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CDS INDEX OPTIONS UNDER INCOMPLETE INFORMATION

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ABSTRACT. We derive practical formulas for CDS index spreads in a credit risk model under incomplete information. The factor process driving the default intensities is not directly observable, and the filtering model of Frey & Schmidt (2012) is used as our setup. In this framework we find a computationally tractable expressions for the payoff of a CDS index option which naturally includes the so-called armageddon correction. A lower bound for the price of the CDS index option is derived and we provide explicit conditions on the strike spread for which this inequality becomes an equality. The bound is computationally feasible and do not depend the noise parameters in the filtering model. We outline how to explicitly compute the quantities involved in the lower bound for the price of the credit index option as well as implement and calibrate this model to market data. A numerical study is performed where we show that the lower bound in our model can be several hundred percent bigger compared with models which assume that the CDS index spreads follows a log-normal process. Also a systematic study is performed in order to understand the impact of various model parameters on CDS index options (and on the index itself).

Keywords: Credit risk; CDS index; CDS index options; intensity-based models; dependence modelling; incomplete information; nonlinear filtering; numerical methods

JEL Classification: G33; G13; C02; C63; G32.

1. INTRODUCTION

The development of liquid markets for synthetic credit index products such as CDS index swaps has led to the creation of derivatives on these products, most notably credit index options, sometimes also denoted CDS index options. Essentially the owner of such an option has the right to enter at the maturity date of the option into a protection buyer position in a swap on the underlying CDS index at a prespecified spread; moreover, upon exercise he obtains the cumulative loss of the index portfolio up to the maturity of the option. Credit index options have gained a lot interest the last turbulent years since they allow investors to hedge themselves against broad movements of CDS index spreads or to trade credit volatility.

To date the pricing and the hedging of these options is largely an unresolved problem. In practice this contract is priced by a fairly ad hoc approach: it is assumed that the loss-adjusted spread of the CDS index at the maturity of the option is lognormally distributed under a martingale measure corresponding to a suitable numeraire, and the price of the option is then computed via the Black formula. Details are described for instance in Morini & Brigo (2011) or Rutkowski & Armstrong (2009). However, beyond convenience there is no justification for the lognormality assumption in the literature. In particular, it is unclear if a dynamic model for the evolution of spreads and credit losses can be constructed that supports the lognormality assumption and the use of the Black formula, and there is no empirical justification for this assumption either.

In this paper we therefore propose a different route for pricing and hedging credit index options, which is based on a full dynamic credit risk model. We use a new, information-based approach to credit risk modelling proposed in Frey & Schmidt (2012) where prices of traded credit derivatives are given by the solution of a nonlinear filtering problem. Frey & Schmidt (2012) solve this problem using the innovations approach to nonlinear filtering and derive in particular the Kushner-Stratonovich SDE describing the dynamics of the filtering probabilities. Moreover, they give interesting theoretical results on the dynamics of the credit spreads and on risk minimizing hedging strategies.

Our paper use the filtering model of Frey & Schmidt (2012) in order to derive computationally practical formulas for a CDS index under the market filtration. The market filtration represents incomplete information since the background factor process driving the default intensities is observed with noise. Furthermore, in this model we derive computationally tractable formula for the payoff of a CDS index option. The formula naturally includes the so-called armageddon correction and is obtained without introducing a change of pricing measure, which is the case in the previous literature, see e.g. in Morini & Brigo (2011) or Rutkowski & Armstrong (2009). We also derive a lower bound for price of the CDS index option and provide explicit conditions on the strike spread for which this inequality becomes an equality. The lower bound is computationally tractable and do not depend on any of the noise parameters in the filtering model. We then outline how to explicitly compute the quantities involved in the lower bound for the price of the credit index option. Furthermore, a systematic study is performed in order to understand the impact of various model parameters on these index options (and on the index itself).

Options on a CDS index have been studied in for example Pedersen (2003), Jackson (2005), Liu & Jäckel (2005), Doctor & Goulden (2007), Rutkowski & Armstrong (2009), Morini & Brigo (2011), Flesaker, Nayakkankuppam & Shkurko (2011) and Martin (2012). In all of these papers it is assumed that either the CDS index spread or the so called loss-adjusted CDS index spread at the maturity of the option is lognormally distributed under a martingale measure corresponding to a suitable numeraire, and the price of the option is then computed via the Black formula. For a nice and compact overview of some of the above mentioned papers, see pp.577-579 in Morini & Brigo (2011).

The idea of using filtering techniques in credit risk modelling to price credit derivatives and defaultable bonds is not new. For example, Capponi & Cvitanic (2009) develops a structural credit risk framework which models the deliberate misreporting by insiders in the firm. In this setting the authors derive formulas for bond and stock prices which lead to a non-linear filtering model. The model is calibrated with Kalman filtering and maximum likelihood methods. The authors then apply their setup to the Parmalat-case and the parameters are calibrated against real data.

The paper Fontana & Runggaldier (2010) considers an intensity based credit risk model where default intensities and interest rates are driven by a partly unobservable factor process. In this setup they state formulas for contingent claims given the filtration generated by the unobservable factor process and the default times. The authors then derive a nonlinear filter system describing the dynamics of the filtering distribution which is needed for pricing the derivatives in their framework. The parameters in the model are obtained via an expected maximum (EM) algorithm which includes solving the nonlinear filter system by using the extended Kalman filter and a linearization of the framework. The model and estimation method is applied on simulated data with successful results.

In Frey & Runggaldier (2010) the authors develops a mathematical framework for handling filtering problems in reduced-form credit risk models.

The rest of the paper is organized as follows. First, in Section 2 we give a brief introduction to how a CDS index works and then present a model independent expression for the so called CDS index spread. Section 2 also introduces options on the CDS index and provides a formula for the payoff such an option which holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. Then, in Section 3 we briefly describe the model used in this paper, originally presented in Frey & Schmidt (2012). Section 4 gives a short recapitulation of the the Kushner-Stratonovich SDE describing the dynamics of the filtering probabilities in the models, where we in particular focus on a homogeneous portfolio. Next, Section 5 describes the main building blocks that will be necessary to find formulas for portfolio credit derivatives such as e.g. the CDS index as well as credit index options. Examples of such building blocks are the conditional survival distribution, the conditional number of defaults and the conditional loss distribution. In Section 6 we use the results from Section 5 to derive computational tractable formulas for the CDS index in the model presented in Section 3. This will be done in a homogeneous portfolio. Continuing, in Section 7 we derive a practical formula for the payoff of a CDS index option in the nonlinear filtering modell. This formula will be used with Monte Carlo simulations in order to find approximations to the price of options on a CDS index in the filtering framework. Further, a lower bound for the price of the CDS index option is derived and we provide explicit conditions on the strike spread for which this inequality becomes an equality. The bound is computationally feasible and do not depend the noise parameters in the filtering model. We then outline how to explicitly compute the quantities involved in the lower bound for the price of the credit index option.

Finally, in Section 8 we discuss how to estimate or calibrate the parameters in the filtering model introduced in Section 3 and also calibrate our model and present different numerical results for prices of options on a CDS index.

2. The CDS index and credit index options

In this section we will discuss the CDS index and options on this index. First, Subsection 2.1 gives a brief introduction to how a CDS index works. Then, in Subsection 2.2 we outline model independent expression for the CDS index spread. Finally, Subsection 2.3 introduces options on the CDS index, sometimes denoted by credit index options, and uses the result form Subsection 2.2 to provide a formula for the payoff such an option which holds for any framework modelling the dynamics of the default times in the underlying credit portfolio.

2.1. **Structure of a CDS index.** Consider a portfolio consisting of m equally weighted obligors. An index Credit Default Swap (often denoted *CDS index* or *index CDS*) for a portfolio of m obligors, entered at time t with maturity T, is a financial contract between a protection buyer $\mathbf{A}$ and protection seller $\mathbf{B}$ with the following structure. The CDS index gives $\mathbf{A}$ protection against all credit losses among the m obligors in the portfolio up to time T where $t < T$. Typically, $T = t + \bar{T}$ for $\bar{T} = 3, 5, 7, 10$ years. More specific, at each default in the portfolio during the period $[t, T]$, $\mathbf{B}$ pays $\mathbf{A}$ the credit suffered loss due to the default. Thus, the accumulated value payed by $\mathbf{B}$ to $\mathbf{A}$ in the period $[t, T]$ is the total credit loss in the portfolio during the period from t to time T. As a compensation for this $\mathbf{A}$ pays $\mathbf{B}$ a fixed fee $S(t, T)$ multiplied what is left in the portfolio at each payment time which are done quarterly in the period $[t, T]$. The fee $S(t, T)$ is set so expected discounted cash-flows between $\mathbf{A}$ and $\mathbf{B}$ is equal at time t and $S(t, T)$ is called the *CDS index spread* with maturity $T - t$. For

$t = 0$ (i.e. "today") so that $T = \bar{T}$ we sometimes denote $S(0, T)$ by $S(T)$ and the quantity $S(T)$ can be observed on a daily basis for standard CDS indexes such as iTraxx Europe and the CDX.NA.IG index, for maturities $T = 3, 5, 7, 10$ years. The quarterly payments from **B** to **A** are done on the IMM dates 20th of March, 20th of June, 20th of September and 20th of December. Standardized indices such as iTraxx are updated twice a year on so called "index-rolls" which takes place on the two IMM dates 20th of March and 20th of September. The most recent rolled CDS index is referred to the "*on-the-run-index*". Indices rolled on previous dates are refereed to as "*off-the-run-indices*". A $\bar{T}$-year on-the-run index issued on 20th of March a given year will mature on 20th of June $\bar{T}$ years later. Similarly, a $\bar{T}$-year on-the-run index issued on 20th of September a given year will mature on 20th of December $\bar{T}$ years later. Thus, the effective protection period will be somewhere between $\bar{T} - 0.25$ and $\bar{T} - 0.25$ years. For example, a 5-year on-the-run CDS index entered on 20th of March will have a maturity of 5.25 years but if it is entered on the 16th of September the same year it will have a maturity of around 4.75 years. As we will see later, these maturity details will play an important role when pricing options on CDS indices. For more on practical details regarding the CDS index, see e.g Markit (2016) or O'Kane (2008).

In order to give a more explicit description of the CDS index spread $S(t, T)$ we need to introduce some further notations and concepts which is done in the next subsection.

2.2. **The CDS index spread.**

In this subsection we give a quantitative description of the CDS index spread. First we need to introduce some notation. Let $(\Omega, \mathcal{G}, \mathbb{Q})$ be the underlying probability space assumed in the rest of this paper. We set $\mathbb{Q}$ to be a risk neutral probability measure which exist (under rather mild condition) if arbitrage possibilities are ruled out. Furthermore, let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtration representing the full market information at each time point t. Consider a portfolio consisting of m equally weighted obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$ adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ and let $\ell_1, \ell_2, \ldots, \ell_m$ be the corresponding individual credit losses at each default time. Typically $\ell_i = (1 - \phi_i)/m$ where ϕ_i is a constant representing the recovery rate for obligor i. The credit loss for this portfolio at time t is then defined as $\sum_{i=1}^m \ell_i 1_{\{\tau_i \leq t\}}$. Similarly, the number of defaults in the portfolio up to time t, denoted by N_t, is $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. Note that if the individual loss is constant and identical for all obligors so that $\ell = \ell_1 = \ell_2 = \ldots = \ell_m$ then the normalized credit loss L_t is given by $L_t = \frac{\ell}{m} N_t$. In the rest of this paper we will assume that the individual loss is constant and identical for all obligors where $1 - \phi = \ell = \ell_1 = \ell_2 = \ldots = \ell_m$ and we therefore have that

$$L_t = \frac{1 - \phi}{m} N_t \quad \text{where} \quad N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}. \tag{2.2.1}$$

Finally, for $t < u$ we let $B(t, u)$ denote the discount factor between t and u, that is $B(t, u) = \frac{B_t}{B_u}$ where B_t is the risk free savings account. Unless explicitly stated, we will assume that the risk free interest rate is constant and given by r so that $B_t = e^{rt}$ and $B(t, u) = e^{-r(u-t)}$.

Let $T > t$ and consider an CDS index entered at time t with maturity T on the portfolio with loss process L_t. In view of the above notation we can now define the (stochastic) discounted payments $V_D(t, T)$ from **A** to **B** during the period $[t, T]$, and $V_P(t, T)$ from **B** to **A** in the timespan $[t, T]$, as follows

$$V_D(t, T) = \int_t^T B(t, s) dL_s \quad \text{and} \quad V_P(t, T) = \frac{1}{4} \sum_{n=n_t}^{\lceil 4T \rceil} B(t, t_n) \left(1 - \frac{N_{t_n}}{m} \right) \tag{2.2.2}$$

where n_t denotes $n_t = \lceil 4t \rceil + 1$ and $t_n = \frac{n}{4}$. We here emphasize that we have dropped the accrued term in $V_P(t, T)$ and also ignored the accrued premium up to the first payment date in $V_P(t, T)$. The expected value of the default and premium legs, conditional on the market information $\mathcal{F}_t$ are given by

$$DL(t, T) = \mathbb{E}\left[V_D(t, T) \mid \mathcal{F}_t \right] \quad \text{and} \quad PV(t, T) = \mathbb{E}\left[V_P(t, T) \mid \mathcal{F}_t \right] \tag{2.2.3}$$

that is

$$DL(t, T) = \mathbb{E}\left[\int_t^T B(t, s) dL_s \,\bigg|\, \mathcal{F}_t \right] \tag{2.2.4}$$

and

$$PV(t, T) = \frac{1}{4} \sum_{n=n_t}^{\lceil 4T \rceil} B(t, t_n) \left(1 - \frac{1}{m} \mathbb{E}\left[N_{t_n} \mid \mathcal{F}_t \right] \right). \tag{2.2.5}$$

In view of structure of a CDS index described in Subsection 2.1, the CDS index spread $S(t, T)$ at time t with maturity T is defined as

$$S(t, T) = \frac{DL(t, T)}{PV(t, T)} \tag{2.2.6}$$

or more explicit, using (2.2.4) and (2.2.5)

$$S(t, T) = \frac{\mathbb{E}\left[\int_t^T B(t, s) dL_s \,\big|\, \mathcal{F}_t \right]}{\frac{1}{4} \sum_{n=n_t}^{\lceil 4T \rceil} B(t, t_n) \left(1 - \frac{1}{m} \mathbb{E}\left[N_{t_n} \mid \mathcal{F}_t \right] \right)}. \tag{2.2.7}$$

The definition of $S(t, T)$ in (2.2.6) is done assuming that not all obligors have defaulted in the portfolio at time t, that is $S(t, T)$ is defined on the event $\{ N_t < m \}$. In the event of a so-called armageddon scenario at time t where $N_t = m$ (i.e. all obligors in the portfolio have defaulted up to time t), we see that the premium leg $V_P(t, T)$ in (2.2.2) is zero at time t, which obviously makes the definition of the spread $S(t, T)$ invalid. Note that for $t = 0$ (i.e. today) the quantity $S(0, T)$ can be observed on a daily basis for standard CDS indexes such as iTraxx Europe and the CDX.NA.IG index, for maturities $T = 3, 5, 7, 10$ years.

We here remark that the outline for the CDS index spread presented in this subsection holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. Consequently, the filtration $\mathcal{F}_t$ used in this subsection can be generated by any credit portfolio model.

2.3. **The CDS index option.** In this subsection we introduce options on the CDS index and discuss how they work. Then we use the result form Subsection 2.2 in order to provide a formula for the payoff of such an option, which holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. First, let us give the definition of a *payer CDS index option*, which is the same as Definition 2.3 in Morini & Brigo (2011) and Definition 2.4 in Rutkowski & Armstrong (2009).

Definition 2.1. A ***payer CDS index option*** (sometimes called a ***put CDS index option***) with strike κ and exercise date t written on a CDS index with maturity T is a financial derivative which gives the protection buyer **A** the right but not the obligation to enter the CDS index with the protection seller **B** at time t with a fixed spread κ and protection period $T - t$. Moreover, at the exercise date t, the protection seller **B** also pays **A** the accumulated credit loss occurred during the period from the inception time of the option (at time 0, i.e.

"today") to the exercise date t, that is **B** pays **A** the loss L_t at time t, which is referred to as the front end protection.

The payoff $\Pi(t, T; \kappa)$ at the exercise time t for a payer CDS index option seen from the protection buyer **A**'s point of view, is given by

$$\Pi(t, T; \kappa) = \left(PV(t, T)\left(S(t, T) - \kappa\right)1_{\{N_t < m\}} + L_t\right)^+ \tag{2.3.1}$$

where $PV(t, T)$ is defined as in (2.2.5). For an analogues expression of (2.3.1), see e.g. Equation (2.18) on p.1045 in Rutkowski & Armstrong (2009) or Equation (2.3) on p.577 in Morini & Brigo (2011). Note that the CDS index at time t is entered only if there are any nondefaulted obligors left in the portfolio at time t, which explains the presence of the indicator function of the event $\{N_t < m\}$ in the expression for the payoff $\Pi(t, T; \kappa)$ in (2.3.1). However, the front end protection L_t will be paid out by **A** at time t even if the event $\{N_t = m\}$ occurs. From (2.2.6) we have that

$$PV(t, T)\left(S(t, T) - \kappa\right)1_{\{N_t < m\}} = DL(t, T)1_{\{N_t < m\}} - \kappa PV(t, T)1_{\{N_t < m\}}. \tag{2.3.2}$$

However, since N_t is a non-decreasing process where $N_t \leq m$ almost surely for all $t \geq 0$ we have from the definitions in (2.2.4) and (2.2.5) that

$$DL(t, T)1_{\{N_t = m\}} = \mathbb{E}\left[\left.\int_t^T B(t, s)dL_s \right| \mathcal{F}_t\right]1_{\{N_t = m\}} = 0 \quad \text{and} \quad PV(t, T)1_{\{N_t = m\}} = 0 \tag{2.3.3}$$

so we can use (2.3.3) to simplify (2.3.2) according to

$$PV(t, T)\left(S(t, T) - \kappa\right)1_{\{N_t < m\}} = DL(t, T) - \kappa PV(t, T). \tag{2.3.4}$$

We here remark that the observations (2.3.3) and (2.3.4) has also been done in Rutkowski & Armstrong (2009) and Morini & Brigo (2011), see e.g Equation (2.6) on p. 1040 in Rutkowski & Armstrong (2009) and Proposition 3.7 on p. 582 in Morini & Brigo (2011). By using (2.3.4) we can rewrite the payoff $\Pi(t, T; \kappa)$ in (2.3.1) as

$$\Pi(t, T; \kappa) = \left(DL(t, T) - \kappa PV(t, T) + L_t\right)^+. \tag{2.3.5}$$

The model outline for payer CDS index option presented in this subsection holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. Consequently, the filtration $\mathcal{F}_t$ used in this subsection can be generated by any credit portfolio model.

