MASTER'S THESIS

The noncommutative Shilov boundary

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Abstract

We introduce Arveson's generalization of the Shilov boundary to the noncommutative case and give a proof based on the work of Hamana of the existence of the Shilov boundary ideal.

Moreover, we describe the Shilov boundary for a noncommutative analog of the algebra of holomorphic functions on the unit polydisk $\mathbb{D}^n$ and for a q-analog of the algebra of holomorphic functions on the unit ball in the space of symmetric complex 2×2 matrices.

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Contents

1 Introduction

The famous Gelfand-Naimark theorem states that any unital commutative C^*-algebra is $*$-isomorphic to a C^*-algebra of continuous functions on a compact Hausdorff space. As a consequence, it can be shown that the topology of a compact Hausdorff space is completely determined by the C^*-algebra of continuous functions defined on it. This observation leads to one of the fundamental ideas of noncommutative geometry, where the duality between compact Hausdorff spaces and commutative C^*-algebras is extended to the noncommutative setting by considering a noncommutative C^*-algebra as an algebra of functions on a noncommutative generalization of a compact Hausdorff space. It is therefore an interesting task to formulate classical geometrical notions solely in terms of the commutative C^*-algebra in order to obtain generalizations to the noncommutative case, which we shall refer to as *noncommutative analogs*.

One such notion is that of the *Shilov boundary*. Let X be a compact Hausdorff space, and let $A \subset C(X)$ be a *uniform algebra*, i.e., a closed subalgebra that contains the constants and separates points of X. The Shilov boundary of X relative to A is defined as the smallest closed subset S of X such that every function in A achieves its maximum modulus on S. The prototypical example of this is of course the maximum modulus principle encountered in the theory of holomorphic functions. For the disk algebra $A(\mathbb{D}) \subset C(\bar{\mathbb{D}})$, consisting of functions holomorphic on the unit disk $\mathbb{D}$ and continuous up to the boundary, it is a well known fact that every function in $A(\mathbb{D})$ achieves its maximum modulus on the unit circle $\mathbb{T}$.

In this thesis we shall explore a noncommutative analog of the Shilov boundary, which was introduced by Arveson in [Arv69]. We shall also describe the Shilov boundary for some concrete situations. In particular we shall describe the Shilov boundary for a noncommutative analog of the holomorphic functions on bounded domains. In other words, this amounts to investigating a noncommutative analog of the maximum modulus principle.

Throughout this thesis we assume that all algebras and homomorphisms are unital with the exception of the C^*-algebra $C_0(X)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space X.

1.1 Gelfand duality and noncommutative geometry

In order to properly motivate the identification of C^*-algebras as noncommutative analogs of algebras of functions, let us begin by elaborating on the claim that the topology of a space is determined by the algebra of functions defined on it.

Let B be a commutative C^*-algebra. A *character* $\chi : B \to \mathbb{C}$ is a nonzero $*$-homomorphism of B into $\mathbb{C}$. The set M_B of all characters on B is called the *maximal ideal space* of B. Endowed with the topology induced by the weak* topology on the dual space of B, M_B is a compact Hausdorff space.

We define the *Gelfand transform* $\Gamma : B \to C(M_B)$ by $\Gamma(x) = \widehat{x}$ where $\widehat{x}$ is defined as $\widehat{x}(\chi) = \chi(x)$, $\chi \in M_B$. Let us now give the statement of the Gelfand-Naimark theorem. For a proof, see e.g. [Dav96].

Theorem 1.1 (Gelfand-Naimark)**.** *Let B be a commutative C^*-algebra, and let M_B be its maximal ideal space. The Gelfand transform is a $*$-isomorphism of B onto $C(M_B)$.*

Given a compact Hausdorff space X, we have a natural way to associate X to a commutative C^*-algebra, namely $C(X)$. This mapping defines a contravariant functor F between the categories of compact Hausdorff spaces and the category of commutative C^*-algebras. The contravariance follows from the fact that if $f : X \to Y$ is a continuous map between two compact Hausdorff spaces X and Y, then we have a $*$-homomorphism $F(f) : C(Y) \to C(X)$ given by $F(f)(g) = g \circ f$.

On the other hand, we also have that a commutative C^*-algebra can be associated to a compact Hausdorff space, namely its maximal ideal space. So we also have a contravariant functor G from the category of commutative C^*-algebras into the category of compact Hausdorff spaces.

Theorem 1.2. *The category of compact Hausdorff spaces with morphisms the continuous maps is dually equivalent to the category of commutative C^*-algebras with morphisms the $*$-homomorphisms.*

Proof. It remains to show that the functors F and G are quasi-inverse to each other, i.e., for any C^*-algebra B and any compact Hausdorff space X, we have natural isomorphisms $B \cong C(M_B)$ and $X \cong M_{C(X)}$. The first case is precisely the statement of the Gelfand-Naimark theorem. For the second case, we define a map $X \to M_{C(X)}$ by $x \mapsto \delta_x$, where δ_x is the evaluation map at x, i.e., $\delta_x(f) = f(x)$. Since $C(X)$ separates points, this map is injective. It also follows readily by the definition of the weak* topology that this map is continuous. To see that the map is surjective, let χ be a character of $C(X)$. Since $C(X)/\operatorname{Ker}\chi \cong \mathbb{C}$, using the characterization of closed ideals from Proposition 1.4 below, it is not difficult to see that $\operatorname{Ker}\chi = \{f \in C(X) : f(x_0) = 0\}$ for some $x_0 \in X$. Then $\chi(f - f(x_0)) = 0$, and hence $\chi(f) = f(x_0)$, showing that $\chi = \delta_{x_0}$. $\square$

One of the fundamental ideas of noncommutative geometry is that the duality between compact Hausdorff spaces and commutative C^*-algebras suggests that we should take noncommutative C^*-algebras as representatives of noncommutative analogs of compact Hausdorff spaces. Since the topology of a compact Hausdorff space is completely determined by its associated C^*-algebra, we take the characterization of geometric notions in terms of C^*-algebras as the definition of these notions in the noncommutative case.

Let us give a couple of examples of these correspondences between geometric notions and their characterization on the algebraic side.

Let X be a compact Hausdorff space. Then we have a correspondence between compact subspaces of X and quotients of $C(X)$. Let $K \subset X$ be a compact subspace, and let J_K denote the following associated closed ideal in $C(X)$:

$$J_K = \{f \in C(X) : f|_K = 0\}. \tag{1.1}$$

From the short exact sequence

$$0 \longrightarrow J_K \longrightarrow C(X) \longrightarrow C(K) \longrightarrow 0, \tag{1.2}$$

we see that $C(K)$ is $*$-isomorphic to a quotient of $C(X)$.

The key to the converse statement is the fact that any closed ideal in $C(X)$ is of the form (1.1) for some closed subspace $K \subset X$. This statement in turn relies on the Stone-Weierstrass theorem, which we now recall.

Theorem 1.3 (Stone-Weierstrass). *Let X be a locally compact Hausdorff space, and let A be a $*$-algebra that separates points and vanishes nowhere. Then A is dense in $C_0(X)$.*

Proposition 1.4. *Let J be a closed ideal in $C(X)$. Then J is of the form*

$$J = \{f \in C(X) : f|_K = 0\}$$

for some closed subspace $K \subset X$.

Proof. Define $K \subset X$ as the set of common zeros of all functions in J, i.e.,

$$K = \{x \in X : f(x) = 0 \text{ for all } f \in J\}.$$

Then K is closed since if x_0 is a limit point of K that is not in K, then there is a function $f \in J$ such that $f|_K = 0$ and $f(x_0) \neq 0$. But then $f \neq 0$ on some neighborhood of x_0, which is a contradiction.

Let now J_K be the closed ideal associated with K as in (1.1). Clearly $J \subset J_K$. Set $M = X \setminus K$, and consider the restriction of J to M, $J|_M = C_0(M)$. It is easy to see that $J|_M$ separates points, and by the definition of K, $J|_M$ vanishes nowhere. By the Stone-Weierstrass theorem, $J|_M$ is dense in $C_0(M)$, and consequently J is dense in J_K, showing that $J = J_K$. □

Thus we get that any quotient of $C(X)$ is of the form $C(X)/J_K$ for an ideal of the form (1.1), and again by the short exact sequence (1.2) we get $C(X)/J_K \cong C(K)$.

Let us now investigate how $C(X \times Y)$, the C^*-algebra of continuous functions on the cartesian product of two compact Hausdorff spaces, is related to $C(X)$ and $C(Y)$.

There are several different ways of defining the tensor product $A \otimes B$ of two C^*-algebras. Let A and B be C^*-algebras, and consider the following norm on the algebraic tensor product of A and B:

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\| = \sup_{\substack{\pi_1 \in \mathrm{Irrep}(A) \\ \pi_2 \in \mathrm{Irrep}(B)}} \left\| \sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i) \right\|.$$

The completion of $A \otimes B$ in this norm is a C^*-algebra and is known as the *minimal tensor product* of A and B.

Let us now show that $C(X \times Y)$ is $*$-isomorphic to $C(X) \otimes C(Y)$. Consider the $*$-homomorphism given on a dense subset of $C(X) \otimes C(Y)$ defined by $f \otimes g \mapsto fg$. As a consequence of the Stone-Weierstrass theorem, it follows that this map is surjective. Moreover, we have

$$\sup_{\substack{\chi_1 \in \mathrm{Irrep}(C(X)) \\ \chi_2 \in \mathrm{Irrep}(C(Y))}} \left\| \sum_{i=1}^{n} \chi_1(f_i) \otimes \chi_2(g_i) \right\| = \sup_{\substack{x \in X \\ y \in Y}} \left\| \sum_{i=1}^{n} f_i(x) g_i(y) \right\| = \left\| \sum_{i=1}^{n} f_i g_i \right\|_\infty,$$

showing that the map is an isometry.

In Section 1.3 we shall give a formulation of the holomorphic functions on a bounded domain in terms of C^*-algebras from which we can formulate a noncommutative analog.

1.2 Representation theory and the Gelfand-Naimark theorem

In this section we review the basic properties of representations for $*$-algebras and C^*-algebras in particular.

A *representation* π of a $*$-algebra A on a Hilbert space H is a $*$-homomorphism $\pi : A \to \mathcal{B}(H)$ of A into the C^*-algebra of bounded operators on H. We shall frequently use the notation (H, π) for a representation π on a Hilbert space H.

If $\pi(A)$ has no proper invariant subspaces, we say that π is *algebraically irreducible*, and if $\pi(A)$ has no proper closed invariant subspaces, we say that π is *topologically irreducible*. In this thesis we shall exclusively be dealing with topologically irreducible representations, whence we shall simply refer to them as *irreducible.* Clearly algebraically irreducible representations are topologically irreducible so there is no ambiguity here. It is worth noting that for C^*-algebras, by a result known as Kadison's Transitivity theorem, these two notions of irreducibility coincide.

Two representations (H_1, π_1) and (H_2, π_2) are said to be *unitarily equivalent* if there exists a unitary isomorphism $U : H_1 \to H_2$ such that $U\pi_1(a)\xi = \pi_2(a)U\xi$ for all $a \in A$ and $\xi \in H_1$. Equivalently, the diagram

$$\begin{tikzcd} H_1 \arrow[r, "U"] \arrow[d, "\pi_1(a)"'] & H_2 \arrow[d, "\pi_2(a)"] \\ H_1 \arrow[r, "U"] & H_2 \end{tikzcd}$$

commutes for all $a \in A$.

For any subset S of $\mathcal{B}(H)$, we define the commutant of S as

$$S' = \{X \in \mathcal{B}(H) : XY = YX \text{ for all } Y \in S\}.$$

Lemma 1.5 (Schur's lemma)**.** *Let (H, π) be a representation of a $*$-algebra A. Then π is irreducible if and only if $\pi(A)' = \mathbb{C}I$, i.e., if $X\pi(a) = \pi(a)X$ for all $a \in A$ then $X = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof. Let (H, π) be irreducible, and suppose that X is a self-adjoint operator in $\pi(A)'$ so that $X\pi(a) = \pi(a)X$ for all $a \in A$ but $X \notin \mathbb{C}I$. Then X has at least two points λ and μ in its spectrum. Let f and g be functions in $C(\sigma(X))$ such that $f(\lambda) \neq 0$ and $g(\mu) \neq 0$ and $fg = 0$. Define $H_f = \overline{f(X)H}$. Then $\pi(a)f(X) = f(X)\pi(a)$ for all $a \in A$, and hence $\pi(a)H_f \subset H_f$. Since $f(X) \neq 0$, we have $H_f \neq \{0\}$, and therefore $H_f = H$ since π is irreducible. Therefore

$$g(X)H = g(X)\overline{f(X)H} \subset \overline{g(X)f(X)H} = \{0\},$$

which implies $g(X) = 0$, a contradiction. The general case follows by writing $X = Y + iZ$, where $Y = (X + X^*)/2$ and $Z = (X - X^*)/2i$ are self-adjoint elements known as the real and imaginary parts of X.

Conversely, suppose $\pi(A)' = \mathbb{C}I$. Let $M \neq \{0\}$ be a closed invariant subspace. The invariance implies that the orthogonal projection P onto M satisfies $\pi(a)P = P\pi(a)P$ for every $a \in A$. But then it follows that

$$P\pi(a) = (\pi(a^*)P)^* = (P\pi(a^*)P)^* = P\pi(a)P = \pi(a)P$$

for every $a \in A$, and hence $P = I$ and $M = H$. □

An immediate consequence of Schur's lemma is that irreducible representations of commutative $*$-algebras are one-dimensional.

A central result in the theory of C^*-algebras is the fact that any C^*-algebra can be faithfully represented as a concrete C^*-algebra of operators in $\mathcal{B}(H)$. This result is also commonly referred to as the Gelfand-Naimark theorem as it in fact is a generalization of Theorem 1.1. Thus when dealing with a C^*-algebra, one can always treat its elements as operators sitting in $\mathcal{B}(H)$ for some Hilbert space H. This is an extremely useful technique when deriving statements about general C^*-algebras, and we shall use it numerous times throughout this thesis.

The key technique in proving the Gelfand-Naimark theorem relies on constructing representations from states, which we shall define momentarily. Let us first briefly recall the notion of positive elements and positive maps defined on C^*-algebras.

A self-adjoint element x of a C^*-algebra B is said to be *positive* if its spectrum $\sigma(x)$ is contained in $[0, \infty)$. It is a well known fact that an element $x \in B$ is positive if and only if it is of the form y^*y for some element $y \in B$.

A linear map $\varphi : A \to B$ defined on a subspace of a C^*-algebra is said to be positive if $\varphi(x)$ is positive whenever $x \in A$ is positive. From the characterization of positive elements above, it follows immediately that $*$-homomorphisms of C^*-algebras are positive maps.

A positive linear functional f on a C^*-algebra B is said to be a *state* if $\|f\| = 1$. If f is an extreme point in the set of all states $\mathcal{S}(B)$, then f is said to be *pure*.

The procedure of constructing representations from states is due to the following result known as the GNS construction, named after Gelfand, Naimark and Segal.

Theorem 1.6 ([Dav96, Theorem I.9.6, I.9.8])**.** *Let f be a state on a C^*-algebra B. Then there exists a representation π_f of B on a Hilbert space H_f and a unit vector ξ_f that is cyclic for $\pi(B)$, and*

$$f(x) = \langle \pi_f(x)\xi_f, \xi_f \rangle$$

for all $x \in B$. Moreover, if f is pure, then (H_f, π_f) is irreducible.

Let us give a brief sketch of how one obtains the Hilbert space H_f and representation π_f from f and B.

It can be shown that $N = \{x \in B : f(y^*x) = 0 \text{ for all } y \in B\}$ is a closed left ideal. It can also be shown that $\langle x + N, y + N \rangle = f(y^*x)$ defines an inner product on the vector space B/N. The Hilbert space H_f is obtained by completing B/N with respect to the norm induced by the inner product, and the representation π_f is obtained by extending the left regular representation of B on B/N: $\pi_f(a)(x + N) = ax + N$.

Lemma 1.7 ([Dav96, Lemma I.9.10]). *Let x be a self-adjoint element of a C^*-algebra B. Then there exists a pure state f on B such that $|f(x)| = \|x\|$.*

This lemma together with the GNS construction yields the following lemma concerning representations of $*$-algebras. Basically, it provides us with a tool which, in many situations, allows us to consider only irreducible representations.

Lemma 1.8. *Let π be a representation of a $*$-algebra A. Then for each element $a \in A$, there exists an irreducible representation ρ of A such that $\|\pi(a)\| = \|\rho(a)\|$.*

Proof. Without loss of generality, we may assume that a is self-adjoint. By the previous lemma, there exists a pure state f on $C^*(\pi(a))$ such that $\|\pi(a)\| = |f(\pi(a))|$. Let π_f and ξ_f be the irreducible representation obtained from the GNS representation applied to f. Then

$$\|\pi(a)\| = |f(\pi(a))| = |\langle \pi_f \circ \pi(a)\xi_f, \xi_f\rangle| \leq \|\pi_f \circ \pi(a)\| \leq \|\pi(a)\|.$$

It is straightforward to verify that $\pi_f \circ \pi$ is irreducible, and hence by defining $\rho = \pi_f \circ \pi$, this proves the lemma. $\square$

We finish this section with the general form of the Gelfand-Naimark theorem.

Theorem 1.9 (Gelfand-Naimark). *Let B be a C^*-algebra. Then B is $*$-isomorphic to a concrete C^*-algebra of operators in $\mathcal{B}(H)$.*

Proof. Define $\pi : B \to \mathcal{B}(H)$ by

$$\pi = \bigoplus_{\substack{f \in \mathcal{S}(B) \\ f \text{ pure}}} \pi_f.$$

Since π is a $*$-homomorphism, $\|\pi(x)\| \leq \|x\|$ so it remains to show that $\|\pi(x)\| \geq \|x\|$ for all $x \in B$.

We claim that for each $x \in B$, there exists a pure state f such that $\|\pi_f(x)\xi_f\| = \|x\|$, where π_f and ξ_f is the representation and unit vector obtained by the GNS construction applied to f. Indeed, by Lemma 1.7, there exists a pure state f such that $f(x^*x) = \|x\|^2$. Then

$$\|\pi_f(x)\xi_f\|^2 = \langle \pi_f(x^*x)\xi_f, \xi_f\rangle = f(x^*x) = \|x\|^2.$$

Using this, we obtain

$$\|\pi(x)\| = \sup_{\|\xi\|=1} \|\pi(x)\xi\| \geq \|\pi_f(x)\xi_f\| = \|x\|.$$

This shows that π is a $*$-isomorphism of B onto the C^*-subalgebra $\pi(B) \subset \mathcal{B}(H)$. $\square$

1.3 Universal enveloping C^*-algebras and noncommutative complex analysis

The central objects of study in this thesis will be noncommutative analogs of C^*-algebras of continuous functions that arise from certain noncommutative analogs of polynomial algebras. Let $P(\mathbb{C}^n)$ denote the $*$-algebra of polynomials defined on $\mathbb{C}^n$. By representing $P(\mathbb{C}^n)$ as functions on the unit polydisk, we obtain the C^*-algebra $C(\bar{\mathbb{D}}^n)$ of continuous functions on the unit polydisk from the completion with respect to the norm. We obtain the holomorphic functions on the unit polydisk that are continuous up to the boundary $A(\mathbb{D}^n)$ as the closed subalgebra generated by the coordinate functions $z_1, \ldots, z_n$.

In order to formulate noncommutative analogs of the algebras of continuous and holomorphic functions respectively, we consider an equivalent characterization of $C(\bar{\mathbb{D}}^n)$ in terms of representations of $P(\mathbb{C}^n)$. Let ρ be the $*$-homomorphism that maps each polynomial in $P(\mathbb{C}^n)$ to its corresponding function in $C(\bar{\mathbb{D}}^n)$. We claim that the pair $(C(\bar{\mathbb{D}}^n), \rho)$ has the following universal property: for every representation π of $P(\mathbb{C}^n)$ that satisfies $\|\pi(z_i)\| \leq 1$, $1 \leq i \leq n$, there exits a unique $*$-homomorphism $\varphi : C(\bar{\mathbb{D}}^n) \to C^*(\pi(P(\mathbb{C}^n)))$ such that $\pi = \varphi \circ \rho$.

$$\begin{tikzcd} P(\mathbb{C}^n) \arrow[r, "\rho"] \arrow[dr, "\pi"'] & C(\bar{\mathbb{D}}^n) \arrow[d, "\varphi"] \\ & C^*(\pi(P(\mathbb{C}^n))) \end{tikzcd}$$

We say that $(C(\bar{\mathbb{D}}^n), \rho)$ is a *universal enveloping* C^*-algebra of $P(\mathbb{C}^n)$.

For $p \in P(\mathbb{C}^n)$, we set $\varphi(\rho(p)) = \pi(p)$. Clearly this map is a well-defined $*$-homomorphism. Moreover, from Lemma 1.8, it readily follows that

$$\|\pi(p)\| \leq \sup_{\substack{\chi \in \mathrm{Irrep}(P(\mathbb{C}^n)) \\ \|\chi(z_i)\| \leq 1}} |\chi(p)| = \sup_{\zeta \in \bar{\mathbb{D}}^n} |p(\zeta)| = \|\rho(p)\|_\infty.$$

Thus φ is bounded on a dense subspace of $C(\bar{\mathbb{D}}^n)$, and hence it extends uniquely to a $*$-homomorphism on $C(\bar{\mathbb{D}}^n)$.

