Students' and Teachers' Jointly Constituted Learning Opportunities

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Students' and Teachers' Jointly Constituted Learning Opportunities

The Case of Linear Equations

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Abstract

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This study emphasises jointly constituted learning opportunities in mathematics instruction by analysing learner contributions, and the attention paid to them, in whole-class teaching. Interaction in mathematics classrooms has been a significant research area for decades and the importance of using a learner perspective in teaching is well recognized. However, few studies have investigated interaction in relation to the opportunities for learning the content of the lesson. The aim of this study is to gain deeper knowledge about the relationship between interaction and the learning opportunities that emerge. Enacted dimensions of variation (e.g. Marton, 2015), the aspects of the content that are made possible to learn, are used as unit of analysis throughout the investigation. Learner contributions are regarded as all the public, content-related utterances from learners in a lesson. This study encompasses 14 video-recorded mathematics lessons, from either grade 9 in compulsory school or from grade 10 or 11 in upper-secondary school in Sweden (ages 15 – 18). All lessons had the same topic, the introduction of linear equations, in order to make learning opportunities comparable. 12 teachers and 14 classes (297 learners) participated. Learner contributions were developed in four different trajectories in the lessons. Depending mainly on different attention from teachers, the learner contributions were disregarded, selected, considered, or explored. Based on this categorisation, the lessons were grouped into three main types. The learning opportunities from a content perspective were thoroughly investigated. Results show that different learning opportunities for concepts like function and slope emerged in different lesson types. In addition, learners and teachers were shown to generate different kinds of aspects of the content taught. Necessary aspects of

linear function, like the separation of b-values as y-intercepts or the fusion of slopes and y-intercepts to the equation of a straight line, were mainly generated by teachers, even though often enacted together with learners. Optional aspects, like the separation of function from a single point or from 'a line between intercepts' were, on the other hand, mainly generated by learners. The optional aspects were, however, greatly dependent on teacher exploration for their enactment. The main conclusion drawn is that the importance of using a learner perspective in instruction also relates to the quality of the learning opportunities that emerge. The enactment of optional aspects of linear equations was greatly dependent on learner contributions but also on teacher exploration. Contrary to what might have been expected, the necessary aspects of linear equations were also enacted in more qualitative ways in lessons in which learner contributions were frequently explored. There seems to be a price for learner silence in instruction. And, furthermore, this price is not only constituted by learners; it also depends on teachers' attentions to learner contributions.

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In accordance with my theoretical framework, I will now mostly in Swedish describe the 18 aspects that were critical for the making of this book.

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Masthugget 171117

Tuula

Då så, nu kan vi börja!

CONTENTS

1 Introduction

A long time ago, in a learning study lesson on subtraction of negative numbers, Oscar asked the teacher Joakim[1] about the operation of subtraction: *couldn't you see it as a difference?* It was evident that Joakim did not understand Oscar's question, and he just mumbled some pointless response to him. Neither did we, Joakim's colleagues in that learning study group, nor our tutor understand the meaning of Oscar's contribution when we watched the video recordings together after the lesson. We did not actually give that contribution much attention at that time. The learning study was at an early phase and we were focused on trying to teach about subtraction of negative numbers by the book, with the help of opposite numbers. Later in the process, that mumbling response became painful for us all. When our understanding of the critical aspect: *discerning subtraction as a difference* had developed and we revised the first lesson, it became evident that Oscar's contribution carried the potential to change the learning opportunities not only in that lesson, but in the whole learning study. Seeing subtraction as a *difference* between for instance (-3) and (-5) is one of the necessary aspects of understanding subtraction with negative numbers (Kullberg, 2010). It would be so easy to discuss how this episode reflects the lack of knowledge of those novice teachers, both about the importance of teaching subtraction with dual meanings ("take away" and "difference") and that there is a point in listening to your students. Fortunately, I was one of the teachers and Joakim was my highly valued colleague. That fact helps me to humbly remember that teaching with the ambition of enhancing learning is one of the most complex activities there is. Even though it has been my main undertaking for more than two decades, there is still much to learn. This study is about those learner contributions, Oscar's and all the others'. I knew as a teacher that they were of importance; I simply wanted to find out more about why.

[1] Oscar and Joakim are real names. The lesson was conducted more than 14 years ago. Oscar is 27 today and does not have any difficulties with negative numbers and Joakim is also much older and has always been the wisest of teachers. By calling them by their real names, I consider that I am paying homage to people I have learnt from. Thank you, Oscar Langenius and Joakim Magnusson. Both have given their consent to be included with real names in the opening of this thesis.

This interaction between Oscar and Joakim on seeing subtraction of negative numbers as a difference was quickly ended. One of the reasons was probably that Joakim did not understand the content of Oscar's contribution. We can only speculate about what could have happened if Oscar's contribution had been given a different kind of attention. This interaction is an every-day occurrence in a classroom; it was just that this one happened to be recorded and analysed further. In educational research, interaction has been one of the main interests during the last 40-50 years (Radford, 2011), and in the beginning of the 1990s, an increasing interest in the socio-cultural aspects of classroom interaction arose in educational research (e.g. Kieran, 2007; Lerman, 2006; Sahlström, 2008). By means of this increased interest, interaction research has evolved an abundance of perspectives, research aims and foci.

The emphasis on interaction in *mathematics* educational research has included interaction between students[2], teachers, and contents. Before the late 80s, interaction did not embrace the students (Radford, 2011). Instead much research effort was placed on how the teacher would present the content – the mathematics – to the students. The German *stoffdidaktik* is an example of a research tradition that did not include students, but only content and teachers. Also the process-product research tradition focused on teacher behaviour, not on students (Fennema & Carpenter, 1991). In contrast, in the constructivist research tradition by Piaget, the emphasis was given to how students understand different concepts, and to how these understandings develop. Hence, this tradition did not take teachers much into account in the research (Radford, 2011). In the beginning of the 90s, the increasing interest in social aspects of teaching and learning did focus immensely on interaction between teachers and students, but in many cases, they left content out of the scope (e.g. Cobb, 2006; Mortimer & Scott, 2003; Steinbring, 2008). This study has the interaction of all three entities – students, teachers and the content – in its objective. Oscar's *contribution*, as well as Joakim's *response*, and furthermore, the possible developments of the content that form the learning opportunities, are analysed in this study.

It is here neither meaningful nor possible to make a fair review of the plethora of different research traditions on interaction that has developed in the last half century. Instead, interaction research will be presented with a distinction between three perspectives: the *forms*, the *functions*, and the *contents* of interaction. Each perspective will be discussed with some examples of results that have implications

[2] Throughout this study the words *students* and *learners* are used as synonyms.

for practice and research today. The most relevant topics will be elaborated on in further detail in Chapter 2. There is a danger of portraying different research traditions as 'a linear, historical sequence of perspectives, each of which overcomes the limitations of its predecessors (Cobb, 2006). In reality, several research traditions develop simultaneously, both contradicting and affecting each other.

1.1 Forms of interaction

In the early 70s, the sociologist Hugh Mehan studied Courtney Cazden's primary classroom in an almost anthropological way. With a linguistic interest, he was trying to understand classroom interaction as a communication system (Cazden, 1988). Almost half a century later, the major findings from this research group[3] still in many ways influence how we see interaction in classrooms. The two most prominent results from these early studies are the QWKA concept, namely what teachers ask: *questions with known answers*, and how: in the instructional *three-part sequence known as the IRE pattern*[4]: Initiation-Reply-Evaluation. Mehan also contributed by showing that the IRE patterns were connected to each other in longer sequences. Another empirical result from these studies is the small delay that often occurs if the evaluation is to be negative, in comparison to the positive evaluation that is on time (Mehan, 1979). Consequently, students can hear the adequacy of their replies in the production of teachers' third turn. Cazden concluded later (1988) that IRE is the default pattern, namely what happens in instruction unless deliberate action to accomplish alternatives is taken. Furthermore, even though the teacher's greater right to speak than the student's was the most important asymmetry found in the interaction, Cazden also discovered other patterns in teacher-student interaction, for instance when students themselves decide to speak. The studies conducted by Mehan's group were not by any means normative but have become a tool for power critique of schools and teachers (Macbeth, 2003). With concepts such as *Questions with known answers*, which begs the question of why teachers ask questions they already know the answer to, the use of the results as a critique might not be too surprising. Even though QWKA is a description of a facet of *naturally occurring discourse* (Macbeth, 2003) not a critical analysis of Discourse, the name itself leaves it open to such a reading. Another example from interaction research, which still has implications in today's school development discussions, is the one second of average *wait time* (Row, 1974) between a teacher question and the expected answer in instruction. From this study

[3] Mehan built on studies by Bellack (1966 in Macbeth, 2003)
[4] Also later called IRF (Initiation-Response-Feedback)

we also know that teachers' reactions to students' responses are on average 0.9 seconds. A longer wait time, for instance 3-5 seconds, could be achieved through training, and this might result in several positive effects in the classroom. This longer wait time affects not only the number of responses, but also the quality and length of the responses. Furthermore, the number of students responding also increases (Rowe, 1986). Even though these conclusions were drawn 4-5 decades ago, they still have implications for teaching today.

The interaction research tradition in Scandinavia has been dominated by conversation analysis studies, and these studies share roots with early interaction studies by Mehan and Cazden. Similarly, conversation analysis (CA) focuses on interaction and micro analyses of how dialogues are conducted. However, the Scandinavian tradition has evolved in a slightly different direction (Sahlström, 2008). CA tradition has generally not focused on learning and development, but on how social life is established, maintained and changed through interaction between people, mostly in contexts outside of school (Sahlström, 2009). When learning has been in focus, the studies have often sought solutions to learning difficulties *in* schools by looking at situations *outside* of school. In many of these environments, learning is a by-product and not the main goal of the activities, as in school (Carlgren, 2009). For instance, when Sahlström (2001) describes the students' dilemma of interaction in whole-class teaching, he emphasises that the students are expected to perform 'acts of listening without the reward of being able to speak'[5]. Evidently interaction is seen as an end in itself from this perspective. The point of listening in whole-class instruction, the actual *reward for listening*, is not learning or anything else; it is the opportunity to talk. Classroom discourse is compared to conversation discourse outside of school, and learning is not highlighted.

These examples are descriptive research studies with the aim of evaluating naturally occurring discourse. Regarding interaction, the answers have been to the question of *how* interaction occurs. This implies that the forms of interaction have been studied. The conclusions drawn concern different outcomes of this interaction and the results are presented in the form of categories of interaction. The classroom interaction is described on its own terms rather than as a tool for other aspects, for instance mathematical learning.

[5] The Swedish original: *Plenarundervisning innebär att eleverna ställs inför ett knepigt interaktionellt dilemma: de skall ägna sig åt lyssnarhandlingar utan att kunna räkna med att få ägna sig åt den förväntade ersättningen för detta, nämligen att själv få prata (Sahlström, 2001, p. 101).*

1.2 Functions of interaction

The diverse functions and/or consequences of interaction in teaching have been widely studied during the last three decades, and this is also nowadays the objective of many research studies. One example is Hall (1997), who analysed how teachers and students jointly created two distinct positions for students to act in. Students in these two positions received different responses and therefore had different opportunities to interact in the lesson. Another example is Lobato, Clark, and Ellis (2005), who analysed teachers' activities in the classroom and described them based on function rather than form, which led to the distinction between *eliciting* and *initiating*. The former embraces activities in which the function is to shed light on students' strategies, images and ways of perceiving the content taught. The latter has the function of initiating new content in teaching. According to the authors, initiating is often preceded by eliciting, as teachers collect information about students' ways of seeing before they make decisions on whether to introduce new content to the discussion. Lobato et al. claim that the interaction between teachers and students needs to change from communicating teachers' mathematics to developing students' mathematics.

Nystrand and Gamoran (1990) made an early contribution to the discussion with a distinction between two functions of classroom discourses, namely *recitation* and *conversation*. These were seen as two ends of a continuum of the quality of the instructional discourse. The former is defined as "normal classroom discourse" and the latter as "high-quality classroom discourse". The main distinction between these two are that in recitation the interaction seems to be driven by a script and in conversation the interaction seems to be largely determined by what has previously been said. Three aspects of high-quality instructional discourse, according to Nystrand and Gamoran (ibid.), are worth mentioning in this context, as they are related to the interaction between teacher and students. In high-quality instructional discourse, teachers take students seriously, acknowledging and building on what they say. Furthermore, what students say in a discussion can affect both the content and focus of instruction, and finally the teacher is the key to creating classrooms where students become engaged in challenging issues and interesting topics. A conclusion is that high-quality classroom discourse involves an exchange of ideas between the teachers and her students (Nystrand & Gamoran, 1990).

An example of a more contemporary study of the function of interaction is a study by Drageset (2015), in which mathematical discourse was studied on a turn-by-turn basis in more than 1800 teacher interventions. Results relevant for this study show that different kinds of teacher interventions are often related to specific student interventions; the actions are intertwined. In the most frequent teacher-students interventions, the teacher controls the process and the students are left to basic task responses. However, Drageset (ibid.) contributes distinctions between several different teacher actions with different functions in instruction. He also problematizes the need to progress within the classroom; the pace of a lesson would decrease if every opportunity to ask for justification was taken.

These four examples are all research studies with the aim of explaining, rather than describing, aspects of interaction. This implies that different mechanisms of interaction are analysed and different variants of the questions of *why* interaction is answered. The conclusions drawn concern various functions of interaction.

1.3 Contents of interaction

Many researchers have pointed out that student talk in itself is not enough to facilitate student learning; both the content and the structure of the discourse has to be considered (Mortimer & Scott, 2003; Stein, Engle, Smith, & Hughes, 2008; Walshaw & Anthony, 2008). In many interaction studies, especially when analysing classroom discourse, the results are reported exclusively in terms of forms or functions of interaction (Sahlström, 2008). The actual content, the *what* that is being taught or discussed, is not regarded as a significant aspect. Hence, the content of interaction is often considered as contextual factor (Mortimer & Scott, 2003). Furthermore, even when the content of interaction *is* considered, it is not always concerning content from a school subject. For instance, Macbeth (2011) explored students' understandings when a teacher instructs in whole-class settings. His conclusions are related to *what is communicated between the lines in conversation*. His study is not of explicit subject content, but there is a focus on *what* is communicated. He argues that in an interaction situation with the teacher, the students are focusing on what is being said implicitly. For example, there is no one who does not understand that the right answer is *yes* to the teacher question of *do you want to change anything there?*

Kullberg (2010) describes how the learning opportunities for content were changed as a result of a student's input in a lesson. Kullberg's study is an intervention study with the aim of probing the validity of *critical aspects*[6] of subtraction of negative numbers, earlier discovered in a learning study. The original plan for one of the lessons was that the teacher would enact only two out of four

[6] Critical aspects will later be thoroughly elaborated on.

critical aspects of the content. However, as a student asked questions about one of the aspects, which according to the plan was not supposed to be enacted, and the teacher attended to these questions, the learning opportunities were affected. In the same study, another lesson in which the plan was successfully implemented was investigated. In this lesson, all four aspects were enacted. When the students from both lessons were tested, it turned out that they had nearly identical results. The conclusion drawn by Kullberg (ibid.) is that both students and teachers contribute to the enactment of the learning opportunities. If a teacher understands what students ask, the opportunity to provide adequate responses to the questions increases. Consequently, Kullberg (2010) emphasises the importance of teachers' knowledge of possible critical aspects of the content taught.

A study of dialogic teaching in science classrooms by Mercer, Dawes, and Staarman (2009) does have the content of the lesson in focus, even though the content as such is not elaborated on in the results. Sociocultural discourse analysis (Mercer, 1995) was used and the dialogue between teachers and pupils was investigated. Case studies of two teachers are used as an illustration of the difficulties in making education 'a cumulative, continuing process for guiding the development of children's understanding' (Mercer et al., 2009, p. 353). The results show that even though both teachers in the study elicited pupils' ideas about the topic, neither of them picked up any of these ideas and built them into the content of the lesson as it developed. The conclusions drawn by Mercer et al. (2009) are, on the one hand, that their study supports the view that better motivation and engagement is found among children whose ideas are sought and used through classroom dialogue. On the other hand, the results show that there is still a need for knowledge development of *how* pupils' ideas are not only elicited, but also built into the content of the lesson.

These three studies (Kullberg, 2010; Macbeth, 2011; Mercer et al., 2009) have the content of interaction as their foci. The questions they answer are *what* the interaction is about, either in between the lines, or more explicitly.

1.4 This study in relation to earlier interaction research

The contents of interaction, and particularly school-subject contents, have not gained much attention in research on interaction (e.g. Mortimer & Scott, 2003). One of the reasons for this is probably that learning is regarded as situated and embedded in social activities in the sociocultural theories that evolved in the 90s. Carlgren (2009) distinguishes between considering social aspects or individual aspects of learning in relation to interaction. From the former perspective, learning is regarded *as interaction*, whereas from the latter perspective, learning is regarded *as rooted in interaction.* Considering only the sociocultural aspects of learning is likely to reduce the phenomenon of learning (Carlgren, 2009). In the first case, *meaning* is constituted in interaction and in the latter case *learning* is constituted in interaction.

As described in this chapter, much interaction research is directed towards the forms and functions of interaction. These studies investigate interaction *as if* the interaction is content-free. Hence, the content of interaction is often considered as contextual factor (Mortimer & Scott, 2003). In this study, social aspects of learning are acknowledged, as learning is regarded as rooted in interaction, not as something that happens unconnected from a context. However, learning is not seen *as* interaction, but as the act of discerning new aspects of a phenomenon. In other words, learning is seen as relational but as a relation between a human being and aspects of the world (content). Therefore, this study has an explicit content perspective. This will be further elaborated on in Chapter 2.

1.5 The structure of this thesis

Chapter 2 aims at giving a research background to the questions asked in this study. Content interaction research is emphasised and the conclusions from this research are discussed in relation to the outset of this study. Chapter 3 is also a background chapter, and here the mathematical content of the study is emphasised. Relevant studies on learning and teaching linear equations/functions are reviewed. Chapter 4 is only a page long, but the aim and the research questions are clarified here. The intention of Chapter 5 is to argue for the theoretical framework. Presumptions and analytical tools are discussed. The purpose of chapter 6 is to give all relevant information on the methods and how the empirical part was carried out. Chapter 7 is the analysis chapter. Here detailed descriptions of both the process of making data ready for analysis and the analyses conducted are given. In Chapter 8, the results are described and the research questions are answered. In Chapter 9 the results are discussed in light of earlier research and the conclusions are drawn. Furthermore, here some implications of the study are discussed.

2 Learner perspectives in teaching

The objective in this study is to investigate learning opportunities against the background of interaction. More specifically, it is about how the *content* of learners' contributions is attended to in the introduction of linear equations in whole-class teaching and, furthermore, what implications this practice may have for the learning opportunities that emerge. This chapter is therefore devoted to exploring and discussing research that emphasises *learner perspective* in instruction. By learner perspective I imply learners' ways of seeing the content taught. The main target is studies of students' and teachers' exchanging of ideas in lessons. This means that the content perspective of the studies has to be acknowledged, either implicitly or explicitly.

2.1 Meanings are negotiated

Even though much education is still founded on different variants of transferring information from teachers to students, the idea of direct transmission of knowledge is no longer much supported. Nowadays the relation between teaching and learning is recognised as much more complicated than that. The concept of *negotiation of meaning* was introduced by a German-American research collaboration in the mid-90s (e.g. Cobb & Bauersfeld, 1995; Voigt, 1994; Wood, 1998) to illustrate that interaction involves subtle shifts in the meaning of the content being communicated. Voigt (1994, 1995, 1996) argues that this negotiation takes place beyond the consciousness of the participants and the focus of Voigt's studies rests in the interactively constituted meanings in a teaching situation. In contrast to many of his contemporaries, he does not see social interaction *as* learning. Instead, he argues that by investigating individuals' meaning-making in ethnographic studies, more and more detailed interpretations of what students are thinking can be made. Voigt addresses the differences in what individuals in a classroom ascribe to a topic, particularly when new a topic is introduced:

> My point is that, especially in introductory situations, we cannot presume that the learner would ascribe specific meanings to the topic by themselves – meanings that are compatible with the mathematical meanings the teacher wants the students to learn. (Voigt, 1996, p. 25)

Negotiation of meaning is constantly taking place in all teaching (Voigt, 1996). Contrastingly, Richards (1996) argues that negotiation of meaning in teaching is only applicable in situations where a willingness to change among both students and teachers exists. Much of what is known as communication in the classroom could be characterised as "talk", he continues. A real negotiation of meaning requires a readiness to change, and in the school mathematical discourse there is not much meaningful negotiation, according to Richards (ibid.), due to the diverse roles that the teacher and students have in negotiation. The teacher is, or should be, a trained negotiator with an agenda that represents a mathematical consensus domain in the classroom, which is a crucial difference compared to the students in relation to negotiation. Voigt (1996) identifies the different backgrounds and agendas in the classroom between teachers and students, and believes that exactly this difference makes the negotiation of meaning into a necessary condition of learning (ibid). My interpretation is that Voigt and Richards discern different aspects of this negotiation of meaning. Richards perceives negotiation as a conscious and formal act, more like the acts of negotiation that diplomats or labour unions are engaged in. Voigt discerns the unconscious and implicit shifts in meaning, which the participants are often unaware of.

The differences between the two ways of perceiving negotiation could also be related to the distinction between *making sense*, a cultural phenomenon, and *making meaning*, an individual aspect of learning (Carlgren, 2009). In this study, learning is seen as *rooted in* interaction, not as the interaction itself. This implies that the idea of an individual meaning making is acknowledged. As Carlgren (2009, p. 206) formulates it: "Even if knowing and acting are one and the same in interaction, the knowing can be taken away and be used in some other interaction."

Voigt (ibid.) describes how teachers and students interactively constitute the content of teaching, like a river that paves its own way, by the negotiation of meaning. Students indicate by their contributions how they interpret the content. The interpretations are responded to by teachers' acceptance or rejections of the contributions. This might appear as a description of an IRE pattern. However, the main distinction between IRE patterns and negotiation of meaning is the same as between recitation and conversation (Nystrand & Gamoran, 1990): the interaction in IRE patterns is driven by a script, whereas in negotiation of meaning, the interaction is determined by what has previously been said.

2.2 Dialogic or authoritative approach

Another way of describing the distinction between IRE and negotiation of meaning is that the content[7] of interaction is unchanging in the first case, and open for modifications in the second. Mortimer and Scott (2003) call this distinction the *dialogic/authoritative dimension*. This, together with their second distinction: the *interactive/non-interactive dimension* has been used as an instrument to select earlier relevant interaction research.

	INTERACTIVE	NON-INTERACTIVE
DIALOGIC	A Interactive/dialogic	B Non-interactive/dialogic
AUTHORITATIVE	C Interactive/authoritative	D Non-interactive/ authoritative

Figure 2.2 Four classes of communicative approach. (Mortimer & Scott, 2003, p. 35)

Mortimer and Scott used this matrix to analyse interaction along two dimensions. The first one, interactive/non-interactive, is the basic construct for much interaction research: does the teaching studied include or exclude the participation of other people? Both an IRE pattern and a negotiation of meaning would belong to the interactive part of this dimension (A/C in Figure 2.2). However, the second distinction in the dialogic/authoritative dimension concerns whether the interaction regards the students' point of view or the science perspective. In an authoritative approach, students' interpretations of the topic of talk are disregarded, whereas in the dialogic approach, different meanings are negotiated. Hence, along this dimension the IRE pattern would belong to the authoritative approach (C) and the negotiation would belong to the dialogic approach (A). Even if classroom interactions are rarely this unambiguous, according to Mortimer & Scott, the two dimensions of interaction are worth reflecting upon. Specifically for the present study, these distinctions will be useful.

It is worth mentioning, that in the intervention study by Mortimer and Scott, several of the participating teachers firmly believed in the beginning of the study that they were taking into account students' ideas because they were always getting the students to talk. Not until later in the development programme, the teachers realised that just because students were heard a lot in the lessons, it did not imply that their ideas had been taken into account.

[7] This word is mine. Mortimer and Scott use "student ideas", "student talk", and "student perspective". However, in my understanding, we refer to the same thing.

2.3 Modes of listening to learner contributions

If interaction is important for learning opportunities, teachers have both the power and the responsibility to create a classroom discourse that enables interaction (Mok, Cai, & Fung, 2008). However, how teachers create this interaction depends on many factors, for instance on how learning is seen. Davis (1997) investigated a middle-school teacher's modes of listening to mathematics lessons. The context was an extension of a collaborative research project with this teacher. He found three different manners of listening to students, and moreover, that these manners are based on fundamentally different rationales. *Evaluative listening* consists of listening for specific answers from the students rather than listening to them. The aim is to check whether they 'stay on the prepared path for the lesson' and consequently the students' contributions have virtually no effect on the continuance of the lesson. *Interpretative listening* has the aim of 'making sense of the sense that students make' and consists of listening for different interpretations of the content taught. Finally, *hermeneutic listening* consists of an actual participation in an inquiry together with the students. Students' contributions are explored and the taken-for-granted aspects of the content are searched for. All three manners of listening were found in one teacher's practice, albeit in different phases of her development and experience as a teacher. Davis (ibid.) studied lessons conducted by this teacher over several months, while also participating in discussions about learning, teaching and mathematics with her. Therefore, he had the chance to build on the teacher's views on mathematics, teaching, and learning. One conclusion by Davis (ibid.) is that the quality of student contributions is closely related to the teacher's ways of attending to them. In lessons in which hermeneutic listening occurred, behaviours and understandings emerged in interaction that would probably not have occurred with the other manners of listening.

Using the two dimensions by Mortimer and Scott (2003) described above, all three listening manners would belong to interactive teaching. However, along the authoritative/dialogic dimension, both evaluative and interpretative listening would be categorised in the authoritative approach, whereas hermeneutic listening would fall into the dialogic approach. This is because in the interpretative manner, one certainly acknowledges that there are different perspectives on the content, but only in the hermeneutic listening are the students' contributions built upon.

2.4 Building on learner contributions

Not many studies have examined how teachers make use of the content of learner contributions. One of the reasons for that could be that this phenomenon is considered as a subtle in-the-moment phenomenon. Rowland and Zazkis (2013) reanalysed three episodes described in earlier research using the question of how mathematics teachers take and miss opportunities to build on students' unexpected contributions. They conclude that there are three possible responses to unexpected contributions from students: *to ignore, to acknowledge but put aside,* and *to acknowledge and incorporate*. They further suggest that the choices teachers make depend both on the sort of mathematical knowledge they possess and also on their perception of teaching as such. Not all teachers attend to students' questions, deal with unexpected ones, or take advantage of opportunities in teaching; it could instead be perceived that they are solely delivering a predetermined curriculum.

Black, Harrison, Lee, Marshall, and Wiliam (2003) studied how teachers can use students' perspectives on the content while teaching. The context was a research project on formative assessment conducted together with British teachers. The main objective of the project was to investigate how teachers could improve their formative assessment skills and thus develop their teaching. Through new insights into their own teaching obtained during the research project, most of the teachers increased the time between question and expected answers, changed their ways of asking questions, and changed the procedures for getting more students to participate in the classroom dialogue. One conclusion from the study was that there seems to be huge differences in teachers' attitudes towards the use of students' perspectives in teaching, when students contribute a wrong answer. Some teachers believed that students' mistakes are at least as valuable as the correct responses, as they may lead to a further development of the content, whereas others described that the reason for not stimulating too much student contribution is the fear of exposing students who answer incorrectly. Hence, this fear seems to control some of the interaction in the classroom.

2.5 A tension between pace and interaction

Another tension in instruction is the one between wanting to use learner perspectives and simultaneously trying to keep a brisk pace in order to cover the syllabus. Jones and Tanner (2002) address the question of what constitutes direct interactive teaching by studying the interpretation of whole-class interactive teaching in eight secondary mathematics teachers' classrooms. The teachers participated in a project which aimed at developing high-quality interactive teaching. Some of the obstacles found relate to the tension described above such as: running out of time since you are debating each contributed method in full; balancing between encouraging pupils even though they contribute wrong answers and the need to progress to more accurate strategies; keeping the pre-planned focus of the lesson while 'going with the pupils'. Some of the benefits found by the teachers were the 'eye-opener' of pupils explaining their own methods, the higher degree of ownership of the mathematical culture for the pupils, and the higher degree of attention to common errors. Jones and Tanner (ibid.) also concluded that in spite of superficial similarities, the quality of the interaction in class varied between teachers. The quality depended on the types of scaffolding (e.g. Wood, Bruner, & Ross, 1979) used, the opportunities created for reflection, and the extent to which thoughts articulated by pupils influenced the classroom processes.

Contemporary assessment strategies are often emphasised as powerful instructional tools. The rationale is that teachers' understanding of learners' misconceptions[8] or errors would inform their instructional decision-making (e.g. Black et al., 2003). Even (2005) examined this conjecture by analysing episodes of teacher-student classroom interactions. Conclusions from the analyses confirm the tension between following up on students' ideas and keeping to the lesson plan. One teacher acted 'as if he had not heard his students at all' (ibid., p. 48) in order to not deviate from his lesson plan. Even (2005) concluded that he was tuned into not hearing his students when their contributions did not match his plan. The teacher was not familiar with common conceptions of the topic taught; therefore, he could neither identify nor accurately address his students' difficulties. Even (ibid.) argues that in order to *hear through* the students' difficulties, you have to sense that there is something to hear, but also to recognise and understand common misconceptions, in order to be able to act on them. Other studies have also described the difficulties of listening to the students' ideas and the case of teachers switching into telling and explaining when the lesson is not going according to the plan. Mason and Davis (2013) conclude that this phenomenon is especially common in teachers' early stages of learning to teach in new ways. It is one thing to understand, for instance, a misconception but quite another thing to use that understanding to make better instructional decisions in teaching (Even, 2005; Mason & Davis, 2013).

[8] For a deeper discussion of misconceptions, mistakes and errors in mathematics, Mason & Johnston-Wilder (2013, pp. 206-213) is recommended.

2.6 Teacher response to learner contributions

Even and Gottlib (2011) investigated how an experienced teacher created her classroom discourse in collaboration with her students. The empirical data in the study consisted of 17 lessons in the teacher's two math classes in grades 9 and 10 in Israel. The focus of the study was the teacher's response to the students' contributions in the lessons. The researchers described different teacher responses as *elaboration*, *accompanying talk*, *opposition* and *puzzlement*. Thereafter, the responses were related to various teaching sequences in the lessons. The categories of teaching sequences were encoded with respect to the purpose of the sequence, based on the TIMSS Video Study (Hiebert, 2003) but with modifications to the teacher's statements in interviews about her teaching. Three of the four categories comprised sequences where the lesson topic was in the foreground: working with the lesson's main content, going back to the previous content and developing beyond the lesson content. The most common forms of teacher response were elaboration and accompanying talk, and these two forms occurred in all whole-class sequences. The analysis also showed that almost all of this teacher's whole-class teaching was generated by or built on the students' contributions of questions, answers, hypotheses and comments. One of the conclusions from the study was that sequences that most often led to the content developing beyond the lesson purpose were initiated by students' contributions. Although the contents of the lesson were not in the analytic focus of the study by Even and Gottlib (2011), one of the conclusions was that the contents of this teacher's lessons developed by means of the learner contributions. The researchers also describe how the teacher made the students' mistakes into the topic of mathematical exploration and how she acknowledged the value of mathematical mistakes in developing understanding. This study by Even and Gottlib emphasises a teacher's awareness of students' ways of thinking. Furthermore, they highlight a teacher's sensitivity to student contributions, and make evident that the lesson content can evolve beyond its original purpose, when the teacher uses her sensitivity.

2.7 A relation between content and interaction

A few studies have used combined theoretical frameworks with the intent of discovering relations between interaction and learning in mathematics classrooms (e.g. Clarke, Emanuelsson, Jablonka, & Mok, 2006; Yackel & Cobb, 1996). Variation theory (Marton & Tsui, 2004) has been used in combination with other theories to reveal connections between interaction and learning opportunities. With the ambition of analysing how interaction can affect learning opportunities from a content perspective, Emanuelsson and Sahlström (2008) compared two lessons, one from the US and one from Sweden, using variation theory to understand *what* learning opportunities were enacted and conversation analysis (CA) to recognise *how* variation emerged in interaction. The lesson setting was whole-class instruction and the two lessons shared the same topic: the slopes of graphs. The results from the study suggest that there might be a price for student participation. The Swedish teacher was attentive to his students and the researchers claim that this led to a watering down of the complexity of the content taught. By contrast, the US teacher kept the interaction with her students to a minimum. The conversation analysis showed that the students had limited opportunities to interact in other ways than just with short answers to and comments on the teacher's questions. The variation theory analysis indicated that the content was elaborated more distinctively in the US lesson, leading to more complex learning opportunities for the content taught. Due to student-teacher interaction in the Swedish lesson, the learning opportunities that emerged had a weaker mathematical focus. Emanuelsson and Sahlström (2008) suggest that interaction may affect learning opportunities negatively.

How can two studies totally contradict each other, such as the ones by Even and Gottlib (2011) and by Emanuelsson and Sahlström, (2008)? The former emphasises interaction for better learning and the latter states that there is a price for participation in terms of learning. Furthermore, both studies embrace content. The explanation lies in what they categorise *as interaction*. Whereas the Israeli teacher, in the study by Even and Gottlib (ibid.), managed to be sensitive to students' content contributions, the Swedish teacher, in the study by Emanuelsson and Sahlström (ibid.), was sensitive to student contributions in general. Participation in the latter study is not specifically defined as *content* interaction, but all teacher-learner interaction. Hence, the differences reside in the attention to content in interaction. A conclusion drawn from a combination of these studies would be that sensitivity to content interaction would enhance learning opportunities whereas general student participation could affect the learning opportunities negatively. Studying relations between participation and learning opportunities is also the purpose of a study by Ryve, Larsson, and Nilsson (2013), in which a combination of frameworks is likewise used. One lesson from an intervention project is analysed in which the content is problem solving with algebraic expressions. The researchers combine three frameworks in the study: variation theory to analyse learning opportunities, a framework of mathematical proficiency to distinguish mathematically important aspects, and the sociocultural concept of semiotic mediation to analyse student participation in the lesson. On the basis of the results from the study, Ryve et al. claim that the explicitness of the content influences the participation of students. When the content is made explicit during the lesson, the students have better opportunities to contribute to the teacher's paths, whereas when the content is kept less explicit, the students are restricted to short responses. How the content is approached in a lesson seems to affect the opportunities for student participation. In these studies (Emanuelsson & Sahlström, 2008; Ryve, Larsson, & Nilsson, 2013), conclusions are drawn about the relationships between student participation and how the content in a lesson is dealt with. Either student participation seems to affect the way the content is enacted, or ways of dealing with the content affect the participation opportunities. In any case, both studies give support to the close connection between student participation and the ways in which the content is enacted in mathematics lessons.

2.8 Exchange of content aspects

Not many studies have examined learning outcomes and the exchange of ideas in instruction. With the same theoretical framework as in this study, Al-Murani (2007) carried out an intervention project with the intention of studying whether deliberate teaching with variation[9] can be associated with better learning outcomes. The intervention comprised co-planning together with five teachers. In addition, five other teachers functioned as a comparison group. Al-Murani studied 80 algebra lessons, and conducted pre-tests, post-tests, and delayed post-tests with the students, along with interviews with the teachers. Both quantitative and qualitative analyses were conducted. The results showed, among other aspects, that all teachers used variation of the content, albeit to different extents. Furthermore, there were no significant differences in the frequency of variations used between the intervention and the comparison groups, although qualitative differences were found. Additionally, the results did not show significant differences in *general* learning outcomes between learners from the intervention and the comparison lessons. However, differences between the two teacher groups were found with regard to how content aspects were exchanged in the classrooms. In the intervention classes, a dynamic exchange of content aspects occurred between the teacher and the students. An assumption in Al-Murani's work is that the students' contributions are linked to their comprehension of the content taught. When the intervention teachers, in contrast to the comparison teachers, responded to the

[9] In this context, teaching with variation implies variation of the content. This will be discussed in Chapter 5.

student-generated variation, they often generated deliberate variation. *Exchange systematicity* (ibid.) is the mechanism through which the teacher, by exchanging aspects of the content taught with their students, expands the shared common ground. All intervention teachers showed some degree of exchange systematicity, whereas there were no signs of exchange systematicity in any of the comparison lessons. Al-Murani concludes that one possible explanation for this is that the variation theoretical intervention may have developed the teachers' awareness for the potential benefits of exchange systematicity. Furthermore, teachers who attended to the contributions from learners and incorporated them into the flow of the lesson were associated with better learning outcomes (Al-Murani, 2007; Al-Murani & Watson, 2009).

2.9 Towards the questions of this study

Exchange systematicity by Al-Murani is a good example of an interactive, but also dialogic approach to teaching by the matrix of Mortimer and Scott (2003). This is because the content is not only *open* for modifications; the core of exchange systematicity is that the content be modified. Would it be possible to systematically exchange every aspect that comes from the students? No, Al-Murani (ibid.) concludes, teachers must assume some knowledge as teaching would otherwise be both inefficient and boring for some students, as some aspects are already well understood. By focusing content aspects of learning and teaching and by acknowledging that learners and teachers, together but probably to different extents, constitute the learning opportunities in a lesson, this study draws a great deal on the ideas of *dialogic approach* by Mortimer & Scott, but even more on *exchange systematicity*[10] by Al-Murani.

The importance of using learner perspective in teaching has in this chapter been emphasised in relation to earlier research. However, rationales behind this importance have not always been clearly elaborated in earlier research. Another interesting facet is that even though high quality aspects of instructional discourse are well researched and emphasised in the last few decades, the implications for practice are not overwhelmingly strong. Could one of the reasons be that we do not know *why* these aspects are so important? This study is devoted to finding out what the use of learner perspective can imply for the learning opportunities from a content perspective. Therefore, the next chapter is committed to discussing the content aspects of this study, namely aspects of linear equations.

[10] This concept will be further elaborated on in Chapter 5.

3 The mathematics in the study

In contrast to a common conception of mathematics as a set of fixed rules, neither changing nor contradicting each other, mathematics is here seen as *a complicated collection of evolving language systems*. Goodstein (1965, in Harper, 1987), from whom the description is borrowed, points out that these various systems have terms in common which are used differently in different systems. One example is the meaning and usage of x, which in some contexts has the meaning of an unknown and in others of a variable. These dual meanings of x will be elaborated on when the main mathematical concepts in the study are discussed (Section 3.1). *The introduction of the equation of a straight line*[11] is the content chosen for the mathematics lessons examined in this study. In Chapter 6, the rationale behind this choice will be elaborated on. In the same chapter this content will be related to the applicable Swedish syllabi.

Research about learning algebra and functions (Section 3.2) has undergone great changes during the last four decades, as has research about teaching of algebra and functions (Section 3.3). The main objective for this chapter is to describe a background to the mathematical content in the study and, additionally, to function as a reference point when qualities of learning opportunities are discussed later.

3.1 Concepts related to linear equations and functions

The difference between linear equations and linear functions is not always easy to distinguish. One reason could be that the *equation* in the equation of a straight line is often accentuated, whereas the functional side of the straight line is deemphasised. Furthermore, the equation of a straight line should actually be denoted as the equation of a linear function as the *straight line* refers to a graphical representation of a function[12]. And *the equation* refers to the algebraic representation of that function. Linear equations and linear functions are first distinguished

[11] This is a direct translation of the Swedish concept: *den räta linjens ekvation*, which was used in communication with the teachers.

[12] There are obviously functions that are not equations and equations that are not functions. But linear equations with two variables, which are depicted by the equation of a straight line, are functions in all cases but one. The one exception is the vertical line, as it does not satisfy the rule of having exactly one value for every argument, and is thus not a function.

theoretically in this chapter, but both concepts are later used as synonyms for: a linear equation with two variables/a polynomial function of the first degree.

3.1.1 The development of algebra and functions

Algebra has, over time, undergone an evolution from mainly a procedural to a structural character. This has had consequences for how algebra is perceived and presented, for example in textbooks (Kieran, 1992). Before the 3rd century, mathematics for instance in Babylonia comprised mainly of operations on known numbers, and additionally, no "algebraic" symbols were used. During the 3rd century, Diophantus introduced letters to represent unknown numbers, making algebra more abstract. However, no *general* methods were used and the focus was on procedures for finding unknown numbers (Charbonneau, 1996; Kieran, 1992; Radford, 2001). During the 16th century, Vièta introduced letters for given numbers in addition to the unknowns, and by this algebra developed beyond generalised arithmetic. The conditions for symbolic algebra now existed and general solutions and proofs could be expressed, and thus algebra became the foundation for analysis (Charbonneau, 1996).

Between the 16th and the 18th century, the concept of function developed through work on analytical geometry done by mathematicians like Fermat, Descartes and Euler. The synthesis of geometry and algebra, as well as the use of independent and dependent variables were crucial for this development, which enabled a change, from an earlier procedural approach, *input and output procedures*, to a more structural approach. Dirichlet modified Euler's procedural concept of function[13] in the middle of the 19th century to become a correspondence rule between numbers, and Bourbaki defined functions in the 1930s as relations between two sets (Kleiner, 1989).

By this definition, the dependent/independent-variable relation was reformulated to a domain/codomain concept and furthermore, the function was no longer necessarily a relationship between numbers, but between any sets, as long as the requirement is fulfilled that each element in the first set corresponds to a another element in a different set (Häggström, 2005).

In contrast to algebra as a part of mathematics, school algebra has not undergone the same explicit evolution of definitions. On the contrary, there is no consensus on what constitutes school algebra (Janvier, 1996; Kieran, 2007), even if it is often seen as a gatekeeper for academic success (Stanton & Moore-Russo,

[13] *A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities* (Kleiner, 1989, p. 3)

2012; Dubinsky & Wilson, 2013). The main reason behind this, I presume, is that the objectives behind algebra and school algebra are slightly different. While algebra has evolved as a field of mathematics, driven by theorems, proofs and definitions, school algebra has been used more as a problem-solving tool. The learning of algebra embraces simplifications as a necessity. As a consequence, concepts and distinctions quite often differ between algebra and school algebra. In this dissertation, school algebra is the topic taught in the analysed lessons; therefore *algebra* will hereafter refer to school algebra if the distinction between algebra and school algebra is not explicitly made.

3.1.2 Equations and functions in school algebra

School algebra can be defined in a variety of ways (Janvier, 1996; Kieran, 2007). Some of the most frequent descriptions of algebra are: generalised arithmetic; a way to express generalisation; relations and formulas; studies of structures; a method of problem solving; a way to represent unknowns and to solve equations; the study of functions, relations, and simultaneous variation; or the language of modelling (Bednarz, Kieran, & Lee, 1996; Kieran, 2007).

Equations and functions, and in particular linear functions, appear in school algebra in various ways. An **equation** is seen as a written statement indicating the equality of two expressions (Kiselman & Mouwitz, 2008), therefore $1 + 2 = 3$ is an equation. However, in algebra equations often contain unknowns, such as the equality $x + 5 = 7$. In this case, x is an unknown, albeit specific number (Küchemann, 1981). A value of x that makes the equation a true statement is said to be a solution to the equation, or put differently, the solution consists of all the values of x that satisfy the equation. Some equations have several solutions, such as $x^2 = 9$, with the two solutions $x = 3$ and $x = -3$.

