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Efficient communication, common knowledge, and consensus*

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Abstract

We study a model of pairwise communication in a finite population of Bayesian agents. We show that, in contrast with claims to the contrary in the existing literature, communication under a fair protocol may not lead to common knowledge of signals. We prove that commonly known signals are achieved if the individuals convey, in addition to their own message, the information about every individual's most recent signal they are aware of. If the signal is a posterior probability about some event, common knowledge implies consensus.

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1. Introduction

Aumann (1976) showed in his seminal paper that if two people have the same priors, and their posteriors for an event are common knowledge, then these posteriors are identical. Milgrom (1981) presented an axiomatic characterization of common knowledge, while Geanakoplos and Polemarchakis (1982) proved that if two people have common prior, different posteriors, and they update their beliefs through communication, they will agree on a common posterior after finitely many steps of communication. The main aim of this literature is to study the problem of reaching a consensus in a group of people who hold different subjective beliefs. This issue is rather central, not only in economics, but in the vast majority of social sciences (Goldman, 1987).

Cave (1983), and Bacharach (1985), independently attempted to generalize Aumann's result to arbitrary signal functions, in place of posterior probabilities. Their setting, though conceptually flawed (Moses and Nachum, 1990), has been the stepping stone for further development of models of communication in populations with Bayesian agents. Parikh and Krasucki (1990) introduced a model of pairwise communication, claiming that under some mild assumptions concerning the communication protocol, consensus would be reached. Weyers (1992) challenged their view, by arguing that their updating process was not sufficient to ensure common knowledge, and therefore even though their result on consensus was correct, the proof was incomplete. Her argument is based on the fact that individuals in their setting refine only their current information set, rather than their whole partition. She proposed that by allowing for such a "rational" refining process, all fair protocols would lead to common knowledge.

The present paper challenges this view, by showing that this condition does not suffice either for common knowledge. A pair of simple counter-examples are used in order to illustrate the validity of this argument. It is also shown that a commonly known signal would be reached if the individuals informed their corresponding counterpart about, not only their own signal, but also everybody's most recent signal they are aware of. The recipient takes the conveyed information into account if and only if she has not previously received more recent information about the transmitted signals. Notice that though the transmitted signal is not correct, if the individual it belongs to has already updated her information, common knowledge will eventually be achieved. In addition, if the signals are of the form of posterior probabilities, consensus will be reached.

In general, the distinction between common knowledge and consensus has to be clear. Though we do not go inside the conversation about the conceptual flaw that Moses and Nachum (1990) discovered, we point out that reaching common knowledge does not depend on the nature of the signals as it was claimed till now, but on the amount of information being transmitted. More specifically, we show that it is not the sure thing principle or any generalization of it that leads to common knowledge, but the transmission of the whole information possessed by the speaker. Of course for consensus to be achieved, a number of additional requirements are needed, but this remains outside the scope of the present paper (Aumann and Hart (2006), Samet (2006)).

The issue of common knowledge is rather central since not commonly known consensus may not be robust against protocol perturbations. In other words, it may be the case that a consensus is preserved *ad infinitum* under some fair protocol, but some slight change in the communication pattern could lead to some totally different behavior. This could never happen, had the signal been commonly known, implying that once the population has agreed, the agreement will remain forever.

2. Not commonly known consensus

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a finite population $N = \{1, ..., n\}$. The measure $\mathbb{P}$ determines the (common) prior beliefs of the individuals in the population about every event $E \in \mathcal{F}$. Every individual is endowed with a finite, non-delusional, information partition $\mathcal{I}_i \subset \mathcal{F}$. Let $\mathcal{J} = \vee_{i=1}^{n} \mathcal{I}_i$, and $\mathcal{M} = \wedge_{i=1}^{n} \mathcal{I}_i$ denote the join (coarsest common refinement), and the meet (finest common coarsening) of the information partitions respectively. Similarly to Geanakoplos and Polemarchakis (1982), we assume[1] that $\mathbb{P}[J] > 0$ for every $J \in \mathcal{J}$. We define knowledge as usual, *ie.* we say that i knows some $E \in \mathcal{F}$ at ω, and we write $\omega \in K_i(E)$, whenever $I_i(\omega) \subseteq E$, where $I_i(\omega)$ denotes the member of $\mathcal{I}_i$ that contains ω . The event E is mutually known if $\omega \in K_i(E)$ for every $i \in N$, and commonly known if $M(\omega) \subseteq E$, where $M(\omega)$ denotes the member of the meet that

