PRICING PORTFOLIO CREDIT DERIVATIVES

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Errata to the thesis "Pricing Portfolio Credit Derivatives"

- Introduction, p. 5, l. -5: The expression "that $\boldsymbol{B}$ suffers" should be "that $\boldsymbol{A}$ suffers".

- Introduction, p. 10, l. 7: The expression "strictly increasing process" should be "increasing process".

- Introduction, p. 11, l. 4: The expression $L_t^{(a,b)}$ should be $L_T^{(a,b)}$.

- Introduction, p. 11, Figure 2.7: The expression "$L_t^{(a,b)}$ i.e. all credit losses in $[a,b]$ up to T" should be "$L_T^{(a,b)}$, i.e. all credit losses in $[a,b]$ up to T".

- Paper 3, p. 5. In Equation (3.1.5), $\boldsymbol{E}_{i_k}$ should everywhere be replaced by $\Delta_{i_k}^C$.

- Paper 3, p. 5. In Equation (3.1.6), $\boldsymbol{E}_{i_m}$ should be $\Delta_{i_m}^C$.

- Paper 3, p. 11, l. -11. The expression "$\mathbb{P}\left[T_k > t\right] = \tilde{\boldsymbol{\alpha}} e^{\boldsymbol{Q} t} \tilde{\boldsymbol{m}}^{(k)}$" should be "$\mathbb{P}\left[T_k > t\right] = \tilde{\boldsymbol{\alpha}} e^{\boldsymbol{T} t} \tilde{\boldsymbol{m}}^{(k)}$".

- Paper 3, p. 15, l. -10. The expression "that each $d_p \in \boldsymbol{d}_-$ only appears once in the matrix $\{D_{i,j}\}$" should be "that each p, such that $d_p \in \boldsymbol{d}_-$, only appears once in the matrix $\{D_{i,j}\}$".

- Paper 3, p. 34.
  In Table 13, the row "$a_1 \quad a_2 \quad \ldots \quad a_{10}$" should be "$a_{\text{VOLV}} \quad a_{\text{BMW}} \quad \ldots \quad a_{\text{VW}}$"

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Abstract

This thesis consist of four papers on dynamic dependence modelling in portfolio credit risk. The emphasis is on valuation of portfolio credit derivatives. The underlying model in all papers is the same, but is split in two different submodels, one for inhomogeneous portfolios, and one for homogeneous ones. The latter framework allows us to work with much bigger portfolios than the former. In both models the default dependence is introduced by letting individual default intensities jump when other defaults occur, but be constant between defaults. The models are translated into Markov jump processes which represents the default status in the credit portfolio. This makes it possible to use matrix-analytic methods to find convenient closed-form expressions for many quantities needed in dynamic credit portfolio management and valuation of portfolio credit derivatives.

Paper one presents formulas for single-name credit default swap spreads and k^{th}-to-default swap spreads in an inhomogeneous model. In a numerical study based on a synthetic portfolio of 15 telecom bonds we study, e.g., how k^{th}-to-default swap spreads depend on the amount of default interaction and on other factors.

Paper two derives computational tractable formulas for synthetic CDO tranche spreads and index CDS spreads. Special attention is given to homogenous portfolios. Such portfolios are calibrated against market spreads for CDO tranches , index CDS-s, the average CDS and FtD baskets, all taken from the iTraxx Europe series. After the calibration, which leads to perfect fits, we compute spreads for tranchelets and k^{th}-to-default swap spreads for different subportfolios of the main portfolio. We also investigate implied tranche-losses and the implied loss distribution in the calibrated portfolios.

Paper three is devoted to derive and study, in an inhomogeneous model, convenient formulas for multivariate default and survival distributions, conditional multivariate distributions, marginal default distributions, multivariate default densities, default correlations, and expected default times. We calibrate the model for two different portfolios (with 10 obligors), one in the European auto sector, the other in the European financial sector, against their market CDS spreads and the corresponding CDS-correlations.

In paper four we perform the same type of studies as in Paper 3, but for a large homogenous portfolio. We use the same market data as in Paper 2. Many of the results differ substantially from the corresponding ones in the inhomogeneous portfolio in Paper 3. Furthermore, these numerical studies indicates that the market CDO tranche spreads implies extreme default clustering in upper tranches.

Preface

About this thesis

This thesis consists of a brief introduction and four appended papers. It is a continuation of my Licentiate of Engineering Degree in Industrial Mathematics from Chalmers University of Technology and is submitted as a partial fulfilment for the degree of Doctor of Philosophy (PhD) in Economics. It corresponds to two years of full time research work and in addition, two years of PhD-level courses are required.

Acknowledgements

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Contents

This thesis consists of a brief introduction to portfolio credit derivative valuation and the following four appended papers:

Paper 1: A. Herbertsson and H. Rootzén. Pricing k^{th}-to-default swaps under Default Contagion: the matrix-analytic approach. *Submitted.*

Paper 2: A. Herbertsson. Pricing synthetic CDO tranches in a model with Default Contagion using the matrix-analytic approach. *Submitted.*

Paper 3: A. Herbertsson. Modelling default contagion using Multivariate Phase-Type distributions. *Submitted.*

Paper 4: A. Herbertsson. Default contagion in large homogeneous portfolios. *Submitted.*

Paper 1 and Paper 3 contain some revised material taken from my Licentiate of Engineering thesis.

1. Introduction

"Modelling dependence between default events and between credit quality changes is, in practice, one of the biggest challenges of credit risk models".

David Lando, [23], p. 213.

"Default correlation and default dependency modelling is probably the most interesting and also the most demanding open problem in the pricing of credit derivatives. While many single-name credit derivatives are very similar to other non-credit related derivatives in the default-free world (e.g. interest-rate swaps, options), basket and portfolio credit derivative have entirely new risks and features."

Philipp Schönbucher, [27], p. 288.

"Empirically reasonable models for correlated defaults are central to the credit risk-management and pricing systems of major financial institutions."

Darrell Duffie and Kenneth Singleton [12], p. 229.

In recent years, understanding and modelling default dependency has attracted much interest. A main reason for this is the growing financial market of products whose payoffs are contingent on the default behavior of a whole credit portfolio consisting of, for example, mortgage loans, corporate bonds or single-name credit default swaps (CDS-s). Another reason is the incentive to optimize regulatory capital in credit portfolios given by regulatory rules such as Basel II.

This thesis consist of four papers treating dynamic dependence modelling in portfolio credit risk. The underlying model in all papers are the same, but is split into two different submodels, one for inhomogeneous portfolios, and one for homogeneous ones. The latter framework allows us to work which much bigger portfolios than the former.

In this introductory part of the thesis we briefly present the underlying concepts and give a short introduction to the models treated in the articles. Subsection 1.1 discuss the market of credit derivatives and some general aspect on credit risk. Chapter 2 gives a introduction to the credit derivatives that are the main object of study in the two first papers, as well as being important calibration instruments in the final two papers. The presentation of these instrument are independent of the underlying model for the default times and introduces notation needed in the rest of the chapters.

Chapter 3 is devoted to describe the model used in all papers and give a brief overview of the main results in the four papers that constitute this thesis.

1.1   The credit derivatives market

Credit risk is the risk that an obligor does not honor his payments. In this thesis a typical example of an obligor is a company that has issued bonds. We say that the company defaults, if, for example:

- The company goes bankrupt.

- The company fails to pay a coupon on time, for some of its issued bonds.

There are standardized and more exact definitions of a credit event, see for example Moodys definition of a credit event.

A credit derivative is a financial instrument that allows banks, insurance companies and other market participants to isolate, manage and trade their credit-sensitive investments. Roughly speaking, credit derivatives are tools that partially or completely remove credit risks. They constitute a very broad class of derivatives and it is hard to give a an exact mathematical definition that covers all the different versions. This in contrast to the case of equity and interest rate derivatives where a precise and short mathematical definition can cover most of these contingent claims, see for example [6].

Sometimes credit derivatives are classified into two different categories (see e.g. in [7] and [4]). The first category is so called *default products*. These are credit derivatives that are intimately connected to one or several specified default events. A default event can for example be the default of one specific obligor or the third default in a portfolio of, say, 10 obligors. In this thesis, we will only study credit derivatives of default product type, and at the writing moment they are the far most dominating type of credit derivative. Examples of such derivatives are single-name credit default swaps (CDS) which roughly speaking is an insurance against

credit losses on one obligor. Credit default swaps are today the most traded credit derivatives and constitute over 65 % of the market. They are used as building blocks for synthetic CDO's and basket default swaps, such as k^{th}-to-default swaps. A kth-to-default swaps is a generalization of the CDS, to a portfolio of several obligors. It offers protection against credit losses on default number k in the portfolio. The most common type of kth-to-default swaps are first-to-default swaps (FtD), i.e $k = 1$, which pay protection on the first default in the portfolio. A collateralized debt obligation (CDO) is a financial instrument with a more complicated protection structure for a big credit portfolio. Today, the most common type of such instruments are synthetic CDO's which are defined on large portfolios of CDS-s. An index CDS is a special version of a synthetic CDO.
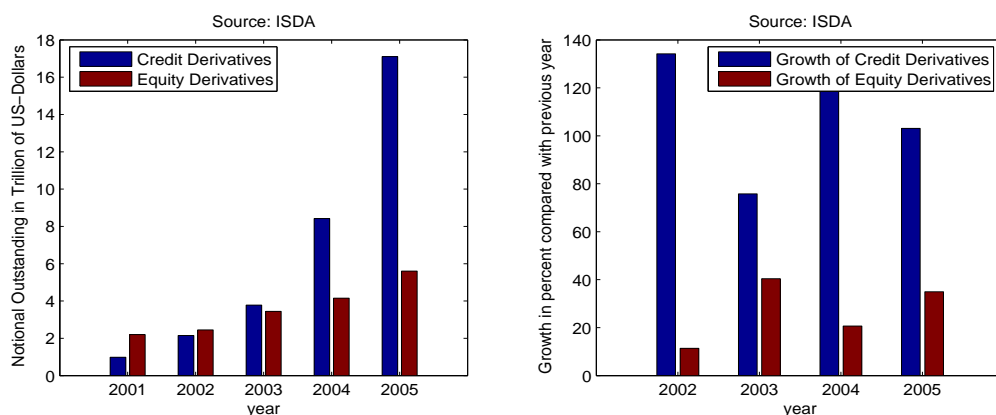

Figure 1.1: The estimated credit and equity derivatives markets in trillions of US-dollars (left) and there annual growth (right). Source: ISDA

The second, and much smaller class of credit derivatives are so called *spread products* which roughly speaking are instruments whose payoff is determined by changes in the credit quality of an asset. Typical examples are default spread options which are standard European put and call options where the underlying asset is the so called credit spread between two bonds. The credit spread is often defined as the yield difference between a sovereign bond, (or a specified interest rate) and a bond issued by a corporate. Today spread products constitute only a small fraction of the credit derivatives market compared with default products.

Credit derivatives are at the writing moment, with few exceptions, not traded on an exchange but are private contracts negotiated between two counterparties, that is, they are so called OTC (over-the-counter) derivatives. An exception are credit futures, which was launched on the European market in the end of March 2007 and are currently traded on an exchange. Despite the fact that most credit derivatives are of OTC-type there exists very liquidly quoted "prices" on CDS-s, standardized synthetic CDO-s, index CDS-s and FtD-s credit derivatives, see e.g. Reuters, Bloomberg or GFI. Further, due to their OTC nature, it is difficult to

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give an accurate estimation of the credit derivatives market size. However, several estimates indicate that the market for credit derivatives has grown explosively during the last 5-6 years, see Figure 1.1.

2. Credit Derivatives and CDO-s

In this chapter we give short descriptions of CDS-s (Section 2.1), k^{th}-to-default swaps (Section 2.2), synthetic CDO tranches (Section 2.3) and index CDS-s (Section 2.4). The presentation is independent of the underlying model for the default times and introduces notation needed later on. In the sequel all computations are assumed to be made under a risk-neutral martingale measure $\mathbb{P}$. Typically such a $\mathbb{P}$ exists if we rule out arbitrage opportunities.

2.1 Credit default swaps

A single-name credit default swap (CDS) with maturity T and where the reference entity is a bond issued by a obligor, is a bilateral contract between two counterparties, $\boldsymbol{A}$ and $\boldsymbol{B}$, where $\boldsymbol{B}$ promises $\boldsymbol{A}$ to pay the credit losses that $\boldsymbol{B}$ suffers if the obligor defaults before time T. As compensation for this $\boldsymbol{A}$ pays be $\boldsymbol{B}$ a fee up to the default time τ or until T, whichever comes first, see Figure 2.1 and Figure 2.2. The fee is determined so that expected discounted cashflows between $\boldsymbol{A}$
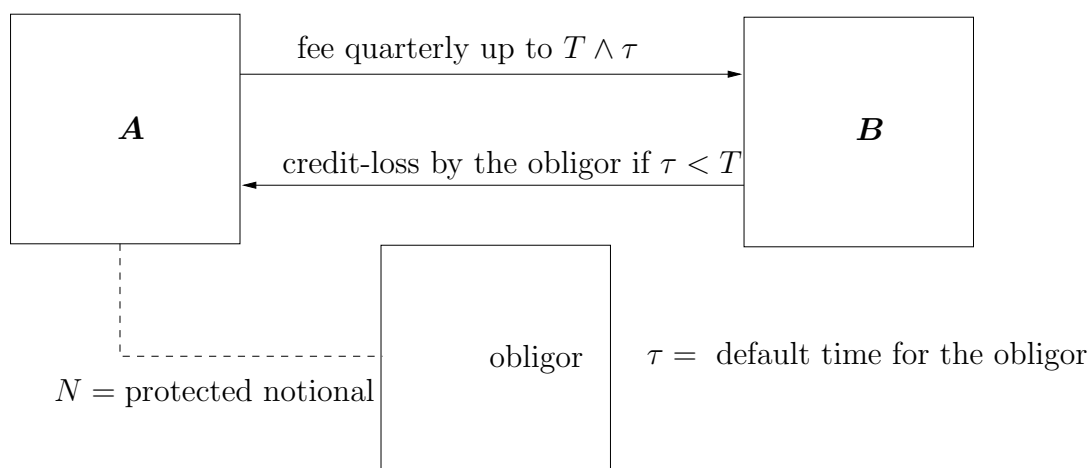

Figure 2.1: The structure of a single-name CDS.

and $\boldsymbol{B}$ are equal when the CDS contract is started. In order to mathematically

express this we need a more detailed description. Let the notional amount on the bond be N. The protection buyer $\boldsymbol{A}$ pays $R(T)N\Delta_n$ to the protection seller $\boldsymbol{B}$, at $0 < t_1 < t_2 < \ldots < t_{n_T} = T$ or until $\tau < T$, where $\Delta_n = t_n - t_{n-1}$. If default happens for some $\tau \in [t_n, t_{n+1}]$, $\boldsymbol{A}$ will also pay $\boldsymbol{B}$ the accrued default premium up to τ. On the other hand, if $\tau < T$, $\boldsymbol{B}$ pays $\boldsymbol{A}$ the amount $N(1 - \phi)$ at τ where ϕ denotes the recovery rate for the obligor in % of the notional bond value. Since $R(T)$ is determined so that expected discounted cashflows between $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal when the CDS contract is settled, we get that

$$R(T) = \frac{\mathbb{E}\left[1_{\{\tau \leq T\}}D(\tau)(1 - \phi)\right]}{\sum_{n=1}^{n_T} \mathbb{E}\left[D(t_n)\Delta_n 1_{\{\tau > t_n\}} + D(0, \tau)(\tau - t_{n-1})1_{\{t_{n-1} < \tau_i \leq t_n\}}\right]}$$

where $D(t) = \exp\left(-\int_0^t r_s ds\right)$ and r_t is the so called short term risk-free interest rate at time t. Note the expression for $R(T)$ it is independent of the amount N that is protected. Assuming that the default time τ and the risk-free interest rate are mutually independent, and that the recovery rate is deterministic, then reduces the above expression to

$$R(T) = \frac{(1 - \phi)\int_0^T B_s dF(s)}{\sum_{n=1}^{n_T}\left(B_{t_n}\Delta_n(1 - F(t_n)) + \int_{t_{n-1}}^{t_n} B_s(s - t_{n-1})dF(s)\right)}$$

where $B_t = \mathbb{E}[D(t)]$ and $F(t) = \mathbb{P}[\tau \leq t]$ is the distribution functions of the default time for obligor. The quantity $R(T)$ is called the T-year CDS spread for the obligor. In order to find $R(T)$, we need a probabilistic model for the default time τ. However,
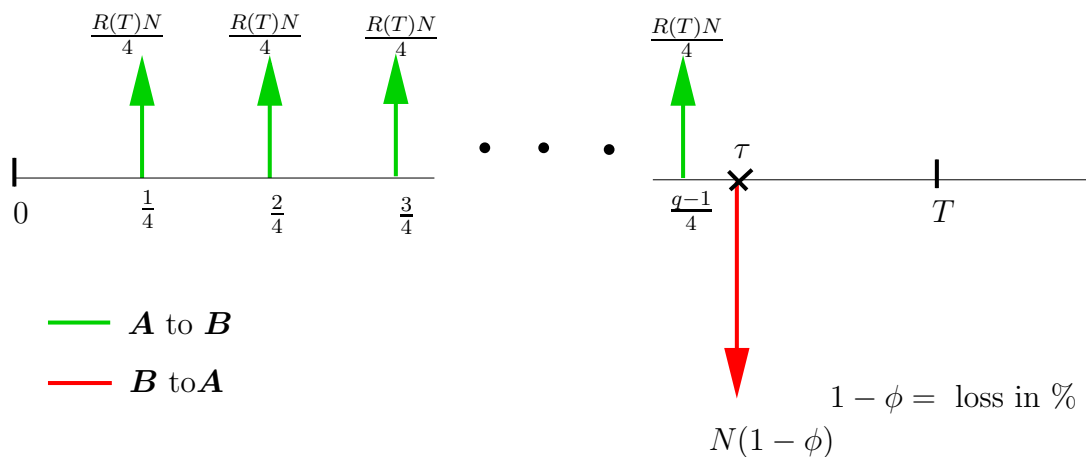

Figure 2.2: The undiscounted cash-flows in a CDS-contract for a scenario where the obligor defaults in the q-th quarter counting from $t = 0$ where $\frac{q}{4} < T$. The fees are quarterly and the accrued premium is ignored.

there exists liquidly quoted CDS spreads on most big companies, and standard

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maturities are $T = 3, 5, 7, 10$. Major corporates have even finer term structures where $T = 1, 2, 3, 5, 7, 9, 10$. Hence, by using these market spreads we can "back out" the implied default distribution $F(t) = \mathbb{P}[\tau \leq t]$ for the obligor. In other words, market CDS-spreads can be used as calibration instruments. Note that these probabilities are measured under the risk-neutral measure and should not be confused with real default probabilities. The latter quantity is difficult to extract on a daily basis and has historically been substantially smaller than implied default probabilities. The market CDS spreads increases as T increases, since the probability of default of a obligor increases with time T, as seen in Table 2.1.

Table 2.1: The bid/ask CDS-spreads for some major Swedish and European companies where $T = 3, 5, 7, 10$ (Reuters 2007-02-15)

	$T = 3$	$T = 5$	$T = 7$	$T = 10$
Volvo	17/21	25/27	34/38	45/49
TeliaSonera	21/24	35/37	55/59	74/78
Ericsson	22/22	26/29		55/ -
StoraEnso	21/25	35/39	52/57	67/72
Vattenfall	6.5/9.5	12/15	15/20	21.5/24.5
Fortum	5/10	10.5/13.5	15.5/20.5	23.5/28.5
Akzo Nobel	14/19	25.5/28	34/39	44.5/49.5
BMW AG	4/8	9/10	12.5/16.5	17/21
Deutsche Telekom	17.5/20.5	31/32.5	43/46	59/62
ABN AMRO	2/3	5/7	7/9	10/13

2.2 k^{th}-to-default swaps

A kth-to default swap is a generalization of a the single-name credit default swap, to a portfolio of m obligors, and pays protection at the k-th default in the portfolio. To be more specific, consider a basket of m bonds each with notional N, issued by m obligors with default times $\tau_1, \tau_2, \ldots, \tau_m$ and recovery rates $\phi_1, \phi_2, \ldots, \phi_m$. Further, let $T_1 < \ldots < T_k$ be the ordering of $\tau_1, \tau_2, \ldots, \tau_m$. A kth-to-default swap with maturity T on this basket is a bilateral contract between two counterparties, **A** and **B**, where **B** promises **A** to pay the credit losses that **B** suffers at T_k if $T_k < T$. Just as in the CDS, **A** pays be **B** a fee up to the default time T_k or until T, whichever comes first, see Figure 2.3. The payments dates and the accrued premium are identical to those in the CDS case and the fee is $R_k(T)N\Delta_n$ where Δ_n is defined as in the CDS contract. The main difference lies in the default payment at T_k. If $T_k < T$, **B** pays **A** $N(1 - \phi_i)$ if it was obligor i which defaulted at time T_k.
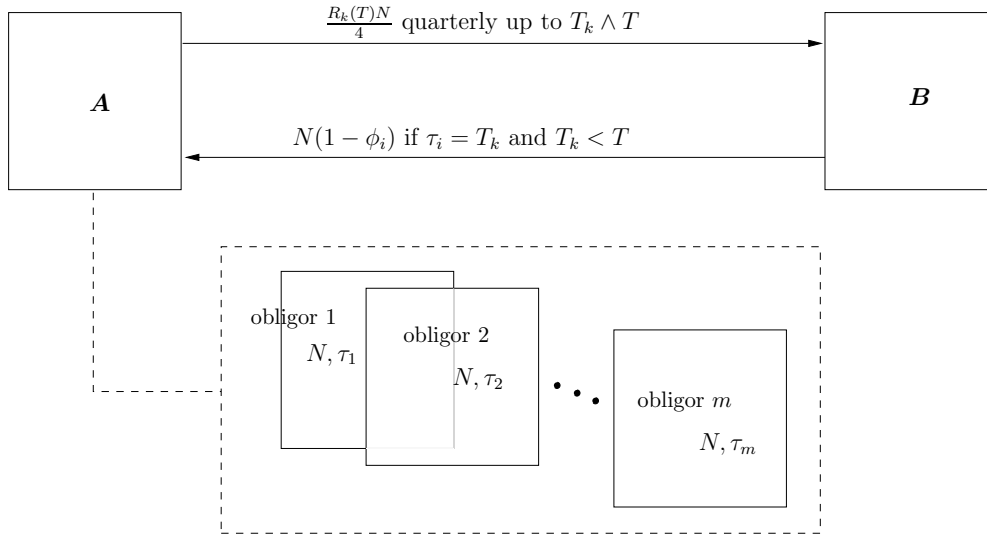
Figure 2.3: The structure of a kth-to-default swap.

A to B

B to A

$\phi_2 = $ recovery rate for obligor 2
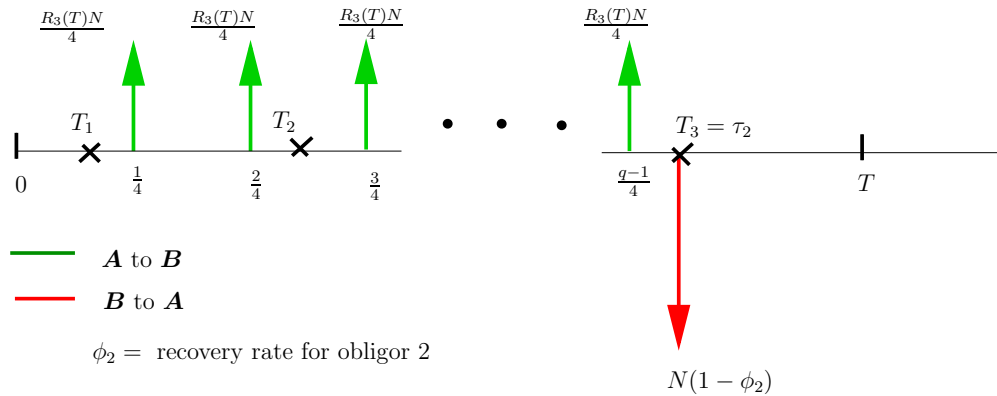
Figure 2.4: The undiscounted cash-flows in a third-to-default swap for a scenario where obligor 2 is the third default in the portfolio which occurs in the q-th quarter counting from $t = 0$ where $\frac{q}{4} < T$. The fees are quarterly and the accrued premium is ignored.

The constant $R_k(T)$, often called k^{th}-to-default spread, is expressed in bp per annum and determined so that the expected discounted cash-flows between A and B coincide at $t = 0$. For an example, see Figure 2.4. Assuming that all the default times and the short time riskfree interest rate are mutually independent, that the recovery rates are deterministic, and following the same arguments as in the CDS-

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case gives that

$$R_k(T) = \frac{\sum_{i=1}^{m} (1 - \phi_i) \int_0^T B_s dF_{k,i}(s)}{\sum_{n=1}^{n_T} \left(B_{t_n} \Delta_n (1 - F_k(t_n)) + \int_{t_{n-1}}^{t_n} B_s (s - t_{n-1}) dF_k(s) \right)}. \qquad (2.2.1)$$

Here are $F_k(t) = \mathbb{P}[T_k \leq t]$ and $F_{k,i}(t) = \mathbb{P}[T_k \leq t, T_k = \tau_i]$ the distribution functions of the ordered default times, and the probability that the k-th default is by obligor i and that it occurs before t, respectively. The rest of the notation are the same as in the CDS contract. Note that again N does not enter into the expression for $R_k(T)$. Furthermore, in the special case when all recovery rates are the same, say $\phi_i = \phi$ the denominator in (2.2.1) can be simplified to $(1 - \phi) \int_0^T B_s F_k(s)$, and hence the $F_{k,i}$ are not needed in this case.

At the writing moment, so called FtD-swaps, (First-to-Default, i.e. $k = 1$) are liquidly traded for standardized portfolios where $m = 5$, see Table 2.2. However, for $k > 1$ and for nonstandardized portfolios we need a model to determine $\mathbb{P}[T_k \leq t]$ and $\mathbb{P}[T_k \leq t, T_k = \tau_i]$ in order to compute $R_k(T)$. This in turn often requires explicit expressions for the joint distribution of $\tau_1, \tau_2, \ldots, \tau_m$.

Table 2.2: The market bid, ask and mid spreads for different FtD spreads on subsectors of iTraxx Europe (Series 6), November 28th, 2006. Each subportfolio have five obligors. We also display the sum of CDS-spreads (SoS) in each basket, as well as the mid FtD spreads in % of SoS.

Sector	bid	ask	mid	SoS	mid/SoS %
Autos	154	166	160	202	79.21 %
Energy	65	71	68	86	79.07 %
Industrial	114	123	118.5	141	84.04 %
TMT	167	188	177.5	217	81.8 %
Consumers	113	122	117.5	140	83.93 %
Financial Sen	30	34	32	43	74.42 %

2.3   Synthetic CDO tranches

A collateralized debt obligation (CDO) is a financial instrument with a protection structure for a big credit portfolio. Depending on the type of credit instrument in the portfolio, CDO-s are sometimes called CLO (L as in loan) if the portfolio consist of loans, CBO-s (B as in bond), or cash-CDO if there are bonds underlying. Today, the far most common CDO-type are so called synthetic CDO-s, which are defined on large portfolios of CDS-s.

The main idea in all kind of CDO-s are roughly the same, which is to offer credit protection on a certain part of the total credit loss in the portfolio. In order to make

this concrete, we will from now on focus on a synthetic CDO. Consider a portfolio consisting of m single-name CDS-s on obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$ and recovery rates $\phi_1, \phi_2, \ldots, \phi_m$. It is standard to assume that the nominal values are the same for all obligors. It is denoted by N. The accumulated credit loss L_t at time t for this portfolio is

$$L_t = \sum_{i=1}^{m} \ell_i 1_{\{\tau_i \leq t\}} \quad \text{where } \ell_i = N(1 - \phi_i). \tag{2.3.1}$$

The loss process L_t is a strictly increasing process, see Figure 2.5.
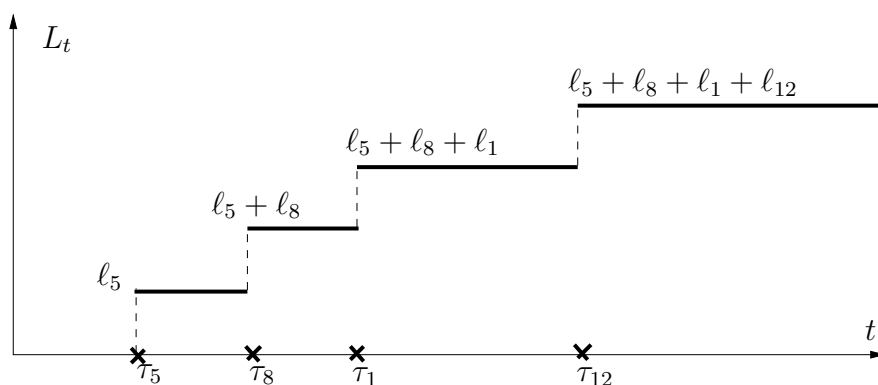
Figure 2.5: A loss scenario where $T_1 = \tau_5$, $T_2 = \tau_8$, $T_3 = \tau_1$ and $T_4 = \tau_{12}$

From now on, we will without loss of generality express the loss L_t in percent of the nominal portfolio value at $t = 0$. Now consider a tranche $[a, b]$ of the loss where $0 \leq a < b \leq 1$, which is a "slice" of the total accumulated loss. The accumulated loss $L_t^{(a,b)}$ of tranche $[a, b]$ at time t is $L_t^{(a,b)} = (L_t - a) \, 1_{\{L_t \in [a,b]\}} + (b - a)) \, 1_{\{L_t > b\}}$, see Figure 2.6.
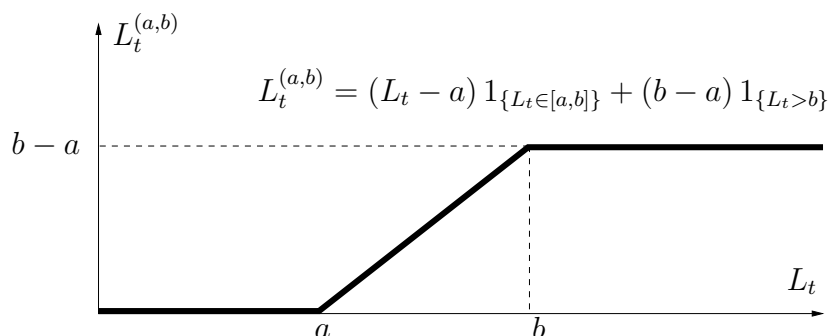
Figure 2.6: The tranche loss for $[a, b]$ as function of the total loss L_t.

The financial instrument that constitutes a synthetic CDO tranche $[a, b]$ with maturity T is a bilateral contract where the protection seller $\mathbf{B}$ agrees to pay the protection buyer $\mathbf{A}$, all losses that occurs in the interval $[a, b]$ derived from L_t up to time T, that is $L_t^{(a,b)}$. The payments are made at the corresponding default times, if they arrive before T, and at T the contract ends. As compensation for this, $\mathbf{A}$ pays $\mathbf{B}$ a periodic fee proportional to the current outstanding (possible reduced due to losses) value on tranche $[a, b]$ up to time T. Thus, if the payments are quarterly, $\mathbf{A}$ pays $\mathbf{B}$

$$\frac{S_{(a,b)}(T)\left((b - a) - L_t^{(a,b)}\right)}{4} \quad \text{for} \quad t = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots, T$$

where we assume that no accrued premiums are paid at the defaults. Note that the contract does not terminate after default time $\tau_i < T$, unless $L_{\tau_i} \geq b$ and $L_{\tau_i-} < b$, since then $(b - a) - L_{\tau_i}^{(a,b)} = 0$ and there is nothing left of the tranche, see Figure 2.7 and Figure 2.8.
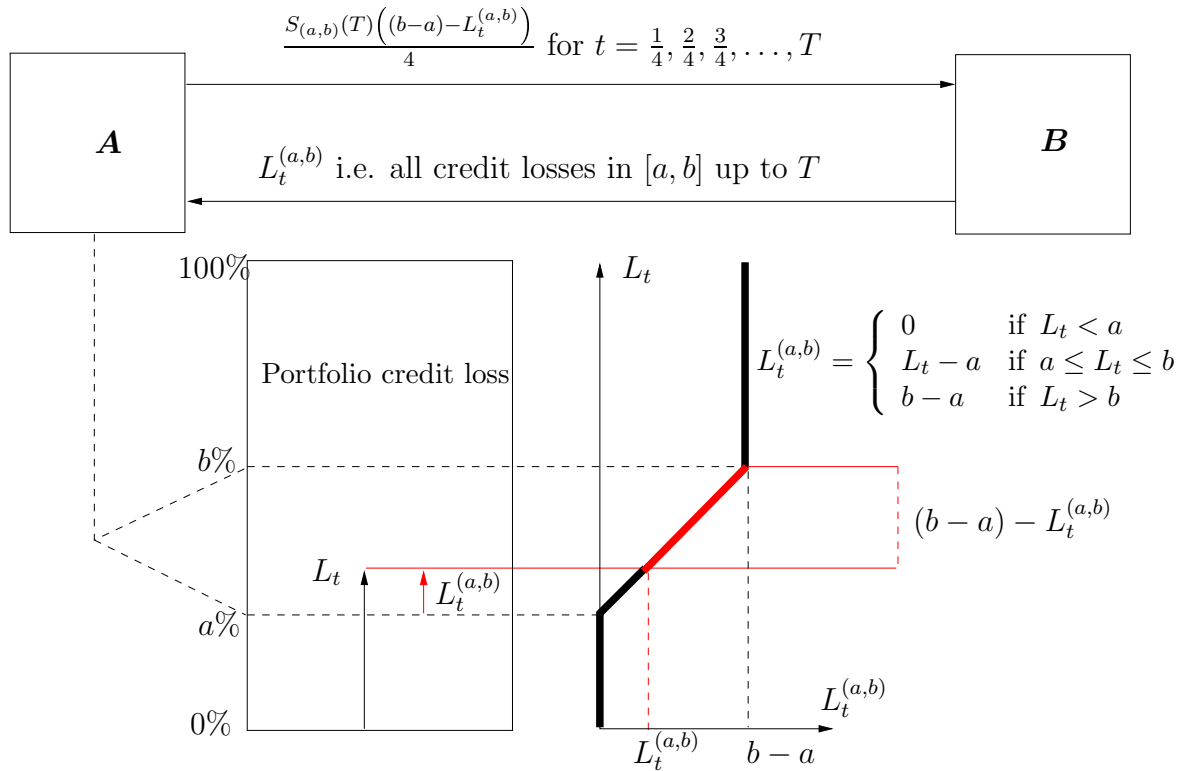
Figure 2.7: The structure of a CDO tranche $[a, b]$.

The expected value of the payment done by $\mathbf{B}$ is sometimes called the *protection leg*, denoted by $V_{(a,b)}(T)$. Further, the expected value of the payment scheme from $\mathbf{A}$ is often refereed to as *premium leg* which we here denote $W_{(a,b)}(T)$. If we assume

$\ell_i =$ loss in % due to obligor i $\qquad \ell_j =$ loss in % due to obligor j
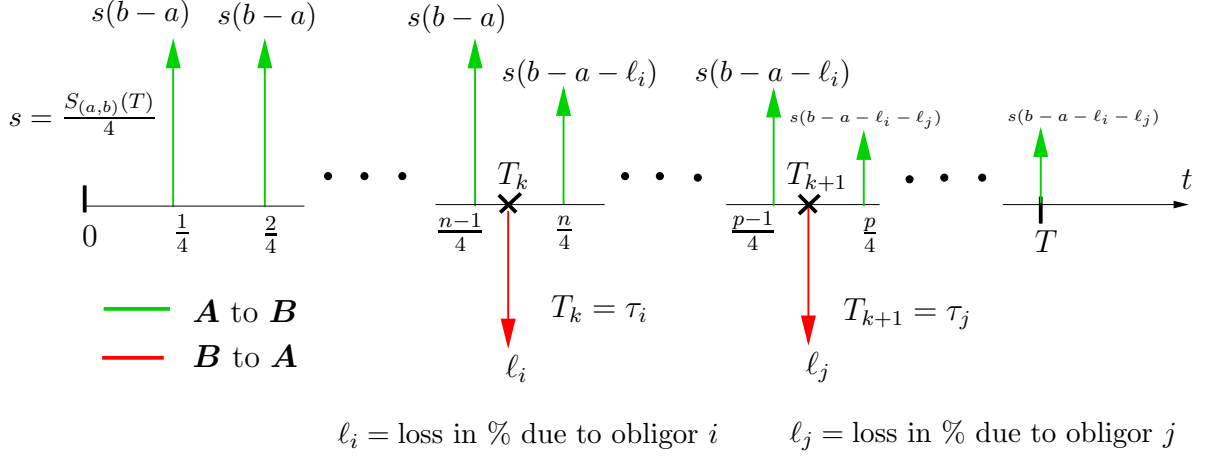
Figure 2.8: The undiscounted cash-flows for a CDO tranche $[a, b]$ where $L_{T_{k-1}} < a < L_{T_k} < L_{T_{k+1}} < b$ and $T_k = \tau_i$, $T_{k+1} = \tau_j$. Furthermore, the ordered default times T_k and $T_{k+1} < T$ arrive in quarter n and p, counting from $t = 0$ where $T_{k+2} > T$. The fees are quarterly and the accrued premium is ignored.

that the interest rate r_t is deterministic, then it is easy to see that

$$V_{(a,b)}(T) = \mathbb{E}\left[\int_0^T B_t dL_t^{(a,b)}\right] = B_T \mathbb{E}\left[L_T^{(a,b)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(a,b)}\right] dt,$$

and

$$W_{(a,b)}(T) = S_{(a,b)}(T) \sum_{n=1}^{n_T} B_{t_n} \left(b - a - \mathbb{E}\left[L_{t_n}^{(a,b)}\right]\right) \Delta_n$$

where $\Delta_n = t_n - t_{n-1}$ denote the times between payments (measured in fractions of a year). The rest of the notation is the same as for the CDS and k^{th}-to-default swap. The constant $S_{(a,b)}(T)$ is called the spread of tranche $[a, b]$ and if $a > 0$ it is determined so that the value of the premium leg equals the value of the corresponding protection leg, that is $V_{(a,b)}(T) = W_{(a,b)}(T)$. For a tranche where $a = 0$, i.e. $[0, b]$, sometimes called a equity tranche, $S_{(0,b)}(T)$ is set to a fixed constant, often 500 bp and a *up-front* fee $S_b^{(u)}(T)$ is added to the premium leg so that $V_{(0,b)}(T) = S_b^{(u)}(T)b + W_{(0,b)}(T)$. Hence, we get that

$$S_{(a,b)}(T) = \frac{B_T \mathbb{E}\left[L_T^{(a,b)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(a,b)}\right] dt}{\sum_{n=1}^{n_T} B_{t_n} \left(b - a - \mathbb{E}\left[L_{t_n}^{(a,b)}\right]\right) \Delta_n} \qquad \text{if } a > 0$$

and for $[a, b] = [0, b]$,

$$S_b^{(u)}(T) = \frac{1}{b}\left[B_T \mathbb{E}\left[L_T^{(1)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(1)}\right] dt - 0.05 \sum_{n=1}^{n_T} B_{t_n} \left(b - \mathbb{E}\left[L_{t_n}^{(0,b)}\right]\right) \Delta_n\right].$$

The spread $S_{(a,b)}(T)$ is quoted in bp per annum while $S_b^{(u)}(T)$ is quoted in percent per annum and they are independent of the nominal size of the portfolio. Today there exists standardized synthetic CDO portfolios, for example, the iTraxx Europe series, which consist of the 125 most liquid traded European CDS-s, equally weighted. On this series, tranche spreads are liquidly traded for $[a,b] = [0,3], [3,6], [6,9], [9,12]$ and $[12,22]$ with $T = 3, 5, 7, 10$, see Table 2.3. Furthermore, a new index is rolled every 6th month. Up to June 2007, 7 series have been rolled since 2004. The index decomposition and default events are currently determined by 38 market makers.

Table 2.3: The bid/ask tranche-spreads for iTraxx Europe Series 6, (Reuters, 15 February 2007) where $T = 3, 5, 7, 10$. The $[0,3]$ spread is quoted in % while the rest of the tranches are quoted in bp.

Tranche	$T = 3$	$T = 5$	$T = 7$	$T = 10$
$[0,3]$	-/1.75	9.75/10.5	23.25/24	38.75/39.5
$[3,6]$	2/-	42/44	106/-	310/313
$[6,9]$	2/6	11.5/13.5	31/35	83/84
$[9,12]$	0.5/3	5/6.5	-/17	37/38
$[12,22]$	-/4	1.5/2.5	4.25/6	12/14

Assume that we want to compute nonstandard tranches in, for example the iTraxx portfolio, where $[a,b] = [0,1], [1,2], \ldots, [11,12]$ and T is arbitrary. In order to price such tranches consistently with the standard tranches, or for risk-management of the CDO portfolio, or computing sensitive test, hedge ratios etc. we need a model. It is crucial that such model can produce model spreads that are consistent with the corresponding market spreads, that is, it should be flexible enough so that it can be calibrated against market spreads.

From the above expressions we see that in order to compute tranche spreads we have to compute $\mathbb{E}\left[L_t^{(a,b)}\right]$, that is, the expected loss of the tranche $[a,b]$ at time t. If we let $F_{L_t}(x) = \mathbb{P}\left[L_t \leq x\right]$, then the definition of the tranche loss implies that

$$\mathbb{E}\left[L_t^{(a,b)}\right] = (b-a)\mathbb{P}\left[L_t > a\right] + \int_a^b (x-a)\, dF_{L_t}(x).$$

Hence, to compute $\mathbb{E}\left[L_t^{(a,b)}\right]$ we must know the loss distribution $F_{L_t}(x)$ at time t. Recall that $L_t = \sum_{i=1}^m \ell_i 1_{\{\tau_i \leq t\}}$, so to find $F_{L_t}(x)$ in our model, we need the joint distribution of $\tau_1, \tau_2, \ldots, \tau_m$. To illustrate this we have in Figure 2.9, displayed two loss scenarios, one with "weak" default dependency, the other with a strong default dependency. In the latter case, the ordered defaults tend to cluster and we see that the loss reaches upper tranches faster, and thus makes them more "riskier" than in the weaker dependency case. Consequently, it we do not take this into account in our model, we will likely have more troubles in the risk-management of

the underlying credit portfolio, and it may be difficult to even calibrate the model against the market spreads.
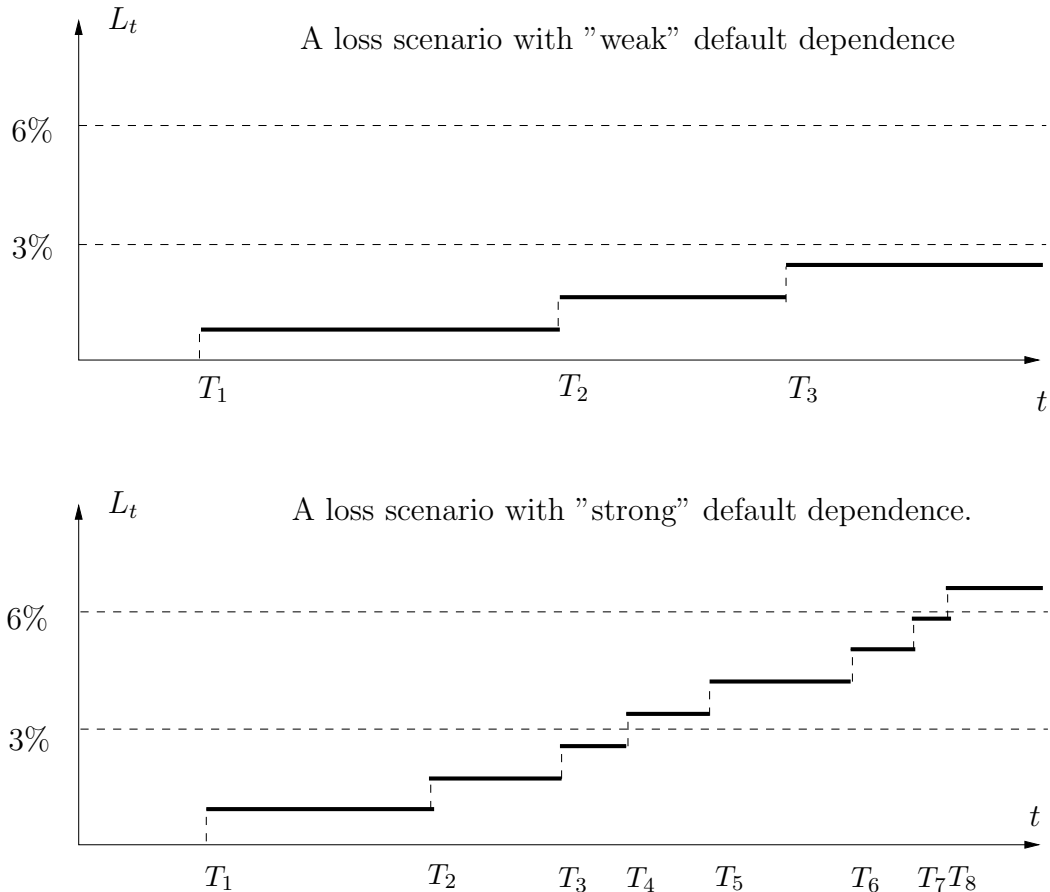
Figure 2.9: Two loss scenarios. Note that upper tranches are reached faster when there is "stronger" default dependency.

2.4   Index CDS-s

Consider the same synthetic CDO as above. An index CDS with maturity T, has almost the same structure as a corresponding CDO tranche, but with two main differences. First, the protection is on *all* credit losses that occurs in the CDO portfolio up to time T, so in the protection leg, the tranche loss $L_t^{(a,b)}$ is replaced by the total loss L_t. Secondly, in the premium leg, the spread is paid on a notional proportional to the number of obligors left in the portfolio at each payment date. Thus, if N_t denotes the number of obligors that have defaulted up to time t, i.e $N_t = \sum_{i=1}^{m} 1_{\{\tau_i \leq t\}}$, then the index CDS spread $S(T)$ is paid on the notional $(1 - \frac{N_t}{m})$.

Since the rest of the contract has the same structure as a CDO tranche, the value of the premium leg $W(T)$ is

$$W(T) = S(T) \sum_{n=1}^{n_T} B_{t_n} \left(1 - \frac{1}{m} \mathbb{E}\left[N_{t_n}\right] \right) \Delta_n$$

and the value of the protection leg, $V(T)$, is given by $V(T) = B_T \mathbb{E}\left[L_T\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t\right] dt$. The index CDS spread $S(T)$ is determined so that $V(T) = W(T)$ which implies

$$S(T) = \frac{B_T \mathbb{E}\left[L_T\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t\right] dt}{\sum_{n=1}^{n_T} B_{t_n} \left(1 - \frac{1}{m} \mathbb{E}\left[N_{t_n}\right] \right) \Delta_n}$$

where $\frac{1}{m} \mathbb{E}\left[N_t\right] = \frac{1}{1-\phi} \mathbb{E}\left[L_t\right]$ if $\phi_1 = \phi_2 = \ldots = \phi_m = \phi$. Here, the rest of the notation is the same as in the CDO-tranche. The spread $S(T)$ is quoted in bp per annum and is independent of the nominal size of the portfolio.

Table 2.4: The bid/ask tranche-spreads for the index on iTraxx Europe Series 6, (Reuters, 15 februari 2007) where $T = 3, 5, 7, 10$. The spreads are quoted in bp.

	$T = 3$	$T = 5$	$T = 7$	$T = 10$
index	11.25/11.75	22.5/22.75	31/31.75	41.5/42.25

3. Modelling dynamic default dependence using matrix-analytic methods

In the previous sections we treated k^{th}-to-default swaps, CDO-s and index CDS-s. To find expressions for the spreads on these instruments, we often need a model for the joint default distribution $\tau_1, \tau_2, \ldots, \tau_m$. The number of articles on dynamic models for portfolio credit risk has grown exponentially during the last years. It is outside the scope of this introduction (and thesis) to treat even some of them.

This chapter gives a description of the intensity based model used in all papers in the thesis. The model is specified by letting the individual default intensities be constant, except at the times when other defaults occur: then the default intensity for each obligor jumps by an amount representing the influence of the defaulted entity on that obligor. If the jump is positive, the likelihood of a default for the obligor increase. This phenomena is often called default contagion since it describes how defaults can "propagate" like a disease in a financial market (see e.g. [10]). Default contagion in an intensity based setting have previously also been studied in for example [2], [3], [4], [5], [8], [9], [11], [10], [13], [14], [15], [16], [21], [22],[23], [24], [25], [26] and [28]. The material in all these papers and books are related to the results discussed in this thesis.

We consider two versions of our model, one for inhomogeneous portfolios, and one for homogeneous ones. Section 3.1 describes the inhomogeneous version, treated in Paper 1 and Paper 3 for small CDS portfolios. This model is difficult to use for larger CDS portfolios, such as synthetic CDO-s. In Section 3.2 we therefore consider a simplification of the framework in Section 3.1, to a homogeneous portfolio where all obligors are exchangeable. Such symmetric models are studied in Paper 2 and Paper 4. Finally, Section 3.3 gives a more detailed description of the four papers that constitute this thesis.

3.1 The inhomogeneous portfolio

For the default times $\tau_1, \tau_2 \ldots, \tau_m$, define the point process $N_{t,i} = 1_{\{\tau_i \leq t\}}$ and introduce the filtrations

$$\mathcal{F}_{t,i} = \sigma\left(N_{s,i}; s \leq t\right), \quad \mathcal{F}_t = \bigvee_{i=1}^{m} \mathcal{F}_{t,i}.$$

Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$. The model studied in Paper 1, 2, and 3 is specified by requiring that the default intensities have the form,

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \qquad \tau_i \geq t, \qquad (3.1.1)$$

and $\lambda_{t,i} = 0$ for $t > \tau_i$. Further, $a_i \geq 0$ and $b_{i,j}$ are constants such that $\lambda_{t,i}$ is non-negative. In this model the default intensity for obligor i jumps by an amount $b_{i,j}$ if it is obligor j which has defaulted. Thus a positive $b_{i,j}$ means that obligor i is put at higher risk by the default of obligor j, while a negative $b_{i,j}$ means that obligor i in fact benefits from the default of j, and finally $b_{i,j} = 0$ if obligor i is unaffected by the default of j, see Figure 3.10.
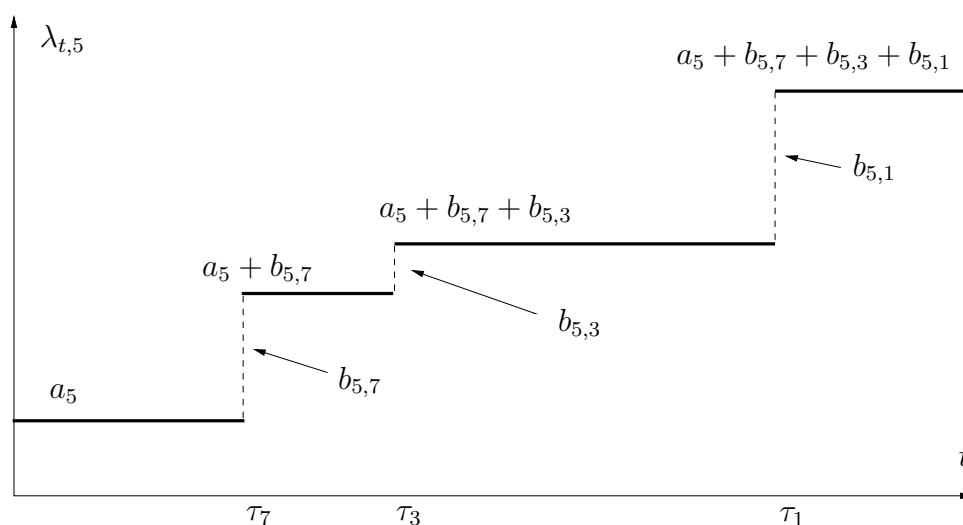
Figure 3.10: The default intensity for obligor 5 when $T_1 = \tau_7$, $T_2 = \tau_3$ and $T_3 = \tau_1$. The defaults put obligor 5 at higher risk.

