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The Fallacy of The Cox, Ingersoll & Ross Model

An empirical study on US bond data

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Abstract

This paper examines the performance of the original one-factor Cox, Ingersoll & Ross model during a contemporary dataset using an OLS procedure. The model is fundamental for many financial models used for investment decisions, it is therefore important to validate its performance during current market conditions characterized by low interest rates. The study is performed on a dataset of US Treasury Notes traded during the period 2005/01/01 - 2016/12/31 and contains 663 unique bonds. In accordance with previous studies, it is concluded that the estimated parameters are unstable and that the model performs worse during periods when short term interest rates are close to zero. Consequently, the assumption of positive interest rates might be too strong.

Key words: Term-structure, CIR, interest rates, bonds, US Treasury Notes

JEL classification: E43, G12

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Contents

1   Introduction

The relationship between yields and time to maturity is a never-ending concern for economists. The term structure contains information regarding present and anticipated future market conditions, and is thus of crucial importance when making investment decisions. As a result, a lot of studies have been conducted to find a suitable model to be able to estimate and explain the dynamics of the term structure. One of the more eminent studies was published by Cox, Ingersoll and Ross (1985a), which resulted in the one-factor Cox-Ingersoll-Ross model. Their proposed model for interest rate dynamics has since been widely discussed and analyzed, and still to this day forms the fundamental basis of many financial valuation models.

Ever since the original article was published by Cox *et al* (1985a), a lot of empirical testing has been done on their proposed model. Only one year after the Cox, Ingersoll and Ross study was published, Brown & Dybvig (1986) estimated the model using monthly data from the U.S. bond market during the time period of December 1952 to December 1983. Their study concludes that, by using the model, it is possible to estimate the implied short and longterm rates and the implied volatility of changes in short rates. However, the model systematically overestimates the implied short rates of return and the pricing errors are not identically distributed across issues of different maturities. In similar fashion the model was estimated by Barone *et al* (1991) by using data from the Italian bond market from 1983/12/30 to 1990/12/31. Additionally, Brown & Schaefer (1994) estimated the model by using British government index-linked bond prices from March 1981 to December 1989. Both of these studies conclude that the CIR model fits the data well with low mean absolute pricing error, but as implied by Brown & Dybvig (1986), the parameters are found to be unstable. The argument of a good fit but with unstable parameters is further strengthened by Rebonato (1996), who analyzes the CIR model in the light of interest-rate option valuation.

Contrary to these older studies, more recent studies argue that the fit of the model is not as good as previously implied. Steely (2008) conducted a study where the CIR model is estimated by using UK government

bonds, STRIPS, during the period 1997/12/08 to 2002/05/15. The details of the CIR parameters are not discussed, but it is concluded that the CIR model produces the largest absolute pricing errors among all the models used in the study. An even more recent dataset is used by Ullah *et al* (2013), who estimate the model using Japanese Treasury Bonds and Bills from January 2000 to December 2011. Their findings show that the model has a poor fit during 2000 to 2006, which they argue is due to the low interest rates in Japan.

The results of previous studies are thus mixed regarding whether the model can fit the term structure of interest or not. The view appears to have changed as the interest rate climate has changed, which suggests that the underlying assumption of the model to preclude negative interest rates may cause problem when applying the model on the present interest rate market. This implies that the model demands further investigation, especially during the present financial climate that has been characterized by lower interest rates than ever previously observed. Nonetheless, the estimated parameters are deemed to be generally unstable and display large standard deviations by all of the mentioned studies. Changes in the term-structure can be perceived as problematic not only in the aspect of assets valuation, but also in measurement of risk exposure. Hence, unstable parameters can give rise to risks unaccounted for and should therefore not be taken lightly. Rebonato (1996) specifically mentions the problem with applying unstable parameters on hedging, but further elaborates that it is a general concern when applying on all other applications than pricing. A solution to the unstable parameters is already suggested in the original article by Cox *et al* (1985a). They suggest that including data from other securities than bonds alone can provide more stable parameters. Example of such data could be option data, which Cox *et al* (1985a) provide a pricing formula for. This is another area that require further examination.

The aim of this study is to test the validity of the CIR model on a contemporary dataset using US Treasury Notes between 2005/01/01 and 2016/12/31. Since the model does not allow for negative interest rates, the chosen market is suitable considering the extremely low, albeit not negative, interest rates compared to other markets in e.g. Europe during the observed period. No previous studies were found using the original CIR

model on a dataset as recent as used in this study; recent studies rather seem to test versions of the CIR model that includes different sets of control variables. Given the recent financial crisis and the subsequently low interest rate climate, it is important to examine the accuracy of the model during a period with such characteristically low interest rates. Since the model is still widely used and forms the basis in many different financial models, it is highly significant to confirm if the model is still relevant.

The model parameters are estimated using an OLS procedure on a dataset consisting of 663 unique US Treasury Notes issued with maturities of either two, three, five, seven, ten, twenty or thirty years. Given the characteristics of the period, evidence from previous studies and potential multicollinearity problems, it is suspected that the fit will vary and the parameters of the model will be inconsistent and display large standard deviations.

The rest of the paper is structured as follows. Section 2 presents relevant previous studies on the subject. Section 3 describes the CIR model and how the estimation is optimized. Section 4 describes the data used conducting the estimation. Section 5 presents the results together with analysis. Conclusions are presented in section 6.

2   Previous studies

Empirical testing of the CIR model began only one year after the original article was published. Brown & Dybvig (1986) estimate the parameters of the CIR model monthly from December 1952 to December 1983 using data from the U.S. bond market. In their work they use non-linear least squares to estimate the parameters, and their results show that the model systematically overestimates the implied short rates of return while the results are a bit more mixed regarding the implied long rate. They further find violation of the assumption that pricing errors are identically distributed across issues of different maturities and significantly differences in pricing of premium- and discount-issues. It is suggested that the last finding is

due to a possibly neglected tax-effect.

The empirical testing continued by Barone *et al* (1991), who use the CIR model to obtain daily estimates of the term structure of the Italian bond market from 1983/12/30 to 1990/12/31. They find that the model fits the data well with a mean absolute pricing error of bonds of 0.29. They find a high correlation between the estimated instantaneous interest rate and the yield of the three month treasury bill, and in accordance with Brown & Dybvig (1986), they also find that the estimated residuals are related to time to maturity. However, the null hypothesis of constant parameter estimates is strongly rejected, although the stability in the parameters increased in the end of their sample when their sample size increased. Further empirical testing was conducted by Brown & Schaefer (1994). They use the CIR model to estimate the term structure of real interest rates. In their paper they use daily data on British government index-linked bond prices between March 1981 and December 1989. Their results show that the CIR model fits the data well, with an absolute price error no larger than 0.20, and that it closely can approximate the shape of the real term structure. They further find that their estimated long-term zero-coupon yield is quite stable, but in accordance with other studies parameter stability is rejected.

The CIR model was scrutinized differently by Rebonato (1996). He analyzes the CIR model in the light of interest-rate option valuation. He concludes that generally, empirical results suggest that the CIR parameters are unstable, but nonetheless are able to provide a good fit. A further argument is made that the model allows for radically different sets of parameters to fit the yield curve, which can be a problem when applying the parameters on other applications than pricing, such as hedging. However, Rebonato (1996) elaborates that many models, Black-Scholes among others, violate the assumption of constant parameters and thus argues that the CIR model still can be of practical use.