A **function** is seen as a relation that describes the link between two sets, so that each element in the first set corresponds to exactly one element in the second set (Kieran, 2007). A function does not always have numerical values. For example, the relation between all the students in a class and their latest test results could be described as a function, although it is not possible to express algebraically. However, functions in the school context most often have numerical values. A linear function is a specific type of numerical function, described as a relation between two variables of the first degree (Kieselman & Mouwitz, 2008). In a graph, a linear function is drawn as a straight line. An equation can have multiple variables without a functional relationship, such as $x^2 + y^2 = 9$.

Functions are often used as mathematical models of real-world processes, such as how the height of a tree changes with time. If the tree grows at a constant[14] rate, it can be described as a linear function and we can study the height and relate it to time. The domain of the function can be the period we measure. The height of the tree can be expressed as a function of time (Janvier, 1996). When Freudenthal (1973) characterised functions, he emphasised the dependency aspect and argued that a function was merely a relation between independent and dependent variables. In many contexts, it is obvious which variable is regarded as the dependent or the independent, but in others it is arbitrary. If we look at the growth of a tree over time as a function, the height of the tree is a dependent variable, because it depends of the time, which is the independent variable. The reverse is not reasonable: that time would depend on the growth of a tree. In other cases, such as the relation between the circumference and the radius of a circle, $C = 2\pi r$ or $r = C/2\,\pi$, the circumference can be considered as a function of the radius, yet the reverse – the radius as a function of the perimeter – is also possible.

A function does not need to have a dependent and independent variable but may be constituted only by the relation between two quantities. For instance, the formula $F = 1{,}8C + 32$ is used to convert values for temperature in Celsius to the values expressed in Fahrenheit. In this case, the F and C are two variables, which have a defined relation to each other, but are not dependent on each other in the way that one value will affect the other. The values can be calculated so that they are expressed in one or the other scale. Both F and C are independent variables (Janvier, 1996). The dependency aspect is nowadays not emphasised in mathematics textbook because function is defined as the correspondence between values in two different sets, not necessarily numerical, or between values in the same amount, so that each element in one set defines exactly one element in the second set (Kieran, 1992). This definition does not include the dependent aspect as a necessity (Häggström, 2005).

Separating an equation from a function is not always easy in practice. For instance, $y = 2$ (a horizontal graph) can be considered a function, as all x-values correspond to exactly one (the same) y-value. Yet, $x = 2$ (a vertical line) is not a function since the x-value of 2 would correspond to several (infinite) y-values. The difference between the two is the meanings of x and y and the conventions of writing functions. In school algebra, a function of the first degree is usually described in terms of x and y, where y depends on x. In algebra the same function

[14] The growth of a tree in real life is naturally not constant as it depends on many factors.

($y = 2$) is represented as $f(x) = 2$. Therefore, the distinction between equations and functions is actually neither on the level of drawing lines or graphs, nor on the level of writing equations, but on the level of meaning of the relation between x and y.

3.1.3 Representations of linear functions

Selden and Selden (1992) identify that functions can be described using: a set of pairs, a correspondence, a graph, a dependent variable, a formula, an event, a process, or an object. Another way of generating functions is to focus the different representations, with which they can be described; a) *geometrically* (including graphs/images) b) *arithmetically* (including figures, tables, and pairs of values) c) *algebraically* (including letters, formulas, and symbols) (Kieran 1992). A typical school algebra context will be used to elaborate briefly on the different representations of linear functions described above:

> The length of your hair is 2 cm from the beginning and it grows 3 cm per month. Show the length of the hair as a function of months passed.

Geometrically: the graphical[15] representation will feature a straight line with the slope 3 which intersects the y-axis at (0,2), see Figure 3.1.

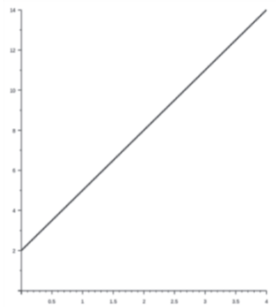

Figure.3.1: Graphical representation

Arithmetically: the relation can be expressed as pairs of numbers: $\{1{:}\,5;\ 2{:}\,8;\ 3{:}\,11;\ 4{:}\,14\}$ or in a table.

[15] The domain is here delimited according to the context.

Algebraically: the function can be expressed with the equation $f(x) = 3x + 2$, where x stands for the number of months that has passed and the lengths of the hair is described by the values of $f(x)$.

In the school algebra context, the function is often expressed in terms of x and y, as $y = 3x + 2$ in the algebraic representation. This keeps the distinction between a function and an equation unclear. This notation is called the slope-intercept form, which is a simplification of the general form: $ax + by + c = 0$, where a and b are separated[16] from 0. This means that the slope-intercept form is $y = -ax/b - c/b$, where the m-value equals $-a/b$ and the y-intercept equals $-c/b$ (Kieselman, & Mouwitz, 2008). The common use of the slope-intercept form might be one of the reasons that the distinction between equations and functions is not clear in the school algebra context.

3.1.4 The equation of a straight line

The algebraic representation $y = mx + b$ denotes *the equation of a straight line*. y and x are variables, m determines the slope and b is described as the y-value of the intercept at the y-axis. In the school algebra context, sometimes b is also labelled as the "start value" for a graph. A straight line is a representation of a relationship between x and y, therefore, straight lines are usually functions, even if they are not always treated as such.

An aspect worth highlighting is that no international standard for this equation has been agreed on[17]. The most common ones probably are: $y = mx + b$ (used in the US, Canada and other countries), $y = mx + c$ (UK, Germany, India, Malaysia and a lot of other countries), $y = ax + b$ (Afghanistan, Denmark, Norway, Romania, South Korea and others) whereas in Sweden (and Latvia) the notation used is $y = kx + m$. Throughout this text the notation used is $y = mx + b$.

3.2 Research on learning algebra and functions

Since the early 20th century, the field of mathematics education research has been influenced by two main perspectives, with different purposes. The cognitive perspective emphasised logical thinking and abstraction, whereas the social perspective stressed the usefulness of mathematics and its relations to an

[16] If $a = 0$, the equation is $y = -c/b$. This can be depicted as a straight line and fulfils the requirements of a function. If $b = 0$, the expression is $x = -c/a$. This can be depicted as a straight line, but it is not a function.

[17] The different notations have been found on a web page: "mathsisfun.com", (n.d)

increasingly technological world. Not until the late 1970s, the number of researchers interested in algebra education reached a level of what could be called a research field. Earlier, mainly behaviouristic research was conducted in search of answers to general questions about learning and memory with a focus on algebra (Kieran, 2007).

During the 1970s and 1980s, Piagetian developmental psychology dominated the research on algebra education and during that period extensive empirical research was conducted on how students understand concepts and procedures in algebra (e.g. Fennema & Carpenter, 1991; Kieran, 2007; Radford, 2011). The focus was often on different *misconceptions*. Misconceptions differ from mistakes, which can be made out of several reasons, such as too hasty reasoning, or a lack of concentration. A misconception is a conceptual understanding that relies on an alternative interpretation of a situation (e.g. Swan, 2001).

This study benefits from knowledge about how students experience different phenomena from the misconception research tradition. Firstly, there is now a vast empirical support for the claim that learners perceive the concepts and procedures of any mathematics topic in diverse ways. Secondly, the literature on misconceptions has revealed an abundance of conceptions of intercepts, slopes, graphs, functions and so on. In many cases, the conceptions are based on intuitive assumptions and pragmatic reasoning. Analogies to everyday life and to other mathematical areas are frequently made (MacGregor & Stacey, 1997). However, students' misconceptions about functions and graphs are intertwined with previous experiences from formal learning, in contrast to science misconceptions that often stem from daily observations of real-world events (Leinhardt, Zaslavsky, & Stein, 1990). As this study emphasises learner contributions in lessons introducing linear equations, results from the research tradition on misconceptions are used as a tool for understanding the rationale behind these contributions.

Some of the misconceptions appear to need a lot of attention on account of being both persistent and common. For instance, the iconic interpretation of a graph and the visual perspective of slope have in common that learners do not *see through* (Mason & Johnston-Wilder, 2013) the picture to the function or situation it represents. This will be elaborated on in detail in this chapter in addition to other topics relevant to the study. The structure of this chapter builds on the classification of properties of a function by (Slavit, 1997) as a way of describing various aspects of linear equations in the context of student thinking.

3.2.1 Functions

Misconceptions about the function concept *as such* are related to whether the function is interpreted pointwise or globally (Bell & Janvier, 1981). Dealing with functions pointwise involves operating with its local properties, for example plotting, reading or dealing with discrete points. The global approach embraces looking at a function's behaviour, for instance by sketching its graph, or finding an extreme point of a graph. The importance of flexibility in using both approaches is emphasised in many studies (e.g. Even, 1998, Leinhardt, Zaslavsky, & Stein, 1990). Overemphasising a pointwise approach in tasks, curricula and teaching 'may result in a conception of a graph as a collection of isolated points rather than as an object or a conceptual entity' (Leinhardt et al., 1990, p.11). Misconceptions following a pointwise approach could be, for instance, discerning function only as a one-to-one correspondence, and not accepting many-to-one correspondences as in constant functions.

Studies have also shown that functions are seen by learners as including at least two variables, so the function of $y = 4$ may not be accepted as a function, as 'it lacks a variable' (Tall & Bakar, 1992). The modern set-theory definition (described in 3.1) seems to cause some problems for learners. For instance, functions not represented by an equation may not be seen as functions (Dubinsky & Harel, 1992), and often only straightforward, non-discrete functions are recognised (Leinhardt et al., 1990).

3.2.2 Graphical and algebraic representations

A large number of studies about students' difficulties with functions focus on multiple representations (e.g. Kieran, 2007; Persson, 2010). A common problem with functions seems to be the ability to transfer information between representations. In particular, the graphical representation causes students concern, and the transition from a graphical to an algebraic representation has been suggested as the most difficult transition (e.g. Kieran, 1992; Markovits, Eylon, & Bruckheimer, 1988).

Misconceptions when it comes to interpreting the information in the graphical representation seem to be a main obstacle (Kieran, 2007). Students' misconceptions about graphical representations of functions have been surveyed by Kerslake (1981). In a study including 1800 British students (13– 15 years old), she found a number of aspects that caused students problems. She identified students' perceptions of the coordinate system, the grading of the axes, their notations of coordinates as well as their understanding of continuity and infinity in the graphical representation. Her results showed that it was not primarily reading or marking coordinates in a coordinate system that the students had difficulties with, but aspects related to the understanding of what a function is, and how it is represented graphically. One example is the question of how many points there are in a graph. Many students discerned only the marked points, whereas others responded that there are points where the graph intercepts the grid of the paper. Very few of the students answered infinitely many. The most common responses suggest that the students did not see other numbers than whole numbers. Misconceptions related to students' over-interpreting correlations in graphs are also described (Kerslake, 1981). One illustration is that in a diagram of the relation between people's heights and waist sizes, students drew lines to connect the people's heights, although no such connection exists, as they were different people. Other difficulties identified were that the notation $(4.6, 10.2)$ was perceived as two points rather than two coordinates identifying one point. The grading of a coordinate system caused concern because the necessity of a uniform grading on the whole axis was not perceived, nor that the grading must be equal on both sides of the origin. Additionally, not perceiving that the origin needs to be zero for both axes caused difficulties.

One well-documented conception of the graphical representation of function is the seeing the graph as a literal picture. This is often called *the iconic interpretation of graphs*. There is a vast literature on this inability to treat a graph as an abstract representation of a relationship (e.g. Clement, 1982; Clement, 1985; Kerslake, 1981; Leinhardt et al., 1990; Monk & Nemirovsky, 1994; Schoenfeld, Burkhardt, Pead, & Swan, 2012; Selden & Selden, 1992). Kerslake (1981) showed that when the axes were switched in a diagram showing the relation between people's heights and waist sizes, the number of correct answers decreased when the y-axis represented waist size and the x-axis height. Higher values of y were interpreted as "up", which meant that students found it easier to read the diagram when people's heights were represented on the y-axis. In a similar way, many students interpreted graphs as representing movements in north and south directions or up and down hills, when the graphs, in fact, represented distance and time (ibid). Making an iconic interpretation of the graph can also mean that a straight line represents a straight path and that negative slope on the graph means that someone walks back or down a hill (Schoenfeld, Burkhardt, Pead, and Swan 2012).

In an interview study by Monk and Nemirovsky (1994), a 12th grade student – "Dan" – was closely studied when he intervened with an "air flow device" connected to a computer that produced graphs of flow rates as well as volumes of air vs. time. Dan focused on the steepness of the graphs, and at the beginning of the interview, he showed an iconic interpretation of graphs. However, he refined his interpretation during the interview and in the end he had developed a more expert-like understanding of graphs. This expertise understanding is reflected by the capability to identify the meaning of visual attributes, to extract information from critical points, to distinguish when the shape of a graph is significant or insignificant and to discern the represented situation through the graph (Monk & Nemirovsky, 1994).

Students' conceptions of algebraic representations of linear functions have not been studied to the same extent as their conceptions of graphical representations. However, some of the misconceptions brought to light by research on algebra in a more general sense are relevant for this context. For instance, Mevarech and Yitschak (1983) concluded that 38 % of college students in a study claimed that k has a larger value than m in the equation $3k = m$. This result points to a lack of understanding of the equality. Although these analyses have been criticised for not taking into account the students' meaning-making of the context (e.g. Aczel, 2001), they show that even with such a seemingly uncomplicated concept as this, many students make interpretations that differ from the standard way of referring to an equation.

Other examples of algebraic difficulties concern understanding and handling brackets (Küchemann, 1981), and perceiving the independency between the m- and b-values of the equation of the straight line (Moschkovich, 1992). The latter implies that students perceived that the change in, for instance, the m-value, would affect the b-value.

3.2.3 Intercepts

Conceptual understanding of intercepts comprises more than using procedures to manipulate equations or graph lines; it involves understanding the connections between the two representations (equation and graph); knowing which aspects are relevant in each representation; and knowing which variables are dependent and independent, as earlier described (Moschkovich, 1992, 1996). In the slope-intercept form ($y = mx + b$), the b-value is often stated to be the y-intercept on the graph (Lobato & Bowers, 2000). For the x-intercept, there is no corresponding value. In fact, the x-intercept in linear functions is mostly an aspect to which little attention is paid. Interestingly, in further learning of functions, the x-intercepts are very useful. The x-intercepts are for instance used in solving equations of the second degree, in which (usually) two x-values are searched for when $y = 0$, i.e. where the graph intercepts the x-axis. Consequently, in linear functions the y-intercept is in focus, whereas in functions of higher degree, the x-intercepts are the important ones.

Understanding the underlying aspects of functions, such as the dual coordinates of every point in a coordinate system, and the correspondence between algebraic and graphical representations, makes switching attention between the intercepts and knowing when to use which quite unproblematic. However, students have been shown not to discern these things and instead make alternative interpretations of intercepts. The results of the study by Kerslake (1981) show that many students aged 13–15 had great difficulties with linear equations. Very few of the students managed to combine a graph with its equation, and Kerslake further describes many alternative interpretations of the relationship between a graph and its equation. Several of these interpretations are procedural. In one case, a student described that she simply determined the equation by using both intercepts of the y- and x-axes as the coefficients b and m respectively.

Almost two decades later, Moschkovich (1998) argues that using the intercepts of the x-axis, like Kerslake's student, is more than just a misconception. Her video recordings and pre-and post-tests of 18 students (15–17 years), who worked in pairs with linear graphs, showed that $72\ \%$ (14) of them at some stage used the x-intercept in discussions and/or on tests. Moschkovich identified three alternative conceptions of the x-intercept: first, as the distance the graph has been moved in the horizontal direction, secondly as the m-value of the equation and, thirdly as the b-value. In light of these conceptions, the students did not perceive that the point of interceptions has two coordinates since $y = 0$ in the x-intercept. These conceptions are common and Moschkovich contends that this shows the complexity of the content. Therefore, she argues for the potential for refinement of this conception. She concludes that this way of experiencing the x-intercept should be considered as a *transitional conception* and not as a superficial error or a misunderstanding, and argues for these transitional concepts as reasonable, useful, and part of learning trajectories (Moschkovich, 1998).

3.2.4 Slopes

Slope is a pivotal concept providing a means to describe a function's behaviour. Students' conceptions of slopes have been vastly researched. Especially concerning slopes in graphical representations and their connection to the rate of change in the corresponding function. Findings show that common misconceptions about slope are: misjudging height for slope (e.g. Leinhardt et al., 1990), seeing slope solely geometrically as an angle (Even & Tirosh, 1995; Zaslavsky, Sela, & Leron, 2002), confusing slope with the total length of the graph (Hadjidemetriou & Williams, 2002), and seeing slope as a physical property of a line (Ayalon, Watson, & Lerman, 2016; Stump, 1999).

In this section, I will, however, discuss one of the conceptions found in much research and seen as an important distinction for further learning, namely seeing slope *visually* or *analytically*. Zaslavsky, Sela, and Leron (2002) closely examined implicit assumptions about slope (of a linear equation) of 124 mathematicians, secondary mathematics teachers, mathematics educators and 11th grade students. The questions were non-standard problems with non-homogeneous coordinate systems and regarded the differences between the geometric and algebraic aspects of slope, angles and scales. Their results showed a distinction between a visual and an analytical approach. By the visual approach, slopes are regarded as a property of the line (graph) which varies if the scale changes non-homogeneously. By the analytic approach, slope is regarded as a property of the function, which remains invariant under changes of scale. Zaslavsky et al. (2002) argue for a less sloppy language concerning for instance slope; the slope of *the line representing* the function is a better formulation than the slope of the function. They also suggest using any opportunity to enhance the learning of slope, by distinguishing between the visual slope (the slope of a line) and the analytic slope (the rate of change of a function).

When slope is perceived visually, the distinction between slope and steepness is not clear. Slope is seen as a characteristic *of the line*, namely the steepness of that line. Lobato and Bowers (2000) argue that slope is better conceived as 'the rate of change in one quantity relative to the change of another quantity, where the two quantities co-vary' (ibid, p. 10). This definition corresponds to the analytical approach by Zaslavsky et al. (2002). Focusing on the steepness of a line as a visual conception could leave the above meaning of slope concealed.

Lobato (2006) made an interview study of generalising activities in school algebra in which the task below (Figure 3.2) was included. This task enhances an analytical conception of the slope, and contrasts to a visual, as the steepness is separated from the slope of the function. Both slopes in the two graphs below are the same, but the steepness varies between them. Hence, the task could be used to separate between a visual and analytical perspective on slopes.

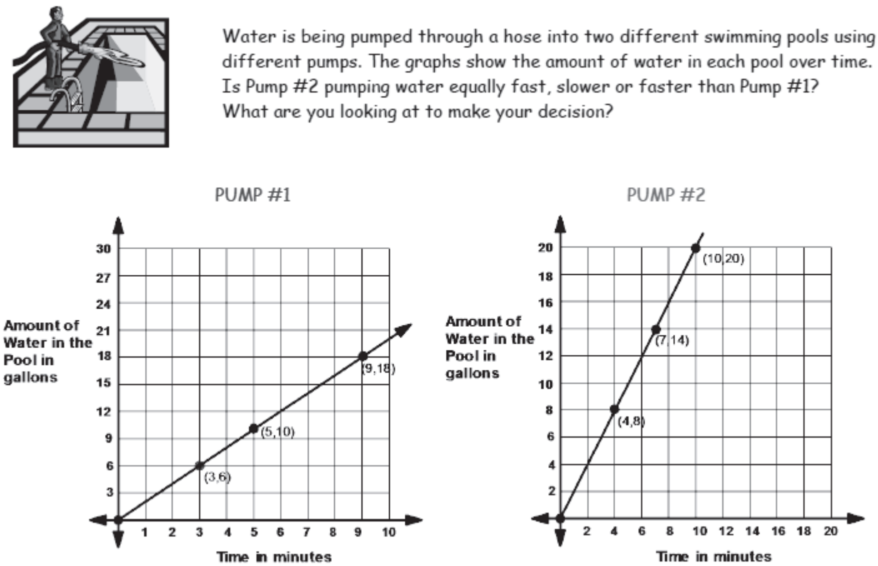

Figure 3.2: A task comparing two lines with the same slope but different steepness.(J. Lobato, 2006): reprinted by permission.

Although research on misconceptions about linear functions has revealed a lot that could be useful in teaching, fewer studies have been carried out on how these conceptions are used, challenged or elaborated on in teaching. Lobato and Thanheiser (2002) strongly argue for the need of developing an understanding of slope as a rate of change in teaching:

> Some people may argue that slope should not be made any more difficult than the slope formula. They are right if we are satisfied when students are only able to solve textbook problems that cue students when and how to use the formula. In contrast, real-world situations involving rates of change are usually messier and more complex. Rather than avoiding complexity, instructional activities should help students learn how to cope with it. (Lobato & Thanheiser, 2002, p. 174)

One of the conclusions by MacGregor and Stacey (1997), in their large-scale study of 2000 pupils (11–15 years), was that the origins of pupils' misconceptions need to be understood in order to improve the teaching of algebra. Therefore, let us now turn to research on *teaching* linear functions.

3.3 Teaching linear equations and functions

The strong sociocultural influence on mathematics education research in the 1990s led to an increasing interest in classroom research and a change in the objects of research. Many studies had earlier focused on the students in algebra and now instead an increasing number of studies focused on the teacher and the teaching (Kieran, 2007). As misconceptions are difficult to study or categorise without conceptions, mathematical content was an essential ingredient in this research. However, after the social turn, the mathematical content studied was more established in different aspects of Pedagogical Content Knowledge (PCK) (Shulman, 1986; 1987), Mathematics Knowledge for Teaching (MKT) (Ball & Bass, 2003) and other ways of analysing teacher knowledge. The student knowledge of content was now given less attention (Sahlström, 2008).

In 2007, Kieran summarised her research review on algebra education, by arguing that we have gained a comprehensive understanding of students' algebra learning and the conditions of the same. What was still missing was to develop an equally deep understanding of algebra education and the teaching practices that are effective in creating opportunities for algebra learning. This account has, also later, been argued for in many studies (e.g. Dubinsky & Wilson, 2013; Kieran, 2007; Radford, 2000; Piccolo, Harbaugh, Carter, Capraro, & Capraro, 2008). However, what we know about algebra teaching is, for instance, that novice students bring meanings from other domains into algebra teaching. As Radford (2000) expresses it:

> Hence, it seems to us, one of the didactic questions with which to deal is not really that of the elaboration of catalogues of students' errors in algebraic manipulations, which may be interesting in itself, but that of understanding how those non-algebraic meanings are progressively transformed by the students up to the point to attain the standards of the complex algebraic meanings of contemporary school mathematics. (Radford, 2000, p. 240)

I will here consider results from research both on teaching linear functions and on the relation between teaching and learning. Furthermore, some relevant studies on interaction in instruction will be discussed.

3.3.1 Conclusions from studies on teaching linear functions

The meaning of algebra is easily lost for students. This loss of meaning is strongly related to how students experience the teaching of algebra (Kieran, 2007). The meaning of algebra can be maintained if the students have the ability to discern the abstract ideas hidden behind the symbols (Sfard & Linchevski, 1994). Even though the meanings of the algebraic structures behind the operations can be elusive, it is still this source of meaning that many mathematics educators and researchers believe is the foundation for learning in algebra (Kieran, 2007). This source of meaning comes from the mathematics itself, either from its underlying structures or from various representations and the transitions between them. There have been some studies on teaching of functions with a perspective on content, yet this is still far from the vast amount of studies on students' conceptualising of functions. Some of the findings are summarised here.

Leinhardt et al. (1990) draw attention to the fact that the algebraic and the graphical representations are two very different symbol systems. Furthermore, the mathematical presentation of function in the school context usually involves going from an algebraic function rule to ordered pairs in tables to a graph, whereas in science most often the direction is the reverse: from observations to ordered pairs of data, to graphs and then perhaps to an algebraic expression. They also conclude that many students who can solve graphing problems in mathematics seem unable to do that in science. Bell and Janvier (1981) argue that there is an overemphasis in traditional instruction on a pointwise approach; students are asked to plot a graph from a table of ordered pairs and are then presented with a series of questions that can be answered from the table alone. Instead, Leinhardt et al. (1990) claim, students 'should be introduced to qualitative graphs of concrete situations and asked to view them globally instead of pointwise' (ibid. p. 28).

Representations of functions, and particularly the transitions between them, have been a major topic in mathematics educational research called the multi-representational perspective (e.g. Kaput, 1989; Lobato & Bowers, 2000). The main representations considered are tables, equations and graphs. These representations are regarded as "the big three" (Nemirovsky, Kaput, & Roschelle, 1998). Conclusions from research on teaching show that students have greater difficulties with the transition from graphical representation to the algebraic than wise versa. Lobato and Bowers critique the multi-representational perspective by questioning whether the different representations are 'multiple representations of anything to students. That is, students may learn to move among the representations, but not understand what is being represented' (Lobato & Bowers, 2000, p. 4).

Two content-specific difficulties with teaching graphs and functions concern slope and intercepts. For the slope, the visual aspect has to be separated from what the slope represents. For the intercepts, it is even harder. In the slope-intercept form[18], only the y-intercept is explicit in the algebraic representation as the b-value. The other, equally visible x-intercept goes unnoticed (Leinhardt et al., 1990). However, when teaching using the general form[19], then both intercepts can be used to find points, which might be the easiest way of drawing a graph. How to handle these aspects has been shown to be challenging (ibid.).

The final concern discussed here from research on teaching functions is the language use. One of the more troublesome aspects of instruction in graphing is the large number of notational conventions. In addition, words such as point or line have particular meanings in both everyday language and in mathematics, and these meanings have both connections and dissociations with each other (Leinhardt et al., 1990). One difficulty is not to overlook the roles of students' natural language while still using, for instance, the language of quantities for slope. Using expressions like "goes up by" (elaborated on below) for slope will leave the way open to many parallel meanings, and allows participants to talk past each other; therefore, a more distinct language is required. Nevertheless, just using formal language may not be the solution as it may 'lack references for the students and therefore represent nothing more than a recipe' (Lobato, Ellis, & Muñoz, 2003, p. 30).

3.3.2 Connecting teaching and learning

There have not been many studies focusing on the relation between learning of explicit mathematical content and the teaching of the same. However, in this section, two studies that share my interest in trying to relate teaching and learning will be discussed. The first was carried out by Dubinsky and Wilson (2013), who evaluated teaching as well as student learning of the concept of function in high school (age 14–16). The students were all at the lowest level[20] of socioeconomic status and academic achievement. The study included a 7-week instructional treatment, immediate assessments after instruction and in-depth interviews several weeks later. The main conclusion of the study was that with appropriate teaching these students were capable of learning high-school mathematics on a high level. Not many of the common misconceptions about function revealed in earlier research (see 3.2) were shown in the in-depth post-intervention interviews. Appropriate teaching in this example was signified by an intensive focus on both the mathematical concepts of function and on the students' mental models of

[18] $y = mx + b$

[19] $ax + by + c = 0$

[20] This was a prerequisite for participation in the study.

these. Dubinsky and Wilson showed that there was a strong relationship between what was taught and what was learnt. This relationship may not come as a surprise to all; however, it is not the most common claim in contemporary educational research, to say the least, and especially not considering the group of students that participated in the project. On the contrary, much research in mathematics education is focused on the differences between what was taught and what was learnt.

The second example of the relation between how learners comprehend a concept and the ways this concept has been dealt with in teaching is an empirical classroom study by Lobato, Ellis, and Muñoz (2003). Lobato et al. argue that there is a requirement for greater attention to what students' attention is directed to when introduced to any new topic. They found strong relations between the meanings for the m of linear equations that high school students expressed and the ways slope as a concept had been taught for those students. All of the students who were able to write an equation for a given line or table, expressed in post-lesson interviews that m is "what it goes up by". They, however, turned out to attach different meanings to that expression. None of them expressed m-value as a rate of change between y- and x-values. Instead it was seen as a difference. This difference could be related to the scale of the x-axis, the change in y-values, or the change in x-values. When the researchers studied the video recordings of the lessons and searched for how the instructional environment afforded each of the meanings expressed by the students, they found that the language used in teaching was mainly in terms of "goes up by". Detailed transcripts showed that the teacher and the students never clarified specific meanings for rate of change. Instead they simultaneously held different meanings for the same phrase. Other aspects having impact on how the students interpreted slope were the usage of graphing calculators, well-ordered tables and uncoordinated sequences and differences. Using Δx as 1 without emphasising that it could vary and considering Δx and Δy as two separate differences, not enacting them as a ratio, also sustained a focus on two uncoordinated quantities. Lobato et al. (ibid.) conjecture that by altering the nature of the ways the content is enacted[21] in a classroom significantly affects the nature of students' generalisations; however, this is far from an automatic and straightforward process. Even though I will use a different theoretical framework for the analysis than the one used by Lobato et al. (2003), it is evident that we share an interest in examining details of how the content is dealt with, in close relation to what students learn.

[21] Lobato et al. call this "focusing phenomena"

In summary, earlier research on the relation between teaching and learning suggests that how the content is interpreted by students is related to how it has been enacted in teaching. Other conclusions from earlier research on the learning of linear equations, propose that students often learn superficial aspects of the concepts instead of understanding deeper facets. For instance, graphs are often interpreted with an iconic interpretation instead of a relational one. Moreover, slopes are often interpreted visually instead of analytically. These distinctions are well established by earlier research and one of the reasons for this superficial learning seems to be that teaching is not always approached to understanding. Another tradition that this study benefits from is the misconception research. When analysing rationales behind learners' contributions in instruction, the results from misconception research will be used to understand diverse ways of understanding the content. However, the concept of misconception will not be used in the analyses. The rationale for this will be discussed in detail in Chapter 9.

4 Aim and research questions

The overall purpose of this study is to contribute to knowledge about the relation between teaching and the opportunities to learn mathematics. However, this purpose needs to be delimited and specified. Therefore, teaching has been limited to whole-class instruction and learning has been limited to the learning opportunities that emerged in different classrooms, from a content perspective. The relation between them remains in focus. For the specification, I turn to an old quote. Long before this study was even thought of, Marton and Booth beautifully and precisely expressed the object of it:

> […] and the essential feature is that the teacher takes the part of the learner, sees the experience through the learner's eyes, becomes aware of the experience throughout the learner's awareness. If we consider the learner to be internally related to the object of learning, and if we consider the teacher to be internally related to the same object of learning, we can see the two, learner and teacher, meet through a shared object of learning. In addition to this, the teacher makes the learner's experience of the object of learning into an object of her own focal awareness: the teacher focuses on the learner's experience of the object of learning. (Marton & Booth, 1997, p.179)

Perceiving interaction as a way of meeting through shared objects of learning omits other sensible reasons for interaction. It also specifies that this study is about content interaction, i.e. the interplay between teachers and learners about the content taught. As comparisons of learning opportunities are conducted, the topic of the lessons has to be the same in some sense. The choice of lesson topic – the introduction of the equation of a straight line – fulfilled three conditions: it is easily referable in communication with teachers, it is delimited, and it occurs in both lower and upper secondary school. The aim of the study is to gain deeper knowledge on relations between interactions and learning opportunities that emerge in instruction when linear equations are introduced. The research questions are:

1. What do teacher attentions to learner contributions in instruction imply for the learning opportunities of linear equations that emerge?

2. What do learners contribute to the enactment of linear equations?

We don't see things as they are; we see them as we are.
Anaïs Nin (1903-1977)

5 Theoretical framework

A presumption in science today[22] is that there are no objective positions to conduct research from (e.g. Latour, 1987). Every research question, method and result emanates from some theoretical perspective(s), which means that some features have been chosen to form the foreground while others have been left in the background. Classroom contexts are highly dynamic and complex; analyses and descriptions of them are always partial and dependent on the theoretical perspective chosen.

The object of this study is content-related learning opportunities. Variation theory addresses learning opportunities in a content perspective and provides theoretical constructs; hence I have chosen variation theory as a framework and some of its main concepts as analytical tools. In this chapter, three basic assumptions for the study are outlined. The first concerns how learners experience the topic taught in a lesson. The second regards the relation between what learners contribute and discern of the topic. The third involves how the learning opportunities in a lesson are constituted. Subsequently, the analytical tools are defined and discussed.

5.1 Perspective

Variation theory is primarily a theory of learning (Marton, 2015; Marton & Booth, 1997), but has also been used as a theory for analysing teaching in diverse ways (Emanuelsson, 2001; Häggström, 2008; Kullberg, 2010; Lo, 2012, Marton & Tsui, 2004; Rovio-Johansson, 2002; Runesson, 1999). Furthermore, variation theory has been applied to activities such as designing tasks, sequencing contents, and other lesson planning[23]. The development of the theory has been driven by an interest in differences in how phenomena are experienced[24] and how these perceptions

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[22] This could be contrasted with a positivistic perspective before the paradigmatic shift in science in the second half of the 20th century (e.g. Molander, 2003))

[23] Variation theory for teaching is not much discussed in this thesis. Interested readers can turn to Lo (2012).

[24] In this study, to perceive or to understand are used as synonyms, as are experiences and understandings. They all have the same signification: ways of seeing something by someone. In literature referred in this chapter to discern, to see, to experience, and conception and discernment are also used.

change through teaching. Variation theory is rooted in phenomenography (Marton, 1981), a research tradition aiming at investigating qualitatively different perceptions of phenomena. The ontological presumption is a non-dualistic position in which perceptions are seen as relations between the world and human beings, including both. These perceptions are understood as ways of seeing something (the object) by someone (the subject). Hence, the human being is seen as an active meaning maker of the relation to the world embracing her (Marton & Booth, 1997).

5.1.1 Experiencing the world in different ways

A point of departure for this study is that we experience the world in different ways. An experiment with a chemistry teacher and her students (Andersson et al., 2003) could serve as an illustration of the central role that our earlier perceptions play in further perceptions. The teacher observed table salt (sodium chloride) in microscopes and sketched her impressions on paper. This was followed by students conducting the same task. The salt grains sketched by the teacher were cubic while the salt grains sketched by students were more circular. An interpretation, made in the study, of the differences between the drawings is that the chemistry teacher sees the salt grains with an understanding of the cubic crystal structure of salt, whereas the students' more circular drawings are connected to their everyday experience of salt grains. Whose drawings are most accurate? Perhaps one could argue that the teacher's views are in line with a natural science model evolved by generations of scientists while the students' views are more influenced by an everyday experiencing. Still, the point made here is that no one can study salt in microscope without *any* perception, and that the aspects we have already discerned do play an important role in what we experience further on. We can, however, conclude that there are at least two different images of salt, seen through a microscope.

The first assumption for this study, in line with the reasoning above, is that learners in a classroom experience the phenomena that constitute the content taught in different ways. Marton and Neuman (1996) elaborate:

> A common assumption adopted in studies of learning in educational contexts is that different learners read the same text, solve the same problem, listen to the same lecture, and then – because they are equipped differently – do different things with the text, problem, lecture they have somehow internalized. Our studies showed this assumption to be invalid. The conclusion we arrived at was that that the learners do not really read the same text, solve the same problem, or listen to the same lecture, even if the experimenter sees them bowed over the same text, struggling with the same problem, or attending the same lecture.
> (Marton & Neuman, 1996, p. 315)

This first assumption builds on the stance that the differences in *how* people experience a phenomenon emanate from differences in *what* they see in that phenomenon, not from general differences in understandings or differences in our logics[25] (Smedslund, 1970, also elaborated on in Marton, 2015). Consequently, the content taught is not *the same* phenomenon for the participants of a lesson. Teachers and their learners will experience different aspects of the content, as will different learners. *Content* is therefore in this study regarded as a concept on a level at which different perspectives are not distinguished. In curricula or lesson plans, for instance, it is accurate to use the word content. However, when studying learning or learning opportunities, the meaning of the content will always differ between different participants or groups of participants in a classroom. Therefore, the concept *object of learning* (e.g. Marton & Booth, 1997; Wernberg, 2009) is used in many variation theory studies to illuminate several angles. The object of learning can be described on several levels, such as the ability to understand how a rainbow can be generated using a prism (Lo, 2012) or the ability to swim (Marton, 2015). Another distinction in relation to the object of learning is what perspective is taken (Häggström, 2008). The *intended* object of learning describes the teacher's/teachers' perspective, i.e. the learning intentions with a lesson. The intended object of learning could be more or less reflected upon (Marton & Tsui, 2004). The *enacted* object of learning describes an observer's perspective and is a result of an analysis of how the content is dealt with during a lesson. The *lived* object of learning describes what the students' actually learnt.

Teaching is more or less constituted in interaction between teachers and their students. Lo (2012) acknowledges the dynamic character of the enacted object of learning, and argues for its unpredictability. The intended and lived objects of learning are beyond the scope of this study, whereas the *unpredictability* of the enacted objects of learning is fundamental. Even though the content of the lessons studied is the same on one level, different aspect of this content will probably be enacted in the lessons, due to the co-constitution by teachers and students. To answer the research questions, comparisons of the aspects enacted of the content between lessons will be conducted.

[25] In contrast to for instance Piaget, who concluded that children's logic differs from adult logic until a certain age (Marton, 2015).

5.1.2 Learning as discernment requires variation

In contrast to theories in which learning is described in terms of enrichment, construction, or participation, variation theory describes learning as differentiation (Marton, 2015). According to variation theory, learning is seen as the discernment of new aspects of phenomena and this discernment presupposes variation. The idea that we discern differences against a background of sameness is central (ibid.). If you have only heard one language in your life, the concept of language has no meaning until you hear a second one. To be able to discern the meaning of language, you have to experience a variation of languages. This is the core theoretical stance in variation theory.

Then again, in real life, different phenomena are intertwined, and a phenomenon such as *language* is not always easy to distinguish clearly. Having heard only different dialects of your own language would make it possible for you to discern what a dialect is, and also to discern your own dialect from another. Nonetheless, you would not be able to discern dialect from language, or notice when a dialect actually turned out to be a new language[26]. Hence, learning is not only about discernment, it is also about building relations between different aspects or between parts and wholes of various phenomena.

5.1.3 Critical aspects discerned or necessary to discern

There are not infinitely many ways of experiencing various phenomena. Phenomenographic studies have numerous times empirically shown that there are limited ways of experiencing a phenomenon (Marton & Booth, 1997). We discern and are aware of different aspects of phenomena, but not in an infinite number of ways. The objectives in a phenomenographic study are the different conceptions of a specific phenomenon. The conceptions are separated by detailed analyses of what aspects are discerned in one conception but not in the other. The distinctions between conceptions are constituted by *critical aspects* (Pang & Ki, 2016). The focus in phenomenographic studies is what critical aspects *have been* discerned, which in turn constitute a conception. Neuman's (1987) phenomenographic study of preschool children's ways of experiencing numbers is an example of research revealing the importance of critical aspects. 105 children were studied regarding their conceptions of the cardinal and the ordinal aspects of numbers. Although the vast majority of the children had experienced both these aspects before starting

[26] This could, for instance, be applied to the question whether Norwegian and Swedish are different languages or dialects.

school, Neuman found other conceptions among the children. Some conceptions could for instance rely on only one of the aspects.

Critical aspects as a concept has dual meanings (Pang & Ki, 2016), both as the aspects that distinguish different conceptions in a phenomenographic study and, furthermore, as the aspects that are *necessary to discern* in order to experience a phenomenon in a specific way (Marton & Booth, 1997; Marton, 2015). Neuman (2013) argued later that the children's alternative conceptions could lead to difficulties for further mathematics learning if not challenged in early school years. The cardinal and the ordinal aspects of numbers were found to be critical aspects, necessary to discern for developing a number sense. A significant number of contributions of the wide-ranging variation theory studies have been in line with the aim of Neuman's study (2013): to find out aspects that it is critical to discern in order to see a phenomenon in more powerful ways. This claim is consistent with Marton's (2015) argument that more powerful ways of seeing lead to more powerful ways of acting.

The second assumption of this study is that a relation exists between how the learners act, in this case by their contributions to the lesson, and what the learners direct their attention to in the content taught. To some extent, learners' contributions express how the content of the lesson is experienced: the acts and the perceptions are related. Al-Murani and Watson (2009) describe the variation generated by the learner as "a partial articulation of the lived object of learning: it may not express everything the student is aware of, but provides a window into some awarenesses" (Al-Murani & Watson, 2009, p.3).

5.1.4 Learning opportunities are co-constituted

The third assumption of this study is that both teachers and learners contribute to the formation of the enacted objects of learning. Depending on what the teachers introduce, what the learners contribute and how the teachers follow up on these contributions in the lessons, the objects of learning will develop in different ways and thus different learning opportunities will be offered in the lessons. The main object of the study is the differences in co-constituted learning opportunities for linear equations in different mathematics classrooms.

Variation theory studies have not so far been very much concerned with the significance of classroom interaction for the learning opportunities. The question has not often been whether necessary variation is shaped by the actions of the teacher or the students, separately or in cooperation (Emanuelsson & Sahlström, 2008). Nonetheless, Runesson (1999) discusses learning opportunities in relation to who introduces the variation in the lessons. She argues in terms of from whose perspective variation is constituted, students' or teachers', and thus emphasises the students' way of experiencing as an important starting point for the co-constitution of learning opportunities. In addition, she proposes seeing variation theory as a complement to learning theories which focus on the interaction itself. When learning opportunities constitute the research object, the limits of what is possible to learn are under study. It is possible to create and change these limits, both for teachers and students (Runesson, 2005).

As variation theory developed into a tool for analysing learning opportunities in terms of enacted objects of learning, more attention was given to the interaction in teaching. Lo (2012) suggests that because the teacher has the most authority in the interaction in the classroom, it is in the teacher's power to open up or close down students' learning opportunities in a lesson. She argues that we can never force students to learn, only offer the best of learning opportunities. In order to make the objects of learning more rewarding for the students, teachers need to apply constant modifications of the content taught based on students' reactions during the lesson. Marton (2015) has recently emphasised the importance of interaction in the classroom, as teaching is interactive: what happens in the classroom does not depend only on the teacher.

In an empirical intervention study[27], which compared learning outcomes from a variation theory perspective, Al-Murani (2007) found that a systematic exchange of variation between the teacher and the students in interaction was a feature of the lessons in which the students performed better in post-tests. Even though teacher-offered and student-generated variation were present in all lessons, the teachers in the intervention study tended to emphasise the students' contributions while the teachers of the comparison group tended to treat the contributions as peripheral. Al-Murani (ibid.) defines this exchange of variation as *exchange systematicity*, which implies that teachers address learners' contributions in a deliberate and systematic way. It manifests itself in the following way: when a new aspect of the content is generated by a learners or when confusion is shown about an existing aspect, the teacher attends to this by a deliberate and systematic variation of this aspect. The rationale behind is that that the learner generated variation is connected to how the learner perceives the content taught and the teacher's response indicates an awareness of a lack of consensus on assumed common grounds. The exposition and variation of learners' contributions in exchange systematicity expands the shared common ground in a lesson.

[27] Earlier described in subsection 2.7.

Al-Murani and Watson (2009) claim that attention to systematic exchange gives insights into how learning opportunities are jointly developed publicly. The last assumption of this study, of the co-constitution of learning opportunities, builds on the conclusions by Al-Murani and Watson.

5.2 Analytical tools

The concepts used in the analysis are discussed and defined in this section.

5.2.1 Learning and learning opportunities

Learning outcomes are not evaluated in this study. Instead, the learning opportunities that emerge and the relation to learner contributions are examined by the use of the tools of variation theory. *The space of learning* (Marton & Morris, 2002; Marton & Tsui, 2004) is a concept to describe what is possible to learn in a situation. The aspects of an object of learning that are enacted in a lesson define the space of learning and constitute the limits for what is made possible to learn (Marton & Tsui, 2004). Runesson (1999) used the term *space of variation* (in an early study of the tradition) to describe how the content was dealt with in the classrooms she studied. One of the conclusions drawn was that different objects were shaped in the classrooms and consequently, the students were offered different opportunities to learn. Although the topic taught was regarded as the same part of school mathematics, the learning opportunities differed (Runesson, 1999). In a similar way, Häggström (2008) examined spaces of learning created for systems of linear equations in 13 lessons carried out in Sweden and in China (Hong Kong and Shanghai). The results showed several substantive differences regarding which aspects of the content that were enacted in different lessons. The conclusions were that different learning opportunities were offered in the Swedish and the Chinese classrooms.