Before ending this section we briefly discuss some properties of CDS index options that are not shared with e.g. standard equity options. First, we note that (2.3.1) or (2.3.5) implies that

$$\lim_{\kappa \to \infty} \Pi(t, T; \kappa)1_{\{N_t < m\}} = 0. \tag{2.3.6}$$

Secondly, since the individual loss $1 - \phi$ is constant and identical for all obligors and since $L_t = \frac{(1-\phi)N_t}{m}$, we have $L_t 1_{\{N_t = m\}} = (1 - \phi)1_{\{N_t = m\}}$ which in (2.3.5) together with (2.3.3) implies that

$$\Pi(t, T; \kappa)1_{\{N_t = m\}} = L_t 1_{\{N_t = m\}} = (1 - \phi)1_{\{N_t = m\}} \quad \text{for all} \quad \kappa \tag{2.3.7}$$

and consequently

$$\lim_{\kappa \to \infty} \Pi(t, T; \kappa)1_{\{N_t = m\}} = L_t 1_{\{N_t = m\}} = (1 - \phi)1_{\{N_t = m\}}. \tag{2.3.8}$$

So combining (2.3.6) and (2.3.8) renders

$$\lim_{\kappa \to \infty} \Pi(t, T; \kappa) = (1 - \phi) 1_{\{N_t = m\}} \quad \text{a.s.} \tag{2.3.9}$$

For $s \leq t$, the price $C_s(t, T; \kappa)$ of a payer CDS index option at time s with strike κ and exercise date t written on a CDS index with maturity T, is due to standard risk neutral pricing theory given by

$$C_s(t, T; \kappa) = e^{-r(t-s)} \mathbb{E}\left[\Pi(t, T; \kappa) \,|\, \mathcal{F}_s\right]. \tag{2.3.10}$$

Furthermore, since

$$\Pi(t, T; \kappa) = \Pi(t, T; \kappa) 1_{\{N_t < m\}} + \Pi(t, T; \kappa) 1_{\{N_t = m\}} = \Pi(t, T; \kappa) 1_{\{N_t < m\}} + (1 - \phi) 1_{\{N_t = m\}}$$

then for $s \leq t$, the price $C_s(t, T; \kappa)$ can be expressed as

$$C_s(t, T; \kappa) = e^{-r(t-s)} \mathbb{E}\left[\Pi(t, T; \kappa) 1_{\{N_t < m\}} \,\big|\, \mathcal{F}_s\right] + (1 - \phi) e^{-r(t-s)} \mathbb{Q}\left[N_t = m \,|\, \mathcal{F}_s\right]. \tag{2.3.11}$$

From (2.3.6) and (2.3.8) together with the dominated convergence theorem, we conclude that if $s \leq t$ then

$$\lim_{\kappa \to \infty} C_s(t, T; \kappa) = (1 - \phi) e^{-r(t-s)} \mathbb{Q}\left[N_t = m \,|\, \mathcal{F}_s\right] \tag{2.3.12}$$

which is in line with the results in (2.3.9). Also note that the results in this section holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. In this paper our numerical examples will be performed for $s = 0$ which in (2.3.12) implies that

$$\lim_{\kappa \to \infty} C_0(t, T; \kappa) = (1 - \phi) e^{-rt} \mathbb{Q}\left[N_t = m\right] \tag{2.3.13}$$

Recall that in the standard Black-Scholes model the call option price converges to zero as the strike price converges to infinity but due to the front end protection this will not hold for payer CDS index option, as is clearly seen in Equation (2.3.11), (2.3.12) and (2.3.13).

2.4. **Some previous models for the CDS index option.** In this subsection we will discuss some previously studied models and one of these models will be used as a benchmark to the framework developed in this paper.

Options on a CDS index have been studied in for example Pedersen (2003), Jackson (2005), Liu & Jäckel (2005), Doctor & Goulden (2007), Rutkowski & Armstrong (2009), Morini & Brigo (2011), Flesaker et al. (2011) and Martin (2012). In all of these papers it is assumed that either the CDS index spread or the so called loss-adjusted CDS index spread at the maturity of the option is lognormally distributed under a martingale measure corresponding to a suitable numeraire, and the price of the option is then computed via the Black formula. For a nice and compact overview of some of the above mentioned papers, see pp.577-579 in Morini & Brigo (2011).

We will here give a very brief review of the results in some of these papers since these will introduce formulas that we will use as a comparison when benchmarking with our model presented in Section 7.

As discussed in Morini & Brigo (2011), in the initial market approach for pricing CDS index options, the price $C_s^{IM}(t, T; \kappa)$ at time $s \leq t$ of a payer CDS index option with strike κ and exercise date t written on a CDS index with maturity T, is modelled as (see also e.g. Equation (2.4) in Morini & Brigo (2011))

$$C_s^{IM}(t, T; \kappa) = e^{-r(t-s)} \mathbb{E}\left[V_P(t, T) \,|\, \mathcal{F}_s\right] C^B\left(S(s, T), \kappa, t, \sigma\right) + e^{-r(t-s)} \mathbb{E}\left[L_t \,|\, \mathcal{F}_s\right] \tag{2.4.1}$$

where we have used the same notation as in Subsection 2.3 and where $C^{(B)}(S, K, T, \sigma)$ is the Black-formula, i.e.

$$C^B(S, K, T, \sigma) = SN(d_1) - KN(d_2)$$
$$d_1 = \frac{\ln(S/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} \qquad (2.4.2)$$

and $N(x)$ is the distribution function for a standard normal random variable. As pointed out by Pedersen (2003), and also emphasized in Morini & Brigo (2011), the formula (2.4.1) does not incorporate the front end protection in a correct way given the payoff expression in Equation (2.3.1). To overcome the problem of a wrong inclusions of the front end protection in the option formula, several papers proposed an improvement of the Black-framework, see for example Doctor & Goulden (2007). The idea is to introduce a so called loss-adjusted market index spread defined, see e.g. Equation (2.6) in Morini & Brigo (2011)). More specific, let t be the exercise date for a CDS index option and for $u < t < T$ let $DL_t(u, T)$ and $PV_t(u, T)$ denote

$$DL_t(u, T) = \mathbb{E}\left[B(u, t)V_D(t, T) \,|\, \mathcal{F}_u\right] \text{ and } PV_t(u, T) = \mathbb{E}\left[B(u, t)V_P(t, T) \,|\, \mathcal{F}_u\right] \qquad (2.4.3)$$

where $V_D(t, T)$ and $V_P(t, T)$ are given by (2.2.2). Next, define loss-adjusted market index spread $\tilde{S}_t(u, T)$ for $u \leq t \leq T$ as

$$\tilde{S}_t(u, T) = \frac{DL_t(u, T) + \mathbb{E}\left[B(u, t)L_t \,|\, \mathcal{F}_u\right]}{PV_t(u, T)}. \qquad (2.4.4)$$

Note that if $u = t$ then $B(t, t) = 1$, $PV_t(t, T) = PV(t, T)$ and L_t is $\mathcal{F}_t$-measurable which reduces $\tilde{S}_t(t, T)$ in (2.4.4) to

$$\tilde{S}_t(t, T) = S(t, T) + \frac{L_t}{PV(t, T)} \qquad (2.4.5)$$

where $S(t, T)$ is defined as in (2.2.6). Also, if $t = 0$ then $L_0 = 0$ so (2.4.5) then gives

$$\tilde{S}_0(0, T) = S(0, T) \qquad (2.4.6)$$

which makes perfect sense. The benefit with using the loss-adjusted market index spread $\tilde{S}_t(u, T)$ in (2.4.4) is that payoff $\Pi(t, T; \kappa)$ at the exercise time $t > 0$ for a payer CDS index option as given in (2.3.5) can via (2.4.5) be rewritten as

$$\Pi(t, T; \kappa) = PV(t, T)\left(\tilde{S}_t(t, T) - \kappa\right)^+. \qquad (2.4.7)$$

Hence, by using $PV_t(u, T)$ as a numeraire for $u \leq t \leq T$ and assuming that $\tilde{S}_t(u, T)$ is lognormally distributed under a martingale measure corresponding to the chosen numeraire, one can at time $s \leq t$ price a payer CDS index option with exercise time t via (2.4.7) and the Black formula according to

$$\widetilde{C}_s(t, T; \kappa) = e^{-r(t-s)}\mathbb{E}\left[V_P(t, T) \,|\, \mathcal{F}_s\right]C^B\left(\tilde{S}_t(s, T), \kappa, t, \tilde{\sigma}\right) \qquad (2.4.8)$$

where we assumed a constant interest rate r. Furthermore, $\tilde{\sigma}$ is the constant volatility of the loss-adjusted market index spread $\tilde{S}_t(u, T)$ and the quantity $C^{(B)}(S, K, T, \sigma)$ is the same as in (2.4.2), see also e.g. Equation (2.8) on p.578 in Morini & Brigo (2011).

Remark 2.2. As pointed out on pp.578-579 in Morini & Brigo (2011), there are three main problems with the formula (2.4.8) and the definition of the loss-adjusted market index spread in (2.4.4). The first problem is that loss-adjusted market index spread $\tilde{S}_t(u,T)$ in (2.4.4) is not defined when $PV_t(u,T) = 0$, i.e. when $N_u = m$. The second problem is that when $PV_t(u,T) = 0$, the formula (2.4.8) is undefined and will not be consistent with the expression in (2.3.12) which must holds for any framework modelling the dynamics of the default times in the underlying credit portfolio for the CDS index. The third problem with (2.4.4) is that since $PV_t(u,T) = 0$ on $\{N_u = m\}$ and if $\mathbb{Q}[N_u = m] > 0$ (which is true for most standard portfolio credit models when $u > 0$), then $PV_t(u,T)$ will not be strictly positive a.s. and will therefore as a numeraire not lead to a pricing measure that is equivalent with the risk-neutral pricing measure $\mathbb{Q}$.

Rutkowski & Armstrong (2009) and Morini & Brigo (2011) have independently developed an approach which overcomes the three problems stated in Remark 2.2 connected to the the loss-adjusted market index spread in (2.4.4) and the pricing formula (2.4.8). The main ideas in Rutkowski & Armstrong (2009) and Morini & Brigo (2011) work as follows (following mainly the notation of Morini & Brigo (2011)). Let $\tau^{(1)} \leq \tau^{(2)} \leq \ldots \leq \tau^{(m)}$ be the ordering of the default times $\tau_1, \tau_2 \ldots, \tau_m$ in the underlying credit portfolio that creates the CDS index. For example, $\tau^{(m)}$ is the maximum of $\{\tau_i\}$, that is

$$\hat{\tau} := \tau^{(m)} = \max\left(\tau_1, \tau_2 \ldots, \tau_m\right) \tag{2.4.9}$$

where we for notational convenience denote $\tau^{(m)}$ by $\hat{\tau}$. So with N_t defined as in previous sections, i.e. $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ we immediately see that

$$\{\hat{\tau} > t\} = \{N_t < m\} \quad \text{and} \quad \{\hat{\tau} \leq t\} = \{N_t = m\}. \tag{2.4.10}$$

Next, both Rutkowski & Armstrong (2009) and Morini & Brigo (2011) assumes the existence of an auxiliary filtration $\hat{\mathcal{H}}_t$ such that underlying full market information $\mathcal{F}_t$ can be decomposed as

$$\mathcal{F}_t = \hat{\mathcal{J}}_t \vee \hat{\mathcal{H}}_t \tag{2.4.11}$$

$$\hat{\mathcal{J}}_t = \sigma\left(\hat{\tau} \leq s; s \leq t\right) \tag{2.4.12}$$

where $\hat{\tau}$ is not a $\hat{\mathcal{H}}_t$-stopping time. Rutkowski & Armstrong (2009) and Morini & Brigo (2011) remarks that one possible construction of (2.4.11)-(2.4.12) is to let $\hat{\mathcal{H}}_t$ be given by

$$\hat{\mathcal{H}}_t = \mathcal{G}_t \vee_{k=1}^{m-1} \mathcal{J}_t^{(k)} \tag{2.4.13}$$

where for each k the filtration $\mathcal{J}_t^{(k)}$ is defined as

$$\mathcal{J}_t^{(k)} = \sigma\left(\tau^{(k)} \leq s; s \leq t\right) \tag{2.4.14}$$

and $\mathcal{G}_t$ in (2.4.13) is a filtration excluding default information, i.e $\mathcal{G}_t$ is the "default free" information. Typically $\mathcal{G}_t$ is a sigma-algebra generated by a d-dimensional stochastic process $(X_t)_{t\geq 0}$ so $\mathcal{G}_t^X = \sigma(X_s; s \leq t)$ where $X_t = (X_{t,1}, X_{t,2}, \ldots, X_{t,d})$ do not contain the random variables $\tau_1, \tau_2 \ldots, \tau_m$ in their dynamics. Such constructions are standard in conditional independent dynamic portfolio credit models, see e.g in Lando (2004) or McNeil, Frey & Embrechts (2005). From the construction in (2.4.11)-(2.4.13) it is clear that $\hat{\tau}$ is not a $\hat{\mathcal{H}}_t$-stopping time. In Remark 3.5 on p.580 in Morini & Brigo (2011) the authors point out that

the construction in (2.4.11)-(2.4.12) may under certain, not unreasonable model assumptions, not be possible to construct. Now, for $u < t < T$ let $\widehat{DL}_t(u,T)$ and $\widehat{PV}_t(u,T)$ denote

$$\widehat{DL}_t(u,T) = \mathbb{E}\left[B(u,t)V_D(t,T) \,|\, \hat{\mathcal{H}}_u\right] \quad \text{and} \quad \widehat{PV}_t(u,T) = \mathbb{E}\left[B(u,t)V_P(t,T) \,|\, \hat{\mathcal{H}}_u\right]$$
$$(2.4.15)$$

where $V_D(t,T)$ and $V_P(t,T)$ are given by (2.2.2). Next, define $\hat{S}_t(u,T)$ as (see Rutkowski & Armstrong (2009) or Morini & Brigo (2011))

$$\hat{S}_t(u,T) = \frac{\widehat{DL}_t(u,T) + \mathbb{E}\left[1_{\{\hat{\tau}>t\}}B(u,t)L_t \,\big|\, \hat{\mathcal{H}}_u\right]}{\widehat{PV}_t(u,T)} \qquad (2.4.16)$$

where t typically is the exercise date for a CDS index option. Furthermore, Morini & Brigo (2011) assumes that

$$\mathbb{Q}\left[\hat{\tau} > s \,|\, \hat{\mathcal{H}}_s\right] > 0 \quad \text{a.s. for any} \quad s > 0 \qquad (2.4.17)$$

and Rutkowski & Armstrong (2009) makes a similar assumption but on a bounded interval for s. The reason for the assumption (2.4.17) is that in the derivations of the formulas for the CDS-index spreads presented in Morini & Brigo (2011) and Rutkowski & Armstrong (2009) the quantity $\mathbb{Q}\left[\hat{\tau} > s \,|\, \hat{\mathcal{H}}_s\right]$ will emerge in the denominator of several expressions. More specific, the choice (2.4.11)-(2.4.12) together with (2.4.17) will for $s \leq t$ make the quantity $\widehat{PV}_t(u,T) = \mathbb{E}\left[B(u,t)V_P(t,T) \,|\, \hat{\mathcal{H}}_u\right]$ to be strictly positive a.s. (see e.g. p.581 in Morini & Brigo (2011)) and can thus be used as a numeraire, which was observed both in Rutkowski & Armstrong (2009) and Morini & Brigo (2011) independently of each other. Furthermore, Morini & Brigo (2011) and Rutkowski & Armstrong (2009) also shows that under the condition (2.4.17) the spread $\hat{S}_t(u,T)$ in (2.4.16) is well defined which thus solves the first and third problem specified in Remark 2.2. By using assumption (2.4.17) together with the assumption that $\hat{S}(u,T)$ in (2.4.16) follows a lognormal distribution under a measure defined via $\widehat{PV}_t(u,T)$, Morini & Brigo (2011) and Rutkowski & Armstrong (2009) prove that for $s \leq t$ the price for a payer CDS index option at time s with exercise date t via (2.4.7) is given by

$$\hat{C}_s(t,T;\kappa) = 1_{\{\hat{\tau}>s\}}e^{-r(t-s)}\mathbb{E}\left[V_P(t,T) \,|\, \mathcal{F}_s\right] C^B\left(\hat{S}_t(s,T), \kappa, t, \hat{\sigma}\right)$$
$$+ \frac{1_{\{\hat{\tau}>s\}}}{\mathbb{Q}\left[\hat{\tau} > s \,|\, \hat{\mathcal{H}}_s\right]}\mathbb{E}\left[1_{\{s<\hat{\tau}\leq t\}}e^{r(t-s)}(1-\phi) \,\Big|\, \hat{\mathcal{H}}_s\right] + 1_{\{\hat{\tau}\leq s\}}(1-\phi)e^{-r(t-s)} \qquad (2.4.18)$$

where $\hat{\sigma}$ is the volatility of $\hat{S}_t(u,T)$ under a suitable measure (see e.g. Corollary 4.3 in Rutkowski & Armstrong (2009)). The quantity $C^{(B)}(S,K,T,\sigma)$ in (2.4.18) is the same as in (2.4.2). We assumed a constant interest rate r while Morini & Brigo (2011) and Rutkowski & Armstrong (2009) allows for a stochastic discount factor in (2.4.18), see e.g. Equation (2.29) in Rutkowski & Armstrong (2009) and Equation (4.1) and (4.4) in Morini & Brigo (2011). We note that if $s > 0$, then the second term in (2.4.18) is nontrivial to compute in practice. However, an important practical case is to compute $\hat{C}_s(t,T;\kappa)$ when $s = 0$, i.e. $\hat{C}_0(t,T;\kappa)$ (the numerical examples in Morini & Brigo (2011) are only done for the case $s = 0$ while Rutkowski & Armstrong (2009) do not provide any numerical examples of their formulas).

So letting $s = 0$ in (2.4.18) implies that $\widehat{C}_0(t, T; \kappa)$ is given by the following expression

$$\widehat{C}_0(t, T; \kappa) = e^{-rt}\mathbb{E}\left[V_P(t, T)\right] C^B\left(\hat{S}_t(0, T), \kappa, t, \hat{\sigma}\right) + e^{-rt}(1 - \phi)\mathbb{Q}\left[N_t = m\right] \qquad (2.4.19)$$

where we used that $\{\hat{\tau} \leq t\} = \{N_t = m\}$. So we clearly see that formula (2.4.19) is consistent with (2.3.13) which must holds for any framework modelling the dynamics of the default times in the underlying credit portfolio for the CDS index. Hence, this solves the second problem pointed out in Remark 2.2. Also note that $\hat{S}_t(0, T)$ will via (2.4.16) simplify to

$$
\begin{aligned}
\hat{S}_t(0, T) &= \frac{\widehat{DL}_t(0, T) + \mathbb{E}\left[1_{\{\hat{\tau} > t\}}B(0, t)L_t\right]}{\widehat{PV}_t(0, T)} \\
&= \frac{DL_t(0, T) + \mathbb{E}\left[1_{\{\hat{\tau} > t\}}B(0, t)L_t\right]}{PV_t(0, T)} \\
&= \frac{DL_t(0, T) + \mathbb{E}\left[B(0, t)L_t\right] - \mathbb{E}\left[1_{\{\hat{\tau} \leq t\}}B(0, t)L_t\right]}{PV_t(0, T)} \\
&= \frac{DL_t(0, T) + \mathbb{E}\left[B(0, t)L_t\right]}{PV_t(0, T)} - \frac{\mathbb{E}\left[1_{\{\hat{\tau} \leq t\}}B(0, t)L_t\right]}{PV_t(0, T)} \\
&= \tilde{S}_t(0, T) - \frac{(1 - \phi)\mathbb{E}\left[B(0, t)1_{\{N_t = m\}}\right]}{PV_t(0, T)}
\end{aligned}
\qquad (2.4.20)
$$

where the second equality follows from (2.4.3) and (2.4.15) with $u = 0$ and last equality is due to the definition of $\tilde{S}_t(u, T)$ in (2.4.4) and the fact that $1_{\{\hat{\tau} \leq t\}}L_t = (1 - \phi)1_{\{N_t = m\}}$. Also note that if $t = 0$ then $1_{\{N_0 = m\}} = 0$ a.s. which together with (2.4.5) gives

$$\hat{S}_t(0, T) = \tilde{S}_0(0, T) = S(0, T) \qquad (2.4.21)$$

which makes perfect sense. Furthermore, if we assume that the interest rate is deterministic we can rewrite (2.4.20) as

$$\hat{S}_t(0, T) = \tilde{S}_t(0, T) - \frac{(1 - \phi)\mathbb{Q}\left[N_t = m\right]}{\mathbb{E}\left[V_P(t, T)\right]} \qquad (2.4.22)$$

where $V_P(t, T)$ is defined in (2.2.2).

