i(\omega)|M(\omega)] > \mathbb{P}[I_i(\omega)|M(\omega)]$. It follows from summing over $M(\omega) \cap R_i^s$ (which is $\sigma(\mathcal{I}_i)$-measurable) that

$$s \sum_{I_i \subseteq M(\omega)\cap R_i^s} \mathbb{P}[G_i|I_i]\mathbb{P}[I_i|M(\omega)] > \sum_{I_i \subseteq M(\omega)\cap R_i^s} \mathbb{P}[I_i|M(\omega)].$$

Since $M(\omega) \cap R_i^s \subseteq M(\omega)$, we obtain

$$s \sum_{I_i \subseteq M(\omega)} \mathbb{P}[G_i|I_i]\mathbb{P}[I_i|M(\omega)] \geq s \sum_{I_i \subseteq M(\omega) \cap R_i^s} \mathbb{P}[G_i|I_i]\mathbb{P}[I_i|M(\omega)],$$

with the equality holding when it is commonly known that i participates in $\mathcal{G}$, *ie.* $M(\omega) \subseteq R_i^s$. Combining the previous two inequalities entails

$$s\mathbb{P}[G_i|M(\omega)] > \mathbb{P}[R_i^s|M(\omega)].$$

Since R_i^s is $\sigma(\mathcal{I}_i)$-measurable, it can be partitioned into disjoint elements $J \in \mathcal{J}$, *ie.* $R_i^s = \cup_{J \subseteq R_i^s} J$. Thus,

$$s\mathbb{P}[G_i|M(\omega)] > \sum_{J \subseteq R_i^s} \mathbb{P}[J|M(\omega)].$$

If we sum over N we obtain

$$s \sum_{i \in N} \mathbb{P}[G_i|M(\omega)] > \sum_{i \in N} \sum_{J \subseteq R_i^s} \mathbb{P}[J|M(\omega)].$$

At every $J \subseteq M(\omega)$, there are exactly s individuals who agree to participate, and therefore there are exactly s individuals with $J \subseteq R_i^s$. Hence $\mathbb{P}[J|M(\omega)]$ appears exactly s times in the sum, for every $J \subseteq M(\omega)$. Therefore,

$$\sum_{i \in N} \sum_{J \subseteq R_i^s} \mathbb{P}[J|M(\omega)] = \sum_{J \subseteq M(\omega)} s\mathbb{P}[J|M(\omega)] = s,$$

implying that $s > s$, which is a contradiction, and completes the proof. □

If the number of the individuals who are willing to participate is not commonly known, the standard updating procedure (Geanakoplos and Polemarchakis, 1982) can be applied. Suppose that at every time $t > 0$, the individuals privately announce their willingness to participate in $\mathcal{G}$, and then the number of participants, which is denoted by s_t, is publicly announced. Every $i \in N$, after having received this signal, implicitly constructs the partition $\mathcal{W}_t = \{W_t, W_t^c\}$, where W_t contains the states ω', at which exactly s_t individuals have positive expected payoff. Then they refine their own current information partition according to

$$\mathcal{I}_i^{t+1} = \mathcal{I}_i^t \vee \mathcal{W}_t. \tag{4}$$

Given this refining scheme, the set of participants will be empty after finitely many rounds of communication.

Proposition 2. *Consider a finite population N, a gamble $\mathcal{G}$, and suppose that the individuals are informed about the number of participants at every time $t > 0$. Then they will eventually agree not to bet.*

Proof. It follows from equation (4) that $\mathcal{I}_i^{t+1}$ is finer than $\mathcal{I}_i^t$ for every $i \in N$, and every $t > 0$. From $\sigma(\mathcal{J})$ being finite, it follows that there is $T > 0$ such that $\mathcal{I}_i^t = \mathcal{I}_i^*$ for every $i \in N$, and every $t > T$. Hence both W^* and W^{*c} are $\sigma(\mathcal{I}_i^*)$-measurable for every $i \in N$, which implies that either $W^* \in \cap_{i \in N} \sigma(\mathcal{I}_i^*) = \sigma(\wedge_{i \in N} \mathcal{I}_i^*) = \sigma(\mathcal{M})$, or $W^{*c} \in \cap_{i \in N} \sigma(\mathcal{I}_i^*) = \sigma(\wedge_{i \in N} \mathcal{I}_i^*) = \sigma(\mathcal{M})$. Since the individuals satisfy non-delusion, it follows that either $M(\omega) \subseteq W^*$, or $M(\omega) \subseteq W^{*c}$. In the first case, it follows from proposition 1 that the individuals cannot agree to bet, while in the second one it is commonly known that nobody bets, which completes the proof. $\square$

The previous results can be extended to risk-neutral individuals. In this case, one would participate if and only if the expected payoff was non-negative (instead of strictly positive). It is rather trivial to show that if the number of participants is commonly known, no individual has a positive expected payoff, and therefore nobody who is involved in $\mathcal{G}$ expects to gain a profit.

3. Concluding discussion

In the present note we have discussed gambles that do take place even when a subset of the individuals who are proposed to participate, refuse to take part in it. In this case the payoff to be distributed is linear with respect to the number of individuals who agree to bet. We show that if the size of the participants is commonly known, the gamble will not take place. In other words, it cannot be commonly known that an exact number of individuals has positive expected payoff. Notice that the it is the size of the group which is revealed, rather than the identity of its members. Repeated public announcement of the number of individuals who accept participation leads to commonly known size of the group, and therefore agreement not to bet.

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