It is well known from point-process theory that the intensities uniquely determine all distributions for a point-process. Hence, Equation (3.1.1) determines the default times through their intensities. However, as discussed in Chapter 2, the expressions for e.g. k^{th}-to-default swap spreads and CDO-tranche spreads are in terms of their joint distributions of the default times. The joint distribution is also needed in credit portfolio management. It is by no means obvious how to find these from (3.1.1). The following result is proved in Paper 1, ([20]).

Proposition 3.1.1. *There exists a Markov jump process $(Y_t)_{t \geq 0}$ on a finite state space $\boldsymbol{E}$ and a family of sets $\{\Delta_i\}_{i=1}^m$ such that the stopping times*

$$\tau_i = \inf \{t > 0 : Y_t \in \Delta_i\}, \qquad i = 1, 2, \ldots, m,$$

have intensities (3.1.1). Hence, any distribution derived from the multivariate stochastic vector $(\tau_1, \tau_2 \ldots, \tau_m)$ can be obtained from $\{Y_t\}_{t \geq 0}$.

In Paper 1 ([20]), Paper 2 ([19]) and Paper 3 ([18]) we us Equation (3.1.1) as the intuitive way of describing the dependencies in a credit portfolio. However, Proposition 3.1.1 is used for computing credit derivatives spreads, expected credit losses, and other related quantities.

The number of states in the Markov jump processes for a nonhomogeneous portfolio is $|\boldsymbol{E}| = 2^m$. In practice, this forces us to work with portfolios of size, say, 25 or smaller. Standard synthetic CDO-s typically contains 125 obligors. We therefore also (in Section 3.2 below) consider a special case of (3.1.1) which leads to a symmetric portfolio where the state space $\boldsymbol{E}$ can be simplified to make $|\boldsymbol{E}| = m + 1$. This allows us to work with portfolios where m is 125 or larger. Using homogeneous models when pricing CDO tranches is currently standard in most credit literature.

3.2 The homogeneous portfolio

The homogeneous model is a special case of (3.1.1) where all obligors have the same default intensities $\lambda_{t,i} = \lambda_t$ specified by parameters a and $b_1, \ldots, b_{m-1}$, as

$$\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}} \tag{3.2.1}$$

where $\{T_k\}$ is the ordering of the default times $\{\tau_i\}$. In this model the obligors are exchangeable. The parameter a is the base intensity for each obligor i, and given that $\tau_i > T_k$, then b_k is how much the default intensity for each remaining obligor jump at default number k in the portfolio.

We know that Equation (3.2.1) determines the default times through their intensities as well as their joint distribution. To find these expressions, Paper 2 ([19]) proves the following simpler version of Proposition 3.1.1.

Corollary 3.2.1. *There exists a Markov jump process $(Y_t)_{t \geq 0}$ on a finite state space $\boldsymbol{E} = \{0, 1, 2, \ldots, m\}$, such that the stopping times*

$$T_k = \inf \{t > 0 : Y_t = k\}, \quad k = 1, \ldots, m$$

are the ordering of m exchangeable stopping times $\tau_1, \ldots, \tau_m$ with intensities (3.2.1).

Hence, in the homogeneous model the states in $\boldsymbol{E}$ can be interpreted as the number of defaulted obligors in the portfolio. Therefore there is no need of keeping track of which obligors that have defaulted, as in the inhomogeneous portfolio. The model (3.2.1) is used in Paper 2 and Paper 4.

3.3 Summary of papers

We here give a short summary of the four papers that constitute this thesis.

Paper 1

In this paper ([20]) we find expressions for single-name credit default swap spreads and k^{th}-to-default swap spreads. This is done in the inhomogeneous model (3.1.1), by using Proposition 3.1.1. We reparameterize the basic description (3.1.1) of the default intensities to the form

$$\lambda_{t,i} = a_i \left(1 + c \sum_{j=1, j\neq i}^{m} \theta_{i,j} 1_{\{\tau_j \leq t\}} \right), \tag{3.3.1}$$

where the a_i are the base default intensities, c measures the general "interaction level" and the $\theta_{i,j}$ measure the "relative dependence structure". The last quantity is assumed to be exogenously given, which makes the number of unknown quantities to be $m+1$. We then "semi-calibrate" a portfolio consisting of 15 telecom companies, against their corresponding 5-year market CDS spreads, for different interaction levels c and two different dependence structures. In all calibrations, the CDS fits where perfect. After this we study the influence of portfolio size on k^{th}-to-default spreads, of changing the interaction level, the impact of using inhomogeneous recovery rates, the sensitivity to the underlying CDS spreads, and finally compare the inhomogeneous model with a non-symmetric dependence to a corresponding symmetric model (i.e. (3.2.1)).

Most of the numerical results where qualitatively as expected. However, it would be difficult to guess the sizes of the effects without actually doing the computations.

Paper 2

The paper ([19]) derives formulas for synthetic CDO spreads and index CDS spreads. This is first done in the inhomogeneous model (3.1.1). Then we show that derivation is identical in the homogeneous model (3.2.1). However, from a practical point of view, the formulas simplify considerable in the latter case. Furthermore, in the homogeneous model, we also give expressions for the average CDS spreads and k^{th}-to-default swap spreads on subportfolios in the CDO portfolio. This problem is different from the corresponding one in Paper 1, since the obligors undergo default contagion both from the subportfolio and from obligors outside the subportfolio, in the main portfolio. Because to the reduction of the state space in the homogeneous model, we use it as a basis for our numerical studies. A homogeneous portfolio is calibrated against CDO tranche spreads, index CDS spread and the average CDS and FtD spreads, all taken from the iTraxx series, for a fixed maturity of five years.

The resulting fits where perfect. The parameter space $\{b_k\}$ in (3.2.1) is reduced by using the following parameterization

$$
b_k = \begin{cases}
b^{(1)} & \text{if} \quad 1 \le k < \mu_1 \\
b^{(2)} & \text{if} \quad \mu_1 \le k < \mu_2 \\
\vdots & \\
b^{(c)} & \text{if} \quad \mu_{c-1} \le k < \mu_c = m
\end{cases}
$$

where $1, \mu_1, \mu_2, \ldots, \mu_c$ is an partition of $\{1, 2, \ldots, m\}$. This means that all jumps in the intensity at the defaults $1, 2, \ldots, \mu_1 - 1$ are same and given by $b^{(1)}$, all jumps in the intensity at the defaults $\mu_1, \ldots, \mu_2 - 1$ are same and given by $b^{(2)}$ and so on. We let $c + 1$ be equal to the number of calibration instruments, that is the number of credit derivatives used in the calibration.

After the calibration, we computed spreads for tranchelets, which are CDO tranches with smaller loss-intervals than standardized tranches. We also investigated k^{th}-to-default swap spreads as function of the size of the underlying subportfolio in main calibrated portfolio. The implied expected loss in the portfolio and the implied expected tranche-losses were studied. Finally, we explored the implied loss-distribution as function of time.

Paper 3

Paper 3 ([18]) is devoted to derive and study multivariate default and survival distributions, conditional multivariate distributions, marginal default distributions, multivariate default densities, default correlations, and expected default times. This is done in the inhomogeneous model (3.1.1). Some of the results in this paper were stated in [1], but without proofs.

After the derivations, we introduce two inhomogeneous CDS portfolios, one in the European auto sector, the other in the European financial sector. Both consist of 10 companies. The baskets are calibrated against their market CDS spreads and corresponding CDS correlations. This gives a perfect fit for the banking case and good fit for the auto case. The major difference in this calibration compared to the one in Paper 1, is that we use a CDS-correlation matrix for each portfolio, retrieved from time-series data on the market spreads in the portfolio. While we in paper 1 only used m "observations", our data sets now consist of m market CDS spreads and their $m(m-1)/2$ pairwise CDS correlations, that is $m(m+1)/2$ market observations. This is still only around half as many as the unknown model parameters $\{a_i\}, \{\theta_{i,j}\}$ in the parametrization (3.3.1) used in Paper 1, with $c = 1$. To overcome this problem, we assume that some of the $\theta_{i,j}$-s are equal. Formally, we make a reduction of the dependence structure $\{\theta_{i,j}\}$ as follows,

$$
\lambda_{t,i} = a_i \left(1 + \sum_{j=1, j \ne i}^{m} \varepsilon d_{D_{i,j}} 1_{\{\tau_j \le t\}} \right),
$$

where $\varepsilon = \pm 1$ and $\{D_{i,j}\}$ is a exogenously given matrix $D_{i,j} \in \left\{ 1, 2 \dots \frac{(m-1)m}{2} \right\}$ and $\{d_1, d_2, \dots, d_{(m-1)m/2}\}$ are $(m-1)m/2$ different nonnegative parameters. The d_q-s will be determined in the calibration, together with the nonnegative base default intensities a_i. The sign $\varepsilon = \pm 1$ for $d_{D_{i,j}}$ is set equal to the sign in the CDS correlation matrix for entry (i, j). Although, $\{D_{i,j}\}$, is fictitious, we avoid to use the phrase "semi-calibration" (as in Paper 1), since we here use $(m+1)m/2$ market observations, compared to m in [20].

In the calibrated portfolios, we study the implied joint default and survival distributions and the implied univariate and bivariate conditional survival distributions. Furthermore, the implied default correlations, the implied expected default times and expected ordered defaults times are also investigated.

Paper 4

In [17], we perform the same type of studies as in Paper 3, but for a homogenous model (3.2.1) and thus a much larger portfolio. Using the same numerical data and the same parameterizations of the homogenous model as in Paper 2, we calibrate the model against CDO tranche spreads, index CDS spread and the average CDS, all taken from the iTraxx Europe series, with a fixed maturity of five years. We study the implied expected ordered defaults times, implied default correlations, and implied multivariate default and survival distributions, both for ordered and unordered default times. Many of the results differ substantially from the ones in the inhomogeneous portfolio in Paper 3. Furthermore, the numerical studies indicates that the market spreads produce extreme default clustering in upper tranches, as illustrated in Figure 2.9.

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Paper I

PRICING k^{th}-TO-DEFAULT SWAPS UNDER DEFAULT CONTAGION: THE MATRIX-ANALYTIC APPROACH

ALEXANDER HERBERTSSON AND HOLGER ROOTZÉN

ABSTRACT. We study a model for default contagion in intensity-based credit risk and its consequences for pricing portfolio credit derivatives. The model is specified through default intensities which are assumed to be constant between defaults, but which can jump at the times of defaults. The model is translated into a Markov jump process which represents the default status in the credit portfolio. This makes it possible to use matrix-analytic methods to derive computationally tractable closed-form expressions for single-name credit default swap spreads and k^{th}-to-default swap spreads. We "semi-calibrate" the model for portfolios (of up to 15 obligors) against market CDS spreads and compute the corresponding k^{th}-to-default spreads. In a numerical study based on a synthetic portfolio of 15 telecom bonds we study a number of questions: how spreads depend on the amount of default interaction; how the values of the underlying market CDS-prices used for calibration influence k^{th}-th-to default spreads; how a portfolio with inhomogeneous recovery rates compares with a portfolio which satisfies the standard assumption of identical recovery rates; and, finally, how well k^{th}-th-to default spreads in a nonsymmetric portfolio can be approximated by spreads in a symmetric portfolio.

1. INTRODUCTION

In this paper we study dynamic dependence modelling in intensity-based credit risk. We focus on the concept of default contagion and its consequences for pricing k^{th}-to-default swaps. The paper is an extension of Chapter 6 of the licentiate thesis [28].

Default dependency has attracted much interest during the last few years. A main reason is the growing financial market of products whose payoffs are contingent on the default behavior of a whole credit portfolio consisting of, for example, corporate bonds or single-name credit default swaps (CDS-s). Example of such instruments that have gained popularity are k^{th}-to-default swaps and (synthetic) CDO-s. These products are designed to manage and trade the risk of default dependencies. We refer to [6], [8], [16], [18], [28], [40], [44] or [53] for more detailed descriptions of the instruments. Models which capture

default dependencies in a realistic way is at the core of pricing, hedging and managing such instruments.

As the name suggest, default contagion, treats the phenomenon of how defaults can "propagate" like a disease in a financial market (see e.g. [13]). There may be many reasons for this kind of domino effect. For a very interesting discussion of sources of default contagion, see pp. 1765-1768 in [38].

It is, of course, important for credit portfolio managers to have a quantitative grasp of default contagion. This paper describes a new numerical approach to handle default interactions. The underlying idea is the same as in [5], [7], [19], [21], which is to model default contagion via a Markov jump process that represents the joint default status in the credit portfolio. The main difference is that [19], [21] use time-varying parameters in their practical examples and solve the Chapman-Kolmogorov equation by using numerical methods for ODE-systems. In [7], the authors implement results from [5] by using Monte Carlo simulations to calibrate and price credit derivatives.

In this article, we focus on intensities which are constant between defaults, but which may jump at the default times. This makes it possible to obtain compact and computationally tractable closed-form expressions for many quantities of interest, including k^{th}-to-default spreads. For this we use the so-called matrix-analytic approach, see e.g. [1]. From a portfolio credit risk modeling point of view, it also turns out that this method posses useful intuitive and practical features, both analytical and computationally. We believe that these features in many senses are at least as attractive as the copula approach which is current a standard for practitioners. (For a critical study of the copula approach in financial mathematics, see [45]).

The number of articles on dynamic models for portfolio credit risk has grown exponentially during the last years. The subtopic of default contagion in intensity based models is not an exception and has been studied in for example [3], [6], [9], [10], [12], [14], [16], [24], [25], [31] [33], [38], [39], [40], [44], [50], [51], [52], [56], [58].

The paper [3] considers a chain where states record if obligors have defaulted or not, and implemented this model for a basket of two bonds. The intensities in the model were calibrated to market data using linear regression. In [14] the authors model default contagion in symmetric portfolio by using a piecewise-deterministic Markov process and find the default distribution. The book [40], pp. 126-128, studies a Markov chain model for two firms that undergo default contagion. Further, [58] treats default contagion using the total hazard construction of [49], [54], as first suggested in [15]. This method allows for general time dependent and stochastic intensities and that the intensities are functionals of the default times. The latter seems difficult to handle in a Markov jump process framework. Given the parameters of the model, the total hazard method gives a way to simulate default events. The total hazard construction seems rather complicated to implement even in simple cases such as piece-wise deterministic intensities considered in this paper.

The paper [38] assumes a so called *primary-secondary structure*, were obligors are divided into two groups called *primary obligors* and *secondary obligors*. The idea is that the default-intensities of primary obligors only depend on macroeconomic market variables while the default intensity for secondary obligors can depend on both the macroeconomic variables

and on the default status of the primary firms, but not on the default status of the other secondary firms. Assuming this structure, [38] derives closed formulas for defaultable bonds, default swaps, etc, also for stochastic intensities. In the article [10] the authors propose a method where one can value defaultable claims without having to use the so called "no-jump condition". This technique is then applied to find survival distributions for a portfolio of two obligors that undergo default contagion. In [56] the author studies counterparty risk in CDS valuation by using a four state Markov process that includes contagion effects. [56] considers time dependent intensities and then uses perturbation techniques to approximately solve the Chapman-Kolmogorov equation. The framework in [56] is similar to [12], where the author treats the same problem in a setup where the intensities are constant.

The rest of this paper is organized as follows. In Section 2 we give a short introduction to pricing of credit k^{th}-to-default swaps. Section 3 contains the formal definition of default contagion used in this paper, given in terms of default intensities. It is then used to construct such default times as hitting times of a Markov jump process.

In Section 5 we use the results of Section 4, for numerical investigation of a number of properties of kth-to-default spreads. Specifically, we semi-calibrate portfolios with up to 15 obligors against market CDS spreads and then compute the corresponding k^{th}-to-default spreads. The results are used to illustrate how k^{th}-to-default spreads depend on the strength of default interaction, on the underlying market CDS-prices used for calibration, and on the amount of inhomogeneity in the portfolios.

Section 6 discusses numerical issues and some possible extensions, and the final section, Section 7 summarizes and discusses the results.

2. Pricing k^{th}-to-default swap spreads

In this section and in the sequel all computations are assumed to be made under a risk-neutral martingale measure $\mathbb{P}$. Typically such a $\mathbb{P}$ exists if we rule out arbitrage opportunities.

Consider a kth-to-default swap with maturity T where the reference entity is a basket of m bonds, or obligors, with default times $\tau_1, \tau_2, \ldots, \tau_m$ and recovery rates $\phi_1, \phi_2, \ldots, \phi_m$. Further, let $T_1 < \ldots < T_k$ be the ordering of $\tau_1, \tau_2, \ldots, \tau_m$. For k^{th}-to-default swaps, it is standard to let the notional amount on each bond in the portfolio have the same value, say, N, so we assume this is the case.

The protection buyer $\boldsymbol{A}$ pays a periodic fee $R_k N \Delta_n$ to the protection seller $\boldsymbol{B}$, up to the time of the k-th default T_k, or to the time T, whichever comes first. The payments are made at times $0 < t_1 < t_2 < \ldots < t_n = T$. Further let $\Delta_j = t_j - t_{j-1}$ denote the times between payments (measured in fractions of a year). Furthermore, if default happens for some $T_k \in [t_j, t_{j+1}]$, $\boldsymbol{A}$ will also pay $\boldsymbol{B}$ the accrued default premium up to T_k. On the other hand, if $T_k < T$, $\boldsymbol{B}$ pays $\boldsymbol{A}$ the loss occurred at T_k, that is $N(1 - \phi_i)$ if it was obligor i which defaulted at time T_k.

The constant R_k, often called k-th-to default spread, is expressed in bp per annum and determined so that the expected discounted cash-flows between $\boldsymbol{A}$ and $\boldsymbol{B}$ coincide at $t = 0$.

This implies that R_k is given by

$$R_k = \frac{\sum_{i=1}^m \mathbb{E}\left[1_{\{T_k \leq T\}} D(T_k)(1-\phi_i)1_{\{T_k=\tau_i\}}\right]}{\sum_{j=1}^n \mathbb{E}\left[D(t_j)\Delta_j 1_{\{T_k>t_j\}} + D(T_k)(T_k-t_{j-1})1_{\{t_{j-1}<\tau\leq t_j\}}\right]}, \tag{2.1}$$

where $D(T) = \exp\left(-\int_0^T r_s ds\right)$, and r_t is the so called short term risk-free interest rate at time t. Note that N does not enter into this expression. Thus, we will from now on without loss of generality let $N = 1$ when discussing spreads on credit swaps.

In the credit derivative literature today, it is standard to assume that the default times and the short time riskfree interest rate are mutually independent, and that the recovery rates are deterministic. Under these assumptions Equation (2.1) can be simplified to

$$R_k = \frac{\sum_{i=1}^m (1-\phi_i)\int_0^T B(s)dF_{k,i}(s)}{\sum_{j=1}^n \left(B(t_j)\Delta_j(1-F_k(t_j)) + \int_{t_{j-1}}^{t_j} B(s)(s-t_{j-1})\,dF_k(s)\right)} \tag{2.2}$$

where $B(t) = \mathbb{E}[D(t)]$ is the expected value of the discount factor, and $F_k(t) = \mathbb{P}[T_k \leq t]$ and $F_{k,i}(t) = \mathbb{P}[T_k \leq t, T_k = \tau_i]$ are the distribution functions of the ordered default times, and the probability that the k-th default is by obligor i and that it occurs before t, respectively. It may be noted that in the special case when all recovery rates are the same, say $\phi_i = \phi$ the denominator in (2.2) can be simplified to $(1-\phi)\int_0^T B(s)dF_k(s)$, and hence the $F_{k,i}$ are not needed in this case.

The latter of course in particular holds if there is only one bond (or obligor) so that $m = 1$. This case gives the most liquidly traded instrument, called a single-name Credit Default Swap (CDS), which has special importance in this paper as our main calibration tool.

3. Intensity based models reinterpreted as Markov jump processes

In this section we define the intensity-based model for default contagion which is used throughout the paper. The model is then reinterpreted in terms of a Markov jump process. This interpretation makes it possible to use a matrix-analytic approach to derive computationally tractable closed-form expressions for single-name CDS spreads and k-th-to default spreads. These matrix analytic methods has largely been developed for queueing theory and reliability applications, and in these context are often called phase-type distributions, or multivariate phase-type distributions in the case of several components (see e.g. [2]).

With $\tau_1, \tau_2 \ldots, \tau_m$ default times as above, define the point process $N_{t,i} = 1_{\{\tau_i \leq t\}}$ and introduce the filtrations

$$\mathcal{F}_{t,i} = \sigma\left(N_{s,i}; s \leq t\right), \quad \mathcal{F}_t = \bigvee_{i=1}^m \mathcal{F}_{t,i}.$$

Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$. Below, we will for convenience often omit the filtration and just write intensity or "default intensity". With a further extension of language we will sometimes also write that the default times $\{\tau_i\}$ have intensities $\{\lambda_{t,i}\}$.

The model studied in this paper is specified by requiring that the default intensities have the following form,

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \qquad t \leq \tau_i, \tag{3.1}$$

and $\lambda_{t,i} = 0$ for $t > \tau_i$. Further, $a_i \geq 0$ and $b_{i,j}$ are constants such that $\lambda_{t,i}$ is non-negative.

The financial interpretation of (3.1) is that the default intensities are constant, except at the times when defaults occur: then the default intensity for obligor i jumps by an amount $b_{i,j}$ if it is obligor j which has defaulted. Thus a positive $b_{i,j}$ means that obligor i is put at higher risk by the default of obligor j, while a negative $b_{i,j}$ means that obligor i in fact benefits from the default of j, and finally $b_{i,j} = 0$ if obligor i is unaffected by the default of j.

The intensities in Equation (3.1) only depend on which obligors that have defaulted, and not by the order in which the defaults have occurred. Thus it is a model for *Unordered Default Contagion*. A more general case is when the intensities also are affected by the order in which defaults have happened. The approach outlined below works equally well for such *Ordered Default Contagion*. We make some further comments on this at the end of the present section.

Equation (3.1) determines the default times through their intensities. However, the expressions (2.1) and (2.2) for the kth-to-default spreads are in terms of their joint distributions. It is by no means obvious how to go from one to the other. Here we will use the following observation.

Proposition 3.1. *There exists a Markov jump process $(Y_t)_{t \geq 0}$ on a finite state space $\boldsymbol{E}$ and a family of sets $\{\Delta_i\}_{i=1}^m$ such that the stopping times*

$$\tau_i = \inf\{t > 0 : Y_t \in \Delta_i\}, \qquad i = 1, 2, \ldots, m, \tag{3.2}$$

have intensities (3.1). Hence, any distribution derived from the multivariate stochastic vector $(\tau_1, \tau_2, \ldots, \tau_m)$ can be obtained from $\{Y_t\}_{t \geq 0}$.

In this paper, Proposition 3.1 is throughout used for computing distributions. However, we still use Equation (3.1) to describe the dependencies in a credit portfolio since it is more compact and intuitive. Proposition 3.1 is rather obvious, and perhaps most easily understood by examples, see below. However, we still give one possible formal construction, since it provides notation which anyhow is needed later on.

Proof of Proposition 3.1. We construct the state space as a union,

$$\boldsymbol{E} = \bigcup_{k=0}^m \boldsymbol{E}_k, \tag{3.3}$$

where $\boldsymbol{E}_k$ is set of states consisting of precisely k elements of $\{1, \ldots m\}$,

$$\boldsymbol{E}_k = \{\boldsymbol{j} = \{j_1, \ldots j_k\} : \quad 1 \leq j_i \leq m, \quad i = 1, \ldots k\}. \tag{3.4}$$

for $k = 1, \ldots m$, and where $\boldsymbol{E}_0 = \{0\}$. The interpretation is that on the set $\boldsymbol{E}_0$ no obligors have defaulted, on $\{j_1, \ldots j_k\}$ the obligors in the set have defaulted, and on $\boldsymbol{E}_m$ all obligors have defaulted.

The Markov jump process $(Y_t)_{t \geq 0}$ on $\boldsymbol{E}$ is specified by making $\{1, \ldots m\}$ absorbing and starting Y in $\{0\}$, and by specifying its intensity matrix $\boldsymbol{Q}$. The latter specification is that transitions are only possible from $\boldsymbol{E}_k$ to $\boldsymbol{E}_{k+1}$, and that for a state $\boldsymbol{j} = \{j_1, j_2, \ldots, j_k\} \in \boldsymbol{E}_k$ a transition can only occur to a state $\boldsymbol{j}' = (\boldsymbol{j}, j_{k+1}) \in \boldsymbol{E}_{k+1}$ where $j_{k+1} \neq j_i$ for $i = 1, 2, \ldots, k$. Further, the intensity for transitions from $\boldsymbol{j} = \{j_1, j_2, \ldots, j_k\} \in \boldsymbol{E}_k$ to such a $\boldsymbol{j}'$ is

$$\boldsymbol{Q}_{\boldsymbol{j}, \boldsymbol{j}'} = a_{j_{k+1}} + \sum_{i=1}^{k} b_{j_{k+1}, j_i}. \tag{3.5}$$

The diagonal elements of $\boldsymbol{Q}$ is determined by the requirement that the row sums of an intensity matrix is zero.

Next, set

$$\Delta_i = \{\boldsymbol{j} \in \boldsymbol{E} : j_n = i \text{ for some } j_n \in \boldsymbol{j}\}$$

and define the hitting times $\tau_1, \ldots, \tau_m$ by

$$\tau_i = \inf \{t > 0 : Y_t \in \Delta_i\}. \tag{3.6}$$

This construction is illustrated in Figure 1 for the case $m = 3$. It is clear from the construction that $\tau_1, \ldots, \tau_m$ have the intensities (3.1), see e.g. [34], Chapter 4. As a final
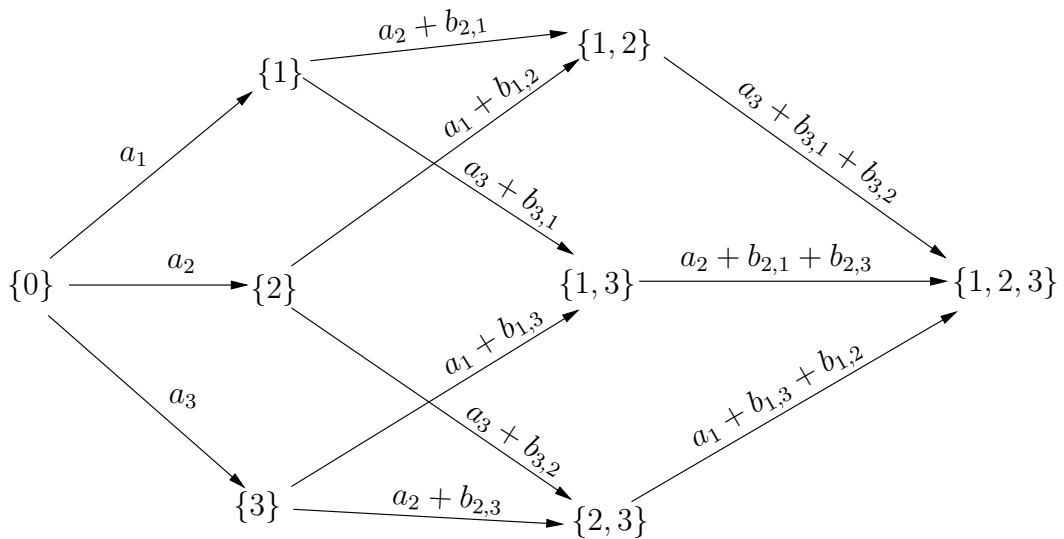
Figure 1: Illustration of the construction for $m = 3$. Arrows indicate possible transitions, and the transition intensities are given on top of the arrows.

aside, when we write down $\boldsymbol{Q}$ as a matrix it is computationally convenient to order the states in $\boldsymbol{E}$ so that $\boldsymbol{Q}$ is upper triangular. This can be done by letting $\{0\}$ be first, then taking the states in $\boldsymbol{E}_1$ in some arbitrary order, followed by the states in $\boldsymbol{E}_2$ in some arbitrary order, and so on. $\qquad \square$

Table 1: The number of states in the unordered and ordered case for different number of obligors, m.

m	unordered	ordered
5	32	326
6	64	1957
7	128	13700
8	256	109601
9	512	986410
10	1024	9864101

So far we have considered Unordered Default Contagion. In Ordered Default Contagion, also the order in which the defaults occur influence default intensities. In our setup, this corresponds to changing the form (3.1) of the intensities to

$$\lambda_{t,i} = a_i + \sum_{\boldsymbol{j} \in \mathcal{P}} b_{i,\boldsymbol{j}} 1_{\left\{ \tau_{j_1} < \ldots < \tau_{j_{|\boldsymbol{j}|}} \leq t \right\}}, \quad \tau_i \geq t, \tag{3.7}$$

and $\lambda_{t,i} = 0$ when $t > \tau_i$. Here $\mathcal{P}$ contains all the ordered subsets $\boldsymbol{j} = \left(j_1, \ldots, j_{|\boldsymbol{j}|} \right)$ of the set $\{1, 2, \ldots, m\}$. Furthermore, a_i and $b_{i,\boldsymbol{j}}$ are constants such that $\lambda_{t,i} \geq 0$.

It is easy to see that the construction for Proposition 3.1 can be extended to the case (3.7). The basic change which has to be made is to change $\boldsymbol{E}_k$ from the set of all subsets of size k of $\{1, \ldots m\}$ to the set of all ordered subsets of size k.

Changing from unordered to ordered default contagion however increases the number of states in $\boldsymbol{E}$ violently. For unordered default contagion

$$|\boldsymbol{E}| = 2^m$$

while for ordered default contagion

$$|\boldsymbol{E}| = \sum_{n=0}^{m} n! \binom{m}{n}.$$

Table 1 shows the number of states in the unordered respectively ordered case for different sizes of the number m of obligors.

It is of course up to the modeler to decide if it is appropriate to use ordered or unordered default contagion. However, from the table we see that in practice it is mainly convenient to work with unordered default contagion. Further, if possible one should for large m try to reduce the number of states in $\boldsymbol{E}$ further, for example by using symmetries.

4. THE MATRIX-ANALYTIC METHOD

We now use the matrix analytic method, see e.g. [1] to find expressions for $F_k(t) = \mathbb{P}[T_k \leq t]$ and $F_{k,i}(t) = \mathbb{P}[T_k \leq t, T_k = \tau_i]$, the distribution functions of the ordered default times, and the probability that the k-th default is by obligor i and that it occurs before t.

The first one is more or less standard, while the second one is less so. These expressions in turn give possibilities to compute the quantities which are at the center of interest in this paper, the kth-to-default spreads. Our development is closely related to so-called multivariate phase type distributions, see e.g. [2].

Define the probability vector $\boldsymbol{p}(t) = (\mathbb{P}[Y_t = \boldsymbol{j}])_{\boldsymbol{j} \in \boldsymbol{E}}$ and let $\boldsymbol{\alpha} = (1, 0, \ldots, 0) \in \mathbb{R}^{|\boldsymbol{E}|}$ be the initial distribution of the Markov jump process and let its generator be $\boldsymbol{Q}$. From Markov theory we know that

$$\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}, \quad \text{and} \quad \mathbb{P}[Y_t = \boldsymbol{j}] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e_j}, \tag{4.1}$$

where $\boldsymbol{e_j} \in \mathbb{R}^{|\boldsymbol{E}|}$ is a column vector where the entry at position $\boldsymbol{j}$ is 1 and the other entries are zero. Furthermore, $e^{\boldsymbol{Q}t}$ is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of $\boldsymbol{Q}$.

Next, define vectors $\boldsymbol{m}^{(k)}$ of length $|\boldsymbol{E}|$ by requiring that

$$\boldsymbol{m}_{\boldsymbol{j}}^{(k)} = \begin{cases} 1 & \text{if} \qquad \boldsymbol{j} \in \cup_{i=0}^{k-1} \boldsymbol{E}_i \\ 0 & \text{otherwise .} \end{cases} \tag{4.2}$$

Then,

$$\mathbb{P}[T_k > t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{m}^{(k)}, \tag{4.3}$$

since $\boldsymbol{m}^{(k)}$ sums the probabilities of states where there has been less than k defaults.

Hence, what is left to compute is $\mathbb{P}[T_k > t, T_k = \tau_i]$. For this we use the imbedded Markov chain $(Y_{T_n})_{n=0}^m$. By definition, the transition probability matrix $\boldsymbol{P}$ for $(Y_{T_n})_{n=0}^m$ is given by

$$\boldsymbol{P}_{\boldsymbol{j},\boldsymbol{j}'} = \mathbb{P}\left[Y_{T_n} = \boldsymbol{j}' \mid Y_{T_{n-1}} = \boldsymbol{j}\right] = \frac{\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}'}}{\sum_{\boldsymbol{k} \neq \boldsymbol{j}} \boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{k}}}, \quad \boldsymbol{j}, \boldsymbol{j}' \in \boldsymbol{E},$$

with the ordering of the states in $\boldsymbol{P}$ the same as for $\boldsymbol{Q}$.

Further, let $\boldsymbol{h}^{i,k}$ be vectors of length $|\boldsymbol{E}|$ and let $\boldsymbol{G}^{i,k}$ be $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrices, defined by

$$\boldsymbol{h}_{\boldsymbol{j}}^{i,k} = \begin{cases} 1 & \text{if} \qquad \boldsymbol{j} \in \Delta_i \cap \boldsymbol{E}_k \\ 0 & \text{otherwise ,} \end{cases}$$

and

$$\boldsymbol{G}_{\boldsymbol{j},\boldsymbol{j}}^{i,k} = \begin{cases} 1 & \text{if} \qquad \boldsymbol{j} \in \Delta_i^C \cap \boldsymbol{E}_k \\ 0 & \text{otherwise .} \end{cases}$$

We now establish the following result.

Proposition 4.1. *With notation as above,*

$$\mathbb{P}[T_k > t, T_k = \tau_i] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \sum_{\ell=0}^{k-1} \left(\prod_{p=\ell}^{k-1} \boldsymbol{G}^{i,p} \boldsymbol{P} \right) \boldsymbol{h}^{i,k}, \tag{4.4}$$

for $k = 1, \ldots m$.

Proof of Proposition 4.1. We will use the following fact, which is straightforward to establish, and standard in Markov chain theory:

If $\{X_n\}$ is a stationary, discrete time, finite state space, Markov chain with initial distribution $\boldsymbol{p}$ and transition matrix $\boldsymbol{P}$, and $\boldsymbol{E}_0, \ldots \boldsymbol{E}_k$ are subsets of the state space, then

$$\mathbb{P}\left[X_0 \in \boldsymbol{E}_0, \ldots X_k \in \boldsymbol{E}_k\right] = \boldsymbol{p}\boldsymbol{G}_0\boldsymbol{P}\boldsymbol{G}_1\boldsymbol{P}\ldots\boldsymbol{G}_{k-1}\boldsymbol{P}\boldsymbol{h}_k,$$

where the $\boldsymbol{G}_\ell$-s are diagonal matrices with diagonal elements equal to one for state $\boldsymbol{j}$ if $\boldsymbol{j} \in \boldsymbol{E}_\ell$ and zero otherwise, for $\ell = 0, \ldots k - 1$, and $\boldsymbol{h}_k$ is a column vector with a 1 in position $\boldsymbol{j}$ if $\boldsymbol{j} \in \boldsymbol{E}_k$.

Let Δ_i^C be the complement of Δ_i in $\boldsymbol{E}$, i.e. the set of all states where i has not defaulted. By an appropriate translation to the situation and notation above, in particular replacing $\boldsymbol{p}$ by $\boldsymbol{\alpha}e^{\boldsymbol{Q}t}$, and if $T_\ell \leq t < T_{\ell+1} < T_k$, we obtain that

$$\mathbb{P}\left[Y_t \in \Delta_i^C \cap \boldsymbol{E}_\ell, Y_{T_{\ell+1}} \in \Delta_i^C \cap \boldsymbol{E}_{\ell+1}, \ldots Y_{T_{k-1}} \in \Delta_i^C \cap \boldsymbol{E}_{k-1}, Y_{T_k} \in \Delta_i \cap E_k\right]$$
$$= \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{G}^{i,\ell}\boldsymbol{P}\boldsymbol{G}^{i,\ell+1}\ldots\boldsymbol{G}^{i,k-1}\boldsymbol{P}\boldsymbol{h}^{i,k}.$$

Since

$$\mathbb{P}\left[T_k > t, T_k = \tau_i\right]$$
$$= \sum_{\ell=0}^{k-1} \quad \mathbb{P}\left[Y_t \in \Delta_i^C \cap \boldsymbol{E}_\ell, Y_{T_{\ell+1}} \in \Delta_i^C \cap \boldsymbol{E}_{\ell+1}, \ldots Y_{T_{k-1}} \in \Delta_i^C \cap \boldsymbol{E}_{k-1}, Y_{T_k} \in \Delta_i \cap E_k\right],$$

this proves (4.4). $\square$

5. Numerical studies

In this section we will use the theory developed in previous sections to study, in a realistic numerical example, how different factors affect the size of k-th to default spreads. For this it is convenient to reparameterize the basic description (3.1) of the default intensities to the form

$$\lambda_{t,i} = a_i\left(1 + c \sum_{j=1, j\neq i}^{m} \theta_{i,j}1_{\{\tau_j \leq t\}}\right), \tag{5.1}$$

which was suggested in [20]. In this parametrization, the a_i are the base default intensities, c measures the general "interaction level" and the $\theta_{i,j}$ measure the "relative dependence structure".

First, in Subsection 5.1 we introduce a portfolio consisting of 15 telecom companies which is used as a basis for the numerical studies. We further "semi-calibrate" our model to this portfolio, using CDS spreads taken from Reuters.

We then study the influence of portfolio size on k^{th}-to-default spreads, (Subsection 5.2), of changing the interaction level (Subsection 5.3), the impact of using inhomogeneous recovery rates (Subsection 5.4), the sensitivity to the underlying CDS spreads (Subsection 5.5), and finally compare a model with non-symmetric dependence to a corresponding symmetric model (Subsection 5.6).

For the rest of this paper we will assume that the $\theta_{i,j}$ are given - hence the term "semi-calibrations", cf. Subsection 5.1. It is a topic for future research to find out how to estimate the $\theta_{i,j}$. For example, using liquid market data on CDO's will give us more information which can be used for some cases. The rapidly increasing market of credit portfolio products may also help. In Section 7 we discuss this topic in more detail.

Numerical studies always carry the risk of programming errors and numerical instability. However, fortunately we have been able to benchmark our numerical methods to an example from [19], pp. 19-20 and [20], which as far as we know, are the only available results on default contagion for nonsymmetric portfolios with more than three bonds.

The paper [19] studies a portfolio with five bonds and time-dependent default intensities, and uses numerical solution of differential equations to compute spreads for a number of cases. Our model for intensities doesn't directly allow for time-dependence, but it was still possible to approximate the portfolio in [19] with our model, calibrate it as discussed in Section 6 below, and compare the spreads thus obtained with those in [20]. The results agreed to at least four significant digits in all cases, which lends some confidence to our numerical implementation.

5.1. **A telecom portfolio.**

Table 2 describes the portfolio which is used in our numerical studies. The data was obtained from `Reuters` at August 23, 2005. We have assumed a fictive recovery rate structure and also a fictive relative dependence structure $\theta_{i,j}$ which is given in Table 8 in Appendix, and we used the interaction level $c = 0.5$. The interest rate was assumed to be constant and set to 3%, and the protection fees were assumed to be paid quarterly. The maturity was 5 years. The a_i-s are obtained by individual calibration to the CDS spreads in Table 2. From Table 8 we see that the intensities can jump up to 284% of their "base values" a_i, when $c = 0.5$. In case both bid and ask prices for the CDS-s were given, we used their average. The calibration is described in more detail in Section 6. We refer to the entire procedure - using the fictive recovery rates, the fictive dependence structure and the calibrated base intensities - as *semi-calibration.*

5.2. **Dependence on portfolio size.**

To study the dependence on portfolio size, we considered 6 different sub-portfolios. The first portfolio consisted of the 10 first bonds from Table 2, the second of the 11 first bonds, and so on, until the last portfolio which contained all the 15 bonds in the table. Each subportfolio with m obligors had a dependence structure given by upper left $m \times m$ submatrix of the matrix given in Table 8. When we calibrated the subportfolios against the market CDS spreads, the corresponding sum of the absolute calibration error never exceeded two hundreds of a bp. For each portfolio the kth-to-default spreads were computed from Equation (2.2). The results are shown in Table 3. The spreads are only shown for $k \leq 5$. The remaining spreads were all less than six hundreds of a basis point.

In the table the spreads increase as the size of the portfolio increases, as they should. Quantitatively, the increase from a portfolio of size 10 to one of size 15 is 47% for a 1st-to-default swap, 92% for a 2nd-to-default swap, 168% for a 3rd-to-default swap, and for a 5th-to-default swap the increase is 700%. Further, for a portfolio of size 10 the price

Table 2: The Telecom companies and their 5 year CDS spreads.

Company	bid	ask	time	recovery %
British Telecom	40	44	23 Aug, 09:33	32%
Deutsche Telecom	34		23 Aug, 19:18	48%
Ericsson	54	54	23 Aug, 18:27	45%
France Telecom	38	42	23 Aug, 17:13	34%
Nokia	21	23	23 Aug, 12:25	42%
Hellenic Telecom		43	23 Aug, 19:18	41%
Telefonica	34	38	23 Aug, 09:34	29%
Telenor	26		23 Aug, 12:25	39%
Telecom Italia	47		23 Aug, 19:34	51%
Telia		35	23 Aug, 12:25	41%
Port Telecom Int	34	38	23 Aug, 12:10	47%
MM02		47	23 Aug, 16:29	33%
Vodafone	24	28	23 Aug, 12:59	35%
KPN	38	42	23 Aug, 09:33	43%
Telekom Aus		35	04 Aug, 19:59	50%

Table 3: The k-th-to default swap premiums in basis points (bp). The first column is for the 10 first obligors in Table 2, the second is for the 11 first obligors, and so on.

k	$m = 10$	$m = 11$	$m = 12$	$m = 13$	$m = 14$	$m = 15$
1	357.7	389.8	432.3	456.6	493.3	526.1
2	55.38	65.27	77.48	84.34	95.96	106.8
3	7.649	9.963	12.84	14.49	17.47	20.40
4	0.8698	1.281	1.814	2.132	2.744	3.366
5	0.08026	0.1373	0.2167	0.2678	0.3701	0.4795

of a 1st-to-default swap is about 4500 times higher than for a 5th-to-default swap. The corresponding ratio for a portfolio of size 15 is about 1100.

5.3. **Dependence on the interaction level.** In this subsection we use a portfolio consisting of the 9 first obligors in Table 2 to study how spreads are affected by the interaction parameter c which was taken to be 0.5 in the previous section. As above, we let the dependence parameters be given by the upper left part of Table 8. We first note that by (5.1) the value of c enters into the calibration of the base intensities a_i: a higher value of c will lead to smaller a_i-s, and hence c affects spreads both directly, and indirectly through its influence on the base intensities.

The dependence of the spreads on the interaction level is illustrated in Figure 2. The 1st-to-default spread decreases with increasing interaction level, and for k larger than 2

the spreads increase. However, it looks as if the 2nd-to-default spread may have a local maxima. To confirm that this is indeed possible, we experimented with different dependence structures. One result was Figure 3, which depicts the same graphs as Figure 2 but for different $\theta_{i,j}$, given by Table 7 where some of elements $\theta_{i,j}$ are much bigger than the corresponding numbers in Table 8. In this figure, the 2nd-to-default spread has a clear local maximum.

It might be worth noting that the graph of the 1st-to-default spread as a function of the interaction level c, roughly had the same structure as the corresponding graph for the intensity for T_1, see Figure 6. However, the same was not true for the 2nd-to-default spread. This can be seen from Figure 7 which shows the intensity $\sum_{i=1, i \neq j}^{9} a_i(1 + c\theta_{i,j})$ for the second default T_2, when the first default was by obligor j, for $j = 1, 2, \ldots 9$.

The case $c = 0$ is of special interest since it means that the defaults are independent of one-another. In particular, Figures 2, 3 and 4 quantifies the errors made in computing spreads as if obligors where independent in cases where there in fact is default contagion. Further, in Figure 3 we note that for very large interaction levels, the spreads for $1 \leq k \leq 5$ tend to converge into a narrow interval, compared with the case with very small interactions. The intuitive explanation for this may be that once one obligor default, several other will quickly follow. Finally, note that as the interaction level increases, the spreads for $6 \leq k \leq 9$ drastically increases and can thus no longer be neglected (see Figure 4), as for example in the Table 3 where $c = 0.5$.
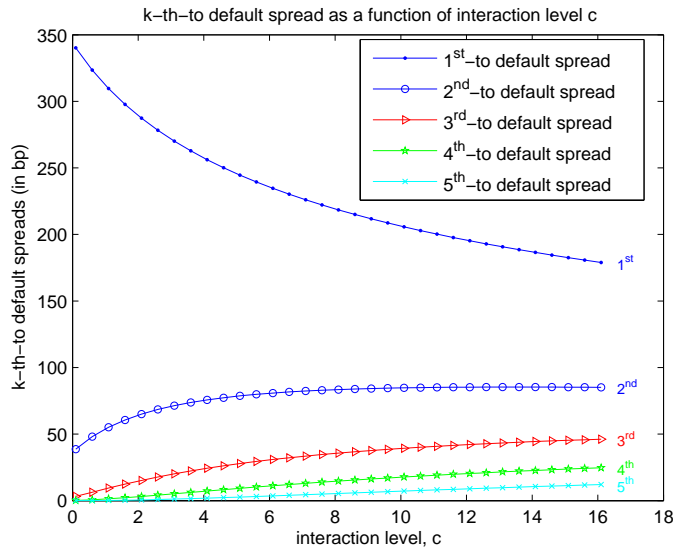
Figure 2: The kth-to-default spreads as a function of the interaction level c, for a portfolio consisting of the first 9 obligors in Table 2.
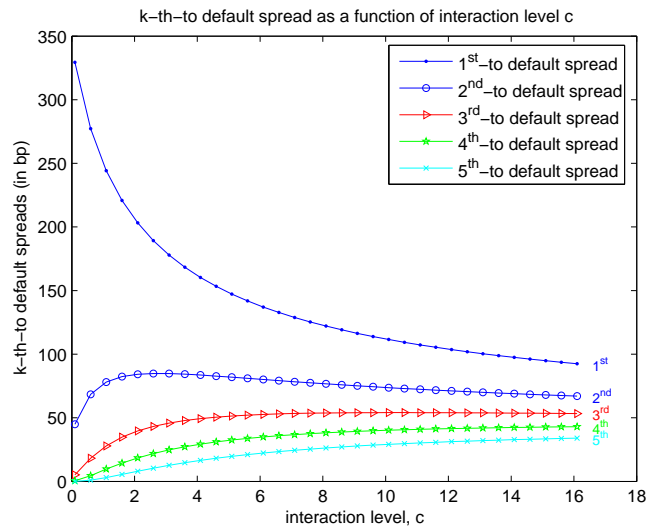
Figure 3: The different k^{th}-to-default spreads for $k \leq 5$ as a function of the interaction level c, for a portfolio consisting of the first 9 obligors in Table 2 with dependence structure given by Table 7.
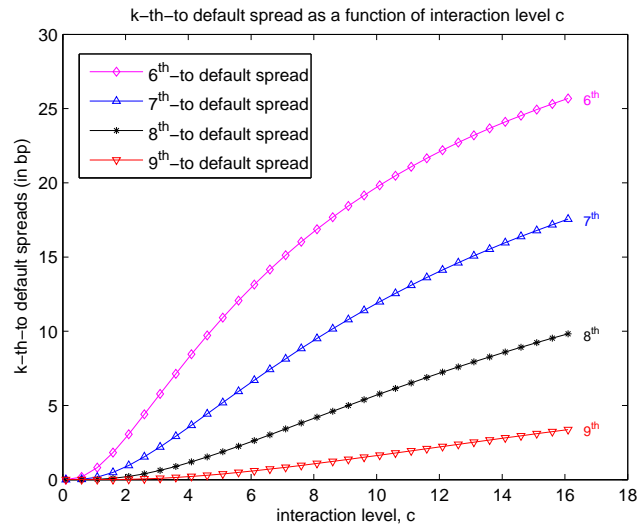
Figure 4: The different k^{th}-to-default spreads for $6 \leq k \leq 9$ as a function of the interaction level c, for a portfolio consisting of the first 9 obligors in Table 2 with dependence structure given by Table 7.
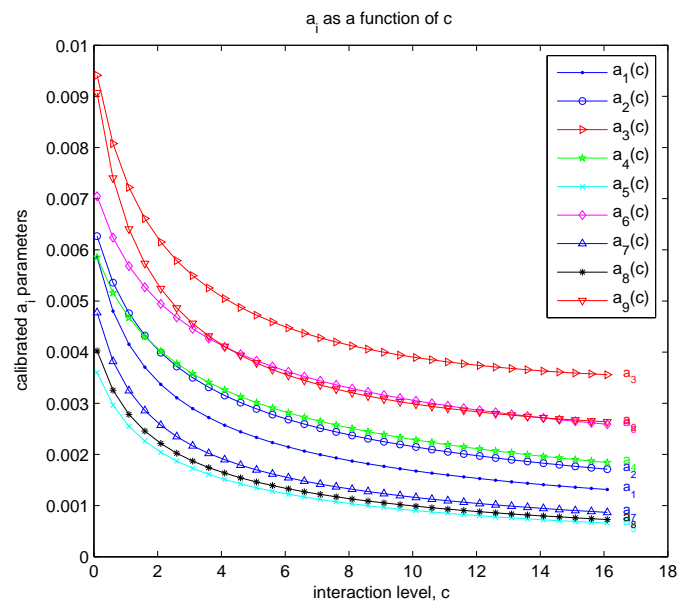
Figure 5: The base intensities a_i as functions of the interaction level c for a portfolio consisting of the first 9 obligors in Table 2 with dependence structure given by Table 7.
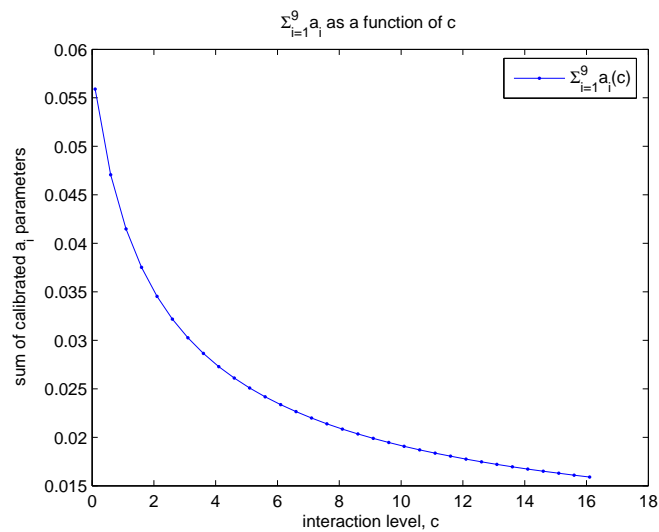
Figure 6: The intensity for T_1, as a function of the interaction level c, for a portfolio consisting of the 9 first obligors in Table 2 with dependence structure given by Table 7.
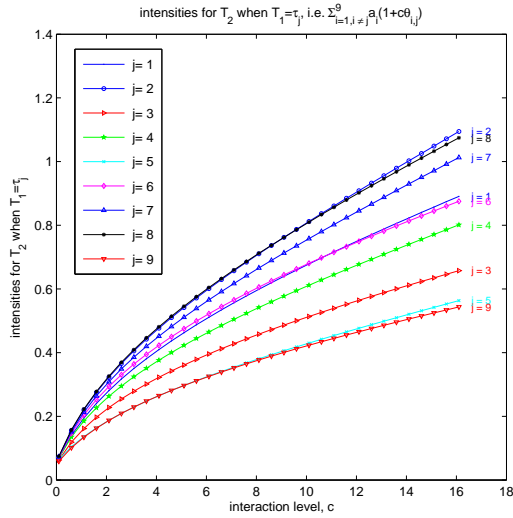
Figure 7: Intensities for T_2 when the first default was by obligor j, for a portfolio consisting of the 9 first obligors in Table 2 with dependence structure given by Table 7.