There are several numerical issues to be considered in (2.4.19). First, as pointed out on p.1051 in Rutkowski & Armstrong (2009), since the loss adjusted spread $\hat{S}_t(u, T)$ is not directly observable on the market at any time point $u \geq 0$, it is quite challenging to estimate the volatility $\hat{\sigma}$ of $\hat{S}_t(u, T)$ where $\hat{\sigma}$ is used in the Black-formula present in (2.4.19). Secondly, computing the quantity $\mathbb{Q}\left[N_t = m\right]$ for large m (for example, $m = 125$ both in the iTraxx Europe and CDX NAG index) is numerically nontrivial and requires special attention even in simple standard portfolio credit models such as the one-factor Gaussian copula model. Note that $\mathbb{Q}\left[N_t = m\right]$ emerges both in the second term of (2.4.19) aswell as in $\hat{S}_t(0, T)$ used in the Black-formula present in (2.4.19), as seen in (2.4.20) or (2.4.22).

While Rutkowski & Armstrong (2009) do not provide any numerical examples, Morini & Brigo (2011) uses a one-factor Gaussian copula model but do not specify which numerical method they use to compute $\mathbb{Q}\left[N_t = m\right]$. There exists many methods for computing $\mathbb{Q}\left[N_t = k\right]$, $0 \leq k \leq m$, in conditional independent models such as copula models, see for example in Gregory & Laurent (2003) and Gregory & Laurent (2005).

In order to numerically benchmark the CDS index model presented in Section 3-7 against Morini & Brigo (2011), we will also implement the model in Morini & Brigo (2011) using a one-factor Gaussian copula model just as Morini & Brigo (2011) do. Our choice of numerical method when computing $\mathbb{Q}[N_t = m]$ in (2.4.19) and (2.4.22) will be based on the normal approximation of the mixed binomial distribution, similar to the method in Frey, Popp & Weber (2008). To be more specific, for any integer $1 \leq k \leq m$ we use the following approximation for $\mathbb{Q}[N_t \leq k]$ in the one-factor Gaussian copula model

$$\mathbb{Q}[N_t \leq k] \approx \int_{-\infty}^{\infty} N\left(\frac{k + 0.5 - mp_t(z)}{\sqrt{mp_t(z)(1 - p_t(z))}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \quad \text{for } k \leq m \qquad (2.4.23)$$

where $p_t(z)$ is given by

$$p_t(z) = N\left(\frac{N^{-1}(\mathbb{Q}[\tau \leq t]) - \sqrt{\rho}z}{\sqrt{1 - \rho}}\right) \qquad (2.4.24)$$

and $N(x)$ is the distribution function for a standard normal random variable, ρ is the correlation parameters and τ has the same distribution as the exchangeable default times $\{\tau_i\}$ in the underlying credit portfolio, see e.g. Corollary 2.5 in Frey et al. (2008). The term 0.5 in (2.4.23) is a so-called "half-correction" which seem to produce better approximations that the ordinary normal approximation of a binomial distribution. Next, since

$$\mathbb{Q}[N_t = m] = \mathbb{Q}[N_t \leq m] - \mathbb{Q}[N_t \leq m - 1] \qquad (2.4.25)$$

we use (2.4.23) with $k = m - 1$ and $k = m$ in the right hand side of (2.4.25) to retrieve an approximation to the quantity $\mathbb{Q}[N_t = m]$ in (2.4.19) and (2.4.22). Next we need to find an expression for $\mathbb{Q}[\tau \leq t]$ used in (2.4.23) via (2.4.24). A standard assumption made in the homogeneous portfolio credit risk one-factor Gaussian copula model is that the default times $\{\tau_i\}$ have constant default intensity λ, that is they are exponentially distributed with parameter λ, i.e. if τ has the same distribution as $\{\tau_i\}$ then

$$\mathbb{Q}[\tau \leq t] = 1 - e^{-\lambda t} \qquad (2.4.26)$$

where λ is given by

$$\lambda = \frac{S_{\mathcal{M}}(\bar{T})}{1 - \phi} \qquad (2.4.27)$$

and $S_{\mathcal{M}}(\bar{T})$ is the market quote for the $\bar{T}$-year CDS-index spread today and ϕ is the recovery rate. The relation (2.4.27) is the so-called credit triangle, frequently used among market practitioners assuming a "flat" CDS term structure, i.e. assuming that the default intensity will be constant for all time points after t.

A derivation of the relation (2.4.27) in the case with quarterly payments is given in Proposition B.1 in Appendix B, since the existing proofs of (2.4.27) found in the litterature are only done in the unrealistic case when the CDS index premium is paid continuously. In practice the CDS premiums are done quarterly.

Furthermore, note that we have used the CDS index spread $S_{\mathcal{M}}(\bar{T})$ in (2.4.27) because this spread will in a homogeneous credit portfolio be identical to the the individual CDS spread for an obligor in the reference portfolio, see e.g. Proposition Lemma 6.1 in Herbertsson, Jang & Schmidt (2011). This ends the specification of how we compute $\mathbb{Q}[N_t = m]$. In Figure 1 we plot $\mathbb{Q}[N_t = m]$ for $t = 9$ months and $m = 125$ as function of the correlation parameter ρ where we used (2.4.23)-(2.4.27) to compute $\mathbb{Q}[N_t = m]$ with $\phi = 40\%$ and $S_{\mathcal{M}}(5) = 200$ bps. As can be seen in Figure 1, the effect of ρ on $\mathbb{Q}[N_t = m]$ will only come in to play when ρ

is bigger than 95% and for smaller ρ, the armageddon probability $\mathbb{Q}[N_t = m]$ will in practice be neglible, see also Figure 5.1 in Morini & Brigo (2011)
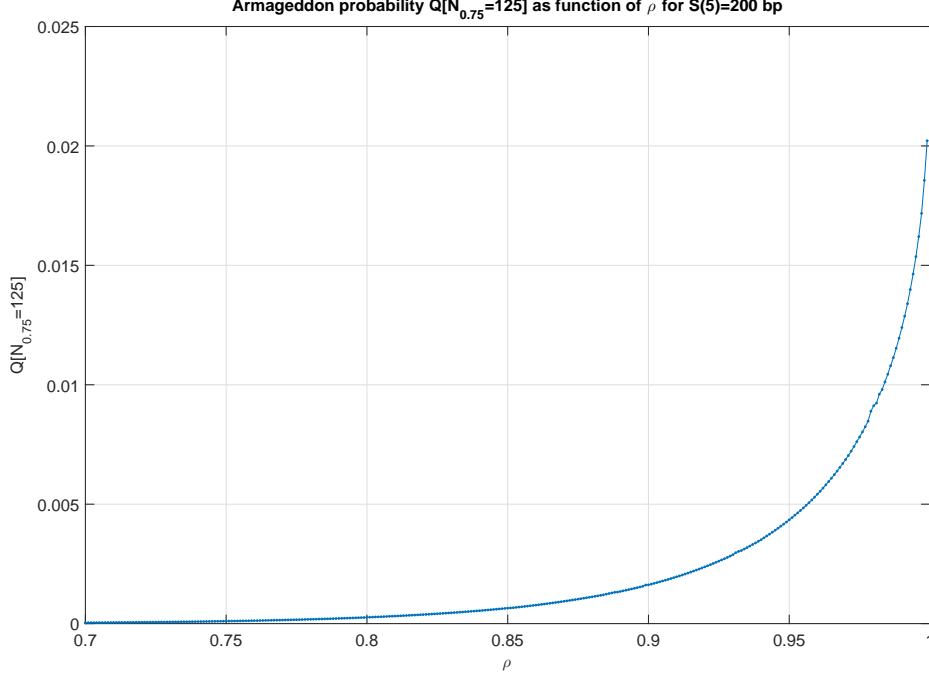
Figure 1. The Armageddon probability $\mathbb{Q}[N_{0.75} = 125]$ as function of the correlation $\rho =$ where $S(0,5) = 200$ and $\phi = 40\%$ bp.

So what is left to compute in (2.4.19) is $\hat{S}_t(0,T)$. This is done in the following proposition.

Proposition 2.3. *Consider a CDS index with maturity T on a homogeneous credit portfolio where the obligors have constant default intensity λ. Then, with notation as above*

$$\hat{S}_t(0,T) = 4(1-\phi)e^{-rt}\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)\frac{\left(\frac{\lambda}{\lambda+r}\left(e^{-(r+\lambda)t} - e^{-(r+\lambda)T}\right) + 1 - e^{-\lambda t} - \mathbb{Q}[N_t = m]\right)}{e^{-\frac{(r+\lambda)n_t}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T \rceil + 1)}{4}}}$$

(2.4.28)

where $n_t = \lceil 4t \rceil + 1$.

Proof. From (2.4.22) we have

$$\hat{S}_t(0,T) = \tilde{S}_t(0,T) - \frac{(1-\phi)\mathbb{Q}[N_t = m]}{\mathbb{E}[V_P(t,T)]}$$

(2.4.29)

so we need explicit expressions for the quantities $\mathbb{E}[V_P(t,T)]$ and $\tilde{S}_t(0,T)$. First, to find $\mathbb{E}[V_P(t,T)]$ we use the exchangeability of the default times $\{\tau_i\}$ all having the same distribution as in (2.4.26), which in the definition of $V_P(t,T)$ given by (2.2.2) with properties for

geometric series and some computations yields

$$\mathbb{E}\left[V_P(t,T)\right] = \frac{e^{rt}\left(e^{-\frac{(r+\lambda)n_t}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}}\right)}{4\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)} \tag{2.4.30}$$

where n_t denotes $n_t = \lceil 4t \rceil + 1$ as in (2.2.2). Next, we provide an explicit expression for $\tilde{S}_t(0,T)$ given by (2.4.4) with $u = 0$ and constant interest rate r, that is

$$\begin{aligned}
\tilde{S}_t(0,T) &= \frac{DL_t(0,T) + e^{-rt}\mathbb{E}\left[L_t\right]}{PV_t(0,T)} \\
&= \frac{DL_t(0,T) + e^{-rt}(1-\phi)\mathbb{Q}\left[\tau \le t\right]}{PV_t(0,T)} \\
&= \frac{\mathbb{E}\left[V_D(t,T)\right]}{\mathbb{E}\left[V_P(t,T)\right]} + \frac{(1-\phi)\mathbb{Q}\left[\tau \le t\right]}{\mathbb{E}\left[V_P(t,T)\right]} \\
&= \frac{e^{rt}\mathbb{E}\left[\int_t^T e^{-rs}dL_s\right]}{\mathbb{E}\left[V_P(t,T)\right]} + \frac{(1-\phi)\mathbb{Q}\left[\tau \le t\right]}{\mathbb{E}\left[V_P(t,T)\right]} \\
&= \frac{(1-\phi)e^{rt}\int_t^T e^{-rs}f_\tau(s)ds}{\mathbb{E}\left[V_P(t,T)\right]} + \frac{(1-\phi)\mathbb{Q}\left[\tau \le t\right]}{\mathbb{E}\left[V_P(t,T)\right]}
\end{aligned} \tag{2.4.31}$$

where the second equality follows the definition of the loss L_t in (2.2.1) together with the exchangeability of the default times $\{\tau_i\}$ all having the same distribution as τ and the third equality comes from the definition of $DL_t(u,T)$ and $PV_t(u,T)$ in (2.4.3) with $u = 0$ using that the interest rate is constant, given by r. The fourth equality is due to the expected value of $V_D(t,T)$ in (2.2.3) and that $B(t,s) = e^{r(s-t)}$ since the interest rate is constant. The last equality in (2.4.31) follows from Equation (6.3.3) in Lemma 6.1, p.1203 in Herbertsson et al. (2011) where $f_\tau(s)$ is the density of the default time τ. So plugging (2.4.31) into (2.4.29) we get that $\hat{S}_t(0,T)$ can be rewritten as

$$\hat{S}_t(0,T) = \frac{1-\phi}{\mathbb{E}\left[V_P(t,T)\right]}\left(e^{rt}\int_t^T e^{-rs}f_\tau(s)ds + \mathbb{Q}\left[\tau \le t\right] - \mathbb{Q}\left[N_t = m\right]\right). \tag{2.4.32}$$

Note that (2.4.32) holds for any distribution of τ, and to make $\hat{S}_t(0,T)$ more explicit we use that τ in this paper (as in most articles treating homogeneous one-factor Gaussian copula models applied to portfolio credit risk) has constant default intensity λ, i.e. τ is exponentially distributed with parameter λ as in (2.4.26) which implies

$$\int_t^T e^{-rs}f_\tau(s)ds = \int_t^T \lambda e^{-(r+\lambda)s}ds = \frac{\lambda}{\lambda+r}\left(e^{-(r+\lambda)t} - e^{-(r+\lambda)T}\right). \tag{2.4.33}$$

So (2.4.26), (2.4.30) and (2.4.33) in (2.4.32) renders an explicit formula for $\hat{S}_t(0,T)$ given by

$$\hat{S}_t(0,T) = 4(1-\phi)e^{-rt}\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)\frac{\left(\frac{\lambda}{\lambda+r}\left(e^{-(r+\lambda)t} - e^{-(r+\lambda)T}\right) + 1 - e^{-\lambda t} - \mathbb{Q}\left[N_t = m\right]\right)}{e^{-\frac{(r+\lambda)n_t}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}}}$$

which concludes the proposition. $\qquad\square$

Note that in the expression for $\hat{S}_t(0,T)$ given by (2.4.28) we will in this paper compute $\mathbb{Q}[N_t = m]$ via the equations (2.4.23)-(2.4.27) as outlined above, and λ will be given by (2.4.27).

In Subsection 8.2 we will use (2.4.19), (2.4.28) and (2.4.23)-(2.4.27) as a benchmark against the model developed in the next sections.

We here remark that Morini & Brigo (2011) do not provide any explicit expression of $\hat{S}_t(0,T)$ given on the form (2.4.28), see e.g. the equation under Table 5.1 on p.589 in Morini & Brigo (2011). But as will be seen in Subsection 8.2, our numerical values for (2.4.19), roughly coincide with those presented in Table 5.1-5.2 in Morini & Brigo (2011). We have not done any numerical benchmark against Rutkowski & Armstrong (2009) since there are no numerical results presented in Rutkowski & Armstrong (2009).

Furthermore, we will also show that the filtering modell presented in this paper will for the same CDS index spread $S(0,T)$ create CDS index option prices that can be several hundred percent, or even several thousands percent bigger (depending on the value of ρ and t and the strike κ) than those given by (2.4.19) with the same CDS index spread $S(0,T)$, and at the same time it will hold that $\mathbb{Q}[N_t = m] = 0$ in the filtering model while $\mathbb{Q}[N_t = m] > 0$ in the one-factor Gaussian copula as used in Morini & Brigo (2011).

3. The model

In this section we shortly recapitulate the model of Frey & Schmidt (2012). Thus, we will consider a reduced-form model driven by an unobservable background factor process X modelling the "true" state of the economy. For tractability reasons X is modelled as finite-state Markov chain. The factor process X is not directly observable. Instead model quantities are given as conditional expectation with respect to the so called market filtration $\mathbb{F}^M = (\mathcal{F}_t^M)_{t \geq 0}$. The filtration $\mathbb{F}^M$ is generated by the factor process X plus noise, which will be specified in detail below. Intuitively speaking, this means that the model quantities are observed given an incomplete history of the state of the economy. Furthermore, in the model of Frey & Schmidt (2012) the default times of all obligors are conditionally independent given the information of the factor process X. This setup is close to the one found in e.g. Graziano & Rogers (2009).

Frey & Schmidt (2012) treat the case with stochastic recoveries in a general theoretical setting. In this paper we will take a simplified approach and only consider deterministic recoveries, which up to the credit crises of 2008-2009 has been considered as standard in the credit literature.

3.1. **The factor process.** In this section we introduce the model that we will consider under the full information.

Let X_t be a finite state continuous time Markov chain on the state space $S^X = \{1, 2, \ldots, K\}$ with generator $\boldsymbol{Q}$. Let $\mathcal{F}_t^X = \sigma(X_s; s \leq t)$ be the filtration generated by the factor process X. Consider m obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$ and let the mappings $\lambda_1, \lambda_2 \ldots, \lambda_m$ be the corresponding $\mathcal{F}_t^X$ default intensities, where $\lambda_i : S^X \mapsto \mathbb{R}^+$ for each obligor i. This means that each default time τ_i is modeled as the first jump of a Cox-process, with intensity $\lambda_i(X_t)$. It is well known (see e.g. Lando (1998)) that given an i.i.d sequence $\{E_i\}$ where E_i is exponentially distributed with parameter one, such that all $\{E_i\}$ are independent of $\mathcal{F}_\infty^X$, then

$$\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_i(X_s) ds \geq E_i \right\}. \tag{3.1.1}$$

Hence, for any $T \geq t$ we have

$$\mathbb{Q}\left[\tau_i > t \,|\, \mathcal{F}_T^X\right] = \exp\left(-\int_0^t \lambda_i(X_s)ds\right) \tag{3.1.2}$$

and thus

$$\mathbb{Q}\left[\tau_i > t\right] = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_i(X_s)ds\right)\right]. \tag{3.1.3}$$

Note that the default times are conditionally independent, given $\mathcal{F}_\infty^X$.

The states in $S^X = \{1, 2, \ldots, K\}$ are ordered so that state 1 represents the best state and K represents the worst state of the economy. Consequently, the mappings $\lambda_i(\cdot)$ are chosen to be strictly increasing in $k \in \{1, 2, \ldots, K\}$, that is $\lambda_i(k) < \lambda_i(k+1)$ for all $k \in \{1, 2, \ldots, K-1\}$ and for every obligor in the portfolio.

3.2. The market filtration and full information.
In this subsection we formally introduce the market filtration, that is the information observed by the market participants. Recall that the prices of all securities are given as conditional expectations with respect to this filtration. We also shortly discuss the full information $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, which is the biggest filtration containing all other filtrations, where $(\Omega, \mathcal{G}, \mathbb{P})$ with $\mathcal{G} = \mathcal{F}_\infty$ will be the underlying probability space assumed in the rest of this paper.

Let $Y_{t,i}$ denote the random variable $Y_{t,i} = 1_{\{\tau_i \leq t\}}$ and Y_t be the vector $Y_t = (Y_{t,1}, \ldots, Y_{t,m})$. The filtration $\mathcal{F}_t^Y = \sigma(Y_s; s \leq t)$ represents the default portfolio information at time t, generated by the process $(Y_s)_{s \geq 0}$. Furthermore, let B_t be a one-dimensional Brownian motion independent of $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ and let $a(\cdot)$ be a function from $\{1, 2, \ldots, K\}$ to $\mathbb{R}$. Next, define the process Z_t as

$$Z_t = \int_0^t a(X_s)ds + B_t. \tag{3.2.1}$$

We here remark that Frey & Schmidt (2012) allows for multivariate Brownian motion B_t in (3.2.1) as well as a vector valued mapping $\boldsymbol{a}(\cdot)$ with same dimension as B_t and in the numerical studies of Frey & Schmidt (2012) they use a one-dimensional Brownian motion B_t. In this paper we restrict ourselves to only one source of randomness in the noise representation (3.2.1). Extending to several sources of randomness in (3.2.1) will in principle not change the main ideas in this paper. Intuitively Z_t represents the noisy history of X_t and the functional form of Z_t given by (3.2.1) is a representation that is standard in the nonlinear filtering theory, see e.g. Davis & Marcus (1981). Following Frey & Schmidt (2012), we define the market filtration $\mathbb{F}^M = (\mathcal{F}_t^M)_{t \geq 0}$ as

$$\mathcal{F}_t^M = \mathcal{F}_t^Y \vee \mathcal{F}_t^Z. \tag{3.2.2}$$

We set the full information $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ to be the biggest filtration containing all other filtrations with $\mathcal{G} = \mathcal{F}_\infty$. We can for example let $\mathcal{F}_t$ be given by

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^Y \vee \mathcal{F}_t^B \tag{3.2.3}$$

where $(\mathcal{F}_t^B)_{t \geq 0}$ is the filtration generated by the Brownian motion B_t. Note that $\mathcal{F}_t^X$ is not a subfiltration of $\mathcal{F}_t^Z$, and similarly, $\mathcal{F}_t^B$ is not contained in $\mathcal{F}_t^Z$.