[1] Nielsen (1984), and Brandenburger and Dekel (1987) relax this assumption.

contains ω.

Let the function $f : \sigma(\mathcal{J}) \to A$, map every possible information set to an action (signal). For simplicity we assume that $A = \mathbb{R}$, and we write $f(I_i(\omega)) = f_i(\omega)$. If the signals are of the form of posterior beliefs about some event E, the function f can be rewritten as $f_i(\omega) = \mathbb{P}[E|I_i(\omega)]$.

The action of the i-th individual is commonly known at ω, if $M(\omega) \subseteq R_i$, where

$$R_i = \{\omega' \in \Omega : f_i(\omega') = f_i(\omega)\}. \tag{1}$$

Aumann (1976) showed that, in populations with two people, if the signals are posterior probabilities indeed, and $R = R_1 \cap R_2$ is commonly known, then $\mathbb{P}[E|I_1(\omega)] = \mathbb{P}[E|I_2(\omega)]$. Geanakoplos and Polemarchakis (1982) consequently proved that indirect communication between two individuals leads to commonly known posterior beliefs.

Cave (1983), and Bacharach (1985) independently attempted to generalize these results to arbitrary signal functions which satisfy the sure thing principle[2]. However, Moses and Nachum (1990) discovered a conceptual flaw in their reasoning. More specifically, they showed that common knowledge, and the sure thing principle do not suffice for agreement. Their argument is based on the fact that the union of two information sets is not an information set in the same partition. A number of solutions to this problem have been proposed ever since (Moses and Nachum (1990), Aumann and Hart (2005), and Samet (2006)). All of them require additional assumptions regarding the signals. However, this is outside the scope of the present paper.

Parikh and Krasucki (1990) extended the Cave-Bacharach framework, by introducing a model of pairwise communication. They defined a protocol as a pair of sequences $(\{s_t\}_{t=1}^{\infty}, \{r_t\}_{t=1}^{\infty})$, with $s_t, r_t \in N$ for every $t > 0$. At time t the receiver (r_t) observes the sender's (s_t) signal, and refines her own information set according to $I_{r_t}^{t+1}(\omega) = I_{r_t}^{t}(\omega) \cap R_{s_t}^{t}$, where $R_{s_t}^{t}$ is defined by equation (1). Every individual $j \neq r_t$ does not revise her information. We say that there is a directed edge between i and j if and only if there are infinitely many $t > 0$ such that $s(t) = i$ and $r(t) = j$.

[2]A function f satisfies the sure thing principle, if for every disjoint $J_1, J_2 \in \sigma(\mathcal{J})$, such that $f(J_1) = f(J_2) = f$, then $f(J_1 \cup J_2) = f$. Cave (1983) calls this property union consistency.

Definition 1. [Parikh and Krasucki, 1990] A protocol is fair whenever the graph of directed edges is strongly connected, *ie.* if there is a path of directed edges passing from all vertexes (individuals), returning to its origin.

They generalized the union consistency property by introducing the concept of convexity: f is convex if $f(J_1 \cup J_2) = \alpha f(J_1) + (1-\alpha) f(J_2)$, for every disjoint $J_1, J_2 \in \sigma(\mathcal{J})$, where $\alpha \in (0,1)$. Then they showed that if f is convex, and the protocol is fair, a consensus is reached. However, as they pointed out, the agreed beliefs are not necessarily commonly known.