Table 4: The relative difference in percent between k^{th}-to-default swap spreads priced with homogeneous recovery rates and nonhomogeneous recovery rates. The different recovery rates are displayed in Table 6 in Appendix 8

k	std = 3.24	std = 4.90	std = 6.92	std = 11.10
1	0.13	0.16	0.20	0.23
2	1.27	1.35	2.05	2.74
3	2.81	3.25	5.13	6.71
4	4.27	5.17	8.45	11.10
5	5.60	6.99	11.69	15.51

5.4. **Dependence on the recovery rates.** In this subsection the numerical experiment aimed at investigating to what extent k^{th}-to-default swaps spreads for portfolios with nonhomogeneous recovery rates differ from the spreads in a corresponding portfolio where all recovery rates are the same and equal to the average of the nonhomogeneous rates.

The experiment was performed on a portfolio consisting of the first 11 obligors in Table 2 with dependence structure given by the upper left 11×11 submatrix of the matrix given in Table 8. We studied five different cases. In the first one all recovery rates were set to 40%. In the other four cases the recovery rates were varied "randomly" around approximately the mean 40%, but with the different standard deviations $3.24, 4.90, 6.92$ and 11.10, respectively. The results are displayed in Table 4. For the 1st- and 2nd-to-default spreads, the inhomogeneous cases differed from the homogenous one by at most

3%, and even the largest difference, for $k = 5$ and for the standard deviation equal to 11.10%, was only 15%. The different recoveries are displayed in Table 6 in Appendix 8

5.5. **Dependence on the market spreads.** In this subsection we investigate in numerical experiments how the kth-to-default swap prices change when the market prices of the underlying single-name CDS prices change.

The first experiment used a portfolio consisting of the 5 first obligors in Table 2. The CDS spreads for obligors $1, 3, 4$ were held at their market values, and while the CDS spreads for obligors 2 and 5 were varied from 10 to 225 in steps of 10 bp. The resulting k^{th}-to-default spreads increased smoothly as the CDS spreads increased, and this increase was more dramatic for larger k-s, see Figure 8.
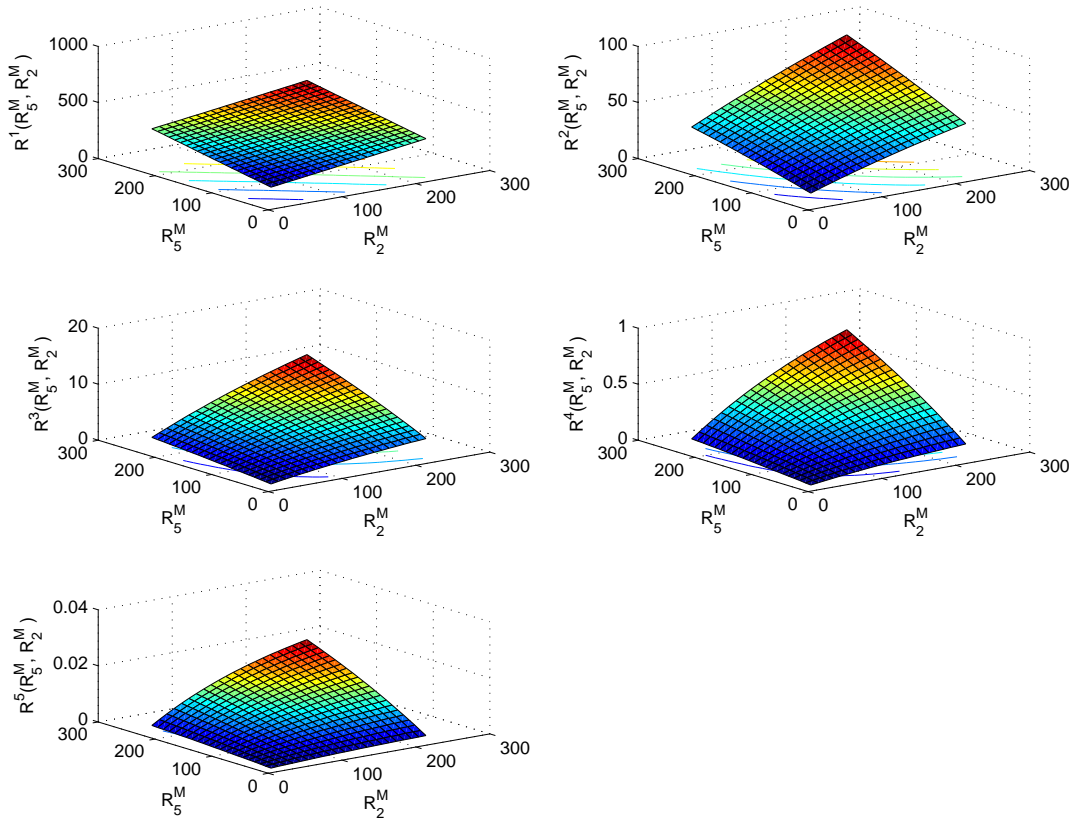
Figure 8: k^{th}-to-default spreads as function of market CDS spreads. The portfolio consisted of the first 5 obligors in Table 2. The CDS spreads for obligors 1 and 5 were varied, while the others were held constant.

5.6. **Approximation by a symmetric portfolio.** A portfolio is symmetric if the obligors are completely interchangeable. In the intensity formulation (5.1) for unordered default contagion, this means that the parameters a_i all are equal, and similarly the $\theta_{i,j}$ are the same for each obligor and the ϕ_i are equal. To compute spreads it then is sufficient to keep track of how many obligors have defaulted, but there is no need to know which ones it was. Here we consider the special case of (3.1) where all obligors have the same default intensities $\lambda_{t,i} = \lambda_t$ specified by $\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}}$ where $\{T_k\}$ is the ordering of the default times $\{\tau_i\}$. For this symmetric case, the Markov jump process constructed in Proposition 3.1 can be collapsed into a chain with the $m+1$ states $\{0\}, \{1\}, \ldots \{m\}$. The interpretation is that the chain is in the state $\{k\}$ if precisely k obligors have defaulted. In the literature such a process is called a death process. This new state space is very much smaller state than the one in Proposition 3.1, which means that it is possible to do numerical computation for much larger portfolios, with hundreds or thousands of obligors, see e.g. [30] where CDO-tranche spreads are computed on portfolios with 125 obligors in such a symmetric model.

It is hence of interest to understand how well non-symmetric portfolios can be approximated by symmetric ones. To explore this we constructed and semi-calibrated three different portfolios, 1) the telecom portfolio from Table 2 with all recovery rates set to 40%, 2) the telecom portfolio with the recovery rates given in Table 2 (the standard deviation of these rates are 6.88%), and 3) the telecom portfolio with the first 11 recovery rates given by the last row in Table 6 and the remaining rates for obligors 12 to 15 set to $30, 32, 40$ and 60% (the rates then have standard deviation 11.71%). In all cases the dependence structure was given by Table 8 and the interaction level was $c = 0.5$.

Each of these three portfolios was then approximated by a symmetric portfolio. In the symmetric approximating portfolio all recovery rates were set equal to the average of the recovery rates in the original portfolio. The jump parameters where assumed to be the same so $b_k = b$ at each default time T_k. Further, the parameters a and b were chosen so that the model CDS spread in the symmetric portfolio coincided with the average market CDS spread in the nonsymmetric counterpart, and so that the first-to-default spreads also agreed.

From Table 5 we see that the relative differences are small. For $k = 2$ and 3 they increase as the standard deviation of the recovery rates increase. However, for $k = 4$ and 5 the relative differences for the second portfolio (std = 6.88) somewhat surprisingly are smaller than for the other two cases. For the nonhomogeneous recovery rate cases, the relative differences are not monotone in k. Finally, for $6 \leq k \leq 15$ the differences can range between 9% up to 130% where the error on average increases with k. However, for each case the sum of these spreads are smaller than one tenth of a bp.

6. Calibration and numerical implementation

In this section we discuss the computational aspects in more detail. Subsection 6.1 gives a short description how to calibrate (or *semi-calibrate*) the model (5.1) against market data and formulas for the CDS and k^{th}-to-default spreads, and Subsection 6.2 discusses how to

Table 5: The difference in k^{th}-to-default swap spreads between an approximating symmetric portfolio and a nonsymmetric portfolio, in percent of the spreads for the nonsymmetric portfolio; for three different cases. For $k = 1$ the difference is by construction (almost) zero.

k	std $= 0$	std $= 6.88$	std $= 11.71$
2	0.153	0.587	1.98
3	0.811	0.850	3.37
4	2.24	0.339	3.82
5	4.65	1.26	3.13

compute the matrix exponential. Subsections 6.3 and 6.4 consider more general models and computation by simulation, respectively.

6.1. **Calibration.** As discussed above, we assume that the relative dependence structure $\theta_{i,j}$, the interaction level c and the recovery rate structure $\phi_1, \ldots, \phi_m$ all are exogenously given - hence the term "semi-calibration". The base intensities a_i are then obtained by individual calibration to the market CDS spreads. The calibration uses a nonlinear least squares method where the model CDS spreads $R^{(i)}$ are matched against the corresponding market CDS spreads $R^{i,M}$. Thus the base intensities are computed as

$$(a_1, \ldots, a_m) = \underset{\tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \hat{a}_m)}{\operatorname{argmin}} \sum_{i=1}^{m} \left(R^{i,M} - R^{(i)}(\hat{\boldsymbol{a}}) \right)^2$$

where we have emphasized that the $R^{(i)}$ are functions of the parameters $\boldsymbol{a} = (a_1, \ldots, a_m)$. We then use the calibrated base intensities a_i with $\{\theta_{i,j}, \phi_i, c\}$ to compute the kth-to default spreads R_k.

Reverting to the notation of (3.1), closed-form expressions for R_k and $R^{(i)}$ may be obtained by inserting (4.3) and (4.4) into (2.2). For ease of reference we exhibit the resulting formulas (detailed proofs can be found in [28] or [29]).

Proposition 6.1. *Consider m obligors with default intensities* (3.1) *and assume that the interest rate r is constant. Then,*

$$R_k = \frac{\boldsymbol{\alpha} \left(\boldsymbol{A}(0) - \boldsymbol{A}(T) \right) \boldsymbol{\phi}^{(k)}}{\boldsymbol{\alpha} \left(\sum_{j=1}^{n} \left(\Delta_j e^{\boldsymbol{Q} t_j} e^{-rt_j} + \boldsymbol{C}(t_{j-1}, t_j) \right) \right) \boldsymbol{m}^{(k)}}$$

and

$$R^{(i)} = \frac{(1 - \phi_i) \boldsymbol{\alpha} \left(\boldsymbol{A}(0) - \boldsymbol{A}(T) \right) \boldsymbol{g}^{(i)}}{\boldsymbol{\alpha} \left(\sum_{j=1}^{n} \left(\Delta_j e^{\boldsymbol{Q} t_j} e^{-rt_j} + \boldsymbol{C}(t_{j-1}, t_j) \right) \right) \boldsymbol{g}^{(i)}}.$$

Here

$$\boldsymbol{\phi}^{(k)} = \sum_{i=1}^{m} (1 - \phi_i) \boldsymbol{R}^{i,k} \boldsymbol{h}^{i,k} \quad and \quad \boldsymbol{R}^{i,k} = \sum_{\ell=0}^{k-1} \left(\prod_{p=\ell}^{k-1} \boldsymbol{G}^{i,p} \boldsymbol{P} \right)$$

and

$$\boldsymbol{C}(s,t) = s\left(\boldsymbol{A}(t) - \boldsymbol{A}(s)\right) - \boldsymbol{B}(t) + \boldsymbol{B}(s),$$

where

$$\boldsymbol{A}(t) = e^{\boldsymbol{Q}t}\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}$$

$$\boldsymbol{B}(t) = e^{\boldsymbol{Q}t}\left(t\boldsymbol{I} + \left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}$$

Finally, $\boldsymbol{g}^{(i)}$ is an $|\boldsymbol{E}|$ column vector such that

$$\boldsymbol{g}_{\boldsymbol{j}}^{(i)} = \begin{cases} 1 & \text{if} & \boldsymbol{j} \in \Delta_i^C \\ 0 & \text{otherwise} . \end{cases}$$

There are several possible computational shortcuts. The quantities $\boldsymbol{g}^{(i)}$, $\boldsymbol{h}^{i,k}$, $\boldsymbol{g}^{i,k}$ and $\boldsymbol{m}^{(k)}$ do not depend on the parametrization, and hence only have to be computed once. The row vectors $\boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)$ and $\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1},t_n)\right)\right)$ are the same for all CDS spreads and all k^{th}-to-default swap spreads and hence only have to be computed once for each parametrization $\{a_i, \theta_{i,j}, \phi_i, c\}$. The same holds for $\boldsymbol{\phi}^{(k)}$ for each k. Furthermore, if Δ_n is constant then $\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1},t_n)\right)\right)$ can be simplified in terms of $e^{\boldsymbol{Q}t_n}$. If the ϕ_i are the same for all obligors, $\boldsymbol{\phi}^{(k)}$ can be replaced by $(1-\phi)\boldsymbol{m}^{(k)}$. Next, if we rewrite the sum $\boldsymbol{R}^{i,k}$ as $\boldsymbol{R}^{i,k} = \sum_{\ell=0}^{k-1}\boldsymbol{M}^{i,\ell,k}$ where $\boldsymbol{M}^{i,\ell,k} = \prod_{p=\ell}^{k-1}\boldsymbol{G}^{i,p}\boldsymbol{P}$ then $\boldsymbol{M}^{i,\ell-1,k} = \boldsymbol{G}^{i,\ell-1}\boldsymbol{P}\boldsymbol{M}^{i,\ell,k}$ which is useful for computation. Also note that $(\boldsymbol{Q} - r\boldsymbol{I})$ is invertible since it is upper diagonal with strictly negative diagonal elements. The conditioning number of $(\boldsymbol{Q} - r\boldsymbol{I})$ is often large, but still we have not encountered numerical problems in computing the inverse.

6.2. **Computation of the matrix exponential.** The main challenge in Proposition 6.1 is to compute the matrix exponential $e^{\boldsymbol{Q}t}$. We have mainly experimented with two different numerical methods; direct series expansion of the matrix exponential and the uniformization method (which sometimes also is called the randomization method). Both were fast and robust for our problems. However, while the series expansion method lacks lower bounds on possible worst case scenarios (see [46]), the uniformization method provides analytical expressions for the residual error. Furthermore, previous studies indicate that it can handle large sparse matrices with remarkable robustness, see e.g. [55], [26] or Appendix C.2.2 in [40]. We have therefore chosen the uniformization method. A probabilistic interpretation of the method can be found in [26] and pure matrix arguments which motivates the method are given in [55] and Appendix C.2.2 in [40].

There are many different methods to compute the matrix exponential ([46] [47]). However, most of the other standard methods are not adapted to very large, but sparse, matrices and don't seem possible when the state space is larger than a few hundred (see [55],[26]). As one example, it is tempting to try eigenvalue decomposition since the eigenvalues are given by the diagonal of $\boldsymbol{Q}$. However, in our examples this method failed already for $m = 9$ since the eigenvector matrices turned out to be ill-conditioned, which introduced large numerical errors in the inversions. Compared to e.g. Krylov-based methods or ODE methods the uniformization method is relatively simple to implement.

The uniformization method works as follows. Let $\Lambda = \max\left\{|\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}}| : \boldsymbol{j} \in \boldsymbol{E}\right\}$ and set $\widetilde{\boldsymbol{P}} = \boldsymbol{Q}/\Lambda + \boldsymbol{I}$. Then

$$e^{\boldsymbol{Q}t} = \sum_{n=0}^{\infty} \widetilde{\boldsymbol{P}}^n e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}. \tag{6.2.1}$$

Recall that $\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}$ and define $\widetilde{\boldsymbol{p}}(t, N) = \boldsymbol{\alpha} \sum_{n=0}^{N} \widetilde{\boldsymbol{P}}^n e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}$. Let $\varepsilon > 0$ be arbitrary and pick $N(\varepsilon)$ so that $1 - \sum_{n=0}^{N(\varepsilon)} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} < \varepsilon$. Then the L_1 error $\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t, N(\varepsilon))\|_1$ is also less than ε, i.e., $\widetilde{\boldsymbol{p}}(t, N(\varepsilon))$ approximates $\boldsymbol{p}(t)$ with an accumulated absolute error which is less than ε. Furthermore, since all entries in $\widetilde{\boldsymbol{p}}(t, N(\varepsilon))$ are positive there are no cancelation effects and the approximation error decreases monotonically with increasing N. Furthermore, for fixed N, the error $\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t, N)\|_1$ is also decreasing in t.

Further useful properties of (6.2.1) are that it separates the computation of $\widetilde{\boldsymbol{P}}^n$ from the time dependent components and that

$$\int_0^T e^{(\boldsymbol{Q}-r\boldsymbol{I})t} dt = \sum_{n=0}^{\infty} \widetilde{\boldsymbol{P}}^n \frac{\Lambda^n}{n!} I_n^{(\Lambda+r)}(T), \tag{6.2.2}$$

where

$$
\begin{aligned}
I_n^{(\beta)}(t) &= \int_0^t s^n e^{-\beta s} ds \\
&= \frac{e^{-\beta t}}{(-\beta)^{n+1}} \left[(-\beta t)^n - n(-\beta t)^{n-1} + n(n-1)(-\beta t)^{n-2} - \ldots + (-1)^{n-1}(n-1)!(-\beta t)\right].
\end{aligned}
$$

Since $I_n^{(\beta)}(t) > 0$, there are no cancelation effects in the approximation of the RHS in (6.2.2). Truncating the sum in the RHS in (6.2.2) gives an approximation to the integral in the LHS. In this case, the error control requires a little more work.

For $m \leq 13$, we used a standard laptop with 1024 MB RAM. For $m = 14$ and $m = 15$ the memory requirements were too big for the laptop so the computations were done on a Sun Solaris, 2x900 MHz UltraSPARC-III with 5GB RAM. As an example if $m = 15$, $c = 0.5$, $T = 5$ and $\theta_{i,j}$ were as in Table 8 and if we put $\varepsilon = 3.33 \cdot 10^{-16}$ (the floating-point relative accuracy for the number 1 in Matlab is $2.22 \cdot 10^{-16}$) then our calibration against Table 2 implied that $\Lambda = 0.2500$ and that $N(\varepsilon) = 19$ terms were needed in the computation of $\widetilde{\boldsymbol{p}}(T, N)$. Furthermore, the calibration errors were neglige: the sum of the individual absolute calibration errors were less than two tenths of a bp.

A further point is that our matrices in general are very large, for example if $m = 15$ then the generator has $2^{15} = 32768$ rows and thus contain $2^{30} \approx 1.$ billion entries. However, at the same time it is extremely sparse and the sparseness is increasing with m. E.g., for $m = 15$ there are only 0.025% nonzero entries in $\boldsymbol{Q}$, and hence only about 280.000 elements have to be stored.

A final point is that we are not interested in finding the matrix exponential itself, but only the probability vector $\boldsymbol{p}(t)$. This is important, since computing $e^{\boldsymbol{Q}t}$ is very time and memory consuming compared with computing $\boldsymbol{\alpha} e^{\boldsymbol{Q}t}$. For example, using the uniformization method with $t = 5$, $m = 14$ and $\varepsilon = 3.331 \cdot 10^{-16}$, which implies that $N(\varepsilon) = 19$ the time

to find $\boldsymbol{p}(t)$ in Matlab, by first computing $e^{\boldsymbol{Q}t}$ with the uniformization method and then multiplying this matrix with $\boldsymbol{\alpha}$, was 42.3 seconds. On the other hand, computing $\boldsymbol{p}(t)$ via the vectors $\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n$ only took 0.14 seconds, and hence was about 300 times faster.

To get the same accuracy with direct Taylor method required 32 terms in the truncated sum. For this case, the corresponding computational times where 108 seconds and 0.25 seconds. The quick method was about 450 times faster than the slow one. Further, the uniformization method was about 2.5 and 1.8 times faster compared to the corresponding slow and quick Taylor methods. The reason was that the Taylor method required 32 terms in the sums, compared to the 19 needed for the uniformization method.

6.3. **Extensions.** It is natural to believe that the default intensity of a obligor in addition to dependence on defaults of other obligors also depends on exogenous macroeconomic and market factors. It is possible generalize Proposition 3.1 in Section 3 to the case when the parameters a_t^i and $b_t^{i,j}$ in the formula (3.1) for the intensity depend on some background random process. The idea is to first condition on the whole realization of the background process and then treat $(Y_t)_{t \geq 0}$ as an inhomogeneous Markov jump process with time-dependent generator. It is usually impossible to find tractable closed-form expressions for distributions derived from $(\tau_1, \ldots, \tau_m)$ for this case. Instead, to use our method one has to rely on a fine discretisation of time.

We also note that Proposition 3.1 does not seem easily applicable if the parameters in the intensities depend of the default times. For example, so called *self-exciting* point processes (or *Hawkes processes*), see e.g. [27] with intensities given by

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} e^{-(t-\tau_j)} 1_{\{\tau_j \leq t\}} \tag{6.3.1}$$

are therefore not suited to our setup. Applications of Hawkes processes in credit risk are discussed in e.g. [23], [22] and [17].

6.4. **Simulation.** An alternative to numerical computation is to use simulation to produce realizations of the random vector $(\tau_1, \tau_2 \ldots, \tau_m)$. If the intensities are given by (3.1) this is straightforward, see e.g. [34].

The so called *total hazard construction* ([49], [54]) can be used in more general circumstances where the intensities may be functionals of the default times as well as of time, see e.g. (6.3.1). The paper, [15], we believe, was the first to point out that the total hazard construction can be used in credit risk. In Chapter 5 of [58], the author used the total-hazard construction to study four cases of default contagion. Three of these cases can be handled without invoking the total-hazard construction, for example using our approach, cf also [49]. The fourth case, which considers stochastic parameters, can actually also be treated without the total hazard construction. The case with a stochastic processes $\boldsymbol{X}_t$ in the parameters as in [58], is handled by doing a straightforward extension of the results in [49], [54].

To simulate one needs to know the values of the parameters. It is far from trivial how to estimate the parameters in general. Although one can repeat the simulations with different parameters until the model is calibrated, this is often very time consuming.

7. DISCUSSION AND CONCLUSIONS

In this paper we considered the intensity based default contagion model (3.1), where the default intensity of one firm is allowed to change when other firms default. The model was reinterpreted in terms of a Markov jump process, and this reinterpretation made it possible to derive closed form expressions for k^{th}-to-default spreads. With the computational resources available to us these expressions were tractable for general portfolios with up to 15 obligors. These are much larger than the general examples treated by other authors.

We used a synthetic telecom portfolio with 15 companies taken from `Reuters` at August 23, 2005 in a numerical study of how default contagion influences k^{th}-to-default spreads. For this we performed a "semi-calibration" of the portfolios where interaction parameters, interest rates and recovery rates were assumed obtained from prior knowledge but where the baseline default intensities were calibrated to market CDS spreads. The questions we tried to illustrate were:

How did the size of the portfolio influence k^{th}-to-default spreads? In our example, the 1th-to-default spread increased by about 50% when the portfolio size increased from 10 to 15 and by about 700% for a 5th-to-default spread. For a portfolio of size 10 the prize of a 1st-to-default spread was about 4500 times higher than for a 5th-to-default spread and for a portfolio of size 15 the spread was about 1100 times higher. Qualitatively this is completely as expected. However it would seem rather impossible to guess the sizes of the effects without computation.

How were k^{th}-to-default spreads influenced by the strength of interaction in the portfolio? We considered two examples. In those the 1th-to-default spread decreased when the interaction became stronger, and the higher order than 2nd-to-default spreads decreased. In the first example there was some indication that 2nd-to-default spreads first increased and then decreased, and this was clear in the second example. This last finding may be somewhat counterintuitive. A possible intuitive explanation may be that as the interaction increases, the 2nd-to-default will have a behavior more like the 1th-to-default spread. This view is also supported by nothing that when the interaction level is extremely big, all swap spreads tend to converge into a narrow interval, compared with the case with very small interaction, see e.g. Figure 3. The results also illustrate the error made if one assumes independence in cases where there in fact is default contagion.

How did k^{th}-to-default spreads depend on the underlying market CDS spreads? The kth-to-default spreads increased smoothly with increasing market spreads. The increase was greater for larger k-s. This first result was complectly as expected, and the second one perhaps slightly less obvious.

How were spreads affected by non-homogeneities in the recovery rates? The spreads were virtually unchanged by moderate inhomogeneities in the recovery rates. This agrees with earlier findings for single-name CDS-s, see e.g. [32].

Does approximation of a non-homogeneous portfolio with a homogenous one work well? In our example the approximation worked quite well. The interest of this comparison is that the computational burden for a homogenous portfolios is very much smaller than for non-homogeneous ones.

As a general comment, qualitatively most of the results summarized above were as was expected beforehand. However, it seems difficult to guess the sizes of the effects without actually doing the computations.

How can we estimate the dependence structure ? There are several possible ways to estimate, or calibrate the dependence matrix θ. One approach is to use historical time series data on the traded CDS spreads and consider the quadratic covariance process between the obligors i and j to get an indication of $\theta_{i,j}$. A similar procedure can also be done on the corresponding bonds. Another approach is to estimate equity correlation and use this as a proxy for default correlation. To be more specific, using Proposition 3.1 it is straightforward to find computational tractable closed-form expression for pairwise default correlation (i.e. $\mathrm{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$) as function of time, the matrix θ, c and the baseline intensities a_i. These analytical expressions for the default correlations can be used to extract θ from the numerical values on the corresponding equity correlations. However, further assumptions have to be done since the equity correlations are only half as many as the entities in the matrix θ. Using equity correlation as a proxy for default correlation has previously been very common when modelling default dependencies, see e.g. [4], [35], [41], [42], [43] and [48]. For example, in [41], the authors value k^{th}-to-default swaps by using two different copulas for a portfolio with 6 obligors. First, a one-factor Gaussian copula is used where the six correlations are estimated from equity returns. The one-factor Gaussian copula lacks correlations in the tails and [41] redoes the computations with a Clayton copula where the lower tail dependence is estimated from equity returns by using Kendall's tau on this data. Similar techniques are also pursued in e.g. [4] and [42].

A third approach to extract θ is first to estimate default correlation in an intensity based model from historical corporate data under the statistical probability measure. This is done in e.g. [11], [36] and [37]. Then, if we know the relationship between the statistical probability measure and the risk-neutral martingale measure, this can be used to extract θ. Determining the relationship between these two measures is equivalent to finding the connection between the intensities when changing the measures. Such a procedure is far from trivial and a discussion of how this can be done is given in e.g. [5], [6], [44] and [57].

Finally, as discussed above the approach of this paper can be generalized to time-dependent and stochastic intensities. Further, it also gives a possibility to price other products, such as CDO-s. This will be presented in further papers by the first author.

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8. Appendix

In this appendix we display the recovery rate cases and the dependence structures used in Section 5.

Table 6: Different recovery rate cases used in Table 4 and Table 5. Recoveries expressed in %.

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}	mean	std
36	43	45	37	41	44	38	37	43	39	43	40.55	3.24
33	47	46	34	44	41	36	39	42	39	46	40.64	4.90
32	48	45	34	42	41	29	39	51	41	47	40.82	6.93
27	52	41	31	45	31	29	35	52	41	61	40.45	11.10

Table 7: The θ matrix describing the alternative dependence structure, rounded to two decimal places.

i	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$	$j = 9$
1	0	3.03	5.13	2.5	4.69	6.36	5.69	6.7	1.32
2	3.84	0	2.96	3.49	1.41	4.18	4.27	1.03	3.44
3	1.84	5.62	0	3.04	1.92	2.31	4.91	6.09	2.87
4	4.18	0.59	0.51	0	4.39	3.35	0.65	5.39	3.25
5	0.35	6.62	3.88	4.32	0	4.18	2.98	3.11	4.28
6	4	6.41	2.05	0.8	0.42	0	2.63	4.35	0.5
7	4.91	4.22	6.01	6.29	0.63	5.81	0	6.66	2.2
8	6.74	1.78	2.35	5.28	1.9	6.69	5.83	0	4.26
9	5.25	6.11	4.76	5.54	2.87	4.17	5.87	1.74	0

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Table 8: The θ matrix describing the dependence structure, rounded to two decimal places. This matrix is used in all examples except in Figure 3 to Figure 7.

i	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$	$j=11$	$j=12$	$j=13$	$j=14$	$j=15$
1	0	5.68	0.11	0.12	0.13	0.16	0.11	0.16	0.13	0.1	0.12	0.12	0.13	0.14	0.15
2	0.15	0	2.91	2.22	0.14	0.2	0.16	0.11	0.2	0.1	0.15	0.11	0.14	0.19	0.14
3	0.13	0.16	0	0.11	4.61	5.21	0.18	0.16	0.11	0.13	0.13	0.15	0.19	0.14	0.16
4	2.5	4.79	0.17	0	0.14	0.13	2.11	1.83	0.14	0.16	0.17	0.12	0.17	0.11	0.19
5	0.1	0.13	0.18	0.18	0	0.2	0.16	0.16	3.87	1.29	0.13	0.17	0.2	0.11	0.19
6	0.1	0.13	0.16	0.2	0.19	0	0.14	0.16	0.11	0.17	1.87	3.28	0.17	0.17	0.12
7	0.19	0.19	0.12	0.13	0.2	0.17	0	0.14	0.1	0.17	0.1	0.18	2.42	2.8	0.15
8	1.57	0.17	2.8	0.15	2.94	0.17	3.36	0	2.04	0.16	2.6	0.18	1.19	0.18	1.38
9	0.18	2.46	0.11	2.76	0.13	2.5	0.14	0.9	0	2.3	0.13	2.22	0.16	1.92	0.15
10	1.62	2.45	1.29	0.12	1.76	2.47	1.87	0.18	0.13	0	1.72	2.6	1.44	0.14	0.18
11	1.59	1.52	0.12	1.28	1.05	0.87	0.13	0.71	0.11	2.15	0	0.15	0.13	0.58	1.4
12	0.1	0.11	0.84	0.12	1.08	0.14	0.8	0.18	1.51	0.14	0.16	0	0.2	0.15	1.14
13	0.19	0.69	0.12	0.41	0.12	0.19	0.11	0.64	0.17	1.32	0.12	0.17	0	0.76	0.11
14	0.5	0.15	0.94	0.16	0.27	0.2	0.74	0.12	0.18	0.12	0.39	0.13	0.7	0	0.16
15	0.2	0.33	0.16	0.16	0.1	0.4	0.15	0.19	0.14	0.14	0.15	0.57	0.14	0.17	0

Paper II

PRICING SYNTHETIC CDO TRANCHES IN A MODEL WITH DEFAULT CONTAGION USING THE MATRIX-ANALYTIC APPROACH

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ABSTRACT. We value synthetic CDO tranche spreads, index CDS spreads, k^{th}-to-default swap spreads and tranchelets in an intensity-based credit risk model with default contagion. The default dependence is modelled by letting individual intensities jump when other defaults occur. The model is reinterpreted as a Markov jump process. This allow us to use a matrix-analytic approach to derive computationally tractable closed-form expressions for the credit derivatives that we want to study. Special attention is given to homogenous portfolios. For a fixed maturity of five years, such a portfolio is calibrated against CDO tranche spreads, index CDS spread and the average CDS and FtD spreads, all taken from the iTraxx Europe series. After the calibration, which render perfect fits, we compute spreads for tranchelets and k^{th}-to-default swap spreads for different subportfolios of the main portfolio. We also investigate implied tranche-losses and the implied loss distribution in the calibrated portfolios.

1. INTRODUCTION

In recent years the market for synthetic CDO tranches and index CDS-s, which are derivatives with a payoff linked to the credit loss in a portfolio of CDS-s, have seen a rapid growth and increased liquidity. This has been followed by an intense research for understanding and modelling the main feature driving these products, namely default dependence.

In this paper we derive computationally tractable closed-form expressions for synthetic CDO tranche spreads and index CDS spreads. This is done in an intensity based model where default dependencies among obligors are expressed in an intuitive, direct and compact way. The financial interpretation is that the individual default intensities are constant, except at the times when other defaults occur: then the default intensity for each obligor jumps by an amount representing the influence of the defaulted entity on that obligor. This phenomena is often called default contagion. The above model is then reinterpreted

in terms of a Markov jump process. This interpretation makes it possible to use a matrix-analytic approach to derive practical formulas for CDO tranche spreads and index CDS spreads. Our approach is the same as in [15] and [17] where the authors study aspects of kth-to default spreads in nonsymmetric as well as in symmetric portfolios. The contribution of this paper is a continuation of this technique to synthetic CDO tranches and index CDS-s.

Except for [15] and [17], the methods presented in [2], [4], [7], [8], [9], [11], [12], Section 5.9 in [22] and Subsection 9.8.3 in [23], are currently closest to the approach of this article. The framework used here (and in [15] and [17]) is the same as in [11], [12] and is related to [2], [4]. The main differences are that [11], [12] use time-varying parameters in their practical examples and then solve the corresponding Chapman-Kolmogorov equation using numerical methods for ODE-systems. Furthermore, in [12], the authors also consider numerical examples where the portfolio is split into homogeneous groups with default contagion both within each group and between groups. [4] use Monte Carlo simulations to calibrate and price the instruments.

Default contagion in an intensity based setting have previously also been studied in for example [1], [3], [6], [13], [14], [19], [21], [25], [26] and [27]. The material in all these papers and books are related to the results discussed here.

This paper is organized as follows. In Section 2 we give an introduction to synthetic CDO tranches and index CDS-s which motivates results and introduces notation needed in the sequel. Section 3 presents the intensity-based model for default contagion. Using a result from [17], the model is reinterpreted in terms of a Markov jump process. The results in Section 4, convenient analytical formulas for synthetic CDO tranche spreads and index CDS spreads, are the main theoretical contribution in this paper. We assume that the recovery rates are deterministic and that the interest rate is constant. In Section 5 we apply the results from Section 4 to a homogenous model. Then, in Section 6, for a fixed maturity of five years, this portfolio is calibrated against CDO tranche spreads, the index CDS spread and the average CDS and FtD spreads, all taken from the iTraxx series, resulting in perfect fits. After the calibration, we compute k^{th}-to-default swap spreads for different subportfolios of the main portfolio. This problem is slightly different from the corresponding one in previous studies, e.g. [15] and [17], since the obligors undergo default contagion both from the subportfolio and from obligors outside the subportfolio, in the main portfolio. Further, we compute spreads on tranchelets which are nonstandard CDO tranches with smaller loss-intervals than standardized tranches. We also investigate implied tranche-losses and the implied loss distribution in the calibrated portfolios. The final section, Section 7 summarizes and discusses the results.

2. Valuation of Synthetic CDO tranche spreads and index CDS spreads

In this section we give a short description of tranche spreads in synthetic CDO-s and of index CDS spreads. It is independent of the underlying model for the default times and introduces notation needed later on. At the end of the section we give a technical

motivation for the main purpose of this article, which, roughly speaking, is to derive practical formulas for functions of the credit loss in a portfolio.

2.1. **The cash-flows in a synthetic CDO.** In this section and in the sequel all computations are assumed to be made under a risk-neutral martingale measure $\mathbb{P}$. Typically such a $\mathbb{P}$ exists if we rule out arbitrage opportunities. Further, we assume the that risk-free interest rate, r_t is deterministic.

A synthetic CDO is defined for a portfolio consisting of m single-name CDS's on obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$ and recovery rates $\phi_1, \phi_2, \ldots, \phi_m$. It is standard to assume that the nominal values are the same for all obligors, denoted by N. The accumulated credit loss L_t at time t for this portfolio is

$$L_t = \sum_{i=1}^{m} N(1 - \phi_i) 1_{\{\tau_i \leq t\}}. \tag{2.1.1}$$

We will without loss of generality express the loss L_t in percent of the nominal portfolio value at $t = 0$. For example, if all obligors in the portfolio have the same constant recovery rate ϕ, then $L_{T_k} = k(1 - \phi)/m$ where $T_1 < \ldots < T_k$ is the ordering of $\tau_1, \tau_2, \ldots, \tau_m$.

A CDO is specified by the attachment points $0 = k_0 < k_1 < k_2 < \ldots k_\kappa = 1$ with corresponding tranches $[k_{\gamma-1}, k_\gamma]$. The financial instrument that constitutes tranche γ with maturity T is a bilateral contract where the protection seller **B** agrees to pay the protection buyer **A**, all losses that occurs in the interval $[k_{\gamma-1}, k_\gamma]$ derived from L_t up to time T. The payments are made at the corresponding default times, if they arrive before T, and at T the contract ends. The expected value of this payment is called the *protection leg*, denoted by $V_\gamma(T)$. As compensation for this, **A** pays **B** a periodic fee proportional to the current outstanding (possible reduced due to losses) value on tranche γ up to time T. The expected value of this payment scheme constitutes the *premium leg* denoted by $W_\gamma(T)$. The accumulated loss $L_t^{(\gamma)}$ of tranche γ at time t is

$$L_t^{(\gamma)} = (L_t - k_{\gamma-1}) 1_{\{L_t \in [k_{\gamma-1}, k_\gamma]\}} + (k_\gamma - k_{\gamma-1}) 1_{\{L_t > k_\gamma\}}. \tag{2.1.2}$$

Let $B_t = \exp\left(-\int_0^t r_s ds\right)$ denote the discount factor where r_t is the risk-free interest rate. The protection leg for tranche γ is then given by

$$V_\gamma(T) = \mathbb{E}\left[\int_0^T B_t dL_t^{(\gamma)}\right] = B_T \mathbb{E}\left[L_T^{(\gamma)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right] dt,$$

where we have used integration by parts for Lebesgue-Stieltjes measures together with Fubini-Tonelli and the fact that r_t is deterministic. Further, if the premiums are paid at $0 < t_1 < t_2 < \ldots < t_{n_T} = T$ and if we ignore the accrued payments at defaults, then the premium leg is given by

$$W_\gamma(T) = S_\gamma(T) \sum_{n=1}^{n_T} B_{t_n} \left(\Delta k_\gamma - \mathbb{E}\left[L_{t_n}^{(\gamma)}\right]\right) \Delta_n$$

where $\Delta_n = t_n - t_{n-1}$ denote the times between payments (measured in fractions of a year) and $\Delta k_\gamma = k_\gamma - k_{\gamma-1}$ is the nominal size of tranche γ (as a fraction of the total nominal value of the portfolio). The constant $S_\gamma(T)$ is called the spread of tranche γ and is determined so that the value of the premium leg equals the value of the corresponding protection leg.

2.2. **The tranche spreads.** By definition, the constant $S_\gamma(T)$ is determined at $t = 0$ so that $V_\gamma(T) = W_\gamma(T)$, that is, so that the value of the premium leg agrees with the corresponding protection leg. Furthermore, for the first tranche, often denoted the *equity* tranche, $S_1(T)$ is set to 500 bp and a so called *up-front* fee $S_1^{(u)}(T)$ is added to the premium leg so that $V_1(T) = S_1^{(u)}(T)k_1 + W_1(T)$. Hence, we get that

$$S_\gamma(T) = \frac{B_T \mathbb{E}\left[L_T^{(\gamma)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right] dt}{\sum_{n=1}^{n_T} B_{t_n}\left(\Delta k_\gamma - \mathbb{E}\left[L_{t_n}^{(\gamma)}\right]\right)\Delta_n} \qquad \gamma = 2, \ldots, \kappa$$

and

$$S_1^{(u)}(T) = \frac{1}{k_1}\left[B_T \mathbb{E}\left[L_T^{(1)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(1)}\right] dt - 0.05\sum_{n=1}^{n_T} B_{t_n}\left(\Delta k_1 - \mathbb{E}\left[L_{t_n}^{(1)}\right]\right)\Delta_n\right].$$

The spreads $S_\gamma(T)$ are quoted in bp per annum while $S_1^{(u)}(T)$ is quoted in percent per annum. Note that spreads are independent of the nominal size of the portfolio.

2.3. **The index CDS spread.** Consider the same synthetic CDO as above. An index CDS with maturity T, has almost the same structure as a corresponding CDO tranche, but with two main differences. First, the protection is on *all* credit losses that occurs in the CDO portfolio up to time T, so in the protection leg, the tranche loss $L_t^{(\gamma)}$ is replaced by the total loss L_t. Secondly, in the premium leg, the spread is paid on a notional proportional to the number of obligors left in the portfolio at each payment date. Thus, if N_t denotes the number of obligors that have defaulted up to time t, i.e $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$, then the index CDS spread $S(T)$ is paid on the notional $(1 - \frac{N_t}{m})$. Since the rest of the contract has the same structure as a CDO tranche, the value of the premium leg $W(T)$ is

$$W(T) = S(T)\sum_{n=1}^{n_T} B_{t_n}\left(1 - \frac{1}{m}\mathbb{E}\left[N_{t_n}\right]\right)\Delta_n$$

and the value of the protection leg, $V(T)$, is given by $V(T) = B_T \mathbb{E}\left[L_T\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t\right] dt$. The index CDS spread $S(T)$ is determined so that $V(T) = W(T)$ which implies

$$S(T) = \frac{B_T \mathbb{E}\left[L_T\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t\right] dt}{\sum_{n=1}^{n_T} B_{t_n}\left(1 - \frac{1}{m}\mathbb{E}\left[N_{t_n}\right]\right)\Delta_n} \tag{2.3.1}$$

where $\frac{1}{m}\mathbb{E}\left[N_t\right] = \frac{1}{1-\phi}\mathbb{E}\left[L_t\right]$ if $\phi_1 = \phi_2 = \ldots = \phi_m = \phi$. The spread $S(T)$ is quoted in bp per annum and is independent of the nominal size of the portfolio.

2.4. **The expected tranche losses.** From Subsection 2.2 we see that to compute tranche spreads we have to compute $\mathbb{E}\left[L_t^{(\gamma)}\right]$, that is, the expected loss of the tranche $[k_{\gamma-1}, k_\gamma]$ at time t. If we let $F_{L_t}(x) = \mathbb{P}\left[L_t \leq x\right]$ then (2.1.2) implies that

$$\mathbb{E}\left[L_t^{(\gamma)}\right] = (k_\gamma - k_{\gamma-1})\,\mathbb{P}\left[L_t > k_\gamma\right] + \int_{k_{\gamma-1}}^{k_\gamma} (x - k_{\gamma-1})\,dF_{L_t}(x). \qquad (2.4.1)$$

Hence, in order to compute $\mathbb{E}\left[L_t^{(\gamma)}\right]$ and $\mathbb{E}\left[L_t\right]$ and we must know the loss distribution $F_{L_t}(x)$ at time t. Furthermore, if the recoveries are nonhomogeneous, then to determine the index CDS spread, we also must compute $\mathbb{E}\left[N_{t_n}\right]$, which is equivalent to finding the default distributions $\mathbb{P}\left[\tau_i \leq t\right]$ for all obligors, or alternatively determining the distributions $\mathbb{P}\left[T_k \leq t\right]$ for all ordered default times T_k.

To find analytical expressions for expected tranche losses, expected losses, and thus for tranche spreads and index CDS spread, is the main objective in this paper.

3. Intensity based models reinterpreted as Markov jump processes

In this section we define the intensity-based model for default contagion which is used throughout the paper. The model is then translated into a Markov jump process. This makes it possible to use a matrix-analytic approach to derive computationally convenient formulas for CDO tranche spreads, index CDS spreads, single-name CDS spreads and k^{th}-to-default spreads. The model presented here is identical to the setup in [17] where the authors study aspects of kth-to-default spreads in nonsymmetric as well as in symmetric portfolios. In this paper we focus on synthetic CDO trances, index CDS and k^{th}-to-default swaps on subportfolios to the CDO portfolio.

With $\tau_1, \tau_2 \ldots, \tau_m$ default times as above, define the point process $N_{t,i} = 1_{\{\tau_i \leq t\}}$ and introduce the filtrations

$$\mathcal{F}_{t,i} = \sigma\left(N_{s,i}; s \leq t\right), \quad \mathcal{F}_t = \bigvee_{i=1}^m \mathcal{F}_{t,i}.$$

Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$. Below, we for convenience often omit the filtration and just write intensity or "default intensity". With a further extension of language we will sometimes also write that the default times $\{\tau_i\}$ have intensities $\{\lambda_{t,i}\}$. The model studied in this paper is specified by requiring that the default intensities have the form,

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \qquad \tau_i \geq t, \qquad (3.1)$$

and $\lambda_{t,i} = 0$ for $t > \tau_i$. Further, $a_i \geq 0$ and $b_{i,j}$ are constants such that $\lambda_{t,i}$ is non-negative.

The financial interpretation of (3.1) is that the default intensities are constant, except at the times when defaults occur: then the default intensity for obligor i jumps by an amount $b_{i,j}$ if it is obligor j which has defaulted. Thus a positive $b_{i,j}$ means that obligor i is put at higher risk by the default of obligor j, while a negative $b_{i,j}$ means that obligor i in fact

benefits from the default of j, and finally $b_{i,j} = 0$ if obligor i is unaffected by the default of j.

Equation (3.1) determines the default times through their intensities. However, the expressions for the loss and tranche losses are in terms of their joint distributions. It is by no means obvious how to go from one to the other. Here we will use the following result, proved in [17].

Proposition 3.1. *There exists a Markov jump process* $(Y_t)_{t \geq 0}$ *on a finite state space* $\boldsymbol{E}$ *and a family of sets* $\{\Delta_i\}_{i=1}^m$ *such that the stopping times*

$$\tau_i = \inf\{t > 0 : Y_t \in \Delta_i\}, \qquad i = 1, 2, \ldots, m,$$

have intensities (3.1). Hence, any distribution derived from the multivariate stochastic vector $(\tau_1, \tau_2, \ldots, \tau_m)$ *can be obtained from* $\{Y_t\}_{t \geq 0}$.

Each state $\boldsymbol{j}$ in $\boldsymbol{E}$ is of the form $\boldsymbol{j} = \{j_1, \ldots j_k\}$ which is a subsequence of $\{1, \ldots m\}$ consisting of k integers, where $1 \leq k \leq m$. The interpretation is that on $\{j_1, \ldots j_k\}$ the obligors in the set have defaulted. The Markov jump process Y_t on $\boldsymbol{E}$ is specified by making $\{1, \ldots m\}$ absorbing and starting in $\{0\}$.

In this paper, Proposition 3.1 is throughout used for computing distributions. However, we still use Equation (3.1) to describe the dependencies in a credit portfolio since it is more compact and intuitive. In the sequel, we let $\boldsymbol{Q}$ and $\boldsymbol{\alpha}$ denote the generator and initial distribution on $\boldsymbol{E}$ for the Markov jump process in Proposition 3.1. The generator $\boldsymbol{Q}$ is found by using the structure of $\boldsymbol{E}$, the definition of the states $\boldsymbol{j}$, and Equation (3.1), see [17]. By construction $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$. Further, if $\boldsymbol{j}$ belongs to $\boldsymbol{E}$ then $\boldsymbol{e_j}$ denotes a column vector in $\mathbb{R}^{|\boldsymbol{E}|}$ where the entry at position $\boldsymbol{j}$ is 1 and the other entries are zero. From Markov theory we know that $\mathbb{P}[Y_t = \boldsymbol{j}] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e_j}$ were $e^{\boldsymbol{Q}t}$ is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of $\boldsymbol{Q}$.

4. Using the matrix-analytic approach to find CDO tranche spreads and index CDS spreads

In this section we derive practical formulas for CDO tranche spreads and index CDS spreads. This is done under (3.1) together with the standard assumption of deterministic recovery rates and constant interest rate r. Although the derivation is done in an inhomogeneous portfolio, we will in Section 5 show that these formulas are almost the same in a homogeneous model.

The following observation is a key to all results in this article. If the obligors in a portfolio satisfy (3.1) and have deterministic recoveries, then Proposition 3.1 implies that the corresponding loss L_t can be represented as a functional of the Markov jump process Y_t, $L_t = L(Y_t)$ where the mapping L goes from $\boldsymbol{E}$ to all possible loss-outcomes determined via (2.1.1). For example, if $\boldsymbol{j} \in \boldsymbol{E}$ where $\boldsymbol{j} = \{j_1, \ldots j_k\}$ then $L(\boldsymbol{j}) = \frac{1}{m} \sum_{n=1}^k (1 - \phi_{j_n})$. The range of L is a finite set since the recoveries are deterministic. This implies that for any mapping $g(x)$ on $\mathbb{R}$ and a set A in $[0, \infty)$, we have

$$\int_A g(x) dF_{L_t}(x) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{h}(g, A)$$

where $\boldsymbol{h}(g, A)$ is a column vector in $\mathbb{R}^{|\boldsymbol{E}|}$ defined by $\boldsymbol{h}(g, A)_{\boldsymbol{j}} = g(L(\boldsymbol{j}))1_{\{L(\boldsymbol{j}) \in A\}}$. From this we obtain the following easy lemma, which is stated since it provides notation which is needed later on.