4. Applying the Kushner-Stratonovic SDE in the credit risk model

In this section we study the Kushner-Stratonovic SDE in our filtering model. We use the same notation as in Frey & Schmidt (2012). First, define π_t^k as the conditional probability of the event $\{X_t = k\}$ given the market information $\mathcal{F}_t^M$ at time t, that is

$$\pi_t^k = \mathbb{Q}\left[X_t = k \,|\, \mathcal{F}_t^M\right] \tag{4.1}$$

and let $\boldsymbol{\pi}_t \in \mathbb{R}^K$ be a row-vector such that $\boldsymbol{\pi}_t = \left(\pi_t^1, \ldots, \pi_t^K\right)$. In the sequel, for any $\mathcal{F}_t$-adapted process U_t we let $\widehat{U_t}$ denote the optional projection of $\widehat{U_t}$ onto the filtration $\mathcal{F}_t^M$, that is $\widehat{U_t} = \mathbb{E}\left[U_t \,|\, \mathcal{F}_t^M\right]$. To this end, we have for example

$$\widehat{\lambda_i(X_t)} = \mathbb{E}\left[\lambda_i(X_t) \,|\, \mathcal{F}_t^M\right] = \sum_{k=1}^K \lambda_i(k)\pi_t^k$$

$$\widehat{a(X_t)} = \mathbb{E}\left[a(X_t) \,|\, \mathcal{F}_t^M\right] = \sum_{k=1}^K a(k)\pi_t^k.$$

Next, define $M_{t,i}$ and μ_t as

$$M_{t,i} = Y_{t,i} - \int_0^{t\wedge\tau_i} \widehat{\lambda_i(X_{s-})}ds \qquad \text{for } i = 1, \ldots, m \tag{4.2}$$

$$\mu_t = Z_t - \int_0^t \widehat{a(X_s)}\, ds$$

In Frey & Schmidt (2012) it is shown that $M_{t,i}$ is an $\mathcal{F}_t^M$-martingale, for $i = 1, 2, \ldots, m$ and that μ_t is a Brownian motion with respect to the filtration $\mathcal{F}_t^M$. Thus, the vector $M_t = (M_{t,1}, \ldots, M_{t,m})$ is an $\mathcal{F}_t^M$-martingale. These results have been proven previously when considered separately, i.e. for pure diffusion filtering problems, see e.g. Davis & Marcus (1981), and pure jump process filtering process, see e.g Brémaud (1981).

Furthermore, Frey & Schmidt (2012) also proves the following proposition, which is a version of the Kushner- Stratonovic equations, adopted to the filtering models presented in this paper (originally developed in Frey & Schmidt (2012)).

Proposition 4.1. *With notation as above, the processes π_t^k satisfies the following K-dimensional system of SDE-s,*

$$d\pi_t^k = \sum_{\ell=1}^K \boldsymbol{Q}_{\ell,k}\pi_t^\ell dt + (\boldsymbol{\gamma}^k(\boldsymbol{\pi}_{t-}))^\top dM_t + \alpha^k(\boldsymbol{\pi}_t)\, d\mu_t\,, \tag{4.3}$$

where $(\boldsymbol{\gamma}^k(\boldsymbol{\pi}))^\top = \left(\gamma_1^k(\boldsymbol{\pi}), \ldots, \gamma_m^k(\boldsymbol{\pi})\right)$ with $\boldsymbol{\pi} = (\pi^1, \pi^2, \ldots, \pi^m)$ and the coefficients $\gamma_i^k(\boldsymbol{\pi})$ are mappings given by

$$\gamma_i^k(\boldsymbol{\pi}) = \pi^k\left(\frac{\lambda_i(k)}{\sum_{n=1}^K \lambda_i(n)\pi^n} - 1\right), \quad 1 \le i \le m \tag{4.4}$$

and

$$\alpha^k(\boldsymbol{\pi}_t) = \pi_t^k\left(a(k) - \sum_{n=1}^K \pi_t^n a(n)\right), \quad 1 \le k \le K. \tag{4.5}$$

The K-dimensional SDE-system partly uses the vector notation for the M_t vector. However, as will be seen below, it will be beneficial to rewrite this SDE on component form, especially when we consider homogeneous credit portfolios. Thus, let us rewrite (4.3) on component form, so that

$$d\pi_t^k = \sum_{\ell=1}^K \boldsymbol{Q}_{\ell,k}\pi_t^\ell dt + \sum_{i=1}^m \gamma_i^k(\boldsymbol{\pi}_{t-})dM_{t,i} + \alpha^k(\boldsymbol{\pi}_t)d\mu_t. \tag{4.6}$$

Next, let us consider a homogeneous credit portfolio, that is, all obligors are exchangeable so that $\lambda_i(X_t) = \lambda(X_t)$ and $\gamma_i^k(\boldsymbol{\pi}_t) = \gamma^k(\boldsymbol{\pi}_t)$ for each obligor i and define N_t as

$$N_t = \sum_{i=1}^m Y_{t,i} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}. \tag{4.7}$$

Furthermore, define $\boldsymbol{\lambda}$ as $\boldsymbol{\lambda} = (\lambda(1), \ldots, \lambda(K))$ and let $\boldsymbol{e}_k \in \mathbb{R}^m$ be a row vector where the entry at position k is 1 and the other entries are zero. For a homogeneous portfolio the results of Proposition 4.1 can be simplified to the following corollary.

Corollary 4.2. *Consider a homogeneous credit portfolio with m obligors. Then, with notation as above, the processes π_t^k satisfy the following K-dimensional system of SDE-s,*

$$d\pi_t^k = \gamma^k(\boldsymbol{\pi}_{t-})dN_t + \boldsymbol{\pi}_{t-}\left(\boldsymbol{Q}\boldsymbol{e}_k^\top - \gamma^k(\boldsymbol{\pi}_{t-})\boldsymbol{\lambda}^\top (m - N_t)\right)dt + \alpha^k(\boldsymbol{\pi}_t)d\mu_t \tag{4.8}$$

where $\gamma^k(\boldsymbol{\pi}_t)$ and $\alpha^k(\boldsymbol{\pi}_t)$ are given by

$$\gamma^k(\boldsymbol{\pi}_t) = \pi_t^k\left(\frac{\lambda(k)}{\boldsymbol{\pi}_t\boldsymbol{\lambda}^\top} - 1\right) \quad and \quad \alpha^k(\boldsymbol{\pi}_t) = \pi_t^k\left(a(k) - \sum_{n=1}^K \pi_t^n a(n)\right). \tag{4.9}$$

Proof. First, from (4.2) we have $dM_{t,i} = dY_{t,i} - 1_{\{\tau_i>t\}}\widehat{\lambda_i(X_t)}dt = dY_{t,i} - 1_{\{\tau_i>t\}}\sum_{k=1}^K \lambda_i(k)\pi_t^k dt$ which in (4.6) implies that

$$d\pi_t^k = \boldsymbol{\pi}_t\boldsymbol{Q}\boldsymbol{e}_k^\top dt + \sum_{i=1}^m \gamma_i^k(\boldsymbol{\pi}_{t-})dY_{t,i} - \sum_{i=1}^m \gamma_i^k(\boldsymbol{\pi}_{t-})1_{\{\tau_i>t\}}\sum_{k=1}^K \lambda_i(k)\pi_t^k dt + \alpha^k(\boldsymbol{\pi}_t)d\mu_t. \tag{4.10}$$

Since $\lambda_i(X_t) = \lambda(X_t)$ and $\gamma_i^k(\boldsymbol{\pi}_t) = \gamma^k(\boldsymbol{\pi}_t)$ for all obligors i, and recalling that N_t denotes $N_t = \sum_{i=1}^m Y_{t,i} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ so that $\sum_{i=1}^m 1_{\{\tau_i>t\}} = m - N_t$, we can after some computations rewrite (4.10) as

$$d\pi_t^k = \gamma^k(\boldsymbol{\pi}_{t-})dN_t + \boldsymbol{\pi}_{t-}\left(\boldsymbol{Q}\boldsymbol{e}_k^\top - \gamma^k(\boldsymbol{\pi}_{t-})\boldsymbol{\lambda}^\top (m - N_t)\right)dt + \alpha^k(\boldsymbol{\pi}_t)d\mu_t$$

where $\gamma^k(\boldsymbol{\pi}_t)$ and $\alpha^k(\boldsymbol{\pi}_t)$ are given by $\gamma^k(\boldsymbol{\pi}_t) = \pi_t^k\left(\frac{\lambda(k)}{\boldsymbol{\pi}_t\boldsymbol{\lambda}^\top} - 1\right)$ and $\alpha^k(\boldsymbol{\pi}_t) = \pi_t^k\left(a(k) - \sum_{n=1}^K \pi_t^n a(n)\right).$ $\square$

From the SDE (4.8) in Corollary 4.2 we clearly see that the dynamics of the conditional probabilities π_t^k contains a drift part, a diffusion part and a jump part. The diffusion part is due to the $d\mu_t$ component and the jump part is due to the defaults in the portfolio, given by the differential dN_t.

Figure 2 visualizes a simulated path of π_t^1 given by (4.8) in Corollary 4.2 in an example where $K = 2$ and $m = 125$, using fictive parameters for $\boldsymbol{Q}$ and $\boldsymbol{\lambda}$ assuming $a(k) = c \cdot \ln \lambda(k)$ for a constant c. From the third Figure 2 we clearly see that π_t^1 has jump, drift and diffusion parts. The first and second subfigures in Figure 2 shows the corresponding trajectories for X_t
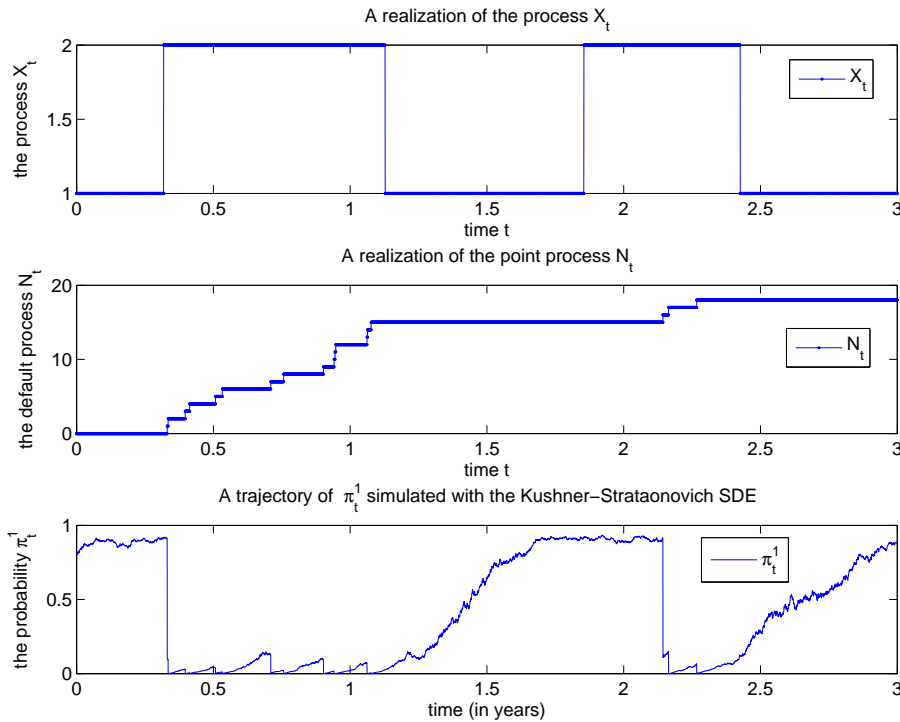
Figure 2. A simulated trajectory of X_t, N_t and π_t^1 where $K = 2$ and $m = 125$.

and N_t. Note how the defaults presented by N_t cluster as X_t switches to state 2, representing the worse economic state among $\{1, 2\}$.

5. THE MAIN BUILDING BLOCKS

In this section we describe the main building blocks that will be necessary to find formulas for portfolio credit derivatives such as e.g. the CDS index. Examples of such building blocks are the conditional survival distribution, the conditional number of defaults and the conditional loss distribution. The conditional expectations are with respect to the market information $\mathcal{F}_t^M$ defined in Equation (3.2.2) in Subsection 3.2. Recall that $Y_{t,i}$ denotes the random variable $Y_{t,i} = 1_{\{\tau_i \leq t\}}$, $Y_t = (Y_{t,1}, \ldots, Y_{t,m})$ and N_t and L_t are given by $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ and $L_t = \frac{1}{m} \sum_{i=1}^m (1 - \phi_i) 1_{\{\tau_i \leq t\}}$ where ϕ_i is the recovery rate for obligor i. Our main task in this section is to find the following quantities

$$\mathbb{Q}\left[\tau_i > T \,|\, \mathcal{F}_t^M\right], \quad \mathbb{E}\left[N_T \,|\, \mathcal{F}_t^M\right] \quad \text{and} \quad \mathbb{E}\left[L_T \,|\, \mathcal{F}_t^M\right]$$

where $T > t$. These expressions will be useful when deriving formulas for the CDS index spread $S(t, T)$ as well as the CDS index option discussed in Section 6.

5.1. **The conditional survival distribution.** In this subsection we study the conditional survival distribution $\mathbb{Q}\left[\tau_i > T \,|\, \mathcal{F}_t^M\right]$ for $T > t$ in the filtering model. To do this we need to introduce some notation. If X_t is a finite state Markov jump process on $S^X = \{1, 2, \ldots, K\}$

with generator $\boldsymbol{Q}$, then, for a function $\lambda(x) : S^X \mapsto \mathbb{R}$ we denote the matrix $\boldsymbol{Q}_\lambda = \boldsymbol{Q} - \boldsymbol{I}_\lambda$ where $\boldsymbol{I}_\lambda$ is a diagonal-matrix such that $(\boldsymbol{I}_\lambda)_{k,k} = \lambda(k)$. Furthermore, we let $\mathbf{1}$ be a column vector in $\mathbb{R}^K$ where all entries are 1. The following theorem is a perquisite for all other results in this paper and is therefore a core result.

Theorem 5.1. *Consider a credit portfolio specified as in Section 3 and let $\lambda_i(X_t)$ be the $\mathcal{F}_t^X$-intensity for obligor i. If $T \geq t$ then, with notation as above*

$$\mathbb{Q}\left[\tau_i > T \,\middle|\, \mathcal{F}_t^M\right] = 1_{\{\tau_i > t\}} \boldsymbol{\pi}_t e^{\boldsymbol{Q}_{\lambda_i}(T-t)} \mathbf{1} \tag{5.1.1}$$

where the matrix $\boldsymbol{Q}_{\lambda_i} = \boldsymbol{Q} - \boldsymbol{I}_{\lambda_i}$ is defined as above.

Proof. Since $T > t$, then

$$\mathbb{E}\left[1_{\{\tau_i > T\}} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[1_{\{\tau_i > T\}} \,\middle|\, \mathcal{F}_t^X \vee \mathcal{F}_t^{Y_i}\right] = 1_{\{\tau_i > t\}} \mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, \mathcal{F}_t^X\right] \tag{5.1.2}$$

where the first equality is due to the fact that conditionally on X, then τ_i is independent of τ_j for $j \neq i$. The second equality follows from a standard result for the first jump time of a Cox-process, see e.g. p.102 in Lando (1998), Corollary 9.1 in McNeil et al. (2005) or Corollary 6.4.2 in Bielecki & Rutkowski (2001). Since $T > t$ and due to the Markov property of X we can rewrite the quantity $\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, \mathcal{F}_t^X\right]$ as

$$\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, \mathcal{F}_t^X\right] = \mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, X_t\right] = \sum_{k=1}^K \mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, X_t = k\right] 1_{\{X_t = k\}}$$

which implies that (recall that $\mathcal{F}_t^X$ is not a subfiltration of $\mathcal{F}_t^M$)

$$\mathbb{E}\left[\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, \mathcal{F}_t^X\right] \,\middle|\, \mathcal{F}_t^M\right] = \sum_{k=1}^K \mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, X_t = k\right] \pi_t^k \tag{5.1.3}$$

where we used the notation $\pi_t^k = \mathbb{Q}\left[X_t = k \,\middle|\, \mathcal{F}_t^M\right]$. By using Theorem A.1 in Appendix A we have that

$$\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, X_t = k\right] = \boldsymbol{e}_k e^{\boldsymbol{Q}_{\lambda_i}(T-t)} \mathbf{1} \tag{5.1.4}$$

where the matrix $\boldsymbol{Q}_{\lambda_i}$ is defined as previously. So (5.1.4) in (5.1.3) yields

$$\mathbb{E}\left[\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, \mathcal{F}_t^X\right] \,\middle|\, \mathcal{F}_t^M\right] = \sum_{k=1}^K \boldsymbol{e}_k e^{\boldsymbol{Q}_{\lambda_i}(T-t)} \mathbf{1} \pi_t^k = \boldsymbol{\pi}_t e^{\boldsymbol{Q}_{\lambda_i}(T-t)} \mathbf{1} \tag{5.1.5}$$

where we recall that $\boldsymbol{\pi}_t$ is a row-vector such that $\boldsymbol{\pi}_t = \left(\pi_t^1, \ldots, \pi_t^K\right)$. Next, note that

$$\begin{aligned}
\mathbb{E}\left[1_{\{\tau_i > T\}} \,\middle|\, \mathcal{F}_t^M\right] &= \mathbb{E}\left[\mathbb{E}\left[1_{\{\tau_i > T\}} \,\middle|\, \mathcal{F}_t\right] \,\middle|\, \mathcal{F}_t^M\right] \\
&= 1_{\{\tau_i > t\}} \mathbb{E}\left[\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\middle|\, \mathcal{F}_t^X\right] \,\middle|\, \mathcal{F}_t^M\right] \\
&= 1_{\{\tau_i > t\}} \boldsymbol{\pi}_t e^{\boldsymbol{Q}_{\lambda_i}(T-t)} \mathbf{1}
\end{aligned}$$

where the second equality is due to (5.1.2) and the third equality follows from (5.1.5). Thus, for $T \geq t$ we conclude that $\mathbb{Q}\left[\tau_i > T \,\middle|\, \mathcal{F}_t^M\right] = 1_{\{\tau_i > t\}} \boldsymbol{\pi}_t e^{\boldsymbol{Q}_{\lambda_i}(T-t)} \mathbf{1}$ which proves the theorem. $\square$

Theorem 5.1 allows us to state credit related derivatives quantizes in very compact and computational convenient formulas, as will seen later in this paper. We also remark that Theorem 5.1 has previously been successfully used in Herbertsson & Frey (2014) in which the theorem was stated without a proof, see Theorem 3.1 p. 1416 in Herbertsson & Frey (2014). Instead Herbertsson & Frey (2014) refers to the proof of Theorem 5.1 in an earlier version of this paper.

5.2. **The conditional number of defaults.** In this subsection we derive practical expressions for $\mathbb{E}\left[N_t \,|\, \mathcal{F}_t^M\right]$. We consider an homogeneous credit portfolios where $\lambda_i(X_t) = \lambda(X_t)$ so that $\boldsymbol{Q}_{\lambda_i} = \boldsymbol{Q}_\lambda$ for each obligor i. Recall that $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. The main message of this subsection is the following proposition.

Proposition 5.2. *Consider an exchangeable credit portfolio with m obligors in a model specified as in Section 3. Then, for $T \geq t$ and with notation as above*

$$\mathbb{E}\left[N_T \,|\, \mathcal{F}_t^M\right] = m - (m - N_t)\,\boldsymbol{\pi}_t e^{\boldsymbol{Q}_\lambda(T-t)}\mathbf{1}. \tag{5.2.1}$$

Proof. Let $T > t$ and first note that

$$\mathbb{E}\left[N_T \,|\, \mathcal{F}_t\right] = m - \sum_{i=1}^m \mathbb{E}\left[1_{\{\tau_i > T\}} \,\Big|\, \mathcal{F}_t\right] = m - \sum_{i=1}^m 1_{\{\tau_i > t\}}\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s)ds} \,\Big|\, \mathcal{F}_t^X\right] \tag{5.2.2}$$

where the last equality is due to Equation (5.1.2) in Theorem 5.1. Furthermore, in a homogeneous portfolio we have $\lambda_i(X_s) = \lambda(X_s)$ for all obligors i and this in (5.2.2) implies that $\mathbb{E}\left[N_T \,|\, \mathcal{F}_t\right] = m - (m - N_t)\mathbb{E}\left[e^{-\int_t^T \lambda(X_s)ds} \,\Big|\, \mathcal{F}_t^X\right]$. Thus, by using $\mathbb{E}\left[N_T \,|\, \mathcal{F}_t^M\right] = \mathbb{E}\left[\mathbb{E}\left[N_T \,|\, \mathcal{F}_t\right] \,|\, \mathcal{F}_t^M\right]$ and following similar arguments as in Theorem 5.1 we conclude after some computations that $\mathbb{E}\left[N_T \,|\, \mathcal{F}_t^M\right] = m - (m - N_t)\,\boldsymbol{\pi}_t e^{\boldsymbol{Q}_\lambda(T-t)}\mathbf{1}$ which proves the proposition. $\square$

A similar proof can be found for inhomogeneous portfolios.