Weyers (1992) challenged their view, by claiming that, though their conclusion is correct, slightly different assumptions to the ones they impose, are needed. She attempted to complete this result in a way to ensure common knowledge. The main point in her model is that rational individuals refine their whole partition, rather than just their current information set, *ie.* the recipient's partition at $t+1$ is given by $\mathcal{I}_{r_t}^{t+1} = \mathcal{I}_{r_t}^{t} \vee \mathcal{R}_{s_t}^{t}$, where $\mathcal{R}_i^t = \{R_i^t, R_i^{t^c}\}$. She argued that, under such a refining scheme, commonly known consensus will be reached, if the function f is convex, and the protocol is fair. However, as it is illustrated in the following example, this is not always the case.

Example 1. Consider a society with 3 individuals, and the corresponding information partitions as shown in figure 1, and assume that the common prior assigns equal probabilities to every state. Let the communication protocol be such that 1 informs 2, who informs 3, who informs 1, and so on, about their posterior beliefs. Obviously, the signal function is convex since it is the conditional probabilities that are being revealed, and the protocol is fair. Let the actual state be ω_2, and the individuals hold beliefs about the event $E = \{\omega_2, \omega_3, \omega_6\}$. By definition, all posteriors are common knowledge at ω_2 if and only if $M(\omega_2) \subseteq R$, where $R = R_1 \cap R_2 \cap R_3$. Clearly this is not the case here, since $M(\omega_2) = \Omega \nsubseteq \{\omega_1, ..., \omega_4\} = R$. Therefore, the existing protocol does not lead to commonly known posterior probabilities, even though the individuals refine their whole information partitions. $\triangleleft$

One might wonder why we consider common knowledge to be so important, even though consensus has been reached. To see this consider the previous example. In the

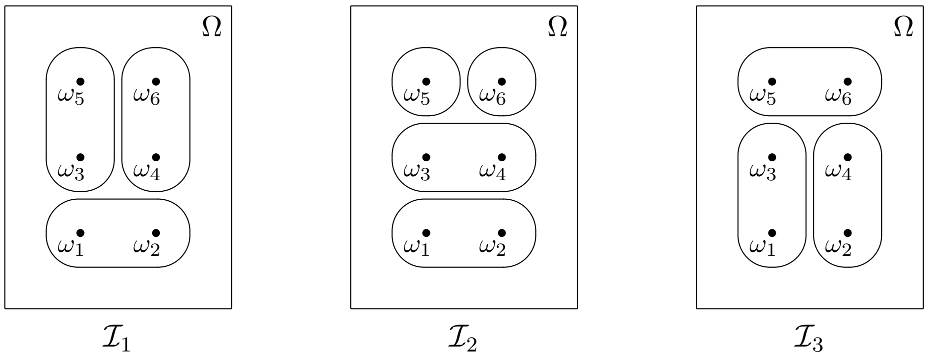

Figure 1: Rational refining does not always lead to common knowledge.

absence of common knowledge, the existing consensus is not robust to protocol perturbations. To see this, suppose that 2 talks to 1 at some point. Then the population's beliefs will eventually converge to 1, which is different from the currently agreed probability. Notice that in the previous example the posterior probabilities are pairwise commonly known, *ie.* $(\mathcal{I}_i \wedge \mathcal{I}_j)(\omega_2) \subseteq R_i \cap R_j$, for every $i, j \in N$. However this does not suffice for common knowledge, and might eventually induce different consensus, as it does indeed in this case.

A second example, which covers both the special cases (the channel, and the star protocol) considered by Krasucki (1996) is presented below. In the channel protocol, 1 talks to 2, who talks to 3, ..., who talks to n, who talks to $n-1$, ..., who talks to 2, who talks to 1, and same process repeats itself *ad infinitum.* In the star protocol, there is a main individual 1 talks to 2, who talks back to 1, who talks to 3, who talks back to 1, ..., who talks to n, who talks back to 1, and so on.