Let us now turn to the noncommutative case. Our goal is to define a noncommutative analog of the continuous functions on the unit polydisk $C(\bar{\mathbb{D}}^n)_q$ as a universal enveloping algebra of some $*$-algebra $P(\mathbb{C}^n)_q$ generated by $z_1, \ldots, z_n$, which we shall refer to as a noncommutative analog of the polynomial algebra on $\mathbb{C}^n$. We interpret $C(\bar{\mathbb{D}}^n)_q$ as a *deformation* of $C(\bar{\mathbb{D}}^n)$ indexed by some *deformation parameter* q with the understanding that, for some specific configuration of q, we recover the classical, i.e., commutative case $C(\bar{\mathbb{D}}^n)$. If q denotes a real number with $0 < q < 1$, such that the commutative case is recovered when q is replaced by 1, then $C(\bar{\mathbb{D}}^n)_q$ is referred to as a *quantum analog* or q-analog of $C(\bar{\mathbb{D}}^n)$.

The key to constructing the universal enveloping C^*-algebra lies in the representations of $P(\mathbb{C}^n)_q$. If $\sup_{\pi \in \mathrm{Irrep}(P(\mathbb{C}^n)_q)} \|\pi(a)\| < \infty$ for all $a \in P(\mathbb{C}^n)_q$, we say that $P(\mathbb{C}^n)_q$ is $*$-bounded, and in this case we define the following seminorm on $P(\mathbb{C}^n)_q$ by

$$\|a\|_0 = \sup_{\pi \in \mathrm{Irrep}(P(\mathbb{C}^n)_q)} \|\pi(a)\|.$$

Let

$$N = \{a \in P(\mathbb{C}^n)_q : \|a\|_0 = 0\}.$$

We define $C(\bar{\mathbb{D}}^n)_q$ to be the completion of $P(\mathbb{C}^n)_q/N$ in the norm induced by $\|\cdot\|_0$, i.e., $\|a+N\| = \|a\|_0$, and ρ is defined as the $*$-homomorphism induced by the quotient map.

Let us now show that the notation $C(\bar{\mathbb{D}}^n)_q$ is justified in the sense that $(C(\bar{\mathbb{D}}^n)_q, \rho)$ satisfies the universal property defined above, i.e., for each representation $\pi : P(\mathbb{C}^n)_q \to C^*(\pi(P(\mathbb{C}^n)_q))$, there exists a unique $*$-homomorphism $\varphi : C(\bar{\mathbb{D}}^n)_q \to C^*(\pi(P(\mathbb{C}^n)_q))$ such that $\pi = \varphi \circ \rho$. If φ exists, it is clear that on $P(\mathbb{C}^n)_q/N$ it has to be given by $\varphi(a+N) = \pi(a)$. As a consequence of Lemma 1.8, we have

$$\|\pi(a)\| \leq \sup_{\omega \in \mathrm{Irrep}(P(\mathbb{C}^n)_q)} \|\omega(a)\| = \|a+N\|,$$

and hence we see that φ is well-defined and bounded on $P(\mathbb{C}^n)_q/N$. Therefore φ extends uniquely to a $*$-homomorphism on the whole of $C(\bar{\mathbb{D}}^n)_q$.

If we identify $z_1, \ldots, z_n \in P(\mathbb{C}^n)$ with their representations in $C(\bar{\mathbb{D}}^n)_q$, completely analogous to the commutative case, we obtain a noncommutative analog of the holomorphic functions $A(\mathbb{D}^n)_q$ as the closed subalgebra generated by $z_1, \ldots, z_n$.

In Chapter 4 we shall study a multidimensional generalization of what is commonly referred to as the quantum unit disk. In order to treat this in proper generality, formally we want to study the universal enveloping C^*-algebra that arises from a $*$-algebra generated by $z_1, \ldots, z_n$ that satisfies the relations

$$z_i^* z_i = f(z_i z_i^*), \quad i = 1, \ldots, n \tag{1.3}$$

$$[z_i, z_j] = 0, \quad [z_i^*, z_j] = 0, \quad i \neq j, \tag{1.4}$$

where $f : [0, \infty) \to \mathbb{R}$ is a continuous function. We shall denote this C^*-algebra by $C(\bar{\mathbb{D}}^n)_f$, and it will be referred to as a noncommutative analog of the algebra of continuous functions on the closed unit polydisk. However, since the expression $z_i^* z_i = f(z_i z_i^*)$ does not make sense for arbitrary functions, we shall need to make an alternative definition of $C(\bar{\mathbb{D}})_f$ than the one above.

This time we start with the free $*$-algebra generated by $z_1, \ldots, z_n$, which we shall denote by P_n. Denote the family of representations π of P_n such that $\pi(z_1), \ldots, \pi(z_n)$ satisfy the relations (1.3) and (1.4) by $\mathcal{F}$. Recall that $f(\pi(z_i)\pi(z_i)^*)$ is well-defined due to the continuous functional calculus for normal elements of a C^*-algebra, see e.g. [Dav96, Corollary I.3.3] for further reference. If P_n is $*$-bounded with respect to these relations, i.e., for each $a \in P_n$ there exists a C_a such that $\|\pi(a)\| \leq C_a$ for all $\pi \in \mathcal{F}$, we define a seminorm on P_n by

$$\|a\|_0 = \sup_{\substack{\pi \in \mathrm{Irrep}(P_n) \\ \pi \in \mathcal{F}}} \|\pi(a)\|.$$

Similar to the previous situation, we define $C(\bar{\mathbb{D}}^n)_f$ to be the completion of P_n/N in the norm induced by $\|\cdot\|_0$. In this case we use f as the deformation parameter, and we note that the commutative case is recovered by defining $f(x) = x$.

From the definition of $C(\bar{\mathbb{D}}^n)_f$, it is clear that any representation $\pi \in \mathcal{F}$ can be extended uniquely to a representation of $C(\bar{\mathbb{D}}^n)_f$.

Let us now verify that $z_1, \ldots, z_n \in C(\bar{\mathbb{D}}^n)_f$ satisfy (1.3) and (1.4). By the Stone-Weierstrass theorem and the fact that the Gelfand transform is an isometry, given $\varepsilon > 0$, there is a polynomials p such that

$$\|p(z_i z_i^*) - f(z_i z_i^*)\| = \|p - f\|_{\sigma(z_i z_i^*)} < \frac{\varepsilon}{2},$$

where $\sigma(z_i z_i^*)$ denotes the spectrum of $z_i z_i^*$ viewed as an element in $C(\bar{\mathbb{D}}^n)_f$. Consequently,

$$\begin{aligned}
\|z_i^* z_i - f(z_i z_i^*)\| &\leq \|z_i^* z_i - p(z_i z_i^*)\| + \|p(z_i z_i^*) - f(z_i z_i^*)\| \\
&= \sup_{\substack{\pi \in \mathrm{Irrep}(P_n) \\ \pi \in \mathcal{F}}} \|\pi(z_i^* z_i - p(z_i z_i^*))\| + \frac{\varepsilon}{2} \\
&= \sup_{\substack{\pi \in \mathrm{Irrep}(P_n) \\ \pi \in \mathcal{F}}} \|f(\pi(z_i z_i^*)) - p(\pi(z_i z_i^*))\| + \frac{\varepsilon}{2} \\
&= \sup_{\substack{\pi \in \mathrm{Irrep}(P_n) \\ \pi \in \mathcal{F}}} \|f - p\|_{\sigma(\pi(z_i z_i^*))} + \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

where the last inequality follows because $\sigma(\pi(z_i z_i^*)) \subset \sigma(z_i z_i^*)$. Since the choice of ε was arbitrary, we get $z_i^* z_i = f(z_i z_i^*)$.

We note that if f is a polynomial, then we have two ways of defining $C(\bar{\mathbb{D}}^n)_f$. Let us show that these definitions are indeed equivalent in this case. Define $C(\bar{\mathbb{D}}^n)_f$ as in the latter case, and let $P(\mathbb{C}^n)_f$ be the $*$-algebra generated by $z_1, \ldots, z_n$ subject to the relations (1.3) and (1.4), which is now well-defined. Since $C(\bar{\mathbb{D}}^n)_f$ also satisfies these relations it follows that we have a $*$-homomorphism $\rho : P(\mathbb{C}^n)_f \to C(\bar{\mathbb{D}}^n)_f$, and hence by the universal property it readily follows that these two constructions give the same result.

2 Subspaces of C^*-algebras

2.1 Operator spaces and operator systems

The theory of subspaces of C^*-algebras can in broad terms be described as a noncommutative analog of the theory of normed spaces. The notion of noncommutativity in a setting without apparent multiplicative structure is motivated by the fact that any normed space is isometrically isomorphic to a subspace of a commutative C^*-algebra. Consider a normed space E, and let X denote the unit ball of the dual space of E, endowed with its weak* topology. Recall that X is a compact Hausdorff space by the Banach-Alaoglu theorem. Define a linear map $E \to C(X)$ by $x \mapsto \widehat{x}$, where $\widehat{x}$ as usual is defined as $\widehat{x}(\varphi) = \varphi(x)$, $\varphi \in X$. Recall that, as a consequence of the Hahn-Banach theorem, for each $x \neq 0$ there exists a linear functional φ with $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$. From this it readily follows that

$$\|\widehat{x}\|_\infty = \sup_{\varphi \in X} |\varphi(x)| = \|x\|,$$

and hence this map is an isometry.

With this identification of a normed space with a subspace of a commutative C^*-algebras, we consider the more general situation of a subspace A of a not necessarily commutative C^*-algebra B. However, as we have just seen, the structure of A as a normed space is not sufficient to discern this situation from the commutative case. In order to proceed, we shall need equip A with some additional structure. We will show that A can be associated with a whole sequence of spaces, namely $M_n(A)$, the spaces of $n \times n$ matrices with entries from A. Just as A inherits its norm from B, each space $M_n(A)$ inherits a norm from the matrix C^*-algebra $M_n(B)$ which we shall now define.

Since B can be identified with a C^*-subalgebra of $\mathcal{B}(H)$, $M_n(B)$ can be naturally identified with a C^*-subalgebra of $\mathcal{B}(H^n)$. The algebraic operations as well as the adjoint operation are defined in complete analogy to the algebraic operations and adjoints of ordinary matrices. This identification allows us to equip $M_n(B)$ with the operator norm, which makes $M_n(B)$ into a C^*-algebra. We note that by the way we have defined the C^*-algebra structure of $M_n(B)$, this norm is unique since C^*-norms are unique.

A linear map $\varphi : A \to C$ into a C^*-algebra C naturally induces a family of linear maps $\varphi_n : M_n(A) \to M_n(C)$ simply by applying φ element by element, i.e., if $(a_{ij}) \in M_n(A)$, then $\varphi_n((a_{ij})) = (\varphi(a_{ij}))$. For $n \geq 1$, we say that φ is n-positive if φ_n is positive and n-contractive if φ_n is contractive. If φ_n is positive and contractive respectively for all n, then φ is said to be *completely* *positive* and *completely contractive* respectively. If φ is bounded, then φ_n is bounded for all n. It need not, however, be the case that all maps φ_n are uniformly bounded. We say that φ is *completely bounded* if

$$\|\varphi\|_{\mathrm{cb}} = \sup_n \|\varphi_n\|$$

is finite.

We define an *operator space* as a subspace A of a C^*-algebra. In order to account for the additional structure associated with A, we define the morphisms in the category of operator spaces as the completely contractive maps.

Closely related to the notion of an operator space is that of an operator system. We define an *operator system* as a unital and self-adjoint subspace V of a C^*-algebra, i.e., $1 \in V$ and $a^* \in V$ whenever $a \in V$. Recall that the positive elements of a C^*-algebra determine a partial order on the self-adjoint elements by setting $a \leq b$ if $b - a$ is positive. Since the presence of the identity element guarantees an abundance of positive elements (any element of the form $\|a\|1 - a$ is positive), it is customary to demand that the maps between operator systems should preserve the order structure on V as well as all associated matrix spaces $M_n(V)$. For this reason, we take the completely positive maps as the morphisms in the category of operator systems. An isomorphism between operator systems is known as a *complete order isomorphism*.

Recall that all $*$-homomorphisms of C^*-algebras are contractive and positive, and since a $*$-homomorphism induces $*$-homomorphisms on the associated matrix C^*-algebras, it is clear that any $*$-homomorphism is completely contractive and completely positive. This implies that the structure as an operator space or operator system is well-defined regardless of how the ambient C^*-algebra is represented. In particular, we can always view an operator space or operator system as a subspace of $\mathcal{B}(H)$.

Let us now verify the claim that operator spaces and operator systems carry more structure than ordinary normed spaces. This amounts to showing that there exists contractive and positive maps which are not completely contractive and positive respectively. We give the following example from [Pau03]. Consider the transpose map $t : M_2(\mathbb{C}) \to M_2(\mathbb{C})$. It is not difficult to see that t is both contractive and positive. However, it is neither completely contractive nor completely positive. Indeed, we have that

$$E = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

is positive in $M_2(M_2(\mathbb{C})) = M_4(\mathbb{C})$, but it is easily seen that

$$t_2(E) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is not. Similarly, one can show that $\|t_2\| \geq 2$, and hence t is not completely contractive either.

The remainder of this section will be devoted to the properties of completely contractive and completely positive maps. In particular we will show that a unital map defined on an operator system is completely contractive if and only if it is completely positive.

Let us begin with a simple but important fact concerning positive maps on operator systems.

Proposition 2.1. *Let V be an operator system, and let $\varphi : V \to B$ be a positive map into a C^*-algebra B. Then φ is self-adjoint, i.e., $\varphi(a^*) = \varphi(a)^*$ for all $a \in V$.*

Proof. First we observe that if $x \in V$ is self-adjoint, then x can be written as a difference of two positive elements in V:

$$x = \frac{1}{2}(\|x\|1 + x) - \frac{1}{2}(\|x\|1 - x).$$

Let now a be an element in V, and write $a = x + iy$, where $x, y \in V$ are self-adjoint. Thus a can be written as $a = p_1 - p_2 + i(q_1 - q_2)$, where $p_1, p_2, q_1, q_2 \in V$ are positive. Since φ is positive, we get

$$\varphi(a^*) = \varphi(p_1) - \varphi(p_2) - i(\varphi(q_1) - \varphi(q_2)) = \varphi(a)^*. \qquad \square$$

Proposition 2.2 ([Pau03, Proposition 2.1])**.** *Let V be an operator system, let B be a C^*-algebra, and let $\varphi : A \to B$ be a positive map. Then φ is bounded with $\|\varphi\| \leq 2\|\varphi(1)\|$.*

Proof. If $a \in V$ is positive, then $0 \leq a \leq \|a\|$, and hence $0 \leq \varphi(a) \leq \|a\|\varphi(1)$, from which it follows that $\|\varphi(a)\| \leq \|\varphi(1)\|\|a\|$.

If $a \in V$ is self-adjoint, then by writing a as a difference of two positive elements in V:

$$a = \frac{1}{2}(\|a\|1 + a) - \frac{1}{2}(\|a\|1 - a),$$

we get that

$$\varphi(a) = \frac{1}{2}\varphi(\|a\|1 + a) - \frac{1}{2}\varphi(\|a\|1 - a)$$

is a difference of two positive elements in B. Note that if p_1 and p_2 are positive, then $\|p_1 - p_2\| \leq \max(\|p_1\|, \|p_2\|)$. Thus $\|\varphi(a)\| \leq \|a\|\|\varphi(1)\|$.

Finally, for an arbitrary element $a \in V$, we write $a = x + iy$ where x and y are self-adjoint elements with $\|x\|, \|y\| \leq \|a\|$, which yields

$$\|\varphi(a)\| \leq \|\varphi(x)\| + \|\varphi(y)\| \leq 2\|\varphi(1)\|\|a\|. \qquad \square$$

It can be shown that this bound is the best possible. However, in the case of unital completely positive maps, much more can be said.

Proposition 2.3 ([Pau03, Proposition 3.2])**.** *Let V be an operator system, and let B be a C^*-algebra. If $\varphi : V \to B$ is a unital 2-positive map, then φ is contractive.*

Proof. For an element $x \in V$, we claim that $\|x\| \leq 1$ if and only if

$$\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} \geq 0.$$

Let π be a faithful representation of B on some Hilbert space H. If $\|x\| \leq 1$, then

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & \pi(x) \\ \pi(x)^* & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right\rangle &= \|\xi_1\|^2 + \langle \pi(x)\xi_2, \xi_1 \rangle + \langle \xi_1, \pi(x)\xi_2 \rangle + \|\xi_2\|^2 \\ &\geq \|\xi_1\|^2 - 2\|\pi(x)\|\|\xi_1\|\|\xi_2\| + \|\xi_2\|^2 \geq 0 \end{aligned}$$

for all $\xi_1, \xi_2 \in H$. Conversely, if $\|x\| > 1$, then with ξ_2 a unit vector and $\xi_1 = -\pi(x)\xi_2/\|\pi(x)\xi_2\|$, the inner product becomes negative.

Let now x be an element in V with $\|x\| \leq 1$. From the first implication, we obtain

$$\begin{pmatrix} 1 & \varphi(x) \\ \varphi(x)^* & 1 \end{pmatrix} \geq 0,$$

which implies $\|\varphi(x)\| \leq 1$ by the reverse implication. $\square$

We note that if $\varphi : V \to B$ is completely positive, then φ is completely contractive. Indeed, since $\varphi_{2n} = (\varphi_n)_2$ is positive, φ_n is contractive by the previous proposition.

Let us now switch our attention to completely contractive maps on unital operator spaces. We shall need the following standard result concerning positive linear functionals.

Proposition 2.4. *Let A be a unital operator space, and let f be a linear functional on A. Then f is positive if $\|f\| = f(1)$.*

Proof. Suppose first that f is unital so that $\|f\| = 1$. We claim that, for a positive element $a \in A$, we have $f(a) \in [0, \|a\|]$. If this were not the case, then there would exist a λ_0 and $r \geq 0$ such that $|f(a) - \lambda_0| > r$ while $\sigma(a) \subset \{\lambda : |\lambda - \lambda_0| \leq r\}$. But this would imply $\|a - \lambda_0\| \leq r$ and $|f(a - \lambda_0)| > \|a - \lambda_0\|$, which is a contradiction.

The general case follows by considering the unital functional given by $g(a) = f(a)/\|f\|$. □

Proposition 2.5. *Let A be a unital operator space. If $\varphi : A \to \mathcal{B}(H)$ is a unital contraction, then φ is positive.*

Proof. Let ξ be a unit vector in H and consider the linear functional f on A given by $f(x) = \langle \varphi(x)\xi, \xi \rangle$. Since ξ is arbitrary, it follows that φ is positive if f is positive. But this follows from the previous proposition since f is unital and $\|f\| = 1$. □

Proposition 2.6 ([Pau03, Proposition 3.4, 3.5])**.** *Let A be a unital operator space, and let B be a C^*-algebra. If $\varphi : A \to B$ is a unital contractive map, then φ has a unique positive extension $\widetilde{\varphi} : A + A^* \to B$ which is given by*

$$\widetilde{\varphi}(x + y^*) = \varphi(x) + \varphi(y)^*.$$

If φ is completely contractive, then $\widetilde{\varphi}$ is completely positive and completely contractive.

Proof. If a positive extension $\widetilde{\varphi}$ of φ exists, then it necessarily satisfies the above equation by Proposition 2.1, so let us define $\widetilde{\varphi}$ by the formula above. In order to show that $\widetilde{\varphi}$ is well-defined, we must show that if both x and x^* belong to A, then $\varphi(x^*) = \varphi(x)^*$. But this follows readily from the fact that φ is contractive on the operator system $S = \{x \in A : x^* \in A\}$ and hence positive by Proposition 2.5.

Finally we show that $\widetilde{\varphi}$ is positive. Let us assume that $B = \mathcal{B}(H)$, and let ξ be a unit vector in H and consider the linear functional $A \to \mathcal{B}(H)$ given by $x \mapsto \langle \varphi(x)\xi, \xi \rangle$. By the Hahn-Banach theorem, this map can be extended to a linear functional f on $A + A^*$ with $\|f\| = 1$. By Proposition 2.4, f is positive, and hence $f(x + y^*) = f(x) + \overline{f(y)} = \langle \widetilde{\varphi}(x + y^*)\xi, \xi \rangle$. Since ξ was arbitrary, $\widetilde{\varphi}$ is positive.

In order to prove the final statement, we observe that $M_n(A + A^*) = M_n(A) + M_n(A)^*$ and that $\widetilde{\varphi_n} = (\widetilde{\varphi})_n$. If φ is completely contractive, then $(\widetilde{\varphi})_n$ is positive. Since $(\widetilde{\varphi})_{2n} = ((\widetilde{\varphi})_n)_2$ is also positive, $(\widetilde{\varphi})_n$ is contractive by the previous proposition. □

We note that if φ is a unital complete isometry, then so is $\widetilde{\varphi}$.

So far we have glossed over the claim that the theory of operator spaces and operator systems truly constitutes a generalization of the theory of normed spaces. This basically amounts to the statement that the word completely, which we attach to certain linear maps, brings nothing new in the commutative case. For a full discussion of this issue, we refer to [Pau03]. However, the following result will be of use in Chapter 4.

Theorem 2.7 ([Pau03, Theorem 3.11])**.** *Let B be a commutative C^*-algebra, and let $\varphi : B \to C$ be a positive map into a C^*-algebra C. Then φ is completely positive.*

Proof. Without loss of generality we may assume that $B = C(X)$. By the uniqueness of C^*-norms, we can define the norm on $M_n(C(X))$ directly by $\|F\| = \sup_{x\in X} \|F(x)\|$. We note that every element $F = (f_{ij})$ in $M_n(C(X))$ is a continuous matrix-valued function defined on X.

Let now F be positive in $M_n(C(X))$ and fix $\varepsilon > 0$. Let $\{P_k\}_{k=1}^m$ be a set of positive scalar matrices, and let $\{U_k\}_{k=1}^m$ be a finite open cover of X such that $\|F(x) - P_k\| < \varepsilon$ for all $x \in U_k$, $1 \le i \le m$. Let $\{u_k\}_{k=1}^m$, $u_k \in C(X)$ be a partition of unity subordinate to this cover, i.e., $u_k(x) \ge 0$, $x \in U_k$, $u_k(x) = 0$, $x \notin U_k$, and

$$\sum_{k=1}^{m} u_k(x) = 1$$

for all $x \in X$.