5.2.2 Dimensions of variation

Häggström (2008) used the concept of *dimension of variation* (DoV) as an analytical tool to compare the learning opportunities in the Swedish and Chinese classrooms. This concept is defined as an aspect of a phenomenon that is enacted by the variation of it. For example, if the slope of a function is focused on in a lesson and several different slopes are (the only aspect) varied then a DoV for slope is opened. As slopes vary, they become possible to discern. This is built on the variation theoretical stance that we discern differences, not sameness. If only one slope is discussed (shown/worked on), no DoV is opened regarding slope as a phenomenon, since no variation in that dimension has been enacted. This could be perfectly in order, as all learners might have already discerned what a slope is and there is therefore no need for a DoV to be opened.

Critical aspects and dimensions of variation are both concepts that can be employed in a variation theoretical analysis. However, a difference between them needs to be pinpointed in relation to this study regarding *critical* in critical aspects. Critical aspects are always critical in relation to a learner and a certain way of experiencing a phenomenon. This study does not include empirical data on the lived objects of learning in the lessons, i.e. what the learners learnt. This exclusion precludes all empirical claims about the critical aspects of linear equations for the learners in the study.

This study does not include any intended object of learning either. The advantage of having an intended object of learning is that presumptive critical aspects can be defined beforehand, as a certain way of seeing a phenomenon is the intent for the lesson. Even though an object of learning has a tendency to develop in a lesson, the critical aspects can still be used as a reference in the analysis of the enacted or lived objects of learning. In this study, there is no such reference; hence all DoVs will have to be considered. Nonetheless, the results will reveal differences in how the content is enacted in different lessons.

Again, by omitting some data, other data can be included. In this case, the omitting of, for instance, pre-lesson interviews enabled the inclusion of more lessons compared to most other variation theoretical studies. This was a necessity in relation to my research questions. Therefore no conclusions will be made about whether the DoVs were critical for students or not. However, learners in a lesson have not ever discerned exactly the same aspects before the lesson, nor will they discern exactly the same ones *in* the lesson. Probably, some DoVs will be critical for some learners, but not for others.

5.2.3 The dynamic nature of critical aspects

Critical aspects cannot be derived from disciplinary knowledge only, but are also dependent on the actual learner. According to the epistemology for variation theory, learning is established as a change of the relation between the world and the learner. Hence, a way of experiencing something is always just a part of the many qualitatively different ways of experiencing the same phenomenon (Pang & Ki, 2016). A study by Lam (2014), on the learning of Chinese characters, serves as an illustration of how critical aspects are dependent on the children you teach.

Children who speak Cantonese must learn to distinguish between two characters as they are homophones, whereas for children with a different dialect it would be absurd to confuse these characters because they do not sound the same in this dialect of spoken Chinese. Lam's example clarifies the dynamic disposition of critical aspects; in this case the mother tongue (dialect) delimits a critical aspect. Lam (ibid.) suggests, in spite of giving an example of how critical aspects are delimited by for instance your mother tongue, that critical aspects of a phenomenon could be infinite in number, because students can experience a phenomenon in a virtually infinite number of ways. Phenomenographic studies have, however, numerous times empirically shown that there are limited ways of experiencing phenomena (Marton & Booth, 1997).

5.2.4 Necessary and optional aspects

Limited ways of experiencing phenomena imply limited number of critical aspects for experiencing a phenomenon in a certain way. These critical aspects are categorised as *necessary* aspects (Marton, 2015) as they are necessary for precisely one way of seeing a phenomenon, often the intended object of learning in a learning situation. These aspects have also been described as the defining aspects (Marton, 2015). From here it would be only a short step to deriving the defining/necessary aspects from the disciplinary knowledge only and to start describing critical aspects as *building bricks* for knowledge. However, Pang and Ki (2016) argue strongly against the idea that critical aspects be derived from the disciplinary knowledge itself, as the learners' perceptions of the object of learning must also be taken into consideration. They conclude:

> In fact, irrelevant aspects must always be seen as part of the nature of any way of experiencing within a discipline. When a new way of experiencing the world is formed, it has an intended purpose and a new vantage point, and differentiates itself from other co-existing ways of experiencing. Hence, although a new way of experiencing may capture important aspects and relations in the world, and thus become generally accepted, it is never a neutral copy of the world.
> (Pang & Ki, 2016, p. 12)

In a classroom, several ways of seeing the same phenomenon co-exist. Learners, already before teaching, experience a phenomenon in different ways and many of the aspects they discern might differ from a disciplinary way of defining a phenomenon. *Optional aspects* (Marton, 2015) are aspects that learners might discern as important but that are not defining or necessary for discerning the phenomenon in a way that corresponds to the discipline. In the quotation above, these optional aspects are referred to as *irrelevant aspects*. However, besides the necessary aspects, it is important to vary optional aspects because in most cases they are critical to discern in order to disregard them. One example is when learners discern *size* of an object as critical for its density when size is actually irrelevant for an object's density (Lybeck, 1981; Magnusson & Maunula, 2013). Another example is when learners subscribe different meanings to different letters (Küchemann, 1981), when they in fact are exchangeable variables. These conceptions build on learners giving meaning to optional aspects of phenomena such as density or variables. Therefore, optional aspects are in a pedagogical sense as important as the necessary aspects.

5.2.5 Patterns of variation

As stated earlier, according to variation theory, learning is considered as the discernment of new aspects, the making of distinctions between aspects or the building of new entities out of new aspects. If we want to systematically help someone to learn something in a specific way, *patterns of variation* have been shown to be powerful tools (Marton, 2015). These patterns are also central analytical tools used in the analysis of the learning opportunities that emerged in this study.

Separation and Fusion

Taking apart (separation) and *bringing together* (fusion) aspects of a phenomenon are the main structures of variation that make learning possible.

> In order to develop a powerful way of seeing something, the learner must decompose the object of learning and bring it together again.
> (Marton, 2015, p. 145)

Consequently, these two structures also shape the learning opportunities in a lesson. Separation is a way to make aspects discernible from a context or to make an aspect distinguishable from other aspects of the phenomenon. Fusion is a way of bringing aspects together in order to reveal relationships between aspects of a phenomenon.

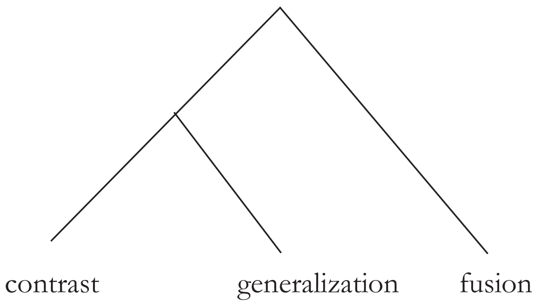

Figure 5.2 Relationships between different patterns of variation and invariance (Marton, 2015, p. 53)

Contrast and Generalization

Separation from background is a key factor for learning, with the aim of discerning aspects that would otherwise remain in the background, unattended to. Separation can be further distinguished into contrast and generalization, with respect to what is varied in the lesson sequence, a focal aspect or a non-focal aspect (Marton, 2015).

Contrast is the pattern of variation when a focal aspect varies. Contrast is used with dual meanings. First, contrast is described as the use of a counterexample (or a non-example) in order to clarify an aspect. To discern what something *is* does include the discernment of what it is not (e.g. Häggström, 2008; Pang & Ki, 2016). A contrast can also be made by the use of two or more conceptions of the same phenomenon opposing each other. This is still regarded as a counterexample. Secondly, contrast is also denoted when a focused aspect varies in several features, which do not have to be constituted as non-examples (e.g. Marton & Häggström, 2017).

Generalization is a pattern of variation that enables the drawing of conclusions over contexts. The focused aspect is kept constant and the background or other aspects vary. Similarities between aspects are aimed at (e.g. Marton & Pang, 2013).

6 The empirical study

The method of each research project is related to both the research questions and the theoretical framework (Silverman, 2010). Gaining an understanding of whether and how learning opportunities can be related to interaction in teaching requires some sort of presence in classrooms. Further, as this study seeks the different meanings co-constituted in these classrooms, and not for instance causalities, interpretative methods are required.

6.1 The setting of the study

Some methodological aspects are defined by the questions and the theoretical assumptions, but other aspects have a more open character. The objective of methodological considerations is to enable a rich empirical data without losing sight of the object of the study. For instance, it would not be possible to capture interaction between a teacher and her students using, for instance, only interviews or questionnaires. Given the research questions, it must be possible to analyse and re-analyse the empirical sources in detail. Therefore field notes alone, for example, would have been insufficient.

The empirical data builds on 19 video-recorded mathematics lessons. These lessons were conducted either in grade 9 (age 16) in compulsory school or in one of the first two years of upper secondary school (age 17–18) in Sweden. In all lessons, the equation of a straight line was introduced. Lesson materials were collected, as were my field notes from the lessons. In total, 13 teachers and 15 classes (307 students) were involved from the beginning. The study is a classroom study in naturalistic settings (Cohen, Manion, & Morrison, 2010).

6.1.1 Use of video recordings

The benefits of video recordings are numerous. Powell, Francisco, and Maher (2003) emphasise the superiority of video recordings in that they go beyond the human capacity to capture aspects of an event. In addition, these events can be watched again and again by researchers. Despite the benefits, one cannot ignore the fact that even video recordings are technology- and theory-laden and do not automatically result in high-quality analyses. Erickson (2011) points out that video recordings as such are not data, but should be seen as a source from which data can be produced. Data production includes choice of video perspective, which sequences are analysed, what is transcribed etc. A difficulty with video recordings is that the material easily becomes so extensive that one loses a sense of it as a whole. To clearly know what you are looking for in a lesson is often a prerequisite for successfully maintaining rigor in analysis, according to Erickson (ibid.).

6.1.2 The designated facets of the lessons

The aim of gaining deeper understanding of the relation between learning opportunities and interaction led to a design with minimal variation in certain aspects and maximal variation in other aspects. The purpose was to capture lessons which were the "same" from a content perspective, while the interaction about this content varied between lessons. The choice of lesson topic – the introduction of the equation of a straight line – fulfilled three conditions. The first was that it was easy for the teachers to understand what content this refers to, the second that the content is delimited, and finally that it occurs in curricula for both lower and upper secondary school. The decisions about the lengths of the introductions, as well as about which lesson(s) to choose, were left to the teachers.

The background factors that possibly affect the nature of interactions taking place in a classroom are complex and probably not easy to capture, nor was this the intention of the study. Nevertheless, there was an ambition to create width in the interaction aspect. Therefore, a number of hypothetical criteria were formulated to guide the selection of lessons to be recorded: teachers' level of education and teaching experience, their age and gender, the pupils' socio-economic status, school type, class size, and the geographic location of the school. A factor that went unnoticed before the implementation, but which turned out to be varied, is the number of times the teacher had taught the specific content before. In the study, two participants introduced the topic for the first time in their careers and one had done it at least 40 times before. This was not well correlated with years of teaching experience. Some participating teachers had taught mathematics for years, but had only just started to teach linear equations, as the topic had recently been highlighted in new curricula.

6.1.3 Linear equations in syllabi

As this study is conducted in both the last year of compulsory school, lower secondary, and the first two years of upper secondary school in Sweden, the syllabi for mathematics concerning linear equations and functions will here be briefly described. In Sweden, compulsory school ends with grade 9 (age 16) and subsequently students attend[28] upper secondary grades for three years (age 17-19). Functions and the equation of a straight line occur in both lower and upper secondary school syllabi. For lower secondary the concepts are described in the *Central contents of mathematics for grades 7– 9*. For upper secondary they are expressed in syllabi for the courses 1b, 1c and 2a (Skolverket, 2011).

Lower secondary

The equation of a straight line is described as a representation of a linear relation, or function, and considered to be part of the central content of mathematics in lower secondary school in Sweden. The concept was not explicitly described the earlier syllabi[29]. However, it has been given a more central role in the latest national syllabus in Sweden, Lgr11:

- Functions and the equation of a straight line. How functions can be used, with as well as without digital tools, to examine change and rate of change and other relationships (Skolverket, 2011a). (Lgr 11, in the central contents of 7-9, Relationships and changes)[30]

In the commentary to the syllabus (Skolverket, 2017) this content is explicated:

- The content regards describing relationships and changes with the use of the function concept. The relationships can be expressed with tables, graphs, coordinate system, or generally as a formula.
- A function that describes a simple proportional relation, such as direct proportionality, is called a linear relationship. *The equation of a straight line*, which is a part of this content, is a representation for such a direct proportional relationship.
- Functions are abstract concepts, but by developing familiarity with different representations and the transition between them, the opportunity to understand the concept increases. Ultimately, the content refers to the ability to express relations numerically, graphically, and algebraically. It can be important in many situations in everyday life and in society.

[28] Even though compulsory school ends at grade 9, today about 98 % of the Swedish students continue to upper secondary school. (www.skolverket.se).

[29] Lpo 94, Lgr 80

[30] Funktioner och räta linjens ekvation. Hur funktioner kan användas för att undersöka förändring, förändringstakt och andra samband. (Lgr 11, i Centralt innehåll för 7 – 9, Samband och förändringar)

Upper secondary

In the latest national syllabus for the Swedish upper secondary schools, Gy 2011, (Skolverket, 2011b) the equation of a straight line and related content, such as linear relations and functions, are described in the syllabi for the various programmes. Classes from different programmes are included in this study, such as vocational programmes (2a), economic and social sciences (1b), natural sciences and technology programmes (1c). Therefore, the description of the equation of a straight line will include different syllabi (2a, 1b, and 1c) with almost identical content[31]:

- The equation of a straight line
- Algebraic and graphical methods for solving linear equations
- The concepts of function, domain and range and properties of linear functions
- Representations of functions, by words, expression of function, tables, and graphs
- Differences between the concepts of equation, algebraic expression and function

Linear equations and functions have been given a more central role in the latest syllabus. The content has been moved to the first mathematics course instead of the second[32] and the syllabus text is more detailed and specific compared to earlier syllabi.

In summary, the equation of a straight line is present in contemporary syllabi for lower and upper secondary schools in Sweden. Additionally, the concept is also emphasised as a function in both syllabi.

6.2 Conducting the study

As I had specific demands on the lesson(s) to be recorded, a systematic search for participants was initiated. The process of finding and recording all 19 lessons took almost 2 years: from January 2012 until December 2013. This chapter discusses the conducting of the empirical part and a few of the difficulties encountered.

[31] I have chosen to report on the comments that are relevant to the content of this study and therefore the text is shortened.

[32] Except for the vocational programmes (2a).

6.2.1 Finding lessons

It turned out to be more difficult to get access to lessons than I expected. Principals of all schools in the municipality where I was working were contacted, and every mathematics teacher that was to teach year 9 in the coming school year received an e-mail. Several school visits were made to inform teachers about the research study. Many teachers responded that they were interested in participating, but in the end it became difficult to find dates, even though my schedule was totally flexible. At one upper secondary school that I visited, most mathematics teachers declared as early as during the first meeting that they would not participate in the study. In some cases, the response was positive to giving access to the lessons, but without video cameras. From this first attempt I had only two teachers who were willing to participate; I had to change strategy.

I now searched everywhere for participants who were to introduce linear equations and after a few months eight participants had accepted. They received regular e-mails during the year, because in some cases the lessons were to be conducted almost a year later. When the recording started, two additional teachers joined since they heard positive comments about the project from their colleagues. When 15 lessons had been recorded, and the original criteria for widening the presumptive interaction factors were met, the search ended. About at this time, more than 18 months after the beginning of the search, three teachers suddenly joined: they had been asked the year before, but had not had appropriate[33] groups until now. All the teachers were informed that it was the introduction of the equation of the straight line that was to be recorded, but they were not informed specifically that the study focused on interaction, in order to minimise the influence of this information. It was pointed out that the project did not have an inspecting role and that participants would be made anonymous in the thesis. This detailed description of the difficulties in finding lessons for the study serves as basis for later reasoning on some of the quality aspects of the study.

6.2.2 Meeting the students

I visited the students a few days before the recording to inform about the research project, to be able to answer questions, and to ask them to complete a consent form[34]. They were told that the participation was voluntary. In the few cases when students did not want to participate, the camera or the students were moved in the

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[33] Appropriate in the sense that linear equations were to be taught.

[34] Due to long distances, two exceptions were made. In these cases, the teacher informed and handled the consent forms ahead of time and I talked to the students before the lesson started.

classroom, so they would not be visible in the recording. They were also informed that voices of non-participating students on the recordings would not be included in the study.

6.2.3 Recordings

The lessons were recorded digitally with two fixed camera perspectives with the purpose of capturing in detail both the public communication in the classroom and how the topic was conducted. One camera was placed at the back and the other at the front of the classroom. In most cases this led to the back cameras recording teachers and white boards, while the front cameras recorded students' faces. However, all the lessons were not organised this way. An external microphone was connected to the back camera, in order to capture all public talk. My placement during the recordings was beside the back camera. I turned the cameras on when the teachers initiated the lessons and turned them off when about half of the students had left the classroom after class[35]. After one lesson (L6), two students were shortly interviewed afterwards because they had expressed views of the content that needed further clarification.

6.2.3 Collecting lesson materials and notes

The teaching materials used in the lessons were collected. It comprised copied pages from teaching materials on any task solved jointly or the digital presentations that teachers used on the smartboard. Also available for analysis were my notes made during the recordings. In these notes, I documented events in which interaction about the content occurred or when learner questions were not attended to.

6.3 The empirical material

Altogether 19 mathematics lessons were videotaped in 15 classrooms. In total, 13 teachers and 307 students participated in the study. Due to ethical considerations that will be argued for later in this text, the lessons will be presented in a way that impedes the identification of individual participants in the study. However, all relevant information for the study regarding lessons, teachers, students, and schools will be described.

[35] In two lessons, the teacher organized the recording in a similar way. This was due to long distance.

6.3.1 Selection of lessons

The teachers interpreted the length of an introduction of the equation of the straight line in different ways. The majority, 10 out of 13, used *one lesson* for the introduction, but two of the upper secondary teachers used *two lessons* and one teacher in grade 9 used *three lessons*. These were all recorded. Later in the analysis, five of the lessons were omitted from the study for two different reasons. Firstly, three consecutive lessons from the same group in grade 9 differed in many ways from all the others. Even though the lessons were rich in interaction and probably offered interesting learning opportunities, the topic dealt with was not linear equations. Instead the lessons were about how to interpret coordinate systems and graphs of real life contexts. Secondly, after more consideration than in the first case, both 'second lessons' of the introductions in upper secondary were omitted because little time was devoted in them to whole-class teaching. Leaving them out of the study still included both teachers and classes, as their first lessons were analysed. In the following, 14 of the original 19 lessons comprised the base for the empirical data production of the study. Now only one lesson per class was included, which also facilitated the comparison.

6.3.2 Teachers

The 14 lessons were conducted by one out of 12 teachers, all qualified mathematics teachers. Two teachers conducted two lessons each in different classes. Seven of them are qualified upper secondary teachers, of whom one has a master's degree in mathematics, four are lower secondary teachers and one of them is a middle school teacher, educated to teach mathematics until grade 5. All of them had at least one other subject in their education; most commonly the subject was either physics or chemistry, but also PE, crafts, languages, geography, and social science were combined with mathematics in their educations. Their experience of teaching varied between 3 and 40 years, with a median of 15 years. The number of times they had introduced linear equations varied greatly. For two of the teachers, this lesson was the first introduction of linear equations conducted and one of them said she had experience of at least 40 introductions. Typically they had introduced the linear equations 7-8 times; this was accurate for half of them. Generally, the upper secondary teachers had introduced linear equation more times than the grade 9 teachers, but there was a variance. The age of the teachers ranged between 29 and 66 years, with a median age of 43.

6.3.3 Students, lessons, classes, and schools

The scheduled lesson lengths varied between 40 and 70 minutes. In real time, from the moment when the teachers indicated a start until an end was indicated, they ranged between 33 and 66 minutes. About 13 hours of video recorded lesson time with dual camera perspective, thus a total of approximately 26 hours of recordings, comprise the empirical material. Altogether 305 students participated in one of the lessons[36]. Eight of them chose not to participate in the study; hence 297 students participated. The number of students participating varied between 13 and 33 per lesson (Table 6.3). All in all, seven classes from grade 9 (from five schools) and seven classes from upper secondary (from four schools) participated. In Table 6.3, different aspects of the empirical material are shown.

Table 6.3 An overview of the participating classes, students and schools

Grades:	**Number of students per class[37]**	**Organisation of classes**	**Programmes (upper secondary)**	**Types of schools**
9th grade: 7 classes	13, 13, 13, 14, 17, 18, 18	4 ordinary classes 3 half classes	N/A	4 public schools 1 independent school[38]
10th grade: 4 classes	22, 24, 29, 31	4 ordinary classes	3 natural science 1 technical Math course: 1c	3 public schools 1 independent school
11th grade: 3 classes	25, 27, 33	1 mixed class 2 ordinary classes	1 Mixed vocational 1 Economics 1 Social science Math courses: 2a and 1b	
In total: 14 classes	**In total: 297 students**			**In total: 9 schools**

Table 6.3 with a description of classes, students, and schools aims at giving an overview of how the original criteria formulated beforehand were met. Facets that

[36] The reason for these numbers differing slightly from the numbers presented earlier is a result of omitting one group of 10 students (with three consecutive lessons).

[37] It has to be remembered that these figures denote participating students, and are close but not identical to class size.

[38] These schools are financed by taxes, but independently run. In 2017, about 18 % of the Swedish students (year 6–19) attend an independent school (Skolverket.se)

varied greatly in the study were: school types, programmes of upper secondary, how the classes were organised, socio-economic status of students, the geographic locations of the school and class sizes. Also teachers' education, teaching experience, their age and gender[39] varied. Consequently, the criteria formulated in the design of the study were fulfilled, even if mostly by pure luck.

6.4 Qualities of producing data

All observations are theory laden and there are no objective observations. To have *perspective awareness* means above all to understand with what theories you study a phenomenon and how those theories determine the limits of what is possible to conclude (Larsson, 2005). In addition, it also implies responsiveness to the possibility that it might be something else, which is not examined, that influences the results. These facets will be discussed in relation to the methods of analysis. But first, the validity of the data and ethical considerations will be discussed in this chapter.

6.4.1 Validity

The concept of validity is a central quality measure for all research (Silverman, 2010). The term originated from quantitative analyses, as a degree of what is studied in relation to the claims (Starrin & Svensson, 1994) yet the term is also used for qualitative analyses, albeit with significant shifts in meaning. Validity in qualitative analyses also regards the question of the depth of the analyses and the range of the data (Cohen et al., 2010). No study has full validity but it is the researcher's role to declare and, where possible, eliminate threats to the validity (Cohen et al., 2010). All investigations affect to different degrees the phenomenon studied and in this way influence the validity. The influence of the investigation on the lessons as a validity feature will be discussed here. Later, the validity of the results will be argued for.

A possible threat to the validity of this study is that participants in class behave differently compared to their usual lessons due to video cameras and my presence. Today's small video cameras, with no memory cards that needed to be replaced during a lesson, probably made it easier for the participants to 'forget' the recordings. Also my position, at the back of the classroom, and the minimal panning, probably contributed to influencing the lesson less. Nevertheless, it cannot be assumed that the participants completely disregarded the cameras and/or

[39] Gender has not yet been discussed; this follows later in the text.

my presence. Even though students today are accustomed to the video medium, it cannot be ruled out that "stupid questions" are less frequent than during lessons without recording. On several occasions, the students showed surprise when I stepped forward after class to collect the equipment; they had obviously forgotten my presence. One teacher said in a post-lesson discussion that her students had been affected greatly by the recordings whereas the others argued that there was no or little impact.

Regarding the teachers' behaviour, there are several factors that might constitute threats to the validity. They most likely have expectations, conscious and unconscious, about what a researcher with video equipment would like to see, which may affect the situation. Two factors most likely reduced this threat to some extent. Firstly, as a teacher I knew the context and was therefore not an outsider. It was not difficult for me to behave as "just another teacher". Secondly, I was not explicit about the object of the study beforehand. This was done first in a post-lesson discussion. The reason for not being explicit about the object of the study was that it might affect the quality of the data. This is of course an ethical dilemma, however, I considered that letting them know the precise objective of the study would have been a crucial threat to validity. In the post-lesson discussions, the teachers' experiences of the recordings were discussed. None of them claimed that they had been completely unaffected by the situation, but neither was the impact considered as very significant. Most of them also said that the impact was more apparent at the beginning of the lesson, as after a while they seemed to forget the cameras. Another aspect that some mentioned was that they had planned this lesson a bit more carefully than their regular lessons. Given the research questions, what this means for the results is not easy to evaluate. Having planned the lesson more carefully as a teacher might of course influence how the content is dealt with, however, it might also influence the willingness to attend to learners' contributions.

6.4.2 Ethical considerations

Ethical considerations are often a question of balancing different dimensions of ethics. Research should not do harm, and at the same time, research contributes to our collective knowledge and has to be truthful and trustworthy. Sometimes these dimensions are in conflict. There are several examples of where research has violated human rights (Silverman, 2010), which has led to the emergence of ethical principles. Some principles are general across all research areas and dictate dimensions such as protecting the informants and being attentive to their constant consent (ibid.). There are also more specific rules and norms. The ethical rules which apply to this study are described in 'The ethical principles for research' by the Swedish Research Council (Vetenskapsrådet, 2002). Three themes are discussed specifically in relation to this study: the information and consent requirements, the anonymity, and the sensitivity for not exposing people to damage.

Information and consent requirements

All participants were informed about the study, its overall intentions and the conditions for participants, such as that it is possible to withdraw consent to participate without explanation at any time. The students had not been consulted about the recording of their lessons as their teachers decided whether she wanted to participate or not. Therefore the students were informed specifically about the terms of their participation and written consent was requested, see Appendix B. As all students were over the age of 15, it was not necessary to obtain consent from their parents. Different levels of consent are recommended, especially when it comes to video recordings (Roschelle, 2000). Thus the students had to indicate whether their participation was to be limited to the research context or whether the recordings also could be used in, for instance, the training of teachers (see Appendix B).

Eight students did not participate in the study, and the cameras were placed so they were not visible in the recordings. In two cases, students were asked to change seats as they were seated centrally in the classroom. Few of these eight students talked during the lessons, but when their voices were heard, they were not included in the transcripts or analyses. Of a total of 305 students, more than 97 % participated. Based on my experience from more than a decade of recording lessons in school development projects, I consider the students involved in this study, generally speaking, not to be very disturbed by the recording of the lesson. Nevertheless, there were individual students who, for various reasons, did not want to participate.

Anonymity

In the case of anonymisation and coding of the material, it is a balancing act between what needs to be identified because it might matter for the results and what cannot be identified because the participants were promised anonymity in the study. Identification markers of students and teachers, such as age, gender and other factors are consistently made anonymous. This study involved seven teachers who identified themselves as men and five that identified themselves as women, yet in the data production the interaction analysed is between students and teachers, not between different teachers. Therefore the distinction she/he is a useful tool, which also can be used in a way to make the identification more difficult.

In research, gender identity is often retained despite anonymisation; however in Swedish the gender-neutral pronoun *hen* is nowadays becoming increasingly common. In Finnish, which is my mother tongue, only one personal pronoun exists, *hän*, and it was from here the idea of using the distinction she/he in the description of the interaction between teacher and students. Vehviläinen (2009) used precisely that distinction, *she* for teachers and *he* for students, in order to make her few participants more anonymous. Each teacher in this study has been given a female alias and is described by using feminine pronouns. All students have been given male aliases and masculine pronouns are used to describe them. Aspects of ethnicity, socio-economic status, and age cannot be discerned for either single students or teachers. The ambition is that it will not be possible for a reader to identify any of the participants.

Sensitivity for not exposing people to damage

My assumption before the study was that mathematics lessons would not be particularly controversial to investigate, yet the difficulties of finding participating teachers showed that this was probably a false conjecture. Something that could possibly give rise to doubts among the participants regards the fear of being categorised as a 'poor' teacher. In this text, perhaps a participating teacher will be able to identify *herself*, even though everything possible has been done to avoid identification of participants. This is difficult to entirely overcome as the study analyses and describes the interaction in the lessons in such detail, and a teacher could have memories of details in these lessons. Even if learning opportunities emerging in interaction might be found to be an important aspect of teaching, it is still at most only a part of all the competencies needed to be a good teacher. In addition, the study captured at most two lessons of each teacher. Therefore, no empirical data in the study can be used to assess general quality of instruction. Even so, there is a risk that a reader will interpret some of the teachers as 'generally poor', which would be a false conjecture. It is, in any case, the participants' experiences that should be taken into account for the assessment of ethics, and sensitivity for this has been present throughout the study. However, there were differences among the teachers in these lessons regarding what the study investigates, which was a fortune, as another side of this ethical dilemma is that research should also contribute to new knowledge.

Another facet of this ethical aspect is the exposure of students' incorrect contributions in the lessons, which are part of the objects for the study. Are students comfortable with their diverse conceptions being explored in the lesson, and in addition recorded for research? There might not be any general answer to this, but there has been a consciousness of this aspect throughout the whole process.

7 From data production to results

In any description of a research analysis there is a danger of describing the process as more straightforward than it actually was. The analysis phase of this study lasted more than 19 months and was anything but straightforward. An overview is given (Figure 7) in which the various steps of the process can be followed. The investigation consisted of several analyses, and many of the descriptions in this chapter are results of analyses that were made in order to *prepare* data (1st stage). These first-stage results were a prerequisite for the main analysis to answer the research questions (2nd stage). The stages are highlighted using different colours in Figure 7: blue for the 1st stage and green for the 2nd stage. The object of this study is neither learning nor interaction per se, but what learning opportunities are enabled by interaction. Therefore, the results of the 1st stage are used as a tool for the 2nd stage, not as results in their own right.

In a qualitative research study, transparency is essential. Therefore, in this chapter the analysis is described in detail. First, I describe how the data was structured (Section 7.1), followed by an elaboration on how the analytic tools were employed. This elaboration is divided into a content part (Section 7.2), and an interaction part (Section 7.3). A need for further distinctions between different kinds of *learner contributions* emerged; hence a section is devoted to that (Section 7.4). During the analysis, it became evident that the 14 lessons could be divided into three different types in accordance with how the learner contributions in them were attended to. Therefore, a restructuring of data was made in this respect (Section 7.5). All dimensions of variation (DoVs) opened in the lessons had to be organised in order to be comparable. Consequently, the DoVs were arranged around different properties of linear functions (Section 7.6). In this way, the analyses from the 1st stage resulted in a Main Table (Appendix A) from which the results of the research questions could be analysed. Finally, I describe the comparisons carried out in the search for relationships between DoVs enacted and learner contributions[40] attended (Section 7.7).

[40] LCv, elaborated on in Section 7.4

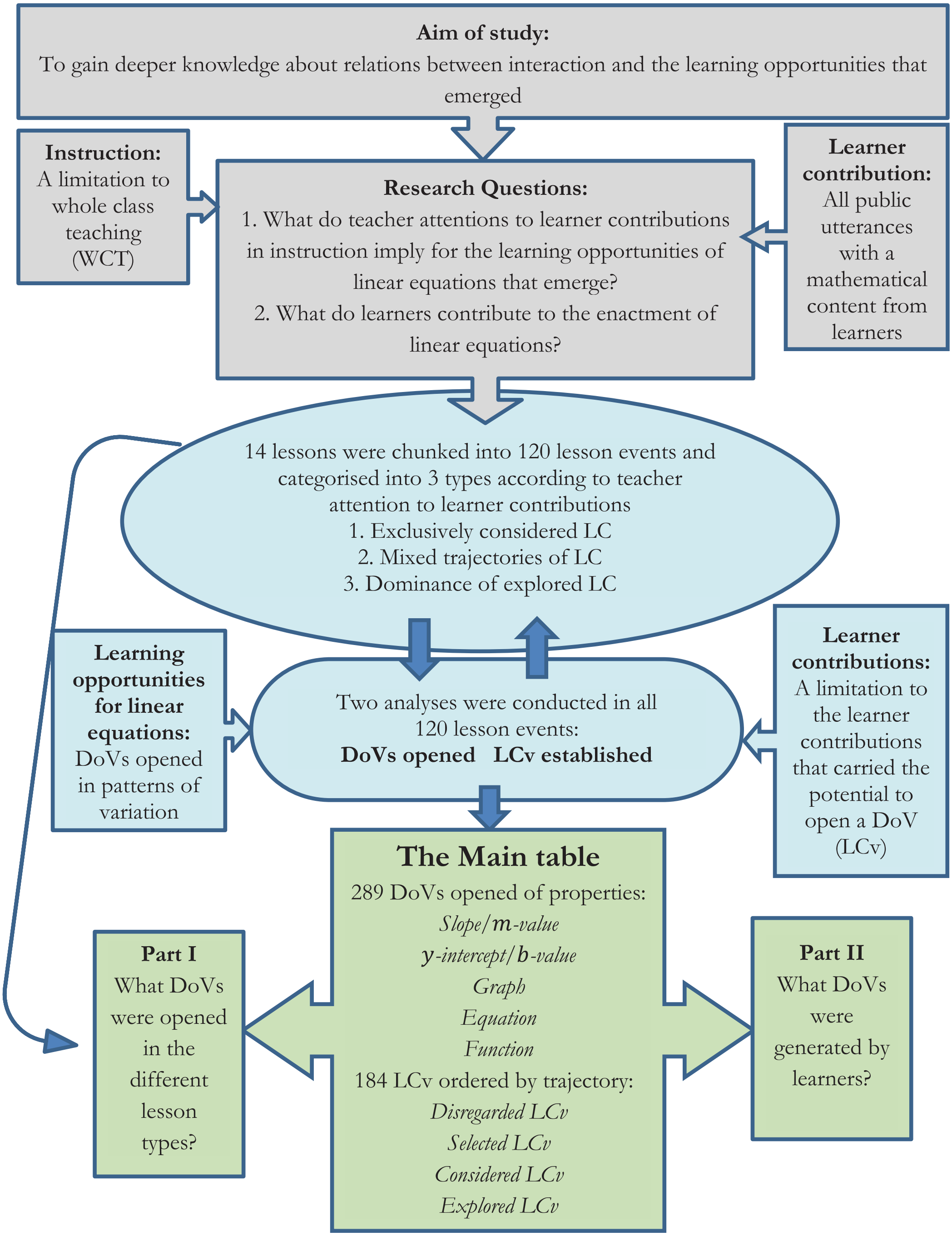

Figure 7: The processes of analysis

7.1 Structuring the recordings

In a transcript, much of the information in the recording is lost, yet the transcript helps to organise and systematise the recordings (Powell, Francisco, & Maher, 2003). The solution was to use transcript and recordings alternately. The first step in structuring the recordings was to transcribe all public talk in whole-class teaching to enable an analysis of what aspects of the content that were enacted and how. Everything written or projected on the whiteboards was included into the transcripts. Gestures, laughter and other non-verbal communication were included when determined to be a part of the public talk, as were silences longer than 5 seconds. The transcripts of the 14 recorded lessons resulted in between 5 and 23 pages[41] per lesson, with a mean of almost 12 pages.

7.1.1 Constructing lesson events

The lesson transcripts needed organising. Inspired by Pillay (2014), the lessons in this study were chunked into lesson events with the intention of producing more manageable data. The start of an event is defined as when a teacher brings a new notion into the lesson. This tool worked well as all teachers in the study showed distinctly when they either closed an event, initiated a new event or both. In contrast to Pillay (ibid.), the events could not be used *as* units of analysis. For a unit of analysis, they were still too incommensurable. However, the lesson events were used in my study as defining parts of the lessons, allowing an analysis of what aspects of the content were enacted, and how.

7.1.2 Translation issues

The translation from verbal language to writing demanded some consideration. Spoken language includes, for instance, pauses, unfinished sentences, overlapping speech and typical words for speech; all normally absent in written language. The ambition to retain the authenticity of the classroom context without making the transcripts incomprehensible required some guiding principles. Pauses, repeated words, interruptions and overlapping speech were noted but spelling was done in a standard way. In the transcripts, the following symbols are included:

41 Text size 12

…	short silence, longer silences than 5 seconds are marked as [10 sec]
//	overlapping speech
T:	teacher
L:	learner
H:	If the learner has been called by name, the first letter (of a false name) is used in the transcript, for instance: *Hampus*? *H: yes*
Ls	several learners speak simultaneously
L1/L2	different learners in same sequence are distinguished
[points at 3x]	actions of participants are marked in brackets
(the graph)	a clarification of what is addressed in the sequence is marked in parentheses

The translation between languages also entailed a few concerns. The language in the recorded lessons, as in the transcripts, was Swedish. One concern was related to how to translate informal Swedish expressions. Often the translation resulted in a more formal language in English, and expressions such as "typ" were translated to "like". Another consideration is that the Swedish school context, from an international perspective, might seem informal. The interaction between teachers and their learners in this study in many respects resembled the interaction between friends. Indications were that forenames were used in all lessons, participants often interrupted each other, and in many lessons, much joking and laughter occurred. This resemblance might just be superficial; however, in the light of a translation from Swedish to English, this is worth mentioning. An advantage of the translation was that the process of transforming the text from one language to another lead to new understandings of the interaction

7.1.3 Analysing lesson events

The division of the lessons into lesson events resulted in 4-20 events, with a mean of 11 events, per lesson. The events that did not include whole-class teaching were excluded from the analysis, leaving a remaining set of 120 events. These events have been analysed with two foci that will be elaborated in greater detail later in this chapter. The role of the events besides organising the material was also to keep the analysis stringent, by which I mean analysing in the same way at the end as in the beginning of the analysis.

In Figures 7.1a and 7.1b, I illustrate how the events served as a tool to maintain the stringency of analysis. In the top box, there is an overview made from the transcript, with some excerpts included, see **(1)** in the figure. Then, an analysis of the enactment of the content in the event took place, see **(2)** and **(3)** in the figure. And finally, the interaction in which the content was enacted was analysed, see **(4)** and **(5)** in the figure.

<table>
<tr><td>N.

INT
WCT
GW
TS

LX</td><td colspan="2">Lesson event number
(1) An overview of what is done and discussed in the event
The kind of activity in the lesson event:
INT: introduction that does not involve any mathematics
WCT: whole-class teaching
GW: students' group work with tasks to be discussed in the lesson
TS: task solving from textbooks/given tasks, not discussed in whole-class setting, but with content related to the topic of the lesson. Often the tasks vary between students.
Lesson id
The duration of the event(s): (00.00-00.00)</td></tr>
<tr><td></td><td>(2) Dimensions of variation opened</td><td>(4) Trajectories for Learner contributions</td></tr>
<tr><td></td><td>(3) Patterns of variation
Features varied in the dimension</td><td>(5) LCv
Learner contributions with the potential to open new dimensions of the content (A), (B), etc.</td></tr>
</table>

Figure 7.1a: A general description of what is analysed in each lesson event

An example of a lesson event from the data is shown in Figure 7.1b. Later in the text (in 7.2.1), a part of the excerpt will be used to show how the patterns of variation (3) and dimensions of variation (2) were determined.

<table>
<tr><td>1.

WCT

L5</td><td colspan="2">(1)Ragnhild begins the lesson by saying that today's topic is functions and graphs and why these are important to know about. She highlights the importance of today's lesson for coming areas like calculus, and functions. She asks whether they remember from previous school years what a coordinate system is. They discuss the designation of axes in a coordinate system drawn on the white board (A). They discuss how to write points and the two coordinates are separated (B) from each other. Ragnhild separates the meanings of the coordinates, of parentheses, of commas. A learner contribution about z-axes is opened, due to a question from another learner (C). Negative coordinates are used as an example.
(00.00-04.27)</td></tr>
<tr><td>1.

L5</td><td>(2) Dimension of variation opened
1. Separation of designation of axes
2. Separation between</td><td>(4) Trajectories for LC
1. Considered LC. Even though the correct answer has been delivered, Ragnhild asks another student for his answer. Then she considers the LC and states that in most</td></tr>
</table>

	coordinates and decimal numbers 3. Separation between parentheses and smileys 4. Separation between a decimal comma[42] and the coordinate comma 5. Separation between x- and y- dimension of coordinates 6. Separation of dimension (2D/3D) 7. Separation of negative coordinates	cases, but not all, one way to designate the axes is chosen. 2.-3. No LC 4. Considered LC 5. No LC 6. Considered LC, an interesting case where two LCs are necessary to open the dimension. A fellow learner determines trajectory for that contribution. 7. Selected LC
1. L5	**(3) Patterns of variation** 1. Generalization of designation of axes • x-axis and y-axis could be changed 2. Contrast of meanings of parentheses • smileys • surrounding coordinates 3. Contrast of meanings of comma • decimal comma • separator of coordinates 4. Contrast of meanings of '3,2' • A little more than 3 • $x = 3$ and $y = 2$ 5. Isolation of dimensions of coordinates • $x = 3, y = 2$	**(5) LCv** **1.** L1: It had x- and y-axis T: x and y-axis…ok, which was which then? L2: x is there and y is in the middle T: [to L1]: Were you going to say the same? L1: No, x is there and y is upwards. **(A)** L2: But that's the same thing. T: Very good! That is x and this is y. And in 999 cases out of 1000 this is what you stick to. **2.- 5.** T: And there is a specific way of noting…of different points in this, what they are called. If we were to describe for instance that point, what it is called, or where it is placed, how would we note that? Several learners answer simultaneously. T: Wait, people have their hands in the air. What do you say, Hampus? Hampus: Three comma two. **(B)** T: Yes and how did you reason? H: Well, first comes x, then the comma and then y. I mean, in the x-position it is three

[42] In Swedish the same sign (,) is used for decimals and coordinates.

	• (3,2) • (x, y) • (horizontal, vertical) • how far to the side, how far up 6. Isolation of dimensions • x and y: 2D • plus z: 3D 7. Generalization to negative coordinates • (-4, -1) • negative coordinates • negative x • negative y	and in the y-position it is two. T: Yes, and if we think like that, then, at least according to Hampus, it would be three comma two. And always when we write coordinates, we write with parentheses like these. They are not always used for smileys, but can be used for other purposes as well. And the comma here is not a decimal comma. It does not stand for a little more than three **[writes 3,2]**, but for the x-coordinate being three. That is if we go straight down to the x-axis we find three there. And this [points at the 2] stands for the height, or the y-dimension being two**,** and that we can see if we go straight out to the y-axis, we can see that it is two there. **6.** L: Will we work with z as well? **(C1)** T: No, we won't. We will not do any 3D, at least for a while. L2: What is z? **(C2)** T: Well, it's if you think that we had an axis out here as well. [T shapes an imaginary axis out from the white board]. L2: Aha T: But we will definitely not deal with those, for a good while, at least.

Figure 7.1b An example of a lesson event from the data

After the first round of analysis of the 120 events, the analysis was re-started with the intention of validating the analysis, i.e. to ensure that there had not been shifts in meaning between the first and the last events, as many months of analysing the data had passed. All events have been revised at least once, and several of the more difficult cases have been analysed repeatedly.