Example 2. Consider a society with 3 individuals, and the corresponding information partitions as shown in figure 2, and assume that the common prior assigns equal probabilities to every state. Let the actual state be ω_1, and the individuals transmit posterior probabilities about $E = \{\omega_1, \omega_5, \omega_6, \omega_7, \omega_9, \omega_{11}\}$, according to the star protocol, with 2 being the distinguished central individual. Notice that since 2 informs 1, who informs 2, who informs 3, who informs 1 about the conditional probabilities, this specific protocol also satisfies the structural property of the channel protocol too. Clearly it does not induce any refinement in the information partitions of any of the three individuals. This

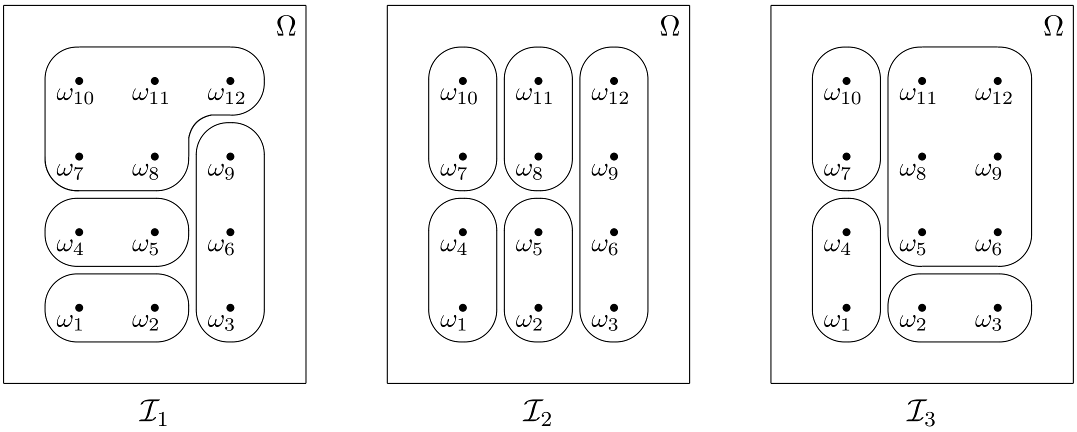

Figure 2: Neither the channel, nor the star protocol enforce common knowledge.

follows from the fact that $R_1 = \{\omega_1, \omega_2, \omega_4, \omega_5\} \in \sigma(\mathcal{I}_2)$, and $R_2 = \Omega \in \sigma(\mathcal{I}_1) \cap \sigma(\mathcal{I}_3)$, and $R_3 = \{\omega_1, \omega_4, \omega_7, \omega_{10}\} \in \sigma(\mathcal{I}_2)$. However, similarly to example 1, the posterior probabilities are not commonly known, since $M(\omega_1) = \Omega \nsubseteq R = \{\omega_1, \omega_4\}$. ◁

Krasucki (1996) showed that in fair protocols with information exchange, consensus on the value of a union-consistent function will be reached. Of course, since the proof is on the same line with Cave (1983), and Bacharach (1985), it suffers from the same flaw that Moses and Nachum (1990) pointed out. Though the proof of this result is incomplete, due to the Moses-Nachum reasoning, a partial result can be established.

Definition 2. A protocol satisfies information exchange whenever for every directed edge from i to j, there is another directed edge from j to i.

Notice that information exchange does not require i to talk to j, and then j talk back to i instantaneously. This could happen later on, after j having received information from someone else. Both the channel, and the star protocol satisfy information exchange. In addition, it is straightforward that every connected protocol with information exchange is fair, since there is a path of directed edges that returns to its origin by going backwards. Though common knowledge is not enforced by information exchange (see example 2), pairwise common knowledge between the connected individuals can be easily shown.

