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Transition Matrices Conditional on Macroeconomic Cycles:
A Portfolio Stress-Test Application

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Abstract

Transition matrices show the probabilities of credit rating migrations for a pool of ratings within a particular industry, geographical area, time-horizon, *etc.* Regulation, in the form of Basel accords, has opted for standards in banking that among other techniques use transition matrices, and thus the probability of default, for internally-based risk-assessment, as well as incorporating the external credit rating in the capital requirement calculation. We address credit-risk through the lens of the recent regulation, IFRS 9, which regulates the immediate recognition of losses on credit for loans entire lifetime if there has been a significant increase in credit-risk from future uncertainty in the macroeconomic environment. Our chosen approach is to simulate Markov chains, for credit ratings, conditional on background information (cycles in the economy). To quantify the effect on losses for a bank, we apply the transition matrices to a portfolio of bonds under the CreditMetrics^{TM}-framework for portfolio stress-tests, and use the pricing formula for defaultable bonds given by Jarrow, Lando, and Turnbull (1997) to value the portfolio. We use data on rating changes from Standard & Poor's for 934 U.S. companies during 1986 – 2018 to estimate the generator matrix, the Weibull-distribution of upgrade and downgrades, and the transition matrix. We compare simulations with a constant rate to the empirical results, to analyze how well the Markov property holds for each rating transition. The rates are then calibrated for macroeconomic cycles in each company's simulated Markov chain. We allow for two cycles in the economy (”expansion” and ”contraction”) and three magnitudes of the cycles (”low”, ”medium” and ”huge”). The transition matrices are applied to stress-tests in discrete-time for 10-years forward, under time-homogeneous models that analyzes consecutive years of economic expansion and contraction, as well as in a Mixture of Markov chains-model, by Fei *et al.* (2012), which mixes a Markov chain for business cycles with the Markov chain for ratings. We find that in scenarios of consecutive years of economic contraction and expansion respectively, the future loss distribution is apparent to be affected by the magnitude of the cycles, for those cycles assumed to be low and huge.

JEL Classification Numbers: G180 G240

Keywords: Risk Management, Migration Analysis, Intensity Models, IFRS 9, Basel Accords, Portfolio Stress Test

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Contents

List of Abbreviations

- BC – Business Cycle
- CDS – Credit Default Swap
- DD – Distance to Default
- ES – Expected Shortfall
- IASB – International Accounting Standards Board
- IFRS – International Financial Reporting Standards
- JLT – Jarrow, Lando and Turnbull
- LGD – Loss Given Default
- MMC – Mixture of Markov Chains
- NBER – National Bureau of Economic Research
- PD – Probability of Default
- PDF – Probability Density Function
- S&P – Standard & Poor's
- TM – Transition Matrix
- VaR – Value-at-Risk
- zcb – Zero-coupon bond

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1 Introduction

The newly adopted accounting standard for recognition and measurement of financial instruments, IFRS 9, started out in 2008 after massive credit losses due to delayed recognition of bad loans during the financial crisis of 2008 − 2009. The rule became effective on January 1, 2018, and aims at preventing some losses on loans, by introducing a forward-looking model that accounts for coming losses as soon as a financial instrument is recognized (International Accounting Standards Board (IASB), 2014). An external credit rating for a company or sovereign debt, are assigned by rating agencies and measures the default-risk. Corporate debt ratings are either ascribed to long-term issues, short-term issues or for a company itself, which are called issuer-ratings. Non-default classes also combine into two broader categories that tell about the quality of an investment in the obligation. These are often used in studies on default-risk and are investment grade, usually AAA to $BBB-$, and speculative (or junk) grade ratings, from BB down to C (Lannotta, 2010).