5.3. **The conditional portfolio loss: The case with constant recovery.** This is trivial for homogeneous portfolios, given the results from Subsection 5.2. To see this, recall that $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ and $L_t = \frac{1}{m}\sum_{i=1}^m (1 - \phi_i)1_{\{\tau_i \leq t\}}$ where ϕ_i are constants and in a homogeneous portfolio we have $\phi_1 = \phi_2 = \ldots = \phi_m = \phi$ so that $L_t = \frac{(1-\phi)}{m}N_t$. Thus,

$$\mathbb{E}\left[L_T \,|\, \mathcal{F}_t^M\right] = \frac{(1-\phi)}{m}\mathbb{E}\left[N_T \,|\, \mathcal{F}_t^M\right] \tag{5.3.1}$$

where $\mathbb{E}\left[N_T \,|\, \mathcal{F}_t^M\right]$ is explicitly given in Subsection 5.2 for homogeneous portfolios. To be more specific, (5.3.1) with Proposition 5.2 yields

$$\mathbb{E}\left[L_T \,|\, \mathcal{F}_t^M\right] = (1-\phi)\left(1 - \left(1 - \frac{N_t}{m}\right)\boldsymbol{\pi}_t e^{\boldsymbol{Q}_\lambda(T-t)}\mathbf{1}\right). \tag{5.3.2}$$

Similar results can also be obtained in an inhomogeneous portfolio both with identical or different recoveries.

6. The CDS index in the filtering model

In this section we apply the results from Section 5 together with Subsection 2.2 to find formulas for the CDS index spreads in the models introduced in Section 3. This will be done in a homogeneous portfolio. We will assume that the risk free interest rate is constant and given by r and for $t < s$ we let $B(t,s)$ denote $B(t,s) = e^{-r(s-t)}$. We can now state the following theorem.

Theorem 6.1. *Consider a CDS index portfolio in the filtering model. Then, with notation as above*

$$DL(t,T) = \mathbb{E}\left[\int_t^T B(t,s)dL_s \,\middle|\, \mathcal{F}_t^M\right] = \left(1 - \frac{N_t}{m}\right)\boldsymbol{\pi}_t\boldsymbol{A}(t,T)\mathbf{1} \tag{6.1}$$

and

$$PV(t,T) = \mathbb{E}\left[V_P(t,T) \,\middle|\, \mathcal{F}_t^M\right] = \left(1 - \frac{N_t}{m}\right)\boldsymbol{\pi}_t\boldsymbol{B}(t,T)\mathbf{1} \tag{6.2}$$

where $\boldsymbol{A}(t,T)$ and $\boldsymbol{B}(t,T)$ are defined as

$$\boldsymbol{A}(t,T) = (1-\phi)\left[\boldsymbol{I} - e^{\boldsymbol{Q}_\lambda(T-t)}\left(\boldsymbol{I} + r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right)e^{-r(T-t)} + r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right] \tag{6.3}$$

$$\boldsymbol{B}(t,T) = \frac{1}{4}\sum_{n=n_t}^{\lceil 4T \rceil} e^{\boldsymbol{Q}_\lambda(t_n-t)}e^{-r(t_n-t)}. \tag{6.4}$$

Furthermore, if $N_t < m$ we have

$$S(t,T) = \frac{\boldsymbol{\pi}_t\boldsymbol{A}(t,T)\mathbf{1}}{\boldsymbol{\pi}_t\boldsymbol{B}(t,T)\mathbf{1}}. \tag{6.5}$$

Proof. First we recall the definitions of $DL(t,T)$, $PV(t,T)$ and $S(t,T)$ from (2.2.3), (2.2.4), (2.2.5) and (2.2.6) with the difference that we now replace $\mathcal{F}_t$ with $\mathcal{F}_t^M$ given by (3.2.2). Next, the term $\int_t^T B(t,s)dL_s$ used in $DL(t,T)$ can be rewritten in a more practical form using integration by parts (see e.g. Theorem 3.36, p.107 in Folland (1999)), so that $\int_t^T B(t,s)dL_s = B(t,T)L_T - L_t + \int_t^T rB(t,s)L_s ds$ and by applying Fubini-Tonelli on this expressions then renders

$$\mathbb{E}\left[\int_t^T B(t,s)dL_s \,\middle|\, \mathcal{F}_t^M\right] = B(t,T)\mathbb{E}\left[L_T \,\middle|\, \mathcal{F}_t^M\right] - L_t + \int_t^T rB(t,s)\mathbb{E}\left[L_s \,\middle|\, \mathcal{F}_t^M\right]ds. \tag{6.6}$$

Furthermore, if $s > t$ then (5.3.2) gives

$$\mathbb{E}\left[L_s \,\middle|\, \mathcal{F}_t^M\right] = (1-\phi)\left(1 - \left(1 - \frac{N_t}{m}\right)\boldsymbol{\pi}_t e^{\boldsymbol{Q}_\lambda(s-t)}\mathbf{1}\right)$$

so using this in (6.6) and recalling that $B(t,s) = e^{-r(s-t)}$ for $s > t$, we get

$$\mathbb{E}\left[\int_t^T B(t,s)dL_s \,\middle|\, \mathcal{F}_t^M\right] = B(t,T)\mathbb{E}\left[L_T \,\middle|\, \mathcal{F}_t^M\right] - L_t + \int_t^T rB(t,s)\mathbb{E}\left[L_s \,\middle|\, \mathcal{F}_t^M\right]ds$$

$$= e^{-r(T-t)}(1-\phi)\left(1 - \left(1 - \frac{N_t}{m}\right)\boldsymbol{\pi}_t e^{\boldsymbol{Q}_\lambda(T-t)}\mathbf{1}\right) - \frac{(1-\phi)}{m}N_t \tag{6.7}$$

$$+ \int_t^T re^{-r(s-t)}(1-\phi)\left(1 - \left(1 - \frac{N_t}{m}\right)\boldsymbol{\pi}_t e^{\boldsymbol{Q}_\lambda(s-t)}\mathbf{1}\right)ds.$$

The integral in the RHS of (6.7) can be simplified according to

$$
\int_t^T r e^{-r(s-t)} (1-\phi) \left(1 - \left(1 - \frac{N_t}{m} \right) \boldsymbol{\pi}_t e^{\boldsymbol{Q}_\lambda (s-t)} \mathbf{1} \right) ds
$$
$$
= (1-\phi) \left(1 - e^{-r(T-t)} \right) \tag{6.8}
$$
$$
- r(1-\phi) \left(1 - \frac{N_t}{m} \right) \boldsymbol{\pi}_t \left(e^{\boldsymbol{Q}_\lambda (T-t)} e^{-r(T-t)} - \boldsymbol{I} \right) (\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1} \mathbf{1}
$$

where the last equality in (6.8) is due to the fact that

$$
\int_t^T e^{-r(s-t)} e^{\boldsymbol{Q}_\lambda (s-t)} ds = \int_t^T e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})(s-t)} ds = \left(e^{\boldsymbol{Q}_\lambda (T-t)} e^{-r(T-t)} - \boldsymbol{I} \right) (\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1}.
$$

Note that $(\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1}$ exists since $\boldsymbol{Q}_\lambda - r\boldsymbol{I}$ by construction is a diagonal dominant matrix, implying that $\det (\boldsymbol{Q}_\lambda - r\boldsymbol{I}) \neq 0$ by the Levy-Desplanques Theorem. By plugging (6.8) into (6.7) and performing some trivial but tedious computations we get

$$
\mathbb{E} \left[\int_t^T B(t,s) dL_s \,\Big|\, \mathcal{F}_t^M \right]
$$
$$
= (1-\phi) \left(1 - \frac{N_t}{m} \right) \left(1 - \boldsymbol{\pi}_t \left(e^{\boldsymbol{Q}_\lambda (T-t)} \left(\boldsymbol{I} + r (\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1} \right) e^{-r(T-t)} - r (\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1} \right) \mathbf{1} \right)
$$
$$
= (1-\phi) \left(1 - \frac{N_t}{m} \right) \boldsymbol{\pi}_t \left[\boldsymbol{I} - e^{\boldsymbol{Q}_\lambda (T-t)} \left(\boldsymbol{I} + r (\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1} \right) e^{-r(T-t)} + r (\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1} \right] \mathbf{1}
$$
$$
= \left(1 - \frac{N_t}{m} \right) \boldsymbol{\pi}_t \boldsymbol{A}(t,T) \mathbf{1}
$$

where we in the second equality used that $1 = \boldsymbol{\pi}_t \mathbf{1} = \boldsymbol{\pi}_t \boldsymbol{I} \mathbf{1}$ and where $\boldsymbol{A}(t,T)$ in the final equality is given by

$$
\boldsymbol{A}(t,T) = (1-\phi) \left[\boldsymbol{I} - e^{\boldsymbol{Q}_\lambda (T-t)} \left(\boldsymbol{I} + r (\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1} \right) e^{-r(T-t)} + r (\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1} \right]
$$

which proves (6.1) and (6.3) where we also used (2.2.4) with $\mathcal{F}_t$ replaced by $\mathcal{F}_t^M$ given in (3.2.2). To derive the expression for the premium leg we use (5.2.1) in Proposition 5.2 with $s > t$ and obtain $1 - \frac{1}{m} \mathbb{E} \left[N_s \,|\, \mathcal{F}_t^M \right] = \left(1 - \frac{N_t}{m} \right) \boldsymbol{\pi}_t e^{\boldsymbol{Q}_\lambda (s-t)} \mathbf{1}$ which in Equation (2.2.5), with $\mathcal{F}_t$ replaced by $\mathcal{F}_t^M$, then renders that

$$
PV(t,T) = \frac{1}{4} \sum_{n=n_t}^{\lceil 4T \rceil} B(t,t_n) \left(1 - \frac{1}{m} \mathbb{E} \left[N_{t_n} \,|\, \mathcal{F}_t^M \right] \right) = \frac{1}{4} \left(1 - \frac{N_t}{m} \right) \sum_{n=n_t}^{\lceil 4T \rceil} \boldsymbol{\pi}_t e^{\boldsymbol{Q}_\lambda (t_n - t)} \mathbf{1} e^{-r(t_n - t)}
$$
$$
= \left(1 - \frac{N_t}{m} \right) \boldsymbol{\pi}_t \boldsymbol{B}(t,T) \mathbf{1}
$$

where $\boldsymbol{B}(t,T) = \frac{1}{4} \sum_{n=n_t}^{\lceil 4T \rceil} e^{\boldsymbol{Q}_\lambda (t_n - t)} e^{-r(t_n - t)}$ and this proves (6.2) and (6.4). Finally, (6.5) follows from the definition in (2.2.6) together with the expressions for the default leg and premium leg in (6.1) and (6.2). $\square$

Note that the term $1 - N_t/m$ in the right hand side of both (6.1) and (6.2) implies that the conditional expectations of the default and premium legs will be zero for the armageddon event $N_t = m$. This fact is in line with the conclusion in (2.3.3) which holds for any model of

the default times $\tau_1, \ldots, \tau_m$. Furthermore, note that the right hand side in (6.5) is still well defined when $N_t = m$.

From Theorem 6.1 we conclude that given the vector $\boldsymbol{\pi}_t$, then the formulas for the default and premium leg in the filtering model as well as the CDS index spread $S(t,T)$ are compact and computationally tractable closed-form expressions in terms of $\boldsymbol{\pi}_t$ and $\boldsymbol{Q}_\lambda$. Furthermore, Theorem 6.1 will also help us to find tractable formulas for the payoff of more exotic derivatives with the CDS index as a underlyer. Example of such derivatives are call options on the CDS index, which we will treat in the next section.

7. CDS index options in the filtering model

In this section we apply the results from Section 6 and Subsection 2.3 to present a highly computationally tractable formula for the payoff of a so called CDS index option in the model presented in Section 3. Furthermore, we derive a lower bound for price of the CDS index option and also provide explicit conditions on the strike spread for which this inequality becomes an equality. The lower bound is computationally tractable and do not depend on any of the "noise" parameters in the filtering model introduced in Section 3. Finally, we outline how to explicitly compute the quantities involved in the lower bound for the price of the CDS index option.

By inserting the explicit expressions for the default and premium legs for the index-CDS spread given by (6.1) and (6.2) in Theorem 6.1 into the expression of the payoff $\Pi(t,T;\kappa)$ for the CDS index option in Equation (2.3.5), that is

$$\Pi(t,T;\kappa) = (DL(t,T) - \kappa PV(t,T) + L_t)^+ .$$

we immediately make the payoff $\Pi(t,T;\kappa)$ very explicit in terms of $\boldsymbol{\pi}_t$, N_t, $\boldsymbol{A}(t,T)$ and $\boldsymbol{B}(t,T)$, as summarized in the following lemma.

Lemma 7.1. *Consider a CDS index portfolio in the filtering model. Then, the payoff $\Pi(t,T;\kappa)$ for an CDS index option with strike κ, exercise date t and maturity T for the underlying CDS index, is given by*

$$\Pi(t,T;\kappa) = \left(\boldsymbol{\pi}_t \Big[\boldsymbol{A}(t,T) - \kappa \boldsymbol{B}(t,T) \Big] \mathbf{1} \left(1 - \frac{N_t}{m} \right) + \frac{(1-\phi)\,N_t}{m} \right)^+ \qquad (7.1)$$

where $\boldsymbol{A}(t,T)$ and $\boldsymbol{B}(t,T)$ are defined as in Theorem 6.1.

Note that on the event $\{N_t = m\}$, the right-hand side in (7.1) reduces to the random variable $(1-\phi)1_{\{N_t=m\}}$ for any strike spread κ, which is consistent with Equation (2.3.7).

In view of Lemma 7.1 and since the price of the CDS index option $C_0(t,T;\kappa)$ at time 0 (i.e. today) is given by $C_0(t,T;\kappa) = \mathbb{E}\left[e^{-rt}\Pi(t,T;\kappa) \right]$ we therefore get

$$C_0(t,T;\kappa) = e^{-rt}\mathbb{E}\left[\left(\boldsymbol{\pi}_t \Big[\boldsymbol{A}(t,T) - \kappa \boldsymbol{B}(t,T) \Big] \mathbf{1} \left(1 - \frac{N_t}{m} \right) + \frac{(1-\phi)\,N_t}{m} \right)^+ \right] . \qquad (7.2)$$

Since no closed formulas are known for the entries in the vector $\boldsymbol{\pi}_t$ it is difficult to find analytical expressions for the formulas in the RHS of Equation (7.2). Instead we rely on Monte Carlo simulations of the filtering probabilities $\boldsymbol{\pi}_t$ together with the compact formula for the payoff function $\Pi(t,T;\kappa)$ given in (7.1).

7.1. **A lower bound for the CDS index option price.** In this subsection we present we derive a lower bound for price of the CDS index option and also provide explicit conditions on the strike spread for which this inequality becomes an equality. The lower bound is computationally tractable and do not depend on any of the "noise" parameters in the filtering model introduced in Section 3.

Even if it does not exists any closed formulas for the expected value in (7.2) we can still derive lower bounds for the price $C_0(t, T; K)$ in our nonlinear filtering model by using Equation (2.3.11). This is done in the following proposition.

Proposition 7.2. *Let $C_0(t, T; \kappa)$ be the price today of an CDS index option with strike κ, exercise date t and maturity T. Then, with notation as above,*

$$C_0(t, T; \kappa) \geq (1 - \phi)e^{-rt}\mathbb{Q}\left[N_t = m\right]$$

$$+ e^{-rt} \sum_{j=0}^{m-1} \left(\sum_{k=1}^{K} p_k(t, T; \kappa) \left(1 - \frac{j}{m}\right) \mathbb{Q}\left[X_t = k, N_t = j\right] + \frac{(1 - \phi)j}{m}\mathbb{Q}\left[N_t = j\right] \right)^+ \quad (7.1.1)$$

where

$$p_k(t, T; \kappa) = \left(\left[\boldsymbol{A}(t, T) - \kappa\boldsymbol{B}(t, T)\right]\mathbf{1} \right)_k \quad (7.1.2)$$

for $\boldsymbol{A}(t, T)$ and $\boldsymbol{B}(t, T)$ defined as in Theorem 6.1.

Proof. From Equation (2.3.11) we have

$$C_0(t, T; \kappa) = e^{-rt}\mathbb{E}\left[\Pi(t, T; \kappa)1_{\{N_t < m\}}\right] + (1 - \phi)e^{-rt}\mathbb{Q}\left[N_t = m\right] \quad (7.1.3)$$

and note that $\mathbb{E}\left[\Pi(t, T; \kappa)1_{\{N_t < m\}}\right]$ can be rewritten as

$$\mathbb{E}\left[\Pi(t, T; \kappa)1_{\{N_t < m\}}\right] = \sum_{j=0}^{m-1} \mathbb{E}\left[\Pi(t, T; \kappa)1_{\{N_t = j\}}\right]. \quad (7.1.4)$$

We now give a lower bound for the quantity $\mathbb{E}\left[\Pi(t, T; \kappa)1_{\{N_t = j\}}\right]$ and for this we need some more notation. For each state k in the state space of the underlying process X_t defined in Section 3, let $p_k(t, T; \kappa)$ denote the k-th component in the vector $\left(\left[\boldsymbol{A}(t, T) - \kappa\boldsymbol{B}(t, T)\right]\mathbf{1} \right)$, that is

$$p_k(t, T; \kappa) = \left(\left[\boldsymbol{A}(t, T) - \kappa\boldsymbol{B}(t, T)\right]\mathbf{1} \right)_k. \quad (7.1.5)$$

Furthermore, we remind the reader that $\boldsymbol{\pi}_t$ is a a row-vector given by $\boldsymbol{\pi}_t = \left(\pi_t^1, \ldots, \pi_t^K\right)$ where each processes π_t^k satisfy the K-dimensional system of SDE-s in Equation (4.8) presented in Corollary 4.2. Hence, this observation together with Equation (7.1) and Equation (7.1.5) then

implies that we can rewrite the quantity $\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t=j\}}\right]$ as follows

$$
\begin{aligned}
\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t=j\}}\right] &= \mathbb{E}\left[\left(\boldsymbol{\pi}_t\left[\boldsymbol{A}(t,T)-\kappa\boldsymbol{B}(t,T)\right]\boldsymbol{1}\left(1-\frac{j}{m}\right)+\frac{(1-\phi)\,j}{m}\right)^+1_{\{N_t=j\}}\right]\\
&= \mathbb{E}\left[\left(\sum_{k=1}^{K}\pi_t^k p_k(t,T;\kappa)\left(1-\frac{j}{m}\right)+\frac{(1-\phi)\,j}{m}\right)^+1_{\{N_t=j\}}\right]\\
&= \mathbb{E}\left[\left(\sum_{k=1}^{K}\pi_t^k p_k(t,T;\kappa)\left(1-\frac{j}{m}\right)1_{\{N_t=j\}}+\frac{(1-\phi)\,j}{m}1_{\{N_t=j\}}\right)^+\right]\\
&\geq \left(\mathbb{E}\left[\sum_{k=1}^{K}\pi_t^k p_k(t,T;\kappa)\left(1-\frac{j}{m}\right)1_{\{N_t=j\}}+\frac{(1-\phi)\,j}{m}1_{\{N_t=j\}}\right]\right)^+
\end{aligned}
$$
(7.1.6)

where the last inequality is due to Jensens inequality. The quantity inside the max expression on the last line in Equation (7.1.6) can be rewritten as

$$
\begin{aligned}
\mathbb{E}&\left[\sum_{k=1}^{K}\pi_t^k p_k(t,T;\kappa)\left(1-\frac{j}{m}\right)1_{\{N_t=j\}}+\frac{(1-\phi)\,j}{m}1_{\{N_t=j\}}\right]\\
&= \sum_{k=1}^{K}p_k(t,T;\kappa)\left(1-\frac{j}{m}\right)\mathbb{E}\left[\pi_t^k 1_{\{N_t=j\}}\right]+\frac{(1-\phi)\,j}{m}\mathbb{Q}\left[N_t=j\right].
\end{aligned}
$$
(7.1.7)