Lemma 4.1. *Consider a synthetic CDO on a portfolio with m obligors that satisfy (3.1). Then, with notation as above,*

$$\mathbb{E}\left[L_t^{(\gamma)}\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{\ell}^{(\gamma)} , \quad \mathbb{E}\left[L_t\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{\ell} \quad and \quad \mathbb{E}\left[N_t\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\sum_{i=1}^{m}\boldsymbol{h}^{(i)}$$

where $\boldsymbol{\ell}^{(\gamma)}$ is a column vector in $\mathbb{R}^{|\boldsymbol{E}|}$ defined by

$$\boldsymbol{\ell}_{\boldsymbol{j}}^{(\gamma)} = \begin{cases} 0 & if \quad L(\boldsymbol{j}) < k_{\gamma-1} \\ L(\boldsymbol{j}) - k_{\gamma-1} & if \quad L(\boldsymbol{j}) \in [k_{\gamma-1}, k_{\gamma}] \\ \Delta k_{\gamma} & if \quad L(\boldsymbol{j}) > k_{\gamma} \end{cases} \tag{4.1}$$

and L is the mapping such that $L_t = L(Y_t)$. Furthermore, $\boldsymbol{\ell}$ and $\boldsymbol{h}^{(i)}$ are column vectors in $\mathbb{R}^{|\boldsymbol{E}|}$ defined by $\boldsymbol{\ell}_{\boldsymbol{j}} = L(\boldsymbol{j})$ and $\boldsymbol{h}_{\boldsymbol{j}}^{(i)} = 1_{\{\boldsymbol{j} \in \Delta_i\}}$ where the sets Δ_i are as in Proposition 3.1.

We now present the main results of this paper.

Proposition 4.2. *Consider a synthetic CDO on a portfolio with m obligors that satisfy (3.1) and assume that the interest rate r is constant. Then, with notation as above,*

$$S_\gamma(T) = \frac{\left(\boldsymbol{\alpha}e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0, T)r\right)\boldsymbol{\ell}^{(\gamma)}}{\sum_{n=1}^{n_T} e^{-rt_n}\left(\Delta k_\gamma - \boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}\boldsymbol{\ell}^{(\gamma)}\right)\Delta_n} \quad \gamma = 2, \ldots, \kappa \tag{4.2}$$

and

$$S_1^{(u)}(T) = \frac{1}{k_1}\left(\boldsymbol{\alpha}e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0, T)r + 0.05\sum_{n=1}^{n_T}\boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}e^{-rt_n}\Delta_n\right)\boldsymbol{\ell}^{(1)} - 0.05\sum_{n=1}^{n_T}e^{-rt_n}\Delta_n \tag{4.3}$$

where

$$\boldsymbol{R}(0, T) = \int_0^T e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt = \left(e^{\boldsymbol{Q}T}e^{-rT} - \boldsymbol{I}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}. \tag{4.4}$$

Furthermore,

$$S(T) = \frac{\left(\boldsymbol{\alpha}e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0, T)r\right)\boldsymbol{\ell}}{\sum_{n=1}^{n_T} e^{-rt_n}\left(1 - \boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}\widehat{\boldsymbol{\ell}}\right)\Delta_n} \tag{4.5}$$

where

$$\widehat{\boldsymbol{\ell}} = \begin{cases} \frac{1}{1-\phi}\boldsymbol{\ell} & if \ \phi_1 = \phi_2 = \ldots = \phi_m = \phi \\ \frac{1}{m}\sum_{i=1}^{m}\boldsymbol{h}^{(i)} & otherwise \end{cases} . \tag{4.6}$$

Proof. Since $r_t = r$, using Lemma 4.1 we have that

$$\int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right] dt = \boldsymbol{\alpha}\int_0^T e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt\boldsymbol{\ell}^{(\gamma)}r = \boldsymbol{\alpha}\boldsymbol{R}(0, T)\boldsymbol{\ell}^{(\gamma)}r$$

where $\boldsymbol{R}(0,T)$ is given by (4.4). So by Lemma 4.1 again, we get

$$V_\gamma(T) = B_T \mathbb{E}\left[L_T^{(\gamma)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right] dt = \left(\boldsymbol{\alpha} e^{\boldsymbol{Q}T} e^{-rT} + \boldsymbol{\alpha} \boldsymbol{R}(0,T)r\right)\boldsymbol{\ell}^{(\gamma)}$$

and

$$W_\gamma(T) = S_\gamma(T) \sum_{n=1}^{n_T} B_{t_n}\left(\Delta k_\gamma - \mathbb{E}\left[L_{t_n}^{(\gamma)}\right]\right)\Delta_n = S_\gamma(T)\sum_{n=1}^{n_T} e^{-rt_n}\left(\Delta k_\gamma - \boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}\boldsymbol{\ell}^{(\gamma)}\right)\Delta_n.$$

Recall that for all tranches γ, except for the equity tranche, the spreads $S_\gamma(T)$ are determined so that $V_\gamma(T) = W_\gamma(T)$. Thus, the equations above prove (4.2). Furthermore, for the equity tranche, $S_1(T)$ is set to 500 bp and the up-front premium $S_1^{(u)}(T)$ is determined so that $V_1(T) = S_1^{(u)}(T)k_1 + W_1(T)$. The expressions for $V_1(T)$ and $W_1(T)$ together with the fact that $\Delta k_1 = k_1$ then imply that $S_1^{(u)}(T)$ is given by

$$S_1^{(u)}(T) = \frac{1}{k_1}\left[B_T \mathbb{E}\left[L_T^{(1)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(1)}\right] dt - 0.05 \sum_{n=1}^{n_T} B_{t_n}\left(\Delta k_1 - \mathbb{E}\left[L_{t_n}^{(1)}\right]\right)\Delta_n\right]$$

$$= \frac{1}{k_1}\left[\left(\boldsymbol{\alpha}e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r\right)\boldsymbol{\ell}^{(1)} - 0.05\sum_{n=1}^{n_T} e^{-rt_n}\left(\Delta k_1 - \boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}\boldsymbol{\ell}^{(1)}\right)\Delta_n\right]$$

$$= \frac{1}{k_1}\left(\boldsymbol{\alpha}e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r + 0.05\sum_{n=1}^{n_T}\boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}e^{-rt_n}\Delta_n\right)\boldsymbol{\ell}^{(1)} - 0.05\sum_{n=1}^{n_T} e^{-rt_n}\Delta_n$$

which establish (4.3). Finally, to find expressions for the index CDS spreads $S(T)$, recall that this contract is almost identical to a CDO tranche (see (2.3.1)), with the differences that $\boldsymbol{\ell}^{(\gamma)}$ is replaced by $\boldsymbol{\ell}$ in the protection leg, and in the premium leg Δk_γ is replaced by 1 and $\boldsymbol{\ell}^{(\gamma)}$ by $\widehat{\boldsymbol{\ell}}$, where

$$\widehat{\boldsymbol{\ell}} = \begin{cases} \frac{1}{1-\phi}\boldsymbol{\ell} & \text{if } \phi_1 = \phi_2 = \ldots = \phi_m = \phi \\ \frac{1}{m}\sum_{i=1}^m \boldsymbol{h}^{(i)} & \text{otherwise} \end{cases}$$

which proves (4.5) and (4.6). $\square$

   The message of Proposition 4.2 is that under (3.1), computations of CDO tranche spreads and index CDS spreads are reduced to compute the matrix exponential. Finding the generator $\boldsymbol{Q}$ and column vectors $\boldsymbol{\ell}^{(\gamma)}$, $\boldsymbol{\ell}$, $\widehat{\boldsymbol{\ell}}$ are straightforward and the matrix $(\boldsymbol{Q} - r\boldsymbol{I})$ is invertible since it is upper diagonal with strictly negative diagonal elements, see [17]. Computing $e^{\boldsymbol{Q}t}$ efficiently is a numerical issue, which for large state spaces requires special treatment, see [17]. For small state spaces, typically less then 150 states, the task is straightforward using standard mathematical software. Several computational shortcuts are possible in Proposition 4.2. The quantities $\boldsymbol{\ell}^{(\gamma)}$, $\boldsymbol{\ell}$ and $\widehat{\boldsymbol{\ell}}$ do not depend on the parametrization, and hence only have to be computed once. The row vectors $\boldsymbol{\alpha}e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r$ and $\sum_{n=1}^{n_T}\boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}e^{-rt_n}\Delta_n$ are the same for all CDO tranche spreads and index CDS spreads and hence only have to be computed once for each parametrization determined by (3.1). In

particular note that $\sum_{n=1}^{n_T} \boldsymbol{\alpha} e^{\boldsymbol{Q} t_n} e^{-r t_n} \Delta_n$ and $(\boldsymbol{Q} - r\boldsymbol{I})^{-1}$ also appears in the expressions for single-name CDS spreads and k^{th}-to-default spreads studied in [17].

In a nonhomogeneous portfolio we have $|\boldsymbol{E}| = 2^m$ which in practice will force us to work with portfolios of size m less or equal to 25, say ([17] used $m = 15$). Standard synthetic CDO portfolios typically contains 125 obligors so we will therefore, in Section 5 below, consider a special case of (3.1) which leads to a symmetric portfolio where the state space $\boldsymbol{E}$ can be simplified to make $|\boldsymbol{E}| = m + 1$. This allows us to practically work with the Markov setup in Proposition 4.2 for large m, where $m \geq 125$ with no further complications. Using homogeneous credit portfolio models when pricing CDO tranches is currently standard in almost all credit literature today.

5. A homogeneous portfolio

In this section we apply the results from Section 4 to a homogenous portfolio. First, Subsection 5.1 introduces a symmetric model and shows how it can be applied to price CDO tranche spreads and index CDS spreads. Subsection 5.2 presents formulas for the single-name CDS spread in this model. Finally, Subsection 5.3 is devoted to formulas for k^{th}-to-default swaps on subportfolios of the main portfolio. This problem is slightly different from the corresponding task in previous studies, e.g. [15] and [17], since the obligors undergo default contagion both from the subportfolio and from obligors outside the subportfolio, in the main portfolio.

5.1. **The homogeneous model for CDO tranches and index CDS-s.** In this subsection we use the results from Section 4 to compute CDO tranche spreads and index CDS spreads in a totally symmetric model. We consider a special case of (3.1) where all obligors have the same default intensities $\lambda_{t,i} = \lambda_t$ specified by parameters a and $b_1, \ldots, b_m$, as

$$\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}} \qquad (5.1.1)$$

where $\{T_k\}$ is the ordering of the default times $\{\tau_i\}$ and $\phi_1 = \ldots = \phi_m = \phi$ where ϕ is constant. In this model the obligors are exchangeable. The parameter a is the base intensity for each obligor i, and given that $\tau_i > T_k$, then b_k is how much the default intensity for each remaining obligor jump at default number k in the portfolio. We start with the simpler version of Proposition 3.1.

Corollary 5.1. *There exists a Markov jump process $(Y_t)_{t \geq 0}$ on a finite state space $\boldsymbol{E} = \{0, 1, 2, \ldots, m\}$, such that the stopping times*

$$T_k = \inf \{t > 0 : Y_t = k\}, \quad k = 1, \ldots, m$$

are the ordering of m exchangeable stopping times $\tau_1, \ldots, \tau_m$ with intensities (5.1.1).

Proof. If $\{T_k\}$ is the ordering of m default times $\{\tau_i\}$ with default intensities $\{\lambda_{t,i}\}$, then the arrival intensity $\lambda_t^{(k)}$ for T_k is zero outside of $\{T_{k-1} \leq t < T_k\}$, otherwise

$$\lambda_t^{(k)} = \left(\sum_{i=1}^{m} \lambda_{t,i} \right) 1_{\{T_{k-1} \leq t < T_k\}}. \tag{5.1.2}$$

Hence, since $\lambda_{t,i} = \lambda_t$ for every obligor i where $\tau_i \geq t$, (5.1.2) implies

$$\lambda_t 1_{\{T_{k-1} \leq t < T_k\}} = \frac{\lambda_t^{(k)}}{m - k + 1}, \quad k = 1, \ldots, m. \tag{5.1.3}$$

Now, let $(Y_t)_{t \geq 0}$ be a Markov jump process on a finite state space $\boldsymbol{E} = \{0, 1, 2, \ldots, m\}$, with generator $\boldsymbol{Q}$ given by

$$\boldsymbol{Q}_{k,k+1} = (m - k) \left(a + \sum_{j=1}^{k} b_j \right) \quad k = 0, 1, \ldots, m-1$$

$$\boldsymbol{Q}_{k,k} = -\boldsymbol{Q}_{k,k+1}, \quad k < m \quad \text{and} \quad \boldsymbol{Q}_{m,m} = 0$$

where the other entries in $\boldsymbol{Q}$ are zero. The Markov process always starts in $\{0\}$ so the initial distribution is $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$. Define the ordered stopping times $\{T_k\}$ as

$$T_k = \inf \{ t > 0 : Y_t = k \}, \quad k = 1, \ldots, m.$$

Then, the intensity $\lambda_t^{(k)}$ for T_k on $\{T_{k-1} \leq t < T_k\}$ is given by $\lambda_t^{(k)} = \boldsymbol{Q}_{k-1,k}$. Further, we can without loss of generality assume that $\{T_k\}$ is the ordering of m exchangeable default times $\{\tau_i\}$, with default intensities $\lambda_{t,i} = \lambda_t$ for every obligor i. Hence, if $\tau_i \geq t$, (5.1.3) implies

$$\lambda_t 1_{\{T_{k-1} \leq t < T_k\}} = \frac{\lambda_t^{(k)}}{m - k + 1} = \frac{\boldsymbol{Q}_{k-1,k}}{m - k + 1} = a + \sum_{j=1}^{k-1} b_j , \quad k = 1, \ldots, m$$

and since $\lambda_t = \sum_{k=1}^{m} \lambda_t 1_{\{T_{k-1} \leq t < T_k\}}$, it must hold that $\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}}$, when $\tau_i \geq t$, which proves the corollary. $\square$

By Corollary 5.1, the states in $\boldsymbol{E}$ can be interpreted as the number of defaulted obligors in the portfolio.

Recall that the formulas for CDO tranche spreads and index CDS spreads in Proposition 4.2 where derived for an inhomogeneous portfolio with default intensities (3.1). However, it is easy to see that these formulas (with identical recoveries) also can be applied in a homogeneous model specified by (5.1.1), but with $\boldsymbol{\ell}^{(\gamma)}$ and $\boldsymbol{\ell}$ slightly refined to match the homogeneous state space $\boldsymbol{E}$. This refinement is shown in the following lemma.

Lemma 5.2. *Consider a portfolio with m obligors that all satisfy (5.1.1) and let $\boldsymbol{E}$, $\boldsymbol{Q}$ and $\boldsymbol{\alpha}$ be as in Corollary 5.1. Then, (4.2), (4.3) and (4.5) hold, for*

$$\boldsymbol{\ell}_k^{(\gamma)} = \begin{cases} 0 & \text{if} \quad k < n_l(k_{\gamma-1}) \\ k(1 - \phi)/m - k_{\gamma-1} & \text{if} \quad n_l(k_{\gamma-1}) \leq k \leq n_u(k_\gamma) \\ \Delta k_\gamma & \text{if} \quad k > n_u(k_\gamma) \end{cases} \tag{5.1.4}$$

where $n_l(x) = \lceil xm/(1 - \phi) \rceil$ and $n_u(x) = \lfloor xm/(1 - \phi) \rfloor$. Furthermore, $\boldsymbol{\ell}_k = k(1 - \phi)/m$.

Proof. Since $L_t = L(Y_t)$ and due to the homogeneous structure, we have

$$\{L_t = k(1 - \phi)/m\} = \{Y_t = k\}$$

for each k in $\boldsymbol{E}$. Hence, the loss process L_t is in one-to-one correspondence with the process Y_t. Define $n_l(x) = \lceil xm/(1 - \phi) \rceil$ and $n_u(x) = \lfloor xm/(1 - \phi) \rfloor$. That is, $n_l(x)$ $(n_u(x))$ is the smallest (biggest) integer bigger (smaller) or equal to $xm/(1 - \phi)$. These observations together with the expression for $\boldsymbol{\ell}^{(\gamma)}$ and $\boldsymbol{\ell}$ in Proposition 4.1, yield (5.1.4). □

In the homogeneous model given by (5.1.1), we have now determined all quantities needed to compute CDO tranche spreads and index CDS spreads as specified in Proposition 4.2.

5.2. **Pricing single-name CDS in a homogeneous model.** If $F(t)$ is the distribution for τ_i, which by exchangeability is the same for all obligors under (5.1.1), then the single-name CDS spread $R(T)$ is given by (see e.g. [17])

$$R(T) = \frac{(1 - \phi) \int_0^T B_t dF(t)}{\sum_{n=1}^{n_T} \left(B_{t_n} \Delta_n (1 - F(t_n)) + \int_{t_{n-1}}^{t_n} B_t (t - t_{n-1}) dF(t) \right)} \tag{5.2.1}$$

where the rest of the notation are the same as in Section 2. Hence, to calibrate, or price single-name CDS-s under (5.1.1), we need the distribution $\mathbb{P}[\tau_i > t]$ (identical for all obligors). This leads to the following lemma.

Lemma 5.3. *Consider m obligors that satisfy (5.1.1). Then, with notation as above*

$$\mathbb{P}[\tau_i > t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{g} \quad \text{and} \quad \mathbb{P}[T_k > t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{m}^{(k)}, \quad k = 1, \ldots, m$$

where $\boldsymbol{m}^{(k)}$ and $\boldsymbol{g}$ are column vectors in $\mathbb{R}^{|\boldsymbol{E}|}$ such that $\boldsymbol{m}_j^{(k)} = 1_{\{j < k\}}$ and $\boldsymbol{g}_j = 1 - j/m$.

Proof. By the construction of T_k in Corollary 5.1, we have

$$\mathbb{P}[T_k > t] = \mathbb{P}[Y_t < k] = \sum_{j=0}^{k-1} \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e}_j = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{m}^{(k)} \quad \text{where} \quad \boldsymbol{m}_j^{(k)} = 1_{\{j < k\}}$$

for $k = 1, \ldots, m$. Furthermore, due to the exchangeability,

$$\mathbb{P}[T_k > t] = \sum_{i=1}^{m} \mathbb{P}[T_k > t, T_k = \tau_i] = m\mathbb{P}[T_k > t, T_k = \tau_i]$$

so

$$\mathbb{P}[\tau_i > t] = \sum_{k=1}^{m} \mathbb{P}[T_k > t, T_k = \tau_i] = \sum_{k=1}^{m} \frac{1}{m} \mathbb{P}[T_k > t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \sum_{k=1}^{m} \frac{1}{m} \boldsymbol{m}^{(k)} = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{g},$$

where $\boldsymbol{g} = \frac{1}{m} \sum_{k=1}^{m} \boldsymbol{m}^{(k)}$. Since $\boldsymbol{m}_j^{(k)} = 1_{\{j < k\}}$ this implies that $\boldsymbol{g}_j = 1 - j/m$ which concludes the proof of the lemma. □

A closed-form expression for $R(T)$ is obtained by using Lemma 5.3 in (5.2.1). For ease of reference we exhibit the resulting formulas (proofs can be found in [15] or [16]).

Proposition 5.4. *Consider m obligors that all satisfies* (5.1.1) *and assume that the interest rate r is constant. Then, with notation as above*

$$R(T) = \frac{(1 - \phi)\boldsymbol{\alpha} \left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)\boldsymbol{g}}{\boldsymbol{\alpha} \left(\sum_{n=1}^{n_T} \left(\Delta_n e^{\boldsymbol{Q}t_n} e^{-rt_n} + \boldsymbol{C}(t_{n-1}, t_n)\right)\right)\boldsymbol{g}}$$

where

$$\boldsymbol{C}(s,t) = s\left(\boldsymbol{A}(t) - \boldsymbol{A}(s)\right) - \boldsymbol{B}(t) + \boldsymbol{B}(s), \qquad \boldsymbol{A}(t) = e^{\boldsymbol{Q}t}\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}$$

and

$$\boldsymbol{B}(t) = e^{\boldsymbol{Q}t}\left(t\boldsymbol{I} + \left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}.$$

For more on the CDS contract, see e.g [10], [15] or [23].

5.3. **Pricing k^{th}-to-default swaps on subportfolios in a homogeneous model.** Consider a homogenous portfolio defined by (5.1.1). Our goal in this subsection is to find expressions for k^{th}-to-default swap spreads on a subportfolio in the main portfolio. The difference in this approach, compared with for example [17] and [12] is that the obligors undergoes default contagion both from entities in the selected basket and from obligors outside the basket, but in the main portfolio.

Let $\boldsymbol{s}$ be a subportfolio of the main portfolio, that is $\boldsymbol{s} \subseteq \{1, 2, \ldots, m\}$ and let $|\boldsymbol{s}|$ denote the number of obligors in $\boldsymbol{s}$ so $|\boldsymbol{s}| \leq m$. The market standard is $|\boldsymbol{s}| = 5$. If the recoveries are homogeneous, it is enough to find the distribution for the ordering of the default times in the basket. Hence, we seek the distributions of the ordered default times in $\boldsymbol{s}$ denoted by $\{T_k^{(\boldsymbol{s})}\}$. The kth-to-default swap spreads $R_k^{(\boldsymbol{s})}(T)$ on $\boldsymbol{s}$ are then given by (see e.g. [17])

$$R_k^{(\boldsymbol{s})}(T) = \frac{(1 - \phi)\int_0^T B_t dF_k^{(\boldsymbol{s})}(t)}{\sum_{n=1}^{n_T}\left(B_{t_n}\Delta_n(1 - F_k^{(\boldsymbol{s})}(t_n)) + \int_{t_{n-1}}^{t_n} B_t\left(t - t_{n-1}\right)dF_k^{(\boldsymbol{s})}(t)\right)} \tag{5.3.1}$$

where $F_k^{(\boldsymbol{s})}(t) = \mathbb{P}\left[T_k^{(\boldsymbol{s})} \leq t\right]$ are the distribution functions for $\{T_k^{\boldsymbol{s}}\}$. The rest of the notation are the same as in Section 2. In Theorem 5.5 below, we derive formulas for the survival distributions of $\{T_k^{(\boldsymbol{s})}\}$. This is done by using the exchangeability, the matrix-analytic approach and the fact that default times in $\boldsymbol{s}$ always coincide with a subsequence of the default times in the main portfolio.

Theorem 5.5. *Consider a portfolio with m obligors that satisfy* (5.1.1) *and let $\boldsymbol{s}$ be an arbitrary subportfolio with $|\boldsymbol{s}|$ obligors. Then, with notation as above*

$$\mathbb{P}\left[T_k^{(\boldsymbol{s})} > t\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{m}^{k,\boldsymbol{s}} \quad for \quad k = 1, 2, \ldots, |\boldsymbol{s}| \tag{5.3.2}$$

where

$$\boldsymbol{m}_j^{k,\boldsymbol{s}} = \begin{cases} 1 & if \quad j < k \\ 1 - \sum_{\ell=k}^{j \wedge |\boldsymbol{s}|} \frac{\binom{|\boldsymbol{s}|}{\ell}\binom{m-|\boldsymbol{s}|}{j-\ell}}{\binom{m}{j}} & if \quad j \geq k. \end{cases} \tag{5.3.3}$$

Proof. The events $\{T_\ell > t\}$ and $\left\{T_k^{(s)} = T_\ell\right\}$ are independent where $k \leq \ell \leq m - |s| + k$. To motivate this, note that since all obligors are exchangeable, the information $\left\{T_k^{(s)} = T_\ell\right\}$ will not influence the event $\{T_\ell > t\}$. Thus, $\mathbb{P}\left[T_\ell > t, T_k^{(s)} = T_\ell\right] = \mathbb{P}\left[T_\ell > t\right] \mathbb{P}\left[T_k^{(s)} = T_\ell\right]$. This observations together with Lemma 5.3 implies that

$$
\begin{aligned}
\mathbb{P}\left[T_k^{(s)} > t\right] &= \sum_{\ell=k}^{m-|s|+k} \mathbb{P}\left[T_k^{(s)} > t, T_k^{(s)} = T_\ell\right] \\
&= \sum_{\ell=k}^{m-|s|+k} \mathbb{P}\left[T_k^{(s)} = T_\ell\right] \mathbb{P}\left[T_\ell > t\right] \\
&= \sum_{\ell=k}^{m-|s|+k} \mathbb{P}\left[T_k^{(s)} = T_\ell\right] \boldsymbol{\alpha} e^{\boldsymbol{Q} t} \boldsymbol{m}^{(\ell)} = \boldsymbol{\alpha} e^{\boldsymbol{Q} t} \boldsymbol{m}^{k,s}
\end{aligned}
$$

where

$$
\boldsymbol{m}^{k,s} = \sum_{\ell=k}^{m-|s|+k} \mathbb{P}\left[T_k^{(s)} = T_\ell\right] \boldsymbol{m}^{(\ell)}.
$$

Using this and the definition of $\boldsymbol{m}_j^{(\ell)}$ renders

$$
\boldsymbol{m}_j^{k,s} = \begin{cases} 1 & \text{if} \quad j < k \\ 1 - \sum_{\ell=k}^{j} \mathbb{P}\left[T_k^{(s)} = T_\ell\right] & \text{if} \quad j \geq k \end{cases}
$$

and in order to compute $\boldsymbol{m}_j^{k,s}$ for $j \geq k$, note that

$$
\bigcup_{\ell=k}^{j} \left\{T_k^{(s)} = T_\ell\right\} = \left\{k \leq N_j^{(s)} \leq j \wedge |s|\right\}
$$

where $N_j^{(s)}$ is defined as $N_j^{(s)} = \sup\left\{n : T_n^{(s)} \leq T_j\right\}$, that is, the number of obligors that have defaulted in the subportfolio s up to the j-th default in the main portfolio. Due to the exchangeability, $N_j^{(s)}$ is a hypergeometric random variable with parameters m, j and $|s|$. Hence,

$$
\sum_{\ell=k}^{j} \mathbb{P}\left[T_k^{(s)} = T_\ell\right] = \sum_{\ell=k}^{j \wedge |s|} \mathbb{P}\left[N_j^{(s)} = \ell\right] = \sum_{\ell=k}^{j \wedge |s|} \frac{\binom{|s|}{\ell}\binom{m-|s|}{j-\ell}}{\binom{m}{j}}.
$$

which proves the theorem. $\qquad\square$

Returning to k^{th}-to-default swap spreads, expressions for $R_k^{(s)}(T)$ may be obtained by inserting (5.3.2) into (5.3.1). The notation and proof are the same as in Proposition 5.4

Corollary 5.6. *Consider a portfolio with m obligors that satisfy* (5.1.1) *and let $\boldsymbol{s}$ be an arbitrary subportfolio with $|\boldsymbol{s}|$ obligors. Assume that the interest rate r is constant. Then, with notation as above,*

$$R_k^{(\boldsymbol{s})}(T) = \frac{(1-\phi)\boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)\boldsymbol{m}^{k,\boldsymbol{s}}}{\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1}, t_n)\right)\right)\boldsymbol{m}^{k,\boldsymbol{s}}}, \quad k = 1, 2, \ldots, |\boldsymbol{s}|.$$

For a more detailed description of kth-to-default swap, see e.g. [10], [15], [17] or [23].

6. NUMERICAL STUDY OF A HOMOGENEOUS PORTFOLIO

In this section we calibrate the homogeneous portfolio to real market data on CDO tranches, index CDS-s, average single-name CDS spreads and average FtD-spreads (i.e average 1th-to-default swaps). We match the theoretical spreads against the corresponding market spreads for individual default intensities given by (5.1.1). First, in Subsection 6.1 we give an outline of the calibration technique used in this paper. Then, in Subsection 6.2 we calibrate our model against an example studied in several articles, e.g [12] and [18], with data from iTraxx Europe, August 4, 2004. The iTraxx Europe spreads has changed drastically in the period between August 2004 and November 2006. We therefore recalibrate our model to a more recent data set, collected at November 28th, 2006. This second calibration also lends some confidence to the robustness of our model.

Having calibrated the portfolio, we can compute spreads for exotic credit derivatives, not liquidly quoted on the market, as well as other quantities relevant for credit portfolio management. In Subsection 6.3 we compute spreads for tranchelets, which are CDO tranches with smaller loss-intervals than standardized tranches. Subsection 6.4 investigates k^{th}-to-default swap spreads as function of the size of the underlying subportfolio in main calibrated portfolio. Continuing, Subsection 6.5 studies the implied expected loss in the portfolio and the implied expected tranche-losses. Finally, Subsection 6.6 is devoted to explore the implied loss-distribution as function of time.

6.1. **Some remarks on the calibration.** The symmetric model (5.1.1) can contain at most m different parameters. Our goal is to achieve a "perfect fit" with as many parameters as there are market spreads used in the calibration for a fixed maturity T. For a standard synthetic CDO such as the iTraxx Europe series, we can have 5 tranche spreads, the index CDS spread, the average single-name CDS spread and the average FtD spread. Hence, for calibration, there is at most 8 market prices with maturity T available. However, all of them do not have to be used. We make the following assumption on the parameters b_k for $1 \leq k \leq m-1$

$$b_k = \begin{cases} b^{(1)} & \text{if} \quad 1 \leq k < \mu_1 \\ b^{(2)} & \text{if} \quad \mu_1 \leq k < \mu_2 \\ \vdots \\ b^{(c)} & \text{if} \quad \mu_{c-1} \leq k < \mu_c = m \end{cases} \tag{6.1.1}$$

where $1, \mu_1, \mu_2, \ldots, \mu_c$ is an partition of $\{1, 2, \ldots, m\}$. This means that all jumps in the intensity at the defaults $1, 2, \ldots, \mu_1 - 1$ are same and given by $b^{(1)}$, all jumps in the intensity

at the defaults $\mu_1, \ldots, \mu_2 - 1$ are same and given by $b^{(2)}$ and so on. This is a simple way of reducing the number of unknown parameters from m to $c + 1$.

If η is the number of calibration-instruments, that is the number of credit derivatives used in the calibration, we set $c = \eta - 1$. Let $\boldsymbol{a} = (a, b^{(1)}, \ldots, b^{(c)})$ denote the parameters describing the model and let $\{C_j(T; \boldsymbol{a})\}$ be the η different model spreads for the instruments used in the calibration and $\{C_{j,M}(T)\}$ the corresponding market spreads. In $C_j(T; \boldsymbol{a})$ we have emphasized that the model spreads are functions of $\boldsymbol{a} = (a, b^{(1)}, \ldots, b^{(c)})$ but suppressed the dependence of interest rate, payment frequency, etc. The vector $\boldsymbol{a}$ is then obtained as

$$\boldsymbol{a} = \operatorname*{argmin}_{\hat{\boldsymbol{a}}} \sum_{j=1}^{\eta} \left(C_j(T; \hat{\boldsymbol{a}}) - C_{j,M}(T)\right)^2 \tag{6.1.2}$$

with the constraint that all elements in $\boldsymbol{a}$ are nonnegative. Note that it would have been possible to let the jump parameters b_k be negative, as long as $\lambda_t > 0$ for all t. In economic terms this would mean that the non-defaulted obligors benefit from the default at T_k.

The model spreads $\{C_j(T; \boldsymbol{a})\}$, such as average CDS spread $R(T; \boldsymbol{a})$, index CDS spread $S(T; \boldsymbol{a})$, CDO tranche spreads $\{S_\gamma(T; \boldsymbol{a})\}$ etc. are given in closed formulas derived in the previous sections. We use Padé-approximation with scaling and squaring, (see [24]) to compute the matrix exponential, since in the present setting, it outperforms all other methods, both in computational time and accuracy. Note that this is not the case for a nonhomogeneous portfolio with a large state space $\boldsymbol{E}$, where the uniformization method is better, see [17]. The reason for the lesser performance of the uniformization method in the homogeneous CDO model is that the quantity $\max\left\{|\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}}| : \boldsymbol{j} \in \boldsymbol{E}\right\}$ is very large, which introduces many terms in the approximation of the matrix exponential.

The initial parameters in the calibration can be rather arbitrary. The "optimal solution" for this first iteration, is taken as a the initial value in a new calibration. Repeating this procedure one, or if needed, two or three times, have in our numerical examples (see next subsections) always lead to perfect calibrations. Finding good initial parameters when using the model in practice is most likely a minor problem. This is due to the fact that calibrations are performed on a daily basis and the initial guess could simply be the optimal solution from the previous calibration.

Finally, it should be mentioned that the calibrated parameters are not likely to be unique. By perturbing the initial guesses, we have been able to get calibrations that are worse, but "close" to the optimal calibration, and where some of the parameters in the calibrated perturbed vector, are very different from the corresponding parameters in the optimal vector. We do not further pursue the discussion of potential nonuniqueness here, but rather conclude that the above phenomena is likely to occur also in other pricing models.

6.2. **Calibration to the iTraxx Europe series.** In this subsection we calibrate our model against credit derivatives on the iTraxx Europe series with maturity of five years. There are five different CDO tranche spreads with tranches $[0, 3], [3, 6], [6, 9], [9, 12]$ and $[12, 22]$, and we also have the index CDS spreads and the average CDS spread.

First, a calibration is done against data taken from iTraxx Europe on August 4, 2004 used in e.g. [12] and [18]. Here, just as in [12] and [18], we set the average CDS spread equal to (i.e. approximated by) the index CDS spread. No market data on FtD spreads are available in this case. The iTraxx Europe spreads has changed drastically since August 2004. We therefore recalibrate our model to a more recent data set, collected at November 28th, 2006. This data also contains the average CDS spread and average FtD spread (see Table 8). All data is taken from `Reuters` on November 28th, 2006 and the bid, ask and mid spreads are displayed in Table 7.

In both calibrations the interest rate is set to 3%, the payment frequency is quarterly and the recovery rate is 40%.

Table 1: iTraxx Europe, August 4th 2004. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0,3]$ spread is quoted in %. All maturities are for five years.

	Market	Model	error (bp)	error (%)
$[0,3]$	27.6	27.6	0.0004514	1.635e-005
$[3,6]$	168	168	0.003321	0.001977
$[6,9]$	70	70.07	0.06661	0.09515
$[9,12]$	43	42.91	0.09382	0.2182
$[12,22]$	20	20.03	0.03304	0.1652
index	42	41.99	0.01487	0.03542
avg CDS	42	41.96	0.04411	0.105
Σ abs.cal.err			0.2562 bp	

Table 2: iTraxx Europe Series 6, November 28th, 2006. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0,3]$ spread is quoted in %. All maturities are for five years.

	Market	Model	error (bp)	error (%)
$[0,3]$	14.5	14.5	0.007266	0.0005011
$[3,6]$	62.5	62.41	0.08523	0.1364
$[6,9]$	18	18.1	0.09727	0.5404
$[9,12]$	7	6.881	0.1193	1.704
$[12,22]$	3	3.398	0.3979	13.26
index	26	26.13	0.1299	0.4997
avg CDS	26.87	26.12	0.7535	2.804
Σ abs.cal.err			1.59 bp	

We choose the partition $\mu_1, \mu_2, \ldots, \mu_6$ so that it roughly coincides with the number of defaults needed to reach the upper attachment point for each tranche, see Table 10 in

Appendix. The numerical values of the calibrated parameters a, obtained via (6.1.2), are shown in Table 9 in Appendix 8.

For both data sets we also performed calibrations where some of the available market spreads were excluded from the fitting and where the model spreads for the omitted instruments were computed with the parameters obtained from the rest of the instruments in the calibration.

There were two reasons for these tests. First, we wanted to explore if the derivatives not used in the calibration, but computed with the parameters obtained from the rest of the instruments, produces model spreads that are close to the corresponding market spreads. Secondly, we wished to investigate the "robustness" of the model, that is, would the model spreads change drastically if we used different calibration instruments. For the August 4th 2004 data set, this was done for two cases. In the first fitting we excluded the index CDS and in the second, the average CDS spread was omitted in the calibration. The sum of the absolute calibration error for the two cases (and sum of total absolute model error, equal to the total calibration error and sum of absolute differences between model and market spreads for instruments not used in the calibration) were approximately 1.14 bp (1.577 bp) and 1.129 bp (1.623 bp) respectively. We can therefore, in all three calibrations, speak of a perfect fit for $T = 5$ years. A superior fit was in this case obtained when both the average CDS spread and index CDS were included, see Table 1

We also performed the same procedure for the November 28th, 2006 data set, but now with one more case since we had one more market observation, the average FtD spread. The sum of the absolute calibration error for the three cases (and sum of total absolute model error) were approximately 1.661 bp (4.096 bp), 0.7527 bp (3.921 bp) and 3.919 bp (3.919 bp), where the last case included the average FtD-spread. Hence, once again, we can in all four cases speak of a perfect fit when $T = 5$. In the 2006-11-28 study we observed that the FtD model spread was very robust, that is, the computed model spreads differed very little after each calibration. This may indicate that the average FtD spread is difficult to calibrate using the model in (5.1.1). To summarize, in both data sets, the best calibrations where obtained when both the index CDS spread and average CDS spread where included, but where the average FtD-spread was excluded, see Tables 1 and 2.

Finally, since the calibrations where performed on two data sets where the corresponding spreads differed substantially, the above observations lend some confidence in the robustness of our model.

6.3. **Pricing tranchelets in a homogeneous model.** As discussed above, a tranchelet is a nonstandard CDO tranche with smaller loss-intervals than standardized tranches, see e.g. [5] or [20]. Tranchelets are typically computed for losses on $[0, 1], [1, 2], \ldots, [5, 6]$. Currently, there are no liquid market for these instruments, so they can still be regarded as somewhat "exotic". Nevertheless, tranchelets have recently become popular and pricing these instruments are done in the same ways as for standard tranches.

In this subsection we compute the five year tranchelet spreads for $[0, 1], \ldots, [11, 12]$, on iTraxx Europe Series 6, November 28th 2006, and iTraxx Europe, August 4th, 2004 as well as the corresponding absolute difference in % of the 2004-08-04 spreads. The computations

Table 3: Tranchelet spreads on iTraxx Europe, November 28th 2006 (Series 6) and August 4th 2004 and the absolute difference in % of the 2004-08-04 spreads. The $[0, 1]$ and $[1, 2]$ spreads are the upfront premiums on the tranche nominals, quoted in % where the running fee is 500 bp. Tranchelets above $[1, 2]$ are expressed in bp. All maturities are five years.

Tranchelet	04/08/04	06/11/28	diff. (in %)
$[0, 1]$	60.85	47.93	21.25
$[1, 2]$	22.43	7.006	68.76
$[2, 3]$	488.9	245.5	49.79
$[3, 4]$	240.9	97.85	59.39
$[4, 5]$	154	54.49	64.61
$[5, 6]$	110.2	35.13	68.12
$[6, 7]$	84.29	24.26	71.22
$[7, 8]$	68.41	17.35	74.65
$[8, 9]$	57.53	12.69	77.94
$[9, 10]$	49.29	9.315	81.1
$[10, 11]$	42.53	6.676	84.3
$[11, 12]$	36.9	4.652	87.39

are done with parameters obtained from the calibrations in the Tables 1 and 2, where all other quantities such as recovery rate, interest rate, payment frequency etc. are the same as in these tables. The $[0, 1]$ and $[1, 2]$ spreads are computed with Equation (4.3) where $\ell^{(1)}$ is replaced by a corresponding column vector adapted for $[0, 1]$, and $[1, 2]$ respectively, given as in Lemma 5.2. Furthermore, in (4.3), k_1 is set to 0.01 for both tranchelets $[0, 1]$ and $[1, 2]$. Tranchelets above $[1, 2]$ are computed with Equation (4.2). It is interesting to note that the average for the three tranchelets between 3 and 6 are 168.4 (2004-08-04) and 62.49 (2006-11-28) which both are close to the corresponding $[3, 6]$ spreads. The same

Table 4: The market spreads (used for calibration) on iTraxx Europe, November 28th 2006 (Series 6) and August 4th 2004 and the absolute difference in % of the 2004-08-04 spreads. All maturities are five years.

	$[0, 3]$	$[3, 6]$	$[6, 9]$	$[9, 12]$	$[12, 22]$	index	avg CDS
04/08/04	27.6	168	70	43	20	42	42
06/11/28	14.5	62.5	18	7	3	26	26.87
diff. (%)	47.46	62.8	74.29	83.72	85	38.1	36.02

holds for the averages of tranchlets between 6 to 9 and 9 to 12, which are 70.08, 18.1 and 42.91, 6.881 respectively. These observations explain why the average of the differences for the three tranchelets between 3 to 6, 6 to 9 and 9 to 12, given by 64 %, 74.6 % and 84.3

%, are close to the corresponding differences in the $[3, 6]$, $[6, 9]$ and $[9, 12]$ tranche spreads, displayed in Table 4.

6.4. **Pricing k^{th}-to-default swaps on subportfolios in a homogeneous model.** In this subsection we price five year k^{th}-to-default spreads $R_k^{(s)}$ with $k = 1, \ldots, 5$ for different subportfolios s, of the main portfolio. The subportfolios have sizes $|s| = 5, 10, 15, 25, 30$ and the computations are done for the two different data sets, iTraxx Europe Series 6, November 28th, 2006 and iTraxx Europe August 4th, 2004. The computations are done with parameters obtained from the calibrations in the Tables 1 and 2 , where all other quantities such as recovery rate, interest rate, payment frequency etc. are the same as in these tables.

Table 5: The five year k^{th}-to-default spreads $R_k^{(s)}$ with $k = 1, \ldots, 5$ for different subportfolios s in the main portfolio calibrated to iTraxx Europe, November 28th 2006 (Series 6) and August 4th 2004 and the absolute difference in % of the 2004-08-04 spreads. We consider $|s| = 5, 10, 15, 25, 30$.

$|s|$	Date	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
5	04/08/04	180.9	25.19	7.002	3.037	1.404
	06/11/28	119	9.597	2.31	1.728	1.59
	diff. (%)	34.19	61.9	67.01	43.09	13.25
10	04/08/04	331	67.94	22.39	10.85	6.35
	06/11/28	226.8	30.6	6.183	2.6	1.937
	diff. (%)	31.47	54.96	72.39	76.03	69.49
15	04/08/04	467.4	117.1	41.91	21.13	12.9
	06/11/28	327.7	58.89	13.69	4.848	2.68
	diff. (%)	29.89	49.7	67.33	77.05	79.22
20	04/08/04	594.6	170.1	64.57	32.96	20.6
	06/11/28	423.1	91.73	24.34	8.69	4.234
	diff. (%)	28.84	46.07	62.31	73.63	79.44
25	04/08/04	714.9	225.5	90.06	46.15	29
	06/11/28	514.1	127.6	37.6	14	6.691
	diff. (%)	28.08	43.42	58.25	69.67	76.93

There exists liquid quoted market spreads on FtD baskets (i.e. $k = 1$) and often the FtD spreads are also quoted in percent of the sum of the individual spreads in the subportfolio s (see Table 8 in Appendix). No market spread on FtD swaps are available for 2004-08-04 but the model FtD-spread is 180.9 bp which is around 86 % of the SoS (sum of spreads) given by $5 \cdot 42 = 210$ bp. As seen in Table 8 in Appendix, this is a very realistic FtD spread in terms of the SoS. Furthermore, for 2006-11-28 we have access to the average FtD market-spread which is 116.8 bp, see Table 8.

From Table 5 we see that, for fixed $\boldsymbol{s}$ and k, the spreads differ substantially between the two dates. Given the difference between the market spreads in the calibration (Table 4), this should not come as a surprise. For example, when $|\boldsymbol{s}| = 5$, $k = 1$ the difference is 34 %, and for $|\boldsymbol{s}| = 15$, $k = 5$ the 2006-11-28 spread is 79 % lower than the 2004-08-04 spread. The spreads increase as the size of the portfolio increases, as they should.

For the 2006-11-28 case, the increase from a portfolio of size 5 to one of size 25 is 432% for a 1st-to-default swap, 1330% for a 2nd-to-default swap, 1628% for a 3rd-to-default swap, and for a 5th-to-default swap the increase is 421%. Further, for a portfolio of size 10 the price of a 1st-to-default swap is about 117 times higher than for a 5th-to-default swap and the corresponding ratio for a portfolio of size 15 is about 122. These ratios are much smaller than for a "isolated" portfolio, which only undergo default contagion from obligors within the basket, see [17]. Qualitatively the above results are completely as expected, however, given market spreads on CDO tranches, index CDS spreads etc. it would seem rather impossible to guess the sizes of the effects without computation.

6.5. **The implied tranche losses and implied loss in a homogeneous portfolios.** In the credit literature today, expected risk-neutral tranche losses are often called *implied* tranche losses. Here "implied" is refering to the fact that the quantities are retrieved from market data via a model. Similarly, the implied portfolio loss refers to the expected risk-neutral portfolio loss. In this subsection we compute the expected risk-neutral portfolio loss and the implied expected tranche losses at different time points.

Table 6: The implied tranche losses in % of tranche nominal, at $t = 3, 5, 7, 10$ for the calibrated CDO portfolios on iTraxx Europe Series 6, November 28th 2006, and iTraxx Europe, August 4th, 2004 and the absolute differences in % of the 2004-08-04 tranche losses.

t	Date	$[0, 3]$	$[3, 6]$	$[6, 9]$	$[9, 12]$	$[12, 22]$
3	04/08/04	26.52	0.7142	0.1014	0.03198	0.005744
	06/11/28	19.31	0.2082	0.01647	0.002157	0.0004121
	diff. (%)	27.18	70.85	83.75	93.26	92.83
5	04/08/04	49.26	8.649	3.67	2.258	1.059
	06/11/28	36.61	3.255	0.954	0.3641	0.1802
	diff. (%)	25.67	62.37	74.01	83.88	82.99
7	04/08/04	69.28	28.61	18.7	14.74	10.13
	06/11/28	54.39	13.7	7.005	4.161	2.9
	diff. (%)	21.49	52.09	62.54	71.78	71.38
10	04/08/04	87.91	63.57	54.27	49.67	43.12
	06/11/28	75.73	40.75	30.24	24.01	20.58
	diff. (%)	13.86	35.89	44.27	51.65	52.28

These are important quantities for a credit manager and Lemma 4.1 and Lemma 5.2 provides formulas for computing them. We study $100 \cdot \mathbb{E}\left[L_t^{(\gamma)}\right] / \Delta k_\gamma$ for $3, 5, 7$ and 10
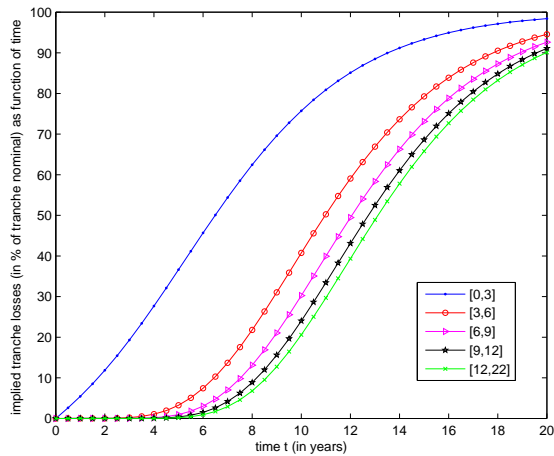
Figure 1: The implied tranche losses in % of tranche nominal for the 2006-11-28 portfolio.
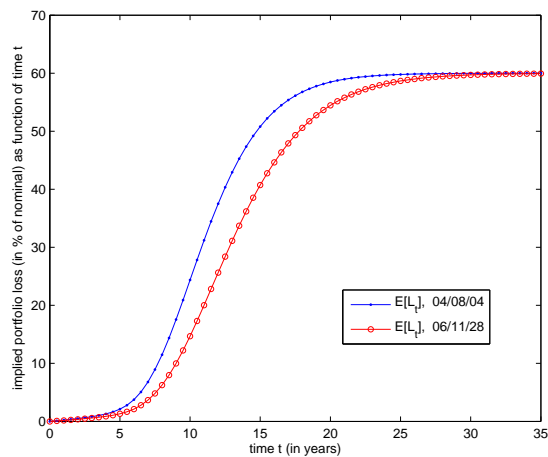
Figure 2: The implied portfolio losses in % of nominal, for the 2004-08-04 and 2006-11-28 portfolios.

years on CDO portfolios calibrated against iTraxx Europe Series 6, November 28th 2006, and iTraxx Europe, August 4th, 2004. Just as for previous computations, the corresponding tranche losses differ substantially between the two dates. For example, in the 2006-11-28 case, the tranche loss on $[0, 3]$ for $t = 3$ is 27 % smaller than the corresponding quantity for the 2004-08-04 collection, but this differences drastically increases for the upper tranches, $[6, 9], [9, 12]$ to 84% and 93%, as seen in Table 6. Further, for the 2006-11-28 case, we clearly

see the effect of default contagion on the upper tranche losses, making them lie close to each other, see Figure 1. From Figure 2 we conclude that our model, with a constant recovery rate of 40%, calibrated to market spreads on the five year iTraxx Europe Series implies that the whole portfolio has defaulted within approximately 30 years (for both data sets). In reality, this will likely not happen, since risk-neutral (implied) default probabilities are substantially larger than the "real", so called actuarial, default probabilities.