More recent empirical testing of the CIR model concludes different results regarding the fit. Steeley (2008) undertakes a study in which the author tests and compares different models for term structure estimation.

The dataset consists of UK government bonds, STRIPS, over the period 1997/12/08 to 2002/05/15. Although much is not said regarding the details of the CIR parameters, the CIR model appears to produce the largest absolute pricing and yield errors among the tested models. Steeley (2008) concludes that models fitted directly to the yield curve is likely to perform better than models fitted to the discount function (as is the case with the CIR model). An even more recent study is done by Ullah *et al* (2013) who evaluate the CIR model on the Japanese bond market. The sample used consists of Japanese treasury bonds and treasury bills between January 2000 and December 2011. Ullah *et al* (2013) find that the CIR model poorly estimates the yield curve, which the authors suggest could be explained by the low interest rates in Japan from year 2000 to 2006.

There seems to be a consensus that the estimated parameters are unstable, but the view regarding the fit of the model varies. More recent studies show that the fit is not as good as previously observed, which could be due to the changed interest rate climate. This implies that the model demands further examination in a more contemporary regard.

3   Methodology

This section presents relevant theory related to the model, together with a description of how the study was conducted.

3.1   The Cox, Ingersoll & Ross Model

The one-factor Cox, Ingersoll & Ross Model (1985a), from now on referred to as the CIR model, is based on a general equilibrium model developed by Cox *et al* (1985b). The CIR model aims to describe the term structure of interest, i.e. the relationship between bonds of different maturities and yields. Following their work, the interest rate dynamics can be written as:

$$d\rho = \kappa(\theta - r)dt + \sigma\sqrt{r}dz_1,$$

where ρ is the current instantaneous interest rate, κ, θ and σ^2 are constants with $\kappa\theta \geq 0$ and $\sigma^2 > 0$, and

z_1 is a Wiener process. The interest rate follows a diffusion process with drift $\kappa(\theta - r)$ and variance $\sigma^2 r$.

The interest rate process is a continuous time first-order autoregressive process, where the interest rate is

pulled toward its long-term value θ. κ determines the speed of adjustment towards the long-term mean. An

important property which is implied by the setup of this model is that the model does not allow for negative

interest rates according to Cox *et al* (1985a).

Cox *et al* (1985a) further provide formulas for pricing zero-coupon bonds based on this interest rate process.

The bond price can be calculated as:

$$P(\rho, t, T) = A(t, T)e^{-B(t,T)\rho}, \tag{1}$$

where

$$A(t, T) = \left[\frac{2\gamma e^{[(\kappa+\lambda+\gamma)(T-t)]/2}}{(\gamma + \kappa + \lambda)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{2\kappa\theta/\sigma^2},$$

$$B(t, T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + \kappa + \lambda)(e^{\gamma(T-t)} - 1) + 2\gamma},$$

$$\gamma = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2},$$

where κ, θ and σ^2 are the same as before and λ is a "market" risk parameter (Cox *et al*, 1985a). Yield-to-

maturity, $R(\rho, t, T)$, can be calculated as:

$$R(\rho, t, T) = \frac{\rho B(t, T) - \log A(t, T)}{(T - t)}$$

As maturity approaches zero, $R(\rho, t, T)$ approaches the current instantaneous interest rate ρ, as maturity goes to infinity yield-to-maturity approaches the limit:

$$R(\rho, t, \infty) = \frac{2\kappa\theta}{\gamma + \kappa + \lambda}. \tag{2}$$

From this limit it follows that the curvature of the yield curve is rising if the current interest rate, ρ, is below $2\kappa\theta/(\gamma + \kappa + \lambda)$, and falling if ρ is larger than $\kappa\theta/(\kappa + \lambda)$. If the interest rate is between these values, the yield curve is humped. The variance of the instantaneous rate is:

$$Var\left(\rho(\tau)\right) = \rho\left(\frac{\sigma^2}{\kappa}\right)(e^{-\kappa\tau} - e^{-2\kappa\tau}) + \sigma\left(\frac{\sigma^2}{2\kappa}\right)(1 - e^{-\kappa\tau})^2. \tag{3}$$

When pricing bonds with coupons one can use the fact that a coupon bond can be considered a portfolio of several zero coupon bonds. The price of the coupon bond, B, is calculated as:

$$B = \sum_{k=1}^{K} P_k(\rho, \kappa, \theta, \sigma, \tau_k)c_k,$$

where $P_k(\rho, \kappa, \theta, \sigma, \tau_k)$ comes from *Equation (1)*, c_k is the coupon or nominal of the bond and K is the number of payments to be made during the remaining lifetime of the bond, including the nominal.

3.1.1   Option Pricing in the Cox, Ingersoll & Ross Model

In extension to the CIR bond pricing formulas, Cox *et al* (1985a) provide formulas for pricing options on zero coupon bonds. Consider a call option of expiration date T and exercise price K on a zero coupon bond of maturity date s. The value of the call option at date t, $C(r, t, T; s, K)$, is subject to the standard terminal condition:

$$C(\rho, t, T; s, K) = \max[P(\rho, T, s) - K, 0].$$

The formulas for the option price are:

$$C(\rho, t, T; s, K) = P(\rho, t, s)\chi^2 \left(2r^*[\phi + \psi + B(T, s)]; \frac{4\kappa\theta}{\sigma^2}, \frac{2\phi^2 \rho e^{\gamma(T-t)}}{\phi + \psi + B(T, s)}\right) -$$
$$-KP(\rho, t, T)\chi^2 \left(2r^*[\phi + \psi]; \frac{4\kappa\theta}{\sigma^2}, \frac{2\phi^2 \rho e^{\gamma(T-t)}}{\phi + \psi}\right), \tag{4}$$

where

$$\gamma = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2},$$

$$\phi = \frac{2\gamma}{\sigma^2(e^{\gamma(T-t)} - 1)},$$

$$\psi = \frac{\kappa + \lambda + \gamma}{\sigma^2},$$

$$r^* = \frac{\log\left(\frac{A(t,s)}{K}\right)}{B(t,s)}.$$

χ^2 is the non-central chi-square distribution and r^* is the critical interest rate such that $K = P(r^*, T, s)$, if the interest rate is below r^* exercise will occur. $P(.)$ is the same as in *Equation (1)*. Note that these formulas are for options on zero coupon bonds only.

3.2   Estimation

The parameters of the CIR model is estimated using the provided bond pricing formula in *Equation (1)*, together with actually observed market prices of US Treasury Notes. Since λ cannot be observed nor estimated separately, this parameter is set to zero. A minimization procedure is performed to find the parameter values of κ, ρ, θ and σ which minimizes the sum of squared residuals between actual bond prices and CIR bond prices:

$$min_{\rho, \kappa, \theta, \sigma} \sum_{i=1}^{n} \left[B_{i,t} - \sum_{k=1}^{K_i} P_k(\rho_t, \kappa_t, \theta_t, \sigma_t, \tau_{i,k})c_{i,k}\right]^2, \tag{5}$$

where $B_{i,t}$ is the observed market price of bond i at time t, $\sum_{k=1}^{K_i} P_k(\rho_t, \kappa_t, \theta_t, \sigma_t, \tau_{i,k})c_{i,k}$ is the CIR bond price of bond i at time t and n is the number of bonds used in the optimization.