Furthermore, since $\pi_t^k=\mathbb{Q}\left[X_t=k\,|\,\mathcal{F}_t^M\right]$ we have

$$
\begin{aligned}
\mathbb{E}\left[\pi_t^k 1_{\{N_t=j\}}\right] &= \mathbb{E}\left[\mathbb{Q}\left[X_t=k\,|\,\mathcal{F}_t^M\right]1_{\{N_t=j\}}\right]\\
&= \mathbb{E}\left[\mathbb{Q}\left[X_t=k,N_t=j\,|\,\mathcal{F}_t^M\right]\right]\\
&= \mathbb{Q}\left[X_t=k,N_t=j\right]
\end{aligned}
$$
(7.1.8)

where the second equality follows from the fact that N_t is $\mathcal{F}_t^M$-measurable since $\mathcal{F}_t^M=\mathcal{F}_t^Y\vee\mathcal{F}_t^Z$ in view of Equation (3.2.2). Hence, inserting (7.1.8) in (7.1.7) and using (7.1.4) and (7.1.6), we retrieve the following lower bound for $\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t<m\}}\right]$

$$
\begin{aligned}
&\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t<m\}}\right]\\
&\geq \sum_{j=0}^{m-1}\left(\sum_{k=1}^{K}p_k(t,T;\kappa)\left(1-\frac{j}{m}\right)\mathbb{Q}\left[X_t=k,N_t=j\right]+\frac{(1-\phi)\,j}{m}\mathbb{Q}\left[N_t=j\right]\right)^+
\end{aligned}
$$
(7.1.9)

where $p_k(t,T;\kappa)$ is given by Equation (7.1.5). Next, plugging (7.1.9) into (7.1.3) finally yields the following lower bound for the option price $C_0(t,T;\kappa)$,

$$
\begin{aligned}
C_0(t,T;\kappa) &\geq (1-\phi)e^{-rt}\mathbb{Q}\left[N_t=m\right]\\
&+ e^{-rt}\sum_{j=0}^{m-1}\left(\sum_{k=1}^{K}p_k(t,T;\kappa)\left(1-\frac{j}{m}\right)\mathbb{Q}\left[X_t=k,N_t=j\right]+\frac{(1-\phi)\,j}{m}\mathbb{Q}\left[N_t=j\right]\right)^+
\end{aligned}
$$

which proves (7.1.1). $\qquad\square$

Thus, Proposition 7.2 establish a lower bound for the option price $C_0(t, T; \kappa)$ as function of the probabilities $\mathbb{Q}[X_t = k, N_t = j]$ and $\mathbb{Q}[N_t = j]$ for each state k and $j = 0, 1, \ldots, m$. Furthermore, we also remark that the quantity in the right hand side of (7.1.1) does not depend on any of the "noise" parameters in the filtering model introduced in Section 3. That is, the expression in the right hand side of (7.1.1) is independent of the mapping $\boldsymbol{a}$: $\{1, 2, \ldots, K\} \mapsto \mathbb{R}^l$ which is used in (3.2.1) to generate the noisy information $\mathcal{F}_t^Z$ and the corresponding noisy market information $\mathcal{F}_t^M$ in (3.2.2), which in turn creates the nonlinear filtering model introduced in Section 3.

The following corollary to Proposition 7.2 gives conditions for the possibilities of having an equality in (7.1.1) instead of an inequality.

Corollary 7.3. *Let $C_0(t, T; \kappa)$ be the price today of an CDS index option with strike κ, exercise date t and maturity T. Then there exists a constant κ^* such that for $\kappa \leq \kappa^*$ it holds*

$$C_0(t, T; \kappa) = (1 - \phi)e^{-rt}\mathbb{Q}[N_t = m]$$

$$+ e^{-rt} \sum_{j=0}^{m-1} \sum_{k=1}^{K} p_k(t, T; \kappa) \left(1 - \frac{j}{m}\right) \mathbb{Q}[X_t = k, N_t = j] + e^{-rt} \sum_{j=0}^{m-1} \frac{(1 - \phi)j}{m} \mathbb{Q}[N_t = j]$$

(7.1.10)

where κ^* is given by

$$\kappa^* = \min_{k=1,\ldots,K} \kappa_k^* \qquad and \qquad \kappa_k^* = \frac{\boldsymbol{e}_k \boldsymbol{A}(t, T)\mathbf{1}}{\boldsymbol{e}_k \boldsymbol{B}(t, T)\mathbf{1}} \tag{7.1.11}$$

with $\boldsymbol{A}(t, T)$, $\boldsymbol{B}(t, T)$ and $p_k(t, T; \kappa)$ defined as in Proposition 7.2.

Proof. First recall the option pricing formula (7.2)

$$C_0(t, T; \kappa) = e^{-rt}\mathbb{E}\left[\left(\boldsymbol{\pi}_t\left[\boldsymbol{A}(t, T) - \kappa\boldsymbol{B}(t, T)\right]\mathbf{1}\left(1 - \frac{N_t}{m}\right) + \frac{(1 - \phi)N_t}{m}\right)^+\right] \tag{7.1.12}$$

where $\boldsymbol{A}(t, T)$ and $\boldsymbol{B}(t, T)$ are given as in Lemma 7.1. From Theorem A.1 in Appendix A, Proposition 5.2 and Equation (6.2) in Theorem 6.1 we conclude that $\boldsymbol{e}_k\boldsymbol{B}(t, T)\mathbf{1} \geq 0$ for each state k. Similarly, from the Equations (6.7) and (6.8) we also conclude that $\boldsymbol{e}_k\boldsymbol{A}(t, T)\mathbf{1} \geq 0$ for every state k. Therefore, for each k the quantity

$$\boldsymbol{e}_k\big(\boldsymbol{A}(t, T) - \kappa\boldsymbol{B}(t, T)\big)\mathbf{1} = \boldsymbol{e}_k\boldsymbol{A}(t, T)\mathbf{1} - \kappa\boldsymbol{e}_k\boldsymbol{B}(t, T)\mathbf{1} \tag{7.1.13}$$

is the difference of two positive expressions when $\kappa \geq 0$. Consequently, for each state k there is a smallest strike spread denoted by κ_k^* (bounded below by zero) for which the payoff in (7.1.13) is non-negative for all $\kappa \leq \kappa_k^*$. More explicit, κ_k^* is defined by

$$\kappa_k^* = \frac{\boldsymbol{e}_k\boldsymbol{A}(t, T)\mathbf{1}}{\boldsymbol{e}_k\boldsymbol{B}(t, T)\mathbf{1}}.$$

Furthermore, let κ^* be

$$\kappa^* = \min_{k=1,\ldots,K} \kappa_k^*.$$

Then, by the construction of κ^*, we conclude that (7.1.13) is non-negative for all states k and all strike spreads κ where $\kappa \leq \kappa^*$ which implies

$$\boldsymbol{\pi}_t\Big(\boldsymbol{A}(t, T) - \kappa\boldsymbol{B}(t, T)\Big)\mathbf{1} \geq 0 \text{ a.s. for } \kappa \leq \kappa^*, \tag{7.1.14}$$

that is

$$\mathbb{Q}\left[\boldsymbol{\pi}_t\Big(\boldsymbol{A}(t,T)-\kappa\boldsymbol{B}(t,T)\Big)\mathbf{1}\geq 0\right]=1 \quad \text{for } \kappa\leq\kappa^*.$$

By using (7.1.14) we conclude that the "max" expression in (7.1.12) is superfluous for $\kappa\leq\kappa^*$ and the option price $C_0(t,T;\kappa)$ can then be rewritten as

$$C_0(t,T;\kappa)=e^{-rt}\mathbb{E}\left[\left(\boldsymbol{\pi}_t\Big[\boldsymbol{A}(t,T)-\kappa\boldsymbol{B}(t,T)\Big]\mathbf{1}\left(1-\frac{N_t}{m}\right)+\frac{(1-\phi)\,N_t}{m}\right)^+\right]$$

$$=e^{-rt}\mathbb{E}\left[\boldsymbol{\pi}_t\Big(\boldsymbol{A}(t,T)-\kappa\boldsymbol{B}(t,T)\Big)\mathbf{1}\left(1-\frac{N_t}{m}\right)+\frac{(1-\phi)\,N_t}{m}\right] \qquad (7.1.15)$$

$$=e^{-rt}\mathbb{E}\left[\boldsymbol{\pi}_t\Big(\boldsymbol{A}(t,T)-\kappa\boldsymbol{B}(t,T)\Big)\mathbf{1}\left(1-\frac{N_t}{m}\right)\right]+e^{-rt}\frac{(1-\phi)}{m}\mathbb{E}\left[N_t\right].$$

Furthermore, by following similar steps as in the Equations (7.1.7)-(7.1.8) we can rewrite the expression $\mathbb{E}\left[\boldsymbol{\pi}_t\Big(\boldsymbol{A}(t,T)-\kappa\boldsymbol{B}(t,T)\Big)\mathbf{1}\left(1-\frac{N_t}{m}\right)\right]$ as

$$\mathbb{E}\left[\boldsymbol{\pi}_t\Big(\boldsymbol{A}(t,T)-\kappa\boldsymbol{B}(t,T)\Big)\mathbf{1}\left(1-\frac{N_t}{m}\right)\right]=\sum_{j=0}^{m-1}\sum_{k=1}^{K}p_k(t,T;\kappa)\left(1-\frac{j}{m}\right)\mathbb{Q}\left[X_t=k,N_t=j\right]$$

$$(7.1.16)$$

where $p_k(t,T;\kappa)$ is defined as in (7.1.2). Note that

$$e^{-rt}\frac{(1-\phi)}{m}\mathbb{E}\left[N_t\right]=(1-\phi)e^{-rt}\mathbb{Q}\left[N_t=m\right]+e^{-rt}\sum_{j=0}^{m-1}\frac{(1-\phi)\,j}{m}\mathbb{Q}\left[N_t=j\right] \qquad (7.1.17)$$

and this observation together with (7.1.16) in (7.1.15) for $\kappa\leq\kappa^*$ then yields that

$$C_0(t,T;\kappa)=(1-\phi)e^{-rt}\mathbb{Q}\left[N_t=m\right]$$

$$+e^{-rt}\sum_{j=0}^{m-1}\sum_{k=1}^{K}p_k(t,T;\kappa)\left(1-\frac{j}{m}\right)\mathbb{Q}\left[X_t=k,N_t=j\right]+e^{-rt}\sum_{j=0}^{m-1}\frac{(1-\phi)\,j}{m}\mathbb{Q}\left[N_t=j\right]$$

which proves (7.1.12). □

Note that Equation (7.1.10) in Corollary 7.3 is simply Equation (7.1.1) with a strict equality and without a "max" function for the second expression in the right hand side of (7.1.1). We therefore conclude that the inequality in (7.1.1) turns into a strict equality when the strike spread κ satisfies $\kappa\leq\kappa^*$ with κ^* defined as in (7.1.11).

It is remarkable that we in the noise model introduced in Section 3 for strike spreads κ such that $\kappa\leq\kappa^*$ have derived an semi-explicit expression for the option price $C_0(t,T;\kappa)$ given by (7.1.10), which is independent of the mapping $\boldsymbol{a}:\{1,2,\ldots,K\}\mapsto\mathbb{R}^l$ used in (3.2.1) to generate the noisy information $\mathcal{F}_t^Z$.

7.2. **Auxiliary tools for computing the lower bounds.** In this subsection we outline how to compute the quantizes involved in the lower bounds presented in Subsection 7.1. The results derived here are utilized in the implementations that are used in our numerical studies in later sections of this paper.

Equation (7.1.1) in Proposition 7.2 provides a lower bound for the option price $C_0(t,T;\kappa)$ as function of the probabilities $\mathbb{Q}\left[X_t=k,N_t=j\right]$ and $\mathbb{Q}\left[N_t=j\right]$ for each state k and $j=0,1,\ldots,m$. Furthermore, (7.1.10) in Corollary 7.3 gives an exact expression for option price

$C_0(t, T; \kappa)$ when $\kappa \leq \kappa^*$ as function of $\mathbb{Q}[X_t = k, N_t = j]$ and $\mathbb{Q}[N_t = j]$. Therefore we will below outline how to compute the probabilities $\mathbb{Q}[X_t = k, N_t = j]$ and $\mathbb{Q}[N_t = j]$ in our model presented in Section 3.

Consider a bivariate Markov process H_t on a state space S^H defined as

$$S^H = \{1, \ldots, K\} \times \{0, 1, \ldots, m\} \tag{7.2.1}$$

where $|S^H| = K(m+1)$. So each state $\boldsymbol{j} \in S^H$ can be written as a pair $\boldsymbol{j} = (k, j)$ where k and j are integers such that $1 \leq k \leq K$ and $0 \leq j \leq m$. The first component of H_t belongs to $\{1, \ldots, K\}$ while the second component of H_t is defined on $\{1, \ldots, m\}$. The intuitive idea behind the bivariate Markov process H_t is of course that the first component of H_t should "mimic" the factor process X_t defined in Section 3.1 while the second component of H_t should represent N_t, i.e. the number of defaulted obligors in the portfolio at time t, as defined in previous sections. More specific, for any pair $(k, j) \in S^H$ and for any time point $t \geq 0$, we want that the events $\{H_t = (k, j)\}$ and $\{X_t = k, N_t = j\}$ should have the same probability under the risk-neutral measure $\mathbb{Q}$, that is

$$\mathbb{Q}[H_t = (k, j)] = \mathbb{Q}[X_t = k, N_t = j] \quad \text{where} \quad (k, j) \in S^H \quad \text{and} \quad t \geq 0. \tag{7.2.2}$$

In view of the above description of the bivariate Markov process H_t we now specify the generator $\boldsymbol{Q}_H$ for H_t on S^H. For a fixed value k of the first component of H_t we can treat the second component of H_t as a pure death process on $\{0, 1, \ldots, m\}$, i.e. a process which counts the number of defaulted obligors in the portfolio given that the underlying economy is in state k, that is $X_t = k$. Therefore, for any $j = 0, 1, \ldots, m-1$ the process H_t can jump from (k, j) to $(k, j+1)$ with intensity $(m-j)\lambda(k)$ where the mapping $\lambda(\cdot)$ is defined as in Corollary 4.2. Recall that $\lambda(k)$ is the individual default intensity when the factor process is in state k, i.e. $X_t = k$. Next, for a fixed value j of the second component of H_t (i.e. the number of defaulted obligors at time t are j) consider two distinct states k and k' in $\{1, \ldots, K\}$. Then, inspired by the construction of the underlying factor process X_t with generator $\boldsymbol{Q}$, we let the bivariate process H_t jump from (k, j) to (k', j) with intensity $\boldsymbol{Q}_{k,k'}$ where $k \neq k'$. These are the only allowed transitions for H_t. Hence, the generator $\boldsymbol{Q}_H$ for H_t is then given by

$$\begin{aligned} (\boldsymbol{Q}_H)_{(k,j),(k,j+1)} &= (m-j)\lambda(k) \quad 0 \leq j \leq m-1, \quad 1 \leq k \leq K \\ (\boldsymbol{Q}_H)_{(k,j),(k',j)} &= \boldsymbol{Q}_{k,k'} \quad\quad\quad 0 \leq j \leq m, \quad 1 \leq k, k' \leq K \quad k \neq k' \end{aligned} \tag{7.2.3}$$

and for each pair k, j we also have that

$$(\boldsymbol{Q}_H)_{(k,j),(k,j)} = -\sum_{(k',j') \in S^H, k' \neq k, j' \neq j} (\boldsymbol{Q}_H)_{(k,j),(k',j')}. \tag{7.2.4}$$

where the other entries in $\boldsymbol{Q}_H$ are zero. In view of this construction it is easy to see that it must hold

$$\mathbb{Q}[H_t = (k, j)] = \mathbb{Q}[X_t = k, N_t = j] \quad \text{where} \quad (k, j) \in S^H \quad \text{and} \quad t \geq 0. \tag{7.2.5}$$

Let $\boldsymbol{\alpha}_H \in \mathbb{R}^{K(m+1)}$ be the initial distribution of the Markov process H_t on the state space S^H with generator $\boldsymbol{Q}_H$ and consider $\boldsymbol{j} \in S^H$. From Markov theory we know that

$$\mathbb{Q}[H_t = \boldsymbol{j}] = \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t} \boldsymbol{e}_{\boldsymbol{j}}, \tag{7.2.6}$$

where $\boldsymbol{e}_{\boldsymbol{j}} \in \mathbb{R}^{K(m+1)}$ is a column vector where the entry at position $\boldsymbol{j}$ is 1 and the other entries are zero. Furthermore, $e^{\boldsymbol{Q}_H t}$ is the matrix exponential which has a closed form expression in

terms of the eigenvalue decomposition of $\boldsymbol{Q}_H$. Thus, in view of (7.2.5) and (7.2.6) we have for any $\boldsymbol{j} = (k, j) \in S^H$ and $t \geq 0$ that

$$\mathbb{Q}\left[X_t = k, N_t = j\right] = \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t} \boldsymbol{e}_{(k,j)}. \tag{7.2.7}$$

So (7.2.7) provides us with an efficient way to compute the probabilities $\mathbb{Q}\left[X_t = k, N_t = j\right]$ for any $t \geq 0$ and any pair $\boldsymbol{j} = (k, j)$ where k and j are integers such that $1 \leq k \leq K$ and $0 \leq j \leq m$. Note that there exist over 20 different ways to compute the matrix exponential, for more on this see e.g in Moeler & Loan (1978) and Moeler & Loan (2003).

Since $\mathbb{Q}\left[N_t = j\right] = \sum_{k=1}^{K} \mathbb{Q}\left[X_t = k, N_t = j\right]$ we retrieve that

$$\mathbb{Q}\left[N_t = j\right] = \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t} \sum_{k=1}^{K} \boldsymbol{e}_{(k,j)} = \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t} \boldsymbol{e}_{(\cdot,j)} \tag{7.2.8}$$

where $\boldsymbol{e}_{(\cdot,j)} \in \mathbb{R}^{K(m+1)}$ is a column vector defined as $\boldsymbol{e}_{(\cdot,j)} = \sum_{k=1}^{K} \boldsymbol{e}_{(k,j)}$. Finally, let us specify the the initial distribution $\boldsymbol{\alpha}_H \in \mathbb{R}^{K(m+1)}$ of the Markov process H_t on the state space S^H, defined as in (7.2.1). First, let $\boldsymbol{\alpha}$ be the initial distribution of the process X_t defined in Section 3.1. Then $\boldsymbol{\alpha}_k = \mathbb{Q}\left[X_0 = k\right]$ but we also know that

$$\pi_0^k = \mathbb{Q}\left[X_0 = k \,|\, \mathcal{F}_0^M\right] = \mathbb{Q}\left[X_0 = k\right] = \boldsymbol{\alpha}_k \tag{7.2.9}$$

which gives a relation between the values π_0^k and $\boldsymbol{\alpha}_k$. Next, let us now the initial distribution $\boldsymbol{\alpha}_H \in \mathbb{R}^{K(m+1)}$. We assume that all obligors in the portfolio are "alive" (non-defaulted) at time $t = 0$, i.e. today, which implies that the second component must be zero for all states of the economy background process modelled by the first component of the bivariate Markov process. Hence, it must hold that

$$\sum_{k=1}^{K} (\boldsymbol{\alpha}_H)_{(k,0)} = 1 \quad \text{and} \quad (\boldsymbol{\alpha}_H)_{(k,j)} = 0 \quad \text{for} \quad j = 1, 2, \ldots, m \tag{7.2.10}$$

which in turn guarantees that the sum of the entries in $\boldsymbol{\alpha}_H$ are one.