Observe that we have $\varphi_n(u_kP_k) = \varphi_n((u_kp_{ij}^k)) = (\varphi(u_k)p_{ij}^k)$, which is easily seen to be positive. Now clearly

$$\left\| F - \sum_{k=1}^{m} u_kP_k \right\| < \varepsilon,$$

and since φ_n is bounded by Proposition 2.2, $\varphi_n(F)$ can be approximated arbitrarily well by a sum of positive elements. Since the set of positive elements in a C^*-algebra is closed, this shows that $\varphi_n(F)$ is positive. □

2.2 Multiplicative domains of completely positive maps

In addition to the fact that completely positive maps preserve the order structure on the associated matrix spaces, the motivation for completely positive maps as the morphisms of operator systems can be further justified by their properties in the case when the domain and codomain are C^*-algebras. The aim of this section is to show that a unital completely positive map

φ between C^*-algebras restricts to a $*$-homomorphism on its multiplicative domain, which can roughly be described as the largest subalgebra where φ is multiplicative. As a consequence we obtain a simple proof of the fact that a unital complete order isomorphism of C^*-algebras is a $*$-isomorphism.

The following theory relies to a large extent on the following generalized Schwarz inequality for unital 2-positive maps.

Proposition 2.8 (Schwarz inequality for unital 2-positive maps [Cho74])**.** *Let $\varphi : V \to B$ be a unital 2-positive map from an operator system V into a C^*-algebra B. Then $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$ for all $a \in V$.*

Proof. Without loss of generality we assume that $B = \mathcal{B}(H)$. Since

$$\begin{pmatrix} 1 & a \\ a^* & a^*a \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \geq 0,$$

we have

$$\begin{pmatrix} 1 & \varphi(a) \\ \varphi(a)^* & \varphi(a^*a) \end{pmatrix} \geq 0,$$

and hence

$$0 \leq \left\langle \begin{pmatrix} 1 & \varphi(a) \\ \varphi(a)^* & \varphi(a^*a) \end{pmatrix} \begin{pmatrix} -\varphi(a)\xi \\ \xi \end{pmatrix}, \begin{pmatrix} -\varphi(a)\xi \\ \xi \end{pmatrix} \right\rangle = \langle (\varphi(a^*a) - \varphi(a)^*\varphi(a))\xi, \xi \rangle$$

for all $\xi \in H$. Consequently $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$ for all $a \in V$. □

The original statement of the following theorem due to Choi [Cho74] relates the multiplicative domain of a 2-positive map to the subset where equality holds in the Schwarz inequality. We prefer to give a somewhat simplified version of the theorem for completely positive maps.

Theorem 2.9 ([Pau03, Theorem 3.18 (iii)])**.** *Let B and C be C^*-algebras, and let $\varphi : B \to C$ be a unital completely positive map. Then the multiplicative domain of φ,*

$$\{x \in B : \varphi(xy) = \varphi(x)\varphi(y) \text{ and } \varphi(yx) = \varphi(y)\varphi(x) \textit{ for all } y \in B\},$$

is equal to the set

$$\{x \in B : \varphi(x^*x) = \varphi(x)^*\varphi(x) \text{ and } \varphi(xx^*) = \varphi(x)\varphi(x)^*\}.$$

Consequently the multiplicative domain of φ is a C^*-subalgebra of B, and φ is a $*$-homomorphism when restricted to this set.

Proof. Clearly the multiplicative domain is contained in the other set. Conversely, suppose $\varphi(x^*x) = \varphi(x)^*\varphi(x)$ and apply the Schwarz inequality to the map φ_2 and the matrix

$$\begin{pmatrix} x & y^* \\ 0 & 0 \end{pmatrix},$$

for an element $y \in B$. This gives

$$\begin{pmatrix} \varphi(x) & \varphi(y)^* \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} \varphi(x) & \varphi(y)^* \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} \varphi(x^*x) & \varphi(x^*y^*) \\ \varphi(yx) & \varphi(yy^*) \end{pmatrix},$$

and hence

$$\begin{pmatrix} \varphi(x^*x) - \varphi(x)^*\varphi(x) & \varphi(x^*y^*) - \varphi(x)^*\varphi(y)^* \\ \varphi(yx) - \varphi(y)\varphi(x) & \varphi(yy^*) - \varphi(y)\varphi(y)^* \end{pmatrix} \geq 0.$$

Since $\varphi(x^*x) - \varphi(x)^*\varphi(x) = 0$, it follows that $\varphi(yx) = \varphi(y)\varphi(x)$. Similarly $\varphi(xy) = \varphi(x)\varphi(y)$, showing that the two sets are equal. The remaining statements follow readily from this. □

Lemma 2.10. *Let B and C be C^*-algebras, and let $\varphi : B \to C$ and $\psi : C \to B$ be unital completely positive maps such that $\varphi \circ \psi = \mathrm{id}_C$. Then $\varphi(\psi(y)^*\psi(y)) = y^*y$ and $\varphi(\psi(y)\psi(y)^*) = yy^*$ for all $y \in C$.*

Proof. By the Schwarz inequality, we have $\psi(y^*y) \geq \psi(y)^*\psi(y)$ for all $y \in C$. Applying φ to this inequality yields

$$y^*y \geq \varphi(\psi(y)^*\psi(y)) \geq \varphi(\psi(y^*))\varphi(\psi(y)) = y^*y,$$

and hence $\varphi(\psi(y)^*\psi(y)) = y^*y$. Similarly $\varphi(\psi(y)\psi(y)^*) = yy^*$. □

Theorem 2.11. *Let B and C be C^*-algebras, and let $\varphi : B \to C$ be a unital complete order isomorphism. Then φ is a $*$-isomorphism.*

Proof. Applying the previous lemma to φ and φ^{-1} yields $\varphi(x^*x) = \varphi(x)^*\varphi(x)$ and $\varphi(xx^*) = \varphi(x)\varphi(x)^*$ for all $x \in B$. By Theorem 2.9, φ is a $*$-isomorphism. □

2.3 The BW-topology

In this section we shall consider some topological aspects of the spaces of completely positive and completely bounded maps into the Banach space $\mathcal{B}(H)$. To be precise we make the following definitions. Let A be an operator space, let V be an operator system, and define

$$\mathcal{B}_r(A, \mathcal{B}(H)) = \{\varphi \in \mathcal{B}(A, \mathcal{B}(H)) : \|\varphi\| \leq r\}$$

$$\mathcal{CB}_r(A, \mathcal{B}(H)) = \{\varphi \in \mathcal{B}(A, \mathcal{B}(H)) : \|\varphi\|_{\text{cb}} \leq r\}$$

$$\mathcal{CP}_r(V, \mathcal{B}(H)) = \{\varphi \in \mathcal{B}(V, \mathcal{B}(H)) : \varphi \text{ is completely positive with } \|\varphi\| \leq r\}.$$

Our goal of this section is to show that these spaces can be equipped with a weak* topology, which will provide us with important compactness arguments for these spaces in subsequent sections. In this case, this topology is known as the *bounded weak topology*, commonly referred to as the *BW-topology*. In order to equip these spaces with the BW-topology, we shall begin with the following quite general result, from which it follows that all spaces of the form $\mathcal{B}(X, Y^*)$, where X and Y are normed spaces, can be equipped with the BW-topology.

Proposition 2.12 ([Pau03, Lemma 7.1])**.** *Let X and Y be normed spaces. Then there exists a Banach space Z such that $\mathcal{B}(X, Y^*)$ is isometrically isomorphic to Z^*.*

Proof. Consider the algebraic tensor product $X \otimes Y$. We define a norm on this space as the operator norm induced by the dual pairing

$$\langle T, x \otimes y \rangle = T(x)(y),$$

$T \in \mathcal{B}(X, Y^*)$. We let Z denote the completion of $X \otimes Y$ with respect to this norm. It is clear that this dual pairing induces an isometric linear map from $\mathcal{B}(X, Y^*)$ into Z^*. To see that it is surjective, we let f be a linear functional in Z^*. For each $x \in X$, we define a linear map $f_x : Y \to \mathbb{C}$ by $f_x(y) = f(x \otimes y)$. Since $|f_x(y)| \leq \|f\|\|x\|\|y\|$, it follows that $f_x \in Y^*$. Define $T_f : X \to Y^*$ by $T_f(x) = f_x$. Clearly T_f is linear and bounded with $\|T_f\| \leq \|f\|$. Since

$$f(x \otimes y) = f_x(y) = T_f(x)(y) = \langle T_f, x \otimes y \rangle$$

for all $x \otimes y \in Z$, it follows that f is the image of the operator T_f. □

The fact that $\mathcal{B}(X, Y^*)$ is isometrically isomorphic to the dual of a Banach space allows us to equip $\mathcal{B}(X, Y^*)$ with the weak* topology, which we shall refer to as the BW-topology.

Proposition 2.13 ([Pau03, Lemma 7.2])**.** *Let $\{\varphi_\lambda\}$ be a bounded net in $\mathcal{B}(X, Y^*)$. Then $\varphi_\lambda \to \varphi$ in the BW-topology if and only if $\varphi_\lambda(x) \xrightarrow{w^*} \varphi(x)$ for all $x \in X$.*

Proof. Suppose $\varphi_\lambda \to \varphi$ in the BW-topology. Then

$$\varphi_\lambda(x)(y) = \langle \varphi_\lambda, x \otimes y \rangle \to \langle \varphi, x \otimes y \rangle = \varphi(x)(y)$$

for all $y \in Y$, and hence $\varphi_\lambda(x) \xrightarrow{w^*} \varphi(x)$ for all $x \in X$.

Conversely, if $\varphi_\lambda(x) \xrightarrow{w^*} \varphi(x)$ for all $x \in X$, then $\lim_\lambda \langle \varphi_\lambda, x \otimes y \rangle = \langle \varphi, x \otimes y \rangle$ for all $x \otimes y \in Z$ and consequently also for linear combinations of elementary tensors in Z. Since the net is bounded, it follows that $\lim_\lambda \langle \varphi_\lambda, z \rangle = \langle \varphi, z \rangle$ for all $z \in Z$. $\square$

If H is a Hilbert space, then it is isometrically isomorphic to the dual of some Hilbert space by the Riesz-Frechét representation theorem. Thus $\mathcal{B}(H) = \mathcal{B}(H, H)$ is isometrically isomorphic to the dual of some normed space by the previous lemma. Consequently, if X is a normed space, $\mathcal{B}(X, \mathcal{B}(H))$ can be equipped with the BW-topology. In order to obtain some useful criteria for convergence in the BW-topology, however, we shall instead realize $\mathcal{B}(H)$ as the dual of the *trace class* operators $\mathcal{S}_1(H)$, i.e., the 1st *Schatten class*. Let us recall the definition and some of its properties.

An operator $T \in \mathcal{B}(H)$ is said to be of *trace class*, $T \in \mathcal{S}_1(H)$, if there exists an orthonormal basis $\{e_\alpha\}$ for H such that

$$\sum_\alpha \langle (T^*T)^{1/2} e_\alpha, e_\alpha \rangle < \infty.$$

It can be shown that for $T \in \mathcal{S}_1(H)$, this sum is finite for any orthonormal basis.

Analogous to the finite-dimensional case, we define the trace of $T \in \mathcal{S}_1(H)$ as

$$\operatorname{tr} T = \sum_\alpha \langle T e_\alpha, e_\alpha \rangle,$$

where $\{e_\alpha\}$ is an orthonormal basis for H. It can be shown that this sum is finite and independent of the choice of orthonormal basis. From this we can construct the following norm on $\mathcal{S}_1(H)$:

$$\|T\|_1 = \operatorname{tr}(T^*T)^{1/2} = \sum_\alpha \langle (T^*T)^{1/2} e_\alpha, e_\alpha \rangle.$$

It can be shown that if $A \in \mathcal{B}(H)$ and $T \in \mathcal{S}_1(H)$, then $\operatorname{tr}(AT) = \operatorname{tr}(TA)$ and $|\operatorname{tr}(AT)| \leq \|A\| \|T\|_1$.

Theorem 2.14. *$(\mathcal{S}_1(H), \|\cdot\|_1)$ is a Banach space containing the finite-rank operators as a dense subspace. For $A \in \mathcal{B}(H)$, the map $A \mapsto \operatorname{tr}(A\,\cdot\,)$ defines an isometric isomorphism $\mathcal{B}(H) \to \mathcal{S}_1(H)^*$.*

For two vectors $\xi, \zeta \in H$, we define the operator $R_{\xi,\zeta}$ by $R_{\xi,\zeta}(\eta) = \langle \eta, \zeta \rangle \xi$. It readily follows that the linear span of these operators is precisely the finite-rank operators on H. We note that if A is an operator in $\mathcal{B}(H)$, then $\operatorname{tr}(AR_{\xi,\zeta}) = \langle A\xi, \zeta \rangle$.

Proposition 2.15 ([Pau03, Lemma 7.3]). *Let X be a normed space, let H be a Hilbert space, and let $\{\varphi_\lambda\}$ be a bounded net in $\mathcal{B}(X, \mathcal{B}(H))$. Then $\varphi_\lambda \to \varphi \in \mathcal{B}(X, \mathcal{B}(H))$ in the BW-topology if and only if $\lim_\lambda \langle \varphi_\lambda(x)\xi, \zeta\rangle = \langle \varphi(x)\xi, \zeta\rangle$ for all $x \in X$ and all $\xi, \zeta \in H$.*

Proof. By the previous theorem and Proposition 2.13, $\varphi_\lambda \to \varphi$ in the BW-topology if and only if $\lim_\lambda \operatorname{tr}(\varphi_\lambda(x)T) = \operatorname{tr}(\varphi(x)T)$ for all $T \in \mathcal{S}_1(H)$ and $x \in X$. By the above remark, the forward implication follows.

For the converse statement we observe that $\lim_\lambda \operatorname{tr}(\varphi_\lambda(x)R) = \operatorname{tr}(\varphi(x)R)$ for all finite-rank operators R by the above remark. Again since the net is bounded, it follows that the convergence holds for all $T \in \mathcal{S}_1(H)$. □

Theorem 2.16 ([Pau03, Theorem 7.4]). *Let A be a operator space and V an operator system. Then $\mathcal{CB}_r(A, \mathcal{B}(H))$, $\mathcal{CB}_r(A, \mathcal{B}(H))$, and $\mathcal{CP}_r(V, \mathcal{B}(H))$ are compact in the BW-topology.*

Proof. Since the BW-topology is a weak* topology, $\mathcal{B}_r(A, \mathcal{B}(H))$ is compact by the Banach-Alaoglu theorem. Since the $\mathcal{CB}_r(A, \mathcal{B}(H))$ and $\mathcal{CP}_r(V, \mathcal{B}(H))$ are subspaces of $\mathcal{B}_r(A, \mathcal{B}(H))$, we must show that they are closed in the BW-topology.

Let $\{\varphi_\lambda\}$ be a net in $\mathcal{CB}_r(A, \mathcal{B}(H))$ with $\lim_\lambda \varphi_\lambda = \varphi$. Suppose (a_{ij}) is a matrix in $M_n(A)$ and ξ and ζ are vectors in H^n. Then by the above proposition,

$$|\langle (\varphi(a_{ij}))\xi, \zeta\rangle| = \lim_\lambda |\langle (\varphi_\lambda(a_{ij}))\xi, \zeta\rangle| \le \liminf_\lambda \|(\varphi_\lambda(a_{ij}))\|\|\xi\|\|\zeta\| \le r\|\xi\|\|\zeta\|,$$

showing that $\|(\varphi_\lambda(a_{ij}))\| \le r\|(a_{ij})\|$. Thus $\|\varphi\|_{\mathrm{cb}} \le r$.

If $\{\varphi_\lambda\}$ is a net in $\mathcal{CP}_r(V, \mathcal{B}(H))$ with $\lim_\lambda \varphi_\lambda = \varphi$, then $\|\varphi\| \le r$ as in the previous step, and

$$\langle (\varphi(a_{ij}))\xi, \xi\rangle = \lim_\lambda \langle (\varphi_\lambda(a_{ij}))\xi, \xi\rangle \ge 0,$$

showing that φ is completely positive. □

2.4 Arveson's extension theorem

In this section we investigate the extension theorems for completely positive and completely contractive maps, which have become known as the non-commutative analogs of the Hahn-Banach theorem. The main result of this section will be Arveson's extension theorem, which asserts the existence of extensions of completely positive maps on operator systems into $\mathcal{B}(H)$ and unital completely contractive maps on unital operator spaces into $\mathcal{B}(H)$.

The original proof of Arveson's extension theorem is found in Arveson's article [Arv69]. We present a simplified proof by the same author which is found in [Arv]. In the same vein as Proposition 2.5 and 2.6, a key technique of the proof will be to deduce properties of linear functionals associated to positive linear maps, which are reflected back onto the corresponding map in question. In particular we shall apply a weak version of Krein's theorem, which concerns extensions of positive linear functionals on operator systems. In order to prove it, we begin with a generalization of Proposition 2.4 in the setting of operator systems.

Proposition 2.17. *Let V be an operator system, and let f be a linear functional on V. Then f is positive if and only if $\|f\| = f(1)$.*

Proof. Suppose that f is positive. Given $a \in V$, it is easy to see that there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $|f(a)| = f(\lambda a)$. Since f is positive and therefore also self-adjoint,

$$|f(a)| = f(\lambda a) = \operatorname{Re} f(\lambda a) = f(\operatorname{Re} \lambda a) \leq f(\|\lambda a\| 1) = \|a\| f(1),$$

and hence $\|f\| \leq f(1)$. Since it is clear that $\|f\| \geq f(1)$, we have that $\|f\| = f(1)$.

The converse statement is precisely Proposition 2.4. $\square$

From this proposition we get the following weak version of Krein's theorem.

Corollary 2.18. *Let B be a C^*-algebra, let $V \subset B$ be an operator system, and let f be a positive linear functional on V. Then f can be extended to a positive linear functional on B.*

Proof. By the previous proposition, we have $\|f\| = f(1)$. By the Hahn-Banach theorem, f has a norm-preserving extension to B, which is also positive by the previous proposition. $\square$

Theorem 2.19 (Arveson's extension theorem)**.** *Let B be a C^*-algebra, let $V \subset B$ be an operator system, and let $\varphi : V \to \mathcal{B}(H)$ be a completely positive map. Then there exists a completely positive map $\psi : B \to \mathcal{B}(H)$ that extends φ.*

Proof. Assume first that H is finite-dimensional. Our aim is to show that there is a Hilbert space K, a representation $\pi : B \to \mathcal{B}(K)$, and an operator $T : H \to K$ such that $\varphi(a) = T^*\pi(a)T$ for all $a \in V$. If we define ψ by $\psi(x) = T^*\pi(x)T$, $x \in B$, it is clear that ψ is completely positive, and hence ψ is the desired extension of φ.

To this end, let $\xi_1, \ldots, \xi_n$ be a basis for H, set $\xi = (\xi_1, \ldots, \xi_n) \in H^n$, and let f be the linear functional defined on $M_n(V)$ by $f((a_{ij})) = \langle \varphi_n((a_{ij}))\xi, \xi\rangle$. Clearly f is positive, and hence there exists a positive linear functional g on $M_n(B)$ that extends f by Corollary 2.18. By the GNS construction applied to g, we obtain a representation $\widetilde{\pi}$ on a Hilbert space $\widetilde{K}$ and a vector $\widetilde{\eta} \in \widetilde{K}$ such that $g((x_{ij})) = \langle \widetilde{\pi}((x_{ij}))\widetilde{\eta}, \widetilde{\eta}\rangle$. Since $\widetilde{\pi}$ is defined as the completion of the left regular representation of $M_n(B)$ on a quotient of $M_n(B)$, it readily follows that there exists a Hilbert space K such that $\widetilde{K} = K^n$ and a representation $\pi : B \to \mathcal{B}(K)$ such that $g((x_{ij})) = \langle (\pi(x_{ij}))\eta, \eta\rangle$, where $\eta = (\eta_1, \ldots, \eta_n)$ is a column vector consisting of the rows of $\widetilde{\eta}$.

If we define $T : H \to K$ by $T\xi_k = \eta_k$, $1 \le k \le n$, and apply g to a matrix $(a_{ij}) \in M_n(V)$, we get

$$\sum_{i,j=1}^{n} \langle \varphi(a_{ij})\xi_j, \xi_i\rangle = g((a_{ij})) = \sum_{i,j=1}^{n} \langle \pi(a_{ij})\eta_j, \eta_i\rangle = \sum_{i,j=1}^{n} \langle T^*\pi(a_{ij})T\xi_j, \xi_i\rangle.$$

Since this holds for all choices of matrices, it readily follows that

$$\langle \varphi(a)\xi_i, \xi_j\rangle = \langle T^*\pi(a)T\xi_i, \xi_j\rangle$$

for all $a \in V$, and consequently $\varphi(a) = T^*\pi(a)T$.

In order to prove the theorem for the general case, we let L be a finite-dimensional subspace of H, and let $\varphi_L : V \to \mathcal{B}(L)$ be the *compression* of φ to L, i.e., $\varphi_L(a) = P_L\varphi(a)|_L$. By the previous discussion, there exists a completely positive map $\psi_L : B \to \mathcal{B}(L)$ that extends φ_L. This map is in turn extended to a completely positive map $\widetilde{\psi}_L : B \to \mathcal{B}(H)$ by defining $\widetilde{\psi}_L(a) = \psi_L(a)$ on L and $\widetilde{\psi}_L(a) = 0$ on $L^\perp$.