Because of potential multicollinearity between parameters, the minimization is performed under some constraints on the parameters. Ideally no constraints would be used, but constraints are necessary. The constraints prohibits the instantaneous rate, ρ, to become negative which is not allowed in the CIR framework. The other constraints are chosen on financial intuition and provides ranges which are considered to be "reasonable" without forcing the model to optimize within a too narrow interval. See Table 1 for the used boundaries.

Table 1: Parameter Boundaries

Parameter	Lower Bound	Upper Bound
κ	0.00001	3
ρ	0.00001	0.15
θ	0.000001	0.15
σ	0.000001	0.7

Together with these linear constraints, non-linear constraints are put on the estimation as well. Given what is known from the section above regarding yield-to-maturity when maturity goes to infinity, $R(\rho, t, \infty)$, *Equation (2)*, is constrained to be greater than or equal to the yield of the bond with the longest maturity in the data set on a given day, which is economically motivated. Further, the variance of the instantaneous rate, *Equation (3)*, is restricted to be no greater than 2%.

For the first day in the sample, the set of parameters are estimated by minimizing the residuals in *Equation (5)*. From the second day and forward, the optimization procedure starts by using the estimated set of parameters from the day before when minimizing the residuals in *Equation (5)*.

Bond prices calculated by using the CIR model are "dirty prices", in other words they include accrued interest. For the estimation to be done correctly, prices retrieved from the data source need to be adjusted

to include accrued interest, since they originally does not. The value of accrued interest is calculated as:

$$AccruedInterest = c\left(\frac{DLC}{DP}\right),$$

where c is the coupon, DLC is the number of days from the last coupon payment to the settlement day and DP is the number of days in the current coupon period. To get the dirty price, accrued interest is simply added on to the clean price retrieved from the Bloomberg database. Yields are calculated using the built-in Matlab function, *bndyield*, which corresponds with Bloomberg's yields.

3.2.1   Estimation including option data

It has previously been observed that the CIR parameters are unstable and display high standard deviations. To account for this it has been proposed already in the original work of Cox *et al* (1985a) to include information from other markets than the bond market alone. Cox *et al* (1985a) state the following on page 396: "[...] applications to other securities may permit richer and more powerful empirical tests than could be done with the bond market alone.". One such approach would be to include data from the option market and use the provided option pricing formulas to estimate the volatility parameter, σ. Considering option prices high sensitivity to volatility, options should be a suitable security type to include in the estimation.

The formula for valuing a call option on a zero coupon bond is given by *Equation (4)*. When valuing a call option on a coupon bond the formula need to be adjusted. Observing *Equation (4)* it consists of two terms where the first term is the expected value of the bond and the second term is the expected value of the strike. To adjust the formula to allow for valuing options on coupon bonds, Longstaff (1993) suggests that the first term changes to:

$$\sum_{n=1}^{N} P(\rho, t, s_n^*) * c_n * \chi^2\left(2r^*[\phi + \psi + B(T, s_n^*)]; \frac{4\kappa\theta}{\sigma^2}, \frac{2\phi^2\rho e^{\gamma(T-t)}}{\phi + \psi + B(T, s_n^*)}\right),$$

where N is the number of coupons plus nominal value left until maturity of the bond, s_n^* is the maturity

date of coupon or nominal n and c_n is the coupon or nominal value. This is accordance with Jamshidian's (1989) proof that an option on a portfolio is equivalent to a portfolio of options when the prices are monotone functions of the same state variable, interest rate in this case. To proceed with the optimization one is in need of option prices. The option pricing formula provided by Black (1976), from now on referred to as the Black-76 option pricing formula, is used to calculate the option prices given the gathered option data.

The starting values for the optimization are the CIR parameters obtained when estimated on the bond market alone. Keeping κ, ρ and θ constant on these values, the CIR option pricing formula is used to estimate σ:

$$min_\sigma \sum_{i=1}^{n} \left[C_{i,t}^* - C_{i,t} \right]^2 ,$$

where $C_{i,t}^*$ is the Black-76 price of bond i at time t, $C_{i,t}$ is the CIR option price of option i at time t and n is the number of options used in the optimization. Values for r^* need to be re-estimated in every iteration for the option minimization program. The resulting value of σ is then held constant while re-estimating κ, ρ and θ using the procedure for optimization with bond data:

$$min_{\rho,\kappa,\theta} \sum_{i=1}^{n} \left[B_{i,t} - \sum_{k=1}^{K_i} P_k(\rho_t, \kappa_t, \theta_t, \sigma_t, \tau_{i,k})c_{i,k} \right]^2 , \tag{6}$$

where $B_{i,t}$ is the observed market price of bond i at time t, $\sum_{k=1}^{K_i} P_k(\rho_t, \kappa_t, \theta_t, \sigma_t, \tau_{i,k})c_{i,k}$ is the CIR bond price of bond i at time period t and n is the number of bonds used in the optimization. This procedure is repeated until the change in SSE (sum of squared errors) between CIR and actual bond prices is sufficiently small (smaller than at least 0.01%), but at least repeated two times.

4   Data

This section gives an description of the data set used in the estimation

4.1   Bond data

Data is retrieved from Bloomberg's database and consists of all US Treasury Notes Bloomberg has available data of, issued with maturities of either two, three, five, seven, ten, twenty or thirty years, traded during the observed period. Time to maturity ranges from one and a half year to thirty years. Shorter time to maturities has been removed due to liquidity reason, which coincides with the discussion to Brown and Dybvig (1986) paper where Ferson (1986) argues that shorter maturities are only used by investment banks for margin calls and are not actively traded. The sample consists of 663 different US Treasury Notes traded during the period 2005/01/01 to 2016/12/31. The US Treasury Note market is chosen since the model does not allow for negative interest rates, which are present in e.g. Europe during the observed period.

The retrieved data consists of clean prices, maturity dates, issue dates and coupon rates. All the US Treasury Notes has coupon payment on a semi-annual basis, and day count convention is actual over actual. The settlement day is the next business day from the observed trade date for all US Treasury Notes. Callable US Treasury Notes and detected erroneous data are omitted from the sample. The number of active notes during a given trading day ranges from 95 in the beginning of the observed period to 233 in the end of the observed period. Coupon rates on these US Treasury Notes ranges from 0.125 to 12 percent. Since only clean prices are retrievable from the Bloomberg database, yields and dirty prices are manually calculated. The bond yields on these US Treasury Notes ranges from 0.1 percent to 5.5 percent.

See Table A1 in appendix for a more detailed summary statistics.

4.2   Option data

During this study, an attempt has been made to gather suitable option data, trying to improve the stability in the estimated parameters. Unfortunately, the quality of the data available on interest rate options in the Bloomberg database has been questioned by the authors of this paper. It has therefore been decided not

to include any results from this attempt in this paper. The market data available consisted of three month implied volatilities for options of 90%-, 100% - and 110%-moneyness strikes on US Treasury Notes Future contracts with different maturities, ranging from 2 years up to 30 years. First of all, inconsistencies were observed between the relationship of implied volatilities for different strikes. Most notably the difference between implied volatility for the 90%- and 110%-moneyness strikes are unreasonably large. Secondly, large "spikes" are observed in the data. Between two days the implied volatility could temporarily increase by several hundred percent, inconsistently across strikes, only to return to a value similar to the one before the spike on the following day.