By using the formulas (7.2.7), (7.2.8) and (7.2.10) in (7.1.1) we can efficiently compute numerical values for the lower bounds of the option price $C_0(t, T; \kappa)$. Similarly, using (7.2.7), (7.2.8) and (7.2.10) in (7.1.10) will render exact values of the option price $C_0(t, T; \kappa)$ when $\kappa \leq \kappa^*$ where κ^* is defined as in (7.1.11).

Since typically m and K are allowed to be large, especially m, we will in general deal with very high dimensional state spaces of size $(m+1) \times K$, which requires special treatment when numerically dealing with the matrix exponential of the generator for H_t. Just computing the matrix exponential with standard algorithms will make the implementation slow and also inaccurate. Instead we will rely on the so-called uniformization method which has successfully been utilized in high-dimensional state space applications of portfolio credit risk, see e.g. in Herbertsson (2007), Herbertsson & Rootzén (2008), Herbertsson (2011), Bielecki, Crépey & Herbertsson (2011) and Lando (2004). In our case we will also exploit the sparseness of the transition matrices for H_t which makes the running times even quicker. With the help of H_t we will also display the loss distribution $\mathbb{Q}\left[N_t = k\right]$ for $k = 0, 1, \ldots, m$ and in particular the armageddon probabilities $\mathbb{Q}\left[N_t = m\right]$ for some calibrated examples in the filtering model outlined in Section 3 - 7. Finally, we have also performed robustness testes in order to increase the reliability of the implemented code. For example, we have checked that $\mathbb{Q}\left[X_t = k\right]$ is the same via $\boldsymbol{Q}$ and $\boldsymbol{Q}_H$.

8. NUMERICAL STUDIES

In this section we perform various numerical studies of the CDS index spread and CDS index option prices presented in Section 6 and 7, which in turn are based on the filtering model outlined in Section 3. The numerical studies are performed by calibration all parameters to market data

First, in Subsection 8.1 we gives a detailed outline of how the matrix Q and vector λ are chosen and then discuss how to estimate/calibrate the parameters $\theta = (Q, \lambda, a(x))$ in the filtering model introduced in Section 3 - 7.

Next, in Subsection 8.2 we use the results of Subsection 8.1 to calibrate, compute and display the simulated CDS index option prices as well as the lower bounds derived in Section 7 as functions of varies parameters such as the strike, the maturity and the spot-spread. We also calibrate the benchmark model 2.4.19 to the same spot CDS index spread. After this we compare the filtering prices, their lower bound prices and the benchmark prises for different values of ρ, $\hat{\sigma}$ and c. In particular, a numerical study is performed where we show that the lower bound in our model can be several hundred percent bigger compared with models which assume that the CDS-spreads follows a log-normal process. Also a systematic study is done in order to understand the impact of various model parameters on CDS index options (and on the index itself)

8.1. **Specifying the parameters and calibrating the filtering model.** In this section we discuss how to estimate/calibrate the parameters $\theta = (Q, \lambda, a(x))$ in the filtering model introduced in Section 3. Recall that $a(x)$ is the mapping used in (3.2.1) when constructing Z_t which is a function of X_t via the integral of the mapping $a(X_t)$ plus noise in form of a Brownian motion. Just as in Herbertsson & Frey (2014) and Frey & Schmidt (2012) we use one source of randomness when modelling the noise and in this paper we choose the mapping $a : \{1, 2, \ldots, K\} \mapsto \mathbb{R}$ in (3.2.1) to be on the form

$$a(X_t) = c \ln \lambda(X_t) \quad \text{that is} \quad a(k) = c \ln \lambda(k) \quad \text{for each state } k \qquad (8.1.1)$$

where c is a constant. The parametrization (8.1.1) have previously also been used in Herbertsson & Frey (2014), Frey & Schmidt (2012) and Frey & Schmidt (2011), see e.g. Subsection 5.3 in Frey & Schmidt (2012) and Example 7.6.1 in Frey & Schmidt (2011). For a detailed motivation of the choice (8.1.1) see on p.1420 in Herbertsson & Frey (2014). Hence, $a(k) = c \cdot \ln \lambda(k)$ for a constant c, so the parameters θ to be estimated are then given by $\theta = (Q, \lambda, c)$.

Next, for $\lambda = (\lambda(1), \ldots, \lambda(K))$ we will use the following piecewise linear parametrization of the mapping $\lambda(k)$ for the individual default intensity,

$$\lambda(k) = \begin{cases} bk & \text{if} \quad k \leq \lceil \frac{K}{2} \rceil \\ b\lceil \frac{K}{2} \rceil(1 - \beta) + \beta bk & \text{if} \quad k > \lceil \frac{K}{2} \rceil \end{cases} \qquad (8.1.2)$$

where β and b are constants such that $\beta > 1$ and $b > 0$. Note that for a continuous version of $\lambda(\cdot)$ we see that the left and right limit are the same which motivates the choice of the parameters for the case $k > \lceil \frac{K}{2} \rceil$, i.e. these are choosen so that

$$\lim_{x \uparrow \lceil \frac{K}{2} \rceil} \lambda(x) = \lim_{x \downarrow \lceil \frac{K}{2} \rceil} \lambda(x).$$

We here remind the reader that the states in $S^X = \{1, 2, \ldots, K\}$ are ordered so that state 1 represents the best state and K represents the worst state of the economy. Consequently, the mapping $\lambda(\cdot)$ is chosen to be strictly increasing in $k \in \{1, 2, \ldots, K\}$. This implies that

$\beta \geq 1$ in (8.1.2) and in our practical implementation we choose strict inequality, i.e. $\beta > 1$ so that the slope of the line $\lambda(k)$ will increase more for $k > \lceil \frac{K}{2} \rceil$ compared with $k \leq \lceil \frac{K}{2} \rceil$. The parametrization of $\lambda(k)$ in (8.1.2) is convenient in the sense that it describes the function $\lambda(k)$ with only two parameters β and b, regardless of the number of states K. The choice of the breakpoint $\lceil \frac{K}{2} \rceil$ can of course be changed, as well as the number of breakpoints, but more breakpoints forces us to use more parameters when describing $\lambda(k)$ (for a piecewise linear function we need a parameter for each slope in each region between the breakpoints etc.).

Next, we will assume that the finite state continuous time Markov chain X_t on the state space $S^X = \{1, 2, \ldots, K\}$ is a birth-death process with identical up and down transition intensities given by q. Hence, the generator $\boldsymbol{Q}$ will satisfy

$$
\boldsymbol{Q}_{i,j} = \begin{cases} q & \text{if } i = j-1 \text{ or } i = j+1 \\ -2q & \text{if } 2 \leq i = j \leq K-1 \\ -q & \text{if } i = j = 1 \text{ or } i = j = K \\ 0 & \text{otherwise} \end{cases} \tag{8.1.3}
$$

where $q > 0$. So (8.1.3) gives us only one parameter describing the generator $\boldsymbol{Q}$ regardless of the number of states K.

Hence, given the mapping $a(x)$ in (8.1.1) and the parametrization of $\boldsymbol{Q}, \boldsymbol{\lambda}$ in (8.1.2)-(8.1.3), the parameters to be estimated/calibrated are then $\boldsymbol{\theta} = (b, \beta, q, c)$. In this paper we will estimate $\boldsymbol{\theta} = (b, \beta, q, c)$ by calibrating the model spot CDS-index spread $S(0, 5)$ towards the corresponding observed market spread by using Equation (6.5) for $t = 0$ and the fact that $\boldsymbol{\pi}_0 = \boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ be the initial distribution of the process X_t defined in Section 3.1. This means that we also need to find the K values of $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K)$ and the parameters to be estimated are then $\boldsymbol{\theta} = (b, \beta, q, \boldsymbol{\alpha}, c)$. By using (6.5) with $\boldsymbol{\pi}_0 = \boldsymbol{\alpha}$ we define the $\bar{T}$-year model spot spread $S(\bar{T}; \boldsymbol{\theta})$ as

$$
S(\bar{T}; \boldsymbol{\theta}) = \frac{\boldsymbol{\alpha} \boldsymbol{A}(0, \bar{T}) \boldsymbol{1}}{\boldsymbol{\alpha} \boldsymbol{B}(0, \bar{T}) \boldsymbol{1}} \tag{8.1.4}
$$

and $\boldsymbol{\theta} = (b, \beta, q, \boldsymbol{\alpha}, c)$ is then calibrated via the following minimization routine

$$
\begin{aligned}
\min_{\boldsymbol{\theta}} \ & \left(\frac{S(\bar{T}; \boldsymbol{\theta}) - S_{\mathcal{M}}(\bar{T})}{S_{\mathcal{M}}(\bar{T})} \right)^2 \\
& \text{subject to} \\
& \boldsymbol{\alpha} \boldsymbol{1} = 1 \\
& 0 \leq \alpha_k \leq 1 \quad \text{for} \quad k = 1, \ldots, K \\
& \beta > 1, b > 0, q > 0
\end{aligned} \tag{8.1.5}
$$

where $S_{\mathcal{M}}(\bar{T})$ is the market quote for the $\bar{T}$-year CDS-index spread today, i.e. at time $t = 0$. We here remark that we can extend the above calibration routine by including market CDS-index spreads $\{S_M(\bar{T})\}_{T \in \mathcal{T}}$ for a several maturities such as e.g. $\bar{T} \in \mathcal{T} = \{3, 5, 7, 10\}$. In such a case the objective function in (8.1.5) is then replaced with

$$
\sum_{\bar{T} \in \mathcal{T}} \left(\frac{S(\bar{T}; \boldsymbol{\theta}) - S_{\mathcal{M}}(\bar{T})}{S_{\mathcal{M}}(\bar{T})} \right)^2 .
$$

We will in this paper only use one maturity $\bar{T} = 5$ which historically has been the most liquidly quoted CDS-index spread, in particular for the individual entries in the index.

The calibration of the parameters $\boldsymbol{\theta} = (b, \beta, q, \boldsymbol{\alpha}, c)$ is thus a constrained nonlinear optimization problem and such routines are mostly solved numerically using standard mathematical software packages such as e.g. matlab. Numerical optimization routines typically requires the user to provide an initial guess $\boldsymbol{\theta}_0 = \left(b_0, \beta_0, q_0, \boldsymbol{\alpha}^{(0)}, c_0\right)$ before running the scheme. In our initial guess $\boldsymbol{\theta}_0$ we let $\boldsymbol{\alpha}_k^{(0)} = \frac{1}{K}$ for each k.

Note that with $t = 0$ and for a given $\boldsymbol{\alpha}$, Equation (6.5), or equivalently (8.1.4) do not contain the parameter c used in (8.1.1). Hence, the calibration routine in (8.1.5) will not provide an estimation of c so this parameter must therefore be found by using another method.

If the mapping $a(x)$ is on the form (8.1.1) one can estimate c, or more generally, the parameters $\boldsymbol{\theta} = (\boldsymbol{Q}, \boldsymbol{\lambda}, c)$ in several different ways. In Herbertsson & Frey (2014), which uses (8.1.1), the authors outline a novel approach for estimating the parameters $\boldsymbol{\theta} = (\boldsymbol{Q}, \boldsymbol{\lambda}, c)$ in the filtering model of Frey & Schmidt (2012) under the physical/real measure by using time-series data on a CDS index and classical maximum-likelihood algorithms. This calibration-approach naturally incorporates the Kushner-Stratonovich SDE for the dynamics of the filtering probabilities. The computationally convenient formula for survival probability stated in Theorem 5.1 is a prerequisite for the estimation algorithm in Herbertsson & Frey (2014). In Frey & Schmidt (2012) the authors calibrate the parameter c under the risk-neutral measure by using the quadratic variation of the diffusion part of the index spread dynamics. Just as in Herbertsson & Frey (2014), the authors Frey & Schmidt (2012) observe that there were no defaults within the iTraxx Europe in their observation period for the time series, so the empirical quadratic variation of the index spreads is an estimate of the continuous part of the quadratic variation on the same index spread. The value of c obtained in Frey & Schmidt (2012) is in the same order as the corresponding value of c found via the MLE-estimation in Herbertsson & Frey (2014) under the real measure, see Section 5 in Herbertsson & Frey (2014).

Herbertsson & Frey (2014) gives a short literature overview of some papers that develops estimation techniques in models with incomplete information applied to credit risk applications. These are, amongst others Hurd & Zhou (2011), Azizpour, Giesecke & Kim (2011), Capponi & Cvitanic (2009), Fontana & Runggaldier (2010), Bhar & Handzic (2011), Duffie, Eckner, Horel & Saita (2009) and Koopman, Lucas & Schwaab (2011). Furthermore, Herbertsson & Frey (2014) also outline alternative methods to estimate $\boldsymbol{\theta} = (\boldsymbol{Q}, \boldsymbol{\lambda}, c)$ using quadratic programming. We refer the reader to Herbertsson & Frey (2014) for more details on all of the above topics.

In this paper the parameter c will be exogenously given and we can without loss of generality assume that c has been estimated with some of the above mentioned methods. Nevertheless, we will perform numerical studies of the price $C_0(t, T; \kappa)$ for an CDS index option in the filtering model presented in Section 3 as function of e.g. the parameter c. In particular we will use Monte-Carlos simulations to study $C_0(t, T; \kappa)$ as functions of the parameters $\boldsymbol{\theta} = (b, \beta, q, \boldsymbol{\alpha})$ and c where $(b, \beta, q, \boldsymbol{\alpha})$ are obtained via the calibration (8.1.5). Furthermore, we will also investigate how far away the simulated prices will deviate from the lower bound of the price derived in Proposition 7.2. A numerical study of when the inequality in Proposition 7.2 turns to the equality in given by Corollary 7.3 will also be performed.

8.2. Computing prices of options on the CDS index and their lower bound prices and comparing with lognormal model prices.
In this Subsection 8.2 we use the results of Subsection 8.1 to calibrate, compute and display the simulated CDS index option prices as well as the lower bounds derived in Section 7 as functions of varies parameters such as

the strike, the maturity and the spot-spread. We also calibrate the benchmark model 2.4.19 to the same spot CDS index spread. After this we compare the filtering prices, their lower bound prices and the benchmark prises for different values of ρ, $\hat{\sigma}$ and c.

First, for $m = 125$, and $r = 1\%$ we calibrate both the filter modelling for $K = 4$ with the parametrization (8.1.2) via (8.1.5) and the benchmark model (2.4.19) via (2.4.27) against a market spot CDS index spread $S(0,5) = 200$ bp. Then we compute the option price (2.4.19) as function of the strike price κ for different maturities of $t = 1, 3, 6, 9$ months and different values of ρ where $\rho = 90\%, 95\%$ and $\rho = 99.9$ for $\hat{\sigma} = 113\%$ and plot these values in Figure 3-5. In the same figures we also display lower bounds prices as function of κ when calibrated to the same market spot CDS index spread $S(0,5) = 200$ bp. In all cases the we have that $\phi = 40\%$. We also display the relative difference in percent between the option prices without absolute value in the numerator, and where the denominator is the Morini-Brigo value. In appendix we also show the difference between these prices without absolute value.

In Figure 3- 5 we clearly see that the lower bound in our model can be several hundred or thosands percent bigger compared with a model which assume that the CDS index spreads follows a log-normal process.

In Figure 3- 5 the volatility is kept constant. In order to view the impact of the volatility $\hat{\sigma}$, Figure 6 displays the Morini-Brigo option price 2.4.19 as function of the volatility $\hat{\sigma}$ for the maturities $t = 1, 3, 6, 9$ months and correlations $\rho = 90\%, 95\%, \rho = 99.9$ where $\S(0,5) = 200, \phi = 40\%$ bp. In the same figure wa also pot the lower bound prices in the filtering model calibrated against the same spot spread $S(0,5)$ and same recovery, as well as the relative difference in percent.