Since the set Γ of finite-dimensional subspaces of H is a directed set under inclusion, $\{\widetilde{\psi}_L\}_{L\in\Gamma}$ is a net in $\mathcal{CP}_r(B, \mathcal{B}(H))$ where $r = 2\|\varphi(1)\|$. Since this set is compact in the BW-topology, there exists a subnet that converges to some map $\psi \in \mathcal{CP}_r(B, \mathcal{B}(H))$. We claim that ψ is the desired extension of φ. Let L be the subspace of H spanned by vectors ξ and ζ. Then for any subspace $F \subset H$ containing L, we have $\langle \varphi(a)\xi, \zeta\rangle = \langle \widetilde{\psi}_F(a)\xi, \zeta\rangle$. Since the set of subspaces containing L is cofinal, we get $\langle \varphi(a)\xi, \zeta\rangle = \langle \psi(a)\xi, \zeta\rangle$. This completes the proof of the theorem. $\square$

2.5 Dilation theory

We end this chapter with a brief introduction to the topic of dilations, which will provide us with an important technique when describing the Shilov boundary for some concrete examples in Chapter 4 and 5.

We define the *dilation* of an operator T on a Hilbert space H as an operator V on a Hilbert space K containing H as a subspace such that

$$T^n = P_H V^n|_H$$

for all $n \geq 0$.

The utility of dilations for our purposes stems from the fact that a dilation of an operator can in some sense be viewed as a realization of it as part of a simpler operator on a larger space. Let us state and prove the main result of this section, which will provide us with a unitary dilation of a contractive operator.

Theorem 2.20 (Sz.-Nagy's dilation theorem). *If $T \in \mathcal{B}(H)$ is a contraction, then there exists a Hilbert space K containing H as a subspace and a unitary operator $U \in \mathcal{B}(H)$ such that*

$$T^n = P_H U^n|_H$$

for all $n \geq 0$.

We follow the proof found in [Pau03], which relies on two intermediate dilations, namely the *isometric dilation of a contraction* and the *unitary dilation of an isometry.*

We define the isometric dilation of a contraction as follows. Let $T \in \mathcal{B}(H)$ be a contraction. Then we can define $D = \sqrt{I - T^*T}$, so that

$$\|T\xi\|^2 + \|D\xi\|^2 = \langle T^*T\xi, \xi\rangle + \langle D^2\xi, \xi\rangle = \|\xi\|^2.$$

Define $V \in \mathcal{B}(\ell^2(H))$ by $V((\xi_1, \xi_2, \dots)) = (T\xi_1, D\xi_1, \xi_2, \dots)$. Since

$$\|V((\xi_1, \xi_2, \dots))\|^2 = \|T\xi_1\|^2 + \|D\xi_1\|^2 + \|\xi_2\|^2 + \cdots = \|(\xi_1, \xi_2, \dots)\|^2,$$

it follows that V is an isometry. Identifying H with $H \oplus 0 \oplus \dots$ gives $T^n = P_H V^n|_H$ for all $n \geq 0$.

Let us now define the unitary dilation of an isometry. Let $V \in \mathcal{B}(H)$ be an isometry, and let $P = I - VV^*$ be the projection onto the orthogonal complement of the range of V. Set $K = H \oplus H$ and define $U \in \mathcal{B}(K)$ as

$$U = \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix}.$$

It is easy to see that $U^*U = UU^* = I$, i.e., U is a unitary operator. Identifying H with $H \oplus 0$, we have $V^n = P_H U^n|_H$ for all $n \geq 0$.

Proof of Theorem 2.20. Set $K = \ell^2(H) \oplus \ell^2(H)$, where we identify H with $(H \oplus 0 \oplus \dots) \oplus 0$. Let V be the isometric dilation of T on $\ell^2(H)$, and let U be the unitary dilation of V on $\ell^2(H) \oplus \ell^2(H)$. Since $H \subset \ell^2(H) \oplus 0$, we have $T^n = P_H V^n|_H = P_H U^n|_H$ for all $n \geq 0$. $\square$

This direct construction of a unitary dilation of a contraction could have been included more or less anywhere in this thesis. However, its existence can also be attributed to a deeper structural theorem for completely positive maps.

Theorem 2.21 (Stinespring's dilation theorem [Pau03, Theorem 4.1])**.** *Let B be a C^*-algebra, and let $\varphi : B \to \mathcal{B}(H)$ be a completely positive map. Then there exists a Hilbert space K, a $*$-homomorphism $\pi : B \to \mathcal{B}(K)$, and an operator $V : H \to K$ with $\|\varphi(1)\| = \|V\|^2$ such that*

$$\varphi(x) = V^*\pi(x)V$$

for all $x \in B$.

Since any map of the form $\varphi(x) = V^*\pi(x)V$ is completely positive, Stinespring's dilation theorem can be seen as a structure theorem that classifies the completely positive maps on C^*-algebras into $\mathcal{B}(H)$.

Let us now give a sketch of how Sz.-Nagy's dilation theorem follows from Stinespring's dilation theorem. We note that if φ is unital, then V is an isometry, and hence we may identify H with the subspace $VH \subset K$, so that V^* becomes the projection of K onto H. With this identification, we see that

$$\varphi(x) = P_H\pi(x)|_H.$$

One can show that, for polynomials p and q, the map $\varphi(p+\bar{q}) = p(T)+q(T)^*$ can be extended to a completely positive map of $C(\mathbb{T})$ into $\mathcal{B}(H)$ [Pau03, Theorem 2.6]. Let now K, π, and V be the Hilbert space, $*$-homomorphism, and operator obtained from Stinespring's dilation theorem applied to this map. Since φ is unital, we may identify H with $VH \subset K$ as above, and hence since $\pi(z) = U$ is unitary, we get

$$T^n = \varphi(z^n) = P_H\pi(z^n)|_H = P_H U^n|_H$$

for all $n \geq 0$.

3 The noncommutative Shilov boundary

3.1 Preliminaries and definition

In this chapter we introduce the noncommutative analog of the Shilov boundary as given by Arveson in [Arv69]. In order to motivate his definition, we shall begin by translating the definition of the Shilov boundary in the commutative case into terms of the C^*-algebra structure.

Let X be a compact Hausdorff space, let A be a uniform algebra of $C(X)$, and let $S \subset X$ be the Shilov boundary relative to A, i.e., the smallest subset such that any function in A achieves its maximum modulus on S. Let J_S be the associated closed ideal as in (1.1). For the quotient map $C(X) \to C(X)/J_S$, we have, for a function f in A,

$$\|f + J_S\| = \sup_{x \in S} |f(x)| = \sup_{x \in X} |f(x)| = \|f\|,$$

by the definition of the Shilov boundary. Thus the quotient map restricted to A is an isometry. A closed ideal with this property will be referred to as a *boundary ideal*. Conversely, if J is a boundary ideal, then by Proposition 1.4, J is of the form (1.1) for some closed subset $K \subset X$. Since J is a boundary ideal,

$$\sup_{x \in X} |f(x)| = \|f\| = \|f + J\| = \sup_{x \in K} |f(x)|.$$

Thus the notion that every function in A achieves its maximum modulus on some closed subset is equivalent to the fact that the corresponding ideal is a boundary ideal. The statement that the Shilov boundary is the smallest closed set is captured by the notion that the corresponding boundary ideal is maximal, i.e., it contains all other boundary ideals.

Let us now give Arveson's definition of the noncommutative analog of the Shilov boundary.

Definition 3.1. Let A be a unital operator space of a C^*-algebra B such that A generates B as a C^*-algebra. An ideal J in B is called a *boundary ideal* for A if the quotient map $q : B \to B/J$ is a complete isometry when restricted to A. A boundary ideal is called the *Shilov boundary* for A if it contains every other boundary ideal.

We see that this definition agrees completely with our previous discussion since the requirement that the quotient map be a *complete isometry* introduces nothing new in the commutative case. Given the subtleties that arise when considering subspaces of noncommutative C^*-algebras, this requirement is quite natural however. Note that, by the definition, if the Shilov boundary exists, then it is unique.

Much of the original motivation for the introduction of a noncommutative analog of the Shilov boundary was the question of to what extent an operator space of a C^*-algebra determine its structure. This question is probably best motivated by the example of the disk algebra. By the Stone-Weierstrass theorem, $A(\mathbb{D})$ generates $C(\bar{\mathbb{D}})$ as a C^*-algebra, i.e., $C(\bar{\mathbb{D}}) = C^*(A(\mathbb{D}))$. On the other hand, the discussion in the beginning of this chapter together with the maximum modulus principle shows that the restriction to the unit circle induces a $*$-homomorphism of $C(\bar{\mathbb{D}})$ onto $C(\mathbb{T})$ which is completely isometric when restricted to $A(\mathbb{D})$. Thus, even though $A(\mathbb{D})$ and $A(\mathbb{D})|_{\mathbb{T}}$ are isomorphic as operator spaces, they generate different C^*-algebras since $C^*(A(\mathbb{D})|_{\mathbb{T}}) = C(\mathbb{T})$.

More generally, let $A_1 \subset B_1$, and $A_2 \subset B_2$ be unital operator spaces such that A_1 and A_2 are unitally completely isometrically isomorphic. The above example shows that, in general, there is no relation between $C^*(A_1)$ and $C^*(A_2)$.

Although Arveson conjectured the existence of the Shilov boundary, he did not prove its existence in full generality as given by Definition 3.1. This question remained open until 1979, when Hamana proved the existence of the Shilov boundary by introducing the notion of the injective envelope and C^*-envelope of an operator system.

Inspired by Hamana's article [Ham79] and some improvements in [Pau03], we shall introduce the notion of the injective envelope of an operator system and prove its existence. Our goal will be to use the injective envelope of an operator system to show that a unital operator space A can be completely isometrically embedded into a minimal C^*-algebra $C^*_e(A+A^*)$ called the C^*-envelope of the operator system $A+A^*$, and that $C^*_e(A+A^*)$ is $*$-isomorphic to $C^*(A)/J$ where J is the Shilov boundary for A. Moreover, addressing the question above, we shall prove that a unital completely isometric isomorphism of A_1 onto A_2 is the restriction of a unique $*$-isomorphism of $C^*_e(A_1 + A_1^*)$ onto $C^*_e(A_2 + A_2^*)$, showing that the structure of a C^*-algebra is determined by a unital subspace if and only if its Shilov boundary is trivial.

3.2 The injective envelope of an operator system

In the most general setting, injectivity can be defined as follows. Let $\mathcal{C}$ be a category. An object I is said to be *injective* in $\mathcal{C}$ if for every pair $E \subset F$ of objects in $\mathcal{C}$ and every morphism $\varphi : E \to I$, there exists a morphism $\psi : F \to I$ that extends φ. As a first familiar example, we note that the Hahn-Banach theorem is equivalent to the statement that $\mathbb{C}$ is injective in the category of normed spaces.

In this section the category under consideration will be the category of operator systems with morphisms the unital completely positive maps or, equivalently, the unital completely contractive maps. In this case we see that Arveson's extension theorem is equivalent to the statement that $\mathcal{B}(H)$ is injective in the category of operator systems.

We will now introduce the notion of the injective envelope of an operator system and prove its existence.

Let V and W be operator systems, and let $\kappa : V \to W$ be a unital completely contractive map. We begin by introducing some terminology for the pair (W, κ). We say that (W, κ) is an *extension* of V if κ is a complete isometry. If in addition W is injective, then (W, κ) is said to be an *injective extension.* An extension (W, κ) of V is said to be a *rigid extension* if for any unital completely contractive map $\varphi : W \to W$ such that $\varphi \circ \kappa = \kappa$, then it holds that $\varphi = \mathrm{id}_W$.

There are several equivalent definitions for the injective envelope of an operator system. We shall take a definition in terms of the properties that will be most convenient for our purposes. We say that an extension (W, κ) is an *injective envelope* of V if it is both an injective and rigid extension of V.

Roughly speaking, the injective envelope of an operator system should correspond to the idea that the injective envelope should be a minimal injective object containing V. To capture this notion, we say that an injective extension (W, κ) is *minimal* if it holds that whenever $W_1 \subset W$ is injective with $\kappa(V) \subset W_1$, then $W_1 = W$. Let us show that this property follows from the definition of the injective envelope.

Proposition 3.2. *Let V be an operator system, and let (W, κ) be an injective envelope of V. Then (W, κ) is a minimal injective extension.*

Proof. Let $W_1 \subset W$ be an injective operator system with $\kappa(V) \subset W_1$. Since W_1 is injective, the inclusion map of $\kappa(V)$ into W_1 can be extended to a unital completely contractive map $\varphi : W \to W_1$. Since $W_1 \subset W$ and $\varphi \circ \kappa = \kappa$, the rigidity of (W, κ) implies that $\varphi = \mathrm{id}_W$, and hence $W_1 = W$. □

The main result of this section will be to show that any operator system has a unique injective envelope. Uniqueness here should be interpreted as unique up to unital completely isometric isomorphism, a notion whose meaning we make precise in the following statement.

Proposition 3.3. *Let V_1 and V_2 be operator systems, and let (W_1, κ_1) and (W_2, κ_2) be injective envelopes of V_1 and V_2 respectively. If $\iota : V_1 \to V_2$ is a unital completely isometric isomorphism, then there exists a unique unital completely isometric isomorphism $\lambda : W_1 \to W_2$ such that $\lambda \circ \kappa_1 = \kappa_2 \circ \iota$.*

Proof. Since W_2 is injective, the map $\kappa_2 \circ \iota \circ \kappa_1^{-1} : \kappa(V_1) \to W_2$ can be extended to a unital completely contractive map $\lambda : W_1 \to W_2$ with $\lambda \circ \kappa_1 = \kappa_2 \circ \iota$. Similarly, $\kappa_1 \circ \iota^{-1} \circ \kappa_2^{-1}$ can be extended to a unital completely contractive map $\mu : W_2 \to W_1$ with $\mu \circ \kappa_2 = \kappa_1 \circ \iota^{-1}$. Since

$$\mu \circ \lambda \circ \kappa_1 = \mu \circ \kappa_2 \circ \iota = \kappa_1,$$

it follows by rigidity that $\mu \circ \lambda = \mathrm{id}_{W_1}$. Similarly, $\lambda \circ \mu = \mathrm{id}_{W_2}$, showing that λ is a unital completely isometric isomorphism of W_1 onto W_2. Also from rigidity, it readily follows that λ is unique.

$$\begin{tikzcd} V_1 \arrow[r, "\kappa_1"] \arrow[d, "\iota"'] & W_1 \arrow[d, shift right, "\lambda"'] \\ V_2 \arrow[r, "\kappa_2"] & W_2 \arrow[u, shift right, "\mu"'] \end{tikzcd}$$

□

The proof of the existence of an injective envelope turns out to be a rather elaborate application of Zorn's lemma. We begin by introducing some more terminology.

Let $V \subset \mathcal{B}(H)$ be an operator system. A unital completely contractive map $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ is called a *V-map* if $\varphi|_V = \mathrm{id}_V$. If φ is also idempotent, i.e., it satisfies $\varphi \circ \varphi = \varphi$, then φ is called a *V-projection.*

For a V-map φ, we define a *V-seminorm* p_φ on $\mathcal{B}(H)$ by $p_\varphi(x) = \|\varphi(x)\|$. We define a partial order on the collection of all V-seminorms by setting $p \le q$ if $p(x) \le q(x)$ for all $x \in \mathcal{B}(H)$.

Let us show that the family of V-seminorms has a minimal element, which in turn will guarantee the existence of a V-projection.

Lemma 3.4 ([Pau03, Proposition 15.3])**.** *Let $V \subset \mathcal{B}(H)$ be an operator system. Then there exists a minimal V-seminorm on $\mathcal{B}(H)$.*

Proof. Let $\{\varphi_\lambda\}$ be V-maps such that $\{p_{\varphi_\lambda}\}$ is a decreasing chain of V-seminorms. Since $\mathcal{CB}_1(\mathcal{B}(H), \mathcal{B}(H))$ is compact in the BW-topology, $\{\varphi_\lambda\}$ has a convergent subnet $\{\varphi_{\lambda_\mu}\}$. Denoting its limit by φ, by Proposition 2.15, we have

$$|\langle \varphi(x)\xi, \zeta\rangle| = \lim_\mu |\langle \varphi_{\lambda_\mu}(x)\xi, \zeta\rangle| \le \liminf_\mu \|\varphi_{\lambda_\mu}(x)\|\|\xi\|\|\zeta\|,$$

and so $\|\varphi(x)\| \le \liminf_\mu \|\varphi_{\lambda_\mu}(x)\|$ for all $x \in \mathcal{B}(H)$. Thus $p_\varphi \le p_{\varphi_\lambda}$ for all λ, and hence every decreasing chain of V-seminorms has a lower bound. By Zorn's lemma, there exists a minimal V-seminorm. □

Lemma 3.5 ([Pau03, Theorem 15.4])**.** *Let $V \subset \mathcal{B}(H)$ be an operator system. If $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a V-map such that p_φ is a minimal V-seminorm, then φ is a V-projection.*

Proof. Since φ is completely contractive, we have $\|\varphi \circ \varphi(x)\| \le \|\varphi(x)\|$ for all $x \in \mathcal{B}(H)$. Since $\varphi \circ \varphi$ is also a V-map and since p_φ is a minimal V-seminorm, we must have $\|\varphi\circ\varphi(x)\| = \|\varphi(x)\|$. Inductively, $\|\varphi^k(x)\| = \|\varphi(x)\|$ for all $k \ge 1$, where $\varphi^k = \varphi \circ \cdots \circ \varphi$ k times. For each $n \ge 1$, define $\psi_n = (\varphi + \cdots + \varphi^n)/n$. Then each ψ_n is a V-map and $\|\psi_n(x)\| \le \|\varphi(x)\|$, and so $\|\psi_n(x)\| = \|\varphi(x)\|$. Combining these observations, we have

$$\begin{aligned}\|(\varphi - \varphi\circ\varphi)(x)\| &= \|\varphi(x - \varphi(x))\| = \|\psi_n(x - \varphi(x))\| \\ &= \left\| \frac{\varphi(x) + \cdots + \varphi^n(x)}{n} - \frac{\varphi^2(x) + \cdots + \varphi^{n+1}(x)}{n} \right\| \\ &\le \frac{2\|\varphi(x)\|}{n} \to 0\end{aligned}$$

as $n \to \infty$. Thus $\varphi \circ \varphi = \varphi$, showing that φ is a V-projection. $\square$

With the existence of a V-projection $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ such that p_φ is a minimal V-seminorm, let us state and prove the main theorem of this section.

Theorem 3.6. *Any operator system V has a unique injective envelope.*

Proof. Without loss of generality, we may assume $V \subset \mathcal{B}(H)$. Let $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a V-projection such that p_φ is a minimal V-seminorm. First we claim that $\varphi(\mathcal{B}(H)) \subset \mathcal{B}(H)$ is an injective operator system. Let $U_1 \subset U_2$ be operator systems, and let $\rho : U_1 \to \varphi(\mathcal{B}(H))$ be a unital completely contractive map. By Arveson's extension theorem, ρ extends to a unital completely contractive map $\psi : U_2 \to \mathcal{B}(H)$. By the previous lemma, φ is idempotent, and so $\varphi \circ \psi : U_2 \to \varphi(\mathcal{B}(H))$ is a unital completely contractive map with $\varphi \circ \psi|_{U_1} = \varphi \circ \rho = \rho$.

We proceed to show that $(\varphi(\mathcal{B}(H)), \mathrm{id}_V)$ is a rigid extension. To this end, suppose that $\gamma : \varphi(\mathcal{B}(H)) \to \varphi(\mathcal{B}(H))$ is a unital completely contractive map with $\gamma|_V = \mathrm{id}_V$. Note that $\gamma \circ \varphi$ is a V-map, and since p_φ is a minimal V-seminorm and $\|\gamma \circ \varphi(x)\| \le \|\varphi(x)\|$ for all $x \in \mathcal{B}(H)$, we have $\|\gamma \circ \varphi(x)\| = \|\varphi(x)\|$, i.e., $p_{\gamma\circ\varphi} = p_\varphi$. By the previous lemma, $\gamma \circ \varphi$ is idempotent, and hence

$$\gamma \circ \varphi = \gamma \circ \varphi \circ \gamma \circ \varphi = \gamma \circ \gamma \circ \varphi,$$

where the last equality follows since the range of γ is $\varphi(\mathcal{B}(H))$ and φ is also idempotent. Since γ is injective, the statement follows. $\square$

3.3 C^*-envelopes and the Shilov boundary

Now that we have established the existence and uniqueness of the injective envelope of an operator system, we turn our attention to the C^*-envelope, which will be the key to proving the existence of the Shilov boundary. We begin with the following general statement about injective operator systems, which we will use to prove a well known theorem due to Choi and Effros, stating that an injective operator system can be equipped with a C^*-algebra structure.

Proposition 3.7 ([Pau03, Theorem 15.1]). *Let $V \subset \mathcal{B}(H)$ be a operator system. Then V is injective if and only if there exists a completely contractive projection $\varphi : \mathcal{B}(H) \to V$ onto V.*

Proof. If V is injective, the identity map on V extends to a unital completely contractive map $\varphi : \mathcal{B}(H) \to V$. Since $\varphi|_V = \mathrm{id}_V$, we have that φ is a projection onto V.