6.6. **The implied loss distribution in a homogeneous portfolio.** In this subsection we study the implied distribution for the loss process L_t at different time points. Since we are considering constant recovery rates, then for every t, the distribution of L_t is discrete and formally the values for $\mathbb{P}[L_t = x]$ should be displayed as bars at $x = k(1-\phi)/m$ where $0 \le k \le m$.
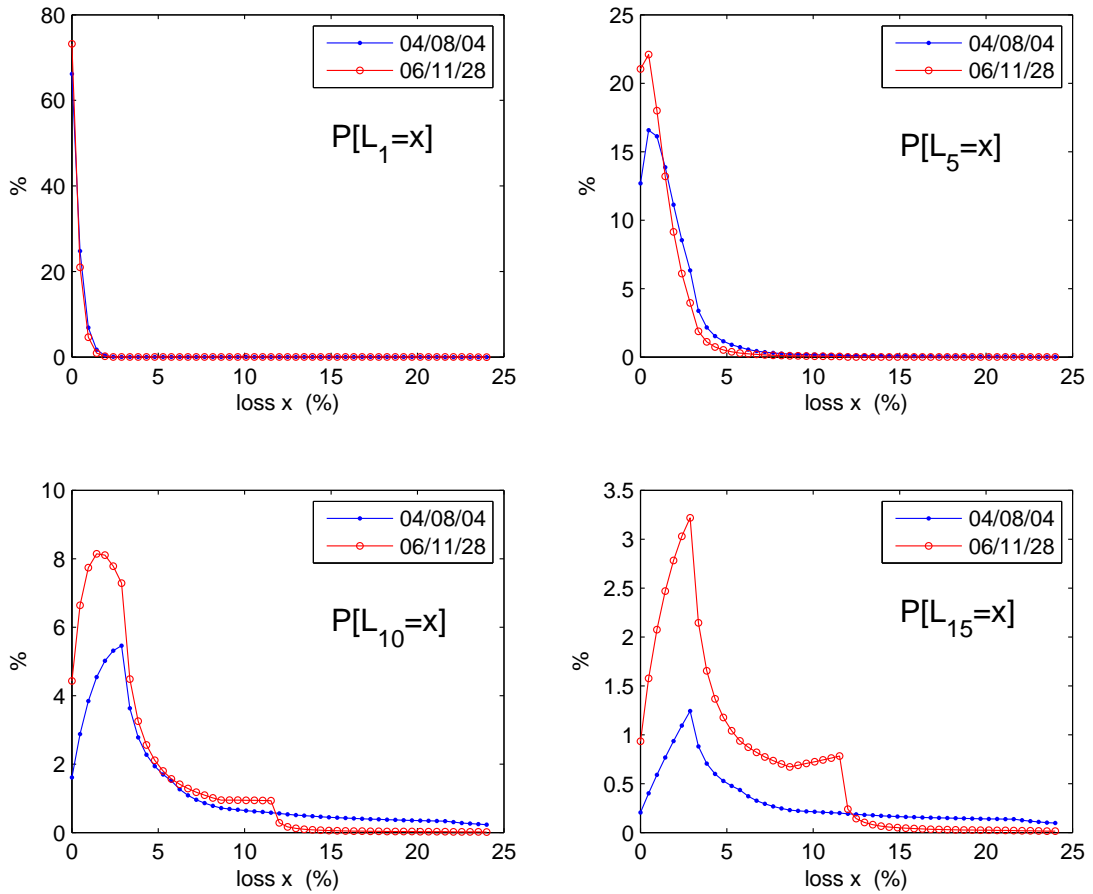
Figure 3: The implied loss distributions for the 2004-08-04 and 2006-11-28 portfolios.
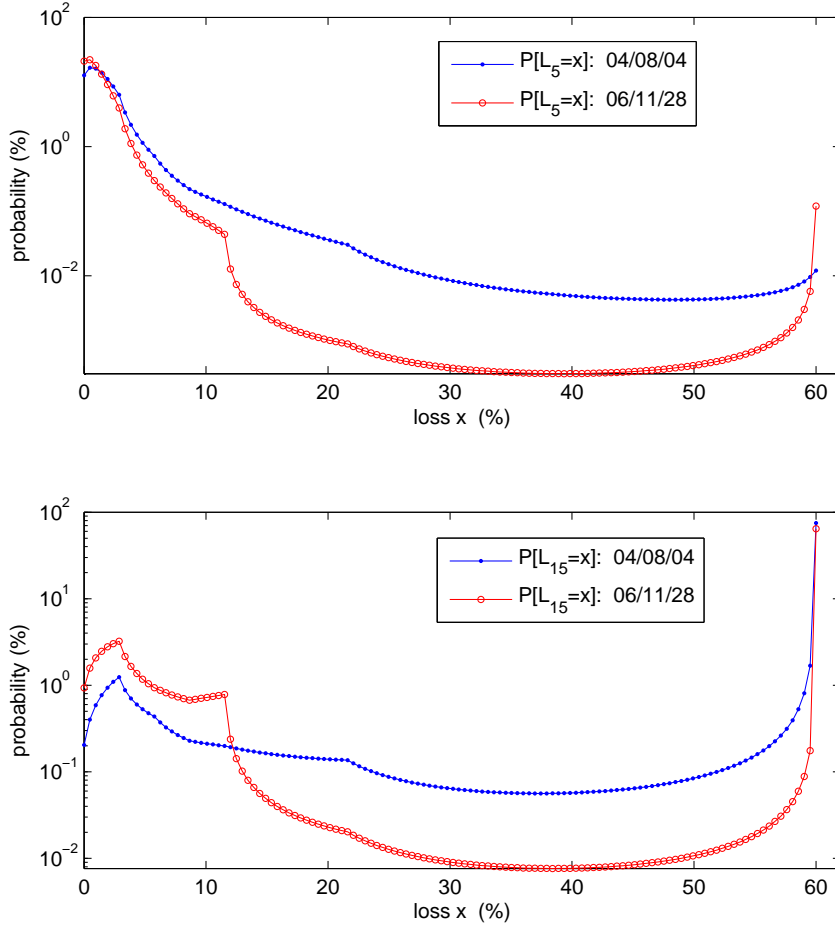
Figure 4: The implied loss distributions (in log-scale) for the 2004-08-04 and 2006-11-28 port-folios.

However, since there are totaly 126 different outcomes we do not bother about this and connect the graph continuously between each discrete probability. The loss probabilities are computed by using that $L_t = L(Y_t)$ so $\mathbb{P}\left[L_t = k(1-\phi)/m\right] = \mathbb{P}\left[Y_t = k\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e}_k$ for $k = 0, 1, \ldots, m$, see Corollary 5.1.

In Figure 3 for $0 < x < 12$, the implied loss probabilities in the 2006-11-28 case are bigger than their 2004-04-28 counterparts, at several occasions in time t, which at first glance may contradict the results in Table 6. However, a more careful study, using a log-scale, shows that for $20 < x < 50$ and at most time points t, the 2004-04-28 loss distribution is about 10 times bigger than the corresponding values for the 2006-11-28 case, see Figure 4. This

supports the results in Table 6 where the expected tranche losses for the 2004-04-28 case are always bigger than in the 2006-11-28 case.

7. Conclusions

In this paper we have derived closed-form expressions for CDO tranche spreads and index CDS spreads. This is done in a inhomogeneous model where dynamic default dependencies among obligors are expressed in an intuitive, direct and compact way. By specializing this model to a homogenous portfolio, we show that the CDO and index CDS formulas simplify considerably in a symmetric model. The same method are used to derive k^{th}-to-default swap spreads for subportfolios in the main CDO portfolio. In this setting, we calibrate a symmetric portfolio against credit derivatives on the iTraxx Europe series for a fixed maturity of five years. We do this at two different dates, where the corresponding market spreads differ substantially. In both cases we obtain perfect fits. These two calibrations therefore lends some confidence to the robustness of our model.

In the calibrated portfolios, we compute tranchelet spreads and investigate k^{th}-to-default swap spreads as function of the portfolio size. Further, the implied tranche losses and the implied loss distributions are also extracted. All these computations and investigations would be difficult to perform without having convenient formulas for the quantities that we want to study. Furthermore, given the recovery rate, the number of model parameters are as many as the market instruments used in the calibration. This implies that all calibrations are performed without inserting "fictitious" numerical values for some of the parameters, making the calibration more realistic.

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8. Appendix

Tables 7 shows the market spreads collected from iTraxx Europe Series 6, November 28th, 2006 and taken from `Reuters`. Table 8 shows the FtD spreads, i.e. 1st-to-defaults spreads for 6 standardized subportfolios on iTraxx Europe Series 6, launched September 20th, 2006. Each basket consist of five obligors that are taken from a sector in the iTraxx Series 6 (Autos, Energy, Industrial, TMT, Consumers and Financial). The names of the obligors in each basket as well as the selection criteria can be found on the webpage for iboxx. In the financial FtD basket, we have used the subordinated FtD spread, since the senior spread is much smaller (30 bp) than the other spreads, which will pull down the average mid FtD spread to 112.25 bp.

The numerical values of the calibrated parameters $\boldsymbol{a}$, obtained via (6.1.2), are shown in Table 9 and the partition (see Equation (6.1.1)) in Table 10.

Table 7: The market bid, ask and mid spreads for iTraxx Europe (Series 6), November 28th, 2006. All data is taken from Reuters. The mid spreads, i.e. average of the bid and ask spread, are used in the calibration in Section 6.

	bid	ask	mid	time
$[0,3]$		14.5	14.5	28 Nov, 18:23
$[3,6]$	60	65	62.5	28 Nov, 17:14
$[6,9]$	16.5	19.5	18	28 Nov, 13:36
$[9,12]$	5.5	8.5	7	28 Nov, 13:36
$[12,22]$	2	4	3	28 Nov, 13:36
index	25.75	26.25	26	28 Nov, 18:34
avg CDS	25.94	27.8	26.87	28 Nov, 19:40

Table 8: The market bid, ask and mid spreads for different FtD spreads on subsectors of iTraxx Europe (Series 6), November 28th, 2006. Each subportfolio have five obligors. We also display the sum of CDS-spreads (SoS) in each basket, as well as the mid FtD spreads in % of SoS. The mid spread is used in the calibration in Section 6.

Sector	bid	ask	mid	SoS	mid/SoS %	time
Autos	154	166	160	202	79.21 %	28 Nov, 10:26
Energy	65	71	68	86	79.07 %	28 Nov, 10:26
Industrial	114	123	118.5	141	84.04 %	28 Nov, 10:26
TMT	167	188	177.5	217	81.8 %	28 Nov, 10:26
Consumers	113	122	117.5	140	83.93 %	28 Nov, 10:26
Financial	55	63	59	79	74.68 %	28 Nov, 10:26
average	111.3	122.2	116.8	144.2	80.98 %	

Table 9: The calibrated parameters that gives the model spreads in the Tables 1 and 2.

	a	$b^{(1)}$	$b^{(2)}$	$b^{(3)}$	$b^{(4)}$	$b^{(5)}$	$b^{(6)}$	
04/08/04	33.0	16.4	84.5	145	86.4	124	514	$\times 10^{-4}$
06/11/28	24.9	13.9	73.6	62.4	0.823	2162	4952	$\times 10^{-4}$

Table 10: The integers $1, \mu_1, \mu_2, \ldots, \mu_c$ are partitions of $\{1, 2, \ldots, m\}$ used in the models that generates the spreads in the Tables 1 and 2

partition	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
	7	13	19	25	46	125

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Paper III

MODELLING DEFAULT CONTAGION USING MULTIVARIATE PHASE-TYPE DISTRIBUTIONS

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ABSTRACT. We model dynamic credit portfolio dependence by using default contagion in an intensity-based framework. Two different portfolios (with 10 obligors), one in the European auto sector, the other in the European financial sector, are calibrated against their market CDS spreads and the corresponding CDS-correlations. After the calibration, which are perfect for the banking portfolio, and good for the auto case, we study several quantities of importance in active credit portfolio management. For example, implied multivariate default and survival distributions, multivariate conditional survival distributions, implied default correlations, expected default times and expected ordered defaults times. The default contagion is modelled by letting individual intensities jump when other defaults occur, but be constant between defaults. This model is translated into a Markov jump process, a so called multivariate phase-type distribution, which represents the default status in the credit portfolio. Matrix-analytic methods are then used to derive expressions for the quantities studied in the calibrated portfolios.

1. INTRODUCTION

In recent years, understanding and modelling default dependency has attracted much interest. A main reason is the incentive to optimize regulatory capital in credit portfolios, provided by new regulatory rules such as Basel II. Another reason is the growing financial market of products whose payoffs are contingent on the default behavior of a whole credit portfolio consisting of, for example, mortgage loans, corporate bonds or single-name credit default swaps (CDS-s).

In this paper we model dynamic credit portfolio dependence by using default contagion and consider two different portfolios, one in the European auto sector, the other in the European financial sector. Both baskets consist of 10 companies which are calibrated against their market CDS spreads and the corresponding CDS correlations, resulting in a

perfect fit for the banking case and good fit for the auto case. We then study the implied joint default and survival distributions, the implied univariate and bivariate conditional survival distributions, the implied default correlations, and the implied expected default times and expected ordered defaults times. These quantities are of importance in active credit portfolio management.

We us an intensity based model where default dependencies among obligors are expressed in an intuitive and compact way. The financial interpretation is that the individual default intensities are constant, except at the times when other defaults occur: then the default intensity for each obligor jumps by an amount representing the influence of the defaulted entity on that obligor. This model is then translated into a Markov jump process, which leads to so called multivariate phase-type distributions, first introduced in [3]. This translation makes it possible to use a matrix-analytic approach to derive practical formulas for all quantities that we want to study. The contribution of this paper is to adapt results from [3] to credit portfolio applications. Special attention is given how to retrieve the model parameters from market CDS spreads and their CDS-correlations.

The framework used here is the same as in [19], where the authors consider CDS and k^{th}-to default spreads and in [18] where the same technique is applied to synthetic CDO tranches and index CDS-s. In this paper however, we focus on multivariate default and survival distributions. As mentioned above, computing such quantities is at the core of active credit portfolio management. The paper is an extension of Chapter 6 in the licentiate thesis [17]. Default contagion in an intensity based setting have previously also been studied in for example [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [21], [22],[23], [24], [25], [26] and [28]. The material in all these papers and books are related to the results discussed here.

The rest of this paper is organized as follows. Section 2 contains the formal definition of default contagion used in this paper, given in terms of default intensities. It is then used to construct such default times as hitting times of a Markov jump process. The joint distribution of these hitting times is called a multivariate phase-type distribution, see [3]. The results in Section 3 give convenient analytical formulas for multivariate default and survival distributions, conditional multivariate distributions, marginal default distributions, multivariate default densities, default correlations, and expected default times. These are the main theoretical contribution of this paper. Some of the results in this section have previously been stated in [3], but without proofs. Section 4 gives formulas for CDS-spreads. They are our main calibration instruments. We provide a detailed description of the calibration against CDS spreads and their correlations. Special attention is given to the relation between market CDS-correlations and the corresponding default correlations. Furthermore, we discuss how to deal with negative jumps in the intensities, which are required if there are negative CDS-correlations. In Section 5 we use the results of Section 3 for our numerical investigations. Two CDS portfolios are calibrated against market CDS spreads and their CDS-correlations. We then study several quantities of interest in credit portfolio management. Section 6 is devoted to numerical issues and the final section, Section 7, summarizes and discusses the results.

2. Intensity based models reinterpreted as Markov jump processes: multivariate phase-type distributions

In this section we define the intensity-based model for default contagion which is used throughout the paper. The model is then reinterpreted in terms of a Markov jump process, a so called multivariate phase-type distribution, introduced in [3]. Such constructions have largely been developed for queueing theory and reliability applications, see e.g. [1] and [3]).

For the default times $\tau_1, \tau_2 \ldots, \tau_m$, define the point process $N_{t,i} = 1_{\{\tau_i \leq t\}}$ and introduce the filtrations

$$\mathcal{F}_{t,i} = \sigma\left(N_{s,i}; s \leq t\right), \quad \mathcal{F}_t = \bigvee_{i=1}^m \mathcal{F}_{t,i}.$$

Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$. Below, we will for convenience often omit the filtration and just write intensity or "default intensity". With a further extension of language we will sometimes also write that the default times $\{\tau_i\}$ have intensities $\{\lambda_{t,i}\}$. The model studied in this paper is specified by requiring that the default intensities have the following form,

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \qquad t \leq \tau_i, \tag{2.1}$$

and $\lambda_{t,i} = 0$ for $t > \tau_i$. Further, $a_i \geq 0$ and $b_{i,j}$ are constants such that $\lambda_{t,i}$ is non-negative.

The financial interpretation of (2.1) is that the default intensities are constant, except at the times when defaults occur: then the default intensity for obligor i jumps by an amount $b_{i,j}$ if it is obligor j which has defaulted. Thus a positive $b_{i,j}$ means that obligor i is put at higher risk by the default of obligor j, while a negative $b_{i,j}$ means that obligor i in fact benefits from the default of j, and finally $b_{i,j} = 0$ if obligor i is unaffected by the default of j.

Equation (2.1) determines the default times through their intensities as well as their joint distribution. However, it is by no means obvious how to find these expressions. Here we will use the following observation, proved in [19].

Proposition 2.1. *There exists a Markov jump process $(Y_t)_{t \geq 0}$ on a finite state space $\boldsymbol{E}$ and a family of sets $\{\Delta_i\}_{i=1}^m$ such that the stopping times*

$$\tau_i = \inf\left\{t > 0 : Y_t \in \Delta_i\right\}, \qquad i = 1, 2, \ldots, m, \tag{2.2}$$

have intensities (2.1). Hence, any distribution derived from the multivariate stochastic vector $(\tau_1, \tau_2, \ldots, \tau_m)$ can be obtained from $\{Y_t\}_{t \geq 0}$.

The joint distribution of $(\tau_1, \tau_2, \ldots, \tau_m)$ is sometimes called a multivariate phase-type distribution (MPH), and was first introduced in [3]. In this paper, Proposition 2.1 is throughout used for computing distributions. However, we still use Equation (2.1) to describe the dependencies in a credit portfolio since it is more compact and intuitive.

Each state $\boldsymbol{j}$ in $\boldsymbol{E}$ is of the form $\boldsymbol{j} = \{j_1, \ldots j_k\}$ which is a subsequence of $\{1, \ldots m\}$ consisting of k integers, where $1 \leq k \leq m$. The interpretation is that on $\{j_1, \ldots j_k\}$ the obligors in the set have defaulted. Before we continue, further notation are needed. In the

sequel, we let $\boldsymbol{Q}$ and $\boldsymbol{\alpha}$ denote the generator and initial distribution on $\boldsymbol{E}$ for the Markov jump process in Proposition 2.1. The generator $\boldsymbol{Q}$ is found by using the structure of $\boldsymbol{E}$, the definition of the states $\boldsymbol{j}$, and Equation (2.1). The states are ordered so that $\boldsymbol{Q}$ is upper triangular, see [19]. In particular, the final state $\{1, \ldots m\}$ is absorbing and $\{0\}$ is always the starting state. The latter implies that $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$. Furthermore, define the probability vector $\boldsymbol{p}(t) = (\mathbb{P}[Y_t = \boldsymbol{j}])_{\boldsymbol{j} \in \boldsymbol{E}}$. From Markov theory we know that

$$\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}, \quad \text{and} \quad \mathbb{P}[Y_t = \boldsymbol{j}] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e_j}, \tag{2.3}$$

where $\boldsymbol{e_j} \in \mathbb{R}^{|\boldsymbol{E}|}$ is a column vector where the entry at position $\boldsymbol{j}$ is 1 and the other entries are zero. Recall that $e^{\boldsymbol{Q}t}$ is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of $\boldsymbol{Q}$.

3. Using Multivariate Phase-type distributions and the matrix-analytic approach to find multivariate default distributions

In this section we derive expressions for various quantities of importance in active credit portfolio management. The portfolio consists of m obligors with default intensities (2.1). Subsection 3.1 presents formulas for multivariate default and survival distributions, conditional multivariate default distributions, and multivariate default densities. In subsection 3.2 we briefly restate some expressions for marginal survival distributions, originally presented in [19]. These distributions are needed in Section 4. Analytical formulas for the default correlations are given in Subsection 3.3. Finally, in Subsection 3.4 we present compact expressions for the moments of the default times and the ordered default times.

3.1. **The multivariate default distributions.** In this subsection we derive formulas for multivariate default and survival distributions, conditional multivariate default distributions, and multivariate default densities. Let $\boldsymbol{G}_i$ be $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrices, defined by

$$(\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}} = 1_{\{\boldsymbol{j} \in \Delta_i^C\}} \quad \text{and} \quad (\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}'} = 0 \quad \text{if} \quad \boldsymbol{j} \neq \boldsymbol{j}'. \tag{3.1.1}$$

Further, for a vector $(t_1, t_2, \ldots, t_m)$ in $\mathbb{R}_+^m = [0, \infty)^m$, let the ordering of $(t_1, t_2, \ldots, t_m)$ be $t_{i_1} < t_{i_2} < \ldots < t_{i_m}$ where $(i_1, i_2, \ldots, i_m)$ is a permutation of $(1, 2, \ldots, m)$. The following proposition was stated in [3], but without a proof.

Proposition 3.1. *Consider m obligors with default intensities (2.1). Let $(t_1, t_2, \ldots, t_m) \in \mathbb{R}_+^m$ and let $t_{i_1} < t_{i_2} < \ldots < t_{i_m}$ be its ordering. Then,*

$$\mathbb{P}[\tau_1 > t_1, \ldots, \tau_m > t_m] = \boldsymbol{\alpha} \left(\prod_{k=1}^m e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} \right) \mathbf{1} \tag{3.1.2}$$

where $t_{i_0} = 0$.

Proof. First, note that

$$
\begin{aligned}
\mathbb{P}\left[\tau_1 > t_1, \ldots, \tau_m > t_m\right] &= \mathbb{P}\left[\tau_{i_1} > t_{i_1}, \ldots, \tau_{i_m} > t_{i_m}\right] \\
&= \mathbb{P}\left[Y_{t_{i_1}} \in \Delta_{i_1}^C, \ldots, Y_{t_{i_m}} \in \Delta_{i_m}^C\right] \\
&= \mathbb{P}\left[Y_0 = \mathbf{0}, Y_{t_{i_1}} \in \Delta_{i_1}^C, \ldots, Y_{t_{i_m}} \in \Delta_{i_m}^C\right] \qquad (3.1.3) \\
&= \sum_{\boldsymbol{j}_{i_1} \in \Delta_{i_1}^C} \cdots \sum_{\boldsymbol{j}_{i_m} \in \Delta_{i_m}^C} \mathbb{P}\left[Y_0 = \mathbf{0}, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \ldots, Y_{t_{i_m}} = \boldsymbol{j}_{i_m}\right]
\end{aligned}
$$

where $\mathbf{0} = \{0\}$ is the state representing that no default have occurred. Further,

$$
\begin{aligned}
&\mathbb{P}\left[Y_0 = \mathbf{0}, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \ldots, Y_{t_{i_m}} = \boldsymbol{j}_{i_m}\right] \\
&= \mathbb{P}\left[Y_0 = \mathbf{0}\right] \mathbb{P}\left[Y_{t_{i_1}} = \boldsymbol{j}_{i_1} \mid Y_0 = \mathbf{0}\right] \cdot \ldots \cdot \mathbb{P}\left[Y_{t_{i_m}} = \boldsymbol{j}_{i_m} \mid Y_{t_{m-1}} = \boldsymbol{j}_{i_{m-1}}\right] \qquad (3.1.4) \\
&= \boldsymbol{\alpha} e^{\boldsymbol{Q} t_{i_1}} \boldsymbol{e}_{\boldsymbol{j}_{i_1}} \boldsymbol{e}_{\boldsymbol{j}_{i_1}}^T e^{\boldsymbol{Q}\left(t_{i_2} - t_{i_1}\right)} \boldsymbol{e}_{\boldsymbol{j}_{i_2}} \boldsymbol{e}_{\boldsymbol{j}_{i_2}}^T \cdot \ldots \cdot \boldsymbol{e}_{\boldsymbol{j}_{i_{m-1}}} \boldsymbol{e}_{\boldsymbol{j}_{i_{m-1}}}^T e^{\boldsymbol{Q}\left(t_{i_m} - t_{i_{m-1}}\right)} \boldsymbol{e}_{\boldsymbol{j}_{i_m}}
\end{aligned}
$$

where the first equality follows from the Markov property of Y_t, and $\mathbb{P}\left[Y_0 = \mathbf{0}\right] = 1$. The second equality is because

$$
\mathbb{P}\left[Y_t = \boldsymbol{j}_{i_k} \mid Y_s = \boldsymbol{j}_{i_{k-1}}\right] = \mathbb{P}\left[Y_{t-s} = \boldsymbol{j}_{i_k} \mid Y_0 = \boldsymbol{j}_{i_{k-1}}\right] = \left(\boldsymbol{e}_{\boldsymbol{j}_{i_{k-1}}}^T e^{\boldsymbol{Q}(t-s)}\right)_{\boldsymbol{j}_{i_k}}
$$

since Y_t is a homogeneous Markov process. Next,

$$
\sum_{\boldsymbol{j}_{i_k} \in \boldsymbol{E}_{i_k}} \boldsymbol{e}_{\boldsymbol{j}_{i_k}} \boldsymbol{e}_{\boldsymbol{j}_{i_k}}^T e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} = \left(\sum_{\boldsymbol{j}_{i_k} \in \boldsymbol{E}_{i_k}} \boldsymbol{e}_{\boldsymbol{j}_{i_k}} \boldsymbol{e}_{\boldsymbol{j}_{i_k}}^T\right) e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} = \boldsymbol{G}_{i_k} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \qquad (3.1.5)
$$

for $k = 1, 2, \ldots m - 1$, and

$$
\sum_{\boldsymbol{j}_{i_m} \in \boldsymbol{E}_{i_m}} e^{\boldsymbol{Q}\left(t_{i_m} - t_{i_{m-1}}\right)} \boldsymbol{e}_{\boldsymbol{j}_{i_m}} = e^{\boldsymbol{Q}\left(t_{i_m} - t_{i_{m-1}}\right)} \boldsymbol{G}_{i_m} \mathbf{1}. \qquad (3.1.6)
$$

Hence, inserting the equations (3.1.4)-(3.1.6) into (3.1.3) shows that (3.1.2) hold. $\qquad \square$

Let $(t_{i_1}, t_{i_2}, \ldots, t_{i_m})$ be the ordering of $(t_1, t_2, \ldots, t_m) \in \mathbb{R}_+^m$ and fix a p, $1 \le p \le m - 1$. We next consider conditional distributions of the types

$$
\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m} \mid \tau_{i_1} \le t_{i_1}, \ldots, \tau_{i_p} \le t_{i_p}\right]
$$

and

$$
\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m} \mid \tau_{i_1} \le t_{i_1}, \ldots, \tau_{i_p} \le t_{i_p}, T_{p+1} > t_{i_p}\right]
$$

There is a subtle but important difference between these two probabilities. The conditioning in the first expression includes the possibility that all obligors have defaulted before t_{i_p}. This is not the case in the second one, where the event excludes the possibility that other obligors than $i_1, \ldots, i_p$ default before t_{i_p}. These probabilities may of course be computed from (3.1.2) without any further use of the structure of the problem. However, using this

structure leads to compact formulas. For this, further notation is needed. Define Δ as the final absorbing state for Y_t, i.e.

$$\Delta = \bigcap_{i=1}^{m} \Delta_i, \tag{3.1.7}$$

and let $\boldsymbol{F}_i$ and $\boldsymbol{H}_i$ be $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrices, defined by

$$(\boldsymbol{F}_i)_{\boldsymbol{j},\boldsymbol{j}} = 1_{\{\boldsymbol{j} \in \Delta_i \backslash \Delta\}} \quad \text{and} \quad (\boldsymbol{F}_i)_{\boldsymbol{j},\boldsymbol{j}'} = 0 \quad \text{if} \quad \boldsymbol{j} \neq \boldsymbol{j}'. \tag{3.1.8}$$

$$(\boldsymbol{H}_i)_{\boldsymbol{j},\boldsymbol{j}} = 1_{\{\boldsymbol{j} \in \Delta_i\}} \quad \text{and} \quad (\boldsymbol{H}_i)_{\boldsymbol{j},\boldsymbol{j}'} = 0 \quad \text{if} \quad \boldsymbol{j} \neq \boldsymbol{j}'. \tag{3.1.9}$$

The following proposition is useful.

Proposition 3.2. *Consider m obligors with default intensities (2.1). Let $(t_1, t_2, \ldots, t_m) \in \mathbb{R}_+^m$ and let $t_{i_1} < t_{i_2} < \ldots < t_{i_m}$ be its ordering. If $1 \leq p \leq m-1$ then,*

$$\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, \tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m}\right]$$
$$= \boldsymbol{\alpha}\left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k}\right)\left(\prod_{k=p+1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k}\right)\boldsymbol{1}. \tag{3.1.10}$$

Further,

$$\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m} \,\middle|\, \tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}\right]$$
$$= \frac{\boldsymbol{\alpha}\left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k}\right)\left(\prod_{k=p+1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k}\right)\boldsymbol{1}}{\boldsymbol{\alpha}\left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{H}_{i_k}\right)\boldsymbol{1}}. \tag{3.1.11}$$

and

$$\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m} \,\middle|\, \tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, T_{p+1} > t_{i_p}\right]$$
$$= \frac{\boldsymbol{\alpha}\left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k}\right)\left(\prod_{k=p+1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k}\right)\boldsymbol{1}}{\boldsymbol{\alpha}\left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k}\right)\boldsymbol{1}} \tag{3.1.12}$$

where $t_{i_0} = 0$.

Proof. First we prove (3.1.10). Similarly as in the proof of Proposition 3.1

$$\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, \tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m}\right]$$
$$= \mathbb{P}\left[Y_0 \in \boldsymbol{E}, Y_{t_{i_1}} \in \Delta_{i_1} \backslash \Delta, \ldots, Y_{t_{i_p}} \in \Delta_{i_p} \backslash \Delta, Y_{t_{i_{p+1}}} \in \Delta_{i_{p+1}}^C, \ldots, Y_{t_{i_m}} \in \Delta_{i_m}^C\right]$$
$$= \sum_{\boldsymbol{j_0} \in \boldsymbol{E}} \sum_{\boldsymbol{j}_{i_1} \in \Delta_{i_1} \backslash \Delta} \cdots \sum_{\boldsymbol{j}_{i_p} \in \Delta_{i_p} \backslash \Delta} \sum_{\boldsymbol{j}_{i_{p+1}} \in \Delta_{i_{p+1}}^C} \cdots \sum_{\boldsymbol{j}_{i_m} \in \Delta_{i_m}^C} \mathbb{P}\left[Y_0 = \boldsymbol{j_0}, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \ldots, Y_{t_{i_m}} = \boldsymbol{j}_{i_m}\right]$$
$$= \boldsymbol{\alpha}\left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k}\right)\left(\prod_{k=p+1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k}\right)\boldsymbol{1}.$$

Here the last equality follows from similar arguments as in the equations (3.1.4)-(3.1.6) in Proposition 3.1, using the definition of the matrix $\boldsymbol{F}_k$.

To prove (3.1.11) it is enough to show that

$$\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}\right] = \boldsymbol{\alpha}\left(\prod_{k=1}^p e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{H}_{i_k}\right)\boldsymbol{1}$$

since Equation (3.1.11) then follows from (3.1.10) and the definition of conditional probabilities. Now,

$$\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}\right] = \mathbb{P}\left[Y_0 \in \boldsymbol{E}, Y_{t_{i_1}} \in \Delta_{i_1}, \ldots, Y_{t_{i_p}} \in \Delta_{i_p}\right]$$

$$= \sum_{\boldsymbol{j_0} \in \boldsymbol{E}} \sum_{\boldsymbol{j}_{i_1} \in \Delta_{i_1}} \cdots \sum_{\boldsymbol{j}_{i_p} \in \Delta_{i_p}} \mathbb{P}\left[Y_0 = \boldsymbol{j}_0, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \ldots, Y_{t_{i_p}} = \boldsymbol{j}_{i_p}\right]$$

$$= \boldsymbol{\alpha}\left(\prod_{k=1}^p e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{H}_{i_k}\right)\boldsymbol{1}$$

where the last equality follows from arguments as in Proposition 3.1, using the definition of the matrix $\boldsymbol{H}_k$. Finally, for Equation (3.1.12), note that

$$\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m} \,\middle|\, \tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, T_{p+1} > t_{i_p}\right]$$

$$= \frac{\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, \tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m}\right]}{\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, T_{p+1} > t_{i_p}\right]}.$$

Hence, by using (3.1.10) it is enough to show that

$$\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, T_{p+1} > t_{i_p}\right] = \boldsymbol{\alpha}\left(\prod_{k=1}^p e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{F}_{i_k}\right)\boldsymbol{1}.$$

Let $\boldsymbol{E}_n$ be the set of states representing precisely n defaults. Then,

$$\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, T_{p+1} > t_{i_p}\right]$$

$$= \mathbb{P}\left[Y_{t_{i_1}} \in \Delta_{i_1}, \ldots, Y_{t_{i_p}} \in \Delta_{i_p}, Y_{t_{i_p}} \in \bigcup_{k=p+1}^m \boldsymbol{E}_{i_k}\right]$$

$$= \mathbb{P}\left[Y_0 \in \boldsymbol{E}, Y_{t_{i_1}} \in \Delta_{i_1} \setminus \Delta, \ldots, Y_{t_{i_p}} \in \Delta_{i_p} \setminus \Delta\right]$$

$$= \sum_{\boldsymbol{j_0} \in \boldsymbol{E}} \sum_{\boldsymbol{j}_{i_1} \in \Delta_{i_1} \setminus \Delta} \cdots \sum_{\boldsymbol{j}_{i_p} \in \Delta_{i_p} \setminus \Delta} \mathbb{P}\left[Y_0 = \boldsymbol{j}_0, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \ldots, Y_{t_{i_p}} = \boldsymbol{j}_{i_p}\right]$$

$$= \boldsymbol{\alpha}\left(\prod_{k=1}^p e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{F}_{i_k}\right)\boldsymbol{1}$$

where the second equality comes from the fact that Δ is an absorbing state representing default of all obligors. The last equality follows from arguments as in Proposition 3.1, using the definition of the matrix $\boldsymbol{F}_k$. $\square$

The following corollary is an immediate consequences of Equation (3.1.10) in Proposition 3.2.

Corollary 3.3. *Consider m obligors with default intensities (2.1). Let $\{i_1, \ldots, i_p\}$ and $\{j_1, \ldots, j_q\}$ be two disjoint subsequences in $\{1, \ldots, m\}$. If $t < s$ then*

$$\mathbb{P}\left[\tau_{i_1} > t, \ldots, \tau_{i_p} > t, \tau_{j_1} < s, \ldots, \tau_{j_q} < s\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\left(\prod_{k=1}^{p}\boldsymbol{G}_{i_k}\right)e^{\boldsymbol{Q}(s-t)}\left(\prod_{k=1}^{q}\boldsymbol{H}_{j_k}\right)\mathbf{1}$$

and for $s < t$

$$\mathbb{P}\left[\tau_{i_1} > t, \ldots, \tau_{i_p} > t, \tau_{j_1} < s, \ldots, \tau_{j_q} < s\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}s}\left(\prod_{k=1}^{q}\boldsymbol{F}_{j_k}\right)e^{\boldsymbol{Q}(t-s)}\left(\prod_{k=1}^{p}\boldsymbol{G}_{i_k}\right)\mathbf{1}.$$

We can of course generalize, the above proposition for three time points $t < s < u$, four time points $t < s < u <$ etc. Using the notation of Corollary 3.3 we conclude that if $t < s$ then

$$\mathbb{P}\left[\tau_{j_1} < s, \ldots, \tau_{j_q} < s \mid \tau_{i_1} > t, \ldots, \tau_{i_p} > t\right] = \frac{\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\left(\prod_{k=1}^{p}\boldsymbol{G}_{i_k}\right)e^{\boldsymbol{Q}(s-t)}\left(\prod_{k=1}^{q}\boldsymbol{H}_{j_k}\right)\mathbf{1}}{\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\left(\prod_{k=1}^{p}\boldsymbol{G}_{i_k}\right)\mathbf{1}}$$

and for $s < t$

$$\mathbb{P}\left[\tau_{i_1} > t, \ldots, \tau_{i_p} > t \mid \tau_{j_1} < s, \ldots, \tau_{j_q} < s\right] = \frac{\boldsymbol{\alpha}e^{\boldsymbol{Q}s}\left(\prod_{k=1}^{q}\boldsymbol{F}_{j_k}\right)e^{\boldsymbol{Q}(t-s)}\left(\prod_{k=1}^{p}\boldsymbol{G}_{i_k}\right)\mathbf{1}}{\boldsymbol{\alpha}e^{\boldsymbol{Q}s}\left(\prod_{k=1}^{q}\boldsymbol{H}_{j_k}\right)\mathbf{1}}.$$

Our next task is to find the probability density $f(t_1, \ldots, t_m)$ of the multivariate random variable $(\tau_1, \ldots, \tau_m)$. For $(t_1, t_2, \ldots, t_m)$, let $(t_{i_1}, t_{i_2}, \ldots, t_{i_m})$ be its ordering where $(i_1, i_2, \ldots, i_m)$ is a permutation of $(1, 2, \ldots, m)$. We denote $(i_1, i_2, \ldots, i_m)$ by $\boldsymbol{i}$, that is, $\boldsymbol{i} = (i_1, i_2, \ldots, i_m)$. Furthermore, in view of the above notation, we let $f_{\boldsymbol{i}}(t_1, \ldots, t_m)$ denote the restriction of $f(t_1, \ldots, t_m)$ to the set $t_{i_1} < t_{i_2} < \ldots < t_{i_m}$. The following proposition was stated in [3], but without a proof.

Proposition 3.4. *Consider m obligors with default intensities (2.1). Let $(t_1, t_2, \ldots, t_m) \in \mathbb{R}_+^m$ and let $t_{i_1} < t_{i_2} < \ldots < t_{i_m}$ be its ordering. Then, with notation as above*

$$f_{\boldsymbol{i}}(t_1, \ldots, t_m) = (-1)^m \boldsymbol{\alpha}\left(\prod_{k=1}^{m-1} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)}\left(\boldsymbol{Q}\boldsymbol{G}_{i_k} - \boldsymbol{G}_{i_k}\boldsymbol{Q}\right)\right)e^{\boldsymbol{Q}\left(t_{i_m} - t_{i_{m-1}}\right)}\boldsymbol{Q}\boldsymbol{G}_{i_m}\mathbf{1}$$

$$(3.1.13)$$

where $t_{i_0} = 0$.

Proof. By Proposition 3.1, since the order of partial differentiation is irrelevant

$$f_{\boldsymbol{i}}(t_1, \ldots, t_m) = (-1)^m \frac{\partial^m}{\partial t_{i_1} \cdots \partial t_{i_m}}\mathbb{P}\left[\tau_{i_1} > t_{i_1}, \ldots, \tau_{i_p} > t_{i_p}\right]$$

$$= (-1)^m \boldsymbol{\alpha}\left(\frac{\partial^m}{\partial t_{i_1} \cdots \partial t_{i_m}}\prod_{k=1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)}\boldsymbol{G}_{i_k}\right)\mathbf{1}$$

$$(3.1.14)$$

where $t_{i_0} = 0$. First, note that

$$\frac{\partial}{\partial t_{i_1}} \prod_{k=1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} = e^{\boldsymbol{Q}t_{i_1}} \boldsymbol{Q}\boldsymbol{G}_{i_1} \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k}$$

$$- e^{\boldsymbol{Q}t_{i_1}} \boldsymbol{G}_{i_1} e^{\boldsymbol{Q}\left(t_{i_2} - t_{i_1}\right)} \boldsymbol{Q}\boldsymbol{G}_{i_2} \prod_{k=3}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k}$$

$$= e^{\boldsymbol{Q}t_{i_1}} \boldsymbol{Q}\boldsymbol{G}_{i_1} \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} \qquad (3.1.15)$$

$$- e^{\boldsymbol{Q}t_{i_1}} \boldsymbol{G}_{i_1} \boldsymbol{Q} \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k}$$

$$= e^{\boldsymbol{Q}t_{i_1}} \left(\boldsymbol{Q}\boldsymbol{G}_{i_1} - \boldsymbol{G}_{i_1}\boldsymbol{Q}\right) \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k}$$

where the second equality is due to the fact that $e^{\boldsymbol{Q}t}\boldsymbol{Q} = \boldsymbol{Q}e^{\boldsymbol{Q}t}$. Next, (3.1.15) implies that

$$\frac{\partial^2}{\partial t_{i_1} \partial t_{i_2}} \prod_{k=1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} = e^{\boldsymbol{Q}t_{i_1}} \left(\boldsymbol{Q}\boldsymbol{G}_{i_1} - \boldsymbol{G}_{i_1}\boldsymbol{Q}\right) \frac{\partial}{\partial t_{i_2}} \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k}. \quad (3.1.16)$$

The derivative of the product in the right-hand side in Equation (3.1.16) is treated exactly as in (3.1.15) but now with t_{i_2} instead of t_{i_1}. Hence, by repeating this procedure for $k = 3, \ldots, m-1$ and noting that

$$\frac{\partial}{\partial t_{i_m}} e^{\boldsymbol{Q}\left(t_{im} - t_{i_{m-1}}\right)} \boldsymbol{G}_{i_m} = e^{\boldsymbol{Q}\left(t_{im} - t_{i_{m-1}}\right)} \boldsymbol{Q}\boldsymbol{G}_{i_m}$$

and inserting the results in Equation (3.1.14) finally yields

$$f_{\boldsymbol{i}}(t_1, \ldots, t_m) = (-1)^m \boldsymbol{\alpha} \left(\prod_{k=1}^{m-1} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \left(\boldsymbol{Q}\boldsymbol{G}_{i_k} - \boldsymbol{G}_{i_k}\boldsymbol{Q}\right)\right) e^{\boldsymbol{Q}\left(t_{im} - t_{i_{m-1}}\right)} \boldsymbol{Q}\boldsymbol{G}_{i_m} \boldsymbol{1}$$

where $t_{i_0} = 0$. This proves (3.1.13). $\qquad\square$

3.2. **The marginal distributions.** In this section we state expressions for the marginal survival distributions $\mathbb{P}\left[\tau_i > t\right]$ and $\mathbb{P}\left[T_k > t\right]$, and for $\mathbb{P}\left[T_k > t, T_k = \tau_i\right]$ which is the probability that the k-th default is by obligor i and that it not occurs before t. The first ones are more or less standard, while the second one is less so. These marginal distributions are needed to compute single-name CDS spreads and k^{th}-to-default spreads, see e.g [17], [19]. Note that CDS-s are used as calibration instruments when pricing portfolio credit derivatives. We come back to this in Section 4. The following lemma is trivial, but stated since it is needed later on.

Lemma 3.5. *Consider m obligors with default intensities* (2.1). *Then,*

$$\mathbb{P}\left[\tau_i > t\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{g}^{(i)} \quad and \quad \mathbb{P}\left[T_k > t\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{m}^{(k)} \qquad (3.2.1)$$

where the column vectors $\boldsymbol{g}^{(i)}$, $\boldsymbol{m}^{(k)}$ of length $|\boldsymbol{E}|$ are defined as

$$\boldsymbol{g}_{\boldsymbol{j}}^{(i)} = 1_{\left\{\boldsymbol{j} \in (\Delta_i)^C\right\}} \quad and \quad \boldsymbol{m}_{\boldsymbol{j}}^{(k)} = 1_{\left\{\boldsymbol{j} \in \cup_{n=0}^{k-1} \boldsymbol{E}_n\right\}}$$

and $\boldsymbol{E}_n$ is set of states consisting of precisely n elements of $\{1, \ldots m\}$ where $\boldsymbol{E}_0 = \{0\}$.

The lemma immediately follows from the definition of τ_i in Proposition 2.1. The same holds for the distribution for T_k, where we also use that $\boldsymbol{m}^{(k)}$ sums the probabilities of states where there has been less than k defaults. We next restate the following result, proved in [19].

Proposition 3.6. *Consider m obligors with default intensities* (2.1). *Then,*

$$\mathbb{P}\left[T_k > t, T_k = \tau_i\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q} t} \sum_{\ell=0}^{k-1} \left(\prod_{p=\ell}^{k-1} \boldsymbol{G}^{i,p} \boldsymbol{P}\right) \boldsymbol{h}^{i,k}, \tag{3.2.2}$$

for $k = 1, \ldots m$, where

$$\boldsymbol{P}_{\boldsymbol{j},\boldsymbol{j}'} = \frac{\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}'}}{\sum_{k \neq \boldsymbol{j}} \boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{k}}}, \quad \boldsymbol{j}, \boldsymbol{j}' \in \boldsymbol{E},$$

and $\boldsymbol{h}^{i,k}$ is column vectors of length $|\boldsymbol{E}|$ and $\boldsymbol{G}^{i,k}$ is $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrices, defined by

$$\boldsymbol{h}_{\boldsymbol{j}}^{i,k} = 1_{\left\{\boldsymbol{j} \in \Delta_i \cap \boldsymbol{E}_k\right\}} \quad and \quad \boldsymbol{G}_{\boldsymbol{j},\boldsymbol{j}}^{i,k} = 1_{\left\{\boldsymbol{j} \in (\Delta_i)^C \cap \boldsymbol{E}_k\right\}} \quad and \quad \boldsymbol{G}_{\boldsymbol{j},\boldsymbol{j}'}^{i,k} = 0 \quad if \quad \boldsymbol{j} \neq \boldsymbol{j}'.$$

Equipped with the above distributions, we can derive closed-form solutions for single-name CDS spreads and k^{th}-to-default swaps for a nonhomogeneous portfolio, see e.g [17], [19]. In the present we focus on CDS spreads as our main calibration tools, see Section 4.

3.3. **The default correlations.** In this subsection we derive expressions for pairwise default correlations, i.e. $\rho_{i,j}(t) = \text{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$ between the obligors $i \neq j$ belonging to a portfolio of m obligors satisfying (2.1).

Lemma 3.7. *Consider m obligors with default intensities* (2.1). *Then, for any pair of obligors $i \neq j$,*

$$\rho_{i,j}(t) = \frac{\boldsymbol{\alpha} e^{\boldsymbol{Q} t} \boldsymbol{c}^{(i,j)} - \boldsymbol{\alpha} e^{\boldsymbol{Q} t} \boldsymbol{h}^{(i)} \boldsymbol{\alpha} e^{\boldsymbol{Q} t} \boldsymbol{h}^{(j)}}{\sqrt{\boldsymbol{\alpha} e^{\boldsymbol{Q} t} \boldsymbol{h}^{(i)} \boldsymbol{\alpha} e^{\boldsymbol{Q} t} \boldsymbol{h}^{(j)} \left(1 - \boldsymbol{\alpha} e^{\boldsymbol{Q} t} \boldsymbol{h}^{(i)}\right) \left(1 - \boldsymbol{\alpha} e^{\boldsymbol{Q} t} \boldsymbol{h}^{(j)}\right)}} \tag{3.3.1}$$

where the column vectors $\boldsymbol{h}^{(i)}$, $\boldsymbol{c}^{(i,j)}$ of length $|\boldsymbol{E}|$ are defined as

$$\boldsymbol{h}_{\boldsymbol{j}}^{(i)} = 1_{\{\boldsymbol{j} \in \Delta_i\}} \quad and \quad \boldsymbol{c}_{\boldsymbol{j}}^{(i,j)} = 1_{\{\boldsymbol{j} \in \Delta_i \cap \Delta_j\}} = \boldsymbol{h}_{\boldsymbol{j}}^{(i)} \boldsymbol{h}_{\boldsymbol{j}}^{(j)}. \tag{3.3.2}$$

Proof. By the definition of covariance and variance

$$\text{Cov}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}}) = \mathbb{P}\left[\tau_i \leq t, \tau_j \leq t\right] - \mathbb{P}\left[\tau_i \leq t\right] \mathbb{P}\left[\tau_j \leq t\right],$$

and $\mathrm{Var}(1_{\{\tau_i \leq t\}}) = \mathbb{P}[\tau_i \leq t](1 - \mathbb{P}[\tau_i \leq t])$. According to Equation (2.2) we have that $\mathbb{P}[\tau_i \leq t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{h}^{(i)}$ where $\boldsymbol{h}_{\boldsymbol{j}}^{(i)} = 1_{\{\boldsymbol{j} \in \Delta_i\}}$, and that

$$\mathbb{P}[\tau_i \leq t, \tau_j \leq t] = \mathbb{P}[Y_t \in \Delta_i \cap \Delta_j] = \sum_{\boldsymbol{j} \in \Delta_i \cap \Delta_j} \mathbb{P}[Y_t = \boldsymbol{j}] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{c}^{(i,j)}$$

where $\boldsymbol{c}_{\boldsymbol{j}}^{(i,j)} = 1_{\{\boldsymbol{j} \in \Delta_i \cap \Delta_j\}} = \boldsymbol{h}_{\boldsymbol{j}}^{(i)} \boldsymbol{h}_{\boldsymbol{j}}^{(j)}$. Inserting these expressions into the definition for correlation between two random variables yields (3.3.1). $\qquad\square$

Note that if we have determined the vector $\boldsymbol{g}^{(i)}$, then $\boldsymbol{h}^{(i)}$ is retrieved from $\boldsymbol{g}^{(i)}$ according to $\boldsymbol{h}^{(i)} = \boldsymbol{1} - \boldsymbol{g}^{(i)}$ which is useful for practical implementation.

3.4. **Expected default times.** By construction (see Proposition 2.1), the intensity matrix $\boldsymbol{Q}$ for the Markov jump process Y_t on $\boldsymbol{E}$ has the form

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{0} & 0 \end{pmatrix}$$

where $\boldsymbol{t}$ is a column vector with $|\boldsymbol{E}| - 1$ rows. The $\boldsymbol{j}$-th element $\boldsymbol{t}_{\boldsymbol{j}}$ is the intensity for Y_t to jump from the state $\boldsymbol{j}$ to the absorbing state $\Delta = \cap_{i=1}^m \Delta_i$. Furthermore, $\boldsymbol{T}$ is invertible since it is upper diagonal with strictly negative diagonal elements. Thus, we have the following standard lemma.

Lemma 3.8. *Consider m obligors with default intensities (2.1). Then, with notation as above*

$$\mathbb{E}[\tau_i^n] = (-1)^n n! \tilde{\boldsymbol{\alpha}} \boldsymbol{T}^{-n} \tilde{\boldsymbol{g}}^{(i)} \quad and \quad \mathbb{E}[T_k^n] = (-1)^n n! \tilde{\boldsymbol{\alpha}} \boldsymbol{T}^{-n} \tilde{\boldsymbol{m}}^{(k)}$$

for $n \in \mathbb{N}$ where $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{g}}^{(i)}, \tilde{\boldsymbol{m}}^{(k)}$ are the restrictions of $\boldsymbol{\alpha}, \boldsymbol{g}^{(i)}, \boldsymbol{m}^{(k)}$ from $\boldsymbol{E}$ to $\boldsymbol{E} \setminus \Delta$.