7.2 Analytical tools employed: learning opportunities

To answer the research questions (what learners contribute to the enactment of linear equations and what teacher attentions to learner contributions imply for the emerging learning opportunities for linear equations), I required several analyses with different tools. The analysis of *learning opportunities* was made in terms of which aspects of the content were enacted in the lesson events. Since both dimension of variation and patterns of variation have been theoretically anchored in a previous chapter, the purpose here is to demonstrate *how* the concepts were used in the analysis.

7.2.1 Dimension of variation

The main unit of analysis in this study was the concept of *dimension of variation* (DoV). Hence, the DoVs were traced through the lesson events. When analysing learning opportunities in a study in which a shared intended object of learning is present, this object is used as a frame of reference for the dimensions of variation opened. In such a case, aspects enacted can in a study be determined out of the scope related to the intended object of learning. In this study, however, there was no such shared intention, only teachers' different interpretations of what constitutes the *introduction of the equation of a straight line*. Subsequently *all* dimensions of variation (DoV) in the 120 lesson events [of 14 lessons] were included in the analysis.

For a dimension of variation to be categorised as opened, at least two features of the dimension had to be present simultaneously, while other aspects stayed invariant. The simplest cases to analyse were the ones in which two (or more) meanings of an aspect varied simultaneously. An example from Lesson 5 is extracted in Excerpt 7.2 in which the teacher distinguishes between a decimal comma and a coordinate comma. The sign comma (,) is separated from the point's position (3,2) and different meanings are varied;

Excerpt 7.2

1. T Ragnhild:	And there is a specific way of noting…of different points in this, what they are called. If we were to describe for instance that point, what it is called, or where it is placed, how would we note that?
2. Several learners answer simultaneously.	
3. T:	Wait, people have their hands in the air. What do you say, Hampus?
4. Hampus:	Three comma two.
5. T:	Yes and how did you reason?

6. H:	Well, first comes x, then the comma and then y. I mean, **in the x-position it is three and in the y-position it is two.**
7. T:	Yes, and if we think like that, then, at least according to Hampus, it would be three comma two. **And always when we write coordinates, we write with parentheses like these. They are not always used for smileys**, but can be used for other purposes as well. **And the comma here is not a decimal comma. It does not stand for a little more than three [writes 3,2], but for the *x*-coordinate being three. That is** if we go straight down to the *x*-axis we find three there. **And this [points at the 2] stands for the height, or the *y*-dimension being two,** and that we can see if we go straight out to the *y*-axis, we can see that it is two there. [L5, min 01.30-02.44]

In these few sentences three dimensions of variation (DoVs) were opened.

Firstly, *the meaning of parentheses* was opened by the variation of two implications of the same symbol:

- smileys
- surrounding coordinates

Secondly, *the meaning of commas* was opened by the variation of two denotations of the same symbol:

- the decimal comma
- a separator between coordinates

Thirdly, *the meaning of 3,2* was opened by the variation of two denotations:

- 3,2 as in a little more than 3
- 3,2 as a point with a x-coordinate of 3 and a y-coordinate of 2

All in all, three DoVs were opened in these 74 seconds. The teacher created a variation by offering two meanings for each of the following: parentheses, commas and numbers with a comma between [3,2]. The different meanings that were varied represent different features in the dimensions. All meanings given in Excerpt 7.2 above are correct in various contexts. However, in the context of coordinates, these meanings were used as contrasts. Consequently, the dimension of variation must also be related to the context in which it is opened.

7.2.2 Keeping track of phenomena, dimensions and features

There were far more complex dimensions opened than commas in the lessons. Regarding *slope*, the analysis of the dimensions of variation opened in the study became more complicated. Four *different* kinds of dimensions of variation regarding slope were opened in the study.

I It could be *a separation of slope*, represented as a steeper and steeper graph. Like when a teacher[43] changed only the slope of a graph that passed through the origin in this particular pattern, see Figure 7.2a.

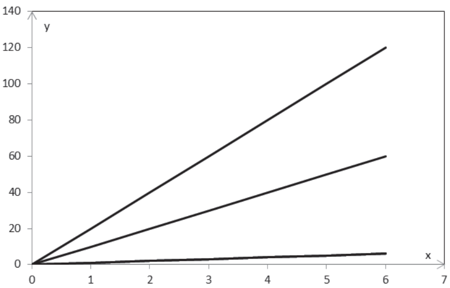

Figure 7.2a: Separation of slope

II It could also be *a separation of the meanings*[44] of slopes:

- as hills in the graphical representation
- as increases of y per x
- as rates of change between x and y

III It could also be a fusion of slopes in graphs to the m-values of equations[45], see Figure 7.2b.

- slopes of graphs vary
- m-values vary (2, 3, 4, 5)

43 In Lesson 9
44 In different lessons, see Results
45 In Lesson 6:4

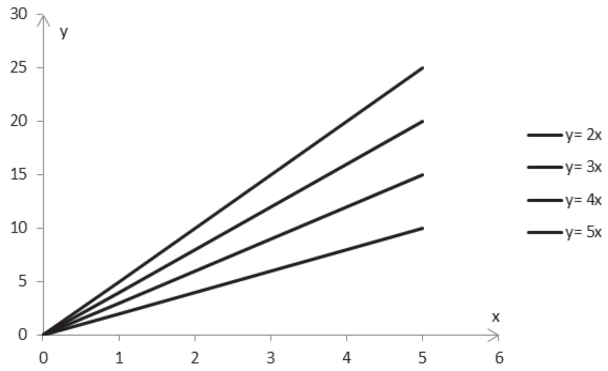

Figure 7.2b: Fusion of slope of graph and m-value in equation

IV Also, *aspects of slopes* could vary, like the meaning of a steep slope[46], in relation to different reference axes:

- 'in relation to x-axis'
- 'in relation to y-axis'

As shown above, phenomena, dimensions, and features are dynamic. A feature opened in a dimension could in another lesson event instead be a dimension in which new features were varied, as in the above examples. In example **I** slope was the *dimension* (of linear equations). In example **IV** instead slope was the *phenomenon* of which new dimensions (reference axis for steepness) were opened. In conclusion, in the analysis of DoVs opened, a fundamental point was to analyse *what features* varied in *what dimension* related to *what phenomenon*. Sometimes the context also mattered, as in the examples with commas described above. To keep track of features, dimensions and phenomena, I used patterns of variation as tools.

7.2.3 Patterns of variation

This part of the text has two intertwined purposes. The first is to describe in detail how the tool of patterns of variation was used in the analysis. The second is to describe a development of the patterns of variation that emerged in the analysis. Conducting a detailed analysis of hundreds of DoVs opened led, unsurprisingly, to a development of the patterns. The two main patterns of variation – separation and fusion – elaborated on earlier in Chapter 5 – were used as a starting point in the

[46] In Lesson 3: T

analysis of opened DoVs. Separation was also distinguished into contrast and generalization, depending on whether the dimension of variation opened was focal or non-focal[47]. So far, these tools are similar to how Marton (2015) describes them. The previous elaboration on two different meanings of *contrast* (see chapter 5) was in this study explicitly used as contrast 1 (the use of non-example, called *contrast* in the analysis) and contrast 2 (the variation of focal and valid features of an aspect, called *isolation* in the analysis). Moreover, a distinct kind of isolation, namely illumination, was distinguished.

In Figure 7.2c, the structure of the patterns of variation used in the analysis is shown. This is followed by a description of how the different patterns were used in the analysis when deciding whether a DoV was opened or not. The description follows the figure from the top downwards.

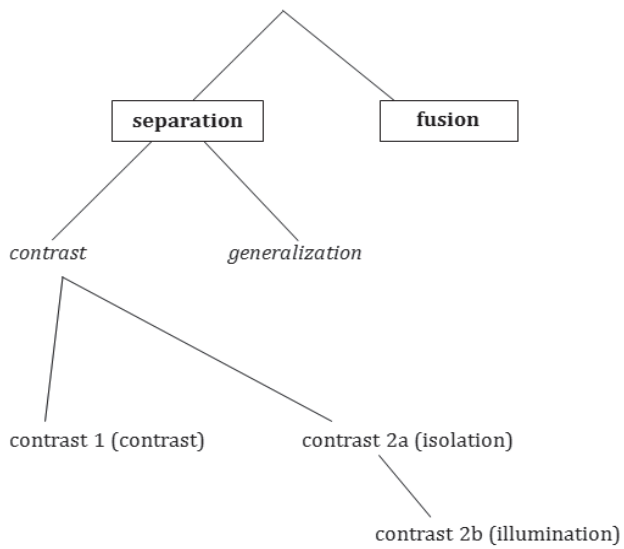

Figure 7.2c: The structure of patterns of variation used in analysis

Steps in analysis of DoVs

To conclude whether a DoV was opened or not, and furthermore *which* DoV, required three steps. These steps are described below, and follow the structure in Figure 7.2c. **The first step** in the analysis of DoVs was to determine whether the variation enacted was a separation of an aspect from the phenomenon or a fusion of aspects.

[47] As discussed previously in chapter 5.

Separation is a way of decomposing aspects from a whole. In separation, one aspect varies by several features, while other aspects remain invariant. An example of separation is when three linear graphs were elaborated on[48], and the only varying aspect was their slopes. The aspect *slope* was made discernible by the variation of several features (values) of it.

Fusion is a way of bringing aspects together in order to reveal relationships between aspects of a phenomenon. An example of fusion is when both slopes and y-intercepts of several graphs varied simultaneously and the task was to find the correct equation for each graph[49]. A prerequisite for this pattern of variation to be successful is that the two aspects varied simultaneously have been discerned beforehand.

If the pattern was found to be separation, **the second step** regarded the question of whether the dimension of variation was focal or non-focal in relation to the discussion in the event. For instance, if *slope* (of a graph) was focal in a lesson event and different slopes were the (only) features varied, then the pattern of variation was found to be contrast. If instead non-focal features, like slopes expressed algebraically and graphically, were varied, the pattern was found to be generalization. Separation was thus distinguished into contrast and generalization depending on whether focal or non-focal features were varied.

Contrast is a pattern of variation that makes the discernment of new aspects possible. The focal aspect is varied and differences in the aspect are created. For instance, if the *slope of a graph* was focused on in a lesson event, and slopes were varied[50], then the DoV was opened in contrast.

Generalization is a pattern of variation that enables the drawing of conclusions across contexts. The focused aspect is kept constant and the background or other aspects vary. Generalization was used in the study relative to questions like *might we instead use*…? or *could you also write a and b (instead of x and y)?*[51] Generalization could also aim at expanding an object of learning to new contexts. An example is when a student asked: *could the line continue down on the negative side as well?*[52] or when in another lesson, the teacher answered *yes, the graph continues very far in both directions*[53] to the same sort of question.

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48 In Lesson 9
49 In Lesson 6
50 In Lesson 12
51 In Lesson 9
52 In Lesson 1
53 In Lesson 5

If the pattern was found to be contrast, **the third step** was to analyse the types of features that were varied in this dimension. Contrast as a pattern of variation was distinguished in the analysis into three different patterns (contrast 1, 2a, 2b, see below) depending on types of features varied.

Contrast 1 (contrast) is a pattern in which a counterexample is used in order to clarify an aspect. To discern what something *is* includes the discernment of what it is *not*. To use a non-linear graph in discussing linearity is an example of contrast. A contrast can also be made by the use of two or more conceptions of the same phenomenon that oppose each other. All three DoVs opened in Excerpt 7.2 above – the parentheses, the commas and the meaning of '3,2' – were opened in this pattern of variation. In all three cases, a contrast was made by counterexamples in the contexts.

Contrast 2 (isolation) is a way of decomposing aspects from a whole. If several valid features of the same aspect are varied, the pattern of variation[54] is categorised as contrast 2. An example of this pattern is when three parallel graphs with different y-intercepts were discussed[55]. The only varying features were the y-intercepts of different values. Thus the y*-intercept* was made discernible by the variation of several features of it. There is also a special case of contrast 2 in this study, namely contrast 2b (also called illumination).

Contrast 2b (illumination) is a pattern of variation when an *invisible*[56] aspect is highlighted and brought to the fore. A focal aspect is varied (thus contrast) but the meaning is illuminated through different representations. In mathematics as a subject there are a lot of aspects "hidden" to learners. This does not only imply the usual discernment of new aspects; these aspects are literally hidden and often taken for granted. Examples are the invisible multiplication sign in between $2x$ $(2 \cdot x)$, the invisible sign for positive numbers ($+3$ as a contrast to 3) and the invisible signs for a number of the power of 1 (x^1 as a contrast to x). All these invisible signs are filled with meanings often taken for granted by those who have already discerned them and they might be obstacles to learning for those who have not.

An example of illumination[57] from earlier research is when the first degree of a variable is illuminated in r and r^1 (Häggström, 2008). Both representations (r and r^1) are valid representations of a variable of the first degree; therefore, this is not a contrast 1. The difference between isolation and illumination consists of the kind

[54] This pattern is referred to as separation in several earlier variation theory studies, see chapter 5.
[55] In Lesson 12
[56] All the aspects not yet discerned are of course invisible. However, invisible in this context of mathematics refers to the many literally invisible aspects that exist and that are often taken for granted.
[57] Although it is not there defined as illumination

of feature varied. In isolation, different features of an aspect are varied, whereas in illumination different representations of the same feature are varied. r and r^1 have the same denotation, but the former representation does not reveal the first degree, while the latter one does. Similarly, in the example of $2x$ and $2 \cdot x$ the meaning stays invariant while the different representations vary. Even though generalization is often categorised as the pattern of variation where representations vary, this is not generalization. In illumination, a focused aspect varies, not backgrounds[58].

The use of *patterns of variation* as an analytical tool revealed what DoV was opened in every lesson sequence and helped keep track of phenomena, dimensions and features. When the DoV were identified as opened, in separation or fusion, they were organised together according to one out of five properties[59] and arranged into the Main table, see Appendix A.

7.3 Analytical tools employed: teacher attentions

The part of research question 1, on teacher attentions to learner contributions and the whole of research question 2, on what learners contribute, required several analyses. These analyses were made in three steps. Firstly, the concept of *learner contributions* was defined. Secondly, learner contributions were traced in the lessons to categorise different developments of them. Thirdly, the learner contributions that carried the potential to open a DoV were distinguished from the other identified learner contribution.

7.3.1 Learner contributions

Since this analysis examines verbal public interaction in whole-class instruction, one of the concepts defined is *learner contributions (LC).* Learners' content-related utterances in a lesson were regarded as LC. The content refers to the mathematical topic of the lesson, thus utterances like *what time is it* or *I haven't done my homework* are excluded. Gestures were considered as LC if they are accompanied by verbal communication. Only public LC made in whole-class settings were considered in this study. This excluded the learners' private conversations, for instance during group work. Exempt from this was when *teachers* make private LC public in a

[58] It is possible to think of situations where this variation could be used as a generalization, when something else is focal in a lesson event, and the both representations ($2x$ and $2 \cdot x$) have already been discerned. However, no events in the study include examples of this kind.

[59] See later in the text. The five properties (slope, y-intercept, graph, equation and function) emerged when the DoVs were compared.

whole-class setting. The rationale behind this is that since the research interest is *what* the learners contribute to public instruction, learners' contributions explicitly and publicly verbalised by teachers count as LC as well. The question in the study is neither whether learners do talk publicly or not, nor how often, so if a teacher declares: *Petter asked me if z could be used instead of x*, this was included in LC, but noted as done by the teacher. Learner contributions came in many different forms: questions, answers to questions, objections, and comments. Finally, LC can be of any length, from utterances as short as one syllable to extensive articulations.

7.3.2 The trajectories of learner contributions

The learner contributions were developed by the teacher[60] in different ways throughout the lessons. These developments were categorised as *trajectories* for learner contributions. A trajectory is defined as the way the content of a learner contribution develops, from the first utterance all through its development in the lesson event. The category system will be presented by means of a description of every type of trajectory using two examples from the lessons. Four distinctively different trajectories for the content of learner contributions were established:

Table 7.3 Comparing trajectories that were established

Trajectory	Content of learner contribution:
Disregarded LC	is not taken into consideration
Selected LC	is rephrased
Considered LC	is changed/applied/contradicted
Explored LC	is made into the topic of discussion

Disregarded learner contributions

When a learner contribution was by the teacher not attended to at all, from a content perspective, the trajectory was categorised as a *disregarded learner contribution*.

Excerpt 7.3a

1. Teacher Åse:	What is 2 – (-1)?
2. L:	One
3. T:	Three

[Åse writes 3 in the expression on the white board]

[Lesson 7: C]

[60] There are very few examples of when a fellow learner attends to a learner contribution. These have been clearly marked in the analysis.

Since the learner's utterance (line 2) has a mathematical content, it was categorised as an LC. This trajectory was categorised as a *disregarded* LC because the learner contribution was not taken into account at all.

Excerpt 7.3b
1. Teacher Ida: What is special about linear graphs?
2. L: They are the impossible ones.
[No attention is paid to this] [Lesson 13: F]

Even if a sentence like *they are the impossible ones*[61] (line 2) is difficult to associate with a mathematical content, it was categorised as an LC, since it probably had a meaning to the learner, even though this was invisible to me in this short dialogue. This LC was not regarded at all; hence its trajectory is a disregarded LC. The categorisation of the trajectories for each learner contribution was based on the development of the *content*[62] of them. Regardless of reasons, in the above excerpts, the utterances from the learners were not taken into account at all. However, later cases will be shown in which learner contributions were responded to, but still categorised as disregarded LC due to the absence of any development of their content.

Selected learner contributions

When a learner contribution was not developed further, yet still taken into account, the trajectory was categorised as a *selected learner contribution*. Selected learner contributions were mainly correct and *known answers*[63] to questions from the teacher. This trajectory is similar to the well-established I-R-E-pattern (Mehan, 1979), discussed in Chapter 1. I-R-E stands for Initiation (by teacher), Response (by learner), and Evaluation (by teacher).

Excerpt 7.3c
In Lesson 5, teacher Ragnhild discusses the graphical representation of a cell-phone subscription with the equation $y = 69x + 29$.
1. T: If we talked for one minute, what would it cost? A one-minute-long call.
2. L1: 29 plus 69
3. T: 29 plus 69 and what would that be? 29 plus 69?

[61] This is one of the very few learner contributions in the study where I have not succeeded in making a hypothesis about a possible meaning.
[62] Earlier defined as related to the mathematical topic of the lesson
[63] As in QWKA, Questions with known answers, Mehan (1979)

4. L2:	98
5. T:	98, that means that after one minute the call costs almost one krona. [Lesson 5: 3]

The teacher initiated a question, which has only one correct and known answer (line 1), the correct answer was given by the response of learner 1 (line 2). The teacher gave an affirmative evaluation, which also initiated a new question (line 3); to which learner 2 gave another correct and known answer (line 4). This sequence ended with the teacher's final affirmative evaluation. According to the framework of Mehan (1979), this would have been described as a double I-R-E-pattern with two learners. In this analysis, it was categorised as a trajectory of two *selected learner contributions* (lines 2 to 3 and 4 to 5).

Selected learner contributions were not always correct, nor were the contents of teachers' questions, but they were treated as though they were. An illustration is shown in Excerpt 7.3d:

Excerpt 7.3d
In Lesson 10, the graph for $y = 2x - 3$ is discussed by teacher Helena and her students.

1. T:	How is the line changed when I move one step to the right on the *x*-axis? How has the *m*-value been affected?
2. L1:	Two *x*
3. T:	It increases by two steps, you see that. One step to the right, two steps up. One step to the right, two steps up. The *m*-value has to be 2, in this case. And where does the green line intercept the *y*-axis?
4. L1:	-3
5. T:	-3 [Lesson 10: 2]

In Helena's questions (line 1), it is not possible to discern *what* was changed when x was changed as the line and the m-value were used as synonyms, while the y-value was absent. The learner answered the unclear question incorrectly, by *two x,* (line 2), but the teacher solved the unclearness by selecting *two steps* as an answer (line 3). This was followed by another selected LC from the same learner (lines 3 to 5).

As in the four examples of selected LCs above, the learner contributions categorised as selected were mostly narrow answers to known questions from teachers. The teachers left it to the learners to express something they could just as well have said themselves, yet it seemed to be a way of including the learner's voice in the lesson. Nonetheless, the categorisation was made in respect to the *trajectory* for the content of the LC throughout the lesson. Selected learner contributions were often repeated (lines 3 and 5 in Excerpt 7.3c, line 5 in Excerpt 7.3d); they were never challenged or changed, and the content of them was not developed in any way. Nothing new was brought into the enactment of the lesson topic in any of the excerpts above.

Considered learner contributions

When learner contributions were used by teachers as an emphasis and the contributions were changed from a content perspective, the trajectory was established as *considered learner contributions*. The contributions were either contradicted or highlighted, for instance with a synonym, but the significant characteristic for considered LC is that some new content was elaborated on in relation to the learner contribution.

Excerpt 7.3e
In Lesson 12, teacher Cecilia projects three parallel lines onto the white board ($y = x - 1, y = x + 1, y = x - 2$)

1. T:	Can you see any similarities between the lines? [8 sec of silence]
2. L:	They are parallel.
3. T:	What does it mean that they are parallel?
4. L:	They all have the same distance between them.
5. T:	All of them have the same slope.

[Lesson 12: C]

The question by Cecilia (line 1) was not a question with only one correct answer; instead she was open to different answers. This might also explain the relatively long silence before a learner contribution appeared (line 2). Instead of just accepting the answer, Cecilia asked for a justification (line 3) and when she got an explanation, *they all have the same distance between them* (line 4), she exposed another way of seeing the same thing: *the lines have the same slopes* (line 5). The original learner contribution of parallelism developed into two more meanings, having the same distance between them and lines having same slope. Consequently, the trajectory was categorised as a considered LC.

It was not always the teacher who provided the development of the learner contribution; the learner himself also developed the contribution further, as a result of teacher questioning.

Excerpt 7.3f
In Lesson 14, teacher Elisabeth discusses what equation a graph showing proportionality has.

1. T:	How would you write it as a formula then?

2. L:	n two
3. T:	You write n two? [instead of two n]
4. L:	Yes, n times two. [Lesson 14: A, B]

Elisabeth's questioning of the order of the factors n and 2 (line 3) leads to a clarification by the learner (line 4). As well the order of the factors (2n/n2), the invisible multiplication sign, between the variable (n) and the constant (2), was revealed by the response from the teacher to the learner contribution. The content of the first learner contribution of *n two* (line 2) was developed to *n times two* (line 4); therefore, the trajectory was categorised a considered LC.

Both Excerpt 7.3e and Excerpt 7.3f show how the teachers took the content of the contributions into consideration by making a clarification or asking for a clarification from the learner. The considered learner contributions were typically developed in quick interactions as in the excerpts above; the content enacted was taken into account and elaborated on further, which means that it was changed in some respect.

Explored learner contributions

With explored learner contributions, the path of the lesson seemed to change. When trajectories were categorised as *explored learner contributions*, the contributions themselves became the focus of discussion in the lesson event. The explored LCs were scrutinised and discussed further. The three previous trajectories of LC were mainly enacted in quick interactions. In contrast, the explored learner contributions took longer time since the contents of the contributions were made into the topic of discussion.

Excerpt 7.3g

In Lesson 14, teacher Elisabeth has been discussing proportionality as a way to determine the slope of a graph. Two right-angled triangles of different sizes have been drawn in the coordinate system where two segments of the graph form the hypotenuses. Both triangles share one angle at the origin between the graph and the x-axis. The proportion of 2:1 and 4:2 ($\Delta y/\Delta x$ of heights/bases of the triangles) is the topic of discussion:

1. T:	It´s the same thing, you just divide both by two. So the relationship in these is two to one.
2. L:	On the whole line? [a whisper]
3. T:	If you consider… if we look at the pattern here… We´ll take it again, the first increased with one step here [points at the first triangle drawn, 2:1]. One step in that direction [follows the base of the triangle to the right with the pen] and how much did it increase by? Two. Then it is two to one. Here it increased between five and seven… that is

two steps, isn´t it? [She draws another identical triangle further up on the line] and here it is just one step [points to the base of the new triangle]. Two to one.
[Lesson 14: H]

The answer to the learner question (line 2) might seem obvious: the slope of a straight line is the same everywhere on the line. Nevertheless, instead of a simple *yes,* Elisabeth drew another identical triangle at a different place on the line and showed that the slope is the same on the whole of a straight line (line 3). There was not much interaction; the learner did not say anything else in the event and Elisabeth both asked and answered the questions (line 3). However, the reason for categorising this trajectory as an explored LC is because the *content* of the learner contribution, the question of whether slope is same on the whole straight line, was explored and elaborated. The LC was explored by using the ratio of the sides of similar triangles at different places on a straight line. Hence, even though the learner contributed only by whispering four words – *on the whole line* – and did not participate orally further in the event, the contribution was categorised as an explored LC. Consequently, the analysis has focused the *development* of the content of the learner contributions, not the contributions themselves.

In the next example, teacher Angelika in Lesson 4 explored several learner contributions and they were made into the topic of discussion. Actually, this was only the beginning of an exploration into what mistakes were made in two learner contributions, but already at the very start the exploration of LC began.

Excerpt 7.3h
The task is: write a function of the form $y = mx + b$ when m is equal to 0 and b is equal to 2 ($y = 2$)

1.	L1:	y equals x plus two
2.	T:	y equals x plus two you say?
3.	L1:	Yes
4.	T:	We write this, are there any other suggestions? [T writes $y = x + 2$]
5.	L2:	y equals two x
6.	T:	y is equal to two x is a suggestion. [T writes $y = 2x$]
7.	T:	Do I have other suggestion?
8.	L3:	y equals two
9.	T:	y is equal to two; could I have one more suggestion? [T writes $y = 2$]
10.	L4:	Hey, what are you doing? (L4 interrupts as he seems not to understand what the task is about)
11.	T:	We're working on this task, write a function of the

	form y equals to mx plus b when k is equal to 0 and m is equal to 2 and I've got all these three proposals.
12. L4:	Oh well!
13. T:	Which is the correct one? Or are they all correct? Can I write this way?
14. L5:	No, the last one.
15. T:	Why should the last one be correct?
16. L5:	There is no x ...
17. T:	What did you say?
18. L5:	There is no x.
19. T:	There is no x, well, really it's like this. m is zero and as it is supposed to stand in front of x, then it's like this y is equal to zero x plus two. [T writes $y = 0x + 2$]
20. T:	And zero x is the same thing as zero times x, right? And zero times x, what will that be?
21. Ls:	Zero
22. T:	Zero yes. I could just as well ignore it and write $y = 2$
23. T:	What kinds of mistakes were made here?

They investigate the mistakes that were made in the suggestions 1 ($y = x + 2$) and 2 ($y = 2x$). [Lesson 4: K]

If only lines 1 to 8 had been considered, this would have appeared as a classical I-R-E-pattern, in which the teacher was waiting for a correct answer to come up, which happened in line 8. By this analytic construct, it would have been regarded as *Selected LC*, since the learner contributions were just rephrased. However, the fact that Angelika wrote the incorrect suggestions on the whiteboard is a clue that something else was going on. In line 9, after the correct answer was given, the crucial question appeared: *y is equal to two; could I have one more suggestion?* Angelika was not just waiting for the correct answer, but she was aiming for <u>many</u> answers. She was interrupted in this by clarifying to a learner what they were doing (lines 10 to 12) but then she continued, even though the request for further suggestions was lost. The most revealing comment that this is an exploration of learner contributions appeared in line 23: *What kinds of mistakes were made here?* The learner contributions were made into the topic of discussion in the lesson event.

In the explored trajectories, the learner contributions could be correct, partial, or incorrect. The teacher might have asked the learner to justify his answer or asked other learners to contribute, but the significant characteristic of explored learner contributions is that they not only contributed to the topic of discussion, but that they at some point *became* the topic of discussion.

7.3.3 Four trajectories established for learner contributions

The categorisation was inspired by Davis's (1997) different manners of *listening* to student contributions (elaborated on in Chapter 2). There are several similarities between this study and the study by Davis (ibid.). Firstly, both acknowledge teacher authority of interaction. Secondly, both investigate differences in teachers' manners in relation to learner contribution. There are also resemblances in the category systems. There are, however, also a few differences between the studies. Whereas Davis (ibid.) focused teacher actions and one teacher's development in a longer perspective, this study investigated the developments for learner contributions in several lessons, but only by one lesson per teacher. Furthermore, this study examined learning opportunities, hence also disregarded learner contributions were of interest in comparisons between lessons. Additionally, the main conclusion drawn concerns differences in learning opportunities in relation to teacher attentions.

Even though a few trajectories for learner contributions were developed (in a few lessons) by fellow learners, the vast majority of the trajectories were established by teachers in the study. When all learner contributions in the 14 lessons had been examined with the focus on what happened to the content of the contribution in the continuance of the lesson, four distinctively different trajectories for learner contributions were identified and categorised.

Disregarded LC is a trajectory in which the content of the contribution is not taken into consideration at all in the lesson.

Selected LC is a trajectory in which the content of the contribution is accepted. The content can be repeated or used as building brick in the remainder of the lesson, but the content is not developed further or changed in any way. Selected LC resembles 'evaluative listening' by Davis (1997).

Considered LC is a trajectory in which the content of the contribution is taken into account. This is done by contradicting or emphasising the content of the contribution, which leads to a development of the topic of discussion.

Finally, in **explored LC**, the content of the contribution develops into the lesson topic. This is done for instance by going back to an earlier question and investigating the content of the contribution in that way or by probing the contribution by several questions. This last trajectory is the only one that seems to change the lesson from its expected path. This, and features such as the inquiry approach to the contribution shows similarities between explored LC and 'hermeneutic listening' by Davis (ibid.).

7.4 Learner contributions with the potential to open new dimensions

If one wants to know the implication of learner contributions for learning opportunities, some learner contributions are more interesting to analyse than others. The number of learner contributions (LC) in the 14 lessons amounted to several hundred. Therefore, a distinction was made between *learner contributions with the potential to open a new dimension of variation of the content taught* (LCv) and those without this potential. These LCv amounted to in total 184 in all 14 lessons. As correct and expected answers to teacher questions seldom carried the potential to open a new DoV of the content, many of the selected learner contributions were omitted by this distinction.

The distinction between LC and LCv

In order to clarify the distinction between learner contributions with and without the potential to open a new dimension of variation, two excerpts with examples of LCv are first described. This is followed by an excerpt from a dialogue in which two learners contribute. One of the contributions is categorised as an LCv and the other is not. Finally, an example is given in which a learner generates a new dimension of variation as he does not seem to experience enough variation in the examples provided by the teacher.

The six examples of LCv and the one example of an LC are then compared in order to distinguish between LC and LCv.

Excerpt 7.4a

In Lesson 15, the way to determine slope is discussed. Teacher Hoda asks:

1. T: How does one measure slope? How can we compare different slopes?

Learners discuss.

2. T: How does one compare slopes?
3. L: One uses angles. One has a right angle to the ground and then one compares it with the slope. **(LCv 1)**
4. T: So with the help of an angle one could measure the slope. That is really good and that's what we intuitively do ... but it is not that simple now when we work with a straight line [Lesson 15: A]

In Excerpt 7.4a, a learner contribution (line 3) carried the potential to open a DoV: to measure slopes using angles. Angles are not used in the canonical way of measuring slopes, which also Hoda claims in line 4, but are often used when measuring for instance steepness in other contexts. It is reasonable to assume that the students here have encountered angles to a greater extent than slopes. As the learner (in line 3) contributes with angles, they can be used as a contrast to slopes. This was also what teacher Hoda did in the following of the lesson. Therefore, the learner contribution carried a potential to open a DoV.

Later in the same lesson, *stairs* were discussed in the context of slopes. A coordinate system with stairs had been drawn on the whiteboard.

Excerpt 7.4b

1.	T:	Let's hear, how do you change the incline[64] of a stair?
2.	L1:	You make the steps like shorter. **(LCv 2)**
3.	T:	So you always have the same here [points to height] and so you change here [points to width of steps]. Was this what you meant?
4.	L1:	Yeah, I think so.

Hoda draws the learner's step model on the whiteboard, with shorter widths but same height as before.

5.	T:	Like that. And if I want less inclination, you tell me to...?
6.	L1:	Longer step **(LCv 3)**

Hoda draws new stairs with longer widths and with the same height.

7.	T:	It is supposed to show the same height in the *y*-direction all the way. Would this work to get different inclines?
8.	Ls:	Yes
9.	T:	Does anybody have any other suggestions?
10.	L2:	You can change the height, too. **(LCv 4)**
11.	T:	I understand, you could change both, instead of keeping this constant you can keep the other (constant).

[Lesson 15: B, C]

In Excerpt 7.4b, a learner contribution (line 2) carried the potential to open a DoV: making the steps shorter is a way of getting a greater incline in the stairs. Teacher Hoda adopted the suggestion and drew steps on the whiteboard with shorter widths, which increases the inclination of the stairs. Later (line 6 and line 10), two additional learner contributions were categorised as LCv because they had the potential to open new DoVs: you could make longer steps and you could also change the height of the steps. It could be discussed whether Hoda "planted" these LCv as at least LCv 2 and LCv 3 seemed to be the kind of answers that Hoda had in mind. However, all three are still contributions from learners with the potential to open a new dimension that was not obvious from Hoda's questions. Later in this event, Δy and Δx were discussed in relation to the widths and heights of the steps.

[64] The Swedish word used – lutning – includes both incline and decline, but in this context inclines are discussed. This word is also the word for slope in Swedish.

In another lesson, the 'start value of a function' was discussed, and two learner contributions appeared:

Excerpt 7.4c

1.	Teacher:	Start value 1, where should I mark then? Can someone come up, or can you explain, Omar?
2.	Omar:	On the *y*-line, but on 1. **(LC 1)**
3.	T:	Exactly, when we started, zero years, for instance, had passed and we had the value of one.
4.	Cornelis:	How do you know it is there? I mean, why don't you start at the *x*-axis? **(LCv 5)**

[Lesson 1: A]

Omar's answer (line 2) was categorised as a learner contribution as it included a mathematical content. Yet it was not characterised as an LCv as it lacked the potential to open a new dimension of the content. It was evidently exactly what the teacher had in mind (line 3) and even though she considered the contribution, it did not lead to the opening of a new dimension of the content. This dimension, the start value at the y-intercept, was already opened by the teacher herself. In contrast to Omar's contribution, Cornelis's contribution (line 4) carried the potential to open a new dimension of the content, namely *why* the start value is at the y-intercept and not for instance at the x-intercept. This contribution was categorised as an LCv as it carried the potential to open a DoV by offering an alternative to the y-axis as a 'start axis'.

Questions like Cornelis's contribution in Excerpt 7.4c were not unusual in the study. Expanding a discussion and searching for limits for or contradictions in the topic of discussion were quite often done by learners in some of the lessons. All these contributions obviously carried the potential to open new dimensions of variation of the content taught as they for instance offered a new value in a specific dimension, hence created variation. In Excerpt 7.4d, an LCv is given in which it seems as though a learner (Hampus) does not find sufficient variation in the tasks given by the teacher. His contribution created a contrast to using only positive b-values in relation to the graph.

Excerpt 7.4d

In Lesson 3, the teacher and the class have just been discussing several tasks with varying *m*- and *b*-values of equations/graphs in different contexts when Hampus interrupts with a question:

1.	Hampus:	Could I just ask about this thing with the positive *b*-value. So far all the examples have been positive, but what happens if you start at the negative part of the *y*-axis? **(LCv 6)**

2.	Teacher:	Good, did you hear what Hampus asked?
3.	Ls:	No/just a bit
4.	T:	Look, a mistake I made with the presentation was that I did not include a single example in which the *b*-value… our constant… is negative. I mean, we have no negative start value in any of the tasks. Then Hampus asks if the slope can still be positive (when having negative *b*-value). Yes, the only thing is that we start, we intercept the *y*-axis at a negative value but the gradient can still be upwards.
5.	H:	So it only has to do with the situation, it does not change how we calculate?
6.	T:	No, the calculation is exactly the same, you just put a minus sign before and the graph will end up below the x-axis in a way… a very good question!

[Lesson 3: Q]

Teacher Angelika quickly understands that all her examples have been with positive *b*-values (line 4) and acknowledges the question by Hampus. As can be noted from the examples above, LCv appeared both as responses to teacher questions and as learner questions (LCv 5 and LCv 6). As I did not know at the beginning of the analysis what would influence the learning opportunities that emerged, all LCv that were generated by a learner himself, without questions from the teacher beforehand, were marked with red in the lesson events. Consequently, a distinction has been made between LCv that appeared in joint discussions (black) and LCv that appeared independently (red), such as LCv 5 and LCv 6 above.

Even though this is not the focus of this study, different teacher questions tend to result in different LC. Often the more open questions: How does one compare slopes (before LCv 1), and Does anybody have any other suggestions (before LCv 4) result in an LCv. This is of course partly due to the categorisation system, as a narrow question like *Start value 1, where should I mark then* (before LC 1) already has a focus and an expected answer.

From the hundreds of learner contributions in the study, only the 184 that carried the potential to open a DoV were analysed further. Consequently, in the following only the LCv of learner contributions will be analysed. All learner contributions mentioned in the following chapters denote LCv.

7.5 Lessons categorised by trajectories for LCv

The 184 learner contributions were established differently in the different lessons. The next step in the methods of analysis was to construct a picture of how the LCv was developed in the lessons.

7.5.1 Trajectories of LCv in each lesson

In Table 7.5a below, the differences in how LCv were established in the lessons are illustrated. The total number of learner contributions in each lesson can be seen in column 2. Some of the contributions were disregarded (column 3); others were selected (column 4), considered (column 5) or explored (column 6). For instance, in Lesson 4 out of 32 LCv in total, two were disregarded, one was selected, 16 were considered and 13 were explored. Lesson 4 could undoubtedly be defined as a highly interactive lesson.

In contrast, in Lesson 7, out of three LCv in total, two were disregarded and one was established as a considered LCv. Neither selected nor explored LCv occurred in Lesson 7. Could it be concluded that Lesson 7 was a non-interactive lesson with silent learners? No, not necessarily. Learner contributions were from the beginning defined as content-related contributions from learners, and given that in addition LCv were distinguished from all learner contribution, in theory, Lesson 7 could have been a highly interactive lesson, just without much LCv. So, Table 7.5a below is not a table of interaction *per se*; it is a table of differences in how the learner contributions with the potential to open new dimensions were developed in the different lessons. There might certainly be some kind of correlation to overall interaction aspects; in fact, in Lesson 7 it was almost exclusively the teacher's voice that was heard. This is, however, out of the scope of the study.

Table 7.5a: Trajectories of LCv in the lessons

L:	LCv In total	Disregarded LCv	Selected LCv	Considered LCv	Explored LCv
1	6	1	0	5	0
2	9	0	3	5	1
3	30	6	1	10	13
4	32	2	1	16	13
5	30	6	0	9	15
6	18	3	0	3	12
7	3	2	0	1	0
9	5	2	0	3	0
10	4	2	0	2	0
11	3	0	0	3	0
12	12	1	2	4	5
13	9	1	3	2	3
14	11	4	1	5	1
15	12	0	0	2	10
Σ	184	30	11	70	73

From Table 7.5a, it could be concluded that learner contributions with the potential to open new dimensions were regarded, i.e. selected, considered or explored, to a high extent. Almost 84 % (154/184) of them were regarded as only 30 were disregarded. It could also be decided that considered LCv was present in all lessons whereas explored LCv only in little more than the half of them. The range of regarded LCv between lessons was considerable, from 1 to 30 per lesson[65].

7.5.2 Three kinds of lessons with reference to trajectories of LCv

The lessons were organised into three groups according to how trajectories for LCv were mainly established, see Table 7.5b.

Table 7.5b: Groups of lessons according to how LCv were mainly established

LCv developed in trajectories:	Dominance of explored LCv	Mixed trajectories of LCv	Only considered LCv
Lessons:	3, 4, 5, 6, 15	2, 12, 13, 14	1, 7, 9, 10, 11

The trajectories were used in classifying the 14 lessons into three different lesson types:

a) Five lessons in which LCv were dominantly established as explored LCv. These are called *explored-LCv lessons* [Lessons 3, 4, 5, 6, and 15].

b) Four lessons in which all three trajectories of LCv occurred. These are called *mixed-LCv lessons*. [Lesson 2, 12, 13, and 14]

c) Five lessons in which only considered LCv occurred. These are called *considered-LCv lessons* [Lesson 1, 7, 9, 11, and 10]

The lessons were organised according to these groups in the Main table (see Appendix A). The classification into these groups of lessons was one of the crucial points in the analysis as it enabled constellations of DoVs to be found. This will be elaborated on in detail in the next section.

[65] One has to remember that the lessons are of different lengths as are the amounts of whole-class teaching, but still this is an interesting observation.

7.6 DoVs collected and organised

The dimensions of variation identified were examined in the same detailed way as the learner contributions. Altogether 289 openings of dimensions of variation (DoVs) were found in the 14 lessons, as shown in Table 7.6. These 289 openings were of 111 distinctively different DoVs. Of the 111 distinct DoVs, 47 were opened only once in the study and subsequently 64 DoVs were opened several times.

7.6.1 DoVs opened in the lessons

As the lessons were of different lengths and also contained various amounts of whole-class teaching (WCT), which is the only form of teaching analysed in this study, these two aspects were taken into account in a quantitative overview. Columns 2 and 3 in Table 7.6 below show the total lesson lengths and the lengths of whole-class teaching, respectively. In column 4, the number of DoVs opened in each lesson is given. The DoVs have then been distinguished into the ones opened by teachers (4a) and the ones opened as a result of regarded LCv (4b).

Table 7.6 Lesson lengths, lengths of whole class teaching (WCT), and number of DoVs opened

1. Lesson	2. Lesson length (min)	3. Length of WCT (min)	4. Total number of DoVs in WCT	4a. Number of DoVs opened by teacher[66]	4b. Number of DoVs opened with LCv
1	51	23	**16**	11	5
2	58	39	**15**	6	9
3	58	51	**33**	14	19
4	63	61	**46**	24	22
5	53	46	**37**	16	21
6	62	39	**23**	8	15
7	53	33	**7**	6	1
9	36	25	**16**	13	3
10	33	33	**7**	5	2
11	38	34	**8**	5	3
12	66	39	**28**	15	13
13	45	22	**13**	5	8
14	45	26	**21**	12	9
15	61	47	**19**	8	11
Σ	722	518	**289**	150	139

[66] This was sometimes done in interaction with learners, but without any LCv

Altogether 289 DoVs were opened, and about half of them following an LCv (139/289). The number of opened DoVs differed considerably between lessons. In two lessons (L7 and L10), seven distinct DoVs were enacted, in another (L4) as many as 46.

7.6.2 DoVs organised

In order to make the qualitative comparisons, all 289 openings of DoVs were sorted into 5 properties of linear equations. These properties were not pre-given but emerged in analysis:

1. slope/m-value
2. y-intercept/b-value
3. Graph
4. Equation
5. Function

In the qualitative analysis of the aspects enacted, this categorisation facilitated the comparison. In some cases, DoVs from different properties were compared, as DoVs could relate to both slopes and graphs simultaneously. That is to say, the five properties are not exclusive, but for sake of clarity all DoVs were organised into one of these properties. Three additional DoVs were opened that were not included in any of the five properties of linear equations as they were regarded as outside the scope[67]. Apart from those three, all DoVs were included in one of the five categories. In addition, 14 disregarded LCv[68] that were difficult to categorise since no DoVs were opened, were left out of the Main table.

7.7 Comparing DoVs and LCvs

The final step in the analysis was the search for relationships between DoVs opened and trajectories for LCv in order to be able to answer the research questions regarding what teacher attentions to learner contributions imply for learning opportunities that emerged and what learners contribute to the enactment of linear equations.