The score has a massive impact on the price of debt securities, as a claim's value decreases if there is a smaller probability that you get back the investment. Rating changes from a rating agency occur when the evaluation of the company, regarding firm-specific or macroeconomic factors, influence the default-risk assessment. When the assigned rating is meant to only survey during a fixed time, taking into account macroeconomic conditions, it is a ”point-in-time” rating, whereas the opposite is a ”through-the-cycle” rating that holds through future business cycles. These different systems govern how rating agencies look at the current and future cycles in credit risk assessments. In general, meetings between rating agencies and corporate managers, after an initial rating, occurs at least once a year. For the company, future outlook, strategies, *etc.* are re-assessed by the rating agency as a way of providing surveillance in changing markets. Changes to external credit ratings occur when the rating agency, for instance during an annual meeting, finds a need to adjust the rated debt instrument, which means they are not periodical (Trueck and Rachev, 2009). Changes to internally-based ratings, however, occur more often as the rating is assessed on the balance sheet by the financial institution each year. This delay for rating agencies means that the external rating lag weeks and sometimes even months after the market has already adjusted its prices of the debt. Therefore, external credit ratings are the slowest to change, in comparison to alternatives that use continuous measures of credit risk (Schönbucher, 2003). In the context of IFRS 9 a discrete-time forward-estimated transition matrix can estimate expected credit losses from defaults, in the long-term, which a portfolio of bonds will have throughout its lifetime (Perederiy, 2017).

An interesting feature of the transition matrix is that it is an *ex-post* measure of the performance of the credit ratings given out by rating agencies because it shows the ability of rating agencies to give out accurate scores that predict changes in creditworthiness after the period has finished (Estrella and Basel Committee on Banking Supervision, 2000). Models for transition matrices, where macroeconomic shocks are assumed to have an impact on the distribution of rating changes, means dealing with some of the same problems as assigning a "through-the-cycle" rating, namely an *ex-ante* estimation of the probability of default of firms in the future.

This thesis examines the effect of credit rating transition matrices (TM) conditional on economic cycles. The transition matrix can be used as a credit risk management tool, since it contains credit rating transition probabilities over a period (*e.g.* 1-year default probabilities). From the viewpoint of banks, which have massive credit exposure, a change in the forecasted probabilities of rating changes can be used to assess the effect it has on the bank's expected losses from default. Bond portfolio stress-tests of credit ratings that use a structural model, based on the Merton (1974) model for contingent claims on the firm's value, assumes that the probability of default (PD) connects to individual asset returns by the distance-to-default (DD) measure. An application of this assumption is CreditMetrics^{TM}, which simulates asset values and assigns new ratings from the defined upper-thresholds using the distance to default. Any modification of the probabilities of default that is calculated conditional on cycles in the economy, therefore, leads to changes in the estimated credit losses of a portfolio of bonds, when using a rating-based pricing formula for risky bonds by Jarrow Lando and Turnbull (JLT) (1997) (Gupton, Finger and Bhatia, 2007).

The relation between rating and default, in the simplest model, only depends on the current rating. These processes for the credit ratings are known as Markov chains, since it is a series of changes to credit ratings with Markov property, which means that the process is entirely dependent on its current state. The assumption of transition probabilities during infinite-small times, called intensities, together with the Markov property leads to the continuous-time Markov chain model for the rating process, which is the model we use for simulating companies' ratings (Schönbucher, 2003). In our portfolio stress-test, all the bonds expire in 10-years, which means Markov chains for each obligor are simulated for 10-years in different cycles (Gupton, Finger and Bhatia, 2007). We allow the intensities to vary in three magnitudes, in time, which we call "low", "medium" and "huge" and we assume that the economy has the following two states:

- "contraction", where upgrades are less common and downgrades are more common, and

- ”expansion”, where upgrades are more common and downgrades are less common for all credit ratings.

The result of our calculated transition matrices from the simulated Markov chains shows that stronger cycles shift the transition probabilities towards added upgrades (downgrades) and fewer downgrades (upgrades) in economic expansions (recessions). These results are in-line with the economic theory about expected credit events in positive and negative cycles, respectively.

Based on the ideas of Bangia *et al.* (2002), Frydman and Schuermann (2008) and Fei *et al.* (2012), we propose a Mixture of Markov Chains (MMC)-model in discrete-time that combines a Markov chain for the business cycle to the credit rating Markov chain. Our first hypothesis is that it is possible to account for future uncertainty in the macroeconomic environment in discrete-time through a mixture of Markov chains-model, which mixes the Markov chain for business cycles with the simulated Markov chains for ratings. The second hypothesis is that the CreditMetrics^{TM} method can quantify the effect that changes in the probability of default, due to cycles, have on the losses for a bank holding a broad portfolio of risky zero-recovery bonds.