Proof. We prove the results for $n = 1$. By Lemma 3.5 we have that

$$\mathbb{P}[\tau_i > t] = \tilde{\boldsymbol{\alpha}} e^{\boldsymbol{T}t} \tilde{\boldsymbol{g}}^{(i)} \quad and \quad \mathbb{P}[T_k > t] = \tilde{\boldsymbol{\alpha}} e^{\boldsymbol{Q}t} \tilde{\boldsymbol{m}}^{(k)}$$

where $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{g}}^{(i)}, \tilde{\boldsymbol{m}}^{(k)}$ are the restrictions of $\boldsymbol{\alpha}, \boldsymbol{g}^{(i)}, \boldsymbol{m}^{(k)}$ from $\boldsymbol{E}$ to $\boldsymbol{E} \setminus \Delta$. If $F_{T_k}(t) = \mathbb{P}[T_k \leq t]$, then $f_{T_k}(t)$ is given by

$$f_{T_k}(t) = \frac{d}{dt} F_{T_k}(t) = -\frac{d}{dt} \mathbb{P}[T_k > t] = -\tilde{\boldsymbol{\alpha}} e^{\boldsymbol{T}t} \boldsymbol{T} \tilde{\boldsymbol{m}}^{(k)}$$

so that

$$\mathbb{E}[T_k] = \int_0^\infty t f_{T_k}(t)\, dt = -\tilde{\boldsymbol{\alpha}} \int_0^\infty t e^{\boldsymbol{T}t} dt\, \boldsymbol{T} \tilde{\boldsymbol{m}}^{(k)} = -\tilde{\boldsymbol{\alpha}} \boldsymbol{T}^{-1} \tilde{\boldsymbol{m}}^{(k)}.$$

To motivate the last equality we use partial integration and the fact that $\boldsymbol{T}$ is invertible to conclude that

$$\int_0^\infty t e^{\boldsymbol{T}t} dt\, \boldsymbol{T} = \lim_{t \to \infty} e^{\boldsymbol{T}t} (t\boldsymbol{I} - \boldsymbol{T}^{-1}) + \boldsymbol{T}^{-1} = \boldsymbol{T}^{-1}$$

since $\lim_{t \to \infty} e^{\boldsymbol{T}t}(t\boldsymbol{I} - \boldsymbol{T}^{-1}) = \boldsymbol{0}$ because the eigenvalues of $\boldsymbol{T}$ are strictly negative. The expression for $\mathbb{E}[T_k^n]$ and $\mathbb{E}[\tau_i^n]$ are derived analogously for $n = 1, 2, 3, \ldots$. $\qquad\square$

The above proof can also be done by using Laplace transforms, see e.g. [2]. From Lemma 3.8 we can determine the risk-neutral, i.e implied, expected default times according to $\mathbb{E}[\tau_i] = -\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{g}}^{(i)}$ and $\mathbb{E}[T_k] = -\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{m}}^{(k)}$. Furthermore, the implied variances of the default times are then given by

$$\text{Var}[\tau_i] = 2\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-2}\tilde{\boldsymbol{g}}^{(i)} - \left(\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{g}}^{(i)}\right)^2 \qquad \text{for } i = 1, 2, \ldots, m$$

$$\text{Var}[T_k] = 2\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-2}\tilde{\boldsymbol{m}}^{(k)} - \left(\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{m}}^{(k)}\right)^2 \quad \text{for } k = 1, 2, \ldots, m.$$

3.5. **Some remarks.** The message in Subsections 3.2-3.3 is that under (2.1), computations of multivariate default and survival distributions, conditional multivariate default and survival distributions, marginal default distributions, multivariate default densities and default correlations can be reduced to compute the matrix exponential. Computing $e^{\boldsymbol{Q}t}$ efficiently is a numerical issue, which for large state spaces requires special treatment. This is discussed in Section 6. Finally, recall that $|\boldsymbol{E}| = 2^m$ which in practice will force us to work with portfolios where m is less or equal to 25, say ([19] used $m = 15$).

4. Calibrating the model parameters against CDS spreads and CDS correlations

In this section we discuss how to find the parameters in the model (2.1). First, Subsection 4.1 derives the model spreads for single-name credit default swaps, CDS-s, which are the most liquid traded credit derivative today. Next, Subsection 4.2 gives a detailed description of the calibration against CDS spreads and the corresponding CDS-correlations. We also discuss how to deal with negative jumps in the intensities, which are required if there are negative CDS-correlations

4.1. **Using the matrix-analytic approach to find CDS spreads.** Given the model (2.1), we will in this subsection derive expressions for CDS-spreads, which constitute our primary calibration instruments. In the sequel all computations are assumed to be made under a risk-neutral martingale measure $\mathbb{P}$. Typically such a $\mathbb{P}$ exists if we rule out arbitrage opportunities.

Consider a single-name credit default swap (CDS) with maturity T where the reference entity is a obligor i with default times τ_i and recovery rates ϕ_i. The protection premiums are paid at $0 < t_1 < t_2 < \ldots < t_{n_T} = T$ if $\tau_i > T$, or until the default time of obligor i, whichever comes first. Assuming that the default time and the risk-free interest rate are independent for each obligor and that the recovery rate is deterministic, one can show that the CDS spread is given by (see e.g. [17] or [19]),

$$R_i(T) = \frac{(1 - \phi_i)\int_0^T B_s dF_i(s)}{\sum_{n=1}^{n_T}\left(B_{t_n}\Delta_n(1 - F_i(t_n)) + \int_{t_{n-1}}^{t_n} B_s\left(s - t_{n-1}\right)dF_i(s)\right)} \tag{4.1.1}$$

where $B_t = \exp\left(-\int_0^t r_s ds\right)$ denote the discount factor, r_t is the risk-free interest rate, and $F_i(t) = \mathbb{P}[\tau_i \leq t]$ is the distribution function of the default time for obligor i. Note that

the CDS spread is independent of the amount that is protected. Expressions for $R_i(T)$ may be obtained by inserting the expression for $\mathbb{P}[\tau_i > t]$ in (3.2.1) into (4.1.1), and have previously been stated in [18], [19], but without proofs. For completeness, this is done in the following proposition.

Proposition 4.1. *Consider m obligors with default intensities (2.1) and assume that the interest rate r is constant. Then,*

$$R_i(T) = \frac{(1 - \phi_i)\boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)\boldsymbol{g}^{(i)}}{\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1}, t_n)\right)\right)\boldsymbol{g}^{(i)}} \tag{4.1.2}$$

where $\boldsymbol{C}(s,t) = s\left(\boldsymbol{A}(t) - \boldsymbol{A}(s)\right) - \boldsymbol{B}(t) + \boldsymbol{B}(s)$ for $\boldsymbol{A}(t) = e^{\boldsymbol{Q}t}\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}$ and

$$\boldsymbol{B}(t) = e^{\boldsymbol{Q}t}\left(t\boldsymbol{I} + \left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}.$$

Proof. Let $f_i(t)$ denote the density for τ_i,

$$f_i(t) = \frac{d}{dt}F_i(t) = -\frac{d}{dt}\mathbb{P}[\tau_i > t] = -\boldsymbol{\alpha}\boldsymbol{Q}e^{\boldsymbol{Q}t}\boldsymbol{g}^{(i)}$$

where the last equality is due to Lemma 3.5. Then,

$$\int_0^T B_t dF_i(t) = \int_0^T e^{-rt}f_i(t)dt = -\boldsymbol{\alpha}\int_0^T \boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt\boldsymbol{g}^{(i)} = \boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)\boldsymbol{g}^{(i)}$$

since

$$\int_a^b \boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt = \boldsymbol{A}(b) - \boldsymbol{A}(a) \quad \text{where} \quad \boldsymbol{A}(t) = e^{\boldsymbol{Q}t}\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}.$$

Furthermore,

$$\int_{t_{n-1}}^{t_n} B_t\left(t - t_{n-1}\right)dF_i(t) = \int_{t_{n-1}}^{t_n} te^{-rt}f_i(t)dt - t_{n-1}\int_{t_{n-1}}^{t_n} e^{-rt}f_i(t)dt$$

$$= -\boldsymbol{\alpha}\left(\int_{t_{n-1}}^{t_n} t\boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt - t_{n-1}\int_{t_{n-1}}^{t_n} \boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt\right)\boldsymbol{g}^{(i)}$$

$$= \boldsymbol{\alpha}\left(t_{n-1}\left(\boldsymbol{A}(t_n) - \boldsymbol{A}(t_{n-1})\right) - \boldsymbol{B}(t_n) + \boldsymbol{B}(t_{n-1})\right)\boldsymbol{g}^{(i)}$$

$$= \boldsymbol{\alpha}\boldsymbol{C}(t_{n-1}, t_n)\boldsymbol{g}^{(i)}$$

where $\boldsymbol{C}(s,t) = s\left(\boldsymbol{A}(t) - \boldsymbol{A}(s)\right) - \boldsymbol{B}(t) + \boldsymbol{B}(s)$ and

$$\int_a^b t\boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt = \boldsymbol{B}(b) - \boldsymbol{B}(a) \quad \text{for} \quad \boldsymbol{B}(t) = e^{\boldsymbol{Q}t}\left(t\boldsymbol{I} + \left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}.$$

Now, inserting the above expressions in Equation (4.1.1) renders (4.1.2). $\qquad\square$

By using the technique in Proposition 4.1 and the expressions for $\mathbb{P}[T_k > t, T_k = \tau_i]$ and $\mathbb{P}[T_k > t]$ in Subsection 3.2, we can derive formulas for k^{th}-to-default swaps, which are generalizations of CDS contracts, to a portfolio of several obligors. These contracts offers protection on the kth default in the portfolio. For more on this, see e.g. [17], [19].

4.2. **The calibration.** The parameters in (2.1) are obtained by calibrating the model against market CDS spreads and market CDS correlations. As in [19] we reparameterize the basic description (2.1) of the default intensities to the form

$$\lambda_{t,i} = a_i \left(1 + \sum_{j=1, j \neq i}^{m} \theta_{i,j} 1_{\{\tau_j \leq t\}} \right), \qquad (4.2.1)$$

where the a_i-s are the base default intensities and the $\theta_{i,j}$ measure the "relative dependence structure". In [19] we assumed that the matrix $\{\theta_{i,j}\}$ was exogenously given and then calibrated the a_i-s against the m market CDS spreads. In this paper we use the m market CDS spreads as in [19] but in addition also determine the $\{\theta_{i,j}\}$ from market data. Let $\rho_{i,j}(T) = \mathrm{Corr}(1_{\{\tau_i \leq T\}}, 1_{\{\tau_j \leq T\}})$ be the default correlation matrix computed under the risk neutral measure. This matrix is a function of the parameters $\{\theta_{i,j}\}$, but is not observable. Instead we use $\beta\{\rho_{i,j}^{(\mathrm{CDS})}(T)\}$ as a proxy for it, where $\{\rho_{i,j}^{(\mathrm{CDS})}(T)\}$ is the observed correlation matrix for the T-years market CDS spreads, and β is a parameter at our disposal. Thus, in the calibration we match $\rho_{i,j}(T)$ against $\beta\{\rho_{i,j}^{(\mathrm{CDS})}(T)\}$.

For standardized portfolios, CDS-correlation matrices can be obtained from e.g. Reuters. However, given times-series for the CDS-spreads on obligors in any portfolio, these matrices can easily be computed using standard mathematical software.

A further issue remains. This is that the CDS correlation matrix is symmetric and thus only contains $m(m-1)/2$ pairwise CDS correlations. Hence, together with the m market CDS spreads we have $m(m+1)/2$ data observations, while there are m^2 unknown parameters in (4.2.1); the $m(m-1)$ different $\theta_{i,j}$-s and the m base intensities $\{a_i\}$. To make the number of model parameters and the number of market observations match, we hence assume that the $\theta_{i,j}$-s are the same for some of the ordered pairs (i, j), so that there are only $m(m-1)/2$ different $\theta_{i,j}$-s.

We now explain the calibration in more detail. First, we reduce the $m(m-1)$ unknown variables $\{\theta_{i,j}\}$ to a set of $(m-1)m/2$ different nonnegative parameters $\{d_q\} = \{d_1, d_2, \ldots d_{(m-1)m/2}\}$, so that the total number of model parameters are as many as the market observations. Secondly, we assume a exogenously given dependence matrix $\{D_{i,j}\}$ where $D_{i,j} \in \{1, 2, \ldots (m-1)m/2\}$ which determines the matrix $\{\theta_{i,j}\}$ according to $\theta_{i,j} = \pm d_{D_{i,j}}$, where the sign is the same as the market CDS correlation $\rho_{i,j}^{(\mathrm{CDS})}(T)$. It is a topic for future research to find methods to estimate the dependence matrix $\{D_{i,j}\}$. For example, from corporate data or from the rapidly increasing market of credit portfolio products, such as CDO's and basket default swaps. In this paper, the matrix $\{D_{i,j}\}$ is determined randomly, see Appendix 8.

Let $\boldsymbol{v} = (\{a_i\}, \{d_q\})$ denote the parameters describing the model and let $\{R_i(T; \boldsymbol{v})\}$ be the m different model T-year CDS spreads and $\{R_{i,M}(T)\}$ the corresponding market spreads. Furthermore, as above, we let $\rho_{i,j}(T; \boldsymbol{v}) = \mathrm{Corr}(1_{\{\tau_i \leq T\}}, 1_{\{\tau_j \leq T\}})$ denote the pairwise T-year default correlations. Here we have emphasized that the model quantities are functions of $\boldsymbol{v} = (\{a_i\}, \{d_q\})$ but suppressed the dependence of the matrix $\{D_{i,j}\}$, interest

rate, payment frequency, etc. The vector $\boldsymbol{v}$ is obtained as

$$\boldsymbol{v} = \operatorname*{argmin}_{\hat{\boldsymbol{v}}} \left[\delta_{\text{CDS}}(T; \hat{\boldsymbol{v}}) + \delta_{\text{corr}}(T; \hat{\boldsymbol{v}}) \right] \qquad (4.2.2)$$

where

$$\delta_{\text{CDS}}(T; \boldsymbol{v}) = F \sum_{i=1}^{m} \left(R_i(T; \boldsymbol{v}) - R_{i,M}(T) \right)^2$$

$$\delta_{\text{corr}}(T; \boldsymbol{v}) = \sum_{i=1}^{m} \sum_{j=i+1}^{m} \left(\rho_{i,j}(T; \boldsymbol{v}) - \beta \rho_{i,j}^{(\text{CDS})}(T) \right)^2 \qquad (4.2.3)$$

with $F > 0$ and $0 < \beta \leq 1$ exogenously chosen. The second expression in (4.2.3) is due to that we use $\beta \{ \rho_{i,j}^{(\text{CDS})}(T) \}$ as a proxy for $\{ \rho_{i,j}(T) \}$. It is possible to include F and β in the unknown parameter vector $\boldsymbol{v}$ and we make some further comments on this at the end of the present subsection.

If all CDS-correlations are positive, the minimization in (4.2.2) is performed with the constraint that all elements in $\boldsymbol{v}$ are nonnegative. However, if there are negative CDS-correlations, that is $\rho_{i,j}^{(\text{CDS})}(T) < 0$ for some pairs (i, j), then we require that $\theta_{i,j} = \text{sign}(\rho_{i,j}^{(\text{CDS})}(T)) d_{D_{i,j}} = -d_{D_{i,j}} < 0$, since it otherwise is difficult to generate negative default correlations. Because $\lambda_{t,i}$ must be positive and all parameters are nonnegative, we have to bound some of the $\{ d_q \}$ if there are negative CDS-correlations. It is then practical to assume that the dependence matrix $\{ D_{i,j} \}$ is constructed so that it splits $\{ d_q \}$ in two disjoint groups, $\{ d_q \} = \boldsymbol{d}_- \cup \boldsymbol{d}_+$ such that if $\rho_{i,j}^{(\text{CDS})}(T) < 0$ then $d_{D_{i,j}} \in \boldsymbol{d}_-$ and if $\rho_{i,j}^{(\text{CDS})}(T) \geq 0$ then $d_{D_{i,j}} \in \boldsymbol{d}_+$. Let N_i denote the sets of obligors $j \neq i$ which are negatively correlated with entity i, that is, where $\rho_{i,j}^{(\text{CDS})}(T) < 0$. Thus, if $j \in N_i$ then $d_{D_{i,j}} \in \boldsymbol{d}_-$ and the following constraints

$$a_i - \sum_{j \in N_i} a_i d_{D_{i,j}} > 0 \quad \text{that is,} \quad 1 > \sum_{j \in N_i} d_{D_{i,j}}, \qquad (4.2.4)$$

must simultaneously hold for every $i = 1, 2, \ldots, m$. These joint bounds finally determine the proper constraints on the parameters in $\boldsymbol{d}_-$, which heavily depend on the elements $D_{i,j}$ and the sign of $\rho_{i,j}^{(\text{CDS})}(T)$. If the number of negative CDS correlations are less than positive CDS correlations, it may be convenient to assume that each $d_p \in \boldsymbol{d}_-$ only appears once in the matrix $\{ D_{i,j} \}$ and use the constraints $d_p < \frac{1}{|N_i|}$ if $\theta_{i,j} = -d_p$ for some $j \in N_i$. Recall that in economic terms, negative CDS correlation, and thus negative jumps in the intensities for a obligor i, means that entity i benefits from defaults of obligors $j \in N_i$.

Let us finally give some remarks on the parameters β and F. A naive first attempt is to let $F = 1$ and $\beta = 1$ in the calibration (4.2.2). However, the market CDS spreads $R_{i,M}(T)$ are about 100 times smaller than $\rho_{i,j}^{(\text{CDS})}(T)$, which then implies unrealistic model CDS spreads. The problem can be avoided by letting $\sqrt{F} = 100$ so that $\sqrt{F} R_{i,M}(T)$ and $\rho_{i,j}^{(\text{CDS})}(T)$ are approximately in the same order. This leads to bad correlation fits, i.e. $\delta_{\text{corr}}(T; \boldsymbol{v})$ is big, when $\beta = 1$. In our examples, the calibrations produce default correlations

much smaller than the corresponding CDS correlations. Motivated by this we assume that $0 < \beta << 1$ and in our numerical studies we let $\beta = 0.05$ and $\sqrt{F} = 100$. This gives perfect correlation calibrations for our data sets where all entities in the CDS-correlation matrix are nonnegative, and reasonable calibrations when the correlation matrix contains both positive and negative entities (see Subsection 5.1). It is possible to include β and F in the parameter vector $\boldsymbol{v}$, and then decrease the set $\{d_q\}$ so that $|\{d_q\}| = m(m-1)/2 - 2$, where the total number of model parameters still are as many as the market observations.

5. Numerical studies

In this section we will use the theory developed in previous sections to study quantities of importance in active credit portfolio management. We consider the same parameterization of (2.1) as in Subsection 4.2, that is

$$\lambda_{t,i} = a_i \left(1 + \sum_{j=1, j\neq i}^{m} \varepsilon_{i,j} d_{D_{i,j}} 1_{\{\tau_j \leq t\}} \right),$$

where $\varepsilon_{i,j}$ is the sign of $\rho_{i,j}^{(\mathrm{CDS})}(T)$, and $\{D_{i,j}\}$ is a exogenously given matrix such that $D_{i,j} \in \left\{ 1, 2 \ldots \frac{(m-1)m}{2} \right\}$. Further, the d_q-s are $(m-1)m/2$ different nonnegative parameters which will be determined in the calibration, together with the base default intensities a_i.

In Subsection 5.1 we introduce two CDS portfolio, one in the European auto sector, the other in the European financial sector. These portfolios, which both consist of 10 companies, are used as a basis for the numerical studies in the rest of this section. For exogenously given dependence matrices $\{D_{i,j}\}$, we calibrate the portfolios against market CDS spreads and their correlations. In the calibrated portfolios, we then study the implied joint default and survival distributions and the implied univariate and bivariate conditional survival distributions (Subsection 5.2), the implied default correlations (Subsection 5.3), and finally the implied expected default times and expected ordered defaults times (Subsection 5.4).

5.1. **Two CDS portfolios.** Table 1 and Table 2 describe the two CDS portfolios which are used in our numerical studies and Table 3 and Table 4 their correlation matrices. The maturity was 5 years and the data was obtained from `Reuters` at February 15, 2007 for the auto portfolio and March 28, 2007 for the financial portfolio.

The correlation matrices are based on rolling 12 months 5-years CDS midpoint market spreads for each obligor, with a daily sampling frequency of the closing level of the spreads. In both portfolios, we have assumed a fictive relative dependence structure $\{D_{i,j}\}$ which are given in Table 11 and Table 12 in Appendix 8 together with a description how they where created. Further, we have also assumed a fictive recovery rate structure which is the same in both baskets. The interest rate was assumed to be constant and set to 3%, and the protection fees were assumed to be paid quarterly.

For each portfolio, the a_i-s and d_q-s are obtained by simultaneously calibrate the CDS spreads in Table 1 and Table 2 and the corresponding correlation matrices in Table 3 and Table 4, as described in Subsection 4.2. In both portfolios the CDS calibrations

Table 1: The auto companies and their 5 year market (2007-02-15) and model CDS spreads, the absolute calibration errors, and the recoveries. The spreads are given in bp.

Company name	Market	Model	abs.error	recovery %
Volvo AB	25.84	25.87	0.03336	32
BMW AG	9.415	9.593	0.178	48
Comp. Fi. Michelin SA	25.34	25.53	0.1915	45
Continental AG	43.66	43.68	0.01789	34
DaimlerChrysler AG	44	43.98	0.02175	42
Fiat SPA	58	58.02	0.016	41
Peugeot SA	24.84	24.9	0.06289	29
Renault SA	28.67	28.72	0.05989	39
Valeo SA	66	65.98	0.01812	51
Volkswagen AG	22.17	22.08	0.08343	41
Σ abs.cal.err			0.6828 bp	

Table 2: The financial companies and their 5 year market (2007-03-28) and model CDS spreads, the absolute calibration errors, and the recoveries. The spreads are given in bp.

Company name	Market	Model	abs.error	recovery %
ABN Amro Bank NV	6.085	6.225	0.1402	32
Barclays Bank PLC	7	6.9	0.1	48
BNP Paribas	6.665	6.562	0.1026	45
Commerzbank AG	9.335	9.41	0.07492	34
Deutsche Bank AG	13.59	13.5	0.08747	42
HSBC Bank PLC	7.25	7.247	0.002626	41
Hypovereinsbank AG	7	7.217	0.2173	29
The Royal Bank of Scotland PLC	7	6.844	0.1556	39
Banco Santander Central Hispano	8.25	8.22	0.02998	51
Unicredito Italiano SPA	9.915	9.989	0.07363	41
Σ abs.cal.err			0.9844 bp	

where perfect. The correlation fit for the financial portfolio was also perfect, as seen in Table 5, while the corresponding calibration for the auto case was mediocre. One possible explanation for the lesser performance in the auto portfolio, is that the negative jumps in the intensities are bounded, which may bound the absolute value of the negative CDS-correlations by a scalar smaller than one.

A quick look in Table 14 reveals that 15 (out of 18) "negative" parameters hit their upper bounds (for more details on this, see Appendix). Such limitations can be avoided by using a different parametrization of the intensities in (2.1), making the jumps-sizes also

Table 3: The auto CDS correlation matrix, based on 5-years CDS midpoint market spreads for each obligor, between 2006-02-15 and 2007-02-15, with a daily sampling frequency of the closing level of the spreads.

	VOLV	BMW	MICH	CONT	DCX	FIAT	PEUG	RENA	VALE	VW
VOLV	1									
BMW	0.63	1								
MICH	0.81	0.64	1							
CONT	-0.5	-0.69	-0.23	1						
DCX	0.12	0.47	0.51	0.13	1					
FIAT	0.67	0.97	0.76	-0.64	0.52	1				
PEUG	0.66	0.28	0.81	0.14	0.34	0.37	1			
RENA	0.55	0.24	0.79	0.1	0.42	0.39	0.82	1		
VALE	0.22	-0.42	0.2	0.44	-0.1	-0.31	0.39	0.41	1	
VW	0.12	0.66	0.47	-0.2	0.77	0.71	0.16	0.34	-0.44	1

Table 4: The financial CDS correlation matrix, based on 5-years CDS midpoint market spreads for each obligor, between 2006-03-28 and 2007-03-28, with a daily sampling frequency of the closing level of the spreads.

	ABN	BACR	BNP	CMZB	DB	HSBC	HVB	RBOS	BSCH	CRDIT
ABN	1									
BACR	0.91	1								
BNP	0.98	0.94	1							
CMZB	0.92	0.95	0.92	1						
DB	0.88	0.84	0.89	0.81	1					
HSBC	0.66	0.96	0.76	0.9	0.88	1				
HVB	0.82	0.9	0.89	0.89	0.8	0.85	1			
RBOS	0.93	0.98	0.95	0.94	0.85	0.98	0.88	1		
BSCH	0.84	0.95	0.89	0.95	0.78	0.88	0.89	0.92	1	
CRDIT	0.78	0.9	0.82	0.91	0.76	0.81	0.87	0.84	0.96	1

be functions of the level of the intensity. To be more specific, the bigger the intensity, the bigger negative jumps are allowed.

From Table 14 and Table 15 in Appendix, we see that in the auto portfolio, the base intensities can have positive jumps up to 589% of their "base values" a_i, and up to 1749 % in the financial portfolio.

5.2. The implied default and survival distributions and the conditional survival distributions.

In the credit literature today, risk-neutral distributions are often called *implied* distributions. Here "implied" is refering to the fact that the quantities are retrieved

Table 5: The average, median, min and max absolute calibration-errors in percent of the scaled market CDS-correlations, i.e. $\{\beta\rho_{i,j}^{(\text{CDS})}(T)\}$, where $\beta = 0.05$

Portfolio	mean	median	min	max
Auto	29.2	18.8	0.347	122
Financial	1.43	0.213	0.00988	13.1

from market data via a model. The implied (joint) default and survival distributions at different time points, are important quantities for a credit manager. The results in Section 3 provides formulas for computing these expressions. In this subsection we use them to find the implied default and survival distributions, as well as conditional survival distributions, for different pairs of obligors, in the calibrated portfolios.

We want to study the bivariate default and survival distributions for the pairs Fiat, BMW and Continental, BMW. Given the CDS spreads and their correlations, it may in general be difficult to draw some qualitative conclusions about these bivariate probabilities and their mutual relations, without actually computing them. The CDS spreads for Fiat and BMW are positively correlated while Continental and BMW are negatively correlated, and the difference in percent between the spreads for Continental and Fiat are $(58 - 43.66)/58 = 24\%$. From this, we intuitively guess that BMW-s bivariate default probabilities with Fiat should be bigger than the bivariate default probabilities with Continental. Conversely, the bivariate survival distributions of the pair Fiat, BMW should be smaller than for Continental, BMW. These hypothesis are confirmed by the Figures 1, 2, 3 and 4. Similar shapes of the bivariate default and survival distributions are obtained by obligors in the financial portfolio, as seen in Figure 5 and 6.

We also note that the CDS spreads for Continental is positively correlated with the spreads for DaimlerChrysler, Peugeot, Renault and Valeo. We therefore suspect that the conditional survival distributions for continental are decreasing with the number of defaults among DaimlerChrysler, Peugeot, Renault and Valeo. For example, when s is fixed, we guess that the survival distribution $\mathbb{P}\left[\tau_{\text{Cont}} > t \,|\, \tau_{\text{DCX}} < s\right]$ as function of t for $t > s$, should lie above the curve $\mathbb{P}\left[\tau_{\text{Cont}} > t \,|\, \tau_{\text{DCX}} < s, \tau_{\text{Peu}} < s\right]$. This claim is supported by Figure 7 for $s = 10$ and $10 \leq t \leq 104$ (and also by Figure 9, for a similar test in the financial portfolio). Furthermore, the CDS spreads for Continental are negatively correlated with the spreads for Volvo, BMW, Michelin, Fiat and Volkswagen. In view of the above results, it is tempting to believe that the conditional survival distributions for continental, are increasing with the number of defaults among for Volvo, BMW, Michelin, Fiat and Volkswagen.

We investigate this for $s = 10$ and $10 \leq t \leq 104$, and note that the claim is only true on the interval $10 \leq t \leq 45$, as seen in Figure 8. For $t > 53$, we see that the curves do not lie in increasing order with increasing amount of negatively correlated defaults. One possible explanation for this is that the negative jumps in the intensities where bounded, in the specification that we use, which implies that the effect of a negative jump will diminish as time progress since several positive jumps then have occurred previously.

Figure 1: The implied bivariate default distribution for Fiat and BMW (left) and Continental and BMW (right) in the auto portfolio.

Figure 2: The isolines for the implied bivariate default distribution for Fiat and BMW (left) and Continental and BMW (right) in the auto portfolio.

We also compare univariate conditional survival distribution, with bivariate conditional survival distribution, in the banking portfolio. In Figure 9 and Figure 10 we see that
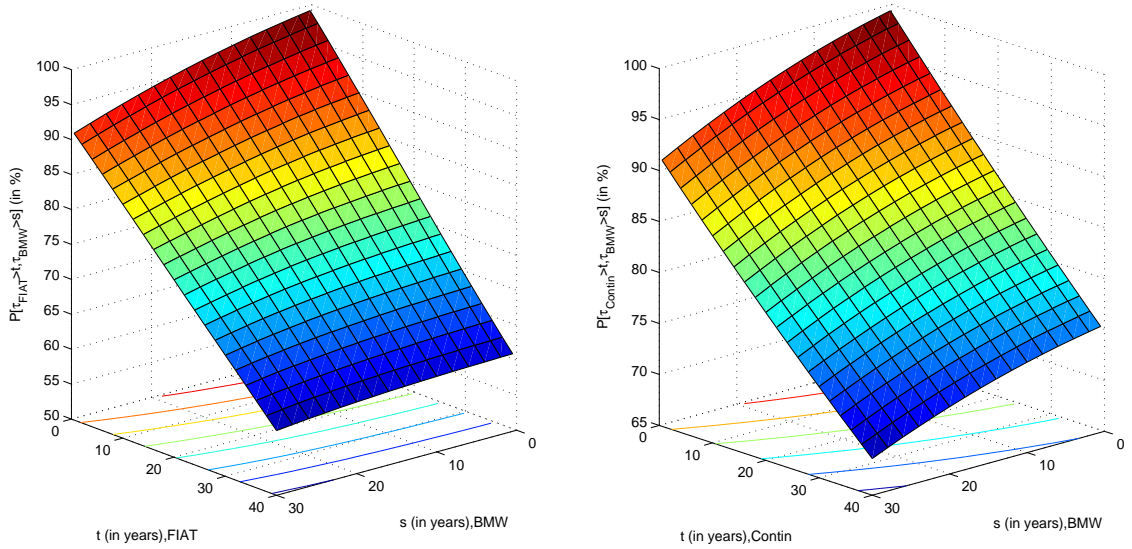
Figure 3: The implied bivariate survival distribution for Fiat and BMW (left) and Continental and BMW (right) in the auto portfolio.

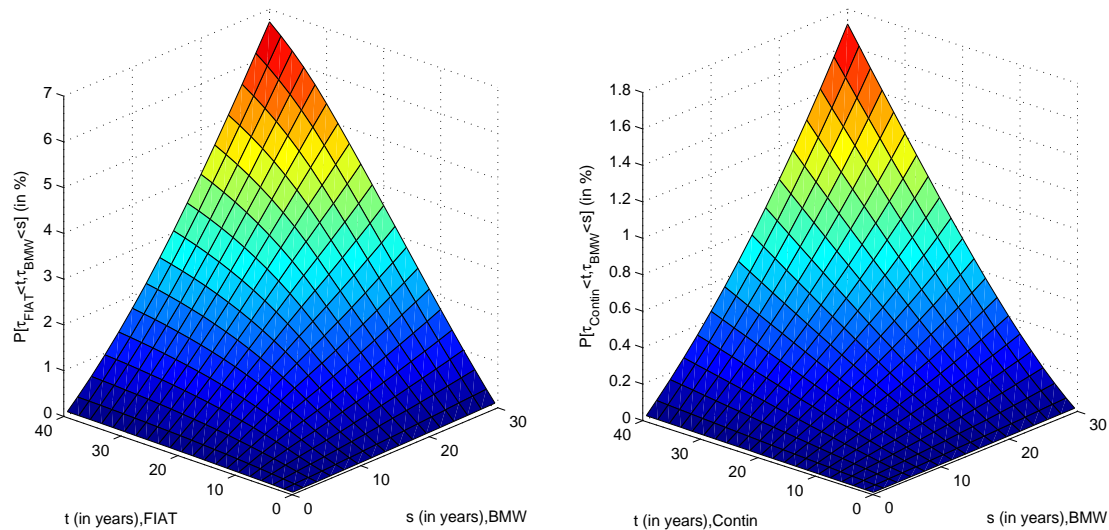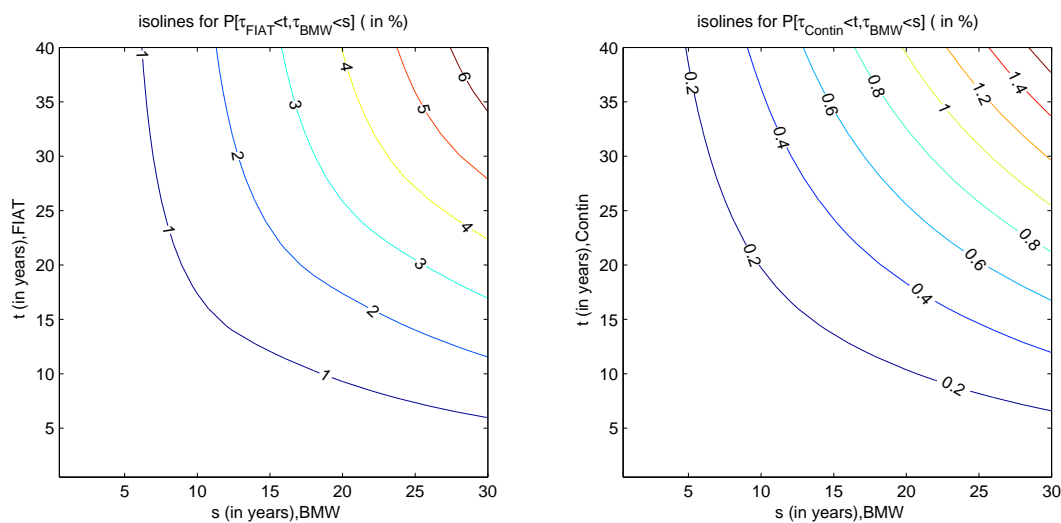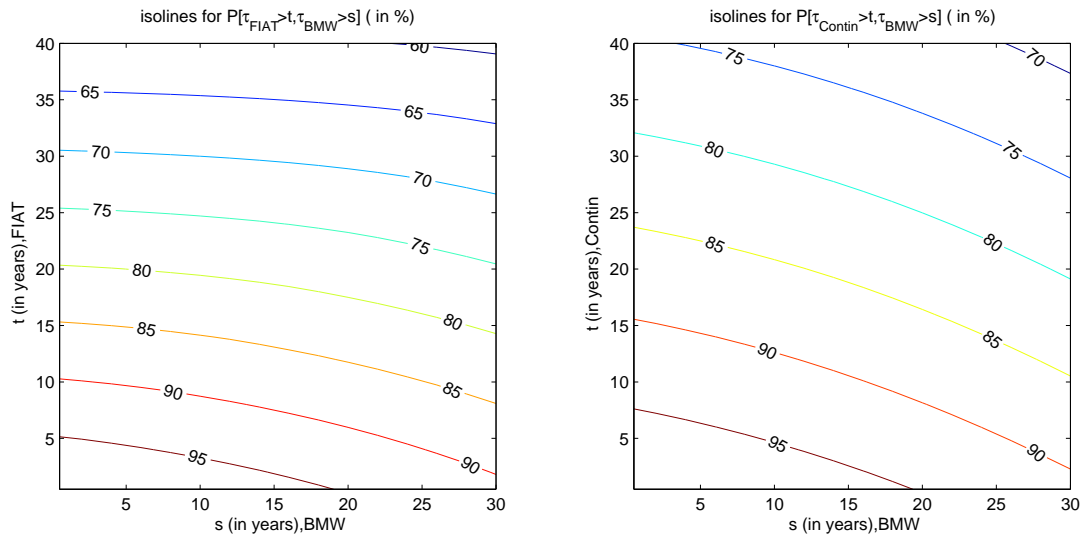**Figure 4:** The isolines for the implied bivariate survival distribution for Fiat and BMW (left) and Continental and BMW (right) in the auto portfolio.

the bivariate conditional survival distribution declines much faster than the corresponding univariate conditional survival distribution.
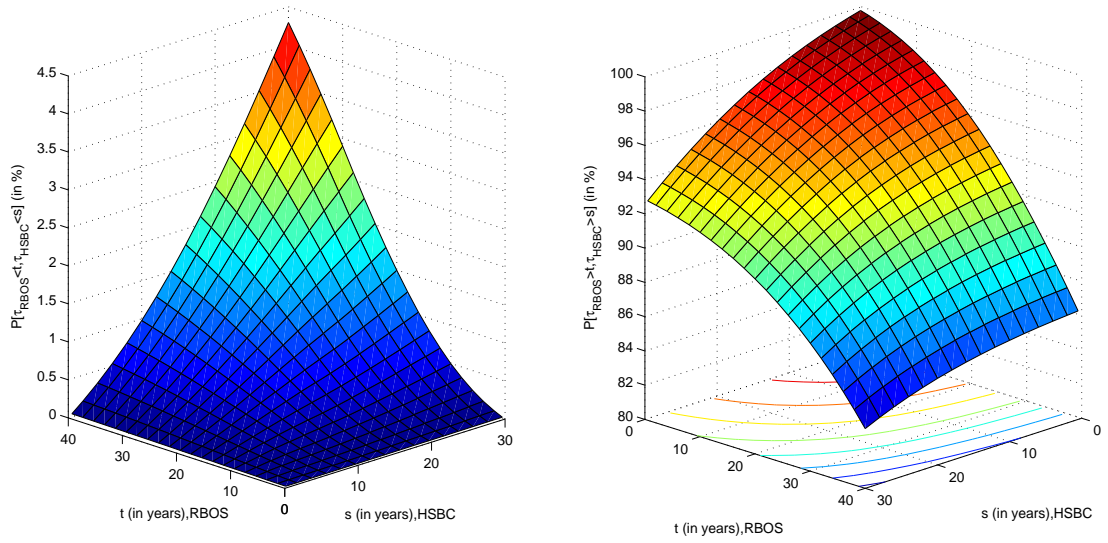
Figure 5: The implied bivariate default (left) and survival (right) distributions for Royal Bank of Scotland and HSBC Bank in the financial portfolio.
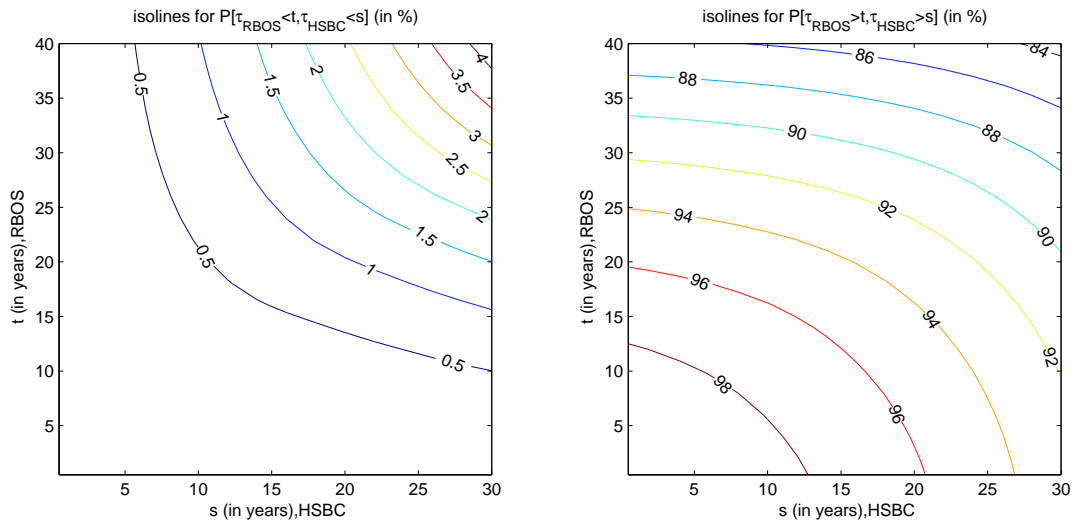
Figure 6: The isolines for the implied bivariate default (left) and survival (right) distributions for Royal Bank of Scotland and HSBC Bank in the financial portfolio.

So far we have only computed joint bivariate distributions, or distributions involving two time points. To show that we can handle distributions with all 10 obligors for 10
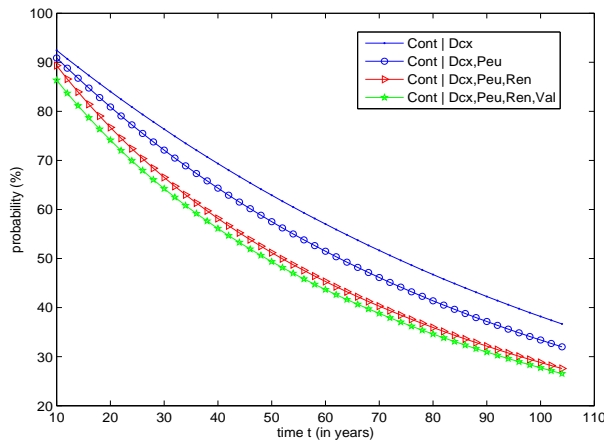
Figure 7: The survival distribution for Continental, conditional on defaults before time 10 years, by firms which are positively correlated with Continental. The firms which have defaulted are indicated in the legend.
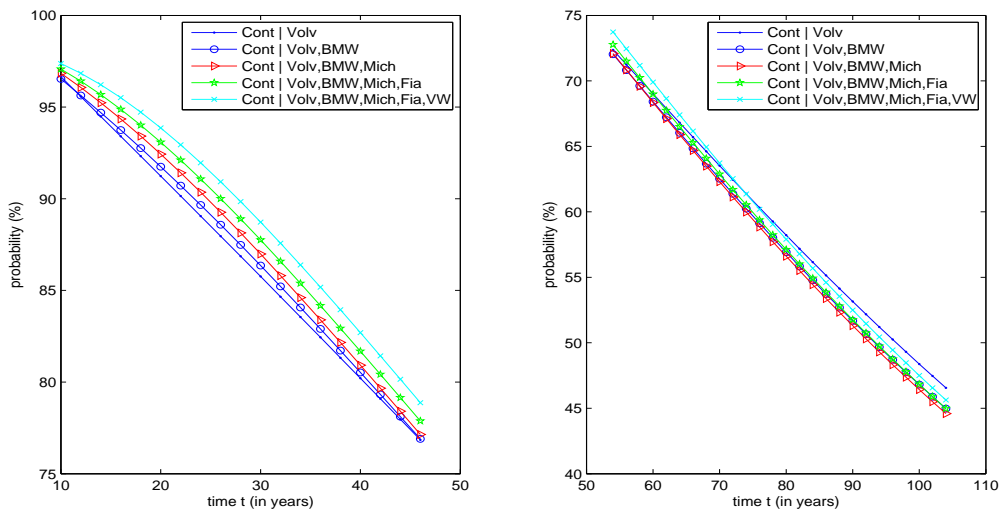
Figure 8: The survival distribution for Continental, conditional on defaults before time 10 years, by firms which are negatively correlated with Continental. Left figure $t < 45$, right figure $t > 52$. The firms which have defaulted are indicated in the legend.

different time points, Table 6 and 7 displays the joint multivariate default and survival distributions for all obligors, in each portfolio. Recall that implied default probabilities are often substantially larger then the "real" so called actuarial default probabilities.

Figure 9: The survival distribution for ABN Amro, conditional on defaults before time 10 years. The firms which have defaulted are indicated in the legend.

Figure 10: The joint survival distribution for ABN Amro and BSCH, conditional on defaults before time 10 years. The firms which have defaulted are indicated in the legend.

Table 6: The multivariate default and survival probabilities $\mathbb{P}\left[\tau_{\mathrm{Volv}} > n, \ldots, \tau_{\mathrm{VW}} > 10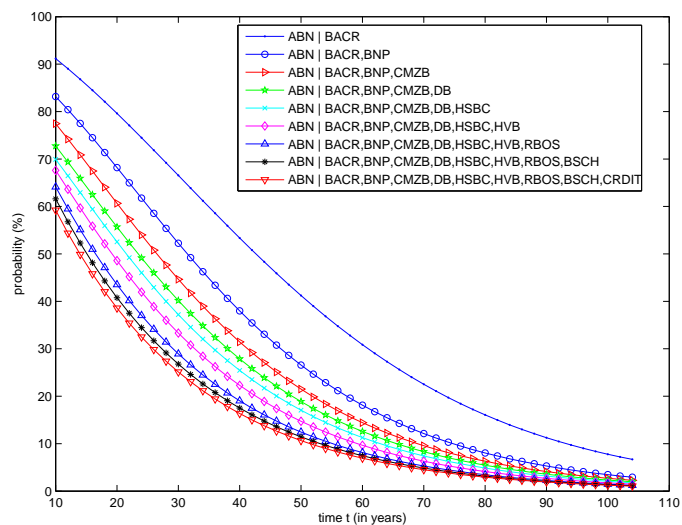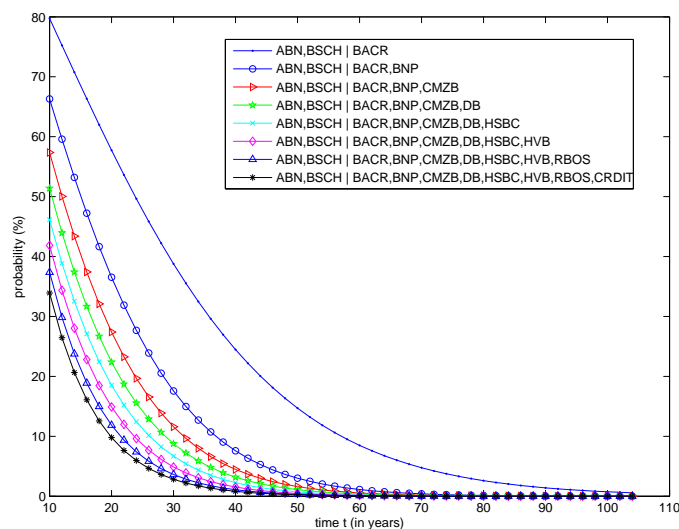n\right]$ and $\mathbb{P}\left[\tau_{\mathrm{Volv}} \leq n, \tau_{\mathrm{BMW}} \leq 2n, \ldots, \tau_{\mathrm{VW}} \leq 10n\right]$ (in %) where $n = 0.5, 1, 1.5, \ldots, 4$, in the auto portfolio.

	$n = 0.5$	$n = 1$	$n = 1.5$	$n = 2$	$n = 2.5$	$n = 3$	$n = 3.5$	$n = 4$
multi.def.prob	11.1	21	29.8	37.6	44.5	50.7	56.2	61
multi.surv.prob	98.3	96.6	94.9	93.1	91.4	89.7	88	86.2

Table 7: The multivariate default and survival probabilities $\mathbb{P}\left[\tau_{\mathrm{ABN}} > n, \ldots, \tau_{\mathrm{CDRIT}} > 10n\right]$ and $\mathbb{P}\left[\tau_{\mathrm{ABN}} \leq n, \tau_{\mathrm{BACR}} \leq 2n, \ldots, \tau_{\mathrm{CDRIT}} \leq 10n\right]$ (in %) where $n = 0.5, 1, 1.5, \ldots, 4$, in the financial portfolio.

	$n = 0.5$	$n = 1$	$n = 1.5$	$n = 2$	$n = 2.5$	$n = 3$	$n = 3.5$	$n = 4$
multi.def.prob	29.4	49.3	63.3	73.3	80.4	85.6	89.5	92.3
multi.surv.prob	99.3	98.6	97.9	97.3	96.7	96.1	95.5	95

5.3. **The implied default correlations.** It may be of interest for a credit manager to have a quantitative grasp of the pairwise default correlations $\rho_{i,j}(t) = \mathrm{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$ between two obligors $i \neq j$, as a function of time t. Especially, if we can study several pairs for a fixed obligor i, simultaneously. Recall that we have calibrated $\rho_{i,j}(T)$ against $0.05\rho_{i,j}^{(\mathrm{CDS})}(T)$ for $T = 5$, as discussed in Subsection 4.2.

We first consider the same example as in the previous subsection, where the CDS spreads for Continental is positively correlated with DaimlerChrysler, Peugeot, Renault and Valeo, and negatively correlated with Volvo, BMW, Michelin, Fiat and Volkswagen. We therefore suspect that for most time points t, the corresponding default correlations $\rho_{\mathrm{Cont},j}(t)$ are positive for $j = \mathrm{Volv}, \mathrm{BMW} \ldots, \mathrm{CW}$ and negative for $j = \mathrm{DCX}, \mathrm{Peu}, \ldots, \mathrm{Valeo}$. This is confirmed by Figure 11. Note that the correlations have parabolic shapes as function of time t.

Furthermore, the CDS-correlation matrix in Table 4 indicate a strong positive correlation among the different CDS-spreads for the banks. This is also the case for the corresponding default correlations, as seen in Figure 12, which displays the correlation between Deutsche Bank and the other banks.

5.4. **The implied expected default times and their ordering.** In this subsection we study implied expected default times $\mathbb{E}\left[\tau_i\right]$ and the implied expected ordered default times $\mathbb{E}\left[T_k\right]$ for the two calibrated portfolios in Subsection 5.1.

If we order the sequence $\{\mathbb{E}\left[\tau_i\right]\}$ in increasing order $\{\mathbb{E}\left[\tau_{i_k}\right]\}$ so that $\mathbb{E}\left[\tau_{i_k}\right] < \mathbb{E}\left[\tau_{i_{k+1}}\right]$ and study the corresponding sequence of model CDS-spreads $\{R_{i_k}\}$, one would expect that $\{R_{i_k}\}$ are strictly decreasing. However, from Table 8 and Table 9 we see that this is far from true.
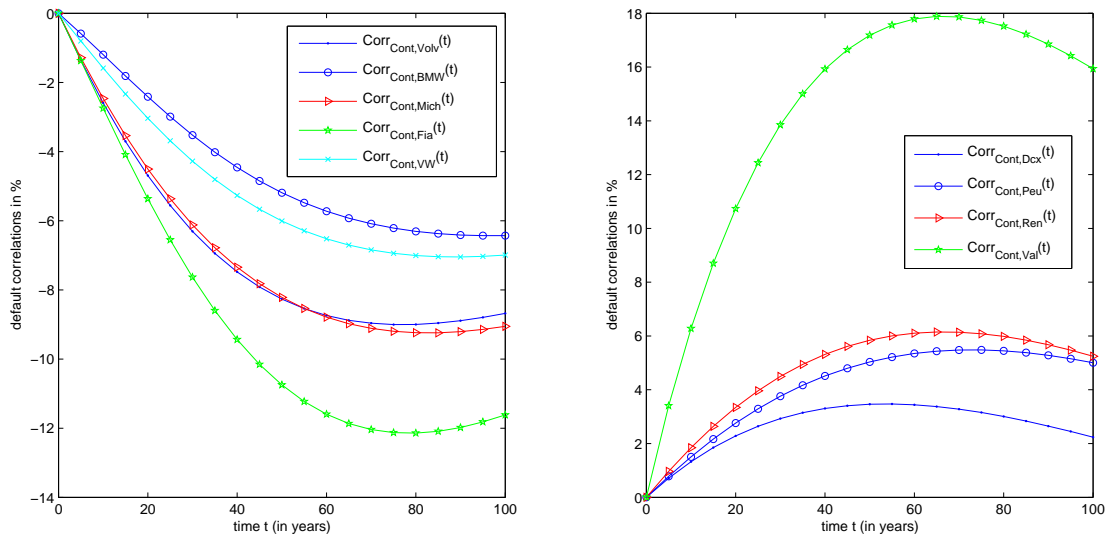
Figure 11: The default correlations between Continental and the companies in the auto portfolio which are negatively correlated (left) and positively correlated (right) with Continental.
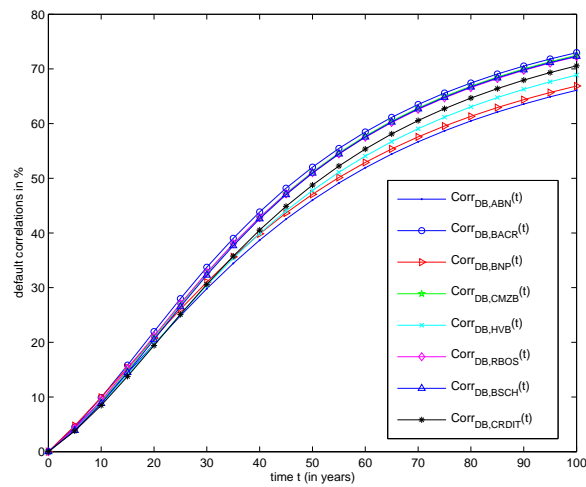

Figure 12: The default correlations between Deutsche Bank and the other banks in the financial portfolio.