Before turning to the results in next chapter, the diversity of the lessons concerning the number of both DoVs opened and LCv regarded in the lessons will

[67] These three DoVs are included in the Main table, see Appendix A.
[68] Such as the above-mentioned: *they are the impossible ones*

be illustrated in a diagram. Thereafter, a construct for the qualitative search for constellations in the Main table will be outlined.

7.7.1 Number of DoVs and LCvs

When the number of DoV opened (from Table 7.6) were combined with the number of LCv regarded in the lessons (from Table 7.5a), in Diagram 7.7 below, differences between the lessons were revealed. In order to make the comparison more accurate, due to the various lengths of whole-class teaching between lessons, the number of LCv as well as of DoV were normed to a one-hour basis. This diagram illustrates both how many learner contributions are attended to by the teacher per hour (vertical axis) and how many DoVs were opened per hour (horizontal axis).

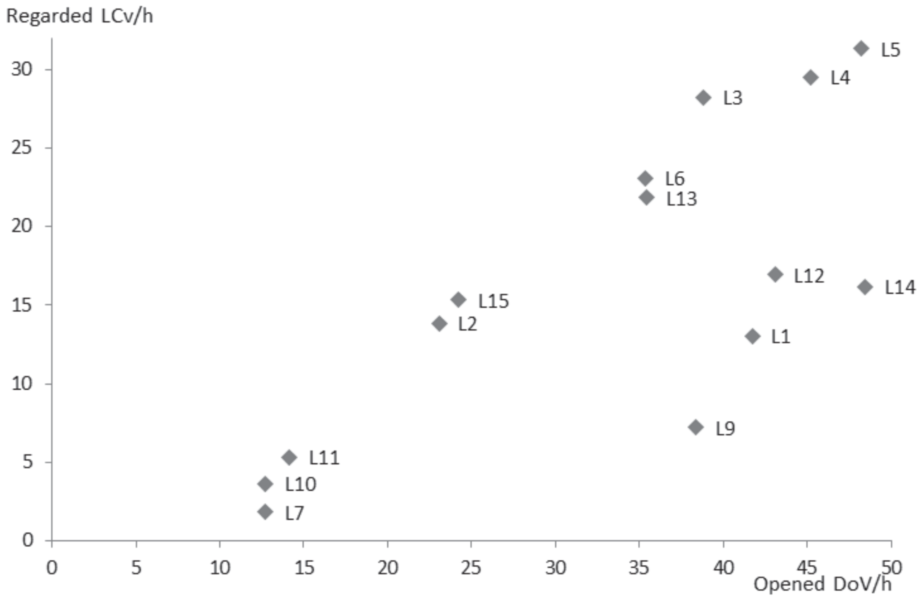

Diagram 7.7: Number of DoVs and LCvs in different lessons[69]

First, it has to be said that this is not a diagram of correlation between LCv and DoVs opened, even though the picture has some similarities to a regression line. The reason for this is twofold; the number of lessons is far too small and the

[69] It has to be noted that for two lessons (L1 and L13), the ratio of whole-class teaching in the lesson is less than 50 %, which affects the comparison in a way that places these two lessons further to the right and up than is reasonable for this comparison as all other lessons have higher ratios of WCT.

lessons have not been chosen to be suitable for statistical analysis. Instead, the diagram serves as an orientation for the differences between lessons in the study. There were the low interactive lessons with few DoVs opened (L11, L10, and L7) and also the opposite: highly interactive lessons with many DoVs opened (L5, L4, and L3). The rest of the lessons are placed in between these extremes concerning DoVs opened and LCv regarded.

The organisation of lessons in the diagram, according to the number of LCv regarded and the number of DoVs opened, fairly well resemble the three types of lessons organised qualitatively according to trajectories (in Table 7.5b). More specifically, the same lessons mentioned above at the boundaries of opened DoVs and regarded LCv, are also at the edges when organised according to trajectories. On the other hand, there are still some lessons that fall outside of this pattern, for instance, in Lesson 9, in which just a few LCv were regarded LCv, all of which were considered LCv, but in which many DoVs were opened. Lesson 15 is an example of the opposite as the trajectories for LCv were mostly explored but as could be seen in Diagram 7.7, not very many DoVs were opened, compared to for instance with Lesson 9. Another observation is that there were no lessons in the study in which the regarding of LCv was high simultaneously with a small number of DoVs being opened. So far there has been only a quantitative overview given, but the different ways of establishing learner contributions in relation to the learning opportunities that emerged require a qualitative analysis.

Does the quantity of DoVs say anything about the quality of learning opportunities? What if in Lessons 7, 10 and 11 a few, but high quality DoVs were really worked through, whereas in Lesson 5 many DoVs with low quality were just rapidly opened? Does the number of DoVs opened measure the richness of learning opportunities? No, the quality of learning opportunities certainly has to do with other aspects than just the number of DoVs opened. This will be further elaborated on in Chapter 9. However, for the possibility of rich learning opportunities, some DoVs have to be opened and the amount of distinct DoVs opened could be a first indicator of the quality of the learning opportunities that emerged.

7.7.2 DoVs opened in relation to lesson types

In the Main table (Appendix A), all 289 openings of DoVs and the 184 LCv in the 14 lessons have been organised. In the table, every opening can be distinguished, as can every trajectory of LCv. Horizontally, the lessons were ordered in the table by lesson type (main trajectories of LCv). Vertically, the DoVs were ordered according to different properties of linear equations (see 7.6.2). Late in the analysis, the DoVs were also organised in the table according to who mainly generated them, teachers or learners. The search for constellations in the Main table comprised the main analysis.

7.8 Limitations

The method of analysis revealed certain facets and delimited others. Chunking the lessons into events was necessary to enable the detailed analysis as well as for maintaining stringency in the analysis of the DoVs and the LCv. However, as the events were not reconnected to each lesson as a whole, the only inferences of the lessons as wholes are the quantitative images of frequency of DoVs and LCv in every lesson (in Diagram 7.7). The choice of DoVs as the unit of analysis, and not for instance lessons, made it possible to discover relations between DoVs and LCv.

8 Results

The research questions of this study concern what contributions learners generate and what teacher attentions to these learner contributions imply for the learning opportunities that emerge. After having arranged both 289 openings of 111 distinct dimensions of variation (DoVs) in 14 lessons and 184 learner contributions (LCv) established in these lessons in the Main table (see Appendix A), relations between DoVs opened and LCv established were sought.

When focusing *how* every DoV was opened, namely whether in interaction or by the teacher alone, no obvious differences appeared. The first analyses showed that it was not the exploring or considering of learner contribution *per se* that made differences for the learning opportunities that emerged. Given that a DoV *was* opened, no differences were found in whether it was opened jointly with learners or by the teacher alone. The differences in the learning opportunities that emerged were related to *whether* a dimension was opened or not; yet the trajectories were found to play an important role at another stage. This stage had significance earlier than at the opening of a dimension in a lesson; the trajectories of LCv were akin to *what* dimensions were opened. Results showed that different aspects of linear equations were enacted in different lesson types. In particular for some properties, such as function and slope, the differences between lessons were related to the main trajectory types of LCv. Therefore, aspects of functions and slopes were chosen to be examined in closer detail.

There were mainly two reasons for choosing functions and slopes to the comparison out of the five properties of linear equations. Firstly, relations in the opening of DoVs regarding y-intercepts, graphs, and equations did not show the same immediate constellations between DoVs opened and lesson types as did functions and slopes. Secondly, the quantity of DoVs opened was extensive but concentrated regarding slopes and functions. In contrast, the DoVs opened regarding graphs and equations were extensive but of many different kinds, whereas the DoVs of the y-intercept were concentrated but few. This can be concluded from the Main table. Nonetheless, all DoVs have been analysed, but to different extents. The results of this analysis will be elaborated on in Part I.

When changing focus and looking closer into what aspects were enacted as a result of *learner contributions* in the lessons, and considering differences between these aspects and the teacher-generated aspects, other results became apparent. These results will be elaborated on in Part II. The qualitative results are mainly based on the analysis of patterns in the Main table (in Appendix A). For reasons of clarity, in each result section only the parts discussed of the Main table are presented.

Part I: Aspects of functions and slopes enacted in different lesson types

Different aspects of linear equations were enacted in different lesson types. In other words, how learner contributions in general were established in the lesson had a relation to *what* aspects of the content were enacted. Particularly, for some properties, such as function and slope, the differences between lessons were related to the main trajectory types of LCv. These will now be closely examined.

8.1 DoVs of function opened

Function as a concept was not present in all lessons. Linear equations can be used to represent functions, but there is nothing to say that function as a concept has to be present in the introduction of linear equations. However, in most of the lessons the concept of function was enacted, that is several different DoVs regarding function were opened. Altogether 13 distinct DoVs regarding function were opened in the lessons, on 33 occasions.

For instance, in Lesson 6 six distinct DoVs of function were opened. The function was enacted as a relationship between sets of x-values and sets of y-values, explicitly by several representations. Specifically, function as a relation was varied by the graphical, the algebraic, the tabular, and the contextual representations. Further, the domain of a function was enacted, as was x as a variable in a function. The distinction between the dependent and the independent variable was also enacted.

The following example of how the *domain of a function* was enacted is taken from Lesson 2.

Excerpt 8.1:
Teacher Görel projects a straight line on the whiteboard and declares that this line shows the function of number of inhabitants (y-axis) of a village in Norrland[70] over a number of years (x-axis). The graph is decreasing and after the negative slope has been discussed, Görel points at the intercept between graph and x-axis (x-value of 38) and asks:

1.	T:	What happened here?
2.	L:	Negative number...
3.	T:	Negative number of people, hum, can a number of people be negative?
4.	L:	No

[70] The Northern half of Sweden

5.	T:	No, so it is quite unreasonable that this would continue here [points at the fourth quadrant], after about 38 years. Then it's a pretty good example of what we talked about in the last lesson, on domain and range. Domain was the *x*-values you could put into a function. It is totally unreasonable that we would put in *x*-values that are greater than 38. This function would not work anymore, because the population cannot be negative as it would be for values greater than 38, so then we have a domain which is between 0 and 38.
6.	L:	**Can't it be less than one?**
7.	T:	Well, now we have defined it with a start value, but it is good that you ask. For sure, it could actually be less than zero. What does it mean that the curve continues upwards, if we go to the left? Svante?
8.	S:	Two years ago…
9.	T:	Yes, two years ago the population was approximately 16000… four years ago it was there somewhere, what is it, something like 17000.

Görel says that it is reasonable to assume that the village had a larger population in the past. The limits of the domain of the function in both directions are enacted. [Lesson 2: G]

In this event, Görel contrasted the domain of a function with a non-example, as the number of people cannot be negative (lines 3 to 5). She focused on the *domain of a function* and showed what would have happened to the y-value of people if the x-value had been more than 38 (line 5). A learner asked about the other limit of the domain, *can't it be less than one,* and Görel answered to the question that *it could be less than zero*, which it is reasonable to believe that the question was about (lines 6 to 9) and by this both limits of the domain of the function were covered. The *domain* was separated from the function as a whole as a DoV was opened by a contrast with a non-example. Notice that even though the word *function* was used in this event and the function was worked on as a relationship between *x* and *y* (line 5), there was no DoV opened of the *function as such*. Neither was any DoV of *range* considered as opened, even though it was mentioned. Actually, the concept function was focused on earlier and varied by different representations in the same event of Lesson 2 as above. Consequently, there are two implications of the analysis of the excerpt. First, in Table 8.1 below a 0 is marked in *Separation of the domain of a function* in the column of Lesson 2. Secondly, I have previously argued that only using words such as *function* or dealing with examples in a "functional way" does not necessarily lead to the opening of a DoV. Thus, in lessons without any 0 in Table 8.1, the concept of function can be present, but no variation has been enacted of the phenomenon of function.

If we turn to see how the DoVs regarding function were opened in all lessons of the study, one conclusion is easy to draw immediately, as the lessons are ordered by trajectory type; not a single DoV of function is opened in the considered LCv lessons. For reasons of clarity, in Table 8.1 below, only DoVs opened (0) are marked. This means that here it can only be revealed *if* a DoV was opened or not. *How* it was opened and on whose initiative can be studied in the Main table, in Appendix A. Also, in Table 8.1 no disregarded LCv are visible, hence it is not possible to detect learner initiatives that were not enacted. These features of learner contributions will be discussed and clarified in Part II in this chapter.

Table 8.1: DoVs of function opened in all lessons

All aspects of function enacted														
Enacted aspects of function	**Type of lessons by LCv trajectories**													
	Explored-LCv lesson type					**Mixed-LCv lesson type**				**Considered-LCv lesson type**				
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of function[71] as a relationship	0	0		0		0	0	0						
of function by representations				0			0	0	0					
of x as a variable in a function		0		0	0									
of relationships in coordinates								0						
of proportional relations[72]								0						
of b-values[73] as y-values at intercept					0									
of function from a line between intercepts	0													
of function from a single point	0					0	0							
of function from an end-point of graph		0		0										
of why y = b if m = 0 in function[74]		0	0											
of the domain of a function	0	0		0					0					
between domain and range		0												
of dependency of variables[75]				0	0									

The comparison between the three types of lesson shows that *no* DoVs of function are opened in the considered-LCv lessons. All opened DoVs of function are found in the explored-LCv lessons and mixed-LCv lessons. This pattern was not obvious when the lessons (in the first analyses) were just ordered by number.

What about the word *function* in the considered-LCv lessons, is it even present? In Lessons 9, 10, and 11 the word is not present at all. In Lesson 7, it is written

[71] Between x and y or sets of x and sets of y
[72] Represented as straight lines
[73] This DoV enacts a relation between x and y
[74] $mx = 0 \; if \; m = 0$
[75] Changing dependency variables ($\Delta x / \Delta y$ or $\Delta y / \Delta x$)

once on the whiteboard when the teacher mentions function, but without any DoVs being opened. Actually, in Lesson 1 the word occurs frequently throughout the lesson, but it is used as a synonym to express the algebraic representation of functions, as a synonym for equation or formula. Examples of utterances from teacher Jenny:

> J: So what we have done now is therefore linear functions. And these are written in the form y equals mx plus b.
>
> [Lesson 1: 12]

> J: I think we are going to look a little on… starting from a graph, and see how you can write a function when you have a graph, so that you get the connection from that as well.
>
> [Lesson 1: 13]

Could DoVs of function not be opened without the use of the word *function*? I consider that it is possible and in a few of the events in the considered-LCv lessons, the relation between x and y is described as a (function) machine: you put something in and you get something out. However, this is done without any openings of DoVs.

In summary, *function* as a concept was not enacted at all in the lessons without *explored* LCv[76], whereas in lessons with explored LCv, several DoVs of functions were opened.

As was mentioned at the beginning of this chapter, it is not necessary to include the concept of function in an introduction of linear equations, as it might perfectly well come up in later lessons. In contrast to function, *slope* is a concept assumed to be present in lesson introducing linear equations. In the next section, the results regarding how the concept of *slope* was enacted in the lesson will be described.

[76] This includes both explored-LCv lessons and mixed-LCv lessons

8.2 DoVs of slope opened

Slope was one of the main concepts in the content taught and, unlike function; it was present in all 14 lessons[77]. In the Main table in Appendix A, all DoVs regarding slope can be detected. In this section, however, results will be reported on the DoVs opened that relate to *the meanings of* slopes, and m-values, namely which denotations were varied of slopes. A well-established construct from Zaslavsky et al. (2002), of comprehending slopes *visually* or *analytically*, was used as a frame in the analysis. This construct has been earlier discussed in Chapter 3.

The results on enacting slopes differently emanate from an early stage of analysis, in which the DoVs were analysed and categorised into the Main table. Difficulties in categorising the DoVs, due to differences between them, led to the insight that slopes were actually enacted with different denotations in the lessons. In Table 8.2, it can be perceived that for Lesson 1, there are no 0 marked for DoVs of slope. This does not imply that slopes were not mentioned and worked on in this lesson; on the contrary, slopes were present to a high degree also in this lesson. However, no DoVs were opened regarding specifically *slope* in Lesson 1. Instead increase and decrease of *graphs* were discussed, while the concept of slope remained unvaried in the background.

[77] And also enacted in 13 of them

Table 8.2: Aspects of slopes/m-values enacted in different lessons

Aspects of slope and *m*-value														
Enacted aspects of slope and *m*-value	**Type of lessons by LCv trajectories**													
	Explored-LCv lesson type					**Mixed-LCv lesson type**				**Considered-LCv lesson type**				
Slopes enacted visually														
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of lines as hills													0	0
of m-values as degree of leaning[78]												0		
of negative lines as slides												0		
between incline and decline of line												0	0	
between uphill and downhill of lines													0	0
Slopes enacted as increases														
Separation:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
of slope as increase per x	0	0	0	0				0	0		0			0
of m-value as increase[79]	0	0	0					0	0					
between increase and decrease[80]						0	0							
of negative slopes[81] as decreases	0	0							0					
Slopes enacted analytically														
Separation:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
of slope as a relationship[82]				0	0		0	0					DIS[83] 10A	
of m-value as rate of change[84]			0	0	0			0						
of why m-value is slope[85]		0	0						0					

DoVs correlating to 12 different aspects of meaning for slope and m-value were opened, on a total of 40 occasions[86] in the lessons. The 12 DoVs can be placed in

[78] Of the line
[79] Per something
[80] Of slope
[81] And m-values
[82] between x and y
[83] As this LCv is taken as an example in the chapter, this square stayed marked even though the DoV was not opened due to the disregarding of LCv 10A.
[84] Between x and y
[85] As a relationship between x and y

three categories: slopes enacted visually, slopes enacted as increases, and slopes enacted analytically. It is quite evident from Table 8.2 that a comparison between DoVs opened and types of lessons results in an overlap of the main type of trajectory for LCv and the different denotations for slopes/m-values enacted. If a lesson did *not* include explored LCv (L1, L9, L11, L10, L7), the DoVs of slopes were enacted mainly visually. There are just two exceptions – in Lesson 7 and Lesson 9, slopes were also enacted once as increases. When the lesson contained explored LCv (the mixed-LCv lessons and explored-LCv lessons), then the DoVs were enacted as increases or analytically as rates of change. In two thirds of these lessons, slopes were enacted as both increases and rates of change. Let us now look more closely at the three categories of aspects of slopes enacted.

8.2.1 Slopes enacted visually

In a few of the lessons, lines and/or slopes were dealt with as if they were pictures, namely stable hills, either uphill or downhill. In these lessons, teachers also talked about increase and sometimes even increase per x. Yet the DoVs opened, i.e. the aspects brought to the fore and enacted in a pattern of variation, regarded lines/slopes visually as hills. One example is from Lesson 7, when teacher Åse made a contrast to the x-direction in the coordinate system, and simultaneously separated slopes as uphill or downhill.

Excerpt 8.2.1a:

1. Åse:	So, what do you say, is this slope a steep one, or… I will do like that [adjusts the graph in GeoGebra]. Is it leaning a lot or a little?
2. L:	A lot
3. Åse:	A lot? How many agree on a lot?

About half of the learners raise their hands.

4. Åse:	Ok, about half of you…
5. Åse:	Is it uphill or downhill?
6. L1:	Both?
7. L2:	You cannot know, there is only one line
8. Åse:	Yes, well, it depends from which direction you come. We have to start by agreeing on that. Normally, we read from left to right in this part of the world. And then, if it is uphill when we go from left to right, we say it is a positive slope. And this (graph) is uphill. We are actually going from left to right. That means this is downhill [Åse changes the direction of the slope of the graph in

[86] In the Main table, several openings of same DoV, with different LCv, in same lesson can be discerned.

GeoGebra]. Then we say it is a negative slope.
[Lesson 7: A]

In this event teacher Åse separated the direction of a slope (line 8), but without varying, or mentioning the reasons for that, namely that x-values increase to the right in a coordinate system (usually). The direction was simply treated as a convention. Even though the word *slope* was used, the graph was dealt with as a hill. Nothing on the relationship between x and y was enacted. Neither was the fact that increasing x gives increasing y if the graph has a positive slope enacted in this event (or in the lesson). Uphill was related to positive slope and downhill to negative slope.

In Lesson 10, the same "hill metaphor" was used frequently for graphs.

Excerpt 8.2.1b:
Teacher Helena is discussing different graphs on the whiteboard, coloured as green, blue and red lines. [The equation of the red graph is $y = -x + 1$]

1. T:	If we look at the blue line, how has my *m*-value changed there?
2. L1:	0.5
3. T:	Hum, I take one step there, and I will end up a half up. I take one step[87] there and a half step up. y has to be 0.5x + …
4. T:	Do we have any *b*-value?
5. L2:	Plus two
6. T:	Plus two, ok, when I moved upwards here, all the time I am moving upwards [Helena moves her entire body and waves with her arms in an upward movement]. When I move from there (left) to the right, the *y*-value is growing. I go up a hill, do you see that, I go up a hill. I go up along the green hill [points along the green graph], I go up along… (the blue). Think like that all the time, upwards a hill.
7. T:	What about the red line? If I again move from left to right… (15 sec) how do I go there then? In the red hill? What happens with my *m*-value, I go one step there, what happens with my *m*-value, no, my *y*-value?
8. L3:	It is minus one.
9. T:	It decreases, yes, I am pulling downhill. I am pulling downhill. Yes, it decreases with one step. The *m*-value is -1

[Lesson 10: 3]

[87] "Step" here and in all following excerpts has the meaning of a unit of whole numbers [Swedish word: steg]. The English way of using commas is used here. In the lesson the Swedish way with 0,5 was used.

First of all, Helena did not make it easy for the learners to discern any relationship between x and y, since the y-value was discussed as an m-value (in line 1, 2, 7, 7, 8, and 9), but also as a y-value (in line 6 and 7). Both learners' answers (in lines 2 and 8) concerned the m-values, not the y-values. The consequences of this blurring of y-and m-value made any attempt to discern relationships between x and y difficult. In addition, the graphs were dealt with as hills, like the green and red hills (in line 7). The hill metaphor appeared several times when a separation of positive and negative slopes was made. In Lesson 11, for example, negative slope is discussed as a slide.

Excerpt 8.2.1c:

1. T:	What is it that makes the red line look different? What makes it differ from the other two? How does the red line differ from the other two?
2. L:	The *m*-value is minus
3. T:	Why?
4. L:	Because it leans in the other direction
5. T:	Yes, it is like a slide [Lesson 11: B]

In summary, slopes were enacted visually (Zaslavsky et al, 2002) as hills. A hill is something motionless, something stable that does not change. By using only this metaphor, it was difficult to discern relationships between variables, and also to discern *the slope* as a phenomenon. By using the denotation of hills, which probably was a way of concretising a mathematical concept, the concept of slope remained in the background. In addition, the difficulty of making distinctions, for instance between y-values and m-values, was shown. Now, let us continue with the second denotation of slopes enacted: as increases.

8.2.2 Slopes enacted as increases

Slopes as increases were enacted on several occasions in about half the lessons. DoVs of slopes as *increases of y per x* as well as m-values as *increases of something (contextual) per something else (contextual)* were opened. This dialogue from Lesson 4 is typical of these openings of slopes/m-values as increases.

Excerpt 8.2.2a:
In Lesson 4, teacher Angelika has taken 6 examples of *m*-value as change per something:
Increase of price per hour: 90 kr/hour
Increase of price per km: 15 kr/km
Decrease of length per hour: -4.5 cm/hour
Increase of length per month: 2 cm/month

Decrease of length per minute: -200 m/min
Decrease of degrees per hour: -4°C/hour

All examples had been given the contextual representation and half of them had also been given the algebraic representation. *m*-values had been focused on. So far, no graphical representation had been present. Angelika asked:

1. T:	But if I only had a graph? Would I have been able to determine how much it inclines[88] only by looking at the graph?
2. L:	Yes
3. T:	How would I have done that then?
4. L:	You would have seen how much ... how much it increases for each step on the *x*-axis.
5. T:	Yep, exactly. So if I take one step in the *x*-direction, I would check how much it increases then.

Angelika returns to the examples used in previous events and finds a context, namely hair that grows two cm per month.

6. T:	Once a month has passed, the length of the hair has increased by two centimetres, hasn't it? So, a month gives an increase of two centimetres. Do you understand that?
7. Ls:	Yes
8. T:	Then the slope is two. So the slope is really the same thing as how fast it increases per unit.

[Lesson 4: V]

Excerpt 8.2.2a shows how slope in the graphical representation was varied as an increase in *y* over *x* (line 4) against the background of various contextual situations (lines 6 to 8 and in relation to what preceded the event). DoVs of slope as increase of y over x was opened.

Lesson 6 had a different start than most of the lessons. Teacher Marita divided the class into groups of 2-3 learners and asked them to solve a task together. The task was to combine three graphs with three equations [$y = 2x$ (red), $y = 3x$ (yellow), $y = 4x$ (green)]. Only slopes (and colours) varied. The five groups had four different ways of determining the slope.

Excerpt 8.2.2b:
A learner from group 2 described that they determined the slope for $y = 4x$ as the increase was 4 squares high on every square[89] to the right. That means they discerned slopes as the increase of *y* per *x*. The next group had another reference point when determining the slope:

1. T:	You at the back, how did you think then?

[88] The Swedish word used (lutar) includes both incline/decline.
[89] The grid of the coordinate system

2. L1:	We thought that for every, as for the green one, there was an increase of ¼ of x for every y
3. T:	A fourth of x for every y, yes.
4. L1:	The same as them [group 2] but we were further down on the graph. And for the yellow one it increased…
5. L2:	By a third.
6. L1:	By a third x for every y.
7. L2:	Yes
8. T:	Did you look down here? [Marita points at the first square in the first quadrant]
9. L1/L2:	Yes
10. T:	Where y is one.
11. L2:	Yes
12. T:	Did you calculate or just estimate?
13. L2:	No, you see that higher up it is three (y) when one (x) has passed.
14. T:	[to the class] Do you understand what they did?

Marita summarises on the whiteboard by sketching only one square of the grid, with the group's ideas about a fourth, a third and a half x for every y.

[Lesson 6: C]

By letting the learners contribute different ways of solving a task, Marita contrasted and compared different solutions. In the excerpt (line 4), the learner expressed that they did the same as the previous group, just further down on the graph. Actually, their solutions differed as they chose different references for the slope. While group 2 determined the slope as increase of y per unit of x, group 3 did it the other way around, increased x per unit of y. This implied that the slopes they determined became inverted values of the "real" slopes: *by a third x for every y* (line 6). As Marita asked if they calculated or just estimated (line 12), we can also conclude that these learners understood that the slope is the same everywhere on the linear graph (line 13) and had no difficulties in connecting the "inverted slope" of $1/3$ to $y = 3x$ (line 13). Experiencing slope as increase of y per unit of x *or* increase of x per unit of y may have seemed as the same for these learners in this early stage of learning linear equations. However, in this event both these ways of defining slope were highlighted by Marita, as a result of exploring LCv, and the construct of using y per unit of x was stressed.

The occurrence of specific words in a lesson does not necessarily imply that slopes are enacted in a way that corresponds to those specific words. Earlier it was mentioned that the word *increase* is not sufficient for the slope to be enacted as an increase of y per x. There has to be DoV opened in that respect. Also, just because a teacher uses a specific word, such as for instance *downhill,* it does not indicate that slopes are enacted as hills. It all has to do with what is done with those words, see Excerpt 8.2.2c when negative slopes are focused on.

Excerpt 8.2.2c:

1. T: How would a curve look? If I drew a graph for this municipality, how would it look?

Some students are waving their hands, a downward motion.

2. T: Good that you sit and wave. Can we describe it in any way?
3. L: It starts in a minus way.
4. T: It starts in a minus way?

Students laugh

5. T: I understand what you mean; can we say it in another way?

Görel shows a graph and discusses the start value and the downward slope.

6. T: What we all are trying to say, is that there will be some sort of downhill. The mathematical term to describe this is that it has negative slope. If it has a negative slope, it decreases. Negative slopes are always related to negative *m*-value, for every year that passes, it decreases.

[Lesson 2: 5]

In Excerpt 8.2.2c, teacher Görel uses downhill, but also generalizes negative slope, an example of decrease and negative m-values. Words alone are not enough to decide how a concept is enacted.

In summary, slopes/m-values as increases were enacted both as increases of y per x or an increase per something else (contextual). In one lesson slopes were enacted both as increase of y per x and as increase of x per y, with "inverted" slopes. Now we will turn to the third and last group of DoVs regarding slope in these lessons: slopes as relations.

8.2.3 Slopes enacted analytically

Before describing results on the analytical enactments of slopes as rates of change, a description of the only disregarded LCv on denotations of slopes will introduce the topic:

Excerpt 8.2.3a:

Teacher Helena in Lesson 10 draws graphs for: $y = 2x, y = 2x + 2$, and $y = 2x - 3$ on the whiteboard. She asks: What is the similarity between these lines? A learner contributes by saying 'they are parallel'. Helena affirms and writes parallel on the white board. She continues:

1. T: What makes them parallel?
2. L: They have the same *m*-value.
3. T: Yes, and what is the *m*-value in this case?

4. L:	**How much *y* changes for every *x*?**
5. T:	Yes, but how much is the value here?
6. L:	2 [Lesson 10: A]

This was an interesting case of disregarded LCv. A learner contributed with an explanation of how the m-value can be seen (line 4), but instead the teacher wanted to know the value in this specific case. Helena narrowed down the complexity of the discussion and no DoVs regarding slope/m-value as change of y for every x, was opened in this event or, in fact, at all in Lesson 10.

The enactment of slopes *analytically* (Zaslavsky et al, 2002) as rates of change or internal relations between x and y requires that the variables are made discernible as a whole, as a relation. When slope was enacted as an increase (or decrease) it was (mostly) done as an increase of y per x. When slope is enacted as a relation between x and y, both variables are discerned simultaneously. For instance, below in Excerpt 8.2.3b, the graph is enacted as infinitely many points and the slope is seen as the simultaneous change between x and y.

Excerpt 8.2.3b:
Teacher Ida is discussing a task that the learners have been working on in groups. Many graphs and equations have been combined when Ida asks:

1. T:	Which one is decreasing, which has a decrease?
2. L:	The blue one.
3. T:	The blue one is decreasing. All the other increase. And then you looked at, all groups did that, you looked at the increase. **How does it change if you ... if you change the *x*-value, how does the *y*-value change then?** And then there was, of course, everyone looked at another...one can find infinitely many points on this line, but there are some points which are particularly interesting… [Lesson 13: 4]

The next case is interesting not only because the m-value is enacted as a rate of change between x and y, but also from an interaction perspective because there is some kind of reversal of traditional roles between the teacher and a learner. Gustav is a learner, of whom teacher Hoda asked a lot of questions, yet not in a traditional way.

Excerpt 8.2.3c:
Hoda says that they are to find a relationship between *y* and *x*, where *x* determines the *y*. She asks the students in groups of four to discuss the question: *how can we describe a change in y if we know that we always take one step in the x-direction?* No comments come from the learners. Hoda presents $y = mx$ and tells the

students to discuss the topic again. Hoda asks how they are coping and says that she does not think that everything is clear to everybody.

1. Gustav:	If x is a step...
2. Teacher:	If we have one step in the x-direction so we change x from 0 to 1 or from 1 to 2, or from 2 to 3... I understand how you think.
3. G:	Yes, and then **m will be the change factor,** how **much larger**, or how many steps y will take. For example, **if the steps are equal,** one y-step and one x-step, then m will be one as well.
4. T:	Wait a minute, one step in the x-direction and m steps in y-direction, did you say that? **So if m was one, they became of equal length?**
5. G:	Yes, and so on that one, the half one, (stairs drawn on the whiteboard), it is one step in the x-direction and just a half in the y-direction, then it will be 0.5...
6. T:	It's like taking 0.5 times ...hum…yes, that's right. Do you see how he reasons?
7. Ls:	Yes/mm
8. T:	How do you make sense of that one (a steeper stair) then? Explain that also, so we get to learn that one as well.
9. G:	Well, there it is one step in the x-direction and two steps in the y-direction, then m will be two.
10. T:	Hum, but these are not straight lines?
11. Hampus:	**What! Is m the steps in y-direction??**
12. T:	Gustav says that. It seems right, I understand how he thinks. [Lesson 15: E]

Gustav described the m-value as a change factor (line 3) and gave three distinct examples in which m-varied ($m = 1, m = 0.5, m = 2$). After the examples were given, a fellow learner Hampus bursts out: *What! Is m the steps in y-direction?* (line 11) In this excerpt (8.2.3c), slopes were enacted as rates of change between x and y. Even though all Δx-values were 1, the change factor (m) was dealt with as a rate. Public joint reasoning, as in this event between Hoda and Gustav, was unusual in the study. Nonetheless, the m-value was enacted as a relation between x and y. Hoda's comment in line 10 relates to the fact that the context for the discussion was actually stairs drawn in a coordinate system, which had been manipulated and changed to make them steeper and flatter. Directly after this event, the stairs were left behind, in favour of lines. Hampus's comment (in line 11) could on the one hand be seen as somewhat procedural as seeing the m-value *only* as the steps in the y-direction would probably be a dead end for further learning. On the other hand, in this context, after Hoda's and Gustav's elaboration on the rate of change between x- and y-values, it might also just be a way of simplifying.

In summary, slopes and m-values were enacted analytically as rates of change or relations. The enactment of slopes, as rates of change or as relations between x and y, require that the variables are made discernible as a whole. Furthermore, by an analytical approach slope is not to be enacted as a property of the graph, but of the function that is illustrated by a graph. When slope was enacted as an internal relation between x and y, both variables were enacted simultaneously.

8.3 Summary: Part I

Results showed that the differences in learning opportunities were related to lesson type, i.e. how learner contributions in general were established in the lesson. Functions as well as slopes were enacted differently in different lesson types.

Regarding the enactment of *function*, the differences between lesson types were even greater. In the explored-LCv lessons several dimensions of variation of the *function* were opened, like varying functions by using different representations, and/or the functions as a relation between sets of x and sets of y. Also dimensions of variation like domains of functions and variables in functions were opened. In the considered-LCv lessons, not a single dimension of variation regarding function was opened.

In explored-LCv lessons, *slopes* were enacted as increases of y per x and/or analytically (Zaslavsky et al, 2002) as relations between x and y whereas in considered-LCv lessons, slopes were enacted visually (Zaslavsky et al., 2002) as hills, if at all.

Table 8.3: Slopes and functions enacted in different lesson types

Lesson type / Aspect	Explored-LCv lessons	Considered-LCv lessons
Function	**Functions enacted:** by different representations and/or as relationships between x and y	No enactments of function
Slope	**Slopes enacted:** as increases of y per x and/or analytically as rates of change or relations	**Slopes enacted:** visually or not at all

The results of the analysis of the differences between DoVs opened in different lesson types were presented in this section. Concepts as functions and slopes were given different meanings in different lessons. These differences were related to the lesson types, specifically how learner contributions *generally* in the lessons were attended to. Nothing has yet been revealed on the differences between teacher- and learner-generated DoVs. This will be the focus of the next section.

Part II: Learner-generated aspects of linear equation

Learner-generated aspects of linear equation turned out to differ a great deal from the aspects that were mostly generated by teachers. Constellations in the Main table of DoVs that were opened as a result of learner contributions, i.e. by attended LCv, have been examined closer. Firstly, the results of this analysis have been organised around functions and slopes, and will be described as the DoVs opened by attended LCv. Secondly an analysis was made to examine what characterises the learner-generated aspects.

Thirty of the 184 LCv were disregarded, which means that no DoVs were opened as a consequence of the contribution. However, all 184 LCv have been registered in the Main table. Certain LCv were disregarded in some lessons and attended to in other and this resulted in different outcomes. These kinds of cases will be described in detail.

8.4 Teacher- and learner-generated aspects of function

Altogether 13 distinct aspects of function were enacted in the study and of these less than half were mainly teacher-generated. The same content as in Table 8.1 is now presented in Table 8.4 below. Yet, three changes have been made. Firstly the trajectories for each LCv are now shown. This means that disregarded LCv are also revealed. Secondly, the aspects have been grouped into either teacher-generated or learner-generated aspects. Thirdly, if several LCv concerned the same DoV, for instance, if an LCv was first disregarded and later another LCv was explored for the same DoV, that has been pointed out in the footnotes.

Table 8.4 provides information about <u>who</u> generated the DoVs. All openings, and disregards (DIS), are coded to depict whether a teacher opened solely (X) or an attended LCv was involved. The trajectories for LCv are also coded by S, C, E, or DIS. This applies to all the tables in the following. Table 8.4a offers information about the teacher-generated DoVs.

Table 8.4a Aspects of function mainly generated by teachers

Aspects of function mainly generated by teachers:														
Enacted aspects of function	**Type of lessons by LCv trajectories:**													
	Explored-LCv lesson type					**Mixed-LCv lesson type**				**Considered-LCv lesson type**				
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of function as relationship[90]	X	X		X		S 12L	X	X						
of function[91] by representations				X			X	C 14D	X					
of x as a variable in a function		X		X	X									
of a relationship in coordinates								X						
proportional relations[92]								S 14G						

Lessons are numbered, LCv are named in order by letters in the alphabet.

X: *Opened by teacher without LCv*

S 14G: *Opened by a trajectory of selected LCv (G) in Lesson 14*

C 14D: *Opened by a trajectory of considered LCv (D) in Lesson 14. An LCv in black means the LCv was an answer/comment to question.*

S 12L: *Opened by a trajectory of selected LCv (L) in Lesson 12. An LCv in red means the LCv was initiated by a learner. This was done most often by a question.*

There were five distinct aspects of function opened mainly by teachers in this study. Altogether these five aspects were opened on 15 occasions. Not much that is surprising appears in Table 8.4a, except from the fact that was already elaborated on in Part I: the absence of aspects of function enacted in one of the lesson types. Part from that, all the DoVs opened seem to be correct and expected aspects of function in a mathematics classroom. If we instead turn to the learner-generated aspects of functions, another picture appears.

[90] Between x and y, and also sets of x and sets of y
[91] Focused on and varied, not only present
[92] Straight lines/lines with $b - value = 0$

Table 8.4b Learner-generated aspects of function

Aspects of function mainly generated by learners:														
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of b-values[93] as y-values at intercept					E 15I					DIS 1A				
of function from a line between intercepts	E 5T		DIS 3F	DIS 6Q							DIS 9E			
of function from a single point	E[94] 5O			DIS 6K		C 12E	E 13D							
of function from an end-point of graph		C 4F		E 6A										
of why y = b if m= 0 in a function[95]		E 4K	C[96] 3P											
of the domain of a function	E[97] 5H	E 4X		E 6R					E 2G					
between domain and range		C 4I												
of dependency of variables[98]	DIS 5M			E 6C	X			DIS 14C						

Lessons are numbered, LCv are named in order by letters in the alphabet.

X: *Opened by teacher without LCv*

DIS 5M: *Disregarded LCv (M) in Lesson 5*

C 15I: *Opened by a trajectory of considered LCv (I) in Lesson 15. An LCv in black means the LCv was an answer/comment to question.*

E 4X: *Opened by a trajectory of explored LCv (X) in Lesson 4. An LCv in red means the LCv was initiated by a learner. This was done most often by a question.*

Table 8.4b provides information about the enactment of the following six aspects: separation of why the b-value can be seen as the y-intercept, of why y equals b if the m-value is zero, of function from a single point, of function from an end-point of the graph, of the domain of a function. Additionally also separations were made between the domain and the range of a function and separations between which of x and y is the dependency variable.

These eight DoVs were enacted on 17 occasions and disregarded on 10 occasions. It can also be concluded that in two of the five lessons in which no aspects of function were enacted, two learners were making attempts, but their contributions were disregarded (Lessons 1 and 9). Additionally, Table 8.4b reveals

[93] This DoV enacts a relation between x and y

[94] Also in 5P

[95] $mx = 0 \ if \ m = 0$

[96] DIS in 3H and 3K

[97] Also DIS later in 5I

[98] Changing dependency variables($\Delta x / \Delta y$ or $\Delta y / \Delta x$)

that the learner-generated aspects were mainly enacted in lessons in which learner contribution were explored, which in one way is a part of the construct of the study. However, it can also be revealed that learners initiated more aspects (red LCv) in lessons in which their contributions are explored, which is not totally self-evident.

The content of the learner-generated aspects of function have two main differences compared to the teacher-generated aspects in Table 8.4a. Firstly, there are aspects that expose *why-questions* of functions, for instance why y equals b when the slope (m-value) is zero. One of these aspects will be described in detail below. Secondly, these eight learner-generated aspects (in Table 8.4b) are not all mathematically correct. However, the results will show that it might be a good idea to enact even incorrect aspects. Also this will be reviewed in greater detail below.

8.4.1 Why the b-value can be seen as the y-intercept

In the lessons, the b-value of the equation of a straight line ($y = mx + b$) was often enacted *as* the y-intercept in the graphical representation. In 11 of the 14 lessons, this was the case, which makes this aspect one of the most frequently separated aspects of linear equations in the study. Only in one lesson this aspect was not present at all (L13, see Table 8.4.1), and in two lessons the aspect was enacted in different ways that will be described below.

To understand how the b −value was dealt with in two of the lessons, namely *why* the b-value can be seen as the y-intercept, assumes that at least two aspects are discerned. Firstly, one has to discern that x equals 0 at the y-intercept. Secondly, one has to be able to see the linear equation as a relationship between x and y in both the algebraic and graphical representations. When x equals 0 in the algebraic form, then mx also equals 0, hence $y = 0 + b,$ i.e. the b-value "ends up" at the y-intercept in the graphical representation as $y = b.$ Nonetheless, in the graphical representation the point where the graph intercepts the y-axis has two coordinates, as all points have both an x- and an y-value, even though x equals 0 and hence is "invisible" in the equation. Consequently, when the b-value is separated in the lesson *as* the y-intercept, the x-value remains unrevealed both in the algebraic and the graphical representation. These two aspects are taken for granted in 11 lessons in which the b-value is enacted *as* the y-intercept.

Table 8.4.1 provides an overview of how the b-value was enacted in different lessons. As stated earlier, in 11 of the 14 lessons, b-values were enacted as y-intercepts. Two learner contributions (in L1 and L15) addressed the issue of *why* of b-value is at the y-intercept. These events will now be more closely examined.

Table 8.4.1 The enactment of b-value in the lessons

Lesson:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of b-values as y-intercepts[99]	X	C 4S	X	E 6M		X		C 14E	X		X	X	X	X
of b-value as the y-value at the y-intercept					E 15I					DIS 1A				

Lessons are numbered, LCv are named in order by letters in the alphabet.
X: *Opened by teacher without LCv*
E 15I: *Opened by a trajectory of explored LCv (I) in Lesson 15*
DIS 1A: *Disregarded LCv (A) in Lesson 1. Disregarded LCv are always marked in red as they are always initiated by learners.*

Early in the first lesson recorded (L1) a learner, Cornelis, posed a question about how to know that the *start value* is placed at the y-axis, and not at the x-axis (1A). Excerpt 8.4.1a shows how the contribution was responded.