From the above studies we also discover that $\mathbb{Q}[N_t = m] = 0$ for all studied scenarios in this paper, and with the same market data such as spot-spread $S(0,5)$, maturity, interest rate, we will show that the corresponding armageddon probability in the one-factor Gaussian copula model is much higher when the correlation parameter ρ goes to one. Despite this fact we will clearly see that the lower bounds for the CDS-index option prices in the filtering modell will almost always be substantially higher (often several hundred percent higher) than the prices computed in the model Morini & Brigo (2011), at least for "at-the-money" strike-spreads, i.e. when $S(0,5) = \kappa$ and the maturity is smaller or equal to 1.5 years. Only in extreme scenarios, such as $\rho \geq 99.5\%$ jointly with volatilizes $\sigma \geq 110\%$ and maturities $t \geq 1.5$ years will generate prices in Morini & Brigo (2011) that are slightly above the lower bounds for the CDS-index option prices in the filtering modell.
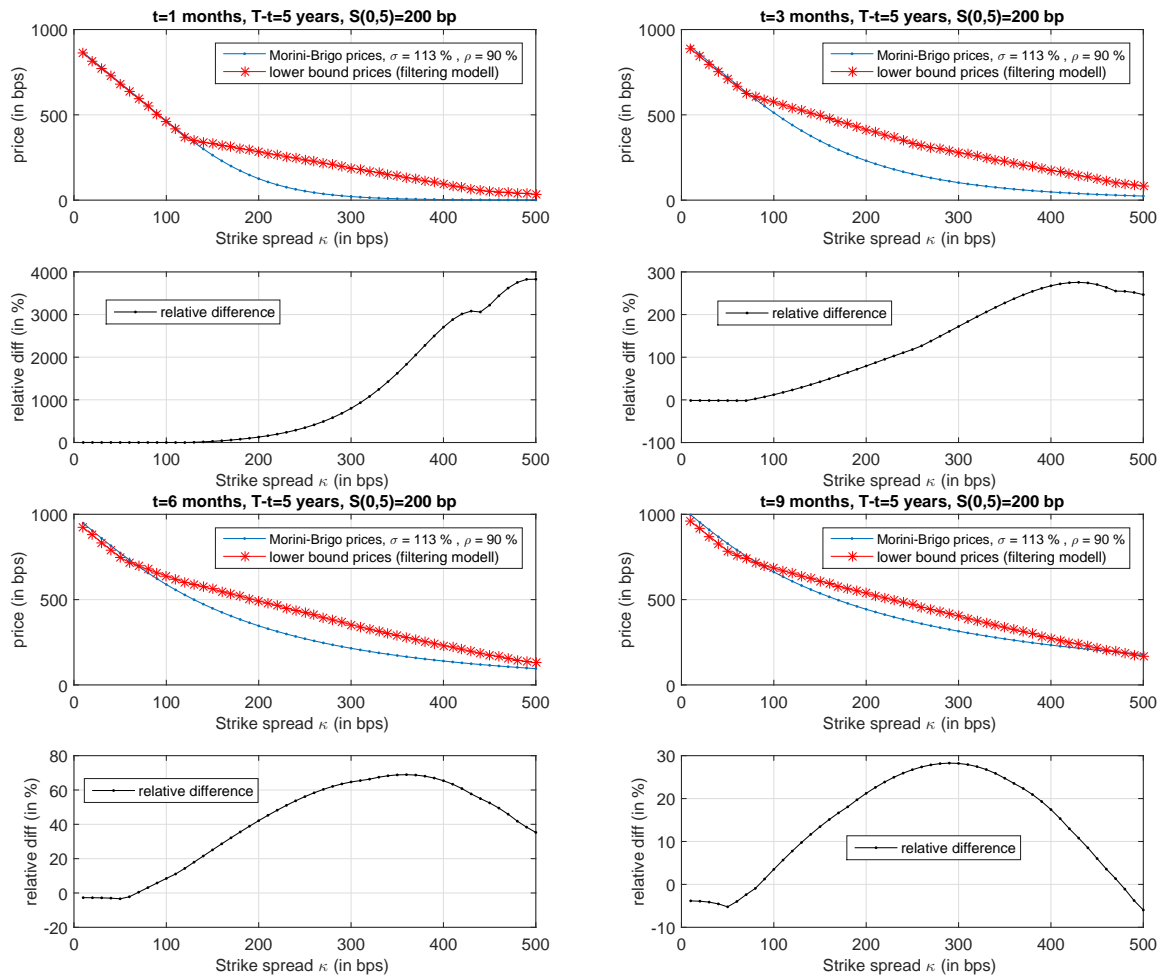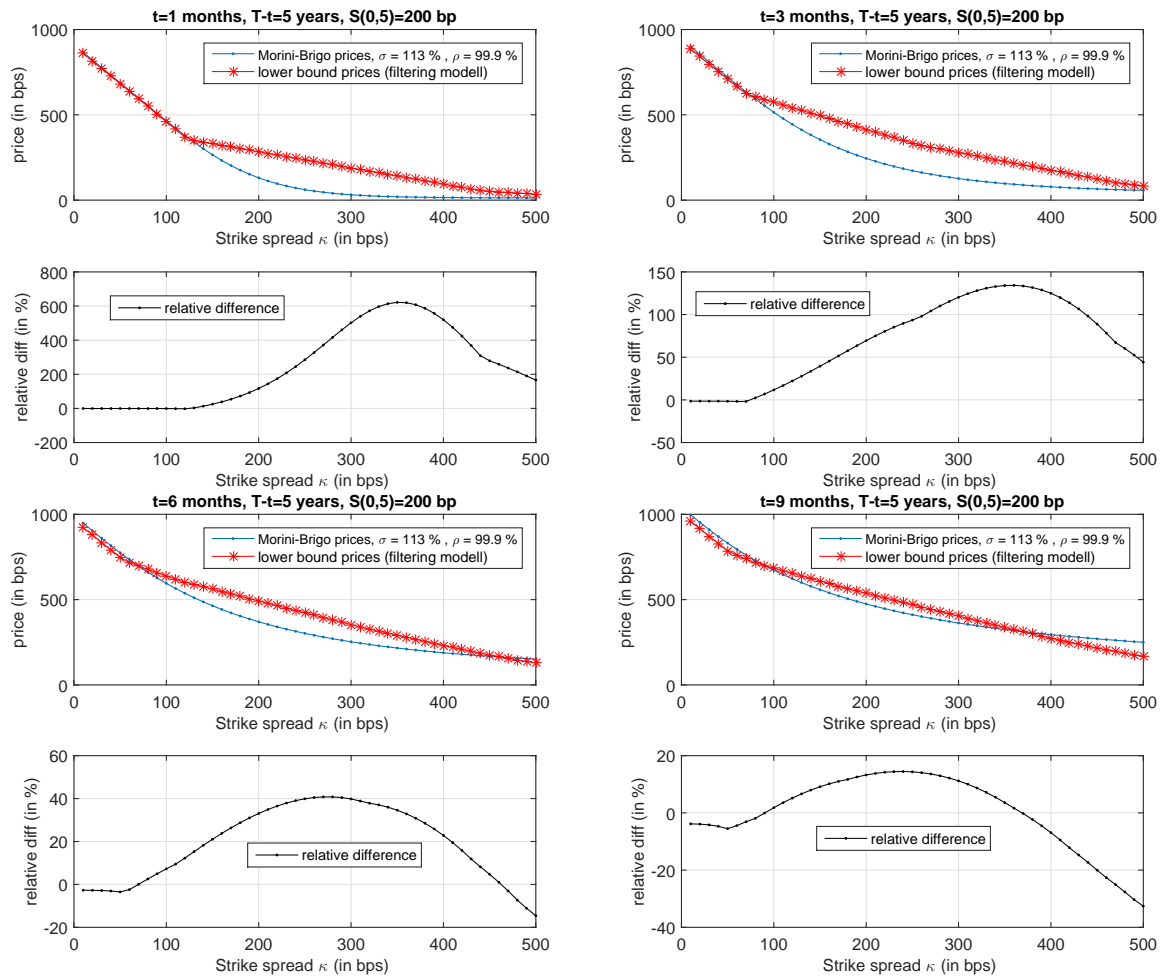
Figure 3. The Morini-Brigo option prices for the maturities $t = 1, 3, 6, 9$ months where $\rho = 90\%, \hat{\sigma} = 113\%, S(0,5) = 200, \phi = 40\%$ bp and the lower bound prices in the filtering model calibrated against the same spot spread $S(0,5)$ and same recovery.
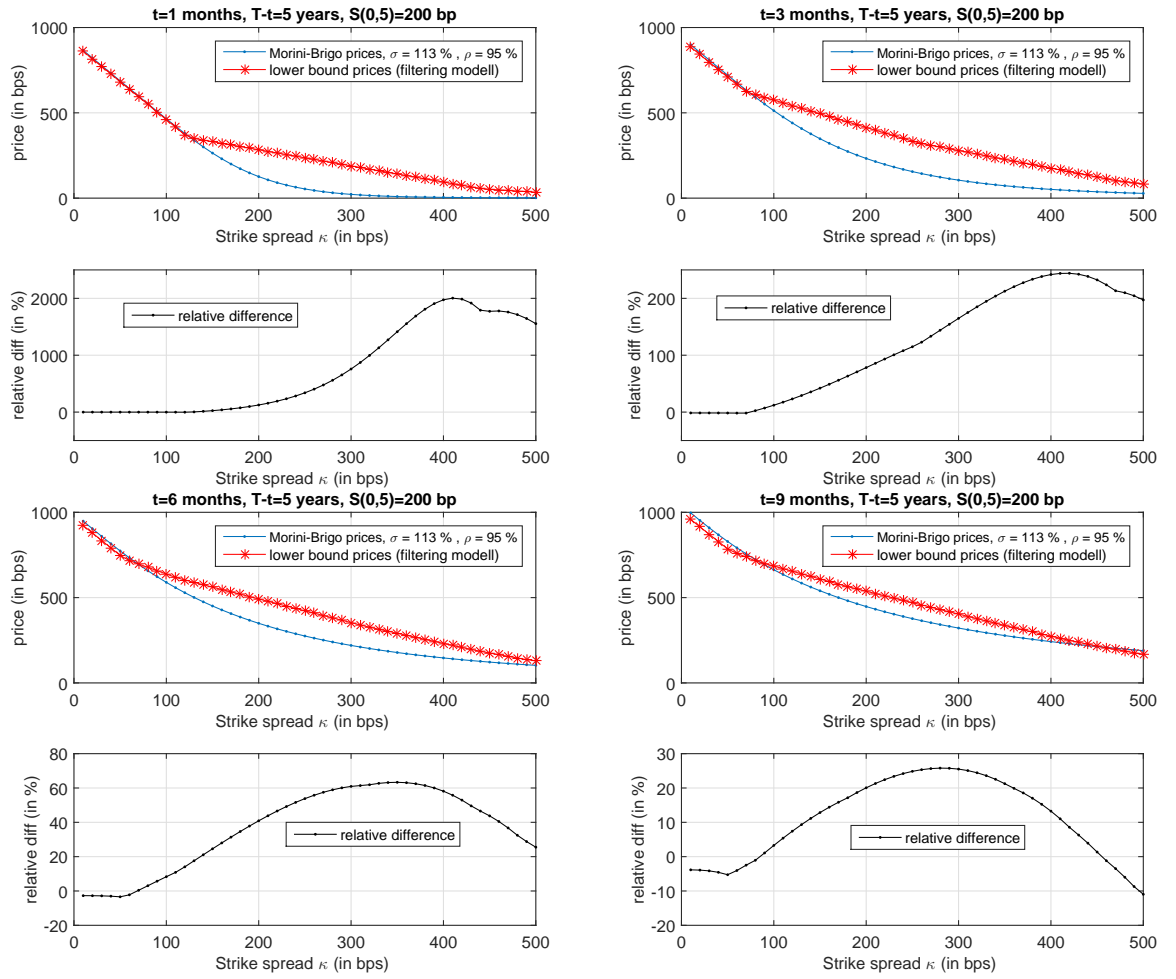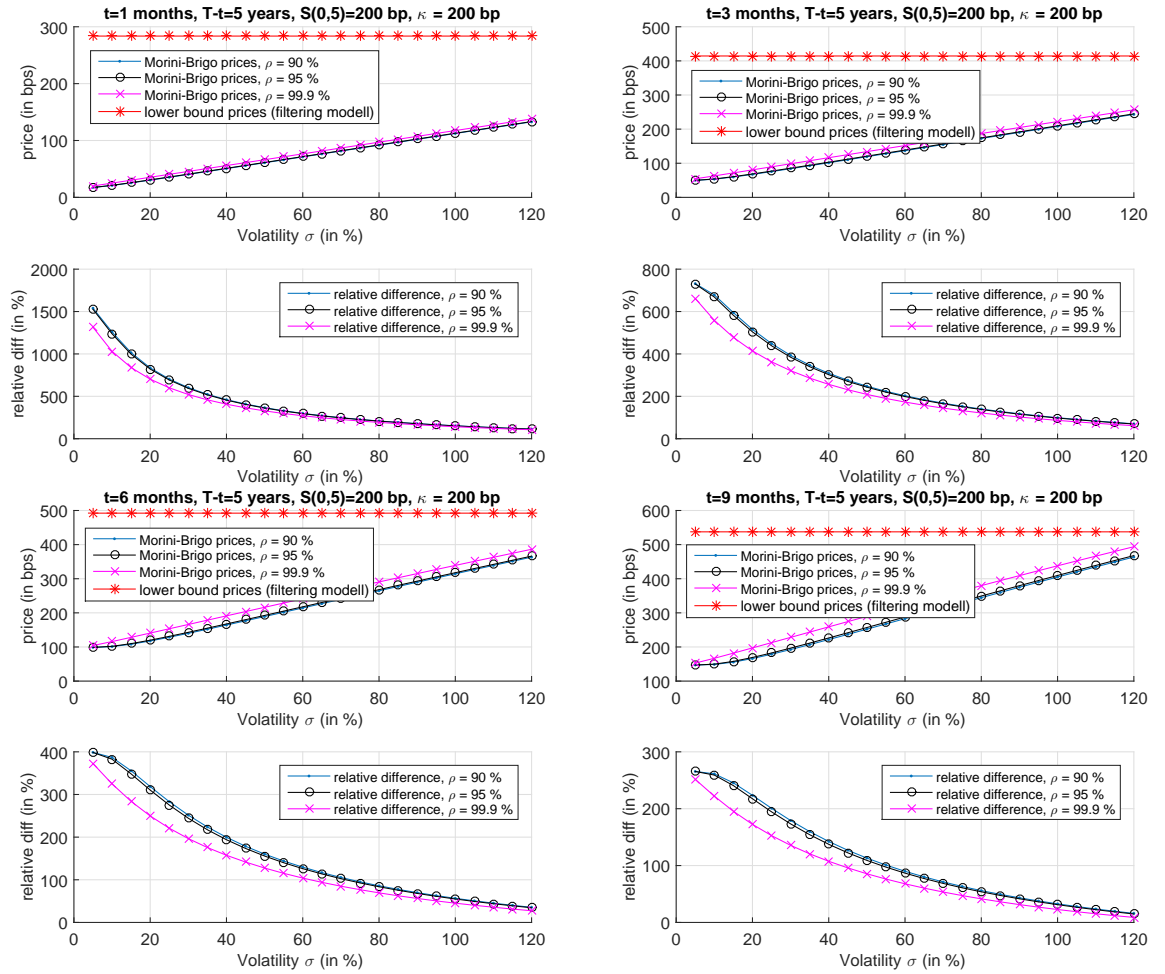
Figure 4. The Morini-Brigo option prices for the maturities $t = 1, 3, 6, 9$ months where $\rho = 95\%, \hat{\sigma} = 113\%, S(0,5) = 200, \phi = 40\%$ bp and the lower bound prices in the filtering model calibrated against the same spot spread $S(0,5)$ and same recovery.

Figure 5. The Morini-Brigo option prices for the maturities $t = 1, 3, 6, 9$ months where $\rho = 99.9\%, \hat{\sigma} = 113\%, S(0,5) = 200, \phi = 40\%$ bp and the lower bound prices in the filtering model calibrated against the same spot spread $S(0,5)$ and same recovery.

Figure 6. The Morini-Brigo option prices as function of the volatility $\hat{\sigma}$ for the maturities $t = 1, 3, 6, 9$ months and correlations $\rho = 90\%, 95\%, \rho = 99.9$ where $S(0,5) = 200, \phi = 40\%$ bp and the lower bound prices in the filtering model calibrated against the same spot spread $S(0,5)$ and same recovery.

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APPENDIX A. FEYMAN-KAC FORMULAS FOR FINITE-STATE MARKOV CHAINS

The purpose of this section is to introduce some useful formulas, that will be used throughout this paper. We first introduce some notation. Let $S^X = \{1, 2, \ldots, K\}$ and consider a functions $f(x) : S^X \mapsto \mathbb{R}$, then we denote $\boldsymbol{f}$ a $\mathbb{R}^K$-column vector with $\boldsymbol{f}_j = f(j)$ for $j \in S^X$. Next, let X_t be a finite state Markov jump process on $S^X = \{1, 2, \ldots, K\}$ with generator $\boldsymbol{Q}$. Then, for a function $\lambda(x) : S^X \mapsto \mathbb{R}$ we denote the matrix $\boldsymbol{Q}_\lambda = \boldsymbol{Q} - \boldsymbol{I}_\lambda$ where $\boldsymbol{I}_\lambda$ is a diagonal-matrix such that $(\boldsymbol{I}_\lambda)_{k,k} = \lambda(k)$ and $\boldsymbol{e}_x$ is a row vector in $\mathbb{R}^K$ where the entry at position x is 1 and the others entries are zero.

We now state with the following result.

Theorem A.1. *Let X_t be a finite state Markov jump process on $S^X = \{1, 2, \ldots, K\}$ with generator $\boldsymbol{Q}$. Consider functions $\lambda(x), f(x) : S^X \mapsto \mathbb{R}$. Then, with notation as above*

$$\mathbb{E}\left[e^{-\int_0^t \lambda(X_s)ds} f(X_t) \,\Big|\, X_0 = x\right] = \boldsymbol{e}_x e^{\boldsymbol{Q}_\lambda(T-t)} \boldsymbol{f}. \tag{A.1}$$

A proof of Proposition A.1 can be found on pp.273-274 in Rogers & Williams (2000). It is easy to extend Theorem A.1 to yield the following equality, for $T \geq t$

$$\mathbb{E}\left[e^{-\int_t^T \lambda(X_T)ds} f(X_T) \,\Big|\, X_t = x\right] = \boldsymbol{e}_x e^{\boldsymbol{Q}_\lambda(T-t)} \boldsymbol{f} \tag{A.2}$$

where the rest of the notation are as in Theorem A.1. The main point in Theorem A.1 is that given the matrix $\boldsymbol{Q}_\lambda$, then the left-hand side in Equation (A.1) (and Equation (A.2) is straightforward to implement using standard mathematical software.

We note that Theorem A.1 does not hold if the functions λ, f also depend on time t, i.e $\lambda(t, x), f(t, x) : [0, \infty) \times S^X \mapsto \mathbb{R}$. In such cases, one generally has to rely on numerical ODE method in order to find the quantity $\mathbb{E}\left[e^{-\int_0^t \lambda(s,X_s)ds} f(t, X_t) \,\Big|\, X_0 = x\right]$.

APPENDIX B. DERIVATION OF THE CREDIT TRIANGLE

The purpose of this section is to derive the relation

$$\lambda = \frac{S(\bar{T})}{1 - \phi} \tag{B.1}$$

where $S(\bar{T})$ is the $\bar{T}$-year CDS index spread for a homogeneous credit portfolio where the default times $\{\tau_i\}$ have constant default intensity λ which means that they are exponentially distributed with parameter λ, i.e. if τ has the same distribution as $\{\tau_i\}$ then

$$\mathbb{Q}\left[\tau \leq t\right] = 1 - e^{-\lambda t}. \tag{B.2}$$

The existing proofs of (B.1) found in the credit literature are only done for the unrealistic case when the CDS premium is paid continuously. In practice the CDS premiums are done

quarterly. Furthermore, formula (B.1) is used repeatedly in portfolio credit risk, see e.g. Equation (9.11) on p.404 in McNeil et al. (2005). Below, we will for notational convenience write T instead of $\bar{T}$.

Proposition B.1. *Consider a CDS index with maturity T on a homogeneous credit portfolio where the obligors have constant default intensity λ and where the interest rate is r. Then,*

$$S(T) = 4(1-\phi)\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)\frac{\lambda}{\lambda+r}\frac{1 - e^{-(r+\lambda)T}}{e^{-\frac{r+\lambda}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}}} \tag{B.3}$$

and if $r + \lambda$ is small it holds that

$$\lambda \approx \frac{S(T)}{1 - \phi}. \tag{B.4}$$

Proof. First recall that $S(T)$ is shorthand notation for $S(0,T)$ and using the definition of $S(0,T)$ in Equation (2.2.6) we have that

$$S(T) = S(0,T) = \frac{DL(0,T)}{PV(0,T)} \tag{B.5}$$

where

$$DL(0,T) = \mathbb{E}\left[\int_0^T e^{-rs}dL_s\right] = (1-\phi)\int_0^T e^{-rs}f_\tau(s)ds = \frac{(1-\phi)\lambda}{\lambda+r}\left(1 - e^{-(r+\lambda)T}\right) \tag{B.6}$$

and the last two equations in (B.6) follows from (B.2) and similar computations as in Equation (2.4.31) and (2.4.33) in Propositon 2.3 with $t = 0$. Furthermore, (2.2.5) implies that

$$PV(0,T) = \frac{1}{4}\sum_{n=1}^{\lceil 4T\rceil} e^{-rt_n}\left(1 - \frac{1}{m}\mathbb{E}\left[N_{t_n}\right]\right) \tag{B.7}$$

where $t_n = \frac{n}{4}$ and which after identical computations as in (2.4.30) with $t = 0$, renders that

$$PV(0,T) = \frac{e^{-\frac{(r+\lambda)}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}}}{4\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)}. \tag{B.8}$$

Hence, (B.6) and (B.8) in (B.5) then gives

$$S(T) = 4(1-\phi)\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)\frac{\lambda}{\lambda+r}\frac{1 - e^{-(r+\lambda)T}}{e^{-\frac{r+\lambda}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}}} \tag{B.9}$$

which proves (B.3). Next, if r and λ are small we can use the following first order Taylor expansion

$$e^{-\frac{r+\lambda}{4}} \approx 1 - \frac{r+\lambda}{4} \tag{B.10}$$

which renders

$$4(1-\phi)\left(1 - e^{-\frac{r+\lambda}{4}}\right)\frac{\lambda}{\lambda+r} \approx (1-\phi)\lambda \tag{B.11}$$

and

$$\frac{1 - e^{-(r+\lambda)T}}{e^{-\frac{r+\lambda}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}}} \approx \frac{1 - e^{-(r+\lambda)T}}{1 - e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}} - \frac{r+\lambda}{4}} \approx 1 \tag{B.12}$$

since $\frac{\lceil 4T \rceil + 1}{4} \approx T$ and $\frac{r+\lambda}{4}$ is small compared to $1 - e^{-\frac{(r+\lambda)(\lceil 4T \rceil + 1)}{4}}$ when T is larger (typically $T = 5$ or $T = 10$). Hence, under (B.10) the approximations (B.11)-(B.12) inserted in (B.9) then imply that

$$S(T) \approx (1 - \phi)\lambda$$

that is

$$\lambda = \frac{S(T)}{1 - \phi}$$

which proves (B.4).   □

We here remark that if one in the single-name CDS spread assumes that the default payment in the default leg is postponed to the end of the quarter in which the default happens, then, assuming (B.2), one can prove (B.4) for a general interest rate which not necessary have to be constant. By Lemma 6.1, p.1203 in Herbertsson et al. (2011) this will therefore also hold for a CDS-index.

In a perfectly calibrated model we have by definition that $S_{\mathcal{M}}(T) = S(0, T)$ which in (B.4) can be used to find a numerical value for λ given that the recovery ϕ is known.

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