5 Results and analysis

In this section, results including an analysis of the results are presented. The first results that are presented represent the entire sample period, but is then analyzed in more detail during three sub-periods. The periods are Period 1, which runs between 2005/01/01 up to 2008/09/30, Period 2, which runs between 2008/10/01 up to 2015/10/30 and Period 3, which runs between 2015/11/02 up to 2016/12/30. The cut-offs for the periods are based on the current market interest rates. In Period 1 interest rates are considered to be normal. In Period 2 interest rates have fallen as a consequence of the financial crisis and are close to, practically equal to, zero. In Period 3 interest rates start to rise again; they are still low but significantly higher than during Period 2. Figure A1 in Appendix provides a graph of market interest rates for the period 2005/01/01 to 2016/12/30.

5.1 Entire Sample

The results are mixed when observing the entire sample period. The parameter values are presented in Table 2 with mean values for κ, ρ, θ and σ of 0.1295, 0.0107, 0.0978 and 0.1232 respectively. The standard deviations of the parameters are high, which indicates unstable parameters. The mean absolute price and yield errors are presented in Table 3 and are 0.6162 and 0.1257 respectively. These values are in accordance

with Steeley (2008), who observed values of 0.67291 and 0.15599 respectively. However, compared to studies conducted with data from earlier time periods, these values are higher. Brown & Schaefer (1994) observed no larger absolute price error than 0.20 for different maturities, while Barone *et al* (1991) estimated a mean absolute price error of 0.29.

Table 4 shows the correlation between the estimated parameters, and as can be seen several parameters display high correlation with each other. This can be an indication that the model is suffering from multi-collinearity. Among the correlations, there seems to be especially strong correlation between κ and θ, ρ and σ, ρ and θ, θ and σ.

Table 2: Parameter Values

Parameters	Min	Max	Average	Std.
κ	0.0023	0.6331	0.1295	0.0988
ρ	1 E-5	0.0504	0.0107	0.0169
θ	0.0285	0.1500	0.0978	0.0485
σ	0.0004	0.3122	0.1232	0.0966

Table 3: Error Statistics

Mean Absolute Price Error, $	Mean Absolute Yield Error, %-units
0.6162	0.1257

Table 4: Correlation Between Parameters

Parameters	κ	ρ	θ	σ
κ	1.0000	0.2954	-0.7346	-0.2670
ρ	0.2954	1.0000	-0.5338	-0.6514
θ	-0.7346	-0.5338	1.0000	0.6463
σ	-0.2670	-0.6514	0.6463	1.0000

Observing Figure 1, which plots the estimated parameters, it is clear that the performance of the model can be divided into three periods. During the first period, 2005/01/01 to 2008/09/30, interest rates are "normal", and hence the values of ρ and θ behave reasonably and lie around actual short and longterm rates. κ and σ are volatile, and low values of κ tend to be associated with high values of σ and vice versa. This is in accordance to Rebonato (1996), who claims that this is a result of the construction of the CIR equations. In the end of this period, interest rates decrease close to zero, which can be observed in Figure A1 in Appendix, and characterizes the next period. Between 2008/10/01 and 2015/10/31, actual interest rates are close to zero during the entire period. This leads to ρ going to zero, while σ and θ increases. θ tend to go to the upper boundary of the optimization, 0.15, which is clearly too high since this indicates the long term mean of the term structure lies around 15%. In the final period, 2015/11/01 to 2016/12/31, actual interest rates start to increase again (see Figure A1 in Appendix), ρ goes up and thus lead to more reasonable values of κ, θ and σ. For more detailed figures regarding estimated parameter values the reader is referred to the Appendix, where figures for κ, ρ, θ and σ are provided separately.
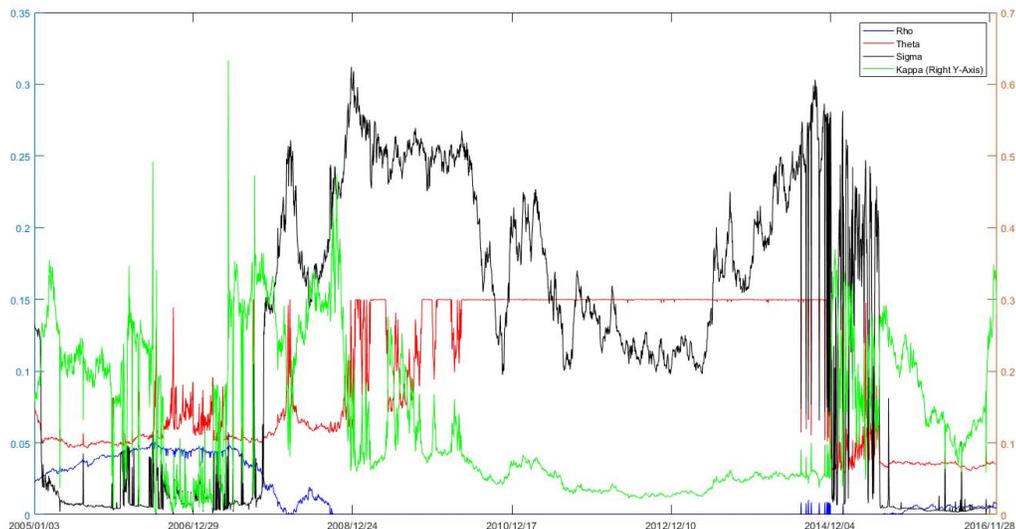
Figure 1: CIR Parameters. Note: Kappa is graphed against the right axis

The big difference between the three periods justifies a deeper analysis of the periods separately, which follows below.

5.2   Period 1, 2005/01/01 - 2008/09/30

The parameter values for Period 1, 2005/01/01 - 2008/09/30, are presented in Table 5. The mean values for κ, ρ, θ and σ are 0.1985, 0.0327, 0.0595 and 0.0553 respectively. The standard deviation for these parameters are 0.1044, 0.0144, 0.0136 and 0.0750. Compared to the entire sample, these values are smaller for ρ, θ and σ, while the standard deviation for κ is slightly larger. The mean absolute price and yield errors are presented in Table 6 and are 0.4435 and 0.0829 respectively. Compared to the corresponding values for the entire sample, both these figures are lower. This implies that during this time period, the CIR model is better at fitting the yield curve relative to the entire sample period. Although the absolute price error is quite small, which indicates that the model is fairly good at fitting the yield curve, the standard deviation of the parameters can still be regarded as high. This implies that the parameters are unstable. Both these results are in accordance with previous studies such as Brown & Schaefer (1994) and Barone *et al* (1991). The mean

absolute price error of 0.4435 is closer to the value of 0.29 estimated by Baron *et al* (1991), nonetheless it is still 53% higher. The mean absolute yield error of 8.29 basis points is a decrease by 4.28 from 12.57 basis point when observing the entire sample. The fact that the model can fit the yield curve despite unstable parameters could imply that the model suffers from multicollinearity. This hypothesis is strengthened by observing Table 4, where the correlations between the parameters are presented. As mentioned earlier, high correlations between parameters can be a sign of multicollinearity.