Conversely, let $U_1 \subset U_2$ be operator systems, and let $\rho : U_1 \to V$ be a unital completely contractive map. By Arveson's extension theorem, there exists a unital completely contractive map $\psi : U_2 \to \mathcal{B}(H)$ that extends ρ. Since φ is a projection, $\varphi \circ \psi : U_2 \to V$ is a unital completely contractive extension of ρ, showing that V is injective. □

Theorem 3.8 ([CE77], [Pau03, Theorem 15.2]). *Let $V \subset \mathcal{B}(H)$ be an injective operator system. Then there exists a unique C^*-algebra $(V, \odot)$ such that the identity map of V into $(V, \odot)$ is a unital completely isometric isomorphism.*

Proof. Let $\varphi : \mathcal{B}(H) \to V$ be a completely contractive projection onto V, which exists by the previous proposition, and define $a \odot b = \varphi(ab)$. We begin by showing that $\odot$ defines a multiplication on V. Clearly $1 \odot a = \varphi(a) = a$ and vice versa. Distributivity is clear since $a \odot (b + c) = \varphi(a(b + c)) = \varphi(ab + bc) = a \odot b + b \odot c$ and vice versa. It remains to show that $\odot$ is associative, i.e., that $\varphi(a\varphi(bc)) = \varphi(\varphi(ab)c)$. We claim, for any $x \in \mathcal{B}(H)$ and $a \in V$, that $\varphi(\varphi(x)a) = \varphi(xa)$ and $\varphi(a\varphi(x)) = \varphi(ax)$. Assuming this, we have

$$\varphi(a\varphi(bc)) = \varphi(abc) = \varphi(\varphi(ab)c),$$

from which associativity follows.

To prove the claim, we apply the Schwarz inequality (Proposition 2.8) to the map φ_2 and the matrix

$$\begin{pmatrix} a^* & x \\ 0 & 0 \end{pmatrix}.$$

This gives

$$\begin{pmatrix} \varphi(aa^*) & \varphi(ax) \\ \varphi(x^*a^*) & \varphi(x^*x) \end{pmatrix} - \begin{pmatrix} aa^* & a\varphi(x) \\ \varphi(x)^*a^* & \varphi(x)^*\varphi(x) \end{pmatrix} \geq 0,$$

and by applying φ_2 once again to this inequality we get

$$\begin{pmatrix} 0 & \varphi(ax) - \varphi(a\varphi(x)) \\ \varphi(x^*a^*) - \varphi(\varphi(x)^*a^*) & \varphi(x^*x) - \varphi(\varphi(x)^*\varphi(x)) \end{pmatrix} \geq 0.$$

By the positivity, $\varphi(a\varphi(x)) = \varphi(ax)$, and since φ is self-adjoint, it readily follows that $\varphi(\varphi(x)a) = \varphi(xa)$ as well.

Next we verify the conditions for the norm. Clearly we have

$$\|a \odot b\| = \|\varphi(ab)\| \leq \|ab\| \leq \|a\|\|b\|.$$

For the C^*-identity, $\|a^* \odot a\| = \|a\|^2$, we observe that

$$\|a^* \odot a\| = \|\varphi(a^*a)\| \leq \|a^*a\| = \|a\|^2.$$

On the other hand, by the Schwarz inequality, $\varphi(a^*a) \geq \varphi(a)^*\varphi(a) = a^*a$, and hence

$$\|a^* \odot a\| = \|\varphi(a^*a)\| \geq \|a^*a\| = \|a\|^2.$$

Finally we show that V is complete with respect to the norm. Since V is a subspace of a C^*-algebra, it suffices to show that V is closed. Since V is injective, the identity map id_V can be extended to a completely positive map $\psi : \mathcal{B}(H) \to V$ with $\psi|_V = \mathrm{id}_V$. Let $\{a_i\} \subset V$ be a convergent sequence with limit $a \in \mathcal{B}(H)$. Since ψ is continuous, $a = \lim_{i\to\infty} a_i = \lim_{i\to\infty} \psi(a_i) = \psi(a)$, showing that a lies in V. Thus we have shown that $(V, \odot)$ is a C^*-algebra.

Let us now show that the inclusion of V into $(V, \odot)$ is a unital complete order isomorphism. Consider the operator system $M_n(V)$. Since φ is a completely positive projection, $\varphi_n : \mathcal{B}(H^n) \to M_n(V)$ is a completely positive projection as well. By Proposition 3.7, $M_n(V)$ is an injective operator system, and hence $M_n(V)$ is a C^*-algebra with product $(a_{ij}) \odot_n (b_{ij}) = \varphi_n((a_{ij})(b_{ij}))$. We have

$$(a_{ij}) \odot_n (b_{ij}) = \sum_{k=1}^{n} \varphi(a_{ik}b_{kj}) = \sum_{k=1}^{n} a_{ik} \odot b_{kj},$$

and hence the identity map induces a $*$-isomorphism of $(M_n(V), \odot_n)$ onto $M_n((V, \odot))$. Consequently the identity map of V into $(V, \odot)$ is a unital complete isometry. By Theorem 2.11, another C^*-algebra $(V, \odot_1)$ with this property would be $*$-isomorphic to $(V, \odot)$, and hence it directly follows that $a \odot_1 b = a \odot b$. □

The following theorem due to Hamana [Ham79] will allow us to define the C^*-envelope of an operator system, which will provide us with the key to proving the existence of the Shilov boundary. Let us first define the notion of a C^*-extension. Let V be an operator system, and let (B, κ) be an extension of V, where B is a C^*-algebra. If $B = C^*(\kappa(V))$, then (B, κ) is said to be a *C^*-extension* of V.

Theorem 3.9 ([Ham79], [Pau03, Theorem 15.16])**.** *Let V be an operator system, let (C, κ) be an injective envelope of V, identified with the C^*-algebra that it is unitally completely isometrically isomorphic to, and let (B, ρ) be a C^*-extension of V. Then there exists a unique surjective $*$-homomorphism $\pi : B \to C^*(\kappa(V))$ with $\pi \circ \rho = \kappa$.*

Proof. Without loss of generality we assume that $B \subset \mathcal{B}(H)$. Since C is injective, $\kappa \circ \rho^{-1} : \rho(V) \to C$ can be extended to a unital completely contractive map $\varphi : \mathcal{B}(H) \to C$ such that $\varphi \circ \rho = \kappa$. On the other hand, by Arveson's extension theorem, $\rho \circ \kappa^{-1}$ extends to a unital completely contractive map $\psi : C \to \mathcal{B}(H)$ with $\psi \circ \kappa = \rho$. This gives $\varphi \circ \psi \circ \kappa = \kappa$, and hence $\varphi \circ \psi = \mathrm{id}_C$ by rigidity.

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[dr, "\kappa"'] & \mathcal{B}(H) \arrow[d, shift right, "\varphi"'] \\ & C \arrow[u, shift right, "\psi"'] \end{tikzcd}$$

By Lemma 2.10, we have that $\varphi(\psi(y)^*\psi(y)) = y^*y$ and $\varphi(\psi(y)\psi(y)^*) = yy^*$ for all $y \in C$. So with $y = \kappa(a) \in C$, we have $\varphi(\psi(\kappa(a))^*\psi(\kappa(a))) = \kappa(a)^*\kappa(a)$, i.e., $\varphi(\rho(a)^*\rho(a)) = \varphi(\rho(a))^*\varphi(\rho(a))$ and similar for $\rho(a)\rho(a)^*$. Defining π as the restriction of φ to B, Theorem 2.9 implies that π a $*$-homomorphism. The fact that the domain and codomain for π are generated by $\rho(V)$ and $\kappa(V)$ respectively, together with $\pi \circ \rho = \kappa$ implies that π is unique and surjective. $\square$

A C^*-extension (D, κ) that satisfies the property that for any C^*-extension (B, ρ) there exists a unique surjective $*$-homomorphism $\pi : B \to D$ with $\pi \circ \rho = \kappa$ is said to be a *C^*-envelope* of V. From Theorem 3.9, we can prove that any operator system has a unique C^*-envelope.

Theorem 3.10. *Any operator system V has a C^*-envelope (κ, D). Moreover, let V_1 and V_2 be operator systems, and let (D_1, κ_1) and (D_2, κ_2) be C^*-envelopes of V_1 and V_2 respectively. If $\iota : V_1 \to V_2$ is a unital completely isometric isomorphism, then there exists a unique $*$-isomorphism $\pi : D_1 \to D_2$ such that $\pi \circ \kappa_1 = \kappa_2 \circ \iota$.*

Proof. The existence follows from the existence of an injective envelope together with Theorem 3.9: if (C, κ) is an injective envelope of V, identified with the C^*-algebra that it is unitally completely isometrically isomorphic to, then $(C^*(\kappa(V)), \kappa)$ is a C^*-envelope of V.

Let us now show that the C^*-envelope is unique in the above sense. It is clear that $(D_1, \kappa_1 \circ \iota^{-1})$ is a C^*-extension of V_2, and hence there exists a surjective $*$-homomorphism $\pi : D_1 \to D_2$ with $\pi \circ \kappa_1 \circ \iota^{-1} = \kappa_2$. Similarly, $(D_2, \kappa_2 \circ \iota)$ is a C^*-extension of V_1, so there exists a surjective $*$-homomorphism $\rho : D_2 \to D_1$ with $\rho \circ \kappa_2 \circ \iota = \kappa_1$. Then we have a surjective $*$-homomorphism $\rho \circ \pi$ such that

$$\rho \circ \pi \circ \kappa_1 = \rho \circ \kappa_2 \circ \iota = \kappa_1.$$

Since id_{D_1} also satisfies this property, the uniqueness implies that $\rho\circ\pi = \mathrm{id}_{D_1}$, and hence π is also injective.

$$\begin{tikzcd} V_1 \arrow[r, "\kappa_1"] \arrow[d, "\iota"'] & D_1 \arrow[d, shift right, "\pi"'] \\ V_2 \arrow[r, "\kappa_2"] & D_2 \arrow[u, shift right, "\rho"'] \end{tikzcd} \qquad \begin{tikzcd} V_1 \arrow[r, "\kappa_1"] \arrow[dr, "\kappa_1"'] & D_1 \arrow[d, "\rho\circ\pi"] \\ & D_1 \end{tikzcd}$$

This completes the proof. $\square$

Theorem 3.11 ([Ham79])**.** *Let A be a unital operator space of a C^*-algebra B such that A generates B as a C^*-algebra. Then the Shilov boundary for A exists.*

Proof. Since a unital completely isometric map on a unital operator space extends uniquely to a unital completely isometric map on $A + A^*$ as a consequence of Proposition 2.5, it readily follows that we may assume that A is an operator system.

Let (D, κ) be a C^*-envelope of A. Since $B = C^*(A)$, there exists a unique surjective $*$-homomorphism $\pi : B \to D$ with $\pi|_A = \kappa$. We claim that $J = \operatorname{Ker}\pi$ is the Shilov boundary for A. Let $q : B \to B/J$ denote the quotient map, and let $\dot{\pi} : B/J \to D$ be the $*$-isomorphism induced by π. Then $\dot{\pi} \circ q|_A = \kappa$, and hence $q|_A = \dot{\pi}^{-1} \circ \kappa$ is a complete isometry. Therefore J is a boundary ideal for A. Moreover, $(B/J, q|_A)$ is a C^*-envelope of A.

Let I be any boundary ideal for A, and let $r : B \to B/I$ be the corresponding quotient map. Since $r|_A$ is a complete isometry and $B/I = C^*(r(A))$, the fact that $(B/J, q|_A)$ is a C^*-envelope of A implies that there exists a surjective $*$-homomorphism $\varphi : B/I \to B/J$ with $\varphi(a + I) = a + J$

for all $a \in A$. Since B is generated by A and φ is a $*$-homomorphism, we in fact have $\varphi(b + I) = b + J$ for all $b \in B$. If x is an element in I, then

$$x + J = \varphi(x + I) = \varphi(0 + I) = 0 + J,$$

i.e., $x \in J$, showing that $I \subset J$. Thus J contains all boundary ideals and we conclude that J is the Shilov boundary for A. $\square$

The fact that $(B/J, q|_A)$ is a C^*-envelope for A allows us to give the precise conditions for a unital operator space so that it determines the structure of its generated C^*-algebra.

Corollary 3.12. *Let A_1 and A_2 be unital operator spaces of C^*-algebras B_1 and B_2 such that $B_1 = C^*(A_1)$ and $B_2 = C^*(A_2)$, and suppose that $\varphi : A_1 \to A_2$ is a unital completely isometric isomorphism. If both A_1 and A_2 have trivial Shilov boundary, then φ is the restriction of a unique $*$-isomorphism $\pi : B_1 \to B_2$.*

Proof. Again, we may assume that A_1 and A_2 are operator systems. Since A_1 and A_2 have trivial Shilov boundary, (B_1, id_{A_1}) and (B_2, id_{A_2}) are C^*-envelopes of A_1 and A_2 respectively. By Theorem 3.10, there exists a unique $*$-isomorphism $\pi : B_1 \to B_2$ such that $\pi|_{A_1} = \varphi$. $\square$

4 The Shilov boundary for a noncommutative analog of the holomorphic functions on the unit polydisk

Let $f : [0,\infty) \to [0,\infty)$ be a continuous function with the unique fixed point 1, and suppose that f satisfies $f(x) \leq 1$ for all $x \in [0,1]$. In this section we will study the universal enveloping C^*-algebra generated by $z_1, \ldots, z_n$ that satisfy

$$z_i^* z_i = f(z_i z_i^*), \quad i = 1, \ldots, n \tag{4.1}$$

$$[z_i, z_j] = 0, \quad [z_i^*, z_j] = 0, \quad i \neq j. \tag{4.2}$$

Recall that P_n is defined as the free $*$-algebra generated by $z_1, \ldots, z_n$. Provided P_n is $*$-bounded with respect to these relations, we define $C(\bar{\mathbb{D}}^n)_f$ as in Section 1.3, and we let $A(\mathbb{D}^n)_f$ be defined as the closed subalgebra generated by $z_1, \ldots, z_n$. Recall that $C(\bar{\mathbb{D}}^n)_f$ and $A(\mathbb{D}^n)_f$ are considered as noncommutative analogs of the continuous and holomorphic functions respectively on the unit polydisk. The main result of this section will be to describe the Shilov boundary for $A(\mathbb{D}^n)_f$, which we interpret as a noncommutative analog of the maximum modulus principle. Our proof relies on classifying the irreducible representations π of P_n such that $\pi(z_1), \ldots, \pi(z_n)$ satisfy (4.1) and (4.2), which will be the objective of the following section.

4.1 Representation theory

Our first result concerns the spectra $\sigma(\pi(z_i)^*\pi(z_i))$ and $\sigma(\pi(z_i)\pi(z_i)^*)$ of the operators $\pi(z_i)^*\pi(z_i)$ and $\pi(z_i)\pi(z_i)^*$, where π is a representation of P_n that satisfies the relations (4.1) and (4.2). From this it will follow that P_n is $*$-bounded with respect to these relations.

Proposition 4.1. *Let (H, π) be a representation of P_n that satisfies (4.1) and (4.2). If $\operatorname{Ker} \pi(z_i)^* \neq \{0\}$, then*

$$\sigma(\pi(z_i)^*\pi(z_i)) = \{f^k(0)\}_{k \geq 1} \cup \{1\}$$
$$\sigma(\pi(z_i)\pi(z_i)^*) = \{f^k(0)\}_{k \geq 0} \cup \{1\},$$

where $f^k = f \circ f^{k-1}$ and $f^0 = \mathrm{id}$. Otherwise

$$\sigma(\pi(z_i)^*\pi(z_i)) = \sigma(\pi(z_i)\pi(z_i)^*) = \{1\}.$$

Proof. We recall the well known fact that

$$\sigma(\pi(z_i)^*\pi(z_i)) \cup \{0\} = \sigma(\pi(z_i)\pi(z_i)^*) \cup \{0\}. \tag{4.3}$$

Moreover, by [Dav96, Corollary I.3.3], we have that

$$\sigma(\pi(z_i)^*\pi(z_i)) = \sigma(f(\pi(z_i)\pi(z_i)^*)) = f(\sigma(\pi(z_i)\pi(z_i)^*)). \tag{4.4}$$

Let us begin by showing that 1 is the largest possible value in the spectra. Suppose to the contrary that $\lambda = \|\pi(z_i)^*\pi(z_i)\| > 1$. Then $\lambda \in \sigma(\pi(z_i)^*\pi(z_i))$ by the spectral radius formula. Consequently λ also belongs to $\sigma(\pi(z_i)\pi(z_i)^*)$ by (4.3), which implies that $f(\lambda) \in \sigma(\pi(z_i)^*\pi(z_i))$ by (4.4). We consider the two separate cases for the behavior of f for $x > 1$. If $f(x) > x$ for all $x > 1$, then $f(\lambda) > \lambda$, which is a contradiction. If $f(x) < x$ for all $x > 1$, we use the fact that $\lambda = f(\mu)$ for some $\mu \in \sigma(\pi(z_i)\pi(z_i)^*)$. Since $f(x) \leq 1$ for all $x < 1$, we must have $\mu > 1$. Then μ belongs to $\sigma(\pi(z_i)^*\pi(z_i))$, but $\lambda = f(\mu) < \mu$, which is also a contradiction.

We proceed to show the inclusion of the spectra into the sets on the right-hand side. Let λ_1 be a value in $\sigma(\pi(z_i)^*\pi(z_i))$ that is not present in $\{f^k(0)\}_{k\geq 1} \cup \{1\}$. Note that $\lambda_1 \neq 0$ since there is no $\mu \in \sigma(\pi(z_i)\pi(z_i)^*)$ such that $f(\mu) = 0$, and hence $\lambda_1 \in \sigma(\pi(z_i)\pi(z_i)^*)$ as well. By (4.4), we have $\lambda_1 = f(\lambda_2)$ for some $\lambda_2 > 0$ in $\sigma(\pi(z_i)\pi(z_i)^*)$ and consequently also in $\sigma(\pi(z_i)^*\pi(z_i))$. Then $\lambda_1 = f(\lambda_2) > \lambda_2$. Arguing inductively, we get a positive decreasing sequence $\{\lambda_i\}$ converging to some $\lambda \geq 0$. But then $f(\lambda) = \lambda$, which is a contradiction since 1 is the unique fixed point.

If $\operatorname{Ker} \pi(z_i)^* \neq \{0\}$, then $0 \in \sigma(\pi(z_i)\pi(z_i)^*)$. Then since $f(0) > 0$, we have that $f(0)$ belongs to both $\sigma(\pi(z_i)^*\pi(z_i))$ and $\sigma(\pi(z_i)\pi(z_i)^*)$. Inductively, $\{f^k(0)\}_{k\geq 1}$ is a subset of both $\sigma(\pi(z_i)^*\pi(z_i))$ and $\sigma(\pi(z_i)\pi(z_i)^*)$. Thus the inclusion of the sets on the right hand side into the spectra follows by this and the fact that a spectrum contains all its limit points.

Suppose now that $\operatorname{Ker} \pi(z_i)^* = \{0\}$. The claim is that $\sigma(\pi(z_i)^*\pi(z_i))$ consists of the single point 1. Indeed, if this were not the case, then by the above, $f(0) \in \sigma(\pi(z_i)^*\pi(z_i))$ and consequently $0 \in \sigma(\pi(z_i)\pi(z_i)^*)$. Since this is an isolated point in the spectrum, it is an eigenvalue and hence there exists a nonzero vector $\eta \in H$ such that $\pi(z_i)\pi(z_i)^*\eta = 0$. But then it follows that η also belongs to $\operatorname{Ker} \pi(z_i)^*$, which is a contradiction. Hence the claim follows. $\square$

We are now in a position where we can begin to classify all irreducible representations of P_n that satisfies (4.1) and (4.2). Given an irreducible representation (H, π) with this property, vectors ζ in $\operatorname{Ker} \pi(z_i)^*$, where i ranges over some subset of $\{1, \ldots, n\}$, will play an important part in describing the representation. The following results will provide us with a family of vectors on which the action of $\pi(P_n)$ is particularly simple.

Lemma 4.2. *If ζ belongs to* $\operatorname{Ker}\pi(z_i)^*$, *then*

$$\pi(z_i)^*\pi(z_i)^k\zeta = \begin{cases} 0, & k=0 \\ f^k(0)\pi(z_i)^{k-1}\zeta, & k \geq 1. \end{cases}$$

Proof. The case $k=0$ follows directly from the definition of ζ. Let k be any positive integer and assume that the result holds for $k-1$. Then

$$(\pi(z_i)\pi(z_i)^*)^m\pi(z_i)^{k-1}\zeta = (\pi(z_i)\pi(z_i)^*)^{m-1}f^{k-1}(0)\pi(z_i)^{k-1}\zeta$$

for all $m \geq 1$. Inductively,

$$(\pi(z_i)\pi(z_i)^*)^m\pi(z_i)^{k-1}\zeta = f^{k-1}(0)^m\pi(z_i)^{k-1}\zeta.$$

If $f(x) = p(x) = a_0x^m + \cdots + a_m$ is a polynomial, then

$$\begin{aligned}\pi(z_i)^*\pi(z_i)^k\zeta &= p(\pi(z_i)\pi(z_i)^*)\pi(z_i)^{k-1}\zeta \\ &= a_0(\pi(z_i)\pi(z_i)^*)^m\pi(z_i)^{k-1}\zeta + \cdots + a_m\pi(z_i)^{k-1}\zeta \\ &= (a_0p^{k-1}(0)^m + \cdots + a_m)\pi(z_i)^{k-1}\zeta \\ &= p(p^{k-1}(0))\pi(z_i)^{k-1}\zeta \\ &= p^k(0)\pi(z_i)^{k-1}\zeta.\end{aligned}$$

For the general case, we approximate f by a polynomial. Fix $\varepsilon > 0$ and let p be a polynomial such that $\|p - f\|_{[0,1]} < \varepsilon/\|\pi(z_i)^{k-1}\zeta\|$. Then

$$\begin{aligned}&\|\pi(z_i)^*\pi(z_i)^k\zeta - f^k(0)\pi(z_i)^{k-1}\zeta\| = \\ &\quad= \|f(\pi(z_i)\pi(z_i)^*)\pi(z_i)^{k-1}\zeta - f^k(0)\pi(z_i)^{k-1}\| \\ &\quad\leq \|f(\pi(z_i)\pi(z_i)^*)\pi(z_i)^{k-1}\zeta - p(\pi(z_i)\pi(z_i)^*)\pi(z_i)^{k-1}\zeta\| \\ &\quad+ \|p(\pi(z_i)\pi(z_i)^*)\pi(z_i)^{k-1}\zeta - p^k(0)\pi(z_i)^{k-1}\zeta\| \\ &\quad+ \|p^k(0)\pi(z_i)^{k-1}\zeta - f^k(0)\pi(z_i)^{k-1}\zeta\| \\ &\quad\leq \|f-p\|_{[0,1]}\|\pi(z_i)^{k-1}\zeta\| + |p^k(0) - f^k(0)|\|\pi(z_i)^{k-1}\zeta\| < \varepsilon,\end{aligned}$$

showing that $\pi(z_i)^*\pi(z_i)^k\zeta = f^k(0)\pi(z_i)^{k-1}\zeta$. $\square$

An immediate consequence of this lemma is that if $k > k'$, then

$$\pi(z_i^*)^k\pi(z_i)^{k'}\zeta = 0.$$

In the following we shall make use of the following standard multi-index notation. Let $l = (l_1, \dots, l_m)$, $1 \leq l_1 < \cdots < l_m \leq n$, and $k = (k_1, \dots, k_m)$, $k_1 \geq 0, \dots, k_m \geq 0$ be m-tuples. Then we define

$$\pi(z_l)^k = \pi(z_{l_1})^{k_1} \dots \pi(z_{l_m})^{k_m}$$

and

$$\widehat{\pi(z_{l_i})^k} = \pi(z_{l_1})^{k_1} \dots \pi(z_{l_{i-1}})^{k_{i-1}}\pi(z_{l_{i+1}})^{k_{i+1}} \dots \pi(z_{l_m})^{k_m}.$$

Proposition 4.3. *Suppose that $\zeta \in H$ is a nonzero vector such that*

$$\zeta \in \bigcap_{i=1}^{m} \operatorname{Ker} \pi(z_{l_i})^*. \tag{4.5}$$

Define $\xi_k = \xi_{k_1,\dots,k_m} = \|\pi(z_l)^k\zeta\|^{-1}\pi(z_l)^k\zeta$. Then $\{\xi_{k_1,\dots,k_m}\}$ is an orthonormal family of vectors.