Excerpt 8.4.1a

1. Teacher: Start value 1, where should I mark then? Can someone come up, or can you explain, Omar?
2. Omar: On the y-line, but on 1.
3. T: Exactly, when we started, zero years, for instance, had passed and we had the value of one.
4. Cornelis: **How do you know it is there? I mean, why don't you start at the x-axis? (LCv 1A)**
5. T: That's because… now, we had no function (=formula) here, but it is the y we are to figure out. And when we have x, if it is actually zero years that have passed, we saw in this task that it is right from the start, right from the beginning, when the price was 2 kronor. And that's what we have in the function (=formula), our initial value, when we still have not moved anything in years…or whatever the x value is. **And when x is zero, then we <u>are</u> always on the y-axis.** So when we start, it is when x is zero. **And you have to see then, what value do we have, and <u>that value ends up</u> on the y axis.** Did you get that, Cornelis, or?
6. C: Well...
7. T: Difficult?
8. C: Yes...
9. T: Yes, but remember that when we start, x is always zero. **And then you know that when x is zero, we have been practicing that, then we <u>are</u> on the y-axis and work.** So when we start from something, we assume that x is

[99] This row is taken from the y-intercept/b-value section in the Main table.

	zero, and then we end up here. Since the change was +2, which means that when we go one year, then we increase by two pieces, and then we go one, two, steps up.
10. T:	Did you understand that?
Silence	[Lesson 1:A]

In this event, teacher Jenny took for granted that Cornelis discerned both x- and y-coordinates in every point and that there is a relation between them. However, what he actually asked: 'why don't you start at the x-axis' (line 4) suggests that he did not discern the relation between x and y. Jenny made a great effort to answer the question, yet the contribution was established as a disregarded LCv. Cornelis asked about *why* we start at the y-axis, and Jenny answered that you have to accept that we are on the y-axis when x is zero (lines 5 and 9). In lines 6 to 8, Cornelis's courage appeared, as it takes guts to not confirm an understanding in that moment, after such a long explanation by his teacher. Jenny made a second try (line 9), but with the same kind of explanation and result. Of the two aspects that are necessary to discern, one of them is present, namely that x equals zero at the y-intercept. However, at least in this section it is not varied. It is only stated as a fact, as something that has to be accepted, practised and remembered (lines 5 and 9).

The other aspect, a relationship between x and y, is not present. There is no doubt that Jenny wanted Cornelis to learn. She heard him and answered his question in a way; however, the tools necessary for learning were not offered in this case. Jenny's answer was in the analysis considered as a disregard of Cornelis's contribution, even if such effort was made. This is due to the interpretation that Jenny did not regard the *content* of Cornelis's contribution, but instead explained the reasons from her own perspective solely.

In the last lesson recorded (L15) and in one of the last learner contributions given (15I), another way of enacting the same aspect was found.

Excerpt 8.4.1b

Teacher Hoda discusses, using a one-meter ruler as a tool, that even though they now can construct all lines in the world, they would all be stuck in the origin; all lines would pass through the origin [as they have only been varying the *m*-value so far]. She asks the students to discuss in pairs how they could move the line to higher/lower positions in the coordinate system. The learners discuss this for about 3 minutes. This is followed by a whole-class discussion, in which a learner contributes the equation of the straight line. Hoda asks the learners to explore the relation between *mx* and *b* and to examine the graph when $x = 0$. After 30 seconds, Hoda clarifies:

1. T:	I will try to help you… *x* can have all values, it's not that, but…I am only curious about the point or the time when

	x is zero. *x* can be all values, you know, it can be 0,1,2,3,4,5,6 and yes, minus, negative numbers also. I am only curious about what…exactly the moment when *x* is zero, so *x* does not always have to be zero, but what happens exactly in this relation, what remains when *x* is zero? That was what I wanted you to talk to your neighbour about. Gustav?
2. Gustav:	**Then it is m times x, which is 0, and then *y* becomes equal to *b* (LCv L15I)**

Hoda re-tells an anecdote about Descartes and his invention, the coordinate system, for 62 seconds with a lot of laughter from the learners. She writes **(0,b)** on the whiteboard.

3. T:	Where do you find this place? Discuss with your neighbour and try to find it. You can try and give *b* some different values and see where you can find what it is… where it is, if we say *b* would be 3, where would you find zero comma three? Where is that point?

Hoda writes **(0,3)** on the whiteboard. Learners talk for 30 seconds.

4. T:	What did you conclude? Did you find zero comma three?
5. L2:	Everything ends up on the *y*-axis.
6. T:	Okay… so depending on what *b* we have, we always end up somewhere at the *y*-axis?
7. L2:	Yes
8. T:	Because exactly here, along the *y*-axis, what value does *x* have?
9. Ls:	Zero
10. T:	Yes, *x* is zero here. So zero comma that *b*-value, and then we have invented the thumbtack we needed.

Hoda shows with the ruler how different *b*-values affect the graph.

[Lesson 15: I]

Hoda did not only explore the learner contribution (in line 2); she also asked the learners to do the same. The relation between x and y was examined and an unusual way of writing the coordinates on the y-axis [(0,b)] revealed both coordinates. The joint exploration of the learner contribution*: then it is m times x, which is 0, and then y becomes equal to b* (line 2) made the rationale visible of why the b-value can be seen as the y-intercept. Consequently, different learning opportunities emerged in these two events as it was not made possible to learn this aspect in Lesson 1.

The learner-generated aspects in these two events both carried the potential of deepening the learning opportunities that emerged compared to the 11 lessons in which the b-value was enacted *as* the y-intercept. However, both the learner contribution and an attention to the contribution were needed for the aspect to be opened. One difference between the learner contributions in Lesson 1 and Lesson

15 is that Cornelis's contribution was a genuine question whereas Gustav's contribution was a result of a task given by the teacher. Hoda deliberately "made" Gustav contribute what was intended by her. In this event, the differences in the learning opportunities did not depend on whether the learner contribution was genuine or "planted". The differences depended on how the content of the contributions was attended to. Nonetheless, learner contributions such as the question: 'why don't you start at the x-axis'[100] carry the potential of deepening the learning opportunities for linear equations. Let us now continue with some results from further cases of attended learner contributions in order to find out what can be learnt.

8.4.2 A function as a line between intercepts

This case is related to the previous one as it also emanates from learners' unconventional ways of comprehending the intercepts. Cornelis's question in the last case (8.4.1) did not reveal how he experienced the function, only that he questioned the role of the y-axis as a starting point. In the present case, several learners make contributions that suggest they have a way of seeing the function as a line drawn between two intercepts. This way of experiencing has been described earlier by Kerslake (1981) and was elaborated on in chapter 3. The rationale behind is that the coefficients of an equation (the m- and b-values) are seen as the intercepts between the graph and the axes. Then a line is simply drawn between the two intercepts. Even though this way of seeing a function or a graph is procedural, it is not hard to see its internal logic. As described in the last example, the b-value was often enacted *as* the y-intercept in the study, without explaining any underlying reason for that. Consequently, using the other coefficient (the m-value) as the x-intercept is quite logical, and yet a dead end if one wants to understand linear equations.

[100] (LCv 1A)

In Table 8.4.2, a segment of the Main table is shown. It concerns the attended LCv of the DoV of seeing the function as a line between intercepts.

Table 8.4.2 Separation of function from a line between intercepts

Lesson: / Separation:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
of function from a line between intercepts	E 5T		DIS 3F	DIS 6Q							DIS 9E			

Lessons are numbered, LCv are named in order by letters in the alphabet.
DIS 3F: Disregarded LCv (F) in Lesson 3
E 5T: Opened by a trajectory of explored LCv (T) in Lesson 5.

Table 8.4.2 provides insights into that this DoV was disregarded in three of the lessons (L3, L6, and L9). Only in Lesson 5 was it opened as a result of an explored LCv. We will now take a closer look at two of those events (from Lesson 3 and Lesson 9) in which this DoV was initiated by an LCv, but disregarded. Thereafter, we will look at how it was explored (in Lesson 5).

Excerpt 8.4.2.a
In Lesson 3, teacher Angelika has just been discussing slopes as *m*-values in the equation when she asks:

1. Teacher: What does that second number determine then?
2. L1: Where it starts at the *y*-axis
3. T: Good, our start value, really good, the start value at the *y*-axis.

Angelika shows the start values of all the graphs.

4. L2: **You said that the start value is on the *y*-axis, but could it have start values on the *x*-axis as well? (LCv 3F)**
5. T: In the functions we talk about, the start value is always in y, because that is where we always start, you could say. [Lesson 3: F]

No explanation is given for the LCv (in line 4); hence this has been categorised as a disregarded LCv. As this LCv is short and moreover disregarded the conclusion that it is about seeing the function as a line between two intercepts is drawn with some uncertainty. More clearly this way of seeing the function was expressed in Lesson 9. In this event the example discussed was starting from a graph and the task was to formulate a corresponding equation.

Excerpt 8.4.2.b
In Lesson 9, the previous task has been to match equations with corresponding graphs by determining either the m-value or the b-value. As the four graphs all had different b-values, the matching was made with a total focus on the b-value versus the y-intercept. The present task is to formulate equations for graphs

projected on the whiteboard. Teacher Rimma shows a graph ($y = x - 2$) without an equation and asks about the equation. (The graph has been re-created in Figure 8.4).

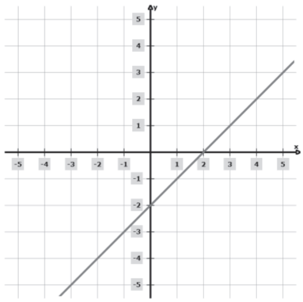

Figure 8.4: the graph related to Lesson 9:E

1. Teacher:	What formula will that one get? Would you like to Vidar?
2. Vidar:	Yes, it will become $y = 2 - \ldots$ no…
3. T:	Do you start with the *b* first?
4. V:	Yes, that is -2.
5. T:	Yes, then I'll write that. It should be minus 2 like that…

Rimma writes on whiteboard: $y = \quad -2$ (she leaves an empty space in between = and -2)

6. V:	**Yes, and equals $2x$ (LCv 9E)**
7. T:	Let's see. We increase by one on the *x*-axis, how much does it increase on *y* then?
8. V:	One
9. T:	So, the formula is…?
10. V:	$1 - 2x$

Rimma waits

11. V:	No, $x - 2$ [Lesson 9: E]

Vidar probably saw the b-value as the y-intercept and the m-value as the x-intercept. His suggestion of *equals* $2x$ (line 6) in front -2 proposes this. Then the equation would have been $y = 2x - 2$ which is in accordance with the internal logic of seeing the function as a line between two intercepts. This was, however, not regarded at all by Rimma. Instead she directed him to the increase, and got the right answer of *one*. Vidar persisted with $2x$ and combined it with the slope of 1 (line 10). After a few seconds of silence and waiting from Rimma by the whiteboard Vidar gave the right answer. In both Lesson 3 and 9, the LCv were categorised as disregarded because the content was not developed further.

In Lesson 5, on the other hand, a similar LCv was instead explored.

Excerpt 8.4.2.c

Teacher Ragnhild discusses learners' drawings of points (instead of graphs) on the whiteboard. Three points have been discussed and the last one, D, is now in focus.

1. Teacher:	Shall we try the last one then?

Ragnhild writes **D)** $\boldsymbol{y = -3x + 5}$ on the whiteboard and wipes out all the previous graphs in the coordinate system, but keeps the point **D** which has been drawn earlier in the coordinate system.

2. L1:	At 5, up there
3. T:	Five, yeah, exactly. You walk five steps up, up there [marks (0,5) with a point] and then it says it should be minus 3x.
4. Elias:	B**ut don't you read on the *x*-axis first? (LCv 5T)**
5. T:	<u>How do you mean?</u>
6. E:	Like it says, -3 + 5, that it starts at -3.
7. L1:	Well no! The start value is 5!
8. E:	Well, and I ... I//I thought…
9. T:	//hum, and <u>it's really important what you say</u>, **it is very common** to think that *okay, then it should intercept the x-axis, that it would go through -3 on the x-axis,* and this [points at -3x] has actually nothing to do with that at all, but how it will continue now. If we say that it increases a minute here (at the *x*-axis), what is this (y) then? What do you say, Joel?
10. Joel:	For each step you go to the right on the *x*-axis, you should go down three steps on the *y*-axis.
11. T:	Yep, if we think like this; we go one step further, times -3. That is, it will drop 3 steps. If we go one forward we go -3, one forward, another three minus, one forward, another -3. And so we get a line that looks something like this instead. [Ragnhild has drawn the graph while she was speaking]
12. Elias:	Aha!! [Lesson 5: T]

It is evident that Ragnhild was aware of the understanding of the function as a line between intercepts, as (in line 9) she referred to the 'very common' way of seeing the $-3x$ as the x-intercept. This also made a difference to Elias as he got a contrast to his own way of seeing negative slope by a more canonical way (lines 10 to 11). His 'aha' (line 12) was uttered with approval. In the study this is the only time the way of seeing the function as a line between intercepts is elaborated on.

Apparently, there are learners in at least three additional lessons that contribute questions indicating this way of experiencing the function. Elias is a learner that contributed to a lot of openings of DoVs throughout Lesson 5 and many of them regarded unusual aspects of functions and slopes.

In addition, there is one more facet shown in Excerpt 8.4.2c that I would like to highlight. In line 5, Ragnhild asked 'how do you mean' and in line 9 she encouraged

Elias and his contribution with 'it's really important what you say'. These two sentences indicate that Ragnhild sees the learner contributions as a resource for her own teaching. This aspect is only touched on here, but it will later be further elaborated on.

In the next section, Ragnhild, Elias, and Joel will return to the stage, in another lesson event, that preceded the above one.

8.4.3 A function as a single point

In the lessons in this study, it was not uncommon for students to express perceptions of a linear equation in the graphical representation as a point instead of as a line. In four lessons, learner contributions indicated that this way of experiencing the function was in play.

Table 8.4.3 Separation of function from a single point

Lesson: / Separation:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
of function from a single point	E[101] 5O			DIS 6K		C 12E	E 13D							

Lessons are numbered, LCv are named in order by letters in the alphabet.

DIS 6K:	*Disregarded LCv (K) in Lesson 6*
C 12E:	*Opened by a trajectory of considered LCv (E) in Lesson 12. An LCv in black means the LCv was an answer/comment to question.*
E 5O:	*Opened by a trajectory of explored LCv (0) in Lesson 5. An LCv in red means the LCv was initiated by a learner. This was done most often by a question.*

In a post-lesson interview, it became evident that a learner, Alvin from Lesson 6, saw the equation $y = 6x$ as a point, not as a graph, in the coordinate system. He used the coefficients as coordinates and distinguished that y stands for $1y$ and considered that this *point* $1y = 6x$ should be placed at (6,1), 'since $x = 6$ and $y = 1$'. An excerpt from the lesson event in Lesson 6 in which Alvin's LCv is disregarded follows:

Excerpt 8.4.3a

The learners have been working on a task, combining three graphs and their corresponding equations. The graphs and equations on the whiteboard are discussed by teacher Marita when Alvin raises his hand:

1. Teacher: Yes, Alvin?
2. Alvin: This feels… I think this is totally illogical.
3. T: Illogical?

[101] Also in 5P

4. A:	It feels like… it is one y equals 6 x, so it should be 6 at the x-axis and 1 at the *y*-axis, I don't understand//
5. T:	//you mean since it is 6x there? [points at $y = 6x$ written on the whiteboard]
6. A:	Yes, and if I had done the task, I would have written it like that. Now I know that it isn´t the case, but I don´t understand why not.
7. T:	x has no value there. That 6 has nothing to do with x, in that way. I hope it will become clear to you later in the lesson. I don't want you to look so unhappy.
8. A:	Ok [Lesson 6: K]

Alvin posed a question (line 4) but the way he saw the function was not taken into consideration by the teacher. Even though Marita tried to bring clarity to the Alvin's question (line 7), she only managed to conclude that *x has no value there* and *6 has nothing to do with x* in 6x. Evidently Alvin and Marita did not understand each other in this dialogue and neither was the *function as a point* made into a topic in the lesson sequence. The question was answered in a nice way, but the content of it was disregarded and no DoVs were opened in this sequence.

In Lesson 12 a similar learner contribution was considered when teacher Cecilia argued that graph A continues and is not placed in a single point, according to the learner contribution:

Excerpt 8.4.3b
Teacher Cecilia and her learners are discussing three graphs [A, B, C] and three formulas that are to be combined.

1. Teacher:	Does anyone have an explanation for how one can see this? Frederic!
2. Frederic:	Yes, A is placed on the positive side.
3. T:	You mean here? [points at the first quadrant] Okay...
4. F:	Yes, and the other two are placed at minus.
5. T:	But A does continue here as well. [points at the graph A in the third quadrant]
6. F:	Yes, but I was thinking of… I mean, at the *y*-axis… [Lesson 12: E]

Whether Frederic actually saw functions as points is not made clear in this sequence; he might just have been speaking of the y-intercept and using the terminology of *graph A* when addressing the y*-intercept* for A. Nevertheless, he said that [graph] A is placed at the positive side (line 2). Cecilia considered the content of the contribution and gave an argument against it (line 5); hence the contribution has been categorised as a considered learner contribution.

Seeing the function as a point, like Alvin in Lesson 6 and perhaps Frederic in Lesson 12, became the topic of discussion in Lesson 5, when learner contributions from Elias and three of his classmates were explored.

Teacher Ragnhild in Lesson 5 had written four equations on the whiteboard and asked the students to draw the graphs of the equations in a coordinate system. [(A) $y = 2x + 1$, (B) $y = 2x - 1$, (C) $y = x$, (D) $y = -3x + 5$]. Elias was the first student to contribute and in total three students sketched their answers on the whiteboard simultaneously. All of them marked points instead of lines in the coordinate system. Graph A was marked as the point (2,1), B as (2,-1), C as (0,0) and D as both (-3,5) and (3,5). In total, 5 points were marked, but after a discussion between two of the students, the point at (3,5) was erased.

Before presenting the lesson excerpt directly following this marking of points, I will clarify the internal logic and resemblance between Alvin's[102] ways of seeing functions as points and the way of marking the points by Elias. I will also highlight two differences. Alvin distinguished that y represents $1y$ and meant that this *point* $1y = 6x$ should be placed at (6,1), 'since $x = 6$ and $y = 1$'. He simply saw the coefficients in the equation $y = 6x$ as coordinates. With the same internal logic, Elias, and two of his fellow learners, marked their points in the coordinate system, using coefficients (the m- and the b-values) as coordinates.

(A) $y = 2x + 1$ is placed at (2,1)
(B) $y = 2x - 1$ is placed at (2,-1)
(C) $y = x$ is placed at the origin (0,0)
(D) $y = -3x + 5$ is placed first at (3,5) but after a short discussion between two learners, on the negative 3, instead at (-3,5).

There are two differences between the rationale of Alvin's way of seeing the points and the rationale way Elias marked points. First, the function Alvin was dealing with was proportional, and thus lacked a b-value ($y = 6x$); hence he used the coefficients of 1 in 1y as the y-coordinate and of 6 in $6x$ as the x-coordinate. Elias instead used the b-values as the y-coordinates and the m-values as the x-coordinates of the point. Secondly, which is revealed in how point C is marked, Elias did not discern the 1 in $1y$ and 1 in $1x$; they seem to lack coefficients and therefore they were probably marked at the origin, at (0,0). Apart from these two differences, the way of drawing functions as points with the coefficients/b-values as coordinates follows the same logic as Alvin expressed through his interpretation.

If we turn to the teacher's role during this lesson sequence, Ragnhild was just standing quietly aside, watching all these points instead of graphs been drawn in the

[102] who was interviewed after the lesson in order to understand the rationale of his contribution

coordinate system. The excerpt begins just when all four points have been drawn on the white board. Ragnhild asks Elias:

Excerpt 8.4.3c

1. Teacher:	Is it just one point or a whole line [refers to the point D in (-3,5)]?
2. Elias:	Well, it is a dot. It should be a dot, not like… I mean a dot.
3. T:	Ok

The teacher is standing aside, watching when a fourth student, Joel, comes up to the whiteboard.

4. Joel:	But that one [refers to the point C drawn in (0,0)] is not a function, it doesn't increase.
5. T:	How would you draw it then?
6. J:	Well, it should increase proportionally, like this [Joel draws a line in the air with a finger]. If x is one, then y would be one. If x is two, then y would be two.
7. Learners:	Just straight ahead// with an angle of 45 degrees.

Joel draws the correct graph for $y = x$

8. T:	Do you agree with what Joel says?

[Lesson 5: O]

The lesson continued with a discussion of what functions are and what they are not: a single point[103]. The remaining points A, B and D were discussed in relation to the graphs and to the context from which the lesson examples originated: cell phone subscriptions. The negative slope of D was discussed in relation to a task on cell phone subscriptions, and negative x-values were discussed in relation to minutes of talking on a cell phone.

In contrast to Alvin's disregarded LCv in Lesson 6, Elias's and his classmates' contributions in Lesson 5 were explored. In this lesson event, Ragnhild asked about the points drawn (line 1) but accepted Elias's answer (line 3). When Joel contradicted him by saying that a point is not a function (line 4), Ragnhild instead of quickly confirming the correct answer, asked Joel how *he* would draw the function (line 5). Still, after Joel's correct explanation, Ragnhild turned to the rest of the learners by saying: 'do you agree with what Joel says' (line 8). The content of Elias's learner contribution was made the topic of discussion; hence it was established as an explored LCv.

103 A function could be defined as one discrete point, but in these examples that is not the case.

8.4.4 Summary: function

Learner-generated aspects of function differed from teacher-generated aspects in a few ways. First of all, the learner-generated aspects showed broader perspectives on linear equations, such as known transitional conceptions (Moschkovich, 1998) of the function as *a line between intercepts*. Secondly, the detailed examination of a few cases have also revealed that learner-generated aspects might contribute to a deeper understanding of linear equations, namely understanding *why* the b-value can be seen as the y*-value* in the y-intercept, not just to fuse the b-value to the y-intercept itself.

Three cases have been closely examined: separation of why the b-value ends up at the y-intercept, separation of function from a line between intercepts, and separation of function from a single point. All three cases share some features; to be able to distinguish why the b-value is at the y-intercepts and to distinguish the function from lines between intercepts or from single points, one needs to discern the function as a relation and the two coordinates in every point in a coordinate system. All learners in the study were not offered the opportunities to discern these and other more unconventional aspects of functions. Results showed that the opening of these DoVs was greatly dependent on learner contributions, and moreover, on teacher attentions to these contributions.

8.5 Learner-generated aspects of slope

Many aspects of slopes might be evident for a mathematics teacher, like the slope being the same everywhere on a straight line or that slopes are not commonly measured using degrees, as angles are. In this study, it became apparent that these dimensions were not evident to all learners. In the same way as for the functions, the learner-generated aspects of slopes were unusual aspects[104]. For instance, learner contributions in three lessons suggested that slopes be seen as angles and in five lessons, the question of whether the slope is the same everywhere on a straight line was discussed. The latter aspect was initiated twice by teachers and three times by learners, as can be seen in Table 8.5. The enactments of learner-generated aspects will now be examined more closely. In Table 8.5, all the DoVs opened and LCv attended can be traced.

[104] For the teacher-generated aspects of slope one could turn to Appendix A, and to the different enactments of slope described earlier in this chapter. These aspects are almost all mainly generated by teachers.

Table 8.5 Learner-generated aspects of slope

1b. Aspects of slope mainly generated by learners:														
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of same slope on straight line		X	C 3W	E[105] 6A				E 14H				X		
between slopes and angles	DIS 5Q				C 15A			DIS 14J						
of the "reference axis to slope"		X[106] (4R)	E 3T	C 6H										
of several ways to determine[107] slope		X	E 3V	E[108] 6D		E 12F			X					
between x and y in negative slope		E 4D									C 9B			
of substitution of the direction of x	E[109] 5V													
of independency[110] of location for slope		E[111] 4W												
between y-value and slope		E 4H												
of 0 as slope of a horizontal line		C 4O												
of same increase/ slope as parallelism		C 4T				C 12C						X		

Lessons are numbered, LCv are named in order by letters in the alphabet.

X: *Opened by teacher without LCv*
DIS 5Q: *Disregarded LCv (Q) in Lesson 5*
C 15A: *Opened by a trajectory of considered LCv (A) in Lesson 15. A black LCv means LCv was an answer/comment to question.*
E 5V: *Opened by a trajectory of explored LCv (V) in Lesson 5. A red LCv means that the LCV was initiated by a learner and most often as a question.*

Table 8.5 makes the 10 learner-generated aspects on 28 occasions available for further consideration. As was the case with learner-generated aspects of *function*, the learner-generated aspects of *slope* also reveal unconventional, not always mathematically correct, understandings of slope. Of the 28 occasions these aspects were enacted, on only six occasions they were generated by the teacher alone. These aspects were therefore highly dependent on learner contributions but also, as can be seen in Table 8.5, on the exploration or consideration of these learner contributions. Two of the 22 learner contributions were disregarded. These were

[105] This LCv opens two DoVs
[106] Angelika discusses an LCv from the previous lesson, L3
[107] If the way of determining slope is varied and the slope is kept invariant, this DoV is opened; it is not enough to give only one example of rise over run.
[108] Also in 6F
[109] Also in 5R
[110] "in 2nd quadrant, slope becomes negative"
[111] Also in 4Y

both about slopes as angles. Now, let us examine a few cases of learner-generated aspects of slopes in order to shed some light on the figures in the table.

8.5.1 A reference axis for slope

If one is familiar with how the axes of a coordinate system are commonly depicted, that is, having positive increasing values to the right on the x-axis and upwards on the y-axis, as well as being familiar with both coordinates for every point and considering the slope as a relation between x and y, then "reference axis for slope" loses all meaning, since both axes are referenced simultaneously. However, this is the case once you have discerned these aspects whereas if the aspects are not discerned, it is not self-evident what the *steepness* of a steep slope refers to, when regarding different straight lines in a coordinate system.

Particularly in a Swedish classroom context, steepness is problematic, due to the fact that the Swedish word for slope [lutning], being the equivalent of "leaning", bears the connotation of referring also to the vertical axis, as in *the Christmas tree is leaning a lot*[112].

Sometimes the task in itself can restrict discernment; this will be described below. The first case described is from Lesson 3 in which a learner contribution (3T) was explored:

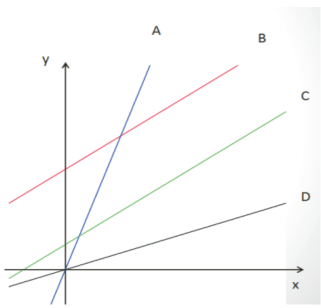

Figure 8.5: The four graphs displayed and discussed graphs in Lesson 3

Excerpt 8.5.1
Teacher Angelika has written four equations on the whiteboard: ($y = 0.5x$; $y = x + 1$; $y = x + 4$; $y = 4x$) to be combined with four graphs [A, B, C, D]. She asks the learners to work with this task. They work in pairs for two minutes.

1. Teacher:	Sorry to interrupt you… I am asking the question to Hannes and Egil…did you manage to combine any of them?

112 Julgranen lutar kraftigt

2. Hannes:	Yes, we combined $y = 0.5x$ with D (which is correct)
3. T:	Why with D?
4. H:	It lacks a b-value and that means it passes through the origin…so, then we saw that it is D.
5. T:	Well, precisely, but that one [points at A] also lacks a b-value?
6. H:	But that is placed at $4x$
7. T:	//But what does that mean?
8. Egil:	//It has a greater slope
9. H:	It has a greater slope
10. T:	Good, it has a greater slope so that is why $y = 0.5x$ must be D. That one has a smaller slope and it passes the origin. So, now we have also stated which one this is [$y = 4x$], haven't we? That must be…A, yes, precisely.
11. L1:	We had them the other way around, because we thought that 4 at x, that was further away.
12. T:	Aha, you mean that it gets further in that direction [points to the right]?
13. L1/L2:	Yes
14. T:	Good that you ask now. [T turns to everyone] The question I got now was like this: 'I think that this one [D] should be $4x$ and this one [A] should be $0.5x$ because it [D] gets further away'. Was that what you asked?
15. L2:	Yes
16. T:	When we talk about slope, we have to consider that this [points horizontally with her arm] is slope 0. The higher we get, the greater the slope [moves her arm upwards in front of the coordinate system along an increasing slope]. I will tell you later how to calculate the slope. It means that this one [$y = 0{,}5x$] does not lean as much as that one [$y = 4x$]. It does get faster to the right on the x-axis, but that is not the slope. Slope is defined in another way, which I will get to later.
17. H:	But you can see it in the way we did, can't you? If you move one step to the right and then up, you can in a way decide that it has a greater slope?
18. T:	Yes, exactly, that's it. That is a way to define slope, if we take one step in the x-direction, how many steps do we have to take up or down, that is our slope. And we will get to that. [Lesson 3: T]

Combining equations and graphs was one of the most common tasks overall in the lessons. In this sequence, two students managed to combine one of the pairs correctly (lines 1 to 2). Yet Angelika asked for a justification (line 3) and was not satisfied with the explanation (line 4), but kept on asking for further justification (lines 5 to 9). When everything seemed settled (line 10), an LCv revealed that two learners have seen the x-axis as the reference axis for slope (line 11), 'since D gets further away to the right'. Angelika made it clear that she had correctly understood how the learners regarded the slope and turned to the whole class and rephrased the LCv (line 14). Hence, the content of the LCv in line 11 was explored and two ways of seeing slope were contrasted (lines 16 to 18). Accordingly, "the reference axis for slope", was enacted as a DoV.

Another important aspect illustrated in this example is revealed in analysing the task as perceived by the learners (line 11). The learners contributed that 'we had them the other way around, because we thought that 4 at x, that was further away'. In the task, the difference between Δx for graph A and graph D is huge. Actually, that is what makes the graph D look like it 'gets faster to the right on the x-axis' as Angelika stated (line 16) or further away to the right. If one does not discern, for instance, both coordinates in every point, i.e. that there are x-coordinates everywhere, it might be easy to think that the x's are only on the x-axis. Then $4x$ is further to the right than $0.5x$. This way of seeing things resembles the 'seeing function as a line between intercepts' (in 8.4.2) described earlier, as that understanding also connected $4x$ to the x-axis. So, when Angelika created the picture with the four graphs, she drew four lines with the same length, but with different slopes, resulting in different Δx for the graphs. In combination with the Swedish word for slope, the picture probably induced a way of seeing the x-axis as "reference axis for the slope". However, without the picture, this optional aspect of slopes would not have been attended to at all. In this lesson, the learner contributions generally were explored to a great extent (see Table 7.5a), which in this case could also be seen as a protection against tasks that do not enable enough discernment.

A few days later, Angelika had the "same" lesson with another class, Lesson 4. In Lesson 4, none of the students contributed alternative ways of seeing "reference axis for slope", yet Angelika discussed this way of seeing slope:

> Angelika: in the other class when we did this, one student said this: *well, why couldn't A be 0.5x as it will get further away, it moves further away here.* And that person thought that the slope has nothing to do with how much it leans like this [shows slope with her arm], it is about how far away it gets and that is not the same thing. But when we're talking about slope we want to know how much it leans, therefore, this is zero slope [arm horizontally] and the higher up we get, the higher the slope is. So it is quite right that $y = 4x$ is A.
>
> [Lesson 4: 17]

From this I interpret that Angelika did not understand the learner contributions in Lesson 3 in the same way as they are analysed here. Still, this is an example of how an LCv was used as a resource for forming the content taught. In this case, an LCv from one class was brought to another since Angelika remarked on a comment from a learner in another class. Like most optional aspects, "the reference axis for slope", tends to vanish as soon as one has discerned and disregarded it.

8.5.2 Negative slopes

As negative slopes seemed to entail trouble for some learners of this study, one more case of slopes will be elaborated on. A positive slope, especially in the first quadrant, was elaborated on without much visible difficulty. In the second quadrant, things tended to change a little. If slope was instead negative, then for some learners, real concerns seemed to appear.

In many of the lessons, learners (and one teacher) had problems with negative slope. This does not imply that there weren't any DoVs opened or LCv attended to. One of the difficulties was to discern <u>what</u> is decreasing in a negative slope, the x-value or the y-value.

Excerpt 8.5.2a
A learner interrupts teacher Rimma (when she is already heading towards the next topic) to ask about the equation of the graph with negative slope just discussed. ($y = 1 - 2x$)

1. L1:	When you take -2, if the formula had not been there and I only had…how could I determine it?
2. T:	How should you reason then? Good! The 1 is easy. But here… before we have always increased. If I increase the *x*-value by 1, we have seen how much the *y*-value has increased by//
3. L1:	**//So you decrease x by one instead?**
4. T:	**Well, or I still think that if I increase the x-value by one, what happens to the y-value?** It decreases by 2. Therefore I write -2x. [Lesson 9: B]

The learner (in line 3) asked if the x-value was decreasing as the slope/m-value was negative. He did not seem to discern the slope as a relation between x and y, which was never enacted in Lesson 9 (see Table 8.2). Consequently, when negative slopes appeared in the lesson, it could also be x that was decreasing. Rimma distinguished that the x-value is still increasing, and that it is y-value that is decreasing, which leads to the negative slope (line 4).

Another similar case was found in Lesson 5, in which Elias did not manage to get Ragnhild to understand what he was asking about until it became evident that he saw the negative slope as a move in "the other x-direction". Teacher Ragnhild, once she understood Elias, in difference to Rimma in Lesson 9, meant that there is no such thing as "x-direction", any direction is fine; slope is about the relation to y.

Excerpt 8.5.2b

1. Elias:	It is easier if you know the *y*-value and maybe not minus or like, yes. Do you see what I mean?
2. Teacher:	No [laughter]
3. E:	Ok, forget it.
4. Learner 2:	Let's hear![113]
5. E:	No, but it is easier when you are to write these… no, it is weird if you don't know the x, should you always begin at zero then? I mean, in this example we begin at zero and up to five…
6. T:	Yes, exactly, that's a good thing you say, the number there [points at the *b*-value in $y = -3x + 5$] that is where we always begin in a way, at the *y*-axis.
7. Learner 3:	At the *b*-value
8. T:	The *b*-value, yes, and then it continues in different ways depending on the number which is multiplied by x here. Let's say it is $+2$, that means that we are to move upwards 2 for every minute or x that proceed.
9. Learner 4:	**So $-3x$ doesn't mean that we are moving backwards, I mean negatively?**
10. T:	**NO, it does not!** Instead imagine that we are constantly heading forward in x and then the *m*-value decides. If it is negative, it should go down, if it is positive, we go up.
11. Elias:	**When do we go the other direction then?**
12. T:	**We can go the other direction all the time really.** If we think like this if...we'll take a new line. We take y equals 2, no, we don't, we take y equals 1.5x minus 1. Where does it start?
13. L:	Minus one.
14. T:	Minus one…there [marks a point at (0,-1)], and so we can think like this… if we were to go forward now, one step forward, then we'd go one and a half step upwards. It would be the same if took a step backwards, like a step in the 'wrong' direction; we would decrease one and a half step down. So we can always think that we are backing or moving forward. [Lesson 5: V]

After some uncertainty, and then thanks to a fellow learner (lines 1 to 8), Elias got Ragnhild to understand what he asked, 'doesn't -3x mean that we are moving backwards?' (lines 9 to 11). Ragnhild (lines 12 and 14) focused on the relation and said that any direction is fine: the m-value determines the slope, but not the direction of x. As can be discerned in Table 8.5 this response from Ragnhild was

[113] One example out of very few in the study in which a fellow learner determines the trajectory for the LCv

categorised as another DoV opened compared to in Lesson 9 by Rimma. The learner contributions in both L9 and L5 were similar, as both concerned whether negative slopes imply negative "x-direction". Rimma separated the decrease of y from the decrease of x in negative slopes, whereas Ragnhild instead separated the substitution of "x-direction" by saying that 'we can go the other direction all the time really' and the example she contrasts with (line 14).

The last case of negative slopes generated by LCv that will be closely examined was enacted in Lesson 4, in which a learner had yet another way of experiencing negative slope. Teacher Angelika had just finished a discussion of a task in which the equation ($y = -2x + 6$) of a graph with negative slope was determined, when a learner raised his hand:

Excerpt 8.5.2c

1. Teacher:	Ah, a question!!
2. Learner:	How does the graph lean on the other side?
3. T:	How do you mean that it would lean?
4. L:	**Because it's negative, why don't you draw it on the other side?**
5. T:	You mean on this side [points at second quadrant]?
6. L:	Yes
7. T:	**What happens now is that you are confusing the slope with the coordinate system. You think that this is the negative part of the coordinate system, right?**
8. L:	Yes
9. T:	Slope has nothing to do with… I mean, <u>where</u> in the coordinate system you are, but it has to do with how the line looks like. It doesn't matter if this line is there, here or there [Angelika "moves" the line]. It still has the slope of minus two. So the slope, we don't determine it on the *y*- or *x*- axis and say ok, now I'm at the side where the slope is negative.
10. L:	Okay
11. T:	Did you understand what I meant there, please ask again otherwise.
12. L:	I have missed some lessons, so therefore...
13. T:	It's totally cool, surely several others thought about this as well. [Lesson 4: W]

From the learner's question about the graph on the other side [second quadrant] and Angelika's assessment of how he experienced it (lines 2 to 6), she concluded that he was confusing the (negative) slope with the coordinate system (line 7). By becoming (or already being) familiar with the idea that there are "negative sides" in which slopes are behaving differently in a coordinate system, she could contrast that idea (line 9). It is not totally clear whether the learner understood Angelika's explanation as some hesitation is apparent in the end of the dialogue (lines 10, and 12). Yet, the way Angelika finished the dialogue: 'it's totally cool, surely several others thought about this as well' shows that learner contributions are accounted for on a regular basis in her teaching.

In five of the lessons (L1, L4, L5, L9, and L12), learners raised questions about negative slope that indicate that slope was not experienced as a relation between x and y. The questions concerned: '*what* is decreasing in negative slope', 'what happens on the negative side (the second quadrant)', 'is there a negative x-direction?' The relation between the variables might be even more important when it comes to negative slopes, as these are more difficult to see as increases, which is actually the only denotation of slope that has been enacted in four of the above-mentioned lessons (L4, L5, L9, and L12). In Lesson 1, no denotation of slope is enacted. This has been described earlier and is shown in Table 8.2. The implication is that enacting *slope as a relation* is particularly important when it comes to negative slopes. In none of the lessons in which slopes were enacted as a relation between the variables[114], were there learner contributions indicating that a negative slope has been mixed up with a "negative x-direction".

8.5.3 Summary: slopes

Learner-generated aspects of slopes were to a high degree aspects that extended the enactment of slope in this study. Two cases with several examples from different lessons have been closely examined: the separation of "reference axis for slope" and the separation of negative slopes. Both cases revealed unconventional ways of seeing slopes, and different ways of enacting aspects of slopes as a result of learner-generated aspects. In comparison to learner-generated aspects of *function*, learner-generated aspects of *slope* were to a lesser extent disregarded in the study. However, similarly to the aspects of functions, these learner-generated aspects of slope were highly dependent on teachers' consideration or exploration of them. And likewise, the amount of teacher attention to learner contributions was not evenly distributed: 17 of the 22 occasions in which learner contributions were attended to, occurred in a third of the lessons: in the explored-LCv lessons. Diverse learning opportunities emerged in different lessons. The results of both examinations suggest that slopes enacted *as a relation* between *variables* might prevent ways of seeing negative slopes

[114] L3, L6, L15, L13, L14

as "negative x-directions" or "negative on the other side of the y-axis" as well as seeing only one axis as the reference axis for slope.

8.6 The characteristics of learner-generated aspects

The optional aspects of linear equations in the study were generated mostly by learner contributions. *Do you have to place* $3x$ *first? Can you write it like* $y = 5 + 3x$ *instead?* This is an example of an optional aspect of linear equations. The teacher in this example had in the lesson event used only the most common denotation of linear equations in the algebraic form. A learner questioned this by opening an optional aspect, namely creating a variation in the order of terms in the equation. Anyone who teaches linear equations knows that the order of terms is irrelevant, but that there is a traditional way to write the equations with the mx-term first. In this study, it was much up to learners to reveal optional aspects such as this one. Learner-generated aspects of function and slope were elaborated on in detail above. A final investigation of the 29 most evident optional aspects enacted led to Table 8.6:

Table 8.6 Optional aspects of all properties

29 optional aspects: mixed properties of linear equations														
	Explored-LCv lesson type					**Mixed-LCv lesson type**				**Considered-LCv lesson type**				
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of letters used for variables			X					C 14I		X	C 9D			
of order of terms (3x + 5 = 5 + 3x)	X	C 4A	S 3A							C 1F	DIS 9A			
of order of terms: (mx + b = b + mx)			S 3O											
of commutativity (of 2n and n2)								C 14A						
of placement of y in equation		C 4B												
of m-value from mx-term	DIS 5L	E 4J				C 12I								
of function from a line between intercepts	E 5T		DIS 3F	DIS 6Q							DIS 9E			
of function from a single point	E[115] 5O			DIS 6K		C 12E	E 13D							
of function from an end-point of graph		C 4F		E 6A										
of the "reference axis to slope"		X[116] (4R)	E 3T	C 6H										
between constants and variables			C 3L	C 6O										
of same slope on straight line		X	C 3W	E 6A				E 14H				X		
of dependency of variables	DIS 5M			E 6C	X			DIS 14C						
of infinity of graphs[117]	C 5W		C 3N	C 6N			S 13G		X					
of dimensions of a coordinate system	C 5C													
between slopes and angles	DIS 5Q				C 15A			DIS 14J						
between first term and term without x			E[118] 3G											
between linear/ non-linear graphs		S 4C	C[119] 3Y				X			C 1E				

[115] Also in 5P

[116] Angelika discusses an LCv from the previous lesson, L3

[117] The domains and ranges of graphs are necessary aspects, but when the graph is infinite, this is an optional aspect easily taken for granted.

[118] This is a self-explored LCv by a learner, not established by teacher

[119] Also in 3M

of invisible m-value of 1	X	E[120] 4L				X				X	X			
of invisible b-value of 0	X	X	DIS 3K			X						X	C 10C	
of invisible[121] plus sign		X	E 3D					X						
of invisible[122] multiplication sign	X							C 14B						
between Δx of 1 and the squares of grid		X	X											
of intercepts from grids						E 12J								
of x-direction from x-axis						E 12K								
of designation of axes	C 5A							C 14C						
between smileys and parentheses	X													
between decimals and coordinates	C 5B											X		
between decimal/ coordinate comma	X													

Table 8.6 provides information about how optional aspects were enacted: by teachers solely (X), by attended learner contributions (for instance C 5B) or by the initiative of learner (for instance C 3N). The 29 most obvious optional aspects in the study were initiated[123] altogether 82 times. However, they were enacted solely by teachers on only 24 of these 82 occasions. Hence, to a great extent (more than 70 % of the occasions) learners were involved in the enactment of optional aspects, either as initiators or co-constituters. Furthermore, the optional aspects were to a high extent (65 %) initiated in the explored-LCv lessons, compared to in the considered-LCv lessons (15 %). Additionally, of the 24 teacher-generated optional aspects, 10 were enacted in the explored-LCv lessons, compared to 6 in the considered-LCv lessons. This implies that although the teachers sometimes enacted optional aspects, these were more frequently enacted in lesson types in which learner contributions were generally explored. However, it can be noted that there are also lessons in which these 29 optional aspects are almost not enacted at all. In L7 none of these optional aspects are enacted, and in L15, L2 and L10 only one or two are enacted in each lesson.

120 Also in 4U
121 For instance (+3000)
122 The examples of course vary in different lessons. Could be 2n, mx, 3x etc.
123 This includes both the enacted and the disregarded aspects.

Table 8.6 also provides information of which optional aspects that teachers do open. About half of the teacher opened optional aspects regard invisibility of either signs or numbers of 1 or 0, such as the "invisible" m-value of 1 in the algebraic representation. Other optional aspects concern the separation between facets of linear equations that are not obviously separated, such as: the intercepts from grids in the coordinate system, the scaling of axes from the squares of the grid, the constants from variables, the m from the mx-term, which of the variables are dependent on the other, and many others.