Table 5: Parameter Values Period 1

Parameter	Min	Max	Average	Std.
κ	0.0023	0.6331	0.1985	0.1044
ρ	1.06 E-5	0.0504	0.0327	0.0144
θ	0.0463	0.1500	0.0595	0.0136
σ	0.0004	0.2611	0.0553	0.0750

Table 6: Error Statistics Period 1

Mean Absolute Price Error, $	Mean Absolute Yield Error, %-units
0.4435	0.0829

5.3   Period 2 2008/10/01 - 2015/10/30

Parameter values for Period 2, 2008/10/01 - 2016/10/30, are presented in Table 7. The average values for κ and ρ have decreased compared to the results for the whole sample, while the average values for θ and σ have increased. The standard deviations for the estimates are somewhat smaller than for the whole sample.

Although the standard deviations of the parameters are lower compared to previous time periods, the parameters obtained during this period are unsound. θ increases a lot, with a mean value of 0.1285, which

should be interpreted as a long term mean of the yield curve around 13%. That is not realistic during this low interest rate climate. This probably has to do with the potential multicollinearity issue discussed earlier. When ρ is pushed down towards zero, it is hard for the model to converge and produce reliable parameter values.

The mean absolute price and yield error during this time period are presented in Table 8, and display values of 0.7504 and 0.1563 respectively. Both of these values are worse than the corresponding figures for both the entire sample and for Period 1 as well. This indicates that during this time period, the CIR model performs worse at fitting the yield curve. The mean absolute pricing error is about 69% higher compared to Period 1, the increase in the mean absolute yield error is about 89%. This raises suspicions that the CIR model is especially bad at estimating the yield curve during periods of low interest rates, which also is suggested by Ullah *et al* (2013) when applying the model on a sample of Japanese bonds from January 2000 to December 2011.

Table 7: Parameter Values Period 2

Parameter	Min	Max	Average	Std.
κ	0.0220	0.4749	0.0889	0.0768
ρ	1 E-5	0.0102	0.0001	0.0007
θ	0.0285	0.1500	0.1285	0.0393
σ	0.0036	0.3122	0.1785	0.0709

Table 8: Error Statistics Period 2

Mean Absolute Price Error, \$	Mean Absolute Yield Error, %-units
0.7504	0.1563

5.4   Period 3 2015/11/02 - 2016/12/30

Parameter values for Period 3, 2015/11/02 . 2016/12/30, are presented in Table 9. The mean value for κ has gone up compared to Period 2, which probably is the reason the mean value for θ has decreased and reached a level of what one would expect. Standard deviations for the estimated parameters are lower compared to the other periods, with one exception. The standard deviation of ρ is somewhat larger during Period 3 compared to Period 2, but this is expected since during Period 2 the short term interest rates are quite constant around zero. That the estimated values for θ begin to behave more reasonably again when short term interest rates increase above zero (consequently ρ goes up), strengthens the evidence that the model works poorly during periods consisting of what are perceived as too low interest rates.

Table 9: Parameter Values Period 3

Parameter	Min	Max	Average	Std.
κ	0.0586	0.3492	0.1565	0.0605
ρ	0.0018	0.0109	0.0052	0.0010
θ	0.0305	0.0522	0.0351	0.0021
σ	0.0004	0.0509	0.0041	0.0042

Table 10: Error Statistics Period 3

Mean Absolute Price Error, $	Mean Absolute Yield Error, %-units
0.2302	0.0427

The estimated parameters and the small standard deviations suggest that Period 3 is the period when the CIR model performs best during the entire sample period. This is also evident by observing Table 10 where the mean absolute price and yield errors are presented. The values are 0.2302 and 0.0427 respectively. This

is a lot lower compared to both Period 1 and 2, and more in accordance with the results of Brown & Schaefer (1994) and Barone *et al* (1991).

5.5   General discussion

Table 11: Results summary

	Period 1				Period 2				Period 3			
Parameter	Min	Max	Average	Std.	Min	Max	Average	Std.	Min	Max	Average	Std.
κ	0.0023	0.6331	0.1985	0.1044	0.0220	0.4749	0.0889	0.0768	0.0586	0.3492	0.1565	0.0605
ρ	1.06 E-5	0.0504	0.0327	0.0144	1 E-5	0.0102	0.0001	0.0007	0.0018	0.0109	0.0052	0.0010
θ	0.0463	0.1500	0.0595	0.0136	0.0285	0.1500	0.1285	0.0393	0.0305	0.0522	0.0351	0.0021
σ	0.0004	0.2611	0.0553	0.0750	0.0036	0.3122	0.1785	0.0709	0.0004	0.0509	0.0041	0.0042
Mean Absolute Price Error, $	0.4435				0.7504				0.2302			
Mean Absolute Yield Error, %-units	0.0829				0.1563				0.0427			

The CIR model produces unstable parameter values, not merely that but also unrealistic values during Period 2. Yet, the fit of the model is *good*, and although the pricing error during Period 2 is larger than during the other periods, the fit can still be considered *decent* in this period as well. This is in line with the hypothesis and corresponds to Rebonato's (1996) argument that the fit can be obtained with radically different sets of parameters, which can be worthless when evaluating e.g. hedge statistics. Rebonato (1996) further argues that the most problematic part of his empirical findings is the lack of stability in the parameters, and thus concludes that "one is trying to force reality into a seriously wrong box". The empirical findings in this study strengthens Rebonato's (1996) discussion regarding unstable parameters. Already in the original article, Cox *et al* (1985a) suggests that the stability of the parameters can be solved by including data from other securities than bonds alone. Since a pricing formula for options is provided by Cox *et al* (1985a), a logical extension would be to include option data. An attempt of this has been made by estimating the

volatility parameter, σ, using option data and thus be able to stabilize the other three parameters. Unfortunately, enough high qualitative data on interest rate options was not found to be able to perform such a test.

An explanation regarding the better performance of the CIR model during Period 1, and especially Period 3, is that during these periods short-term interest rates are significantly higher than during Period 2. The fact that one can observe this pattern suggests that the CIR model is not performing well during periods of very low interest rates. This implies that the model's assumption of positive interest rates may be too strong. Cox *et al* (1985a) state the following on page 391: "The interest rate behavior implied by this structure thus has the following empirically relevant properties: (i) Negative interest rates are precluded.". First of all, negative interest rates are not precluded in the financial theory for equilibrium in financial markets, see for example Sharpe (1964). Secondly, this might have been an empirically relevant property for the time when the article was published, but considering the recent interest rate climate across Europe one can observe that negative interest rates do exist. The assumptions of the model might thus be fundamentally wrong and could affect the results when interest rates are as close to zero as in the sample used in this study. Given the evidence from this study it appears that the model produces nonsensical parameters when interest rates are below around 25 basis points.

It is also worth to mention the rising pressure on the financial system during period 2, which lead to increased government and central bank interventions as explained by IMF (2009). The interventions on the interest rate market during the period could have have affected the market to behave in an abnormal way, which can not be explained by the model. The results suggest that one should tread carefully when using the CIR model, since even though the fit of the model is *good* the parameters may be nonsensical in other applications.

The number of bonds used in this study is sufficient, however the quality of the bonds could be discussed. The traded volume of every bond is not observable in Bloomberg and could thus imply a liquidity problem if the volume is not large enough. On the other hand, some bonds are traded more than others as a result of

being on the active curve or being part of future/option delivery tables. This could give rise to "jumps" in the yield curve for specific bonds and maturities. However, one could argue that the large number of included bonds in the dataset accounts for this and reduces the "noise" that comes from bonds with particular low or high liquidity.