We organize the rest of this thesis as follows. Section 2 outlines the definitions of credit ratings, the distance-to-default measure, regulatory framework for a stress-test on credit ratings and connection between the external rating in a Markov process to the term-structure to obtain a rating based pricing formula for defaultable bonds. Section 3 details the rating-models for the analysis, which includes a Markov chain with constant intensities and time-varying intensities as well as a discrete and time-homogeneous calculation of the transition matrix several years into the future in a mixture of Markov chains process for the business cycle. Section 4 is a formalization of how you apply CreditMetrics^{TM} for simulating the expected losses for a portfolio of risky bonds, by applying the assumption of distance to default for the underlying asset values. Section 5 describes the models and data that we use in the analysis, and the portfolio we use in the simulated stress-test. In Section 6, we present our results of the empirical transition matrices, the transition matrices from simulated Markov chains conditional on cycles, and the loss distributions from the portfolio stress-tests, which applies the probabilities of default that are calculated from the simulations of Markov chains. We end with the conclusion in Section 7.

2 Credit Ratings

In the broad perspective of capital markets, the external credit score is known to have a massive impact on the price of financial instruments and the score is mainly an assessment of the default-risk for the rated instrument (Lannotta, 2010). A credit rating does however also take into account some binding conditions of the financial obligation, such as what the recovery is if a default would occur (S&P, 2018). Credit scores on corporate debt instruments issued by a rating agency typically consist of ratings in the form of letter-scale. All three of the most prominent rating agencies, Moody's, S&P and Fitch adopts a letter scale for assigning a score on the credit quality of the obligor, or bond issue. The definitions of the letters-score in each of the different groups of rating issues, such as long-term or short-term obligations by the rating agencies are publically available on their respective websites.[1] A good external rating should mean that investments in the company are less risky since they have a considerable chance of surviving until the end of the contract.

For instance, the definitions for Standard and Poor's scale maintains that if you include all the rating modifiers for relative strength in creditworthiness, denoted: (+) and (−), a scale of 22 different possible ratings (including default) applies for long-term issuer ratings. Table 1 shows an example of the relative strength modifiers being assigned to speculative ratings for a company's long-term issuer rating. The short-term issuer ratings, on the other hand, adopts a range of eight possible ratings in a letter-scale, hence the difference in scales depending on the nature of the classification. By having cleared out the broad definitions that external ratings are supposed to cover

Date	Rating
2015–02–02	B+
2011–11–28	BB-
2011–10–27	BB
2010–10–29	BB+
2009–10–28	BB-

Table 1: Observed Rating Migrations for a Corporate Bond. Rating history of Netflix Inc. long-term issuer ratings from Standard and Poor's. The table illustrates the frequency of relative strength modifiers added and removed yearly between 2009 – 2011 to speculative rating BB, and then one the following month. The last rating migration occurred $3^{1/6}$ years later and was a downgrade in rating.

on debt markets, we now turn to the calculation of the distance to default measure and show the relation to the upper-threshold that determines a rating in the basic structural model by Merton (1974), the regulatory framework necessary to consider in

[1] Rating Scale Definitions on Rating Agencies Websites
Moody's: https://www.moodys.com/researchdocumentcontentpage.aspx?docid=PBC_79004
Standard and Poor's: https://www.standardandpoors.com/en_US/web/guest/article/-/view/sourceId/504352
Fitch: https://www.fitchratings.com/site/dam/jcr:6b03c4cd-611d-47ec-b8f1-183c01b51b08/Rating%20Definitions%2021%20March%202018.pdf

a stress-test under the assumption of distance to default, as well as utilizing a Markov chain for ratings to price defaultable bonds using the model by Jarrow, Lando and Turnbull (1997). Specifically, in Subsection 2.1, we formulate the distance to default measure, which is an industry standard for linking probabilities of default to rating threshold. Then in Subsection 2.2 we present some empirical research on the effect of macroeconomic variables on the transition probabilities for credit ratings. Subsection 2.3 explains the regulatory framework for a portfolio stress-test using external credit ratings as the changing variable. Then in Subsection 2.4 we detail how defaultable bonds are priced using the Markov model for term-structure by Jarrow, Lando and Turnbull (1997).