In summary, to a great extent, the optional aspects of linear equations were generated by learner contributions in the study. Furthermore, the majority of them were enacted in the explored-LCv lesson type.

8.7 Answers to the research questions

The aim with this study was to gain deeper knowledge on relations between interactions and learning opportunities emerging in mathematics instruction. To examine that, detailed qualitative analyses of all dimensions of variation opened in 14 lessons were conducted. The results of the analyses displayed great differences in the learning opportunities that emerged. These differences were related to how the learner contributions in the lessons were attended to. The relations between learning opportunities and attentions to learner contributions will now be described by answering the two research questions.

What do teacher attentions to learner contributions in instruction imply for the learning opportunities of linear equations that emerge? (RQ 1)

The 14 lessons were categorised into three different lesson types depending on the trajectories for learner contributions: *explored-LCv lessons*, *mixed-LCv lessons*, or *considered-LCv lessons*. These trajectories were almost exclusively established by teachers' different attentions.

An analysis of the 289 openings of DoVs exposed great differences between lessons in both the number of DoVs opened and, more importantly, *what* DoVs were opened. The differences were related to the lesson type, i.e. how learner contributions *in general* were established in the lessons.

Regarding fundamental aspects of linear equations, as slope and function, the results show that in *explored/mixed-LCv lessons*, other learning opportunities emerged compared to *considered-LCv lessons*. The differences regarded both the presence of and the kind of aspects enacted. Table 8.7a provides further details.

Table 8.7a: Slopes and functions enacted in different lesson types

Lesson type / Aspect	Explored/Mixed-LCv lessons	Considered-LCv lessons
Function	*Function enacted:* by different representations as relationships between x and y	No enactments of function
Slope	*Slopes enacted:* as increases of y per x analytically as rate of change or as a relation	*Slopes enacted:* visually not at all

Functions as well as slopes were enacted differently in different lesson types. Regarding the enactment of *function*, the differences between lesson types were found to be extensive. In the *explored/mixed LCv lessons* several DoVs regarding function were opened, namely enacting functions by different representations, and/or enacting the functions as a relation between (sets of) x and (sets of) y. Also DoVs like the domains of functions and variables in functions were opened. In contrast, in the *considered LCv lessons*, not a single dimension of variation regarding function was opened. This means that the learning opportunities for function as a concept were not enacted at all in the lessons in which learner contributions were mainly considered.

Regarding the enactment of *slope,* in *explored/mixed lessons* slopes were enacted as increases of y per x and/or analytically as rates of change or relations between x and y, whereas in *considered LCv lessons*, slopes were enacted visually, if at all enacted[124]. The learning opportunities that emerged were related to the general way of attending to learner contributions.

As many learner contributions *were* explored in the study, despite great differences between lesson types, the next research question was possible to answer.

What do learners contribute to the enactment of linear equations? (RQ 2)

The comparison of learner-generated and teacher-generated DoVs displayed great differences. Teachers mainly initiated aspects of linear equations like the separation of b-values as y-intercepts and the fusion of slopes and y-intercepts with the equation of a straight line, although the aspects were often enacted jointly with learners. Most of the teacher generated aspects were necessary aspects of linear equations.

Also the aspects mainly generated by learners had some common characteristics. First of all, learner contributions revealed alternative ways of seeing linear equations. Such examples are seeing the function as a single point, seeing the x-axis as a reference axis for slope, seeing the function as a line between intercepts, and seeing negative slopes as something occurring at the "other side of the y-axis". All these alternative ways were challenged in the lessons when explored.

Secondly, learner generated aspects were to a great extent optional aspects of linear equations. Examples of these are: the separation of coefficients from intercepts[125], and the order of terms in the equation. Additionally, optional aspects

[124] Two exceptions to this exist, see earlier in Chapter 8.

[125] This is the rationale behind seeing the function as a line between intercepts, and thus using the coefficients as intercepts.

as the linearity of linear equations, the two dimensions of two-dimensional coordinate systems, and the separation between the grids of a coordinate system and intercepts were greatly dependent of learner contributions to be enacted. All these aspects share the characteristic of being optional aspects, and are therefore difficult for the learners to discern if taken for granted and left in the background in teaching. However, not all optional aspects were learner generated; some optional aspects were also generated by teachers. Examples are: the variation of letters used in equations, the separation of parenthesis from smileys, and decimal commas from coordinate commas. Moreover, many of the "invisibles" in mathematics were also generated by teachers, as the invisible slope of 1 ($y = x + 3$) or the invisible b-value of 0 ($y = 2x$). Other optional aspects concern the separation between facets of linear equations that are not obviously separated, as the intercepts from grids in the coordinate system, the $\Delta 1$ from the squares of the grid, the constants from variables, the x from the mx-term, which of the variables are dependent on the other, and many others. Regardless of this, to a great extent, the optional aspects of linear equations were generated by learner contributions in the study.

Thirdly, a detailed case was shown in which learners and a teacher jointly generated a development of common ways to treat the content in the study. This case concerned the separation of b-value as the $\underline{y\text{-value}}$ in the y-intercept in contrast to the b-value *as* the y-intercept. In the former case, more profound learning opportunities emerged as the question of *why* the b-value can be seen as the y-intercept was elaborated on. Table 8.7b provides a summary of teacher- and learner-generated aspects.

Table 8.7b: Teacher and learner generated aspects enacted

Teacher-generated aspects	**Learner-generated aspects**
Teachers had the main impact on the emergence of learning opportunities for necessary aspects of linear equations.	Learner contributions generated aspects that indicated alternative ways of seeing linear equations.
	Optional aspects of linear equations were enacted mostly as a result of learner contributions.
	Learner contributions generated aspects that developed common ways of denoting concepts.

9 Conclusions and discussions

The results from this study suggest that both learner's contributions and teacher attentions to the contributions played important roles for the learning opportunities that emerged. Here, the results will be discussed in relation to earlier research.

Different learning opportunities emerged in different lesson types. These differences were explained from two perspectives in the study. Firstly, the learner-generated aspects of linear equations differed from the teacher-generated ones. Secondly, the results showed that teachers' general attention to learner contributions, i.e. the lesson type in this respect, was related to the learning opportunities that emerged. Table 9 is a way of considering the two research questions simultaneously, in order to draw conclusions related to the aim of the study: to gain deeper knowledge on relations between interactions and learning opportunities. However, the results will first be discussed separately (in Section 9.1 and 9.2) in relation to earlier results from the mathematics education research field. This is followed by a discussion of other results in the study, critical reflections, and implications for theoretical development, for further research, and for practice.

Table 9: Content enacted in relation to lesson type and to teacher-/learner-generated aspects

	Lesson type	
9.1	**Explored/Mixed-LCv lessons**	**Considered-LCv lessons**
Teacher-generated aspects	**Slope was enacted:** as increase of y per x analytically as a rate of change **Function was enacted:** by different representations as a relationship between x and y	Slope was not enacted at all Slope was enacted visually Function was not enacted
9.2		
Learner-generated aspects	Alternative ways of seeing linear equations were enacted	-
	Optional aspects were enacted	Few optional aspects were enacted
	An aspect that developed common ways of denoting a concept in the study was enacted	-

9.1 Qualitatively different learning opportunities

The differences in learning opportunities for linear equations were shown in this study to be related to differences in interaction. Concepts such as slope and function were enacted in qualitatively different ways in lessons in which learner contributions were explored compared to the lessons in which they were not explored. These differences will now be related to earlier research.

9.1.1 Slope

The results from this study show that slope was enacted in three qualitatively different ways, and this can be seen in Table 9. In some lessons, graphs (or in some cases lines) were enacted visually as hills, and in these cases slope as a property of a function remained invisible. In other lessons, slope was enacted as increases of y per x and/or analytically as rates of changes or explicit relations between x and y. As was described in Chapter 8, these different enactments of slope were strongly related to lesson type, namely to how learner contributions were generally attended to in the lessons. This suggests that whether learner contributions in instruction are explored or not, has implications for the quality of the learning opportunities of slope.

Zaslavsky et al. (2002) distinguish between understanding slopes by a visual or by an analytical approach. By the former, slopes are perceived as a property of the graph, i.e. the steepness of the line, whereas by the latter, slopes are understood as property of the function, i.e. the rate of change in one quantity relative to the change in another quantity, where the two co-vary (Lobato & Bowers, 2000). The visual approach is closely related to the well-documented conception of the graphical representation of a function called the *iconic interpretation of graphs* (see elaboration in Chapter 3). This conception builds on the inability to treat a graph as an abstract representation of a relationship but instead seeing the graph/slope as a literal picture (e.g. Monk & Nemirovsky, 1994). When slope is perceived visually, the distinction between slope and steepness is not clear. Slope is seen as the steepness of a line. For instance, by an iconic interpretation of the graph, a negative slope can represent that someone walks back or down a hill (Schoenfeld, Burkhardt, Pead, and Swan 2012).

Most of the earlier studies on this matter are detailed studies of how learners perceive slope. However, they all emphasise the importance of how the concept is treated in teaching. For instance, Zaslavsky et al. (2002) argue for less sloppy language concerning slope and they promote the distinction between visual slope (the slope of a line) and analytic slope (the rate of change of a function) to enhance understanding of slope.

In one of the lesson types in this study in which no learner contributions were explored (considered-LCv type), slope was enacted mainly visually (Zaslavsky et al. 2002). This implies that graphs (lines) were enacted as hills, and slopes were discussed as uphill/downhill. Even if the teachers probably do not have an iconic interpretation of graphs (Monk & Nemirovsky, 1994), the enactments of slope apply to this interpretation. In the two other lesson types in which learner contributions were explored (explored-LCv type and mixed-LCv type), slope was enacted analytically (Lobato & Bowers, 2000; Zaslavsky et al. 2002) as a rate of change or a relation between sets of x and sets of y and/or as an increase of y per x. Having a visual approach to slope is definitively a disadvantage for further learning of functions. Lobato and Thanheiser (2002) strongly argue that instructional activities should help students to cope with complexity– such as the analytical approach to slope – rather than avoiding it. Against the background of earlier research it can be concluded that richer learning opportunities emerged in the lessons in which learner contributions were generally explored.

9.1.2 Function

The results show that the concept of function was not enacted in all lessons, even though graphs were worked on. In fact, the concept of function was not enacted at all in the considered-LCv lessons, whereas the concept was enacted at least once in all the other lessons. In most explored-LCv lessons, the concept of function was enacted several times and mostly as a relation between sets of x and sets of y. As there were no exceptions, the overlap between enactments of function and lesson type was stronger compared to the enactments of slope. However, in contrast to the concept of slope which was enacted in all lessons but one, albeit in qualitatively different ways, the concept of function was clearly either enacted or not. This makes comparisons a bit harder as there might be rational reasons for not enacting the concept of function in the introductory lesson on linear equations. Nonetheless, what can be said is that in lessons in which learner contributions were explored, functions were enacted as relations.

Earlier research (e.g. Bell & Janvier, 1981; Even, 1998; Leinhardt et al., 1990) has argued for a distinction between *a pointwise* and a *global* approach to functions (see elaboration in Chapter 3). Dealing with a function pointwise involves operating with its local properties, which includes for example plotting, reading or dealing with discrete points. The global approach embraces looking at a function's behaviour, for instance by sketching its graph, or finding an extreme point of a graph. Even though the importance of both approaches has been stressed in many studies, traditional instruction has been criticised for having an overemphasis on the pointwise approach (Bell & Javier, 1981). When both approaches are considered to be important, it is specifically the flexibility in using both that is emphasised. Overemphasising a pointwise approach in tasks, curricula and teaching might result in functions and graphs being seen as isolated points rather than objects (Leinhardt et al., 1990).

The results of this study show a strong overlap between the enactment of the concept of function as a relation and the explored-LCv lesson type. Enacting the concept of function as a relation between sets of x and sets of y involves a global approach to functions. Accordingly, it is fair to conclude that the learning opportunities for the concept of function that emerged in these first lessons of linear equations are richer for the explored-LCv lessons.

In conclusion, both the concept of slope and function were enacted in different ways in lessons in which learner contributions were generally explored compared to lessons in which they were not. In light of earlier research on the understanding and the teaching of these concepts, it is concluded that the learning opportunities for slope and function that emerged in explored-LCv lessons were of higher quality compared with the learning opportunities in considered-LCv lessons. In these 14 introductory lessons on linear equations, the exploration of learning contributions implies richer learning opportunities.

9.2 The potential in learner contributions

Research question 2 concerns the qualitative differences between teacher-generated DoVs and learner-generated DoVs. The results showed that there was a potential in learner contributions for what learning opportunities that emerged. This potential was described in detail in Chapter 8 and three facets will now be discussed.

9.2.1 Learners generate alternative ways of seeing linear equations

In contrast to misconceptions in science, which often stem from daily observations of real-world events (e.g. Mortimer & Scott, 2003), misconceptions about linear equations are intertwined with previous experiences from formal learning (Leinhardt et al., 1990). When a learner contribution indicated an alternative way of seeing in this study, and this contribution was explored, the result was often that many different aspects were enacted directly after. For instance, when the learner contribution of *the function as a single point* was explored in Lesson 5, nine different DoVs were enacted immediately afterwards. Examples of these are: *separation of the line as points*, *separation of infinity of the graph,* and *separation of negative slope from negative x-direction*. The explored learner contributions that indicated alternative ways of seeing various aspects generated several openings of DoVs, both by the initiative of teachers and of learners. Earlier research has shown many times that alternative conceptions are quite resistant against teaching (e.g. Mortimer & Scott, 2003). However, several examples from this study show teachers' and learners' joint efforts to challenge alternative ways of seeing. The point is, however, that the contributions of alternative ways of seeing did not bring disorder or confusion into the discussions. On the contrary, they worked as counter-examples. In several lesson events, both teachers and learners participated in the search for such contrasts.

The results have also revealed the internal logic (Smedslund, 1970) of the vast majority of the learner contributions regarding linear equations. Even though some contributions at first sight seemed to be superficial errors, the analyses showed that in all cases but one, it was possible to find the internal logic of them. Consequently, I suggest that alternative ways of seeing the content should be considered as more than just misconceptions or errors by learners. This is in line with the arguments by Moschkovich (1998) about *transitional conceptions*, i.e. common conceptions that show the complexity of the content. Many of the alternative ways of seeing in this study indicate that they emanate from transitional conceptions, which are both well-documented and shown to be common.

9.2.2 Learners generate optional aspects

Conceptions are in a phenomenographic sense constituted by the discernment of different aspects. This includes both necessary and optional aspects (Marton, 2015). In a few of the lessons, some of the learner contributions were explored to the extent that they could be concluded as alternative ways of seeing the content[126]. For many other learner contributions this was not possible. In the latter cases, the analysis focused on the enactment of necessary and optional aspects, i.e. the opening of different DoVs.

[126] For instance, Alvin's way of seeing the function as a single point was found in an interview after Lesson 6. The same alternative conception by Cornelis was extensively explored in Lesson 5. However, not all contributions were as deeply investigated.

Most of the optional aspects are easily taken for granted once they have been discerned. Aspects such as the designation of axes, the order of terms in an equation, the placement of y in the equation, and the letters used for variables are probably not aspects commonly thought of in the planning of a lesson. Furthermore, optional aspects such as "invisible" slopes[127] or "invisible" b-values[128] are probably not often illuminated in lessons. Nonetheless, these optional aspects might become obstacles to learning if they are not attended to, and the learners in this study were definitely eager to question these taken-for-granted aspects when allowed or encouraged.

Optional aspects of linear equations were not frequently initiated by teachers; instead they were shown to be vastly dependent on the contributions generated by learners. Learners were involved in more than 70 % of the occasions in which optional aspects were enacted. Moreover, of the 12 teacher-generated optional aspects, eight were enacted in one of the five explored-LCv lessons. This implies that despite the fact that the teachers sometimes initiated optional aspects, they were more frequently opened in lesson types in which learner contributions were generally explored. In other words, the teachers in this study who did enact optional aspects were the same teachers that also explored LCv. Furthermore, there were lessons in which optional aspects were not enacted at all, and these were all considered-LCv type lessons.

What would have happened with the enacted learning opportunities if none of the learner contributions had been explored in the lessons? Well, a specific lesson type in the study answers that: the considered-LCv type. The answer is that the vast majority of all optional aspects of linear equations would not have been enacted. Optional aspects are not often highlighted in textbooks for mathematics; instead they are often taken for granted. For many of the learners in the study, the enactment of optional aspects of linear equations is probably a necessary condition for learning. In any case, without them the learning opportunities that emerged in the lessons would have been poorer.

The enactments of optional aspects are not only dependent on learner contributions, but also determined by the attention to learner contributions from teachers. This attention has been shown to either enable or hinder the enactment of optional aspects. The vast majority of all optional aspects of linear equations would not have been enacted in the lessons if learner contributions had not been attended to by the teachers.

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[127] Compare $y = x + 2$ with $y = 1x + 2$

[128] Compare $y = 3x$ with $y = 3x + 0$

Optional aspects are perhaps not always mathematically motivated, but this study suggests that they certainly are pedagogically motivated. So many learner-generated aspects regard optional aspects, and this shows that many learners discern other aspects than the teachers, in general.

9.2.3 Learners contribute to the development of instruction

Examining learner contributions closely and acknowledging the internal logic of them rather than regarding them as just errors could also advance our knowledge about what is taken for granted in instruction about linear equations. This could enhance the development of mathematics instruction.

The most detailed example of this, concerns the common teaching of the b-value as the y-intercept in contrast to the b-value as the y-value at the y-intercept. As this aspect was present in 13 of the 14 lessons in the study, it is reasonable to consider the aspect an important one. In most lessons the b-value was enacted *as* the y-intercept, which did not reveal that it is the y-value at the y-intercept that equals the b-value, not the whole y-intercept. The x-value (zero) is taken for granted, both in the graphical and the algebraic representation. There is also some educational research where the x-value of zero is taken for granted and the y-intercept of a graph is expressed as the b-value in the equation (e.g. Leinhardt et al., 1990). This taking-for-granted of the x-value of zero is criticised by Lobato and Bowers (2000), who argue for the clarification of the b-value. "The b-value as the y-intercept" is also common in Swedish mathematics textbooks[129].

In this study, two learners in different lessons contributed to a potential reinforcement of students' understanding of *why* the y-intercept can be seen as the b-value. One of the contributions was disregarded; hence no learning opportunities emerged for that aspect in the lesson. The other contribution was explored, and – in the *only* lesson of the 14 – the teacher and a few learners jointly constituted the b-value as the y-value at the y-intercept. Hence, this aspect was offered as an opportunity for learning to the learners in that class. In 12 of the other 13 lessons, the b-value was enacted as the y-intercept. In one lesson, it was not enacted at all.

Failure to illuminate the x-value, which was the case in 12 of the 13 lessons in which the b-value was enacted *as* the y-intercept, could lead to an acceptance of the b-value as the start value or the y-intercept, without any understanding of why. This could also lead to seeing the x-intercept as the *m*-value, as in fact several

[129] For instance: Formula 9 (Gleerups), Matte Direkt (Bonniers).

learners expressed in the study. Seeing the x-intercept as the m-value has earlier been found as a conception in several studies (Kerslake, 1981; Moschkovich, 1998). In conclusion, learner contributions could also lead to the development of how concepts are commonly treated in mathematics instruction.

9.3 Conclusions

The qualitative differences in learning opportunities for linear equations were shown to be related to differences in interaction in the studied lessons. Firstly, learners and teachers generate different aspects of linear equations. In the lessons in which learner contributions were explored, many optional aspects and alternative ways of seeing the content emerged. This was not the case in the lessons without the exploring of learner contributions. Secondly, the results also showed that teachers' general attention to learner contributions, i.e. the lesson type in this respect, were crucial for the learning opportunities that emerged. Lessons in which learner contributions were explored offered richer learning opportunities compared to the lessons in which learner contributions were not explored. Aspects like slope and function were enacted in deeper and broader ways compared to lessons in which learner contributions were not explored. Two overall conclusions drawn from this study are: that there is a cost of learner silence for learning opportunities, and, furthermore, that the quality of instruction benefits from exploring learner contributions.

Both listening to and using learner contributions in instruction seem like good ideas. But does it come at a price? Will other aspects be suppressed, such as the necessary aspects of a concept? No, this study suggests otherwise, as the lessons in which learner contributions were explored were the same lessons in which the necessary aspects were also enacted in more qualitative ways.

9.4 Using learner contributions as a resource

Do learner contributions work for some teachers as a resource for instruction? Much research has been conducted about teachers' decision making in teaching. Many of the lesson events in this study could contribute to that discussion. The results showed a clear overlap between lesson types which offered richer learning opportunities and lesson types in which learner contributions were generally explored. In addition, there is an indication of another overlap: the teachers who explored learner contributions in the study are the same ones who generated optional aspects. This is in sharp contrast to the lesson types without any explored learner contributions, in which very few optional aspects were enacted. Furthermore, in some of the lessons it was not unusual that the teachers encouraged learners to contribute a wide range of aspects of linear equations and also the contribution of wrong answers was encouraged. These lessons belong without exceptions to the explored-LCv lesson type. This indicates that some teachers see and use learner contributions as a resource for instruction. The learner contributions function as an assessment tool to direct the path of the lesson. The fact that teachers use learner contributions as a source of information as to how learners see different concepts and procedures in mathematics align with findings from Al-Murani and Watson (2009), who discuss learner-generated variation as a source for teachers' decisions about which aspects of linear equations to open.

Also, a few examples in the study suggest that learner contributions were used for expanding the learning opportunities beyond the horizon of the teacher. Only two teachers in the study participated with lessons in two different classes; hence there is no excess of examples in which a disregarded learner contribution in one class is explored by the same teacher in a subsequent lesson with the same or other learners. However, one such existing example is when Angelika in Lesson 4 presents alternative ways of seeing "the x-axis as a reference axis for slope" to her students[130]. She explains the rationale behind this to the learners in Lesson 4; in this way she also creates a contrast to the more conventional way of seeing slope as a rate between co-varying x- and y-values. This was, however, not elaborated in the same way in Lesson 3 the day before, in which the learner contribution about "the x-axis as a reference axis for slope" originally arose. Therefore, this example can be seen as a teacher's way of developing the learning opportunities in new lessons with the input from earlier lessons.

MacGregor and Stacey (1997) proposed, in their large-scale study of 2000 pupils, that the origins of pupils' misconceptions need to be understood in order to improve the teaching of algebra. They point to the importance of research on students' ways of understanding and how this can be utilised in instruction. However, in this study the example with Angelika shows that also teachers can generate such knowledge and use it in teaching.

9.5 Critical reflections

The generalisations that can be made from a study rely on the design. Different kinds of research designs have different kinds of generalisation problems (Larsson,

130 This case is described in detail in sub section 8.5.1

2009). The generalisations that can be made also rely on the claims themselves. Methods, findings and the theoretical frame will now be discussed in light of generalisation, reliability and validity.

9.5.1 Methods

The selection of participating teachers and classes was not made randomly but with the intention of maximising variation (earlier described in Chapter 6). This means that the aim was to cover a variation of qualitatively different cases of content interaction in instruction. Moreover, it implies that the uncommon cases are as important as the common ones (Larsson, 2009). This is in sharp contrast to a representative sample of cases, which quantitative studies are most often based on. The analyses showed that as well attention to learner contributions as the learning opportunities that emerged varied a lot, which implies that a wide range of the phenomenon studied has been encountered. However, there are no possibilities from this study for making generalisations such as how common different kinds of attention to learner contributions are.

Furthermore, the great difficulties in finding participating teachers (earlier described in Chapter 6) indicate that the 14 lessons in this study probably are not very illustrative of typical Swedish mathematics lessons of today. My interpretation is that the 12 teachers that did participate are more confident and open compared to many of the about hundred other teachers that received the request of participation. However, the only bearing this has to the validity of the results is that caution should be taken not to discuss these lessons as typical.

The *inter-rater reliability* (Cohen, Manion, & Morrison, 2010) of the analysis would have improved if another researcher's analysis of the lesson events could have been compared to. Due to the extensive work load that this would have entailed, it was not done. Instead the methods of analysis have been described in considerable detail. Many of the analyses throughout the entire process of analysis were also discussed in different research seminars. I have also limited the analysis to episodes of whole-class instruction. Other forms of teaching would possibly reveal other ways of using learner contributions and highlight various ways of exposing how learners perceive different mathematical concepts.

9.5.2 Findings

It is probably quite safe to claim that the four trajectories for learner contributions identified here could be found in other lessons as well. However, the age of the learners probably plays a central role for the attentions to the contributions. The age of the learners was 16–18. This is an age when most learners already have an idea of what teaching and learning of mathematics is, as many of the socio-mathematical norms (Yackel & Cobb, 1996) are established by then. With younger learners, probably both contributions and attentions to these would be different.

Also, there might be great differences if the lessons are not, for instance, introductory lessons. In this context I propose that some caution has to be taken of the conclusions concerning the enactment of functions. The concept of function was not enacted at all in the considered-LCv type lessons. Nevertheless, as all the lessons were *introductory lesson* of linear equations, and nothing says that function has to be introduced in the first lesson, it could perfectly well be established later on. Compared to the enactment of slope, which was enacted in all lessons but one, the findings regarding functions are weaker.

Another facet that delimits the conclusions is the fact that because of the data load, the whole-lesson perspective was omitted in the analysis. This has left a comparison of stringency and pace of introducing linear equations between lessons out of the scope. As this perspective would have been more holistic and in a sense more true to the diverse intentions of lessons, that could certainly have contributed to deepening the knowledge of what teacher attentions to learner contributions imply for learning opportunities. Moreover, any tool to conclude learning opportunities is obviously a limitation itself. However powerful, my strict and detailed use of dimensions of variation omitted other learning opportunities that might have been visible with other tools, for instance, problem-solving strategies.

The claims of generalisation for the relation between interaction and learning opportunities are through 'the recognition of patterns' (Larsson, 2009). This means that I have investigated a phenomenon and deepened the knowledge of it. The contribution is at this stage primarily an 'interpretational tool for identifying patterns in the everyday world and making better sense of the world around us' (Larsson, 2009, p. 41). This tool can also be used to interpret classrooms other than the ones in this study. In that sense, the study can be generalised as deeper knowledge of a phenomenon has been gained.

9.6 Theoretical contributions

All studies contribute more or less to the development of the theories used in them. Therefore, here some contributions to two research fields related to this study – Variation theory of learning and interaction research – will be discussed.

9.6.1 Variation theory

Variation theory is epistemologically founded on the position that people perceive different aspects of phenomena and, furthermore, that learning is the discernment of new aspects. Despite this, the tradition has not been much interested in interaction. One conclusion drawn from this study is that by analysing *how* the DoVs are opened, jointly or not, and on whose initiative, interactional features of teaching can also be closely investigated without losing sight of learning opportunities. Al-Murani & Watson (2009) describe that in some lessons the teacher controls the learning opportunities, whereas in others teachers and learners jointly construct them. Through learner contributions, the lived object of learning moves from the private domain into the public one, and simultaneously becomes available for all learners in the class (ibid.). What happens to the learning opportunities when learners are encouraged to share parts of their lived objects of learning, compared to when this is not done, is precisely what I have investigated. My results are aligned to the conjectures of Al-Murani and Watson (2009) and contribute details of the complex of interaction and learning opportunities.

Although the identification of the enacted aspects in the lesson events was productively conducted with a variation theoretical lens, complementary research from mathematical education studies was needed to compare qualities of the aspects, so that qualitatively different learning opportunities could be distinguished. This is in line with the argument of Ryve et al. (2013), who used a frame of mathematical proficiencies and research findings from teaching problem-solving, in order to compare enacted learning opportunities between lessons.

The challenges of the analysis were encountered not so much in the interactional aspects as in the 120 lessons events not always being comparable because the content taught in the lessons was sometimes too broad. Despite this, in this study variation theory has been shown to be a powerful analytical tool through which learning opportunities can be identified. The patterns of variation were crucial when 111 distinct DoVs were identified as opened or disregarded on 289 occasions. This extensive and detailed analysis led to the distinction of dual meanings of the pattern of contrast: contrast 1 (counter-example) and contrast 2 (isolation). These dual meanings of contrast have been used in various variation theoretical studies, so the contribution from this study is merely an elucidation. Nonetheless, a new useful pattern of variation arose during the analysis: *illumination*. This pattern may be limited to the mathematical context, as the rationale behind illumination is the highlighting of all "invisible" signs and numbers that mathematics is full of. Consequently, in the context of mathematical education, illumination is both useful and clarifying. Both these facets are elaborated on in great detail in Chapter 5.

9.6.2 The importance of optional aspects

Another contribution from this study to the teaching and learning of mathematics is the emphasis on *optional aspects*. This construct originates from variation theory and was shown in the results to be central in understanding what learners generate to the lessons. Moreover, the optional aspects were fundamentally related to the learning opportunities that emerged. Therefore, I would suggest the optional aspects to be regarded at least as necessary as the *necessary aspects* (Marton, 2015).

9.6.3 Why not misconceptions?

This study has benefitted vastly from the misconception research from the 80s and 90s (e.g. Hart, 1981; Kerslake, 1981; Leinhardt et al., 1990) in the search for possible rationales behind the learner contributions. Yet, I do not use *misconceptions* to denote these possible rationales. There are three reasons for this. The first reason is a theoretical one. Variation theory is rooted in phenomenography, a research tradition in which different conceptions of phenomena are studied. The results from these studies often show that conceptions can be hierarchically ordered in relation to some norm. However, the epistemological assumptions behind the phenomenographic conceptions differ from the epistemological assumptions behind misconceptions. The latter regards conceptions as something cognitive, whereas the former regards conceptions as something relational.

The second reason is a methodological one. Only in one case, in which a student was interviewed after the lesson, can I say with at least some certainty how this student, Alvin, perceived the function as a point in the coordinate system[131]. All the other cases are built on the assumption that a learner contribution 'provides a window into some awareness' (Al-Murani & Watson, 2006, p.3). Conceptions of diverse phenomena, regardless of whether they are the results from a constructivist or phenomenography study, are the results of thorough interviews and not classroom studies. Therefore, *alternative ways of seeing* is a more vague and uncertain concept than a misconception or a conception in a phenomenographic stance.

The third reason is the most important one: the word *misconception* indicates that there are either correct conceptions or wrong conceptions and nothing in between. One of the assumptions in this study is that there are several ways to perceive a

131 This is elaborated on in sub section 8.4.3.

phenomenon, and the results illustrate that not all of them can be defined as correct or wrong. For instance, when participants in this study put emphasis on the b-value as the y-value at the y-intercept, it indicates a correct conception. Nevertheless, the more common way, namely emphasising the b-value as the y-intercept does not indicate a misconception. It rather indicates that something has been taken for granted, not that something has been misunderstood. Or, at least, it is not possible here to see the differences between misconceptions and aspects that are taken-for-granted. Also, these differences are not of much value for the study. Furthermore, from this study I claim that learner contributions are both rational and useful in teaching, beyond their degree of correctness. This claim is in line with the construct of *transitional concept* by Moschkovich (1998).

9.6.4 Interaction research

Interaction research comprises about 50 years of research from school contexts. Teachers' diverse actions in classrooms have been studied extensively. Furthermore, the importance of learner perspectives in teaching has long been acknowledged. However, there has not always been a clear rationale behind this importance. Here, the contribution of this study will be related to both interaction and rationales for learner perspectives.

Rowland and Zazkis (2013) describe three possible responses to unexpected learner contributions: *to ignore*, *to acknowledge but put aside*, and *to acknowledge and incorporate*. They use different lesson sequences with different contents, and perhaps that is why our category systems differ quite a bit. By my categorisation of the trajectories for learner contributions, both *to ignore* and *to acknowledge but put aside* would be regarded as "disregarded learner contributions", whereas *to acknowledge and incorporate* could be distinguished into three different trajectories depending on the development of the content of the contribution: "selected LCv", "considered LCv" and "explored LCv". My category system is more fine-grained, which might be due to the fact that I had such large empirical data and the same lesson content in all cases. One conclusion that this study shares with the conclusions of Rowland and Zazkis (ibid.) is that not all teachers attend to students' questions, deal with unexpected ones, or take advantage of opportunities in teaching; teachers act differently. What this study adds is primarily a relation of the learner attentions and learning opportunities.

It is one thing to understand, for instance, a misconception but quite another thing to use that understanding to make better instructional decisions in teaching (Even, 2005; Mason & Davis, 2013). When Even and Gottlib (2011) studied an experienced teacher's responses to learner contributions: two results showed that this teacher knew how to make decisions 'on her feet'. Firstly, the analysis showed that almost all of this teacher's whole-class teaching was generated by or built on the students' contributions of questions, answers, hypotheses and comments. Secondly, the content of this teacher's lessons developed by means of the learner contributions, as they most often led to the content developing beyond the lesson purpose. The study by Even and Gottlib emphasises a teacher's awareness of students' ways of thinking. The results of the present study are in line with the claims of Even and Gottlib (ibid.). However, instead of studying several lessons by one teacher, I have studied one lesson by several teachers. This design made it possible to focus on content and contributed to a detailed comparison between teachers' actions and learning opportunities. Furthermore, it added a rationale to the importance of teacher awareness of student thinking by investigating also the learning opportunities with the variation theoretical tools.

Al-Murani's (2007) work emphasised the importance of the exchange of dimensions of variation to the analysis of classroom interactions, and he found evidence that this improved learning. His study suggests that attention to the dimensions of variation, and how they are handled by the teacher, can distinguish between similar lessons which were different in terms of learning outcomes. The present study has added how this exchange of dimensions of variations is constituted in interaction.

9.7 Limitations and suggestions for further research

Empirical studies often accumulate more question than they answer. This is also the case here. I have categorised the suggestions for further research according to those questions.

Firstly, due to the research questions, this study is a small-scale investigation of details in interactions between teachers and their learners. As the results show such clear connections between qualities of learning opportunities and attentions to learner contributions, a similar study in another context would be interesting. Questions that concern the phenomenon as such could be answered by comparing to other classrooms. Which are the similarities and differences of the trajectories for learner contributions in other kinds of lessons, compared to introductory lessons with considerable instruction? Much effort in this study was put into recognising and creating a category system for the different trajectories. That system could now be used in other studies with new questions.

Secondly, another suggestion for further research is to investigate the teacher perspective on the phenomenon studied. The results showed what learning opportunities emerged when teachers explored learner contributions, compared to when they did not. However, nothing can here be said about these teachers' rationales for the different ways of attending learner contributions. Why do teachers attend learner contributions differently? Are these lessons typical of each teacher? In this study there is no data from which such conclusions can be drawn. On the one hand, it is not unrealistic to suppose that the 12 teachers have different views on as well teaching, learning, as on mathematics. Davis (1997) clearly related manners of listening to ways of comprehending learning, teaching, and mathematics. On the other hand, the teachers probably had different intentions with their introductory lesson of linear equation. These intentions could also have an impact on the different modes of attending to the learner contributions. It would be fruitful to investigate the connections between different attentions to learner contributions and views on teaching, learning and mathematics. There is a considerable research field of teacher knowledge and beliefs, which could contribute to answering these questions. Moreover, some of the teachers seemed to use learner contributions as a resource for instruction. It would be valuable to know how the decisions regarding this usage are made. As there was disregarding of learner contributions in all lessons, and given that these actions are deliberately performed, it would be beneficial to learn how teachers distinguish between what to attend and what to disregard.

Finally, and perhaps the most significant suggestion for further research, is the *learner perspective* on the phenomenon of attention to learner contributions. I will discuss three different angles of this suggestion. Firstly, a necessary delimitation was to not investigate the learning outcomes. Instead, offered learning opportunities were studied. With actual data about learning outcomes, the claims would have been stronger, and perhaps slightly different. The relation between attentions to learner contributions and actual learning would therefore be valuable to study.

Secondly, even though learner contributions and the internal logic of them was considered closely in this study, and this undoubtedly provided a learner perspective on the content, very little attention was paid to the learners' perspective on the different types of attention *per se*. Some of the learners in this study – Alvin, Cornelis, Elias, Petter and a few others – play a pivotal role as they contribute immensely to the lessons. What makes them do that? And what makes so many of their classmates stay silent?

Thirdly, there is cautiousness in exploring learners' wrong answers, both in most of the teaching discourses I have encountered and also in relevant research. But do we actually have a learner perspective on this issue? Is it something we take for granted since we ourselves have grown up learning that in mathematics teaching wrong is wrong, and not inevitable in learning? From this study, it is not possible to conclude answers to this question, since I cannot know how many of the students that stayed silent for this reason. Yet, some students' willingness to share totally wrong ideas in some of the lessons, made me reflect more deeply on the learner perspectives on this phenomenon.

9.8 Implications for practice

I will now discuss a few implications for practice from this thesis. I will start with a statement: exploring learner contributions is not a recipe for success. The ambition of enhancing learning is a most complex activity, and success in the classroom is always a result of multifaceted efforts. There is no such thing as one recipe for success in teaching. The implications for teaching that a single thesis can propose is, from my point of view, at most, to offer the discernment of new aspects of something we thought we already knew. Even so, I will venture to make some implications for teachers and other colleagues.

It is old news that we ought to listen to our students, and to tell less in instruction. Further, earlier research has repeatedly demonstrated that the idea of direct transmission of knowledge works poorly. What this thesis has to offer for practice is a discussion of *why* we should listen to our students, beyond commendable reasons concerning, for instance, democracy and agency of learning. The results from this thesis suggest there is a cost to *not* taking your students' ways of comprehending the content into account. In this study, this cost concerned both learning opportunities for optional aspects, which are most often learner-generated and, perhaps more surprisingly, the quality of the aspects which teachers generated.

Is it really a good idea to always pursue students' questions and comments in instruction? Do not students come up with a lot of diverse ideas without any real logic attached? What might be lost in pace and stringency if you go along with all those different ideas that come up? These questions illustrate, at least partly, my preconceptions at the beginning of the study. I thought, that there was more to capture in the learner contributions than teachers usually think, but I would have said that it regarded only a few contributions. My results suggest the opposite. In the analysis, I found an internal logic to almost every contribution initiated by a learner. That was an eye-opener for me. So this study leads more to the question:

what might be the cost if you do not pursue students' contributions? However, it must be remembered that I have not examined the pace and stringency of each lesson. That question remains unanswered.

What this study adds to practice, is above all a support of the idea that teachers need to take into account the learners' ways of comprehending the content, in order to be better prepared for teaching. But also, to be able to adjust to what comes from learners, still with the content to the fore. This adjustment, however, can have many forms.

So, it is not enough to be well-prepared; teachers should also predict the unpredictability in lessons. In that way, the learning opportunities for the content can be enriched. One plausible explanation for this may be that learners afford perspectives on the content that teachers have forgotten a long time ago. And thus, the multi-voiced classroom consists of multiple voices as well as of multiple ways of comprehending the content taught.

10 Summary in Swedish

Elevers och lärares gemensamt skapade lärandemöjligheter – exemplet linjära ekvationer

1 Inledning

I denna avhandling undersöks hur lärandemöjligheter av linjära ekvationer samkonstitueras av elever och lärare när den räta linjens ekvation introduceras.

Interaktion i klassrummet har varit i forskningsfokus i snart ett halvsekel (se t ex Radford, 2011) och skilda perspektiv, syften och metoder har utvecklats till många olika forskningsfält. Redan under tidigt 70-tal kom forskningsresultat som fortfarande är relevanta för hur vi ser på klassrumsinteraktion. Ett exempel är IRE-mönstret (Mehan, 1979) som beskriver lärares och elevers interaktionsmönster i tre turtagningar: initiering, respons och evaluering. Ett annat exempel på tidiga forskningsresultat är den genomsnittsliga väntetiden av en sekund mellan en lärarfråga och ett förväntat elevsvar (Row, 1974). Den skandinaviska interaktionsforskningen har dominerats av konversationsanalytiska studier, som visserligen delar sina rötter med dessa tidiga klassrumsstudier, men som har utvecklats i en något annan riktning (Sahlström, 2008). Dessa studier fokuserar på hur socialt liv etableras, upprätthålls och förändras genom interaktion mellan människor. I de flesta fall har inte studierna genomförts i klassrum utan i kontexter utanför skolan (Sahlström, 2009). Såväl de tidiga studierna från klassrum som de senare från andra kontexter är deskriptiva studier med syftet att beskriva *hur* interaktion sker, alltså har man studerat interaktionens *former*.

Andra studier har haft *funktionen* av interaktion som forskningsobjekt och olika underliggande mekanismer av interaktion har analyserats. Ett exempel är Lobato et al. (2005), som skiljer mellan olika funktioner av klassrumsinteraktion: att framkalla elevers strategier, bilder och sätt att se innehållet eller att initiera nytt innehåll i undervisningen. Ett behov som Lobato et al. identifierar i sin studie är att interaktionen mellan elever och lärare behöver skifta funktion ifrån att kommunicera lärares matematik till att utveckla elevers matematik. En tidigare studie av Nystrand och Gamoran (1990) beskriver en distinktion av klassrumsinteraktion i *recitation* och *konversation*, där den senare rubriceras som högre kvalitet eftersom den bestäms av vad som har sagts tidigare och inte av ett färdigt skript, som i recitation. I konversation påverkas både innehåll och fokus i undervisning av vad eleverna bidrar med. En slutsats från studien är att hög-kvalitativ interaktion inbegriper ett utbyte av idéer mellan en lärare och hennes elever (Nystrand & Gamoran, 1990).

När det gäller interaktion i matematikundervisning har interaktionen ofta fokuserats kring två av entiteterna *elev-innehåll-lärare*. Sedan i början av 1990-talet, då sociokulturella och situerade aspekter av lärande fick starkt genomslag, har ett starkt fokus i interaktionsforskning legat på elev-lärare. I många fall har interaktionens innehåll, i synnerhet om man med innehåll betecknar ett skolämne, lämnats utanför forskningsintresset eller betraktats enbart som en kontextuell faktor (se t ex Cobb, 2006; Mortimer & Scott, 2003; Steinbring, 2008).

I denna studie erkänns de sociala aspekterna av lärande eftersom lärande betraktas som *rotad i interaktion* (Carlgren, 2009) och inte något som sker oberoende av en kontext. Däremot ses inte lärande som interaktion i sig, utan som urskiljandet av nya aspekter av ett fenomen. Därför har denna studie ett explicit innehållsfokus och även ett fokus på interaktionen mellan alla tre entiteter: elever, lärare och innehåll.

2 Bakgrund: interaktion

Forskningsintresset i denna studie är att utifrån ett innehållsperspektiv undersöka de skilda lärandemöjligheter som erbjuds i olika lektioner när den räta linjens ekvation introduceras i helklassundervisning. Därutöver finns ett intresse av att analysera hur dessa lärandemöjligheter konstitueras av elever och lärare. Utifrån det innehållsperspektiv som finns i studien diskuteras här tidigare forskning som utgör bakgrund för studien. Kriteriet för urvalet av studier har varit att det finns ett, explicit eller implicit, fokus på utbytet av idéer[132] mellan elever och lärare i undervisning.