In an attempt to stabilize the parameters one could possibly estimate only three of the parameters by holding the fourth parameter fixed at a certain value. This has been tested during the study in two separate estimations. The first estimation is done by holding ρ, the instantaneous interest rate, fixed at 1% and the second estimation is done by letting κ, the speed of adjustment toward the long-term mean, equal to the moving average of the past hundred days. The results of these two estimations can be found in Table A2 and Table A3, as well as in Figure A6 and Figure A7, in the Appendix. As can be observed, holding ρ constant at 1% yields about the same stability in the parameters as the original estimation, but with a considerably worse fit. Constraining κ on the other hand do lead to more stable parameters. It is worth to mention that in this estimation, θ, the long term mean of the yield curve, obtains values that could be regarded as more reasonable compared to the original estimation. However, the fit of this estimation is significantly worse when observing the mean absolute price error.

It can be concluded that constraining either κ or ρ might lead to more stable parameters but with a worse fit. Nonetheless, constraining one of the parameters along these lines does not really make sense, for example holding ρ fixed at 1% implies that the short term rate is constantly 1% during the entire sample period, which is not the case when observing the real US Treasury yields in Figure A1. Regarding κ, it can be argued that the curvature of the yield curve should not change much, however it can still be improper to hold this parameter at a constant value. Constant parameters makes even less sense when applied to other applications than pricing, and thus the parameters are rendered practically unusable. Considering these results and the discussion above, it is deemed that constraining one parameter in this way is not suitable and is not a valid approach to obtain better parameter values.

Further studies on this subject is encouraged using a similar dataset for bonds, but expanding the estimation by also including option data in an attempt to stabilize the parameters. The Bloomberg database, as of date, provides data on implied volatility for interest-rate future options. However, the quality of this data is questioned during the study. Ideally one would prefer actual traded prices of interest-rate options of different maturities and strikes. A drawback of using OLS, as done in this study, to estimate the parameters when using both bond and option data is partly that one have to optimize the bond and option pricing equations separately, but also that the estimation process can be extremely time consuming. Other studies have estimated the model by an MLE-approach, which could be a more suitable alternative when including option data.

6   Conclusion

This paper examines the performance of the one factor CIR model to estimate the yield curve of US Treasury Notes during the period 2005/01/01 to 2016/12/31. During the sample period actual interest rates have varied significantly and it is observed that the performance of the CIR model can be analyzed in three different subsets of the observed period. The most important difference between these periods are the short term actual interest rate, which during Period 2, 2008/10/01 - 2015/10/30, is much lower than in Period 1 and 3. One of the assumptions underlying the CIR model is that interest rates are not allowed to be negative. During the observed period this criteria is fulfilled, although the interest rates levels during Period 2 are extremely low and practically equal to zero.

The results from this study indicate that the CIR model performs poorly during Period 2, with unrealistic parameter values and higher pricing errors compared to Period 1 and especially Period 3. The extremely low interest rate climate during Period 2 lead to ρ, the short term interest rate parameter, going to zero. This seems to give rise to problems estimating the other parameters. The poor performance of the CIR model

during periods of low interest rates has also previously been observed by Ullah *et al* (2013). The results suggests that the assumption of positive interest rates might be too strong. Further, it appears that the model requires interest rates around at least 25 basis points.

In accordance with other studies made on the CIR model, it can be concluded from this study that the CIR model can fit the yield curve reasonably well although the estimated parameters are unstable and display large standard deviations. Unstable parameters of the model can make the model impractical to use in real life investment decisions, since the parameters can differ significantly from one day to another. An explanation to this result could be that the model is suffering from multicollinearity. The estimated parameters show high correlation with each other which strengthens the suspicions of multicollinearity. It is previously proposed, by Cox *et al* (1985a), that this problem could be accounted for by including data from other securities than bonds alone. An attempt has been made to test whether including option data to estimate the volatility parameter, σ, can stabilize the other three parameters. Unfortunately, not enough high qualitative data on interest rate options was found to perform such a test and is thus encouraged for further examination.

The CIR model is a well-established model and forms the fundamental basis of many valuation models widely used in the investment universe. The indications from this study, that the assumption of positive interest rates is too strong, is therefore crucial and strengthened by the fact that negative interest rates are not precluded in the financial theory for equilibrium in financial markets and is not applicable on todays interest rate market. The results suggest the CIR model should not be used during periods when short term interest rates are close to zero. If these results holds, important models for valuation and credit risk including CIR processes might be unreliable and give rise to risks unaccounted for. Further studies on this area should therefore be conducted in a profound way to examine the performance of the CIR model during periods of short term interest rates close to zero.

Bibliography

Barone, Emilio, Domenico Cuoco, and Emerico Zautzik. "Term structure estimation using the Cox, Ingersoll, and Ross model: The case of Italian Treasury bonds." *Journal of Fixed Income* 1.3 (1991): 87-95.

Black, Fischer. "The pricing of commodity contracts." *Journal of financial economics* 3.1-2 (1976): 167-179.

Brown, Roger H., and Stephen M. Schaefer. "The term structure of real interest rates and the Cox, Ingersoll, and Ross model." *Journal of Financial economics* 35.1 (1994): 3-42.

Brown, Stephen J., and Philip H. Dybvig. "The empirical implications of the Cox, Ingersoll, Ross theory of the term structure of interest rates." *The Journal of Finance* 41.3 (1986): 617-630.

Cox, John C., Jonathan E. Ingersoll Jr, and Stephen A. Ross. "A theory of the term structure of interest rates." *Econometrica: Journal of the Econometric Society* (1985a): 385-407.

Cox, John C., Jonathan E. Ingersoll Jr, and Stephen A. Ross. "An intertemporal general equilibrium model of asset prices." *Econometrica: Journal of the Econometric Society* (1985b): 363-384.

Ferson, Wayne E. "The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates: Discussion." *The Journal of Finance*, vol. 41, no. 3, 1986, pp. 630–632.

International Monetary Fund. (2009) Market interventions during the financial crisis: How effective and how disengage?. https://www.imf.org/external/pubs/ft/gfsr/2009/02/pdf/chap3.pdf (Accessed May 22, 2017)

Jamshidian, Farshid. "An exact bond option formula." *The journal of Finance* 44.1 (1989): 205-209.

Longstaff, Francis A. "The valuation of options on coupon bonds." *Journal of Banking & Finance* 17.1 (1993): 27-42.

Rebonato, Riccardo. *Interest-rate option models: understanding, analysing and using models for exotic interest-rate options.* John Wiley & Sons, 1996.

Sharpe, William F. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *The Journal of Finance*, vol. 19, no. 3, 1964, pp. 425–442.

Steeley, James M. "Testing Term Structure Estimation Methods: Evidence from the UK STRIPS Market." *Journal of Money, Credit and Banking*, vol. 40, no. 7, 2008, pp. 1489–1512.

Ullah, Wali, Yasumasa Matsuda, and Yoshihiko Tsukuda. "Term Structure Modeling and Forecasting of Government Bond Yields." *Economic Papers: A journal of applied economics and policy* 32.4 (2013): 535-560.