2.1 The Distance to Default

In this subsection, we use the Merton (1974) model, to define an equation for the probability of default for an individual firm using the distance to default. Loosely speaking, the rating is intended to represent the PD, which all companies within a group of classification should face equally (S&P, 2018). The default-time is, however, non-trivial to model, even over a relative short time-horizon due to unexpected cyclical effects, and only being observed after it has occurred (Trueck and Rachev, 2009). The KMV-method by Moody's for default-risk gives a much quicker credit risk assessment of a company since this model builds on firms capital structure using a continuous measure of the distance to default (McNeil, Frey, and Embrechts, 2010). In the Merton (1974) model, it is possible do derive an explicit expression for the default probability of a company as function of parameters describing the dynamics of the asset-value of the firm. More specificaly, using a structural model, *e.g.* the model by Merton (1974), a single company's default will occur when the value of their assets, V_T, at time T falls below the value of their debt, F, or some other specified default-threshold. The default-threshold's connection to a normaly distributed value of the firm's assets are illustrated in Figure 1. Using a standard geometric Brownian motion model for the asset value, we can write the default probability as a function of the distance-to-default, from the evolution of the asset value from time, 0, to, T, as (Chan *et al.* 2012):

$$P(\tau \leq T) = P(V_T < F) = P(V_0 \exp((\mu - \frac{\sigma^2}{2})T + \sigma Z_T) < F) = N(-DD(T)), \quad (1)$$

where

$$DD(T) = \frac{\ln(\frac{V_0}{F}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad (2)$$

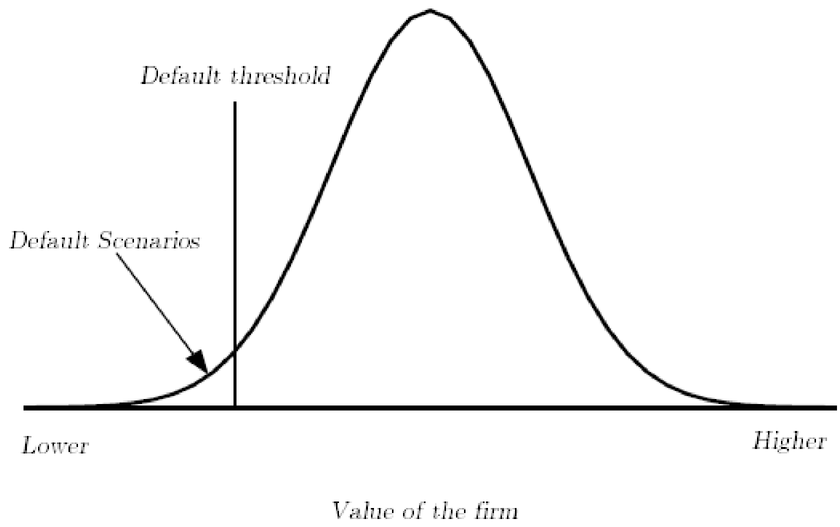


Figure 1: Illustration of the Distance to Default. Illustration of the default threshold defined on normally distributed asset-values in CreditMetrics^{TM} (from Gupton, Finger and Bhatia, 2007, p.37).

μ is the asset return drift, σ is the volatility, Z_T is standard Brownian motion, $N(\cdot)$ is the normal cumulative distribution function and $P(\tau \leq T)$ is the default probability of the company under real probabilities. Moody's KMV-method connects a specific rating with default through thresholds, z, for each rating class. Gordy and Heitfield (2001) extend the Merton (1974) model by assuming other alpha-stable distributions for the distance-to-default, that accounts for errors in otherwise normally distributed asset returns. In their model, ratings are assumed to be independently assigned, both cross-sectionally between a group of obligors and through time and the underlying stochastic process of the asset value:

$$V_{t+1} = V_t + \omega x + \sqrt{(1-\omega^2)}\epsilon, \tag{3}$$

where, x, is the systematic risk factor arising from macroeconomic conditions, ϵ, is an idiosyncratic risk factor, and, ω, is the factor loading parameter. The systematic effect, x, on asset return can assume to have an alpha-stable distribution, such as normal distribution, or other thick-tails.