Proof. By Lemma 4.2, we have the following recursive formula for the norms of $\pi(z_l)^k$.

$$\begin{aligned}
\|\pi(z_l)^k\pi(z_{l_i})\zeta\|^2 &= \langle \pi(z_l)^k\pi(z_{l_i})\zeta, \pi(z_l)^k\pi(z_{l_i})\zeta\rangle \\
&= \langle \pi(z_l)^k\zeta, \widehat{\pi(z_{l_i})^k}\pi(z_{l_i})^*\pi(z_{l_i})^{k_i+1}\zeta\rangle \\
&= f^{k_i+1}(0)\langle \pi(z_l)^k\zeta, \pi(z_l)^k\zeta\rangle \\
&= f^{k_i+1}(0)\|\pi(z_l)^k\zeta\|^2.
\end{aligned}$$

Arguing recursively, we see that all vectors in $\{\xi_{k_1,\dots,k_m}\}$ are well-defined and of norm 1. To show orthogonality, we let ξ_k and $\xi_{k'}$ be two vectors with $k \neq k'$. Assume $k_i > k_i'$ for some i. Again by Lemma 4.2,

$$\begin{aligned}
\langle \xi_k, \xi_{k'}\rangle &= \langle \pi(z_l)^k\zeta, \pi(z_l)^{k'}\zeta\rangle \\
&= \langle \widehat{\pi(z_{l_i})^k}\pi(z_{l_i})^{k_i}\zeta, \widehat{\pi(z_{l_i})^{k'}}\pi(z_{l_i})^{k_i'}\zeta\rangle \\
&= \langle \widehat{\pi(z_{l_i})^k}\zeta, \widehat{\pi(z_{l_i})^{k'}}\pi(z_{l_i}^*)^{k_i}\pi(z_{l_i})^{k_i'}\zeta\rangle \\
&= 0,
\end{aligned}$$

where the equalities hold up to some multiplicative constant. $\square$

Thus, given a vector ζ satisfying (4.5) for some m-tuple l, Proposition 4.3 provides us with an associated family of orthonormal vectors. The action of $\pi(P_n)$ on this family of vectors is given by the following proposition.

Proposition 4.4. *Let $\{\xi_{k_1,\dots,k_m}\}$ be as in Proposition 4.3. Then*

$$\begin{aligned}
\pi(z_{l_i})\xi_{k_1,\dots,k_i,\dots,k_m} &= \sqrt{f^{k_i+1}(0)}\xi_{k_1,\dots,k_i+1,\dots,k_m} \\
\pi(z_{l_i})^*\xi_{k_1,\dots,k_i,\dots,k_m} &= \begin{cases} 0, & k_i = 0 \\ \sqrt{f^{k_i}(0)}\xi_{k_1,\dots,k_i-1,\dots,k_m}, & k_i \geq 1. \end{cases}
\end{aligned}$$

Proof. This follows immediately from the recursive formula for the norm computed above. We have

$$\begin{aligned}
\pi(z_{l_i})\xi_{k_1,\dots,k_i,\dots,k_m} &= \|\pi(z_l)^k\zeta\|^{-1}\pi(z_l)^k\pi(z_{l_i})\zeta \\
&= \|\pi(z_l)^k\zeta\|^{-1}\|\pi(z_l)^k\pi(z_{l_i})\zeta\|\xi_{k_1,\dots,k_{i+1},\dots,k_m} \\
&= \sqrt{f^{k_i+1}(0)}\xi_{k_1,\dots,k_i+1,\dots,k_m}.
\end{aligned}$$

The formula for the adjoint follows as a consequence of this. $\square$

In order to classify the irreducible representations of P_n, we shall argue by induction on the number of generators of P_n. The following theorem lists the irreducible representations of $P = P_1$, which will serve as the basis of induction.

Theorem 4.5. *The irreducible representations of P up to unitary equivalence that satisfy (4.1) and (4.2) are given by*

(i) one-dimensional representations, $\pi_\varphi(z) = e^{i\varphi}$, $\varphi \in [0, 2\pi)$;

(ii) $(\ell^2(\mathbb{Z}_{\geq 0}), \pi)$ defined as follows. Let $\{e_k\}_{k\geq 0}$ be the standard orthonormal basis for $\ell^2(\mathbb{Z}_{\geq 0})$. Then

$$\pi(z)e_k = \sqrt{f^{k+1}(0)}e_{k+1}$$
$$\pi(z)^*e_k = \begin{cases} 0, & k = 0 \\ \sqrt{f^k(0)}e_{k-1}, & k \geq 1. \end{cases}$$

Proof. Suppose first that $\operatorname{Ker}\pi(z)^* \neq \{0\}$. Let ζ be a unit vector in $\operatorname{Ker}\pi(z)^*$, and define $U \subset H$ as the closed subspace generated by $\{\xi_k\}$ from Proposition 4.3. Proposition 4.4 shows that U is invariant under the action of $\pi(P)$ and hence $H = U$.

If $\operatorname{Ker}\pi(z)^* \neq \{0\}$, we have $\pi(z)^*\pi(z) = \pi(z)\pi(z)^* = I$ as a consequence of Proposition 4.1, showing that $\pi(P)$ is commutative. By Schur's lemma, $\pi(P) = \mathbb{C}I$, and hence any subspace is invariant. H must therefore be one-dimensional. $\square$

Let us now turn to the general case where n is any positive integer.

Theorem 4.6. *The irreducible representations of P_n up to unitary equivalence that satisfy (4.1) and (4.2) are given by $(\ell^2(\mathbb{Z}_{\geq 0})^{\otimes m}, \pi_{l,m})$, $0 \leq m \leq n$, $l = (l_1, \ldots, l_m)$, $1 \leq l_1 < \cdots < l_m \leq n$, defined as follows. Let $\{e_k\}_{k\geq 0}$ be the standard orthonormal basis for $\ell^2(\mathbb{Z}_{\geq 0})$, and let the operator $T \in \mathcal{B}(\ell^2(\mathbb{Z}_{\geq 0}))$ be given by*

$$Te_k = \sqrt{f^{k+1}(0)}e_{k+1}$$
$$T^*e_k = \begin{cases} 0, & k = 0 \\ \sqrt{f^k(0)}e_{k-1}, & k \geq 1. \end{cases}$$

Then

$$\pi(z_{l_i}) = I^{\otimes i-1} \otimes T \otimes I^{\otimes m-i}, \quad 1 \leq i \leq m,$$

and $\pi(z_j) = e^{i\varphi_j} I^{\otimes m}$ for those indices j not present in l. (Note that l is empty if $m = 0$ and that $(\ell^2(\mathbb{Z}_{\geq 0})^{\otimes m}, \pi_{l,m})$ is one-dimensional in this case.)

Proof. Let (H, π) be an irreducible representation of P_n that satisfies (4.1) and (4.2). The case $n = 1$ follows from Theorem 4.5. Suppose $n \geq 2$ and assume that the theorem holds for P_{n-1}. Suppose first that $\operatorname{Ker} \pi(z_n)^* \neq \{0\}$. Let ζ_n be a nonzero vector in $\operatorname{Ker} \pi(z_n)^*$ and define V as the closed subspace generated by all expressions

$$p(\pi(z_1), \pi(z_1)^*, \ldots, \pi(z_{n-1}), \pi(z_{n-1})^*)\zeta_n,$$

where p is a polynomial. Now define $\rho_V(z_i) = \pi(z_i)|_V$, $i = 1, \ldots, n-1$. We claim that ρ_V is an irreducible representation of the subalgebra $P_{n-1} \subset P_n$ on V. Suppose $V = V_1 \oplus V_2$ where V_1 and V_2 are two invariant subspaces such that $V_1 \perp V_2$. Then

$$V_i \oplus \pi(z_n)V_i \oplus \pi(z_n)^2 V_i \oplus \ldots, \quad i = 1, 2,$$

are two mutually orthogonal subspaces. As a consequence of Lemma 4.2, we see that both are invariant under the action of $\pi(P_n)$. But since π is irreducible, they must be equal, contradicting the fact that V_1 and V_2 were chosen to be mutually orthogonal. By the induction hypothesis, ρ_V is unitarily equivalent to one of the representations $(\ell^2(\mathbb{Z}_{\geq 0})^{\otimes m}, \pi_{l,m})$, $0 \leq m \leq n-1$, listed above. If $m \geq 1$, define $\zeta \in H$ as the vector corresponding to $e_0 \otimes \cdots \otimes e_0 \in \ell^2(\mathbb{Z}_{\geq 0})^{\otimes m}$ under this isomorphism, and define l' as the $m+1$-tuple (l, n). If $m = 0$, let ζ be any unit vector in V and set $l' = n$. (Note that V is one-dimensional in this case.) Since $\pi(z_n)^*$ commutes with $\pi(z_i)$ and $\pi(z_i)^*$ for all $i \neq n$, we have $V \subset \operatorname{Ker} \pi(z_n)^*$. Thus

$$\zeta \in \bigcap_{i=1}^{m+1} \operatorname{Ker} \pi(z_{l'_i})^*.$$

By Proposition 4.3, we obtain an orthonormal family of vectors $\{\xi_{k_1,\ldots,k_{m+1}}\}$, and we define W as the closed subspace generated by these vectors. By Proposition 4.4, we see that W is an invariant under $\pi(P_n)$ and hence $H = W$.

If $\operatorname{Ker} \pi(z_n)^* = \{0\}$, then as a consequence of Proposition 4.1, we have that $\pi(z_n)^*\pi(z_n) = \pi(z_n)\pi(z_n)^* = I$. From this we get that the commutant $\pi(P_n)'$ contains the subalgebra generated by $\pi(z_n)$ and $\pi(z_n)^*$. By Schur's lemma, we get $\pi(z_n) = e^{i\varphi_n} I$ for some $\varphi_n \in [0, 2\pi)$. As a consequence, $(H, \pi|_{P_{n-1}})$ is an irreducible representation and is therefore unitarily equivalent to one of the representations $(\ell^2(\mathbb{Z}_{\geq 0})^{\otimes m}, \pi_{l,m})$, $0 \leq m \leq n-1$, listed above, by the induction hypothesis. □

4.2 The Shilov boundary for $A(\mathbb{D}^n)_f$

Using the irreducible representations of P_n obtained in the previous section, we can now give a description of the Shilov boundary for $A(\mathbb{D}^n)_f$.

Theorem 4.7. *Let J be the closed ideal generated by*

$$z_i^* z_i - 1, \quad z_i z_i^* - 1, \quad i = 1, \dots, n. \tag{4.6}$$

Then J is the Shilov boundary for $A(\mathbb{D}^n)_f$.

Proof. We claim that $C(\bar{\mathbb{D}}^n)_f/J$ is $*$-isomorphic to $C(\mathbb{T}^n)$. Consider the $*$-homomorphism $\varphi : P_n \to C(\mathbb{T}^n)$ that represents each polynomial as its corresponding function on $\mathbb{T}^n$. It is straightforward to verify that $\operatorname{Ker}\varphi$ is the ideal generated by the elements $z_i^* z_i - 1$, $z_i z_i^* - 1$, $i = 1, \dots, n$. Since 1 is a fixed point for f, φ satisfies the relations (4.1) and (4.2), and hence φ extends uniquely to a $*$-homomorphism on $C(\bar{\mathbb{D}}^n)_f$, which we shall still denote by φ. Moreover, since $\varphi(P_n)$ is dense in $C(\mathbb{T}^n)$, we have that $\varphi : (\bar{\mathbb{D}}^n)_f \to C(\mathbb{T}^n)$ is surjective. Since $C(\bar{\mathbb{D}}^n)_f$ is a faithful representation of P_n, it is clear that $\operatorname{Ker}\varphi = J$, and hence the claim follows.

Let us now show that J is a boundary ideal, i.e., the quotient map $C(\bar{\mathbb{D}}^n)_f \to C(\bar{\mathbb{D}}^n)_f/J \cong C(\mathbb{T}^n)$ is a complete isometry when restricted to $A(\mathbb{D}^n)_f$. Using the one-dimensional representations, we obtain

$$\|g(z_1, \dots, z_n)\| \geq \sup_{\varphi_1, \dots, \varphi_n \in [0, 2\pi)} |g(e^{i\varphi_1}, \dots, e^{i\varphi_n})|,$$

where g is a polynomial.

On the other hand, since $T \in \mathcal{B}(\ell^2(\mathbb{Z}_{\geq 0}))$ from Theorem 4.5 is a contraction, there exists a Hilbert space K containing $\ell^2(\mathbb{Z}_{\geq 0})$ as a subspace and a unitary operator U on K such that $T^k = P_{\ell^2(\mathbb{Z}_{\geq 0})} U^k|_{\ell^2(\mathbb{Z}_{\geq 0})}$ by Sz.-Nagy's dilation theorem. If (H, π) is an irreducible representation, by Theorem 4.5, $\pi(z_i)$ is either of the form

$$\pi(z_i) = P_H(I \otimes \dots \otimes I \otimes U \otimes I \otimes \dots \otimes I)|_H$$

or $\pi(z_i) = e^{i\varphi_i} I \otimes \dots \otimes I$. In both cases we can write $\pi(z_i)^k = P_H U_{\pi,i}^k|_H$, where $U_{\pi,1}, \dots U_{\pi,n}$ are unitary and mutually commuting operators. From this it is not difficult to see that for a polynomial g

$$\begin{aligned}
\|g(z_1, \dots, z_n)\| &= \sup_{\pi} \|g(\pi(z_1), \dots, \pi(z_n))\| \\
&= \sup_{\pi} \|P_H g(U_{\pi,1}, \dots, U_{\pi,n})|_H\| \\
&\leq \sup_{\pi} \|g(U_{\pi,1}, \dots, U_{\pi,n})\| \\
&\leq \sup_{\varphi_1, \dots, \varphi_n \in [0, 2\pi)} |g(e^{i\varphi_1}, \dots, e^{i\varphi_n})|,
\end{aligned}$$

where the supremum ranges over all irreducible representations that satisfy (4.1) and (4.2). Thus we have shown that, for any polynomial g,

$$\|g(z_1,\ldots,z_n)\| = \sup_{\varphi_1,\ldots,\varphi_n\in[0,2\pi)} |g(e^{i\varphi_1},\ldots,e^{i\varphi_n})| = \|g(z_1,\ldots,z_n)\|_{\mathbb{T}^n},$$

and hence the quotient map is an isometry when restricted to $A(\mathbb{D}^n)_f$. Since the range is commutative, it follows by Theorem 2.7 that this map is also a complete isometry.

Finally we show that J is the Shilov boundary, i.e, that it contains all other boundary ideals. Let I be a boundary ideal such that $I \supset J$, and consider the surjective $*$-homomorphism $C(\bar{\mathbb{D}}^n)_f/J \to C(\bar{\mathbb{D}}^n)_f/I$. Since both I and J are boundary ideals, we have

$$\|h(z_1,\ldots,z_n)+I\| = \|h(z_1,\ldots,z_n)\| = \|h(z_1,\ldots,z_n)+J\| \tag{4.7}$$

for every polynomial h. Since J is given by (4.6), for each polynomial g, there exists a polynomial h such that

$$g(z_1,z_1^*,\ldots,z_n,z_n^*)+J = z_1^{*m_1}\ldots z_n^{*m_n}h(z_1,\ldots,z_n)+J.$$

This fact together with (4.7) gives

$$\begin{aligned}
&\|g(z_1,z_1^*,\ldots,z_n,z_n^*)+J\| = \\
&\quad = \|h(z_1,\ldots,z_n)^* z_1^{m_1}\ldots z_n^{m_n} z_1^{*m_1}\ldots z_n^{*m_n} h(z_1,\ldots,z_n)+J\|^{1/2} \\
&\quad = \|h(z_1,\ldots,z_n)+J\| = \|h(z_1,\ldots,z_n)+I\| \\
&\quad = \|h(z_1,\ldots,z_n)^* h(z_1,\ldots,z_n)+I\|^{1/2} \\
&\quad = \|g(z_1,z_1^*,\ldots,z_n,z_n^*)+I\|.
\end{aligned}$$

So $C(\bar{\mathbb{D}}^n)_f/J$ is $*$-isomorphic to $C(\bar{\mathbb{D}}^n)_f/I$, which implies that $I = J$. $\square$

We define $C(S(\bar{\mathbb{D}}^n))_f = C(\bar{\mathbb{D}}^n)_f/J$, which we interpret as a noncommutative analog of the continuous functions on the Shilov boundary.

Recall that, by the maximum modulus principle, the Shilov boundary for the holomorphic functions on the unit disk is the unit circle. From this it is not difficult to deduce that the Shilov boundary for the holomorphic functions on the unit polydisk is given by $(\partial\mathbb{D})^n = \mathbb{T}^n$, which is known as the *distinguished boundary*. From the discussion in Section 1.1, we have

$$C(S(\bar{\mathbb{D}}^n))_f = C(\mathbb{T}^n) = C(\mathbb{T})^{\otimes n} = C(S(\bar{\mathbb{D}}))_f^{\otimes n},$$

and hence we see that the noncommutative analog of the maximum modulus modulus principle satisfies the same property in the multidimensional case.

4.3 The quantum unit disk

We end this chapter with a few examples that have occurred in the literature and which fit into our theory developed above. Both are examples of what is commonly referred to as the *quantum unit disk*.

In [Vak10], the quantum unit disk was defined as the universal enveloping algebra of the polynomial algebra $P(\mathbb{C}^n)_q$ given by the generator z satisfying

$$z^*z = q^2zz^* + 1 - q^2,$$

with $0 < q < 1$. The continuous function in (4.1) is therefore given by

$$f(x) = q^2x + 1 - q^2,$$

which possesses the unique fixed point 1. It readily follows that the iterations for $k = 0, 1, \ldots,$ are given by

$$f^k(0) = 1 - q^{2k}.$$

Since f has the unique fixed point 1 and $f(x) \leq 1$ for all $x \in [0, 1]$, we get that the Shilov boundary for $A(\mathbb{D})_f$ is given by the ideal in Theorem 4.7.

In this case, the Shilov boundary can also be described in terms of operator algebras. Let S denote the unilateral shift on $\ell^2(\mathbb{Z}_{\geq 0})$, i.e., $Se_k = e_{k+1}$, where $\{e_k\}_{k\geq 0}$ is the standard orthonormal basis for $\ell^2(\mathbb{Z}_{\geq 0})$. We claim that $C(\bar{\mathbb{D}})_f$ is $*$-isomorphic to $C^*(S)$, which is known as the *Toeplitz algebra*, and that, under this isomorphism, the Shilov boundary is given by $\mathcal{K}$, the compact operators on $\ell^2(\mathbb{Z}_{\geq 0})$.

We shall prove this statement by first showing that $C^*(\pi(z)) = C^*(S)$ and then show that $C(\bar{\mathbb{D}})_f$ is $*$-isomorphic to $C^*(\pi(z))$. Recall that the two irreducible representations of $P(\mathbb{C}^n)_q$ are given by $\pi_\varphi(z)e_k = e^{i\varphi}$, $\varphi \in [0, 2\pi)$, and $\pi(z)e_k = \sqrt{1 - q^{2(k+1)}}e_{k+1}$. If we let C_2 denote the operator defined by $C_2e_k = \sqrt{1 - q^{2k}}e_k$, it is clear that $\pi(z) = C_2S$. Observe that $C_2 \in C^*(S)$ since

$$C_2^2 = (1 - q^2)\sum_{k=0}^{\infty} q^{2k}S^{k+1}(S^*)^{k+1}, \tag{4.8}$$

and consequently $C(\pi(z)) \subset C^*(S)$. To see the reverse inclusion, we define the function g by $g(0) = 0$ and $g(\zeta) = \zeta^{-1}$ for $\zeta \neq 0$. Since 0 is an isolated point in the spectrum of C_2, g is continuous on $\sigma(C_2)$. Since $S = f(C_2)\pi(z)$ and $C_2 = \sqrt{\pi(z)\pi(z)^*}$, we get that $C^*(S) \subset C^*(\pi(z))$.