Proposition 1. *Consider a protocol with information exchange. Then the signals of any pair of connected individuals eventually become commonly known between them.*

Notice that the previous result does not impose any requirement on the signal function, such as the sure thing principle, or convexity. It is rather straightforward then to show that if the signals are probabilities, they will eventually become equal.

Corollary 1. *Consider a connected protocol with information exchange, and let the individuals transmit posterior probabilities about an event E. Then the population eventually reaches a consensus.*

As we have already discussed this consensus may not be robust to protocol perturbations, since the posterior probabilities are not commonly known. Consider for instance example 2. If 3 talks to 1 at some point, 1 will refine her partition in such a way that $I_1(\omega_1) = \{\omega_1\}$, and the posterior probability will become equal to 1, which is different than 1/2, which is agreed after the communication imposed by the protocol has taken place. In the next section we discuss the conditions that enforce common knowledge for every fair protocol.

3. Efficient communication, and common knowledge

One might naively suspect that the reason why both Parikh and Krasucki (1990), and Weyers' (1992) model fail to converge to common knowledge, could possibly be explained by the flaws that arise when arbitrary signal functions are used (Moses and Nachum, 1990). However, both the previous examples use probabilities as signals, and therefore such a reasoning can be ruled out. Indeed, the underlying reason for the absence of a commonly known consensus is different.

The main result in Weyers (1992) is based on the following argument. If there is $T > 0$ such that $R^t_{s_t} \in \sigma(\mathcal{I}^t_{r_t})$ for every $t > T$, then the signals are commonly known. In this case $\mathcal{I}^t_{r_t}$ will not be refined, since $\mathcal{I}^{t+1}_{r_t} = \mathcal{I}^t_{r_t} \vee R^t_{s_t} = \mathcal{I}^t_{r_t}$. However as the previous example illustrates such a situation can occur even without common knowledge. It would be interesting thus to explore the underlying reasons for such limiting behavior.

Though the information partitions are refined rationally, individuals do not transmit the whole information they possess. They just reveal the value that the function f assigns to their current information set. Thus, it is implicitly assumed that the whole information is embodied in the single signal which is transmitted. However, as it is shown in the previous example, this is not always true. When 1 hears that 3 assigns probability 1/2 to E, she does not learn anything about 2, since she does not even know that 2 and 3 have talked. Not being able to embody this information in the transmitted signal, accounts for the failure to reach common knowledge.

Consider some arbitrary $t > 0$, and let $T_i(t) = \{t' \leq t : r_{t'} = i\}$ be the periods when i has been assigned to receive information according to the protocol. If $T_i(t) = \emptyset$, then i has not been a recipient until t. The individual who informed i at some $t' \leq t$ had already been assigned to be a recipient at some periods prior to t', which are denoted by $T_{s_{t'}}(t') = \{t'' \leq t' : r_{t''} = s_{t'}\}$. Then we define the set of periods when i's informer was informed by someone else by $T_i^2(t) = \cup_{t' \in T_i(t)} T_{s_{t'}}(t')$. By using this iterative process, we determine when information is indirectly transmitted to i at time t, *ie.*

$$T_i^{\infty}(t) = \bigcup_{k=1}^{\infty} T_i^k(t). \tag{2}$$

Assume that the sender at time t informs r_t, not only about the own signal, but also about *every individual's latest signal she is aware of.* Thus, if we consider j's latest signal that i could be aware of, this has occurred at $\tau_i^j(t) = \max_{t' \in T_i^{\infty}(t)} \{t' \leq t : s_{t'} = j\}$.

Definition 3. We say that the individuals communicate efficiently if s_t reports $f_{s_{t'}}^{t'}(\omega)$, for every $t' \in \tau_{s_t}(t) = \{\tau_{s_t}^j(t),\ j \in N\}$, *ie.* if s_t reports every individual's latest signal she is aware of.