Appendix

Table A1: Summary statistics

Parameters	Min	Max	Average	Std.
Number of active bonds	95	233	166	-
Bond yields (%)	0.14	5.50	2.19	1.47
Coupon rate (%)	0.125	11.25	3.804	2.42
Time to maturity (years)	1.5	30.038	7.9078	7.42
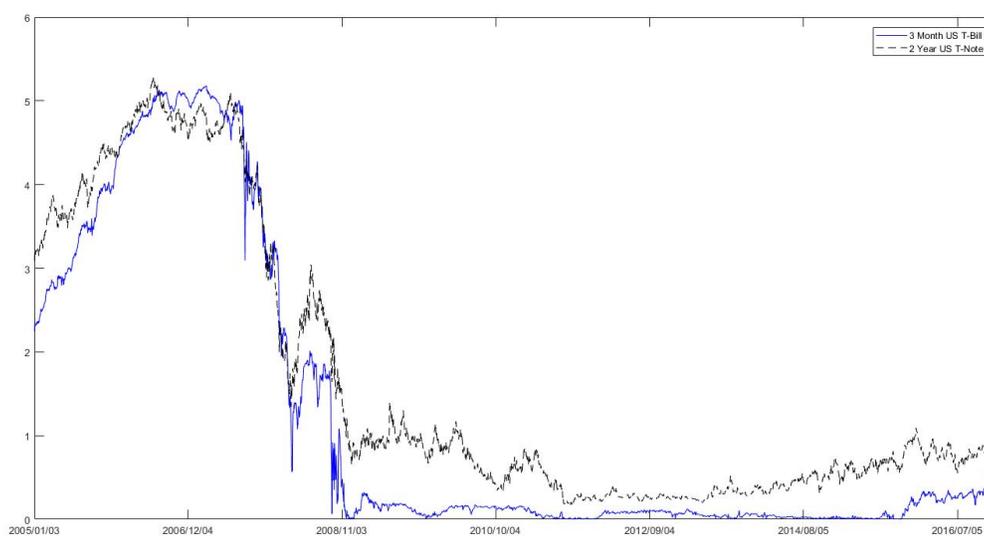
Figure A1: US Treasury Yields for 3 month bill and 2 year note
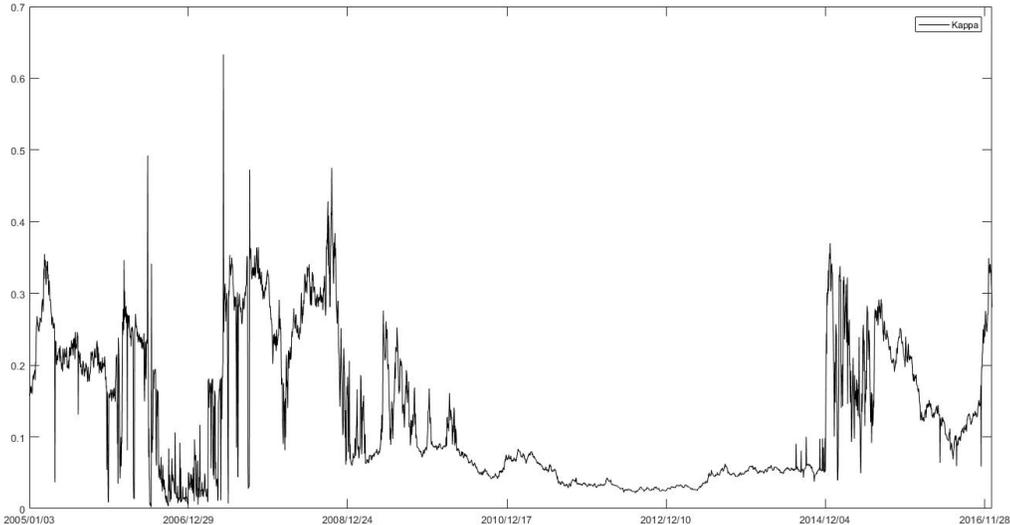
Figure A2: Estimated values, κ
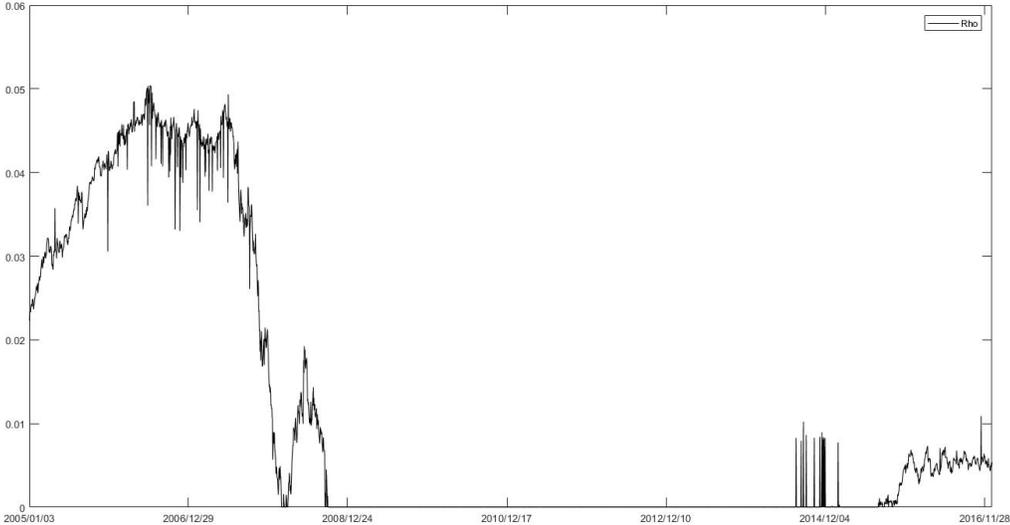
Figure A3: Estimated values, ρ

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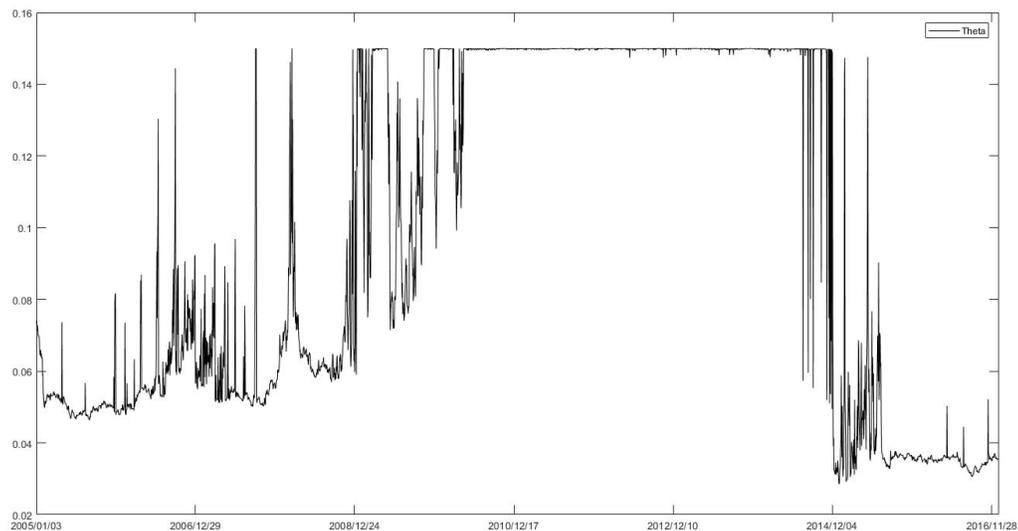
Figure A4: Estimated values, θ
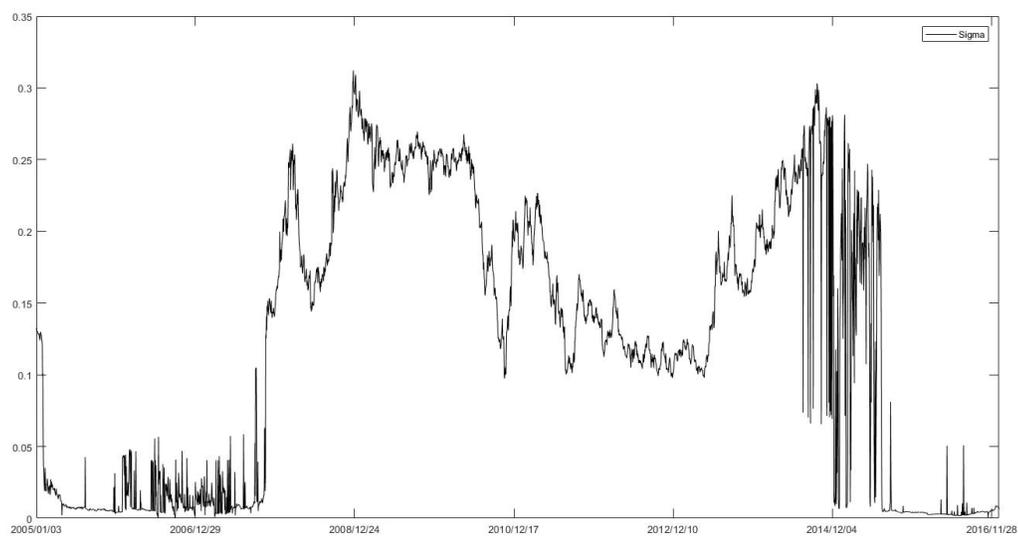
Figure A5: Estimated values, σ