For the second step, consider the $*$-homomorphism $\Theta_\varphi : C^*(S) \to \mathbb{C}$ given by $\Theta_\varphi(S) = e^{i\varphi}$. (We postpone the proof of the claim that Θ_φ is actually a well-defined $*$-homomorphism until the next chapter.) From (4.8) it follows that $\Theta_\varphi(C_2) = 1$, and hence Θ_φ induces a $*$-homomorphism of $C^*(\pi(z))$ into $C^*(\pi_\varphi(z))$. Since $*$-homomorphisms of C^*-algebras are contractive, we get that $\|\pi_\varphi(x)\| \leq \|\pi(x)\|$ for all $x \in P(\mathbb{C}^n)_q$ and $\varphi \in [0, 2\pi)$. Therefore, from the definition of $C(\bar{\mathbb{D}})_f$, it follows that $\pi : C(\bar{\mathbb{D}})_f \to C^*(\pi(z))$ is a $*$-isomorphism.

From the theory developed in Chapter V of [Dav96], it follows that $C^*(S)/\mathcal{K}$ is $*$-isomorphic to $C(\mathbb{T})$, and since $C(\bar{\mathbb{D}})_f/J \cong C(\mathbb{T})$, we find that the Shilov boundary under the isomorphism described above is given by $\mathcal{K}$.

In [KL92], the quantum unit disk was defined as the universal enveloping algebra of the polynomial algebra $P(\mathbb{C}^n)_\mu$ given by the generator z satisfying

$$[z^*, z] = \mu(1 - z^*z)(1 - zz^*), \tag{4.9}$$

with $0 < \mu < 1$. Rearranging, this is equivalent to

$$z^*z(1 + \mu - \mu zz^*) = (1 - \mu)zz^* + \mu. \tag{4.10}$$

Denote $x = z^*z$ and $y = zz^*$. From (4.10) and its adjoint, it is easy to see that $[x, y] = 0$. Given a representation (H, π), $\pi(x)$ and $\pi(y)$ generate a commutative C^*-algebra $C^*(\pi(x), \pi(y))$. By the Gelfand-Naimark theorem, $C^*(\pi(x), \pi(y))$ is $*$-isomorphic to $C(X)$, the C^*-algebra of continuous functions on the maximal ideal space, X, of $C^*(\pi(x), \pi(y))$. Let $\widehat{x}$ and $\widehat{y}$ denote the images of $\pi(x)$ and $\pi(y)$ under this isomorphism. Since both $\pi(x)$ and $\pi(y)$ are positive, $\widehat{x}(\chi) \geq 0$ and $\widehat{y}(\chi) \geq 0$ for all $\chi \in X$. Since

$$\widehat{x}(1 + \mu - \mu\widehat{y}) = (1 - \mu)\widehat{y} + \mu,$$

we have $1 + \mu - \mu\widehat{y}(\chi) > 0$ for all $\chi \in X$, and

$$\widehat{x}(\chi) = \frac{(1 - \mu)\widehat{y}(\chi) + \mu}{1 + \mu - \mu\widehat{y}(\chi)}.$$

The function $t \mapsto ((1 - \mu)t + \mu)(1 + \mu - \mu t)^{-1}$ is strictly increasing for $t \in [0, 1 + 1/\mu)$ and negative for $t > 1 + \mu^{-1}$. Consequently,

$$\|\pi(x)\| = \frac{(1 - \mu)\|\pi(y)\| + \mu}{1 + \mu - \mu\|\pi(y)\|}. \tag{4.11}$$

Since $\|\pi(x)\| = \|\pi(y)\| = \|\pi(z)\|^2$, (4.11) implies $\|\pi(z)\| = 1$, and hence all representations satisfy

$$\pi(z)^*\pi(z) = ((1 - \mu)\pi(z)\pi(z)^* + \mu)(1 + \mu - \mu\pi(z)\pi(z)^*)^{-1}.$$

Thus we can define f in (4.1) as

$$f(x) = \frac{(1-\mu)x + \mu}{1 + \mu - \mu x}, \quad x \in [0,1].$$

Note that we have only defined f on an interval in this case. However, we can either extend f continuously such that no other fixed points are introduced or check that the theory holds also in this case. The iterations for $k = 0, 1, \dots,$ are given by

$$f^k(0) = \frac{k\mu}{1 + k\mu}.$$

Indeed, we have

$$f^{k+1}(0) = f\left(\frac{k\mu}{1+k\mu}\right) = \frac{(1-\mu)k\mu + \mu(1+k\mu)}{(1+\mu)(1+k\mu) - k\mu^2} = \frac{(k+1)\mu}{1+(k+1)\mu}.$$

Again, we see that f has the unique fixed point 1 and that $f(x) \leq 1$ for all $x \in [0,1]$. Therefore the Shilov boundary for $A(\mathbb{D})_f$ is given by the ideal in Theorem 4.7.

5 The Shilov boundary for a noncommutative analog of the holomorphic functions on the unit ball of symmetric matrices

In [PT15], within the subject of quantum bounded symmetric domains, the authors gave a description of the Shilov boundary for a noncommutative analog of the algebra of holomorphic functions on the unit ball of the space of complex 2×2 matrices. In this chapter we show that similar methods can be used to describe the Shilov boundary for a q-analog of the algebra of holomorphic functions on the unit ball of the space of symmetric complex 2×2 matrices.

Throughout this chapter we let S, C_n, $D \in \mathcal{B}(\ell^2(\mathbb{Z}_{\geq 0}))$ denote the operators given by

$$Se_k = e_{k+1}, \quad C_n e_k = \sqrt{1 - q^{nk}} e_k, \quad De_k = q^k e_k. \tag{5.1}$$

Moreover, q denotes a real number such that $0 < q < 1$.

5.1 Quantum groups

Quantum bounded symmetric domains is part of the realm of the theory of quantum groups, which is a rich and vast subject with numerous applications in mathematics and physics. It would be impossible to give a proper treatment of this subject within the scope of this thesis. Instead we direct the reader to the comprehensive monographs [Vak10] and [KS97]. For the sake of motivation, however, let us give a brief demonstration, following [Vak01], on how some of these concepts arise starting from a q-analog of the algebra of polynomials on the space of complex $n \times n$ matrices $\mathbb{C}[M_n]_q$. It is worth to keep in mind that if we set $q = 1$, these procedures yield the classical constructions.

$\mathbb{C}[M_n]_q$ is defined by its generators z_a^α, $a, \alpha = 1, \ldots, n$ and the following relations:

$$\begin{aligned}
z_a^\alpha z_b^\beta - q z_b^\beta z_a^\alpha &= 0, && a = b,\ \alpha < \beta \text{ or } a < b,\ \alpha = \beta \\
z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha &= 0, && a > b,\ \alpha < \beta \\
z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha &= (q - q^{-1}) z_a^\beta z_b^\alpha, && a < b,\ \alpha < \beta.
\end{aligned}$$

Our first goal is to define a q-analog of the regular functions on SU_2. We begin by introducing the *quantum determinant* for the matrix (z_a^α):

$$\det_q(z_a^\alpha) = \sum_{\sigma \in S_n} (-q)^{\operatorname{sgn} \sigma} z_1^{\sigma(1)} \ldots z_n^{\sigma(n)}.$$

It can be shown that $\det_q(z_a^\alpha)$ lies in the center of $\mathbb{C}[M_n]_q$, and hence $\mathbb{C}[M_n]_q$ can be localized with respect to the multiplicative system $\{\det_q(z_a^\alpha)^k : k \geq 0\}$. This algebra is known as the algebra of regular functions on the quantum GL_n and is denoted by $\mathbb{C}[GL_n]_q$.

Next, we define an involution on $\mathbb{C}[GL_n]_q$ by

$$(z_b^\beta)^* = (-q)^{b-\beta} \det_q(z_a^\alpha)^{-1} \det_q \widehat{(z_a^\alpha)}_b^\beta,$$

where $\widehat{(z_a^\alpha)}_b^\beta$ denotes the matrix given by omitting the row β and column b from the matrix (z_a^α). The $*$-algebra $\mathbb{C}[U_n]_q = (\mathbb{C}[GL_n]_q, *)$ is known as the $*$-algebra of regular functions on the quantum U_n. Finally, we obtain the $*$-algebra of regular functions on the quantum SU_n by defining $\mathbb{C}[SU_n]_q = \mathbb{C}[U_n]_q/(\det_q(z_a^\alpha) - 1)$.

In the case $n = 2$, it can be shown that $\mathbb{C}[SU_2]_q$ can be defined in terms of the generators t_{ij}, $i, j = 1, 2$, subject to the relations

$$\begin{aligned}
&t_{11}t_{21} = qt_{21}t_{11}, \quad t_{11}t_{12} = qt_{12}t_{11}, \quad t_{12}t_{21} = t_{21}t_{12} \\
&t_{22}t_{21} = q^{-1}t_{21}t_{11}, \quad t_{22}t_{12} = q^{-1}t_{12}t_{22} \\
&t_{11}t_{22} - t_{22}t_{11} = (q - q^{-1})t_{12}t_{21}, \quad t_{11}t_{22} - qt_{12}t_{21} = 1 \\
&t_{11}^* = t_{22}, \quad t_{12}^* = -qt_{21}.
\end{aligned}$$

It is well known (see, e.g., [Ber14] and references therein) that $\mathbb{C}[SU_2]_q$ admits the irreducible representations π_φ, $\varphi \in [0, 2\pi)$, acting on $\ell^2(\mathbb{Z}_{\geq 0})$, which are given by

$$\begin{aligned}
&\pi_\varphi(t_{11}) = S^*C_2, \quad \pi_\varphi(t_{12}) = -qe^{-i\varphi}D \\
&\pi_\varphi(t_{21}) = e^{i\varphi}D, \quad \pi_\varphi(t_{22}) = C_2S.
\end{aligned}$$

Another q-analogue that we shall encounter is the $*$-algebra $P(\mathbb{C}^n)_q$, a q-analog of the $*$-algebra of polynomials on $\mathbb{C}^n$. $P(\mathbb{C}^n)_q$ is generated by $z_1, \ldots, z_n$ subject to the relations

$$\begin{aligned}
&z_i z_j = qz_j z_i, \quad i < j \\
&z_i^* z_j = qz_j z_i^*, \quad i \neq j \\
&z_i^* z_i = q^2 z_i z_i^* + \left(1 - q^2\right) \left(1 - \sum_{i<j} z_j z_j^*\right).
\end{aligned}$$

The irreducible representations of $P(\mathbb{C}^n)_q$ are well known. For reasons that become evident later, the representations that will be of interest to us is for the particular case $P(\mathbb{C})_{q^2}$. We have the following list of irreducible representations of $P(\mathbb{C})_{q^2}$, up to unitary equivalence:

(i) the Fock representation ρ_F acting on $\ell^2(\mathbb{Z}_{\geq 0})$: $\rho_F(z) = C_4 S$;

(ii) one-dimensional representations ρ_φ, $\varphi \in [0, 2\pi)$: $\rho_\varphi(z) = e^{i\varphi}$.

5.2 The $*$-algebra $P(\mathrm{Sym}_2)_q$ and its representations

The $*$-algebra $P(\mathrm{Sym}_2)_q$ is a q-analog of the $*$-algebra of polynomials on the space Sym_2 of symmetric complex 2×2 matrices. By an analogous procedure to the construction of $\mathbb{C}[SU_2]_q$, $P(\mathrm{Sym}_2)_q$ can be obtained starting from $\mathbb{C}[M_2]_q$. In [Ber14], $P(\mathrm{Sym}_2)_q$ was directly defined in terms of the generators z_{11}, z_{21}, z_{22} and the following list of relations:

$$
\begin{aligned}
&z_{11}z_{21} = q^2 z_{21}z_{11}, \quad z_{21}z_{22} = q^2 z_{22}z_{21}\\
&z_{11}z_{22} - z_{22}z_{11} = q(q^2 - q^{-2})z_{21}^2\\
&z_{11}^* z_{11} = q^4 z_{11}z_{11}^* - q(q^{-1} - q)(1+q^2)^2 z_{21}z_{21}^* + (q^{-1}-q)^2(1+q^2)z_{22}z_{22}^* + 1 - q^4\\
&z_{11}^* z_{21} = q^2 z_{21}z_{11}^* - q(q^{-2} - q^2)z_{22}z_{21}^*\\
&z_{11}^* z_{22} = z_{22}z_{11}^*, \quad z_{21}^* z_{22} = q^2 z_{22}z_{21}^*\\
&z_{21}^* z_{21} = q^2 z_{21}z_{21}^* - (1-q^2)z_{22}z_{22}^* + 1 - q^2\\
&z_{22}^* z_{22} = q^4 z_{22}z_{22}^* + 1 - q^4.
\end{aligned}
$$

It should be noted that this notation differs slightly from the above discussion as well as the definition of $P(M_n)_q$, a q-analog of the polynomials on the space of complex 2×2 matrices, found in e.g. [PT15]. First of all, we see that q has been replaced by q^2. Moreover, since this definition concerns polynomials defined on the space of symmetric matrices, the generator z_{12} is superfluous. For the sake of symmetry, however, one may include z_{12} as a generator together with the relation $z_{12} = qz_{21}$. By comparing the two first rows with the relations for $\mathbb{C}[M_2]_q$, we get the correspondence $z_{ij} = z^i_j$. In particular, we note that $z^1_2 = z_{12} = qz_{21}$.

The irreducible representations of $P(\mathrm{Sym}_2)_q$, which we present in the following theorem, were classified in [Ber14].

Theorem 5.1. *The irreducible representations of $P(\mathrm{Sym}_2)_q$ up to unitary equivalence are given by*

(i) the Fock representation acting on $\ell^2(\mathbb{Z}_{\geq 0})^{\otimes 3}$:

$$
\begin{aligned}
\pi_F(z_{11}) &= I\otimes D^2 \otimes C_4 S - q^{-1}S^* C_4 \otimes C_2 S C_2 S \otimes I\\
\pi_F(z_{21}) &= D^2 \otimes C_2 S \otimes I\\
\pi_F(z_{22}) &= C_4 S \otimes I \otimes I;
\end{aligned}
$$

(ii) representations τ_φ, $\varphi \in [0, 2\pi)$, acting on $\ell^2(\mathbb{Z}_{\geq 0})^{\otimes 2}$:

$$
\begin{aligned}
\tau_\varphi(z_{11}) &= e^{i\varphi} I \otimes D^2 - q^{-1} S^* C_4 \otimes C_2 S C_2 S\\
\tau_\varphi(z_{21}) &= D^2 \otimes C_2 S\\
\tau_\varphi(z_{22}) &= C_4 S \otimes I;
\end{aligned}
$$

(iii) representations ω_φ, $\varphi \in [0, 2\pi)$, acting on $\ell^2(\mathbb{Z}_{\geq 0})$:

$$\begin{aligned}\omega_\varphi(z_{11}) &= -q^{-1}e^{2i\varphi}S^*C_4\\ \omega_\varphi(z_{21}) &= e^{i\varphi}D^2\\ \omega_\varphi(z_{22}) &= C_4S;\end{aligned}$$

(iv) representations ν_φ, $\varphi \in [0, 2\pi)$, acting on $\ell^2(\mathbb{Z}_{\geq 0})$:

$$\begin{aligned}\nu_\varphi(z_{11}) &= q^{-1}C_4S\\ \nu_\varphi(z_{21}) &= 0\\ \nu_\varphi(z_{22}) &= e^{i\varphi}I;\end{aligned}$$

(v) one-dimensional representations $\theta_{\varphi_1,\varphi_2}$, $\varphi_1, \varphi_2 \in [0, 2\pi)$:

$$\begin{aligned}\theta_{\varphi_1,\varphi_2}(z_{11}) &= q^{-1}e^{i\varphi_1}\\ \theta_{\varphi_1,\varphi_2}(z_{21}) &= 0\\ \theta_{\varphi_1,\varphi_2}(z_{22}) &= e^{i\varphi_2}.\end{aligned}$$

From the above list, it readily follows that $P(\mathrm{Sym}_2)_q$ is $*$-bounded. We let $C(\bar{\mathbb{D}}_2)_q$ denote the universal enveloping C^*-algebra of $P(\mathrm{Sym}_2)_q$ and $A(\mathbb{D}_2)_q$ the closed subalgebra generated by z_{11}, z_{21}, and z_{22}. The notation is chosen since $C(\bar{\mathbb{D}}_2)_q$ is a q-analog of the C^*-algebra of continuous functions on the closed unit disk of symmetric complex 2×2 matrices $\bar{\mathbb{D}}_2 = \{x \in \mathrm{Sym}_2 : x^*x \leq 1\}$.

We will now consider an alternative way of constructing representations of $P(\mathrm{Sym}_2)_q$ which was presented in [Ber14], where representations were constructed by composing maps from $P(\mathrm{Sym}_2)_q$ to $*$-algebras whose representations are well known. The $*$-algebras that we have in mind are $\mathbb{C}[SU_2]_q$ and $P(\mathbb{C})_{q^2}$, which were discussed above.

The connection between representations of $P(\mathrm{Sym}_2)_q$ and $\mathbb{C}[SU_2]_q$ is given by the following $*$-homomorphism of a coaction whose existence was indicated in [Ber14].

Lemma 5.2. *There is a $*$-homomorphism*

$$\Delta : P(\mathrm{Sym}_2)_q \to P(\mathrm{Sym}_2)_q \otimes \mathbb{C}[SU_2]_q$$

given by

$$\Delta(z_{ij}) = z_{11} \otimes t_{1i}t_{1j} + qz_{21} \otimes t_{1i}t_{2j} + z_{21} \otimes t_{2i}t_{1j} + z_{22} \otimes t_{2i}t_{2j}.$$

From the commutation relations it follows that the family of maps

$$\Pi_\varphi : P(\mathrm{Sym}_2)_q \to P(\mathbb{C})_{q^2},$$

$\varphi \in [0, 2\pi)$, defined on the generators of $P(\mathrm{Sym}_2)_q$ by

$$\Pi_\varphi(z_{11}) = q^{-1} z, \quad \Pi_\varphi(z_{21}) = 0, \quad \Pi_\varphi(z_{22}) = e^{i\varphi}$$

is a $*$-homomorphism. Defining

$$F_\varphi = \rho_F \circ \Pi_\varphi, \quad \chi_{\varphi_1,\varphi_2} = \rho_{\varphi_1} \circ \Pi_{\varphi_2},$$

we obtain two families of representations of $P(\mathrm{Sym}_2)_q$:

$$(F_\varphi \otimes \pi_0) \circ \Delta, \quad (\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta,$$

where $\varphi, \varphi_1, \varphi_2 \in [0, 2\pi)$. Evaluated on the generators, we have

$$\begin{aligned}(F_\varphi \otimes \pi_0) \circ \Delta(z_{11}) &= q^{-1} \rho_F(z) \otimes \pi_0(t_{11})^2 + e^{i\varphi} I \otimes \pi_0(t_{21})^2 \\ &= q^{-1} C_4 S \otimes S^* C_2 S^* C_2 + e^{i\varphi} I \otimes D^2\end{aligned}$$

$$\begin{aligned}(F_\varphi \otimes \pi_0) \circ \Delta(z_{21}) &= q^{-1} \rho_F(z) \otimes \pi_0(t_{12}) \pi_0(t_{11}) + e^{i\varphi} I \otimes \pi_0(t_{22}) \pi_0(t_{21}) \\ &= -q^{-1} C_4 S \otimes S^* C_2 D + e^{i\varphi} I \otimes C_2 S D\end{aligned}$$

$$\begin{aligned}(F_\varphi \otimes \pi_0) \circ \Delta(z_{22}) &= q^{-1} \rho_F(z) \otimes \pi_0(t_{12})^2 + e^{i\varphi} I \otimes \pi_0(t_{22})^2 \\ &= q C_4 S \otimes D^2 + e^{i\varphi} I \otimes C_2 S C_2 S\end{aligned}$$

and

$$\begin{aligned}(\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(z_{11}) &= q^{-1} e^{i\varphi_1} S^* C_2 S^* C_2 + e^{i\varphi_2} D^2 \\ (\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(z_{21}) &= -q^{-1} e^{i\varphi_1} S^* C_2 D + e^{i\varphi_2} C_2 S D \\ (\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(z_{22}) &= q e^{i\varphi_1} D^2 + e^{i\varphi_2} C_2 S C_2 S.\end{aligned} \tag{5.2}$$

Lemma 5.3. *The representation* $(F_\varphi \otimes \pi_0) \circ \Delta$*,* $\varphi \in [0, 2\pi)$*, is irreducible and unitarily equivalent to* τ_φ*.*

Proof. In order to prove this result, we shall utilize the properties of *coherent representations* of $*$-algebras allowing Wick ordering, which were investigated in [JSW95]. Define $\Omega = e_0 \otimes e_0$. It is straightforward to verify that Ω is cyclic for all representations τ_φ and $(F_\varphi \otimes \pi_0) \circ \Delta$, $\varphi \in [0, 2\pi)$, and

$$\begin{aligned}\tau_\varphi(z_{11})^* \Omega &= (F_\varphi \otimes \pi_0) \circ \Delta(z_{11})^* \Omega = e^{-i\varphi} \Omega \\ \tau_\varphi(z_{21})^* \Omega &= (F_\varphi \otimes \pi_0) \circ \Delta(z_{21})^* \Omega = 0 \\ \tau_\varphi(z_{22})^* \Omega &= (F_\varphi \otimes \pi_0) \circ \Delta(z_{22})^* \Omega = 0.\end{aligned}$$

Therefore both τ_φ and $(F_\varphi \otimes \pi_0) \circ \Delta$ determine coherent representations of the Wick algebra corresponding to $P(\mathrm{Sym}_2)_q$ with equal coherent state. Since a coherent representation of a Wick algebra is irreducible and unique up to unitary equivalence, this proves the lemma. $\square$

This observation will allow us to prove the following result, which will reveal a lot of information of the structure of $C(\bar{\mathbb{D}}_2)_q$. We will show that the Fock representation is in fact faithful, and as a consequence it follows that $C(\bar{\mathbb{D}}_2)_q$ is $*$-isomorphic to the C^*-algebra generated by the Fock representation.