Table 8: The expected default times in the auto portfolio, sorted in increasing order $\mathbb{E}[\tau_{i_k}] < \mathbb{E}[\tau_{i_{k+1}}]$, and the corresponding model CDS-spreads.

	VALE	FIAT	DCX	RENA	PEUG	MICH	VOLV	VW	CONT	BMW
$\mathbb{E}[\tau_{i_k}]$	47.3	66.9	68.1	78.9	86.3	86.6	89.2	91.8	116	118
R_{i_k}	66	58	44	28.7	24.9	25.5	25.9	22.1	43.7	9.59

Table 9: The expected default times in the financial portfolio, sorted in increasing order $\mathbb{E}[\tau_{i_k}] < \mathbb{E}[\tau_{i_{k+1}}]$, and the corresponding model CDS-spreads.

	DB	BSCH	CMZB	BACR	CRDIT	RBOS	HSBC	HVB	BNP	ABN
$\mathbb{E}[\tau_{i_k}]$	113	114	116	117	120	120	126	127	131	133
R_{i_k}	13.5	8.22	9.41	6.9	9.99	6.84	7.25	7.22	6.56	6.23

In the financial portfolio, the spreads $\{R_{i_k}\}$ are not decreasing. The auto portfolio has a decreasing trend in the sequence $\{R_{i_k}\}$, except for the Continental spread, $R_{\text{Cont}} = 43.7$ which is the forth biggest spread, while $\mathbb{E}[\tau_{\text{Cont}}] = 116$ years, is the ninth biggest expected default time in the auto portfolio. These irregularities are likely due to the dependence structure in (2.1), (4.2.1), which plays a major roll in the calibration. For example, in the auto case, the CDS spread for Continental is negatively correlated with the spreads for Volvo, BMW, Michelin, Fiat and Volkswagen which means that Continental will benefit from defaults on these firms. In Table 3 we also note that for Continental, the average of the absolute value for the negative correlations is bigger than the corresponding quantity for the positive correlations and no other car company has so many negative default correlations. These observations *may* explain why $\mathbb{E}[\tau_{\text{Cont}}]$ is the ninth biggest in the sequence $\{\mathbb{E}[\tau_{i_k}]\}$. Hence, from an average default timing point of view, Continental is the second less riskiest company in the auto portfolio, even though the CDS spread is the third biggest. Note however that the base intensity a_{Cont} is the third biggest in the auto basket, see Table 13 in Appendix.

These examples indicate that it can be misleading to use the reverse ordering of the CDS spreads as a measure for the relative default riskiness among the obligors in the portfolio.

Table 10: The expected ordered default times $\mathbb{E}[T_k]$.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
Auto	17.8	33.7	50.2	62.9	74.1	85.1	96.8	111	131	186
Financial	85.3	98.7	107	113	118	124	129	136	145	162

Other observations are that the difference between the smallest and biggest expected default time, is 19.5 years in the financial portfolio and 71 years in the auto portfolio.

Also note that in the banking portfolio, the smallest expected default time lie between the expected value of the fourth and fifth ordered default time and the biggest between the seventh and eight. The corresponding quantities in the auto case lie between the second and third, and between the eight and ninth expected ordered default time.

6. COMPUTATION OF THE MATRIX EXPONENTIAL

All results derived in this paper include computations of the matrix exponential. In this section we describe the method for computing $e^{\boldsymbol{Q}t}$ that is used throughout this article, the so called uniformization method (sometimes also is called the randomization method). It works as follows. Let $\Lambda = \max\left\{|\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}}| : \boldsymbol{j} \in \boldsymbol{E}\right\}$ and set $\widetilde{\boldsymbol{P}} = \boldsymbol{Q}/\Lambda + \boldsymbol{I}$. Then, $e^{\widetilde{\boldsymbol{P}}\Lambda t} = e^{\boldsymbol{Q}t}e^{\Lambda t}$ since $\boldsymbol{I}$ commutes with all matrices, and using the definition of the matrix exponential renders

$$e^{\boldsymbol{Q}t} = \sum_{n=0}^{\infty} \widetilde{\boldsymbol{P}}^n e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}. \tag{6.1}$$

Recall that $\boldsymbol{p}(t) = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}$ and define $\widetilde{\boldsymbol{p}}(t, N) = \boldsymbol{\alpha}\sum_{n=0}^{N}\widetilde{\boldsymbol{P}}^n e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}$. Furthermore, for a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\|x\|_1$ denote the L_1 norm, that is $\|x\|_1 = \sum_{i=1}^{n}|x_i|$. Given $\boldsymbol{Q}$, the uniformization method allows us to find the L_1 approximation error for $\widetilde{\boldsymbol{p}}(t, N)$ apriori, as shown in the following lemma, stated in e.g. [16] and [27], but without a proof.

Lemma 6.1. *Let $\varepsilon > 0$ and pick $N(\varepsilon)$ so that $1 - \sum_{n=0}^{N(\varepsilon)} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} < \varepsilon$. Then,*

$$\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t, N(\varepsilon))\|_1 < \varepsilon. \tag{6.2}$$

Proof. By construction, all elements in $\widetilde{\boldsymbol{P}}$ are in $[0, 1]$ and all rows in $\widetilde{\boldsymbol{P}}$ sums up to one. We can therefore view $\widetilde{\boldsymbol{P}}$ as a transition matrix for a discrete time Markov chain on $\boldsymbol{E}$. Since $\boldsymbol{\alpha}$ is a probability distribution on $\boldsymbol{E}$ we conclude that $\|\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n\|_1 = 1$ for all $n \in \mathbb{N}$. These observations imply

$$\begin{aligned}
\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t, N(\varepsilon))\|_1 &= \left\|\sum_{n=N(\varepsilon)+1}^{\infty}\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}\right\|_1 \\
&\leq \sum_{n=N(\varepsilon)+1}^{\infty}\|\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n\|_1 e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} \\
&= 1 - \sum_{n=0}^{N(\varepsilon)}e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} < \varepsilon
\end{aligned} \tag{6.3}$$

which proves the lemma. $\qquad\square$

The lemma implies that, given $\boldsymbol{Q}$, we can for any $\varepsilon > 0$ find a $N(\varepsilon)$ so that $\widetilde{\boldsymbol{p}}(t, N(\varepsilon))$ approximates $\boldsymbol{p}(t)$ with an accumulated absolute error which is less than ε. Note that the sharp error estimation in Lemma 6.1 relies on a probabilistic argument leading to $\|\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n\|_1 = 1$ for all $n \in \mathbb{N}$. It is tempting to try to prove (6.2) without this observation, by using that $\|\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n\|_1 \leq \|\boldsymbol{\alpha}\|_1\|\widetilde{\boldsymbol{P}}\|_1^n = \|\widetilde{\boldsymbol{P}}\|_1^n$ where $\|\widetilde{\boldsymbol{P}}\|_1$ is the corresponding matrix

norm, and then try show that $\|\widetilde{\boldsymbol{P}}\|_1$ is smaller or equal to one. However, it is easy to see that $\|\widetilde{\boldsymbol{P}}\|_1 > 1$ for $\widetilde{\boldsymbol{P}} = \boldsymbol{Q}/\Lambda + \boldsymbol{I}$ when $\boldsymbol{Q}$ is the generator of a transient Markov process on a finite state $\boldsymbol{E}$ with a final absorbing state, where $\Lambda = \max\left\{|\boldsymbol{Q}_{\boldsymbol{j,j}}| : \boldsymbol{j} \in \boldsymbol{E}\right\}$. This implies that if we use the uniformization method for an arbitrary matrix $\boldsymbol{Q}$, which is not a generator, it may be difficult to find effective apriori error estimates. For such matrices the elements in $\widetilde{\boldsymbol{P}}$ may not even be positive, which makes this method no better than the standard Taylor-series expansion method.

The probabilistic argument for the matrix $\widetilde{\boldsymbol{P}}$ in Lemma 6.1 is no coincidence. The following result can be found in [20].

Theorem 6.2. *Let $(Y_t)_{t\geq 0}$ be a Markov jump process on a finite state $\boldsymbol{E}$ with generator $\boldsymbol{Q}$ where $\Lambda = \max\left\{|\boldsymbol{Q}_{\boldsymbol{j,j}}| : \boldsymbol{j} \in \boldsymbol{E}\right\} < \infty$. Then there exists a discrete time Markov chain $(X_n)_{n=0}^{\infty}$ on $\boldsymbol{E}$ with transition matrix $\widetilde{\boldsymbol{P}} = \boldsymbol{Q}/\Lambda + \boldsymbol{I}$ and a Poisson process N_t with intensity Λ, independent of $(X_n)_{n=0}^{\infty}$, such that the processes $(X_{N_t})_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ have the same finite dimensional distributions.*

Recall that the $\boldsymbol{p}(t) = (\mathbb{P}[Y_t = \boldsymbol{j}])_{\boldsymbol{j}\in\boldsymbol{E}}$ so $\boldsymbol{p_j}(t) = \mathbb{P}[Y_t = \boldsymbol{j}]$ and Theorem 6.2 implies that

$$
\begin{aligned}
\boldsymbol{p_j}(t) &= \mathbb{P}[Y_t = \boldsymbol{j}] \\
&= \mathbb{P}[X_{N_t} = \boldsymbol{j}] \\
&= \sum_{n=0}^{\infty} \mathbb{P}[X_{N_t} = \boldsymbol{j} \mid N_t = n]\,\mathbb{P}[N_t = n] \\
&= \sum_{n=0}^{\infty} \mathbb{P}[X_n = \boldsymbol{j}]\,e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}.
\end{aligned}
\tag{6.4}
$$

Define the row vectors $\boldsymbol{\varphi}(n) = \left(\boldsymbol{\varphi_j}(n)\right)_{\boldsymbol{j}\in\boldsymbol{E}}$ for $n \in \mathbb{N}$ as $\boldsymbol{\varphi_j}(n) = \mathbb{P}[X_n = \boldsymbol{j}]$ when $n \in \mathbb{N}\setminus\{0\}$ and $\boldsymbol{\varphi}(0) = \boldsymbol{\alpha}$. Theorem 6.2 then implies that $\boldsymbol{\varphi}(n) = \boldsymbol{\varphi}(n-1)\widetilde{\boldsymbol{P}}$ which together with Equation (6.4) renders

$$
\boldsymbol{p}(t) = \sum_{n=0}^{\infty} \boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}.
\tag{6.5}
$$

But we also know that $\boldsymbol{p}(t) = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}$, so (6.5) is therefore Equation (6.1) restated.

Further benefits with the uniformization method is that all entries in $\widetilde{\boldsymbol{p}}(t, N(\varepsilon))$ are positive so there are no cancelation effects and the approximation error decreases monotonically with increasing N. If we set $f(t, N) = 1 - \sum_{n=0}^{N} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}$ then $\frac{\partial f(t,N)}{\partial t} = e^{-\Lambda t}\frac{(\Lambda t)^N}{N!} > 0$ so for a fixed N, the approximation error is bounded by a strictly increasing function in t. This is practical, since we then only have to compute one error tolerance for T, that will uniformly bound the error $\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t, N)\|_1$ for all $t \leq T$. For example, when approximating $\sum_{n=1}^{n_T} \boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}e^{-rt_n}$ where $t_1 < \ldots < t_{n_T} = T$, we choose $N(\varepsilon/n_T)$ so $1 - \sum_{n=0}^{N(\varepsilon/n_T)} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} < \frac{\varepsilon}{n_T}$ which implies that the total approximation error for the sum $\sum_{n=1}^{n_T} \boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}e^{-rt_n}$ is smaller than ε (since $e^{-rt_n} \leq 1$ for every n).
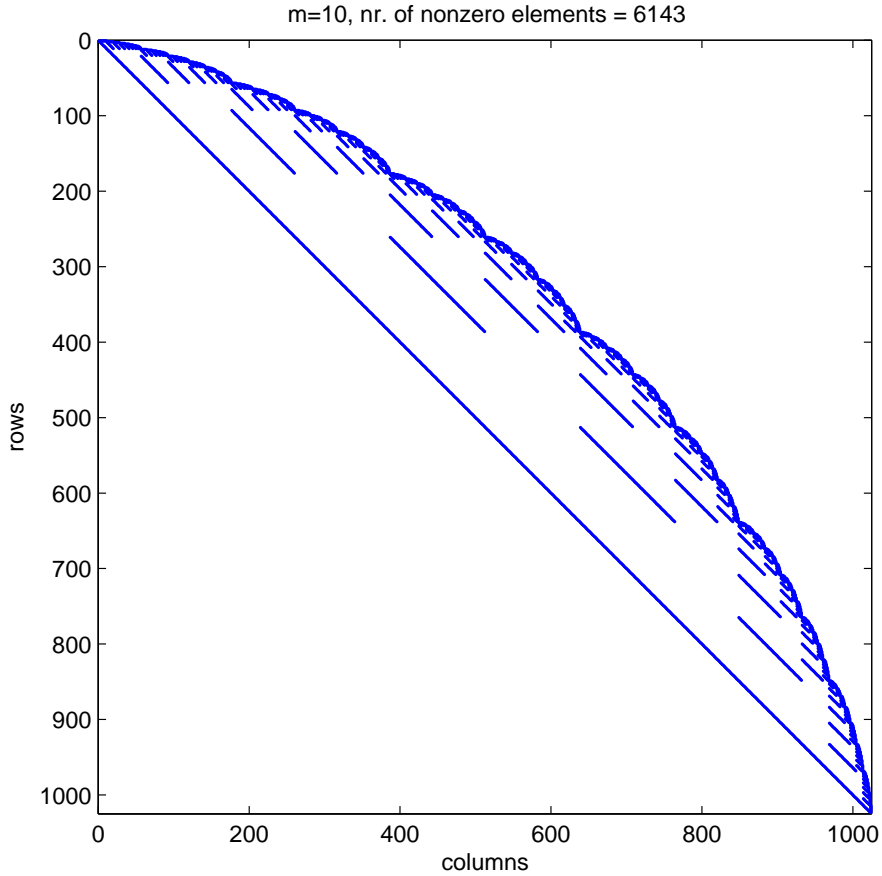
Figure 13: The structure of the nonzero elements in the sparse matrix $\boldsymbol{Q}$ where $m = 10$.

A further point is that our matrices in general are very large, for example if $m = 10$ then the generator has $2^{10} = 1024$ rows and thus contain $2^{20} \approx 1$. millon entries. However, at the same time it is extremely sparse, see Figure 13. For $m = 10$ there are only 0.59% nonzero entries in $\boldsymbol{Q}$, and hence only about 6100 elements have to be stored, which roughly is the same as storing a full quadratic matrix with 78 rows.

A final point is that we are not interested in finding the matrix exponential itself, but only the probability vector $\boldsymbol{p}(t)$, or a subvector of $\boldsymbol{p}(t)$. This is important, since computing $e^{\boldsymbol{Q}t}$ is very time and memory consuming compared with computing $\boldsymbol{\alpha}e^{\boldsymbol{Q}t}$.

For more on the uniformization method with applications in credit derivatives valuations and credit risk, see e.g. [19] and [23].

7. DISCUSSION AND CONCLUSIONS

In this paper we considered the intensity based default contagion model (2.1), where the default intensity of one firm is allowed to change when other firms default. The model was reinterpreted in terms of a Markov jump process, a so called multivariate phase-type distribution. This reinterpretation made it possible to derive practical formulas for many quantities, such as multivariate default and survival distributions, conditional multivariate distributions, marginal distributions, multivariate densities, correlations, expected default times, CDS-spreads and so on.

In the model we used two CDS portfolios for numerical studies, one in the European auto sector, the other in the European financial sector. Both baskets contained 10 companies. For an exogenously given dependence matrices $\{D_{i,j}\}$, we calibrated the portfolios against their market CDS spreads and the corresponding CDS-correlations. In both portfolios the CDS-fits where perfect, and in the financial case the correlation fit was also perfect, while the autos correlation matching was mediocre.

We then computed the implied joint default and survival distributions, the implied univariate and bivariate conditional survival distributions, the implied default correlations, and the implied expected default times and expected ordered defaults times. Qualitatively, many of the results where as expected. However it would seem rather impossible to guess the sizes of the probabilities and other quantities, without computation.

Future extensions of the model (2.1) is for example to include first-to-default swaps, other portfolio credit derivatives and corporate information, so that $\{D_{i,j}\}$ can be determined more realistically. Further empirical investigations of the approximation $\rho_{i,j}(T) \sim \beta \rho_{i,j}^{(\text{CDS})}(T)$ is also needed.

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8. APPENDIX

The dependence matrix $\{D_{i,j}\}$ for the financial portfolio was generated by drawing a random matrix where the entities lie in the interval $1, 2, \ldots, 45$. If some of the elements in $1, 2, \ldots, 45$ are not present in the sampling, we removed doublets in $D_{i,j}$ until all integers between 1 and 45 were present.

The $\{D_{i,j}\}$ matrix for the auto portfolio was created in the same way. However, we also made sure that $\{D_{i,j}\}$ was constructed so that it split $\{d_q\}$ in two disjoint groups, $\{d_q\} = \boldsymbol{d}_- \cup \boldsymbol{d}_+$ such that if $\rho_{i,j}^{(\text{CDS})}(T)) < 0$ then $d_{D_{i,j}} \in \boldsymbol{d}_-$ and if $\rho_{i,j}^{(\text{CDS})}(T) \geq 0$ then $d_{D_{i,j}} \in \boldsymbol{d}_+$, where $\rho_{i,j}^{(\text{CDS})}(T))$ is the CDS-correlation matrix, retrieved from market data. Furthermore, we also constructed $\boldsymbol{d}_-$ so that $d_p \in \boldsymbol{d}_-$ only appeared once in the matrix

$\{D_{i,j}\}$. Then we constrained the parameters in $\boldsymbol{d}_-$ as follows

$$d_p \leq \frac{1}{|N_i|} - 0.005 \quad \text{if} \quad \theta_{i,j} = -d_p \quad \text{for some} \quad j \in N_i,$$

where N_i is the set of obligors $j \neq i$ which are negatively correlated with entity i, that is, where $\rho_{i,j}^{(\text{CDS})}(T) < 0$.

Table 11: The dependence matrix $D_{i,j}$ for the autos portfolio. Entries with a negative subscript indicates that the corresponding entry in the correlation matrix is negative.

	VOLV	BMW	MICH	CONT	DCX	FIAT	PEUG	RENA	VALE	VW
VOLV	0	28	45	5_	29	25	10	26	36	2
BMW	10	0	16	34_	35	31	40	7	43_	1
MICH	27	41	0	20_	31	7	37	29	39	40
CONT	22_	6_	4_	0	17	24_	29	26	40	9_
DCX	40	15	2	21	0	7	37	17	8_	13
FIAT	45	42	25	19_	23	0	23	32	44_	23
PEUG	21	42	25	29	32	17	0	25	12	13
RENA	2	41	27	39	35	39	13	0	11	21
VALE	37	18_	12	25	14_	38_	15	31	0	3_
VW	16	40	25	30_	25	27	39	28	33_	0

Table 12: The dependence matrix $D_{i,j}$ for the financial portfolio.

	ABN	BACR	BNP	CMZB	DB	HSBC	HVB	RBOS	BSCH	CRDIT
ABN	0	10	31	39	22	28	15	20	1	2
BACR	13	0	34	4	29	17	35	12	35	41
BNP	8	6	0	22	2	15	35	19	21	44
CMZB	41	24	24	0	15	40	5	31	2	30
DB	12	31	15	21	0	11	15	38	4	33
HSBC	17	32	3	27	4	0	12	37	43	22
HVB	29	42	12	9	31	14	0	18	27	31
RBOS	14	1	45	23	1	8	40	0	11	16
BSCH	9	26	17	7	14	32	35	7	0	18
CRDIT	10	25	35	33	2	36	11	15	19	0

Table 13: The calibrated base intensities a_i.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
Auto	34.7	16.5	43.2	65.4	71.5	94.4	31.5	43.3	128	33.7	$\times 10^{-4}$
Financial	7.85	10.5	10.5	11.4	20.7	10.6	8.48	8.70	13.8	14.8	$\times 10^{-4}$

Table 14: The dependence variables d_q s.t $\theta_{i,j} = \pm d_{D_{i,j}}$ for the autos portfolio. Entries with a negative subscript indicates that $\theta_{i,j} = -d_{D_{i,j}}$.

$d_1, \ldots, d_9$	5.49	0.903	0.245_	0.195_	0.995_	0.195_	0.675	0.0191_	0.195_
$d_{10}, \ldots, d_{18}$	3.93	0	0.434	1.78	0.245_	0.946	0.383	0.445	0.245_
$d_{19}, \ldots, d_{27}$	0.495_	0.813_	0	0.195_	0.394	0.195_	1.62	0.758	2.83
$d_{28}, \ldots, d_{36}$	1.53	0.355	0.495_	1.11	0.826	0.495_	0.495_	1.17	0
$d_{37}, \ldots, d_{45}$	1	0.142_	0	0	2.66	3.53	0.495_	0.495_	0.936

Table 15: The dependence variables d_q s.t $\theta_{i,j} = d_{D_{i,j}}$ for the financial portfolio.

$d_1, \ldots, d_9$	7.66	4.56	5.43	1.63	9.52	0	7.15	17.9	6.76
$d_{10}, \ldots, d_{18}$	8.68	9.39	10.4	9.68	6.18	7.61	5.2	12.2	11
$d_{19}, \ldots, d_{27}$	1.45	14.3	1.47	2.42	13.4	13.3	0	6.4	7.48
$d_{28}, \ldots, d_{36}$	0.482	7.88	9.77	3.07	4.72	2.98	17	7.34	9.32
$d_{37}, \ldots, d_{45}$	1.47	3.37	4.83	7.21	13.3	9.7	8.22	4.76	17.5

Table 16: The absolute calibration errors for the default correlation matrices, in percent of matrix $\{0.05\rho_{i,j}^{(\text{CDS})}(T)\}$, for the auto portfolio.

	VOLV	BMW	MICH	CONT	DCX	FIAT	PEUG	RENA	VALE	VW
VOLV	0									
BMW	3.4	0								
MICH	9.6	32	0							
CONT	46	83	12	0						
DCX	12	17	9.1	9.4	0					
FIAT	8.9	7.9	2.8	57	11	0				
PEUG	2	49	39	11	31	27	0			
RENA	35	89	14	93	9.1	20	23	0		
VALE	45	64	19	55	7.3	4.7	9.1	0.41	0	
VW	120	0.35	26	20	5.1	24	97	6.8	47	0

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Paper IV

DEFAULT CONTAGION IN LARGE HOMOGENEOUS PORTFOLIOS

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ABSTRACT. We study default contagion in large homogeneous credit portfolios. Using data from the iTraxx Europe series, two synthetic CDO portfolios are calibrated against their tranche spreads, index CDS spreads and average CDS spreads, all with five year maturity. After the calibrations, which render perfect fits, we investigate the implied expected ordered defaults times, implied default correlations, and implied multivariate default and survival distributions, both for ordered and unordered default times. Many of the numerical results differ substantially from the corresponding quantities in a smaller inhomogeneous CDS portfolio. Furthermore, the studies indicate that market CDO spreads imply extreme default clustering in upper tranches. The default contagion is introduced by letting individual intensities jump when other defaults occur, but be constant between defaults. The model is translated into a Markov jump process. Expressions for the investigated quantities are derived by using matrix-analytic methods.

1. INTRODUCTION

In this paper we model dynamic credit dependence in a large homogeneous portfolio with default contagion. The approach is the same as in [15] where the author studies a smaller inhomogeneous credit portfolio, in [17], where the authors focus on k^{th}-to default spreads and in [16] where the same technique are applied to synthetic CDO tranches and index CDS-s. Here we focus on multivariate default and survival distributions and related quantities in a large homogeneous portfolio. Many of the numerical results differ substantially from the corresponding quantities in a smaller inhomogeneous CDS portfolio. Default contagion in an intensity based setting has been studied in for example [1], [2], [3], [4], [5], [6], [8], [7], [10], [11], [12], [13], [19], [20],[21], [22], [25], [26] and [27]. The material in all these papers and books are related to the results discussed here.

The paper is organized as follows. Subsection 2 gives the definition of the intensity based model used in this paper. This framework is then translated into a Markov jump process. In Subsection 3 we present formulas for multivariate default and survival distributions, marginal default distributions, default correlations, and expected default times. Section 4

restates expressions, taken from [16], for our calibration instruments. These are synthetic CDO tranche spreads, index CDS-spread and average CDS-spreads. A short description how to reduce the parameters space is then given. Section 5 is devoted to numerical investigations of several portfolio quantities, derived in Section 3. We use data from the iTraxx series and calibrate two homogeneous CDO portfolios against their CDO tranche spreads, index CDS spread and the average CDS spreads. The maturity is five years and the fits are perfect. We then study the implied expected ordered defaults times, implied default correlations, and implied multivariate default and survival distributions, both for ordered and unordered default times. The numerical studies indicates that the market spreads imply extreme default clustering in upper tranches. The final section summarizes and discusses the results.

2. Intensity based models in a homogeneous model reinterpreted as Markov jump processes

The model we use in this paper is a simplification of the one in [15], [17] to the case where the obligors are exchangeable. It is defined in terms of intensities and then reinterpreted as a Markov jump process. Closely following the cited references, for m exchangeable default times $\tau_1, \tau_2 \ldots, \tau_m$, define the point process $N_{t,i} = 1_{\{\tau_i \leq t\}}$ and introduce the filtrations

$$\mathcal{F}_{t,i} = \sigma\left(N_{s,i}; s \leq t\right), \quad \mathcal{F}_t = \bigvee_{i=1}^{m} \mathcal{F}_{t,i}.$$

Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$, which we refer to just as "intensity" or "default intensity". By exchangeability we have that all the intensities are the same $\lambda_{t,i} = \lambda_t$, if $\tau_i \geq t$ and $\lambda_{t,i} = 0$ if $\tau_i < t$. The model that we study is specified by

$$\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}} \tag{2.1}$$

where $\{T_k\}$ is the ordering of the default times $\{\tau_i\}$. Further, $a > 0$ and $b_1, \ldots, b_{m-1}$, are constants such that λ_t is non-negative. Thus, the default intensities are constant, except at the times when defaults occur: the parameter a is the base intensity for each obligor i, and given that $\tau_i > T_k$, then b_k is how much the default intensity for each remaining obligor jump at default number k in the portfolio. A positive b_k means that all remaining obligors are put at higher risk by the k-th default in the portfolio, while a negative b_k means that the nondefaulted obligors in fact benefits from the k-th default in the basket, and finally $b_k = 0$ if the remaining obligors are unaffected by the k-th default

Equation (2.1) determines the joint distribution of the default times. We will use the following observation, originally proved in [16], but restated here since it provide us with notation needed later on.

Proposition 2.1. *There exists a Markov jump process* $(Y_t)_{t \geq 0}$ *on a finite state space* $\boldsymbol{E} = \{0, 1, 2, \ldots, m\}$, *such that the stopping times*

$$T_k = \inf\{t > 0 : Y_t = k\}, \quad k = 1, \ldots, m$$

are the ordering of m exchangeable stopping times $\tau_1, \ldots, \tau_m$ with intensities (2.1). The generator $\boldsymbol{Q}$ to Y_t is given by

$$\boldsymbol{Q}_{k,k+1} = (m-k)\left(a + \sum_{j=1}^{k} b_j\right) \quad and \quad \boldsymbol{Q}_{k,k} = -\boldsymbol{Q}_{k,k+1} \quad for \quad k = 0, 1, \ldots, m-1$$

where the other entries in $\boldsymbol{Q}$ are zero. The Markov process always starts in $\{0\}$.

The states in $\boldsymbol{E}$ can be interpreted as the number of defaulted obligors in the portfolio. In the sequel, we let $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$ denote the initial distribution on $\boldsymbol{E}$. Further, if k belongs to $\boldsymbol{E}$ then $\boldsymbol{e}_k$ denotes a column vector in $\mathbb{R}^{m+1}$ where the entry at position k is 1 and the other entries are zero. From Markov theory we know that $\mathbb{P}\left[Y_t = k\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e}_k$ where $e^{\boldsymbol{Q}t}$ is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of $\boldsymbol{Q}$.

3. USING THE MATRIX-ANALYTIC APPROACH TO FIND MULTIVARIATE DEFAULT DISTRIBUTIONS AND RELATED QUANTITIES

In this section we derive formulas for multivariate default and survival distributions, both for ordered and unordered default times (Subsection 3.1). The marginal survival distributions are then easily retrieved as special cases (Subsection 3.2). Analytical formulas for the default correlations are given in Subsection 3.3 and the moments of the default times and the ordered default times are presented in Subsection 3.4.

3.1. **The multivariate distributions.** In this subsection we derive formulas for multivariate default and survival distributions both for ordered as well as unordered default times. We start with the latter. Let $\boldsymbol{M}_k$ and $\boldsymbol{N}_k$ be $(m+1) \times (m+1)$ diagonal matrices, defined by $(\boldsymbol{M}_k)_{j,j} = 1_{\{j<k\}}$ and $(\boldsymbol{N}_k)_{j,j} = 1_{\{j\geq k\}}$ where $(\boldsymbol{N}_k)_{j,j'} = (\boldsymbol{M}_k)_{j,j'} = 0$ if $j \neq j'$. Note that $\boldsymbol{M}_k = \boldsymbol{I} - \boldsymbol{N}_k$. The following proposition is similar to Proposition 3.1 in [15].

Proposition 3.1. *Consider m obligors with default intensities (2.1) and let $k_1 < \ldots < k_q$ be an increasing subsequence in $\{1, \ldots, m\}$ where $1 \leq q \leq m$. Furthermore, let $t_1 < t_2 < \ldots < t_q$. Then,*

$$\mathbb{P}\left[T_{k_1} > t_1, \ldots, T_{k_q} > t_q\right] = \boldsymbol{\alpha}\left(\prod_{i=1}^{q} e^{\boldsymbol{Q}(t_i - t_{i-1})} \boldsymbol{M}_{k_i}\right)\mathbf{1} \tag{3.1.1}$$

and

$$\mathbb{P}\left[T_{k_1} \leq t_1, \ldots, T_{k_q} \leq t_q\right] = \boldsymbol{\alpha}\left(\prod_{i=1}^{q} e^{\boldsymbol{Q}(t_i - t_{i-1})} \boldsymbol{N}_{k_i}\right)\mathbf{1} \tag{3.1.2}$$

where $t_{i_0} = 0$.

Proof. By Proposition 2.1,

$$
\begin{aligned}
\mathbb{P}\left[T_{k_1} > t_1, \ldots, T_{k_q} > t_q\right] &= \mathbb{P}\left[Y_{t_1} < k_1, \ldots, Y_{t_q} < k_q\right] \\
&= \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_q=0}^{k_q-1} \mathbb{P}\left[Y_0 = 0, Y_{t_1} = j_1, \ldots, Y_{t_q} = j_q\right] \\
&= \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_q=0}^{k_q-1} \mathbb{P}\left[Y_0 = 0\right] \mathbb{P}\left[Y_{t_1} = j_1 \mid Y_0 = 0\right] \cdot \ldots \cdot \mathbb{P}\left[Y_{t_q} = j_q \mid Y_{t_{q-1}} = j_{q-1}\right] \\
&= \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_q=0}^{k_q-1} \boldsymbol{\alpha} e^{\boldsymbol{Q}t_1} \boldsymbol{e}_{j_1} \boldsymbol{e}_{j_1}^T e^{\boldsymbol{Q}(t_2-t_1)} \boldsymbol{e}_{j_2} \boldsymbol{e}_{j_2}^T \cdot \ldots \cdot \boldsymbol{e}_{j_{q-1}} \boldsymbol{e}_{j_{q-1}}^T e^{\boldsymbol{Q}(t_q-t_{q-1})} \boldsymbol{e}_{j_q} \\
&= \boldsymbol{\alpha} e^{\boldsymbol{Q}t_1} \Big(\sum_{j_1=0}^{k_1-1} \boldsymbol{e}_{j_1} \boldsymbol{e}_{j_1}^T\Big) e^{\boldsymbol{Q}(t_2-t_1)} \Big(\sum_{j_2=1}^{k_2-1} \boldsymbol{e}_{j_2} \boldsymbol{e}_{j_2}^T\Big) \cdots e^{\boldsymbol{Q}(t_q-t_{q-1})} \sum_{j_q=0}^{k_q-1} \boldsymbol{e}_{j_q} \\
&= \boldsymbol{\alpha} e^{\boldsymbol{Q}t_1} \boldsymbol{M}_{k_1} e^{\boldsymbol{Q}(t_2-t_1)} \boldsymbol{M}_{k_2} \cdots e^{\boldsymbol{Q}(t_q-t_{q-1})} \boldsymbol{M}_{k_q} \mathbf{1}
\end{aligned}
$$

which proves (3.1.1). The third equality follows from the Markov property of Y_t, the fourth since Y_t is a homogeneous Markov process and that $\mathbb{P}\left[Y_0 = 0\right] = 1$. The final equality is due to the definition of the matrix $\boldsymbol{M}_k$. Equation (3.1.2) is proved in the same way. $\square$

Finding joint distributions for $\{\tau_i\}$ in a homogeneous model with default intensities (2.1) is a more complicated task than in an inhomogeneous model. For $1 \leq q \leq m$, fix a vector $t_1, \ldots, t_q \in \mathbb{R}_+^q$. For a set of q distinct obligors $i_1, i_2, \ldots, i_q$, the probability $\mathbb{P}\left[\tau_{i_1} \leq t_1, \ldots, \tau_{i_q} \leq t_q\right]$ is by exchangeability the same for any such distinct sequence of q obligors. Therefore we will in this section without loss of generality only consider $\mathbb{P}\left[\tau_1 \leq t_1, \ldots, \tau_q \leq t_q\right]$ where $t_1 \leq \ldots \leq t_q$ and similarly for $\mathbb{P}\left[\tau_{i_1} > t_1, \ldots, \tau_{i_q} > t_q\right]$. To exemplify, we start with the following proposition, where we let $q = 2$ and $t_1 < t_2$.

Proposition 3.2. *Consider m obligors with default intensities* (2.1) *and let $t_1 < t_2$. Then,*

$$
\mathbb{P}\left[\tau_1 \leq t_1, \tau_2 \leq t_2\right] = \frac{(m-2)!}{m!} \boldsymbol{\alpha} e^{\boldsymbol{Q}t_1} \boldsymbol{n} + \frac{(m-2)!}{m!} \sum_{k_1=1}^{m} \sum_{k_2=k_1+1}^{m} \boldsymbol{\alpha} e^{\boldsymbol{Q}t_1} \boldsymbol{N}_{k_1} e^{\boldsymbol{Q}(t_2-t_1)} \boldsymbol{N}_{k_2} \mathbf{1}.
$$

$$(3.1.3)$$

where $\boldsymbol{n}$ is a column vector in $\mathbb{R}^{m+1}$ such that $\boldsymbol{n}_j = \frac{j(j-1)}{2}$.

Proof. First, note that

$$
\mathbb{P}\left[\tau_1 \le t_1, \tau_2 \le t_2\right] = \sum_{k_1=1}^{m} \sum_{k_2=1, k_2 \ne k_1}^{m} \mathbb{P}\left[\tau_1 = T_{k_1}, \tau_2 = T_{k_2}, \tau_1 \le t_1, \tau_2 \le t_2\right]
$$

$$
= \sum_{k_1=1}^{m} \sum_{k_2=1, k_2 \ne k_1}^{m} \mathbb{P}\left[\tau_1 = T_{k_1}, \tau_2 = T_{k_2}, T_{k_1} \le t_1, T_{k_2} \le t_2\right]
$$

$$
= \sum_{k_1=1}^{m} \sum_{k_2=1, k_2 \ne k_1}^{m} \mathbb{P}\left[\tau_1 = T_{k_1}, \tau_2 = T_{k_2}\right] \mathbb{P}\left[T_{k_1} \le t_1, T_{k_2} \le t_2\right]
$$

$$
= \frac{(m-2)!}{m!} \sum_{k_1=1}^{m} \sum_{k_2=1, k_2 \ne k_1}^{m} \mathbb{P}\left[T_{k_1} \le t_1, T_{k_2} \le t_2\right]
$$

(3.1.4)

where the third and fourth equalities are due to the exchangeability in the portfolio. Now, if $k_1 < k_2$, then Proposition 3.1 renders $\mathbb{P}\left[T_{k_1} \le t_1, T_{k_2} \le t_2\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q} t_1} \boldsymbol{N}_{k_1} e^{\boldsymbol{Q}(t_2 - t_1)} \boldsymbol{N}_{k_2} \mathbf{1}$. However, if $k_1 > k_2$, we can no longer use this arguments. To see this, note that since $t_1 < t_2$ and $T_{k_1} > T_{k_2}$ we get

$$
\mathbb{P}\left[T_{k_1} \le t_1, T_{k_2} \le t_2\right] = \mathbb{P}\left[T_{k_1} \le t_1\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q} t_1} \boldsymbol{N}_{k_1} \mathbf{1} \ne \boldsymbol{\alpha} e^{\boldsymbol{Q} t_1} \boldsymbol{N}_{k_1} e^{\boldsymbol{Q}(t_2 - t_1)} \boldsymbol{N}_{k_2} \mathbf{1}.
$$

Hence,

$$
\sum_{k_1=1}^{m} \sum_{k_2=1, k_2 \ne k_1}^{m} \mathbb{P}\left[T_{k_1} \le t_1, T_{k_2} \le t_2\right]
$$

$$
= \sum_{k_1=1}^{m} \left(\sum_{1 \le k_2 < k_1} \mathbb{P}\left[T_{k_1} \le t_1, T_{k_2} \le t_2\right] + \sum_{k_1 < k_2 \le m} \mathbb{P}\left[T_{k_1} \le t_1, T_{k_2} \le t_2\right] \right)
$$

$$
= \sum_{k_1=1}^{m} \left(\sum_{k_2=1}^{k_1-1} \mathbb{P}\left[T_{k_1} \le t_1\right] + \sum_{k_2=k_1+1}^{m} \mathbb{P}\left[T_{k_1} \le t_1, T_{k_2} \le t_2\right] \right)
$$

$$
= \sum_{k_1=1}^{m} \left((k_1-1) \mathbb{P}\left[T_{k_1} \le t_1\right] + \sum_{k_2=k_1+1}^{m} \mathbb{P}\left[T_{k_1} \le t_1, T_{k_2} \le t_2\right] \right)
$$

$$
= \sum_{k_1=1}^{m} (k_1-1) \boldsymbol{\alpha} e^{\boldsymbol{Q} t_1} \boldsymbol{N}_{k_1} \mathbf{1} + \sum_{k_1=1}^{m} \sum_{k_2=k_1+1}^{m} \boldsymbol{\alpha} e^{\boldsymbol{Q} t_1} \boldsymbol{N}_{k_1} e^{\boldsymbol{Q}(t_2 - t_1)} \boldsymbol{N}_{k_2} \mathbf{1}
$$

$$
= \boldsymbol{\alpha} e^{\boldsymbol{Q} t_1} \left(\sum_{k_1=1}^{m} (k_1-1) \boldsymbol{N}_{k_1} \right) \mathbf{1} + \sum_{k_1=1}^{m} \sum_{k_2=k_1+1}^{m} \boldsymbol{\alpha} e^{\boldsymbol{Q} t_1} \boldsymbol{N}_{k_1} e^{\boldsymbol{Q}(t_2 - t_1)} \boldsymbol{N}_{k_2} \mathbf{1}.
$$

Note that the column vector $\boldsymbol{n} = \left(\sum_{k_1=1}^{m} (k_1-1) \boldsymbol{N}_{k_1} \right) \mathbf{1}$ can be simplified according to

$$
\boldsymbol{n}_j = \sum_{k_1=1}^{m} (k_1-1) (\boldsymbol{N}_{k_1})_j = \sum_{k_1=1}^{m} (k_1-1) 1_{\{j \ge k_1\}} = \sum_{k_1=1}^{j} (k_1-1) = \frac{j(j-1)}{2}.
$$

Inserting the above expressions in (3.1.4) renders Equation (3.1.3).   □

By using the same technique as in Proposition 3.2 we can state the following corollary.

Corollary 3.3. *Consider m obligors with default intensities (2.1) and let $t_1 < t_2$. Then,*

$$\mathbb{P}\left[\tau_1 > t_1, \tau_2 > t_2\right] = \frac{(m-2)!}{m!}\boldsymbol{\alpha}e^{\boldsymbol{Q}t_2}\boldsymbol{m} + \frac{(m-2)!}{m!}\sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}\boldsymbol{\alpha}e^{\boldsymbol{Q}t_1}\boldsymbol{M}_{k_1}e^{\boldsymbol{Q}(t_2-t_1)}\boldsymbol{M}_{k_2}\boldsymbol{1}.$$
(3.1.5)

where $\boldsymbol{m}$ is a column vector in $\mathbb{R}^{m+1}$ such that $\boldsymbol{m}_j = \frac{(m-j)(m-j-1)}{2}$.

It is possible to generalize Proposition 3.2 and Corollary 3.3 to more that two default times. These expressions do not seem to be easily simplified. However, if $t_1 = \ldots = t_q = t$ we can find compact formulas.

Proposition 3.4. *Consider m obligors with default intensities (2.1) and let q be a integer where $1 \le q \le m$. Then,*

$$\mathbb{P}\left[\tau_1 \le t, \ldots, \tau_q \le t\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{d}^{(q)} \quad and \quad \mathbb{P}\left[\tau_1 > t, \ldots, \tau_q > t\right] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{s}^{(q)} \qquad (3.1.6)$$

where $\boldsymbol{d}^{(q)}$ and $\boldsymbol{s}^{(q)}$ are column vectors in $\mathbb{R}^{m+1}$ defined by

$$\boldsymbol{d}_j^{(q)} = \frac{\binom{j}{q}}{\binom{m}{q}}1_{\{j \ge q\}} \quad and \quad \boldsymbol{s}_j^{(q)} = \frac{\binom{m-j}{q}}{\binom{m}{q}}1_{\{j \le m-q\}}.$$
(3.1.7)

Proof. By Proposition 2.1,

$$\begin{aligned}
\mathbb{P}\left[\tau_1 \le t, \ldots, \tau_q \le t\right] &= \sum_{j=q}^{m}\mathbb{P}\left[\tau_1 \le t, \ldots, \tau_q \le t, Y_t = j\right] \\
&= \sum_{j=q}^{m}\mathbb{P}\left[\tau_1 \le t, \ldots, \tau_q \le t \,|\, Y_t = j\right]\mathbb{P}\left[Y_t = j\right] \\
&= \sum_{j=q}^{m}\frac{\binom{j}{q}}{\binom{m}{q}}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{e}_j \\
&= \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{d}^{(q)}
\end{aligned}$$
(3.1.8)

where $\boldsymbol{d}^{(q)}$ is a column vector in $\mathbb{R}^{m+1}$ defined by $\boldsymbol{d}_j^{(q)} = \frac{\binom{j}{q}}{\binom{m}{q}}1_{\{j \ge q\}}$. To motivate the third equality in (3.1.8), note that $\mathbb{P}\left[\tau_1 \le t, \ldots, \tau_q \le t \,|\, Y_t = j\right]$ is the probability that q specified obligors have defaulted before t given that exactly j obligors have defaulted until time t where $j \ge q$. Since the portfolio consist of m obligors, and by exchangeability in the model, there are $\binom{m-q}{j-q}$ ways to choose a group of j obligors that contains our q specified obligors, where $j \ge q$. Further, there are $\binom{m}{j}$ ways to pick out a set containing j obligors. Hence,

$$\mathbb{P}\left[\tau_1 \le t, \ldots, \tau_q \le t \,|\, Y_t = j\right] = \frac{\binom{m-q}{j-q}}{\binom{m}{j}} = \frac{\binom{j}{q}}{\binom{m}{q}} \quad \text{for} \quad j \ge q,$$

where the last equality follows from straightforward calculations. This proves the first equality in Equation (3.1.6). Next, by Proposition 2.1 again,

$$
\begin{aligned}
\mathbb{P}\left[\tau_1 > t, \ldots, \tau_q > t\right] &= \sum_{j=0}^{m-q} \mathbb{P}\left[\tau_1 > t, \ldots, \tau_q > t, Y_t = j\right] \\
&= \sum_{j=0}^{m-q} \mathbb{P}\left[\tau_1 > t, \ldots, \tau_q > t \,|\, Y_t = j\right] \mathbb{P}\left[Y_t = j\right] \\
&= \sum_{j=0}^{m-q} \frac{\binom{m-j}{q}}{\binom{m}{q}} \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e}_j \\
&= \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{s}^{(q)}
\end{aligned}
\tag{3.1.9}
$$

where $\boldsymbol{s}^{(q)}$ is a column vector in $\mathbb{R}^{m+1}$ defined by $\boldsymbol{s}_j^{(q)} = \frac{\binom{m-j}{q}}{\binom{m}{q}} 1_{\{j \leq m-q\}}$. To motivate the third equality in (3.1.9), note that $\mathbb{P}\left[\tau_1 > t, \ldots, \tau_q > t \,|\, Y_t = j\right]$ is the probability that q specified obligors have survived before t given that exactly $m-j$ obligors have survived until time t where $m - j \geq q$. Since the portfolio consist of m obligors, and by exchangeability in the model, there are $\binom{m-q}{m-j-q}$ ways to choose a group of $m - j$ obligors that contains our q specified obligors, where $m - j \geq q$. Further, there are $\binom{m}{m-j}$ ways to pick out a set containing $m - j$ obligors. Hence,

$$
\mathbb{P}\left[\tau_1 > t, \ldots, \tau_q > t \,|\, Y_t = j\right] = \frac{\binom{m-q}{m-j-q}}{\binom{m}{m-j}} = \frac{\binom{m-j}{q}}{\binom{m}{q}} \quad \text{for} \quad j \leq m - q,
$$

where the last equality follows from straightforward calculations. This proves the second equality in Equation (3.1.6). $\qquad\square$

Note that the indicator functions $1_{\{j \geq q\}}$ and $1_{\{j \leq m-q\}}$ can be dropped in (3.1.7) since

$$
\binom{j}{q} = \frac{j(j-1)(j-2)\cdots(j-q+1)}{q!} = 0 \quad \text{for} \quad j = 0, 1, \ldots, q-1
$$

and

$$
\binom{m-j}{q} = \frac{(m-j)(m-j-1)(m-j-2)\cdots(m-j-q+1)}{q!} = 0
$$

for $j = m, m-1, \ldots, m-q+1$. We can now check that Proposition 3.2 and Corollary 3.3 are consistent with Proposition 3.4. Letting $t_1 = t_2 = t$ in Equation (3.1.3) in Proposition

3.2 yields that

$$\mathbb{P}\left[\tau_1 \leq t, \tau_2 \leq t\right] = \frac{(m-2)!}{m!}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{n} + \frac{(m-2)!}{m!}\sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{N}_{k_1}\boldsymbol{N}_{k_2}\boldsymbol{1}$$

$$= \frac{(m-2)!}{m!}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\left(\boldsymbol{n} + \sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}\boldsymbol{N}_{k_2}\boldsymbol{1}\right) \tag{3.1.10}$$

$$= \frac{(m-2)!}{m!}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\left(\boldsymbol{n} + \boldsymbol{n}\right) = \frac{1}{\binom{m-2}{2}}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{n} = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{d}^{(2)}$$

where the second equality follows from $\boldsymbol{N}_{k_1}\boldsymbol{N}_{k_2} = \boldsymbol{N}_{k_2}$ since $k_2 > k_1$. To prove the third equality in (3.1.10), note that

$$\left(\sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}\boldsymbol{N}_{k_2}\boldsymbol{1}\right)_j = \sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}\left(\boldsymbol{N}_{k_2}\boldsymbol{1}\right)_j = \sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}1_{\{j \geq k_2\}}$$

$$= \sum_{k_1=1}^{m}\left(\sum_{k_2=k_1+1}^{j}1\right)1_{\{j \geq k_1+1\}} = \sum_{k_1=1}^{m}(j-k_1)1_{\{j \geq k_1\}} = \sum_{k_1=1}^{j}(j-k_1) = \frac{j(j-1)}{2} = \boldsymbol{n}_j$$

and the final equality in (3.1.10) is due to the definition of $\boldsymbol{d}^{(q)}$ in Equation (3.1.7), $\boldsymbol{d}_j^{(q)} = \frac{\binom{j}{q}}{\binom{m}{q}}$ which for $q=2$ implies that $\boldsymbol{d}^{(2)} = \frac{1}{\binom{m-2}{2}}\boldsymbol{n}$. Hence, Proposition 3.2 is consistent with Proposition 3.4 for $q=2$ and the bivariate default distribution. Now, letting $t_1 = t_2 = t$ in Equation (3.1.5) in Corollary 3.3 we get

$$\mathbb{P}\left[\tau_1 > t, \tau_2 > t\right] = \frac{(m-2)!}{m!}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{m} + \frac{(m-2)!}{m!}\sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{M}_{k_1}\boldsymbol{M}_{k_2}\boldsymbol{1}$$

$$= \frac{(m-2)!}{m!}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\left(\boldsymbol{m} + \sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}\boldsymbol{M}_{k_1}\boldsymbol{1}\right) \tag{3.1.11}$$

$$= \frac{(m-2)!}{m!}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\left(\boldsymbol{m} + \boldsymbol{m}\right) = \frac{1}{\binom{m-2}{2}}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{m} = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{s}^{(2)}$$

where the second equality follows from $\boldsymbol{M}_{k_1}\boldsymbol{M}_{k_2} = \boldsymbol{M}_{k_1}$ since $k_2 > k_1$. To prove the third equality in (3.1.11), note that

$$\left(\sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}\boldsymbol{M}_{k_1}\boldsymbol{1}\right)_j = \left(\sum_{k_1=1}^{m}(m-k_1)\boldsymbol{M}_{k_1}\boldsymbol{1}\right)_j = \sum_{k_1=1}^{m}(m-k_1)\left(\boldsymbol{M}_{k_1}\boldsymbol{1}\right)_j$$

$$= \sum_{k_1=1}^{m}(m-k_1)1_{\{j < k_1\}} = \sum_{k_1=j+1}^{m}(m-k_1) = \frac{(m-j)(m-j-1)}{2} = \boldsymbol{m}_j$$

and the final equality in (3.1.11) is due to the definition of $\boldsymbol{s}^{(q)}$ in Equation (3.1.7), $\boldsymbol{s}_j^{(q)} = \frac{\binom{m-j}{q}}{\binom{m}{q}}$, which for $q = 2$ implies that $\boldsymbol{s}^{(2)} = \frac{1}{\binom{m-2}{2}}\boldsymbol{m}$. Hence, Corollary 3.3 is consistent with Proposition 3.4 for $q = 2$ and the bivariate survival distribution.