Table A2: Results summary - Constant Rho, 1%

Parameter	Period 1				Period 2				Period 3			
	Min	Max	Average	Std.	Min	Max	Average	Std.	Min	Max	Average	Std.
κ	0.0536	2.999	1.3249	0.9867	0.0087	0.3554	0.0372	0.0375	0.0064	0.1061	0.0188	0.0140
ρ	0.01	0.01	0.01	0.0000	0.01	0.01	0.01	0.0000	0.01	0.01	0.01	0.0000
θ	0.0439	0.1500	0.0561	0.0148	0.0458	0.1500	0.1448	0.0177	0.0504	0.1500	0.1414	0.0238
σ	1 E-6	0.2326	0.0850	0.0801	1.5 E-6	0.1845	0.0844	0.0589	1.6 E-5	0.1847	0.0652	0.0480
Mean Absolute Price Error, \$		1.4238				1.5364				0.5669		
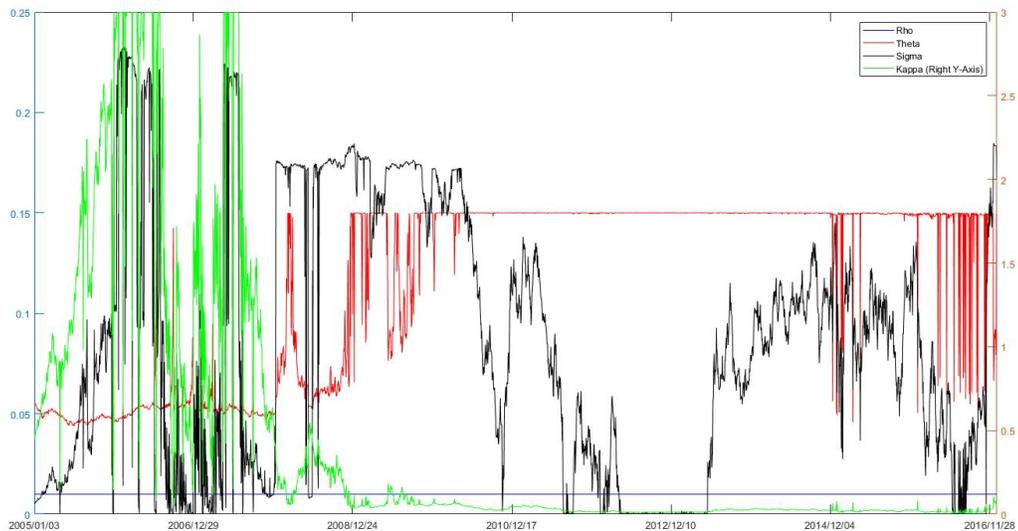
Figure A6: CIR Parameters - Constant Rho 1%

Note: Kappa is graphed against the right axis

Table A3: Results summary - Kappa Moving Average

Parameter	Period 1				Period 2				Period 3			
	Min	Max	Average	Std.	Min	Max	Average	Std.	Min	Max	Average	Std.
κ	0.0366	0.3548	0.2536	0.0188	0.2541	0.2541	0.2541	0.0000	0.2541	0.2541	0.2541	0.0000
ρ	1.06 E-5	0.0501	0.0303	0.0137	1 E-5	0.0108	4.9 E-5	0.0004	1 E-5	0.0063	0.0013	0.0018
θ	0.0457	0.1425	0.0561	0.0082	0.0275	0.0771	0.0452	0.0111	0.0242	0.0522	0.0315	0.0040
σ	1.43 E-6	0.2410	0.0536	0.0704	1 E-6	0.2700	0.0468	0.0725	4.2 E-6	0.1872	0.0101	0.0315
Mean Absolute Price Error, \$	0.7538				1.4675				0.3769			
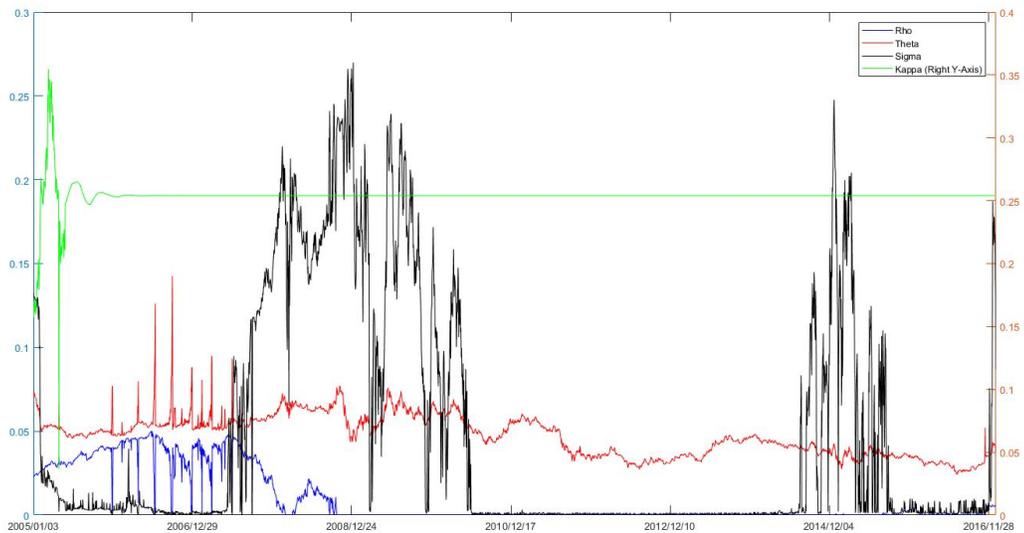
Figure A7: CIR Parameters - Kappa Moving Average

Note: Kappa is graphed against the right axis