The key observation to the proof is that the above lemma allows us to construct $*$-homomorphisms from $C^*(\pi_F(P(\mathrm{Sym}_2)_q))$ into $C^*(\pi(P(\mathrm{Sym}_2)_q))$, where π is any irreducible representation of $P(\mathrm{Sym}_2)_q$. This will imply $\|\pi(x)\| \leq \|\pi_F(x)\|$, from which the result follows.

Theorem 5.4. *The Fock representation π_F of $C(\bar{\mathbb{D}}_2)_q$ is faithful, and consequently $C(\bar{\mathbb{D}}_2)_q$ is $*$-isomorphic to $C^*(\pi_F(P(\mathrm{Sym}_2)_q))$.*

Proof. Consider the closed subspace H of $L^2(\mathbb{T})$ spanned by the orthonormal basis $\{z^n : n \geq 0\}$, and let T_z be the operator on H defined by $T_z h = zh$, $h \in H$. Since $T_z e_k = e_{k+1}$, we see that T_z is unitarily equivalent to S. Thus for $\varphi \in [0, 2\pi)$, we can define a $*$-homomorphism $\Theta_\varphi : C^*(S) \to \mathbb{C}$ by $\Theta_\varphi(S) = e^{i\varphi}$ which is given by the composition

$$C^*(S) \longrightarrow C(\mathbb{T}) \xrightarrow{\varphi} \mathbb{T},$$

where the last arrow corresponds to evaluation at φ.

The operators in (5.1) satisfy

$$\begin{aligned} C_n^2 &= (1 - q^n) \sum_{k=0}^{\infty} q^{nk} S^{k+1} (S^*)^{k+1} \\ D &= \sum_{k=0}^{\infty} q^k \left(S^k (S^*)^k - S^{k+1} (S^*)^{k+1} \right), \end{aligned} \tag{5.3}$$

and hence $C_n, D \in C^*(S)$. Moreover, we have $\Theta_\varphi(C_n) = 1$ and $\Theta_\varphi(D) = 0$.

We note that $C^*(\pi_F(P(\mathrm{Sym}_2)_q)) \subset C^*(S)^{\otimes 3}$ and similarly for the other representations. By letting Θ_φ act on the last factor in the tensor products, we get the $*$-homomorphisms

$$C^*(\pi_F(P(\mathrm{Sym}_2)_q)) \xrightarrow{I \otimes I \otimes \Theta_\varphi} C^*(\tau_\varphi(P(\mathrm{Sym}_2)_q)) \xrightarrow{I \otimes \Theta_\varphi} C^*(\omega_\varphi(P(\mathrm{Sym}_2)_q)).$$

By letting Θ_0 act on the last factor in the tensor product for $(F_\varphi \otimes \pi_0) \circ \Delta$, by Lemma 5.3, we get a $*$-homomorphism

$$C^*(\tau_\varphi(P(\mathrm{Sym}_2)_q)) \to C^*((F_\varphi \otimes \pi_0) \circ \Delta(P(\mathrm{Sym}_2)_q))) \xrightarrow{I \otimes \Theta_0} C^*(\nu_\varphi(P(\mathrm{Sym}_2)_q)),$$

and by letting Θ_{φ_1} act on ν_{φ_2}, $\varphi_1, \varphi_2 \in [0, 2\pi)$, we get a $*$-homomorphism

$$C^*(\nu_{\varphi_2}(P(\mathrm{Sym}_2)_q)) \xrightarrow{\Theta_{\varphi_1}} C^*(\theta_{\varphi_1,\varphi_2}(P(\mathrm{Sym}_2)_q)).$$

Thus for all $x \in P(\mathrm{Sym}_2)_q$ and all irreducible representations π of $P(\mathrm{Sym}_2)_q$, $\|\pi(x)\| \leq \|\pi_F(x)\|$ since $*$-homomorphisms of C^*-algebras are contractive. By the definition of $C(\bar{\mathbb{D}}_2)_q$, it readily follows that the $*$-homomorphism

$$\pi_F : C(\bar{\mathbb{D}}_2)_q \to C^*(\pi_F(P(\mathrm{Sym}_2)_q))$$

is a $*$-isomorphism. □

5.3 The Shilov boundary for $A(\mathbb{D}_2)_q$

We are now ready to state and prove the main result of this chapter.

Theorem 5.5. *Let J be the closed ideal in $C(\bar{\mathbb{D}}_2)_q$ generated by*

$$\begin{aligned} q^2 z_{11} z_{11}^* + q^4 z_{21} z_{21}^* - 1 \\ z_{21} z_{21}^* + z_{22} z_{22}^* - 1 \\ z_{21} z_{11}^* + q z_{22} z_{21}^*. \end{aligned}$$

Then J is the Shilov boundary for $A(\mathbb{D}_2)_q$.

From the above discussion of representations of $P(\mathrm{Sym}_2)_q$, we have the following result on which representations annihilate the ideal J, whose proof is a straightforward verification.

Lemma 5.6. *The representations ω_φ and $\theta_{\varphi_1,\varphi_2}$, $\varphi, \varphi_1, \varphi_2 \in [0, 2\pi)$, are the only irreducible representations of $P(\mathrm{Sym}_2)_q$ that annihilate the ideal J. Moreover, any representation $(\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta$, $\varphi_1, \varphi_2 \in [0, 2\pi)$, annihilates J.*

We can now prove the first part of Theorem 5.5.

Theorem 5.7. *The ideal J is a boundary ideal for $A(\mathbb{D}_2)_q$.*

Proof. Since $C(\bar{\mathbb{D}}_2)_q/J$ is a noncommutative analog of the C^*-algebra of continuous functions on the Shilov boundary, we use the suggestive notation $C(S(\bar{\mathbb{D}}_2))_q = C(\bar{\mathbb{D}}_2)_q/J$. By the previous lemma, any representation $(\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta$, $\varphi_1, \varphi_2 \in [0, 2\pi)$, annihilates J. Thus we have a family of $*$-homomorphisms

$$C(S(\bar{\mathbb{D}}_2))_q \to C^*((\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(P(\mathrm{Sym}_2)_q))$$

given by $b + J \mapsto (\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(b)$, and consequently

$$\sup_{\varphi_1,\varphi_2 \in [0,2\pi)} \|((\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(b_{ij}))\| \leq \|(b_{ij} + J)\|$$

for all $(b_{ij}) \in M_n(C(\bar{\mathbb{D}}_2)_q)$. Since the quotient map $q : C(\bar{\mathbb{D}}_2)_q \to C(S(\bar{\mathbb{D}}_2))_q$ is a $*$-homomorphism, q and consequently $q|_{A(\mathbb{D}_2)_q}$ is a complete contraction. It is therefore sufficient to prove that

$$\|(a_{ij})\| = \|(\pi_F(a_{ij}))\| \leq \sup_{\varphi_1,\varphi_2 \in [0,2\pi)} \|((\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(a_{ij}))\|$$

for all $(a_{ij}) \in M_n(A(\mathbb{D}_2)_q)$.

We note that the operator C_4S is a contraction on $H = \ell^2(\mathbb{Z}_{\geq 0})$. By Sz.-Nagy's dilation theorem, there exists a unitary operator U on a Hilbert space K containing H as a subspace such that $(C_4S)^n = P_H U^n|_H$ for all $n \geq 0$. Consider the map Ψ into $\mathcal{B}(H^{\otimes 2} \otimes K)$ defined on the generators of $P(\mathrm{Sym}_2)_q$ by

$$\begin{aligned}
\Psi(z_{11}) &= I \otimes D^2 \otimes U - q^{-1}S^*C_4 \otimes C_2SC_2S \otimes I \\
\Psi(z_{21}) &= D^2 \otimes C_2S \otimes I \\
\Psi(z_{22}) &= C_4S \otimes I \otimes I.
\end{aligned}$$

It is readily verified that this map extends uniquely to a representation of $P(\mathrm{Sym}_2)_q$ on $H^{\otimes 2} \otimes K$. By the spectral theorem, Ψ can be written as a direct integral representation of the field of representations $\{\tau_\varphi : \varphi \in [0, 2\pi)\}$, i.e.,

$$\Psi = \int_{[0,2\pi)}^{\oplus} \tau_\varphi \otimes I_\varphi \, d\mu(\varphi).$$

For $\xi \in H^{\otimes 2} \otimes K$, we have

$$\|\Psi(b)\xi\|^2 = \int_0^{2\pi} \|\tau_\varphi \otimes I_\varphi(b)\xi(\varphi)\|^2 \, d\mu(\varphi) \leq \sup_{\varphi \in [0,2\pi)} \|\tau_\varphi(b)\|^2 \|\xi\|^2.$$

Thus $\|\Psi(b)\| \leq \sup_{\varphi\in[0,2\pi)} \|\tau_\varphi(b)\|$ for all $b \in C(\bar{\mathbb{D}}_2)_q$, and since Ψ induces a representation on $M_n(C(\bar{\mathbb{D}}_2)_q)$, similar arguments show that

$$\|(\Psi(b_{ij}))\| \leq \sup_{\varphi\in[0,2\pi)} \|(\tau_\varphi(b_{ij}))\|$$

for all $(b_{ij}) \in M_n(C(\bar{\mathbb{D}}_2)_q)$. Since $\pi_F(a) = (I \otimes I \otimes P_H)\Psi(a)|_{H^{\otimes 3}}$, we get

$$\|(\pi_F(a_{ij}))\| \leq \sup_{\varphi\in[0,2\pi)} \|(\tau_\varphi(a_{ij}))\| \tag{5.4}$$

for all $(a_{ij}) \in M_n(A(\mathbb{D}_2)_q)$.

Our next step is to show that, for all $\varphi \in [0, 2\pi)$,

$$\|(\tau_\varphi(a_{ij}))\| \leq \sup_{\varphi_1,\varphi_2\in[0,2\pi)} \|((\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(a_{ij}))\|$$

for all $(a_{ij}) \in M_n(A(\mathbb{D}_2)_q)$. Similar to the previous step, we consider the map Ψ_φ into $\mathcal{B}(K \otimes H)$ defined on the generators of $P(\mathrm{Sym}_2)_q$ by

$$\begin{aligned}
\Psi_\varphi(z_{11}) &= q^{-1}U \otimes S^*C_2S^*C_2 + e^{i\varphi}I \otimes D^2 \\
\Psi_\varphi(z_{21}) &= -q^{-1}U \otimes S^*C_2D + e^{i\varphi}I \otimes C_2SD \\
\Psi_\varphi(z_{22}) &= qU \otimes D^2 + e^{i\varphi}I \otimes C_2SC_2S.
\end{aligned}$$

It is readily verified that Ψ_φ extends to a representation of $C(\bar{\mathbb{D}}_2)_q$ on $K \otimes H$. By (5.2) and the spectral theorem, Ψ_φ can be written as a direct integral representation of the field of representations $\{(\chi_{\varphi_1,\varphi} \otimes \pi_0) \circ \Delta : \varphi_1 \in [0, 2\pi)\}$, i.e.,

$$\Psi_\varphi = \int_{\varphi_1\in[0,2\pi)}^{\oplus} (\chi_{\varphi_1,\varphi} \otimes \pi_0) \circ \Delta \otimes I_{\varphi_1}\, d\mu(\varphi_1).$$

For $\xi \in K \otimes H$, we have

$$\begin{aligned}
\|\Psi_\varphi(b)\xi\|^2 &= \int_0^{2\pi} \|(\chi_{\varphi_1,\varphi} \otimes \pi_0) \circ \Delta \otimes I_{\varphi_1}(b)\xi(\varphi_1)\|^2\, d\mu(\varphi_1) \\
&\leq \sup_{\varphi_1\in[0,2\pi)} \|(\chi_{\varphi_1,\varphi} \otimes \pi_0) \circ \Delta(b)\|^2 \int_0^{2\pi} \|\xi(\varphi_1)\|^2\, d\mu(\varphi_1) \\
&= \sup_{\varphi_1\in[0,2\pi)} \|(\chi_{\varphi_1,\varphi} \otimes \pi_0) \circ \Delta(b)\|^2 \|\xi\|^2.
\end{aligned}$$

Thus

$$\|\Psi_\varphi(b)\| \leq \sup_{\varphi_1\in[0,2\pi)} \|(\chi_{\varphi_1,\varphi} \otimes \pi_0) \circ \Delta(b)\|$$

for all $b \in C(\bar{\mathbb{D}}_2)_q$. Since Ψ induces a representation on $M_n(C(\bar{\mathbb{D}}_2)_q)$, similar arguments show that

$$\|(\Psi_\varphi(b_{ij}))\| \leq \sup_{\varphi_1 \in [0,2\pi)} \|((\chi_{\varphi_1,\varphi} \otimes \pi_0) \circ \Delta(b_{ij}))\|$$

for all $(b_{ij}) \in M_n(C(\bar{\mathbb{D}}_2)_q)$. Since

$$(F_\varphi \otimes \pi_0) \circ \Delta(a) = (P_H \otimes I)\Psi_\varphi(a)|_{H^{\otimes 2}}$$

and

$$\|\tau_\varphi(a)\| = \|(F_\varphi \otimes \pi_0) \circ \Delta(a)\|$$

for all $a \in A(\mathbb{D}_2)_q$, we have

$$\|\tau_\varphi(a)\| \leq \|\Psi_\varphi(a)\| \leq \sup_{\varphi_1,\varphi_2 \in [0,2\pi)} \|(\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(a)\|.$$

By a similar argument, we have

$$\|(\tau_\varphi(a_{ij}))\| \leq \sup_{\varphi_1,\varphi_2 \in [0,2\pi)} \|((\chi_{\varphi_1,\varphi_2} \otimes \pi_0) \circ \Delta(a_{ij}))\| \tag{5.5}$$

for all $(a_{ij}) \in M_n(A(\mathbb{D}_2)_q)$. By combining the inequalities (5.4) and (5.5), we get the desired statement. □

Lemma 5.8. *If π is a representation of $P(\mathrm{Sym}_2)_q$ that annihilates J, then*

$$\|\pi(x)\| \leq \sup_{\varphi \in [0,2\pi)} \|\omega_\varphi(x)\|$$

for all $x \in P(\mathrm{Sym}_2)_q$.

Proof. Let x be an element in $P(\mathrm{Sym}_2)_q$. Without loss of generality, we may assume that x is self-adjoint. Let g be a pure state such that $|g(\pi(x))| = \|\pi(x)\|$, and let π_g and ξ_g be the irreducible representation and unit vector obtained from the GNS construction applied to g. This gives

$$\|\pi(x)\| = |g(\pi(x))| = |\langle \pi_g \circ \pi(x)\xi_g, \xi_g\rangle| \leq \|\pi_g \circ \pi(x)\|,$$

and since $\pi_g \circ \pi$ is an irreducible representation that annihilates J, $\pi_g \circ \pi$ is unitarily equivalent to either ω_φ for some $\varphi \in [0, 2\pi)$ or $\theta_{\varphi_1,\varphi_2}$ for some $\varphi_1, \varphi_2 \in [0, 2\pi)$. Thus it is sufficient to prove that

$$|\theta_{\varphi_1,\varphi_2}(x)| \leq \sup_{\varphi \in [0,2\pi)} \|\omega_\varphi(x)\|$$

for all $x \in P(\mathrm{Sym}_2)_q$ and $\varphi_1, \varphi_2 \in [0, 2\pi)$.

It is readily verified that Θ_{φ_2} induces a $*$-homomorphism

$$C^*(\omega_{(\varphi_1+\varphi_2+\pi)/2}(P(\mathrm{Sym}_2)_q)) \to C^*(\theta_{\varphi_1,\varphi_2}(P(\mathrm{Sym}_2)_q)),$$

where each generator $\omega_{(\varphi_1+\varphi_2+\pi)/2}(z_{ij})$ is mapped to $\theta_{\varphi_1,\varphi_2}(z_{ij})$. Thus

$$|\theta_{\varphi_1,\varphi_2}(x)| \leq \|\omega_{(\varphi_1+\varphi_2+\pi)/2}(x)\| \leq \sup_{\varphi \in [0,2\pi)} \|\omega_\varphi(x)\|$$

for all $x \in P(\mathrm{Sym}_2)_q$ and $\varphi_1, \varphi_2 \in [0, 2\pi)$, which proves the lemma. □

Theorem 5.9. *The ideal J contains all other boundary ideals.*

Proof. Let I be a boundary ideal such that $I \supset J$. Since any representation ω_φ, $\varphi \in [0, 2\pi)$, annihilates J, ω_φ induces a representation $\dot{\omega}_\varphi$ of $C(S(\bar{\mathbb{D}}_2))_q$ given by $\dot{\omega}_\varphi(z_{ij} + J) = \omega_\varphi(z_{ij})$. Define

$$K = \{\varphi \in [0, 2\pi) : \omega_\varphi(I) = 0\}.$$

Since $I/J \subset \cap_{\varphi \in K} \operatorname{Ker} \dot{\omega}_\varphi$, we have $I = J$ if

$$\bigcap_{\varphi \in K} \operatorname{Ker} \dot{\omega}_\varphi = \{0\}.$$

We claim that it is sufficient to prove that K is dense in $[0, 2\pi)$. Indeed, suppose that x lies in $\operatorname{Ker} \omega_\varphi$ for all $\varphi \in K$. If K is dense in $[0, 2\pi)$, it follows by Lemma 5.8 that $\pi(x) = 0$ for all representations π that annihilate J. If we identify $C(S(\bar{\mathbb{D}}_2))_q$ with a concrete C^*-algebra of operators, then the quotient map $q : C(\bar{\mathbb{D}}_2)_q \to C(S(\bar{\mathbb{D}}_2))_q$ can be considered as a representation of $C(\bar{\mathbb{D}}_2)_q$ that annihilates J, which implies $q(x) = 0$, and hence $x \in J$.

Since the quotient maps $q : C(\bar{\mathbb{D}}_2)_q \to C(S(\bar{\mathbb{D}}_2))_q$ and $r : C(\bar{\mathbb{D}}_2)_q \to C(\bar{\mathbb{D}}_2)_q/I \cong C(S(\bar{\mathbb{D}}_2))_q/(I/J)$ are isometries when restricted to $A(\mathbb{D}_2)_q$, we have that for all $a \in A(\mathbb{D}_2)_q$,

$$\|a + J\| = \|a\| = \|a + J + I/J\|, \tag{5.6}$$

and hence the quotient map $s : C(S(\bar{\mathbb{D}}_2))_q \to C(S(\bar{\mathbb{D}}_2))_q/(I/J)$ is an isometry when restricted to $A(\mathbb{D}_2)_q/J$.

From the list of representations it is easy to see that $\|z_{21} + J\| \leq 1$. Thus for a holomorphic function $f \in A(\mathbb{D}_2)$, we have $f(z_{21} + J) \in A(\mathbb{D}_2)_q/J$, and hence, by (5.6),

$$\|f(z_{21} + J)\| = \|f(z_{21} + J + I/J)\|. \tag{5.7}$$

Since ω_φ and $\theta_{\varphi_1,\varphi_2}$, $\varphi, \varphi_1, \varphi_2 \in [0, 2\pi)$, are the only irreducible representations that annihilate J, Lemma 5.8 gives

$$\|f(z_{21} + J)\| = \sup_{\varphi\in[0,2\pi)} \|f(\omega_\varphi(z_{21}))\| =$$
$$= \sup \left\{ |f(\zeta)| : \zeta \in \bigcup_{k\geq 0} q^{2k}\mathbb{T} \right\} = \sup_{\zeta\in\mathbb{T}} |f(\zeta)| = \sup_{\zeta\in\bar{\mathbb{D}}} |f(\zeta)|,$$

where the last two equalities follows from the maximum modulus principle.

If π is an irreducible representation of $C(S(\bar{\mathbb{D}}_2))_q/(I/J)$ which does not vanish on $z_{21}+J+I/J$, then $\pi\circ s$ is an irreducible representation of $C(S(\bar{\mathbb{D}}_2))_q$ which does not vanish on $z_{21} + J$. Since $\pi \circ s(I/J) = 0$, $\pi \circ s$ is unitarily equivalent to $\dot{\omega}_\varphi$ for some $\varphi \in K$. Thus

$$\|f(z_{21} + J + I/J)\| =$$
$$= \sup_\pi \{\|f(\pi \circ s(z_{21}))\|\} = \max \left\{ \sup_{\varphi\in K} \|f(\omega_\varphi(z_{21}))\|,\ |f(0)| \right\}$$
$$= \sup \left\{ |f(\zeta)| : \zeta \in \bigcup_{k\geq 0} q^{2k} X_K \right\},$$

where π ranges over the irreducible representations of $C(S(\bar{\mathbb{D}}_2))_q/(I/J)$ and $X_K = \{e^{i\varphi} : \varphi \in K\}$. By (5.7),

$$\sup \left\{ |f(\zeta)| : \zeta \in \bigcup_{k\geq 0} q^{2k} X_K \right\} = \sup_{\zeta\in\bar{\mathbb{D}}} |f(\zeta)|,$$

and hence, by the maximum modulus principle, $\overline{\cup_{k\geq 0} q^{2k} X_K}$ contains $\mathbb{T}$. Therefore $\mathbb{T} \subset \overline{X}_K \subset \mathbb{T}$, and thus K is dense in $[0, 2\pi)$. □

Theorem 5.7 and 5.9 together proves Theorem 5.5.

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