3.2. **The marginal distributions.** By Proposition 3.4 with $q = 1$ we get $\mathbb{P}[\tau_i > t] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{s}^{(1)}$ where $\boldsymbol{s}_j^{(1)} = (m-j)/m = 1 - j/m$. Furthermore, letting $\boldsymbol{m}^{(k)}$ denote $\boldsymbol{m}^{(k)} = \boldsymbol{M}_k\boldsymbol{1}$, then Proposition 3.1 with $q = 1$ for any $1 \leq k \leq m$, renders that $\mathbb{P}[T_k > t] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{m}^{(k)}$ where $\boldsymbol{m}_j^{(k)} = 1_{\{j<k\}}$.

3.3. **The default correlations.** In this subsection we use Proposition 3.4 to give expressions for pairwise default correlations between two different obligors belonging to a homogeneous portfolio of m obligors satisfying (2.1). By exchangeability, $\mathrm{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$ is the same for all pairs $i \neq j$ and therefore, we write $\rho(t)$ to denote $\mathrm{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$.

Lemma 3.5. *Consider m obligors with default intensities* (2.1). *Then, with notation as in Subsection* 3.1

$$\rho(t) = \frac{\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{d}^{(2)} - \left(\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{d}^{(1)}\right)^2}{\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{d}^{(1)}\left(1 - \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{d}^{(1)}\right)}. \tag{3.3.1}$$

Proof. By exchangeability we have that $\mathrm{Var}(1_{\{\tau_j \leq t\}}) = \mathrm{Var}(1_{\{\tau_i \leq t\}})$. Further, using the definition of covariance and variance we get that

$$\mathrm{Cov}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}}) = \mathbb{P}[\tau_i \leq t, \tau_j \leq t] - \mathbb{P}[\tau_i \leq t]\mathbb{P}[\tau_j \leq t]$$
$$\mathrm{Var}(1_{\{\tau_i \leq t\}}) = \mathbb{P}[\tau_i \leq t](1 - \mathbb{P}[\tau_i \leq t])$$

and by Proposition 3.4, $\mathbb{P}[\tau_i \leq t, \tau_j \leq t] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{d}^{(2)}$ and $\mathbb{P}[\tau_i \leq t] = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{d}^{(1)}$. Inserting this into the definition for correlation between two random variables yields (3.3.1). $\square$

3.4. **Expected default times.** In this subsection we present formulas for the moments for default times and ordered default times. By construction, the intensity matrix $\boldsymbol{Q}$ for the Markov jump process (see Proposition 2.1) has the form

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{0} & 0 \end{pmatrix}$$

where $\boldsymbol{t}$ is a column vector such that $\boldsymbol{t}_{m-1}$ is nonzero and $\boldsymbol{t}_k = 0$ for $k = 0, 1, \ldots, m-2$, because the k-th element $\boldsymbol{t}_k$, $k \leq m-1$ is the intensity for the Markov jump process Y_t to jump from the state k to the absorbing state $\{m\}$. Furthermore, $\boldsymbol{T}$ is invertible since it is upper diagonal with strictly negative diagonal elements. The following lemma is standard.

Lemma 3.6. *Consider m obligors with default intensities* (2.1). *Then,*

$$\mathbb{E}[\tau_i^n] = (-1)^n n! \tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-n}\tilde{\boldsymbol{s}}^{(1)} \quad and \quad \mathbb{E}[T_k^n] = (-1)^n n! \tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-n}\tilde{\boldsymbol{m}}^{(k)}$$

for $n \in \mathbb{N}$ where $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{s}}^{(1)}, \tilde{\boldsymbol{m}}^{(k)}$ are the restrictions of $\boldsymbol{\alpha}, \boldsymbol{s}^{(1)}, \boldsymbol{m}^{(k)}$ from $\boldsymbol{E}$ to $\boldsymbol{E} \setminus \{m\}$.

For a proof, see Lemma 3.8 in [15]. The implied variances of the default times can now be computed as

$$\mathrm{Var}[T_k] = 2\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-2}\tilde{\boldsymbol{m}}^{(k)} - \left(\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{m}}^{(k)}\right)^2 \quad \text{for } k = 1, 2, \ldots, m,$$

and in the same way $\mathrm{Var}[\tau_i] = 2\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-2}\tilde{\boldsymbol{s}}^{(i)} - \left(\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{s}}^{(i)}\right)^2$, which by the exchangeability is identical for all obligors.

3.5. **A remark.** The main message in this section is that under (2.1), computations of multivariate default and survival distributions, marginal default distributions, default correlations, expected default times, and so on have been reduced to compute the matrix exponential. Computing $e^{\boldsymbol{Q}t}$ efficiently for large state spaces, is a numerical issue which requires special treatment, see [17]. For small state spaces, perhaps less then 150 states, there are many different methods to compute the matrix exponential ([23], [24]). Most of them are straightforward to implement using standard mathematical software. Following [16], we will in this paper use Padé approximation with scaling and squaring, see [23]. In the model (2.1) with $m = 125$, this approach outperforms all other methods which we have tried, both in computational time and accuracy. The robustness of the Padé approximation with scaling and squaring have previously also been verified in [23], [24].

Finally, recall that $e^{\boldsymbol{Q}t}$ has a closed form expression in terms of the eigenvalue decomposition of $\boldsymbol{Q}$. Thus, if the eigenvalues of $\boldsymbol{Q}$ are distinct and letting $\boldsymbol{D}$ be a diagonal matrix containing them, then $e^{\boldsymbol{Q}t} = \boldsymbol{U}e^{\boldsymbol{D}t}\boldsymbol{U}^{-1}$ where $\boldsymbol{U}$ is the matrix whose rows are the corresponding eigenvectors. In this paper $\boldsymbol{Q}$ is upper diagonal, so the eigenvalues are given by its diagonal. It is therefore tempting to use this decomposition. However, the method is numerically unstable even for moderate sizes of m ([23]) since $\boldsymbol{U}$ is often ill-conditioned, making it difficult to compute its inverse $\boldsymbol{U}^{-1}$ without introducing large numerical errors.

4. Calibrating the model parameters against CDO tranche spreads, index CDS spreads and average CDS spreads.

In this section we discuss how to find the parameters in the model (2.1). First, Subsection 4.1 presents formulas for the single-name CDS spread in this model. Then, Subsection 4.2 gives expressions for CDO tranche spreads and index CDS spreads. Finally, Subsection 4.3 is devoted to a short description how to calibrate the model spreads against the corresponding market spreads. In the sequel all computations are assumed to be made under a risk-neutral martingale measure $\mathbb{P}$. Typically such a $\mathbb{P}$ exists if we rule out arbitrage opportunities. Further, we assume the that risk-free interest rate is a deterministic constant given by r.

4.1. **The single-name CDS spread.** In this subsection we give a short description of a single-name credit default swap, which is one of our calibration instruments.

Consider a obligor i with default time τ_i and recovery rate ϕ_i. A single-name credit default swap (CDS) with maturity T where the reference entity is obligor i, is a bilateral contract between two counterparties, $\boldsymbol{A}$ and $\boldsymbol{B}$, where $\boldsymbol{B}$ promises to pay $\boldsymbol{A}$ the credit

losses $(1 - \phi_i)$ at τ_i if obligor i defaults before time T. As compensation for this, **A** pays $R_i(T)\Delta_n$ to the protection seller **B**, at $0 < t_1 < t_2 < \ldots < t_{n_T} = T$ or until $\tau_i < T$, where $\Delta_n = t_n - t_{n-1}$. If default happens for some $\tau_i \in [t_n, t_{n+1}]$, **A** will also pay **B** the accrued default premium up to τ_i. By exchangeability in the model (2.1), τ_i has the same distribution for all obligors and $\phi_1 = \phi_2 = \ldots = \phi_m = \phi$ so $R_1(T) = R_1(T) = \ldots = R_m(T) = R(T)$. The CDS spread $R(T)$ is determined so that expected discounted cashflows between **A** and **B** are equal when the CDS contract is settled at $t = 0$. It is expressed in bp per annum and independent of the nominal size protected. Closed-form expression for $R(T)$ is obtained by using the expression for $\mathbb{P}[\tau_i > t]$ in Subsection 3.2. For ease of reference we exhibit the resulting formulas (proofs can be found in [14] or [15]).

Proposition 4.1. *Consider m obligors that all satisfies (2.1) and assume that the interest rate r is constant. Then, with notation as above*

$$R(T) = \frac{(1 - \phi)\boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)\boldsymbol{s}^{(1)}}{\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1}, t_n)\right)\right)\boldsymbol{s}^{(1)}}$$

where $\boldsymbol{C}(s, t) = s\left(\boldsymbol{A}(t) - \boldsymbol{A}(s)\right) - \boldsymbol{B}(t) + \boldsymbol{B}(s)$ and

$$\boldsymbol{A}(t) = e^{\boldsymbol{Q}t}\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt} \qquad \boldsymbol{B}(t) = e^{\boldsymbol{Q}t}\left(t\boldsymbol{I} + \left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}.$$

For more on the CDS contract, see e.g [9], [14], [15] or [22].

4.2. **CDO tranche spreads and index CDS spreads.** In this subsection we present formulas for CDO tranche spreads and index CDS spreads in a model given by (2.1). These expression are then used in the calibration of the model. Our outline is a shorter version of the one presented in Section 2 in [16]. We restate it here in order to make our paper self-contained.

A synthetic CDO is defined for a portfolio consisting of m single-name CDS's on obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$ and recovery rates $\phi_1, \phi_2, \ldots, \phi_m$. It is standard to assume that the nominal values are the same for all obligors. Here we focus on the model (2.1) where all obligors are exchangeable and thus $\phi_1 = \phi_2 = \ldots = \phi_m = \phi$. The credit loss L_t for this portfolio at time t, in percent of the nominal portfolio value at $t = 0$, is given by

$$L_t = \frac{1 - \phi}{m}\sum_{i=1}^{m} 1_{\{\tau_i \leq t\}} = \frac{1 - \phi}{m}\sum_{k=1}^{m} 1_{\{T_k \leq t\}} \tag{4.2.1}$$

where $\{T_k\}$ is the ordering of the default times $\{\tau_i\}$.

Recall that a CDO is specified by the attachment points $0 = k_0 < k_1 < k_2 < \ldots k_\kappa = 1$ with corresponding tranches $[k_{\gamma-1}, k_\gamma]$. The financial instrument that constitutes tranche γ with maturity T is a bilateral contract where the protection seller **B** agrees to pay the protection buyer **A**, all losses that occurs in the interval $[k_{\gamma-1}, k_\gamma]$ derived from L_t up to time T. The payments are made at the corresponding default times, if they arrive before T, and at T the contract ends. As compensation for this, **A** pays **B** a periodic fee at $0 < t_1 < t_2 < \ldots < t_{n_T} = T$, given by $S_\gamma(T)\left(\Delta k_\gamma - L_t^{(\gamma)}\right)\Delta_n$ where $\Delta k_\gamma = k_\gamma - k_{\gamma-1}$, $\Delta_n = t_n - t_{n-1}$ and $L_t^{(\gamma)} = (L_t - k_{\gamma-1})1_{\{L_t \in [k_{\gamma-1}, k_\gamma]\}} + \Delta k_\gamma 1_{\{L_t > k_\gamma\}}$. Note that $\Delta k_\gamma - L_t^{(\gamma)}$

is what is left of the tranche γ at time t. For upper tranches $\gamma > 1$, the tranche spread $S_\gamma(T)$ is determined so that the expected discounted cashflows between **A** and **B** are the same at $t = 0$. It is quoted in bp per annum. Furthermore, for the first tranche, often denoted the *equity* tranche, $S_1(T)$ is set to 500 bp and a so called *up-front* fee $S_1^{(u)}(T)$ is added to the protection payments so that the expected discounted cashflows between **A** and **B** are equal at $t = 0$. It is quoted in percent. Note that the spreads $S_\gamma(T)$, $S_1(T)$ are independent of the nominal size of the portfolio.

Consider the same synthetic CDO as above. An index CDS with maturity T, has almost the same structure as a corresponding CDO tranche, but with two main differences. First, the protection is on *all* credit losses that occurs in the CDO portfolio up to time T, so in the protection leg, the tranche loss $L_t^{(\gamma)}$ is replaced by the total loss L_t. Secondly, in the protection payments, the index spread $S(T)$ is paid on a notional proportional to the number of obligors left in the portfolio at each payment date, that is $1 - \frac{1}{m}\sum_{k=1}^m 1_{\{T_k \leq t\}}$. The rest of the contract has the same structure as a CDO tranche. Hence, the index CDS spread $S(T)$ is determined so that the expected value of cashflows between **A** and **B** are the same at $t = 0$.

From Proposition 2.1 and Equation (4.2.1) it is clear that the loss L_t can be represented as a functional of the Markov jump process Y_t, $L_t = L(Y_t)$, see Lemma 5.2 in [16]. The mapping L goes from $\boldsymbol{E} = \{0, 1, \ldots, m\}$ to all possible loss-outcomes determined via (4.2.1). For example, if $k \in \{0, 1, \ldots, m\}$ then $L(k) = \frac{1-\phi}{m}k$. In view of these observations, we state the following result, proved in [16].

Proposition 4.2. *Consider a synthetic CDO on a portfolio with m obligors that satisfy (2.1) and assume that the interest rate r is constant. Then, with notation as above,*

$$S_\gamma(T) = \frac{\left(\boldsymbol{\alpha}e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r\right)\boldsymbol{\ell}^{(\gamma)}}{\sum_{n=1}^{n_T} e^{-rt_n}\left(\Delta k_\gamma - \boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}\boldsymbol{\ell}^{(\gamma)}\right)\Delta_n} \qquad \gamma = 2, \ldots, \kappa \qquad (4.2.2)$$

and

$$S_1^{(u)}(T) = \frac{1}{k_1}\left(\boldsymbol{\alpha}e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r + 0.05\sum_{n=1}^{n_T}\boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}e^{-rt_n}\Delta_n\right)\boldsymbol{\ell}^{(1)} - 0.05\sum_{n=1}^{n_T}e^{-rt_n}\Delta_n. \tag{4.2.3}$$

Furthermore,

$$S(T) = \frac{\left(\boldsymbol{\alpha}e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r\right)\boldsymbol{\ell}}{\sum_{n=1}^{n_T}e^{-rt_n}\left(1 - \boldsymbol{\alpha}e^{\boldsymbol{Q}t_n}\boldsymbol{\ell}\frac{1}{1-\phi}\right)\Delta_n} \tag{4.2.4}$$

where

$$\boldsymbol{R}(0,T) = \int_0^T e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt = \left(e^{\boldsymbol{Q}T}e^{-rT} - \boldsymbol{I}\right)\left(\boldsymbol{Q}-r\boldsymbol{I}\right)^{-1}.$$

Here $\boldsymbol{\ell}^{(\gamma)}$ is a column vector in $\mathbb{R}^{m+1}$, defined by

$$\boldsymbol{\ell}_k^{(\gamma)} = \begin{cases} 0 & if \quad k < n_l(k_{\gamma-1}) \\ k(1-\phi)/m - k_{\gamma-1} & if \quad n_l(k_{\gamma-1}) \leq k \leq n_u(k_\gamma) \\ \Delta k_\gamma & if \quad k > n_u(k_\gamma) \end{cases} \tag{4.2.5}$$

where $n_l(x) = \lceil xm/(1-\phi) \rceil$ and $n_u(x) = \lfloor xm/(1-\phi) \rfloor$. Finally, $\boldsymbol{\ell}$ is a column vector in $\mathbb{R}^{m+1}$, defined by $\boldsymbol{\ell}_k = k(1-\phi)/m$.

4.3. The calibration.
In this subsection we show how to calibrate the model (2.1) against the credit instruments described in the previous subsections. Let $\boldsymbol{a} = (a, b_1, b_2, \ldots, b_{m-1})$ denote the m parameters in (2.1). Furthermore, let $\{C_j(T; \boldsymbol{a})\}$ be the $\kappa + 2$ model spreads for the instruments used in the calibration. These are the average CDS spread $R(T; \boldsymbol{a})$, the index CDS spread $S(T; \boldsymbol{a})$ and the κ different CDO tranche spreads $\{S_\gamma(T; \boldsymbol{a})\}$, $S_1^{(u)}(T; \boldsymbol{a})$. We let $\{C_{j,M}(T)\}$ denote the corresponding market spreads. In $C_j(T; \boldsymbol{a})$ we have emphasized that the model spreads are functions of $\boldsymbol{a} = (a, b_1, b_2, \ldots, b_{m-1})$ but suppressed the dependence of interest rate, payment frequency, etc. The vector $\boldsymbol{a}$ is then obtained as

$$\boldsymbol{a} = \operatorname*{argmin}_{\hat{\boldsymbol{a}}} \sum_{j=1}^{\eta} \left(C_j(T; \hat{\boldsymbol{a}}) - C_{j,M}(T)\right)^2 \tag{4.3.1}$$

with the constraint that all elements in $\boldsymbol{a}$ are nonnegative. For a fixed maturity T, we use $\kappa = 5$ tranche spreads. This gives us 7 market observations, while the model can contain up to $m = 125$ parameters. In order to reduce the number of unknown parameters to as many as the market observations, we make following assumption on the parameters b_k for $1 \leq k \leq m - 1$

$$b_k = \begin{cases} b^{(1)} & if \quad 1 \leq k < \mu_1 \\ b^{(2)} & if \quad \mu_1 \leq k < \mu_2 \\ \vdots \\ b^{(c)} & if \quad \mu_5 \leq k < \mu_6 = m \end{cases} \tag{4.3.2}$$

where $1, \mu_1, \mu_2, \ldots, \mu_6$ is an partition of $\{1, 2, \ldots, m\}$. This means that all jumps in the intensity at the defaults $1, 2, \ldots, \mu_1 - 1$ are same and given by $b^{(1)}$, all jumps in the intensity at the defaults $\mu_1, \ldots, \mu_2 - 1$ are same and given by $b^{(2)}$ and so on. Hence, in (4.3.1) we now minimize over the unknown vector $\boldsymbol{a} = (a, b^{(1)}, \ldots, b^{(6)})$.

5. NUMERICAL STUDIES

In this section we will, in a homogeneous CDO portfolio, study several quantities of importance in active credit portfolio management. In Section 5.1 we calibrate the portfolio against market data on CDO tranches, index CDS-s and average single-name CDS spreads. We then investigate the implied expected ordered default times (Subsection 5.2), the implied default correlation (Subsection 5.3) and finally various kinds of implied multivariate default and survival distributions, both for ordered as well as unordered default times (Subsection 5.4).

5.1. **Calibration of a homogeneous portfolio.** In this subsection we calibrate our homogenous model against data on the iTraxx Europe Series collected from `Reuters` at August 4th, 2004 and November 28th, 2006. For each date, the data contains five different CDO tranche spreads with tranches $[0,3], [3,6], [6,9], [9,12]$ and $[12,22]$, the index CDS spreads and the average CDS spread. For the 2004-08-04 portfolio, we set the average CDS spread equal to (i.e. approximated by) the index CDS spread, as in [11] and [18]. All maturities are for five years, and these instruments are used in the calibration described in Subsection 4.3. The interest rate is set to 3%, the payment frequency is quarterly and the recovery rate is 40%.

Table 1: iTraxx Europe, August 4th, 2004. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0,3]$ spread is quoted in %. All maturities are for five years.

	Market	Model	error (bp)	error (%)
$[0,3]$	27.6	27.6	0.0004514	1.635e-005
$[3,6]$	168	168	0.003321	0.001977
$[6,9]$	70	70.07	0.06661	0.09515
$[9,12]$	43	42.91	0.09382	0.2182
$[12,22]$	20	20.03	0.03304	0.1652
index	42	41.99	0.01487	0.03542
avg CDS	42	41.96	0.04411	0.105
Σ abs.cal.err			0.2562 bp	

Table 2: iTraxx Europe Series 6, November 28th, 2006. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0,3]$ spread is quoted in %. All maturities are for five years.

	Market	Model	error (bp)	error (%)
$[0,3]$	14.5	14.5	0.007266	0.0005011
$[3,6]$	62.5	62.41	0.08523	0.1364
$[6,9]$	18	18.1	0.09727	0.5404
$[9,12]$	7	6.881	0.1193	1.704
$[12,22]$	3	3.398	0.3979	13.26
index	26	26.13	0.1299	0.4997
avg CDS	26.87	26.12	0.7535	2.804
Σ abs.cal.err			1.59 bp	

We choose the partition $\mu_1, \mu_2, \ldots, \mu_6$ so that it roughly coincides with the number of defaults needed to reach the upper attachment point for each tranche, see Table 4. The

sum of the absolute calibration error where 0.2562 bp for the 2004-08-04 case and 1.59 bp
for the 2006-11-28 set, so in both portfolios we can therefore speak of a perfect fit for $T = 5$
years, see Table 1 and Table 2.

Table 3: The calibrated parameters that gives the model spreads in the Tables 1 and 2.

	a	$b^{(1)}$	$b^{(2)}$	$b^{(3)}$	$b^{(4)}$	$b^{(5)}$	$b^{(6)}$	
04/08/04	33.0	16.4	84.5	145	86.4	124	514	$\times 10^{-4}$
06/11/28	24.9	13.9	73.6	62.4	0.823	2162	4952	$\times 10^{-4}$

Table 4: The integers $1, \mu_1, \mu_2, \ldots, \mu_c$ are partitions of $\{1, 2, \ldots, m\}$ used in the models that generates the spreads in Table 1 and Table 2

partition	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
	7	13	19	25	46	125

The numerical values of the calibrated parameters $\boldsymbol{a}$, obtained via (4.3.1), show that in
both portfolios the parameters $a, b^{(1)}$ and $b^{(2)}$ are approximately in the same order while
$b^{(3)}$ is around two times bigger for the 2004-08-04 case compared with the 2006-11-28
collection. However, the parameters $b^{(4)}, b^{(5)}$ and $b^{(5)}$ differ substantially between the two
portfolios and given the big difference among the corresponding market spreads, this may
not come as a surprise. In the 2004-08-04 case, the variables $b^{(4)}, b^{(5)}$ and $b^{(5)}$ varies rather
smoothly, while for the 2006-11-28 collection, the jump parameters virtually explodes after
default number 25, see Table 3. This drastic increase of the jumps after the 26th default
will have a big impact on the different distributions and other quantities, as will be seen
in the next subsections. We want to remind the reader that the intensities are implied, or
so called risk-neutral intensities, measured under a risk-neutral martingale measure, which
exists if we rule out arbitrage opportunities in our model. Here "implied" is refereing to
the fact that the quantities are retrieved from market data via a model. Recall that risk-
neutral default intensities are substantially larger than the real, so called actuarial, default
probabilities.

5.2. **The implied expected ordered default times.** Given the implied distributions
we can compute important quantities for a credit manager. In this subsection we study
the expected ordered default times $\mathbb{E}[T_k]$ and their standard deviations. Further, the
expected default times $\mathbb{E}[\tau_i]$, which by exchangeability are the same for all obligors, are
also computed. The formulas for all these quantities are given in Subsection 3.4.

In Figure 1, left, we note that the implied expected ordered default times take values
roughly between 3.5 years and 14 years. A striking feature in the 2006-11-28 portfolio
is that after the 25 default, the $\mathbb{E}[T_k]$ cluster around 14 years. This is a consequence of
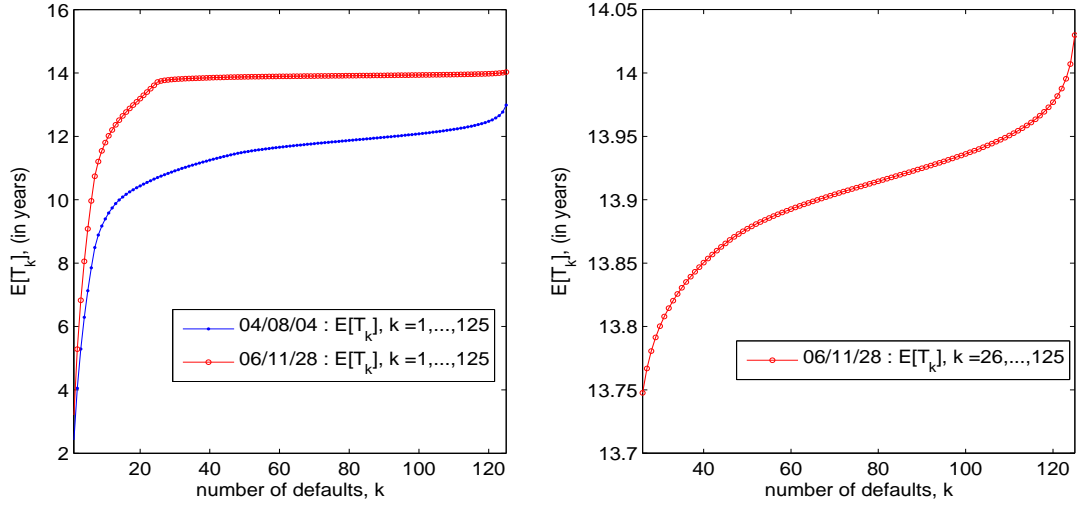
Figure 1: The implied expected ordered default times $\mathbb{E}[T_k]$ for the 2004-08-04 and 2006-11-28 portfolios where $k = 1, \ldots, 125$ (left) and $k = 26, \ldots, 125$ (right, for 2006-11-28).
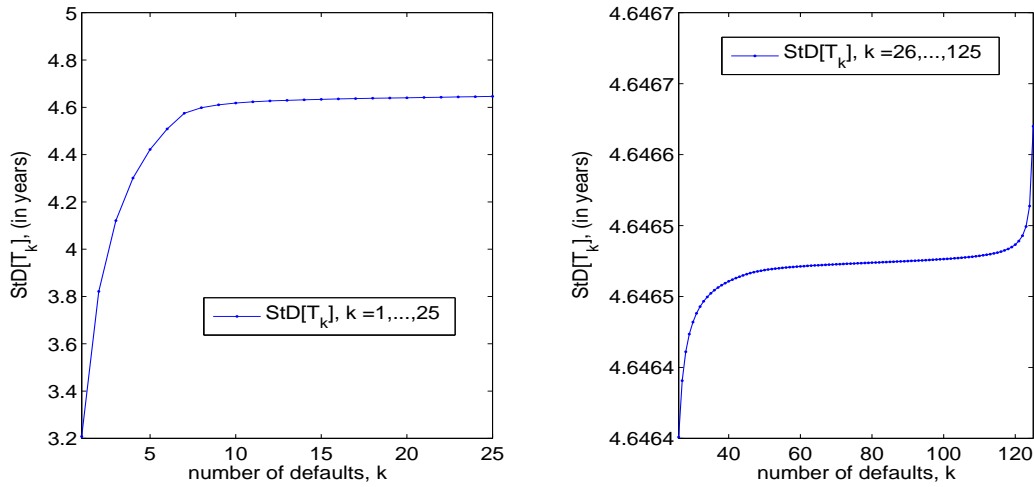

Figure 2: The implied standard deviations $\mathrm{StD}[T_k]$ for the ordered default times where $k = 1, \ldots, 25$ (left) and $k = 26, \ldots, 125$ (right) for the 2006-11-28 portfolio.

the explosion in the jump intensities for $k \geq 25$, as discussed in Subsection 4.3. Under the risk-neutral measure, implied by the market data in Table 2, this clustering of $\mathbb{E}[T_k]$ means that we expect extreme losses in year 13 and 14 for the 2006-11-28 portfolio. This is confirmed by computation of the loss probability, see [16], which renders $\mathbb{P}[L_{15} > 11.52\%] =$

$\mathbb{P}\left[Y_{15} > 24\right] = 66.62\%$, where a loss of 11.52% corresponds to 24 defaults when the recovery is 40%. As a matter of fact, $\mathbb{P}\left[L_{15} = 60\%\right] = \mathbb{P}\left[Y_{15} = 125\right] = 64.256\%$. Again, recall that all computations are under the risk-neutral measure, and should not be confused with real default probabilities and their expectations. These are likely to be substantially smaller for the loss probability and much bigger for the expected ordered default times. Since $\{T_k\}$ are strictly increasing, so are the $\mathbb{E}\left[T_k\right]$.

It is interesting to note that the curves for the standard deviation of the T_k roughly have the same shape as for $\mathbb{E}\left[T_k\right]$. Finally, the implied expected default times, which by the exchangeability are the same for all obligors i, have the values $\mathbb{E}\left[\tau_i\right] = 11.21$, $\mathrm{StD}[\tau_i] = 3.927$ for the 2004-08-04 portfolio and $\mathbb{E}\left[\tau_i\right] = 13.38$, $\mathrm{StD}[\tau_i] = 4.890$ for the 2006-11-28 portfolio

5.3. **The implied default correlation.** It may be of interest for a credit manager to have a quantitative grasp of the implied pairwise default correlation $\rho(t) = \mathrm{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$ for two distinct obligors i, j, as function of time t. In this subsection, we study $\rho(t)$ in the calibrated portfolios in Table 1 and Table 2.s
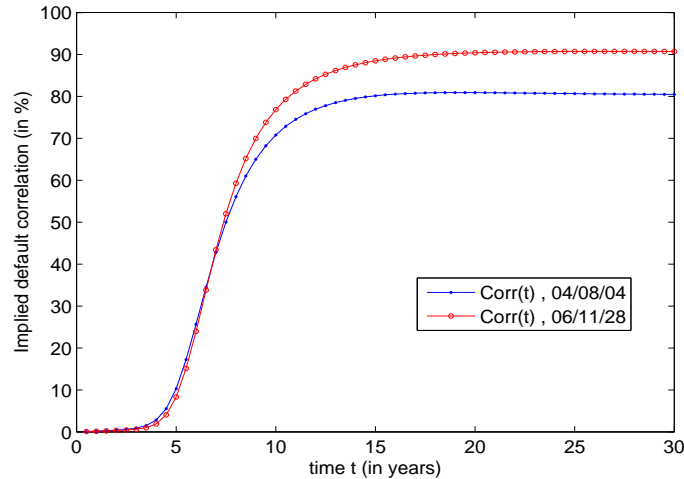

Figure 3: The implied default correlation $\rho(t) = \mathrm{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$, $i \neq j$ as function of time for the 2004-08-04 and 2006-11-28 portfolios.

In the 2006-11-28 portfolio, we see that $\rho(t)$ is less than 2% when $t \leq 4$, but then starts to increase rapidly, first to 4% for $t = 4.5$, then to 77% for $t = 10$ and reaches 88% at $t = 15$. After this drastic development, the implied default correlation flattens out and converges to 91% as time increases against 30 years. The explosive increase of $\rho(t)$ from 2% to 88% in the time interval $[4.5, 15]$ is due to the default contagion and is also consistent with the clustering of $\{T_k\}$ around $t = 14$. We also note that the implied default correlation for the 2004-08-04 portfolio follows an almost identical trend up to 8 years. This is consistent with the jump-to-default parameters for the first 13 defaults, which are in the same order
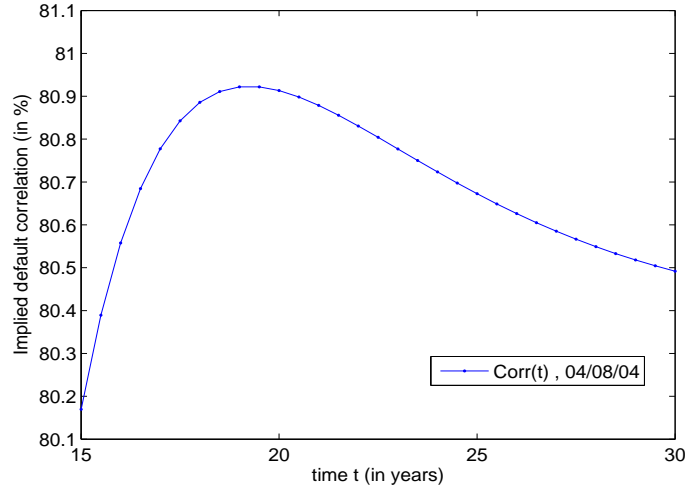
Figure 4: The implied default correlation $\rho(t) = \text{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$, $i \neq j$ as function of time t, where $15 \leq t \leq 30$, for the 2004-08-04 portfolio.

as in 2006-11-28 case, see also Figure 1. Even though there is a big difference between the corresponding contagious parameters for $k > 13$ in the two portfolios, the implied default correlations never differ more than $10\% - 12\%$ during the first 30 years. Furthermore, the 2004-08-04 portfolio seem to have a global maxima of 80.2% around 19 years, as seen in Figure 4.

We observe that implied default correlations around 90% which occur already for $t = 16$ are quite big. In [15] the author studied implied default correlations in an inhomogeneous portfolio consisting of 10 obligors and with very high 5 years CDS-spread correlations. This gave rise to implied default correlations around 70%, but first when $t \geq 95$ years. The corresponding correlation when $t = 15$ was around 12%, see [15].

5.4. **The implied multivariate default and survival distributions.** In this subsection we study the implied bivariate default and survival distributions, the implied survival and default distributions for a fixed t and the joint implied survival and default distributions for the ordered default times.

All computations are done with parameters obtained from the calibrated portfolio in Table 2. From Figure 5 and Figure 6 we note some interesting features in the 2006-11-28 portfolio. For example, the bivariate default distribution $\mathbb{P}[\tau_1 \leq t_1, \tau_2 \leq t_2]$ is approximately constant on the lines (t_1, t_2) where t_1 is fixed, and $6 < t_1 < t_2$. By exchangeability, this is also the case for the lines (t_1, t_2) where t_2 is fixed, and $6 < t_2 < t_1$. Intuitively, this observation imply that the default events $\{\tau_i \leq t_1\}$ and $\{\tau_j \leq t_2\}$ are approximately independent for $(t_1, t_2) \in [6, \infty] \times [6, \infty]$. This property is not present in the region $(t_1, t_2) \in [0, 6] \times [0, 6,]$, see Figure 7. It does not hold for the bivariate survival distribution $\mathbb{P}[\tau_1 > t_1, \tau_2 > t_2]$ either.
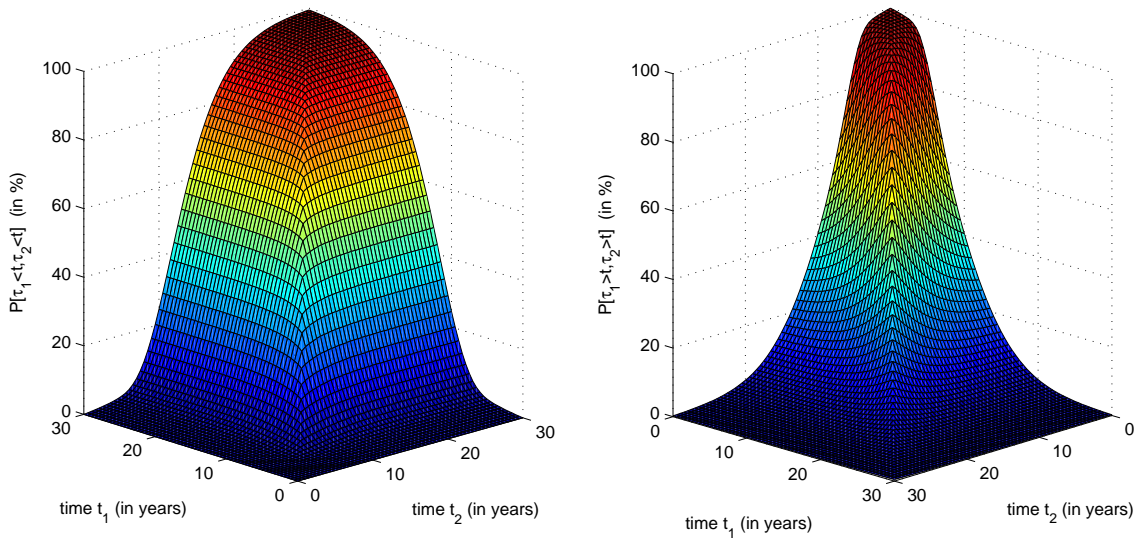
Figure 5: The implied bivariate default (left) and survival (right) distribution for two obligors in the 2006-11-28 portfolio.
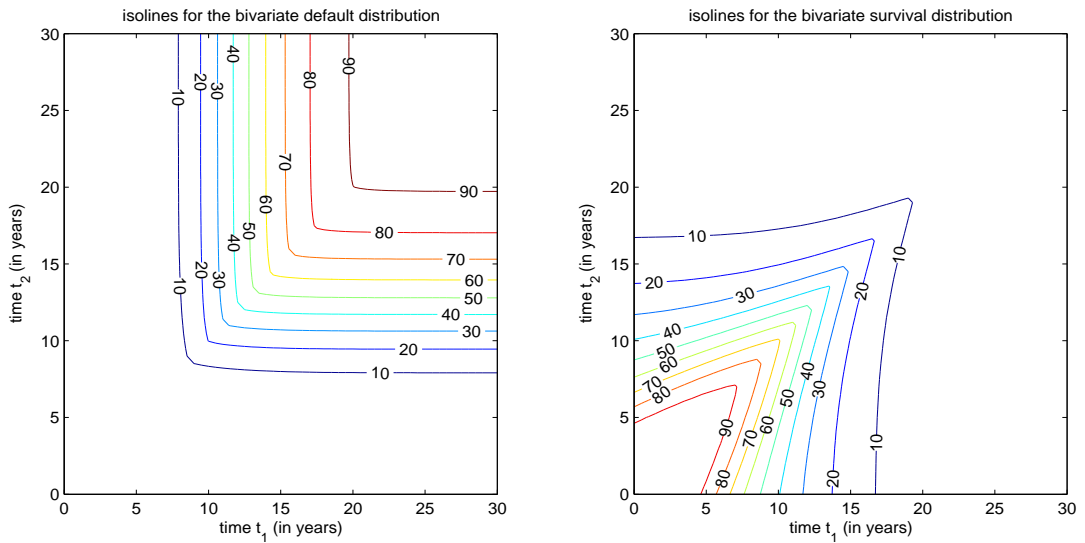

Figure 6: The implied bivariate default (left) and survival (right) distribution for two obligors in the 2006-11-28 portfolio.

Figure 8 shows that $\mathbb{P}[\tau_1 \leq t, \ldots, \tau_q \leq t]$ seems to be independent of q. Similarly, $\mathbb{P}[T_k > t]$ appear to be unchanged for $k > 25$. However, a closer study reveals that the
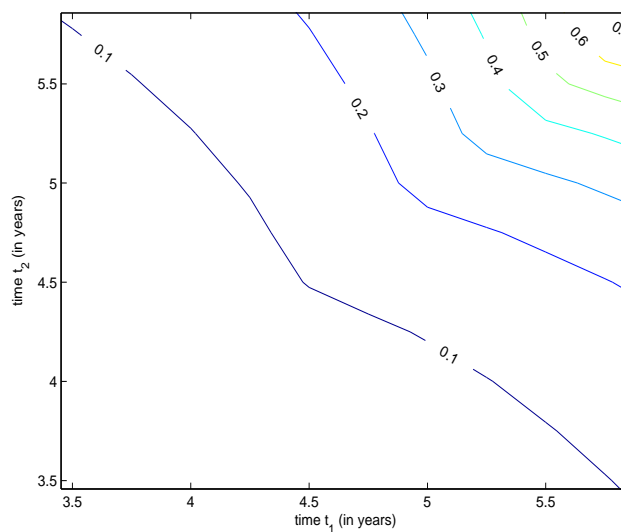
Figure 7: The isolines for the implied bivariate default distributions, on the square $[3.5, 5.5] \times [3.5, 5.5]$, for two obligors in the 2006-11-28 portfolio.
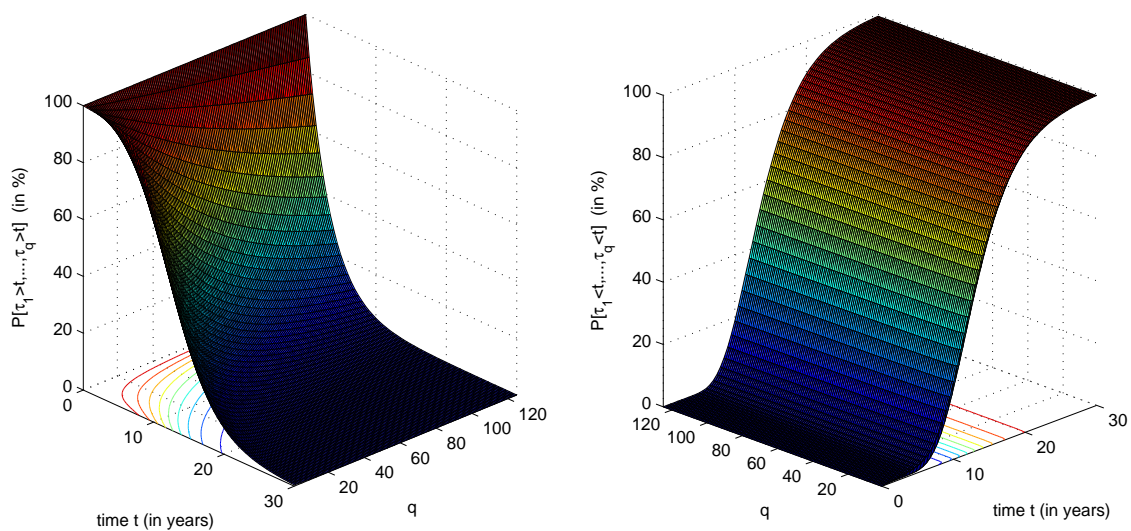

Figure 8: The implied survival distributions $\mathbb{P}\left[\tau_1 > t, \dots, \tau_q > t\right]$ (left) and default distribution $\mathbb{P}\left[\tau_1 \leq t, \dots, \tau_q \leq t\right]$, as functions of q and time t, for the 2006-11-28 portfolio.
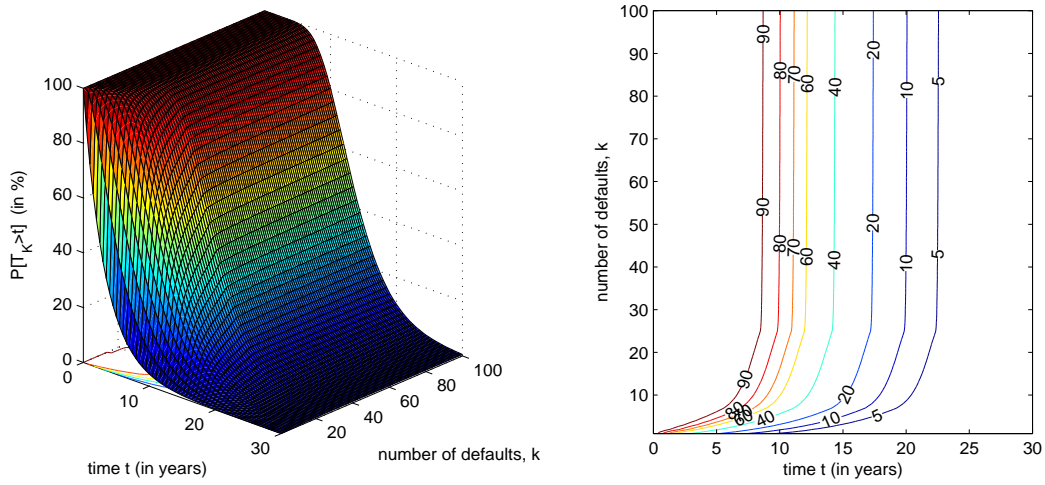
Figure 9: The implied ordered survival distributions $\mathbb{P}\left[T_k > t\right]$ (left) and its isolines (right) as functions of k and time t, for the 2006-11-28 portfolio.
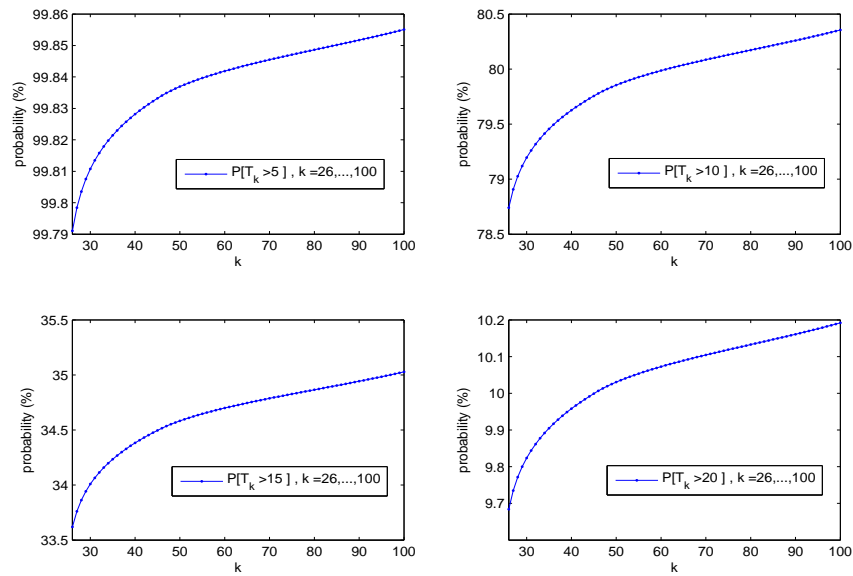

Figure 10: The implied ordered survival distributions $\mathbb{P}\left[T_k > t\right]$ for fixed t as function of k, $k = 26, \ldots, 100$ for the 2006-11-28 portfolio.

computed survival distributions $\mathbb{P}[T_k > t]$ are strictly increasing with k, as the should be, although this is on very narrow intervals, see Figure 10.

Table 5 shows that the effect of the default contagion is clear. For example, in the 2006-11-28 portfolio, $\mathbb{P}[T_1 \leq 1, \ldots, T_{25} \leq 25] = 6\%$ while $\mathbb{P}[T_1 \leq 14, \ldots, T_{25} \leq 38] = 95\%$ and $\mathbb{P}[T_{101} \leq 1, \ldots, T_{125} \leq 25] = 0\%$ while $\mathbb{P}[T_{101} \leq 14, \ldots, T_{125} \leq 38] = 57\%$.

Furthermore, we also see some big differences between the two portfolios. For example, $\mathbb{P}[T_1 > 1, \ldots, T_{25} > 25]$ and $\mathbb{P}[T_{26} \leq 13.5, \ldots, T_{125} \leq 38.5]$ are 0.25%, 2.3% and 80%, 54% respectively, for the 2004-08-05 and 2006-11-28 portfolio.

Table 5: The multivariate default (D) and survival (S) probabilities $\mathbb{P}\left[T_{k_1} \leq nt_1, \ldots, T_{k_q} \leq nt_q\right]$ and $\mathbb{P}\left[T_{k_1} > nt_1, \ldots, T_{k_q} > nt_q\right]$ (in %), for the 2004-08-04 (04) and 2006-11-28 (06) portfolios where $n = 0.25, 0.5, 0.75, 1$ for different sequences $k_1 < \ldots < k_q$ and $t_1 < t_2 < \ldots < t_q$ with $t_i - t_{i-1}$ constant.

$t_1, t_2, \ldots, t_q$	$k_1, \ldots, k_q$		$n = 0.25$	$n = 0.5$	$n = 0.75$	$n = 1$
$1, 2, \ldots, 25$	$1, \ldots, 25$	D, 04	0.021	1	5.1	12
$1, 2, \ldots, 25$	$1, \ldots, 25$	D, 06	0.0048	0.35	2.2	6
$1, 2, \ldots, 25$	$1, \ldots, 25$	S, 04	84	25	2.9	0.25
$1, 2, \ldots, 25$	$1, \ldots, 25$	S, 06	91	51	13	2.3
$1, 2, \ldots, 25$	$101, \ldots, 125$	D, 04	0	0	0	0
$1, 2, \ldots, 25$	$101, \ldots, 125$	D, 06	0	0	0	0
$1, 2, \ldots, 25$	$101, \ldots, 125$	S, 04	100	49	7.1	0.65
$1, 2, \ldots, 25$	$101, \ldots, 125$	S, 06	99	58	15	2.5
$14, 15, \ldots, 38$	$1, \ldots, 25$	D, 04	11	68	93	99
$14, 15, \ldots, 38$	$1, \ldots, 25$	D, 06	4.2	46	81	95
$14, 15, \ldots, 38$	$1, \ldots, 25$	S, 04	22	1.7	0.052	0.0012
$14, 15, \ldots, 38$	$1, \ldots, 25$	S,06	33	6.7	0.62	0.039
$14, 15, \ldots, 38$	$101, \ldots, 125$	D, 04	0.00046	4.1	37	74
$14, 15, \ldots, 38$	$101, \ldots, 125$	D, 06	0.0022	2.6	24	57
$14, 15, \ldots, 38$	$101, \ldots, 125$	S, 04	84	6.4	0.16	0.0032
$14, 15, \ldots, 38$	$101, \ldots, 125$	S, 06	85	14	0.89	0.048
$13.5, 13.75, \ldots, 38.5$	$26, \ldots, 125$	D, 04	0.055	11	49	80
$13.5, 13.75, \ldots, 38.5$	$26, \ldots, 125$	D, 06	0.0027	2.4	22	54
$13.5, 13.75, \ldots, 38.5$	$26, \ldots, 125$	S, 04	81	5.8	0.14	0.0028
$13.5, 13.75, \ldots, 38.5$	$26, \ldots, 125$	S, 06	85	13	0.84	0.045

6. DISCUSSION AND CONCLUSIONS

In this paper we considered the intensity based default contagion model (2.1), where the default intensity of one obligor is allowed to change when other firms default. The portfolio was homogenous so that all obligors were exchangeable. This implied that the individual intensives were expressed using the ordered default times. The model was translated into

a Markov jump process. This made it possible to derive computationally tractable closed-form expressions for many quantities of importance in credit portfolio management.

In the above framework we calibrated two CDO portfolios containing 125 obligors, against market data for CDO tranches, index CDS and average CDS spread. In both cases we obtained perfect fits. In the calibrated portfolios, we then studied the implied expected ordered default times, the implied default correlations and the implied joint default and survival distributions both for ordered and unordered default times. Some of the results where surprising, other not so. For example, in the 2006-11-28 portfolio, the bivariate default distributions revealed that the corresponding marginal events was approximately independent after 10 years.

The calibrated default intensities for the 2006-11-28 portfolio, exploded after 25 defaults, which gave rise to heavy default clustering after 14 years (under the risk-neutral measure). This had profound effect on several of the studied quantities.

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