Även om mycket undervisning fortfarande vilar på olika varianter av direkt överföring av kunskaper mellan läraren och den lärande, är vi numera medvetna om att relationen mellan undervisning och lärande är mer komplex än så. Begreppet ”meningsförhandling”[133] (Cobb & Bauersfeld, 1995; Voigt, 1994; Wood, 1998) illustrerar att interaktion inbegriper subtila skiftningar i mening av det innehåll som kommuniceras, ofta bortanför deltagarnas medvetande. Voigt (1996) menar att vi inte kan utgå ifrån att eleverna tillskriver en specifik mening till

[132] Det är inte idéer i största allmänhet som avses utan ett utbyte av innehåll.
[133] Negotiation of meaning

innehållet av sig själva och att denna mening skulle överensstämma med den som läraren vill att eleverna ska lära sig. Detta gäller i synnerhet i undervisningssituationer där ett nytt innehåll introduceras. Istället menar Voigt (ibid.) att elever och lärare i interaktion konstituerar innehållet under tiden som undervisningen fortgår. Elever indikerar sin förståelse genom sina inspel och lärarens respons blir antingen ett accepterande eller ett avslag.

Vilken interaktion lärare skapar i sina klassrum beror på många olika faktorer, till exempel på hur man ser på lärande. Davis (1997) beskriver tre olika sätt att lyssna till elevinspel: *utvärderande lyssnande*, *tolkande lyssnande* och *hermeneutiskt lyssnande*. Vidare beskriver han hur dessa sätt bygger på helt olika antaganden om lärande och undervisning. Det utvärderande lyssnandet har som funktion att kontrollera att eleverna är med och man letar efter specifika svar. Det tolkande lyssnandet har som funktion att förstå elevers olika förståelse av det innehåll som undervisas. Det hermeneutiska lyssnandet har som funktion att delta i en undersökning av något tillsammans med eleverna. Elevinspel undersöks och man letar efter för-givet-tagna aspekter av innehållet. En slutsats från Davis studie är att kvalitén på elevinspelen är nära relaterade till hur lärare lyssnar på dem. I lektioner där hermeneutiskt lyssnande förekom upptäckte han både förståelser och beteenden som inte fanns i lektioner med de övriga lyssnarsätten.

Hur elevinspel används i undervisning är inte enkelt att studera när det sker, då fenomenet kan betraktas som både subtilt och flyktigt. Rowland och Zazkis (2013) re-analyserar istället tidigare empirisk forskning och beskriver att det finns tre olika sätt att respondera på oväntade elevinspel: *att ignorera dem, att bekräfta men inte använda dem* samt *att bekräfta och bygga in dem i undervisningen*. Det finns även lärare som inte använder elevinspel över huvud taget, menar Rowland och Zazkis (ibid.). Black et al. (2003) undersöker, tillsammans med brittiska lärare, hur lärare kan använda elevperspektiv på innehållet i undervisning. Huvudsyftet med studien var att utveckla formativa bedömningskompetenser. Ett av resultaten från studien var den stora skillnaden mellan olika lärares attityder till elevers felaktiga svar. Några lärare menade att de felaktiga svaren var minst lika betydelsefulla som de korrekta eftersom de kunde leda till en utveckling av lektionens innehåll medan andra beskrev skälet till att inte stimulera till för mycket elevdeltagande berodde på rädslan att exponera elever som svarar felaktigt. Följaktligen tycktes denna rädsla styra över en del av interaktionen i klassrummet.

Ett annat dilemma i undervisning är den mellan tid och interaktion. Denna beskrivs som att balansera mellan att vilja använda elevperspektiv och att samtidigt lyckas ”hinna med kursen”. Interaktion tycks ta tid och några av svårigheterna är: att lektionstiden inte räcker till för att gräva i varenda elevinspel, att uppmuntra elever att bidra med sina strategier samtidigt som man vill att de ska utveckla mer framgångsrika strategier samt att balansera mellan intentionen med lektionen och att följa elevinspel (Even, 2005; Jones & Tanner, 2002).

En utgångspunkt för nutida formativa bedömningsprocesser är att elevers uppfattningar och felsvar skulle kunna informera lärarens beslutsfattande i undervisning (t ex Black et al., 2003). Denna utgångspunkt har problematiserat senare; det handlar inte enbart om att höra felsvar, man måste även kunna *höra igenom* (Even, 2005) vad som sägs. Det innebär att förutom att förstå att det finns något att höra i elevinspelen, så behöver man kunna känna igen och förstå vanliga missuppfattningar för att kunna använda dem (Even, 2005; Mason & Davis, 2013).

Vad innebär det för lektionens innehåll att läraren bygger på elevers frågor, svar, hypoteser och kommentarer? Even och Gottlib (2011) drar slutsatsen, efter att ha studerat 17 lektioner av en erkänt duktig lärare, att de sekvenser där innehållet oftast utvecklades bortom lektionens syfte var initierade av elevinspel. Läraren hade en medvetenhet om elevers olika sätt att tänka, men även en känslighet inför elevers bidrag till lektionen.

Relationen mellan elevdeltagande och innehållsliga lärandemöjligheter har undersökts i flera studier. Emanuelsson och Sahlström (2008) menar att interaktion kan påverka lärandemöjligheterna negativt. Släpper man in eleverna för mycket riskerar lärandemöjligheterna att bli mindre komplexa än när man begränsar elevinspelen. Det kan tyckas som att resultaten från de båda studierna (Even & Gottlib, 2011 och Emanuelsson & Sahlström, 2008) motsäger varandra, men i själva verket har de undersökt olika aspekter av interaktion. Den förra studien har ett innehållsligt perspektiv och betraktar därmed innehållsliga elevinspel medan den senare studien studerar all elev-lärar-interaktion. En kombinerad slutsats kan vara att en lyhördhet för elevinspel om *innehållet* kan utveckla lektionerna, medan en lyhördhet för elevinspel i allmänhet kan leda till mindre komplexa lärandemöjligheter. En tredje studie som undersöker relationen mellan elevdeltagande och lärandemöjligheter är Ryve et al. (2013). Deras slutsats är att hur innehållet behandlas påverkar vilka interaktionsmöjligheter som skapas. När innehållet är explicit i en lektion har eleverna större möjligheter till interaktion jämfört med när innehållet är mer implicit. Alla dessa tre studier tyder på att det finns relationer mellan interaktion och innehållets behandling som är värda att undersöka vidare.

Med samma teoretiska ramverk som i denna studie, men med såväl kvantitativa som kvalitativa analyser, undersökte Al-Murani (2007) 80 algebralektioner samt genomförde för- och eftertester av eleverna. Lärarna deltog i ett undervisningsutvecklingsprojekt där syftet var att studera ifall medveten variation kunde leda till bättre elevresultat. Ett av Al-Muranis resultat, som kan relateras till denna studie, är den skillnad mellan interventionsgruppen och kontrollgruppen som upptäcktes och som innebar att i interventionslektionerna utbyttes innehållsliga aspekter mellan elever och lärare systematiskt. När interventionslärarna behandlade elevinspel gjordes detta med systematisk variation, i kontrast till kontrollärargruppen. Al-Murani (ibid.) menar att en möjlig förklaring är att den variationsteoretiska interventionen i sig har utvecklat lärares medvetenhet om potentialen i elevers inspel. Han kunde även associera detta utbyte av innehållsliga aspekter till bättre elevresultat (Al-Murani, 2007; Al-Murani & Watson, 2009).

Denna studies objekt är behandlingen av innehållet i elevinspel i undervisning. Därtill är ett av syftena att undersöka vad olika sätt att hantera dem kan ha för betydelse för de innehållsliga lärandemöjligheterna som utvecklas i matematiklektioner.

3 Bakgrund: linjära ekvationer

Det matematiska innehåll som studiens lektioner behandlar, introduktionen av den räta linjens ekvation, kan ingå i både skolalgebran och funktionsläran, beroende på var fokus läggs i undervisningen. Ekvationen $y = kx + m$ kan behandlas som en del av skolalgebran medan den grafiska representationen ofta är ett första steg in i funktionsläran. Funktioner kan representeras geometriskt (som grafer), aritmetiskt (i värdetabeller eller i talpar) eller algebraiskt (som formler med variabler) (Kieran 1992). Ekvationen/den algebraiska representationen av en förstagradsfunktion ($y = kx + m$) beskriver ekvationen till en rät linje. y och x är variabler, k beskriver riktningskoefficienten och m beskriver y-värdet i skärningspunkten mellan grafen och y-axeln. Den räta linjen är en representation av relationen mellan x- och y-värden och därför är räta linjer oftast funktioner. Det kan vara värt att notera att det saknas en internationell standard för att beteckna den räta linjens ekvation. I den engelska delen av denna text används $y = mx + b$ konsekvent.

Då studiens lektioner innehåller introduktionen av den räta linjens ekvation bygger denna studie delvis på tidigare forskning om missuppfattningar av olika begrepp inom skolmatematiken. En missuppfattning skiljer sig från ett misstag, som kan följa av att man exempelvis varit för snabb eller okoncentrerad. En missuppfattning är en begreppslig uppfattning som bygger på att man förstår en situation, ett begrepp eller ett fenomen på ett alternativt sätt jämfört med disciplinen (Swan, 2001). Eftersom studien fokuserar elevinspel, används resultaten från denna tidigare forskning för att förstå de uppfattningar som kan ligga bakom elevinspelen.

Vanliga missuppfattningar när det gäller *funktionen* som sådan är relaterade till om funktionen uppfattas punktvis eller globalt (Bell & Janvier, 1981). Att hantera funktionen punktvis, genom att till exempel plotta eller hantera enbart diskreta punkter, kan leda till att uteslutande lokala egenskaper uppfattas. Att även uppfatta funktionen globalt innebär att man ser funktionen som en egen entitet och det kan göras genom att till exempel skissa grafen eller att finna dess extrempunkter. Vikten av att uppfatta funktionen både lokalt och globalt poängteras i flera tidigare studier (t ex Even, 1998; Leinhardt et al., 1990).

En väldokumenterad missuppfattning av den *grafiska representationen* är att se grafen som bild, vilket kan innebära att den räta linjen representerar en rak väg och att negativ lutning tolkas till exempel som att någon går i en nedförsbacke. Denna oförmåga att uppfatta grafen som en abstrakt representation av en relation är bekräftad i flera studier (Clement, 1982; Clement, 1985; Kerslake, 1981; Leinhardt et al., 1990; Monk & Nemirovsky, 1994; Schoenfeld et al., 2012; Selden & Selden, 1992). Tidigare forskning om missuppfattningar av *skärningspunkter* har resulterat i att man poängterat vikten av att förstå underliggande aspekter, såsom de dubbla koordinaterna i varje punkt, att förstå sambandet mellan olika representationer (algebraisk och grafisk), samt att förstå vilka aspekter som är relevanta i vilken representation (Lobato & Bowers, 2000; Moschkovich, 1992, 1996). När det gäller undervisning om den räta linjens ekvation ($y = kx + m$), beskrivs ofta k-värdet som grafens skärningspunkt i y-axeln. Skärningspunkten i x-axeln ges sällan någon uppmärksamhet i linjära ekvationer. Däremot är denna skärningspunkt i x-axeln i stort fokus när det gäller senare lärande av funktioner (Leinhardt et al., 1990).

Lutning[134] är en grundläggande begrepp för att beskriva en funktions beteende och elevers uppfattningar om lutning har kartlagts omfattande. I synnerhet gäller detta lutning i den grafiska representationen och sambandet till riktningskoefficienten i funktionen. En viktig distinktion för fortsatt lärande beskriver två olika uppfattningar om lutning: att se lutning *visuellt* eller *analytiskt* (Zaslavsky et al, 2002). Skillnaden mellan dem är huruvida man applicerar lutningen på grafen som sådan (visuellt) eller på funktionen (analytiskt). När lutning uppfattas visuellt uppfattas inte skillnaden mellan linjens lutning (steepness) och funktionens lutning (slope) (Lobato & Bowers, 2000). Undervisning som enbart har en visuell approach riskerar att utelämna lärandemöjligheterna av lutning som en

[134] Det engelska ordet *slope* har översatts till lutning, eftersom det är det ord som används oftast i både forskningslitteratur och i läromedel. Däremot finns en konnotation i ordet lutning som snarare överensstämmer med *steepness*. Jag menar att det på svenska saknas ett ord för lutning i den grafiska representationen, som skulle kunna ha konnotationen av en "rate of change", här översatt något ofullständigt till *förändringstakt*.

förändringstakt mellan x- och y-värden, det vill säga som en relation mellan två mängder som samvarierar (Lobato & Bowers, 2000; Zaslavsky et al, 2002).

4 Syfte och forskningsfrågor

Det övergripande syftet med denna studie är att bidra med kunskap om relationen mellan undervisning och lärandemöjligheter i matematik. För att avgränsa och specificera detta syfte har undervisning avgränsats till helklassundervisning och lärandemöjligheterna betraktas ur ett innehållsligt perspektiv. Det som undersöks i helklassundervisningen är hur det publika innehållsliga samspelet går till och detta relateras till de innehållsliga lärandemöjligheterna. Av detta skäl har lektionsinnehållet valts att vara ”samma” i någon bemärkelse. Valet av lektionsinnehåll – introduktionen av den räta linjens ekvation – uppfyller tre kriterier: det är kommunicerbart med lärare, det är avgränsat, och det förekommer som innehåll i både grundskolan och gymnasieskolan. Syften med studien är att fördjupa kunskaper om relationen mellan innehållsligt samspel och de lärandemöjligheter som utvecklas i lektioner där den räta linjens ekvation introduceras. Forskningsfrågorna är:

1. Vad betyder lärares behandling av elevinspel i helklassundervisning för de lärandemöjligheter av linjära ekvationer som utvecklas?
2. Vad bidrar eleverna med till iscensättandet av linjära ekvationer?

5 Teoretiskt ramverk

Eftersom forskningsobjektet i studien är innehållsrelaterade lärandemöjligheter, erbjuder variationsteorin (Marton, 2015; Marton & Booth, 1997) både utgångspunkter och analytiska verktyg.

Studien bygger på tre utgångspunkter. Den första är epistemologisk och beskriver hur vi lär. Lärande ses som en förändrad relation mellan människan och hennes omvärld. Enligt variationsteorin urskiljer vi skillnader mot bakgrund av likheter. Lärande innebär att vi urskiljer nya aspekter av ett fenomen, och förändrar därmed relationen till detta fenomen. Synen på lärandet har några konsekvenser för denna studie. För det första betyder det att vi uppfattar fenomen på skilda sätt, det vill säga vi har olika relationer till vår omvärld. I en klassrumskontext innebär det att eleverna redan före undervisning har uppfattningar om undervisningsinnehållet som kommer att påverka deras fortsatta lärande av detsamma. Det finns alltså ingen position inom variationsteorin där man kan utgå ifrån att eleverna inte vet *något* om innehållet. Även om de kan vara okunniga om ett begrepp som sådant, finns det alltid aspekter av detta begrepp som de har någon förståelse av. För det andra har eleverna olika uppfattningar om aspekter av innehållet. Den andra utgångspunkten beskriver relationen mellan elevers inspel och vad de riktar sitt medvetande mot i det innehåll som behandlas. Här bygger studien vidare på tidigare variationsteoretisk forskning där elevers inspel ses som ett "fönster till en del av medvetetandet" (Al-Murani & Watson, 2009, s. 3) eller uttryckt annorlunda: elevers inspel kan visa hur eleverna uppfattar aspekter av innehållet. Den tredje utgångspunkten handlar om att lärandemöjligheterna i en lektion går att analysera genom att studera vilka aspekter av innehållet som varieras (se t ex Häggström, 2008; Marton & Tsui, 2004; Runesson, 1999). Dessa lärandemöjligheter är i de allra flesta fall samkonstituerade i undervisning och om både elever och lärare genererar aspekter till lektionen bestäms iscensättandet av innehållet av hur detta samspel går till.

Variationsteorin erbjuder även teoretiska verktyg för att analysera de lärandemöjligheter av linjära ekvationer som detta samspel utvecklar. De variationsteoretiska analytiska verktyg som används i studien är framför allt: dimension av variation, variationsmönster samt nödvändiga och möjliga aspekter.

Dimension av variation är i denna studie en aspekt av innehållet linjära ekvationer som varieras med minst två drag. Om exempelvis lutning är i fokus i en lektionssekvens, behöver minst två olika lutningar varieras för att en dimension av variation ska betecknas som öppnad. *Variationsmönster* betecknar hur en dimension av variation öppnas, till exempel i en kontrast eller generalisering. De *nödvändiga* (necessary) aspekterna och *möjliga* (optional) aspekterna av ett innehåll betecknar aspekter som är kritiska att urskilja för att lära sig ett fenomen på ett visst sätt. Skillnaden mellan dem är att de nödvändiga aspekterna kan definieras utifrån ett ämnesperspektiv medan de möjliga aspekterna är sådana som elever tillskriver betydelse när de i själva verket ofta borde bortses ifrån. Dessa aspekter är dynamiska och gränsen mellan nödvändiga och möjliga aspekter är långt ifrån förutbestämd, utan istället förändras den till exempel med discipliners utveckling (Pang & Ki, 2016).

6 Den empiriska studien

För att få veta mer om hur elevinspel och lärandemöjligheter utvecklas i undervisning behöver man studera lektioner i sin vanliga kontext. Ska man dessutom analysera vilka dimensioner av innehållet som öppnas samt huruvida de öppnas i interaktion eller inte, behöver lektionerna kunna ses flera gånger. Denna studie bygger på videoinspelningar av 14 matematiklektioner i år 9 eller i någon av gymnasiets första två år. Tolv lärare deltog i studien och samtliga var utbildade matematiklärare, men skillnaderna var stora i andra aspekter, såsom ålder, undervisningserfarenhet, och kön. Två av lärarna deltog med två klasser vardera, därför är antalet deltagande klasser tolv. 97 % av eleverna i dessa klasser gav sitt medgivande till studien och det ledde till att 297 elever från nio olika skolor deltog i studien. Samtliga lektioner hade av den undervisande läraren för klassen valts ut som ”introduktionen av den räta linjens ekvation”. Lektionslängderna varierade mellan 33 och 66 minuter och dubbla kameror spelade in lektionerna, från olika perspektiv. Alla deltagare blev informerade om syftet med studien och samtliga har anonymiserats för att inte kunna identifieras. Samtliga lärare benämns med kvinnonamn och kvinnliga pronomen medan samtliga elever benämns med mansnamn och manliga pronomen.

7 Analysmetod

Analyserna som genomfördes av datamaterialet kan delas upp i två skilda faser, där resultaten från den första fasen möjliggjorde analysen av den andra fasen, som ledde till att forskningsfrågorna kunde besvaras. Studiens forskningsobjekt är varken interaktion eller lärande som sådant, utan vilka lärandemöjligheter som skapas genom olika interaktioner. Därför har resultaten från den första analysfasen enbart använts som ett verktyg för att besvara forskningsfrågorna i den andra fasen, inte som ett resultat i sig. Den första fasen utgjordes av att videoinspelningarna strukturerades, transkriberades, och delades upp i *lektionsevent* (Pillay, 2013) utifrån hur lärarna organiserade lektionen innehållsligt. Därefter sorterades alla event bort som inte innehöll helklassundervisning. De återstående 120 eventen undersöktes på två olika sätt: dels undersöktes hur samtliga innehållsliga elevinspel utvecklades i dem, dels analyserades vilka dimensioner av variation som öppnades i dem och av vem. Resultatet av denna undersökning resulterade i att fyra olika utvecklingar av elevinspel etablerades i materialet: ignorerade elevinspel (disregarded), valda elevinspel (selected), bekräftade elevinspel (considered) samt utforskade elevinspel (explored). Utifrån hur elevinspelen utvecklades sorterades lektionerna i tre skilda lektionstyper: 1. En dominans av utforskade elevinspel (UE), 2. En blandning av utveckling för elevinspel (ME) samt 3. En dominans av bekräftade elevinspel (BE). Resultaten visade även att det främst var lärarens bemötande av elevinspelen som avgjorde vilken utveckling ett elevinspel skulle få, även om det fanns enstaka exempel på att även klasskamrater bidrog till utforskandet av elevinspel. En annan följd från denna första analysfas var att de elevinspel som bar på en potential att öppna en ny dimension av innehållet sorterades ut och de var 184 stycken totalt. I denna sortering försvann många av de elevinspel som enbart blev valda (selected) eftersom de sällan hade potential att öppna en *ny* dimension av innehållet. Efter detta var den första analysfasen över. En huvudtabell med samtliga öppnade dimensioner av variation och samtliga elevinspel med potential att öppna en ny dimension var nu bokförda och sorterade utifrån fem aspekter av linjära ekvationer (se Appendix A). Ur denna tabell genomfördes sedan jämförelser mellan lektioner, mellan lektionsevent, samt mellan olika aspekter av linjära ekvationer för att förstå både elevinspelen, deras olika utvecklingar i lektionerna beroende på lärares respons samt vad detta hade för betydelse för de skilda lärandemöjligheter som utvecklades.

8 Resultat

Resultatet av analyserna visade att lärandemöjligheterna skilde sig åt i de olika lektionerna. Dessa skillnader kunde relateras till hur elevinspelen behandlades i lektionerna. Relationen mellan behandlingen av elevinspelen och de lärandemöjligheter som utvecklades beskrivs genom svaren till de två forskningsfrågorna.

1. Vad betyder lärares behandling av elevinspel i helklassundervisning för de lärandemöjligheter av linjära ekvationer som utvecklas?

Analysen av de totalt 289 öppningar av dimensioner av variation i samtliga 14 lektioner visade stora skillnader i erbjudna lärandemöjligheter mellan lektionstyperna, alltså med avseende på hur elevinspelen generellt utvecklades i lektionerna. Dessa skillnader låg både i antal öppnade dimensioner och, mer signifikant, i *vilka* dimensioner som öppnades. När det gäller fundamentala aspekter av linjära ekvationer, såsom *lutning* och *funktion*, visade analyserna att olika lärandemöjligheter erbjöds i lektionstyperna UE/ME jämfört med lektionstypen BE.

Begreppet funktion iscensattes på flera sätt i lektionstyperna UE/ME. Till exempel varierades funktion med flera olika representationer och/eller som en relation mellan x- och y-värden. I stark kontrast öppnades inte en enda dimension av funktionsbegreppet i lektionstypen BE. Detta innebär att lärandemöjligheterna för funktion inte iscensattes över huvud taget i de lektionstyper där elevinspelen enbart blev bekräftade.

Begreppet lutning blev iscensatt i samtliga lektioner utom en, däremot visade analyserna att lärandemöjligheterna utvecklades olika i skilda lektionstyper. I lektionstyperna UE/ME iscensattes lutning som ökning av y per x och/eller analytiskt som en förändringstakt[135] i funktionen. I lektionstypen BE iscensattes lutning visuellt, som lutning av grafen.

[135] Rate of change

2. Vad bidrar eleverna med till iscensättandet av linjära ekvationer?
Jämförelsen mellan elevgenererade och lärargenererade dimensioner av variation i alla 14 lektioner exponerade stora skillnader. Lärarna genererade i huvudsak aspekter av innehållet som kan betraktas som nödvändiga aspekter[136]. Exempel på dessa är att variera m-värden som skärningspunkter för y-axeln och att bygga ihop lutningen och skärningspunkten för y-axeln till den räta linjens ekvation. Trots att dessa aspekter i huvudsak initierades av lärare utvecklades de ofta i samspel mellan elever och lärare.

De elevgenererade aspekterna hade en del gemensamma drag. För det första bidrog elevinspelen, när de blev utforskade, till alternativa sätt att se på linjära ekvationer. Exempel på dessa är: att se funktion som en enda punkt, att utgå ifrån x-axeln som referens för lutning, att se ekvationen som en linje mellan skärningspunkterna[137], och att se negativ lutning ”som något som sker på andra sidan y-axeln”. Samtliga dessa alternativa sätt att se blev utmanade i lektionerna när elevinspelen utforskades.

För det andra var de elevgenererade aspekterna i hög utsträckning *möjliga aspekter*[138] av linjära ekvationer. Exempel på dessa är: att separera koefficienterna i ekvationen från skärningspunkter i grafen eller att förstå att ordningen på termerna i ekvationen är utbytbar. Dessutom, andra möjliga aspekter som eleverna genererade var det *linjära* i linjära ekvationer, det *tvådimensionella* i koordinatsystem, separationen av rutmönstret i koordinatsystemet från skärningspunkterna. Alla dessa aspekter delar draget att vara möjliga aspekter av linjära ekvationer och är därför svåra för eleven att urskilja ifall de tas för givet i undervisning och lämnas i bakgrunden. Dessa aspekter var beroende av elevinspel för att iscensättas. Däremot fanns det möjliga aspekter som även lärare genererade. Exempel på dessa är att variera vilka bokstäver som användes för variabler, att särskilja decimalkomma och koordinatkomma, att separera ut parentesers roll för koordinater från ”smileys”. Dessutom genererade vissa lärare även ”osynliga” möjliga aspekter av linjära ekvationer, såsom k-värdet 1 ($y = x + 3$) eller m-värdet 0 ($y = 2x$). I hög utsträckning var dock iscensättandet av möjliga aspekter i lektionerna beroende av elevinspel.

För det tredje, några elevinspel bidrog även till att ifrågasätta och utveckla vanliga sätt att undervisa en av aspekterna av linjära ekvationer. I de allra flesta lektionerna behandlades m-värdet som skärningspunkt för y-axeln. I

136 Neccesary aspects (Marton, 2015)
137 Här ses alltså komponenterna i ekvationen, k- och m-värdet, som skärningspunkterna med de båda axlarna.
138 Optional aspect (Marton, 2015).

resultatkapitlet beskrivs två detaljerat fall där elever ifrågasätter detta innehåll och i en av lektionerna utforskas elevinspelen, vilket leder till att läraren och några av eleverna fördjupar innehållet så att m-värdet dels blir associerat med y-värdet i y-axeln (x-värdet av 0 i samma punkt tas inte för givet) och att logiken bakom blir synlig.

9 Slutsatser och diskussion

De kvalitativa skillnaderna i lärandemöjligheter av linjära aspekter mellan olika lektioner visade sig vara relaterade till hur elevinspel behandlades. För det första genererade elever och lärare olika sorts aspekter i lektionerna. Lärares bidrag var i huvudsak nödvändiga (necessary) aspekter av linjära ekvationer medan de möjliga (optional) var beroende av elevinspel i stor utsträckning. För det andra genererades möjliga aspekter och alternativa uppfattningar av innehållet i mycket större utsträckning i de lektionstyper där elevinspel blev utforskade. För det tredje, den lektionstypen där elevinspel i huvudsak blev utforskade var även den lektionstyp som erbjöd de mest komplexa lärandemöjligheterna. Aspekter som både *funktion* och *lutning* gavs andra innebörder i dessa lektioner jämför med de lektioner som inte hade något utforskade av elevinspel. Tidigare forskning (t ex Even, 1998, Leinhardt et al., 1990; Lobato & Bowers, 2000; Moschkovich, 1992, 1996; Zaslavsky et al, 2002) har använts för att jämföra kvalitéer av lärandemöjligheter i lektionerna.

Två övergripande slutsatser dras i denna studie: för det första verkar det finnas ett pris för elevers tystnad när det gäller vilka lärandemöjligheter som utvecklades och för det andra drog undervisningen nytta av lärarnas utforskande av elevinspel eftersom lärandemöjligheterna var mer komplexa i dessa lektioner. Att uppmuntra, men framför allt att utforska elevinspel verkade därför vara en god idé i lektionerna.

I studien finns några begränsningar värda att diskutera. Den viktigaste är att jag inte har analyserat lärandemöjligheterna i lektionerna var för sig. Skälet är att mängden empiri är så pass omfattande i studien att vissa begränsningar behövde göras. Det medför dock att jag saknar ett helhetsperspektiv på varje enskild lektion. Istället har lektionerna delats upp i lektionsevent och varje event med helklassundervisning har noga analyserats och jämförts. Jag uttalar mig i kapitel 7 om mängden aspekter av linjära ekvationer som iscensätts och jag uttalar mig om kvalitén av dem i kapitel 8, men det finns till exempel inga analyser gjorda av hur de iscensatta aspekterna hänger ihop i en lektion.

Forskningsbidraget från denna avhandling omfattar olika aspekter. För det första finns ett teoretiskt bidrag till variationsteorin i form av utvecklade variationsmönster. För det andra finns ett metodologiskt bidrag i form av detaljerade beskrivningar av hur analyserna av lärandemöjligheterna växte fram. För det tredje finns några empiriska bidrag till forskningen om undervisning av matematik. Jag ser bidraget till diskussionen om interaktionens roll för det iscensatta lärandeobjektet samt bidraget till diskussionen om vilken betydelse det kan ha att undersöka innehållets behandling i interaktion som två sidor av samma mynt. Denna avhandling erbjuder empiriska resultat från detta mynt som en helhet eftersom den undersökt interaktionens betydelse för lärandemöjligheterna.

Vilken betydelse har denna avhandling för praktiken? Nå, ingen enskild avhandling ger recept för praktiken. Vi vet att vi bör lyssna på eleverna och att envägskommunikation inte är en effektiv undervisningsform när det gäller vad elever lär sig. Vad denna avhandling bidrar med är framför allt empiriskt stöd till diskussionen om *varför* vi bör lyssna, och utforska, elevernas inspel bortom andra rimliga skäl som delaktighet, demokrati och agentskap över lärandet. Resultaten från denna studie pekar på att det kan finnas ett pris för lärandemöjligheterna att *inte* behandla elevers inspel. Jag har visat att detta pris inbegriper iscensättandet av möjliga aspekter och alternativa sätt att se innehållet, men även att de aspekter som lärare genererade i studien var mer komplexa i lektioner där elevinspel utforskades. Däremot visade en tidig analys att det inte spelade någon roll *vem* som genererade en aspekt av innehållet när den väl genererades. Det var alltså ingen skillnad på de lärandemöjligheter som utvecklades i en lektion av en aspekt som genererades av en lärare jämfört med om samma aspekt genererades av en elev i en annan lektion.

Ett ytterligare resultat som förvånade var att det gick att redogöra för en intern logik bakom de allra flesta elevinspelen. Denna studie motsäger tesen om att elever bara slänger ur sig något slumpmässigt. Om denna studie har något att erbjuda lärare så är det ett stöd för att i undervisning våga utforska elevers bidrag, även sådana som dyker upp spontant under själva lektionen. Troligtvis har eleverna perspektiv på innehållet i våra matematiklektioner som vi lärare har glömt för länge sedan. På så sätt kan det flerstämmiga klassrummet komma att bestå av såväl flera stämmor som flera perspektiv på innehållet som behandlas.

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Appendix A Main Table: All DoV and LCv collected

1a. Enacted aspects of slope/*m*-value														
	Type of lessons by LCv trajectories													
	Explored-LCv lesson type					**Mixed-LCv lesson type**				**Considered-LCv lesson type**				
Slopes enacted visually														
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of lines as hills													X	X
of m-values as degree of leaning[139]												C 11A		
of negative lines as slides												C 11B		
between incline and decline of line												C 11C	X	
between uphill and downhill of lines													X	C 7A
Slopes enacted as increases														
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of slope as increase per x	X	C 4V	X	E 6B				X	C 2A		X			X
of m-value as increase[140]	X	C S	X					X	S 2B					
between increase/ decrease of slope						X	S 13H							
of negative slopes as decreases[141]	C 5U	X							X					
Slopes enacted analytically														
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of slope as a relationship[142]				E 6A	E[143] 15B		X	X					DIS 10A	
of m-value as a rate of change[144]			X	E[145] 6I	E[146] 15E			X						
of why m-value is slope[147]		X	E 3S						S 2D					

[139] Of the line
[140] Per something
[141] And m-values
[142] between x and y
[143] Also in 15C
[144] Between x and y
[145] Also in 6J
[146] Also in 15D
[147] As a relationship between x and y

Additional aspects of slope mainly generated by teachers														
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
between negative/ positive slopes[148]		X			E 15F						X			
of $\Delta y/\Delta x$ as a way to determine slope			E 3X		X									
of slope from everyday leaning[149]														X
Fusion: of m-values as slopes[150]		C[151] 4E	X	C 6O	X	X					X			X

1b. Aspects of slope mainly generated by learners

Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of same slope on straight line		X	C 3W	E[152] 6A				E 14H				X		
between slopes and angles	DIS 5Q				C 15A			DIS 14J						
of the "reference axis to slope"		X[153] (4R)	E 3T	C 6H										
of several ways to determine[154] slope		X	E 3V	E[155] 6D		E 12F			X					
between x and y in negative slope		E 4D									C 9B			
of substitution of the direction of x	E[156] 5V													
of independency[157] of location for slope		E[158] 4W												
between y-value and slope		E 4H												
of 0 as slope of a horizontal line		C 4O												
of same increase/ slope as parallelism		C 4T				C 12C						X		

[148] As m-values
[149] Swedish word: lutning
[150] or slopes as m-values
[151] Also in 4Q
[152] This LCv opens two DoVs
[153] Angelika discusses an LCv from the previous lesson, L3
[154] If the way of determining slope is varied and the slope is kept invariant, this DoV is opened; it is not enough to give only one example of rise over run.
[155] Also in 6F
[156] Also in 5R
[157] "in 2nd quadrant, slope becomes negative"
[158] Also in 4Y

2a. Aspects of y-intercept/b-value mainly generated by teachers

	Type of lessons by LCv trajectories													
	Explored-LCv lesson type					Mixed-LCv lesson type				Considered-LCv lesson type				
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of b-values as y-intercepts[159]	X	C 4S	X	E 6M	[160]	X		C 14E	X	[161]	X	X	X	X
of general b-value as y-intercept	E 5N			C 6P		S 12H		X			X			
of b-values[162] as start/fixed values	X	X	X			E 12A			S 2C		X			
of meanings of b as start value	X [163]	X	X					C[164] 14E						
of start value as the y-intercept[165]										X				
of negative b-values[166]		X	E 3Q	X						X	X			
Fusion: of b-value and y-intercept[167]						X								

2b. Aspects of b-value/y-intercept mainly generated by learners

Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of proportionality as graph passing the origin[168]		E[169] 4N	E 3R	E 6L		C 12B			X				X	
of ways to determine b-value						E 12D								

[159] Also y-intercepts as b-values
[160] In this lesson, a DoV of reasons for b-value equals y-intercept is opened, see under function.
[161] In this lesson, a DoV of reasons for b-value equals y-intercept is disregarded, see under functions.
[162] Or y-intercepts
[163] Two DoVs are opened here
[164] Two DoVs are included in this
[165] The x-value is invisible
[166] Not only existence of, but focused on
[167] When both y-intercepts and b-values vary (everything else kept invariant) it is a fusion. Compare to: separation of y-intercept as b-value.
[168] Or lacking b-value
[169] Also in 4P

3a. Aspects of equation mainly generated by teachers														
	Type of lessons by LCv trajectories													
	Explored-LCv lesson type					Mixed-LCv lesson type				Considered-LCv lesson type				
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
between general m-value/b-value	X	X	X	X	E 15H									
of general m-value as slope	E 5J					X		X			X			
of letters used for variables			X					C 14I		X	C 9D			
of x as all numbers also rational[170]					X									
between increase/ decrease[171]		X	X											
of x = 0 in equation and context	X													
of invisible m-value of 1	X	E[172] 4L				X				X	X			
of invisible b-value of 0	X	X	DIS 3K			X						X	C 10C	
of invisible[173] multiplication sign	X							C 14B						
of invisible[174] plus sign		X	E 3D					X						
Fusion: of signs and values to numbers[175]		X	E 3C											
Fusion: of m-value and b-value to equation	DIS 5D	X	X	X	E 15J	X			C 2H		C 9C			
Fusion: of m-value to slope/b-value to y-intercept	E[176] 5W				X	X		X						

[170] Not only present, but also focused on
[171] as plus/minus signs in equation
[172] Also in 4U
[173] The examples of course vary in different lessons. Could be 2n, mx, 3x etc.
[174] For instance (+3000)
[175] For instance (+3000) (-200x)
[176] Also in 5Y, 5Z and 5Å: four different DoVs are opened.

3b. Aspects of equation mainly generated by learners:

Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of order of terms (3x + 5 = 5 + 3x)	X	C[177] 4A	S 3A							C 1F	DIS 9A			
of order of terms: mx + b = b + mx			S 3O											
of commutativity of 2n and n2								C 14A						
of placement for y in equation[178]		C 4B												
of how to name[179] y-variable [as f(x)]			C 3B											
of m-value from mx-term	DIS 5L	E 4J				C 12I								
between first term and term without x			E[180] 3G											
of coefficients from constants		E 4E												
between constants and variables			C 3L	C 6O										
between specific/ general m-value	C 5K													
between specific/ general b- value	E 5Z													
of relation between coordinates and equation	E 5S													

[177] DoVs opened thrice. For positive terms, negative terms and mixed terms.
[178] y = mx + b or mx + b = y
[179] Nothing on differences between y and f(x)
[180] This is a self-explored LCv by a learner, not established by teacher

4a. Aspects of graph mainly generated by teachers														
	Type of lessons by LCv trajectories													
	Explored-LCv lesson type					Mixed-LCv lesson type				Considered-LCv lesson type				
Lesson:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of x-value as 0 in y-intercept/y-axis[181]	C 5E				X		C 13I	X		X				
of Δx of 1 and the squares of the grid		X	X											
of the graph/line as points	X						X		C 2I					
of the indifference of points[182] in $\Delta y/\Delta x$														X
of direction of slope /negative direction						X[183]								
between x-direction backward/forward										X				C 7A
between smileys and parentheses	X													
between coordinates and decimals	C 5B											X		
between decimal/ coordinate comma	X													
of meaning of negative coordinate	X													
of grid from scales of axes											X			
of representations of graphs											X			
between increase/ decrease of graphs										X[184]				
between x- and y-coordinates	E 5F			X					S 2E					
of methods of finding x-/y-values									S 2F					
Fusion: of decrease and downwards										X[185]				

[181] no ref to function
[182] which is used as point 1 or point 2 in formula to determine the m-value
[183] NOTE: an incorrect meaning of negative slope is used.
[184] Slopes are not discussed, focus is on increase/decrease
[185] in graph/price decrease, slope is not present

4b. Aspects of graph mainly generated by learners

Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of infinity of graphs	C 5W		C 3N	C 6N			S 13G		X					
between linear/ non-linear graphs		S 4C	C[186] 3Y				X			C 1E				
between vertical line[187] and horizontal graph		C 4G	E 3I											
Increasing x from decreasing x										C 1B				
of dimensions[188] of a coordinate system	C 5C													
between a graph and the cross[189]							C 13C							
of proportional and parallel lines[190]													C 10B	
of designation of axes	C 5A							C 14C						
of Δx as variable (not only 1)					C 15K									
of intercepts from grids						E 12J								
of x-direction from x-axis						E 12K								
of domain of graph										C 1C	X			
of range of graph										C 1D				
of graph-equation as bi-relational				E 6E										
between graphs (relations)/lines							C 13A							
of appearance of graphs							E 13B							
of number of points to draw a line							E 13E							
of ways of finding new points	E 5G													

[186] Also in 3M
[187] No defined slope
[188] 2D/3D
[189] in a coordinate system
[190] as names of straight lines

5a. Aspects of function mainly generated by teachers														
	Type of lessons by LCv trajectories:													
	Explored-LCv lesson type					Mixed-LCv lesson type				Considered-LCv lesson type				
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of function as relationship[191]	X	X		X		S 12L	X	X						
of function[192] by representations				X			X	C 14D	X					
of x as a variable in a function		X		X	X									
of a relationship in coordinates								X						
proportional relations[193]								S 14G						

5b. Aspects of function mainly generated by learners														
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
of b-values[194] as y-values at intercept					E 15I					DIS 1A				
of function from a line between intercepts	E 5T		DIS 3F	DIS 6Q							DIS 9E			
of function from a single point	E[195] 5O			DIS 6K		C 12E	E 13D							
of function from an end-point of graph		C 4F		E 6A										
of why y = b if m= 0 in function[196]		E 4K	C[197] 3P											
of the domain of a function	E[198] 5H	E 4X		E 6R					E 2G					
between domain and range[199]		C 4I												
of dependency of variables[200]	DIS 5M			E 6C	X			DIS 14C						

[191] Between x and y, and also sets of x and sets of y

[192] Focused on and varied, not only present

[193] Straight lines/lines with $b - value = 0$

[194] This DoV enacts a relation between x and y

[195] Also in 5P

[196] $mx = \ 0 \ if \ m = 0$

[197] DIS in 3H and 3K

[198] Also DIS later in 5I

[199] of function

[200] Changing dependency variables(Δx /Δy or Δy/Δ x)

Aspects out of scope enacted														
Lessons:	L5	L4	L3	L6	L15	L12	L13	L14	L2	L1	L9	L11	L10	L7
Separation:														
between two meanings of positive[201]										X				
between linear and non-linear increases										X				
That + (-) equals -						X								DIS
Disregarded LCv not in table	2	2	2	1	0	1	1	3	0	0	0	0	1	1

[201] Between a real-life meaning and a mathematical meaning

Appendix B: the consent of participation

Hej! Göteborg 120926

Jag är doktorand i pedagogiskt arbete på Göteborgs universitet och arbetar med ett forskningsprojekt, som handlar om matematikundervisning. Du kanske känner till att svensk matematikundervisning debatteras mycket i media och att fokus ligger på att alltför många elever lär sig för lite matematik. För att kunna utveckla matematikundervisningen behöver vi som forskare få veta mer om den och det är i detta sammanhang som jag planerar att filma en eller två av matematiklektionerna i din klass. Därefter ska jag analysera lektionerna för att på sikt kunna bidra med kunskap om vad i undervisning som möjliggör lärande. När studien (om flera år) är klar kommer den att presenteras i bokform för pedagoger och forskare i pedagogik. Materialet kan även användas i utbildning av lärare, om du godkänner det. Du garanteras anonymitet och din medverkan är frivillig.

Jag hoppas att du ställer dig positiv till att delta i detta forskningsprojekt. Har du frågor eller undrar över något, går det bra att kontakta mig via telefon eller mail.

Vänligen

Tuula Maunula
doktorand vid Institutionen för didaktik och pedagogisk profession (IDPP), Göteborgs universitet

Telefon: xxxx xx xx xx
Mail: tuula.maunula@gu.se

Mina handledare är:
Professor Ulla Runesson
Fil.dr. Johan Häggström

Ditt namn:________________________________ Din klass: ____________________

□ Ja, jag deltar i forskningsprojektet. Filmen med mig får användas både i forskning och i utbildning av lärare.

□ Ja, jag deltar i forskningsprojektet, men filmen med mig får enbart användas i forskningssammanhang.

□ Nej, jag deltar inte i forskningsprojektet.

Underskrift:____

Translation from Swedish original:

Hi! *Göteborg 120926*

I am a doctoral student in pedagogical work at the University of Gothenburg. I work with a research study about mathematics teaching. You might be aware of the fact that Swedish mathematics education is under much debate and that the focus is that too many students learn too little mathematics. To be able to develop mathematics education, we need to gain more knowledge about it. This is the context in which I plan to record one or two lessons with your class. Thereafter I will analyse the lessons in order to contribute to the knowledge on learning opportunities. When the study will be finished (in several years) it will be presented as a book for teachers and researchers of pedagogy. The material will also be possible to use in teacher education, if you approve to that. You will remain anonymous in the study and your participation is voluntary.

I hope that you will approve to participate in this research study. If you have any questions or anything you would to discuss, contact me by phone or mail.

Kind regards

Tuula Maunula
Doctoral student at the Department of Pedagogical, Curricular and Professional Studies, University of Gothenburg

Phone: xxxx xx xx xx
Mail: tuula.maunula@gu.se

My supervisors are:
Professor Ulla Runesson
PhD Johan Häggström

----

*Your name:*________________________________ *Your class:* ____________________

□ *Yes, I do participate in this research study. The recordings may be used both in research and in teacher education.*

□ *Yes, I do participate in the research study, but the recordings may only be used in research contexts.*

□ *No, I do not participate in the research study.*

*Signature:*____

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