It is beyond the scope of this paper to examine the firm-specific factors on credit ratings, and as such, in the remaining part firms within a score will always be treated as homogeneous. The firm-homogeneity assumption also holds for the CreditMetrics^{TM}-model (stress-test application), where the asset returns are assumed to be normally distributed, with mean, 0, and standard deviation, σ (Gupton, Finger and Bhatia, 2007). The thresholds that determine the ratings are therefore the number of standard deviations from the mean, which is 0 in CreditMetrics^{TM}, that the obligor's assets need to move to change in rating. For example, the case of a single obligor, h, using the formulation provided in CreditMetrics$^{\mathrm{TM}}$, has an upper-threshold for default, z_0, that is defined as:

$$\begin{cases} \text{Default:} & \text{if } z_h \leq \frac{z_0}{\sigma} \\ \text{No Default:} & \text{if } z_h > \frac{z_0}{\sigma} \end{cases}.$$

The CreditMetrics$^{\mathrm{TM}}$-model of only moving different standard deviations, depending on rating, in order for a default to occur is realized by looking at Figure 1 if the mean of the function is 0. The default-probability for obligor, h, associated with the threshold then becomes:

$$P(\text{Default}) = P(z_h \leq \frac{z_0}{\sigma}) = N(\frac{z_0}{\sigma}). \tag{4}$$

In other models, *e.g.* the one by Gordy and Heitfield (2001), the defaults do not have to be normally distributed and can have any alpha-stable cumulative distribution. In CreditMetrics$^{\mathrm{TM}}$, asset returns are however assumed to be normally distributed and correlated, *e.g.* between industries (Gupton, Finger and Bhatia, 2007). The needed amount of standard deviations that has to change, z_0, in order for an obligor, h, with a certain rating to default, becomes independent of its own asset return, z_h in Equation (4). As such, obligors within the same rating are homogeneous in the structural models, in terms of thresholds. Therefore, analogously to Equation (4), probabilities of default for any rating can be converted into thresholds by taking the inverse of the normally distributed cumulative function and multiplicating with the standard deviation (Gupton, Finger and Bhatia, 2007):

$$z_0 = N^{-1}(P(\text{Default}))\sigma. \tag{5}$$

The risk of any rating change and or defaults can hence be represented, by using either transition matrices consisting of probabilities, or by a matrix consisting of the upper-thresholds, z. A convenient way to convert from probabilities to thresholds is to use the function: TRANSPROBTOTHRESHOLDS in MATLAB, which converts utilizing the convention established in CreditMetrics$^{\mathrm{TM}}$ (MATLAB and Financial Toolbox Release 2018a, The MathWorks, Inc.). The structural models and its ability to approximate thresholds are a crucial assumption in CreditMetrics$^{\mathrm{TM}}$. There is, however, some benefits to instead use intensity models for the credit rating dynamics, instead of a structural model. As is shown later, the times spent in ratings can be assumed to be exponentially distributed. Under intensity models for the rating process, we do not need to model the evolution of the asset value. Instead rating changes are exogenous and depend on each of the transition rates. The structural model will, however, be revisited in the application of CreditMetrics$^{\mathrm{TM}}$, since it is a part of the stress-test to simulate asset returns and let the upper-threshold determine the new ratings (Gupton, Finger and Bhatia, 2007).

2.2 Macroeconomic Factors on Default Probabilities

It is no doubt that Moody's, S&P and other rating agencies assign their ratings depending on cycles in the economy. Moody's private rating process, known as RiskCalc, looks at several relevant factors individually to account for how they can change depending on the macroeconomic environment (Dwyer *et al.* 2004). Nickell *et al.* (2000) have studied the transition matrix of Moody's long-term bonds between the period 1970 – 1997 in three different cycles, referred to as: ”trough”, ”normal” and ”peak”. They find statistically significant results that Moody's rating transition rates depended on the stage of the business cycle during the period in which the bonds ratings got published. There is a significant increase in non-investment grade bonds defaulting during recessions and default rates for all ratings increased (Nickell *et al.* 2000). Research has also shown that periods of high economic growth tends to swing the transition probabilities downwards, resulting in more upgrades. In Figlewski *et al.* (2012) they conclude that macroeconomic variables which mainly impacted the intensity of upgrades and default of credit ratings were GDP growth rate, unemployment, and inflation.