When r_t hears the sender's most recent information about everybody, she updates her own set $\tau_{r_t}(t)$. Notice that $\tau_{r_t}(t) \neq \emptyset$, since at least $\tau_{r_t}^{s_t}(t) = t$ belongs to it. In other words, the recipient at time t, is aware of at least the sender's signal at t. Notice also that $T_{r_t}^{\infty}(t)$ does not contain all $t' < t$. That is, r_t may not hear j's last (and therefore, current) signal from s_t, implying that r_t may be misinformed about $f_j^t(\omega)$. To see this, consider the fair protocol depicted in figure 3, where $k \geq 0$. When 3 speaks (at $t = 4$), she transmits a wrong signal about 2 (which she received at $t = 1$). Given that in the meantime (at $t = 3$) 2 has revealed her current signal, 1 neglects the information she receives from 3 about 2 at $t = 4$, and the wrong piece of information is actually not taken into account. On the other hand, when 2 speaks to 3 at time $t = 6$, she reveals a wrong

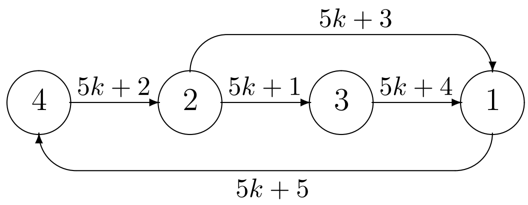

Figure 3: Efficient communication does not rule out misinformation.

signal about 4, *ie.* the one she received at $t = 2$. The difference is that in this case 3 has not received more recent information about 4, and therefore takes into consideration the wrong signal, being thus misinformed about 4's signal. In general, when s_t talks to r_t, the later will take into account what she hears only if she does not already possess more recent information.

Given efficient communication, the recipient r_t infers that the set of possible states at time t is

$$\hat{R}^t_{r_t} = \bigcap_{t' \in \tau_{r_t}(t)} R^{t'}_{s_{t'}}. \tag{3}$$

In other words, $\hat{R}^t_{r_t}$ contains the states that can entail the signals that, according to the most recent information which is available to her, are transmitted by the individuals in the population. Then, she refines her information partition according to the rule

$$\mathcal{I}^{t+1}_{r_t} = \mathcal{I}^t_{r_t} \vee \hat{\mathcal{R}}^t_{r_t}, \tag{4}$$

with $\hat{\mathcal{R}}^t_i = \{\hat{R}^t_i, \hat{R}^{t^c}_i\}$. Clearly, $\hat{R}^{t+1}_j = \hat{R}^t_j$ whenever $j \neq r_t$, and therefore $\mathcal{I}^{t+1}_j = \mathcal{I}^t_j$, implying that the only individual who refines her partition is the recipient at time t.

Theorem 1. *Consider a fair protocol with efficient communication. Then the signals eventually become commonly known in the population.*

As we have already discussed above, efficient communication does not preclude misinformation. However, it is straightforward from the previous result, that wrong information will eventually stop being transmitted after some $T > 0$.

Notice that the previous theorem does not require convex signal functions. This is quite important since, reaching common knowledge, and achieving consensus are actually different problems. In other words, different communication schemes affect the level of knowledge, while the nature of the signals determine whether the commonly known actions ensure consensus or not.

Corollary 2. *Consider a fair protocol, and let the individuals efficiently transmit posterior probabilities about an event E. Then the population eventually reaches a commonly known consensus.*

In order to generalize the previous result to arbitrary signal functions we need to introduce additional requirements (Moses and Nachum (1990), Aumann and Hart (2006), Samet (2006)). It follows from the previous discussion that common knowledge and consensus have to be analyzed separately. Common knowledge is achieved on the basis of how much information is transmitted, whilst consensus is the result of the type of information that is being communicated. Notice that both characteristics of the state of knowledge are important in models of communication, since not commonly known consensus may not be robust against perturbations in the protocol of communication (see examples 1 and 2).

4. Concluding discussion

We have shown that in a model of pairwise communication, the signals eventually become commonly known if the individuals transmit, not only their private signal, but also the information they have conveyed from their past informers. In other words, for common knowledge to be achieved, the individuals cannot presume that their recipients are able to infer what they themselves have heard in the past. We show that common knowledge is a quite important component of consensus, since otherwise agreements might not be robust against protocol perturbations. That is, a slight change in the communication pattern might lead to a totally different consensus.

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Appendix

Proof of Proposition 1. Since $\sigma(\mathcal{J})$ is finite, and $\mathcal{I}_i^{t+1}$ is finer than $\mathcal{I}_i^t$ for every $i \in N$ and every $t > 0$, there is $T > 0$ such that $\mathcal{I}_i^t = \mathcal{I}_i^*$ for every $t > T$, and every $i \in N$. Then it follows that no information refinement occurs after T, and therefore $R_i^t = R_i^*$, for every $t > T$, and every $i \in N$. Consider two arbitrary connected individuals $i, j \in N$. Then there are $t, t' > 0$ such that $s_t = r_{t'} = i$, and $s_{t'} = r_t = j$. Since no refinement occurs after T, it follows that $R_i^* \in \sigma(\mathcal{I}_j^*)$. At the same time we know that $R_i^* \in \sigma(\mathcal{I}_i^*)$, for every $i \in N$. Hence, $R_i^* \in \sigma(\mathcal{I}_i^*) \cap \sigma(\mathcal{I}_j^*) = \sigma(\mathcal{I}_i^* \wedge \mathcal{I}_j^*)$. From non-delusion it follows that $(\mathcal{I}_i^* \wedge \mathcal{I}_j^*)(\omega) \subseteq R_i^*$. The same thing can be proven for R_j^*, which completes the proof. □

Proof of Corollary 1. Since the protocol is connected, and satisfies information exchange, if we sequentially apply proposition 1, it follows from Aumann (1976) that consensus is eventually achieved. □

Proof of Theorem 1. Since $\sigma(\mathcal{J})$ is finite, and $\mathcal{I}_i^{t+1}$ is finer than $\mathcal{I}_i^t$ for every $i \in N$ and every $t > 0$, there is $T > 0$ such that $\mathcal{I}_i^t = \mathcal{I}_i^*$ for every $t > T$, and every $i \in N$. Then it follows that no information refinement occurs after T, and therefore $R_i^t = R_i^*$, for every $t > T$, and every $i \in N$. By definition in a fair protocol there is a path of directed edges that passes from every vertex, and ends up at the origin. Since a directed edge from i to j implies that i talks to j infinitely often, it will be the case that the protocol $(\{\tilde{r}_t\}_{t=1}^{\infty}, \{\tilde{s}_t\}_{t=1}^{\infty})$, defined as $\tilde{r}_t = r_{T+t}$, and $\tilde{s}_t = s_{T+t}$ is also fair. Then, there is some $T_i \geq T$, such that $t \geq T$ for every $t \in \tau_i(T_i)$, and the cardinality of $\tau_i(T)$ is equal to n, implying that i knows everybody's correct signal. Hence,

$\hat{R}_i^t = \cap_{j\in N} R_j^*$, for every $t > T^i$, and every $i \in N$. Therefore, there is $T^* > T$ such that $\hat{R}_i^t = R^*$, for every $i \in N$. Since no refinement occurs it follows that $R^* \in \sigma(\mathcal{I}_i^*)$, for every $i \in N$. Hence, $R^* \in \cap_{i\in N}\sigma(\mathcal{I}_i^*) = \sigma(\wedge_{i\in N}\mathcal{I}_i^*) = \sigma(\mathcal{M}^*)$. From non-delusion it follows that $M^*(\omega) \subseteq R^*$, which proves the theorem. □

Proof of Corollary 2. It follows directly from theorem 1 and Aumann (1976). □