Even though cyclical patterns are observable for credit ratings from all rating agencies, the dominating method in risk-management aims for an approach towards ”through-the-cycle” ratings, so that the evaluation withstands cycles in the economy, which is evident in Basel II accords (The Basel Committee on Banking Supervision, 2003):

> ”A borrower rating must represent the bank's assessment of the borrower's ability and willingness to contractually perform despite adverse economic conditions or the occurrence of unexpected events.” (Basel II, paragraph 376, Basel Committee on Banking Supervision, 2003)

With this in mind, there seems to be a motivation to study if the use of recent data, can estimate a transition matrix that shows notable differences in probabilities for default and rating transitions, due to the recent financial crisis of 2008 – 2009, compared to what the transition matrices published by rating agencies over a larger observation period usually indicates. Our analysis will, however, be limited in this regard as we look at the distribution of upgrades and downgrades over a period that spans multiple cycles, which should not be compared to the external transition matrices estimated from rating agencies over other periods characterized by another business cycle.

2.3 Regulatory Framework for a Portfolio Stress-Test Under Macroeconomic Cycles

The first Basel accord (The Basel Committee on Banking Supervision, 1987) regulated bank's capital in a relatively simplistic way by assigning risk-weights (10%, 20%, 50%, 100%) for claims according to their relative risk, which incorporated credit risk. However, the external ratings played no role in the claims risk-weight. Thus, in the starting phase of the Basel accords, the amount of bank capital needed was only based on these risk-weighted assets. Lando and Skødeberg (2002) describe how the second revision of the accords, Basel II (The Basel Committee on Banking Supervision, 2003), aimed to set out a system where vast differences in risk for financial instruments can be well-recognized and quantified. Technically, there were three-pillars in Basel II. However, the connection to ratings and hence the TM became stronger in the overall aims since more requirement in risk-management were established. Basel II also introduced the internal ratings-based approach for banks to have supervisors, within their bank, to see over creditworthiness of the bank's assets and credit exposures. Supervisors were required to evaluate the risks according to Basel standards and principals, which were:

> ”The overarching principle behind these requirements is that rating and risk estimation systems (...) provide for a meaningful differentiation of risk and accurate and consistent quantitative estimates of risk.”(Basel II, paragraph 351, Basel Committee on Banking Supervision, 2003)

> ”Banks must have a robust system in place to validate the accuracy and consistency of rating systems, processes, and the estimation of all relevant risk components.” (Basel II, paragraph 463, Basel Committee on Banking Supervision, 2003)

Regulators hence advocated for internal credit risk models, which had an advantage for the stability of international banking sectors, compared to the previous accord of only using risk-weighted assets. In many ways, the increased regulation was due to the complexity of financial derivatives. Basel II, implemented in 2007, now meant ratings for a bank's portfolios of assets, should be priced and weighted accordingly with the inherent credit risk represented in the external credit score. Basel III (The Basel Committee on Banking Supervision, 2017), which is the newest accord, outlines more standards for loss distributions and measures for including a credit value adjustments into the price of derivatives. The new strengthened rules thus regulate further how to adopt internally based loss-models for capital and liquidity requirements, which is the focus in our analysis of a credit rating transition matrix model for stress-testing.

Together with evidence from previous studies that macroeconomic covariates influence the creditworthiness assessment, the aims of the two most recent Basel accords suggests that a bank's capital allocation process, to a varying extent, is up to a bank's internal model, *e.g.* one similar to the model of CreditMetrics™. The standards in the Basel accords taken together with an appropriate model for the credit rating TM, could, for example, approximate economic capital needed in a bank. Thus, by analyzing the 95%-confidence-level you get in a CreditMetrics™ loss-distribution from a simulation it would be possible to measure the required capital needed to cushion against future losses on a 95% certainty-level (Gupton, Finger and Bhatia, 2007).

2.4 Ratings Changes Implications on Portfolios

By modeling the TM conditional on business cycles, it becomes a tool for risk managers to account for sensitivity to cycles in their calculations of the credit exposure (Bangia *et al.* 2002). CreditMetrics™ provides a method for quantifying the effect of different transition probabilities in the transition matrix on losses for a portfolio (Gupton, Finger and Bhatia, 2007). Using a transition probability-weighted price for a bond is not correct. Therefore, we will apply the Jarrow, Lando, and Turnbull (JLT) (1997)-model for pricing bonds in our stress-test. We explain the connection between external ratings and risk-neutral bond pricing using the JLT-model in Subsection 2.4.1, as well as limitations of any external ratings based term-structure model in Subsection 2.4.2.

2.4.1 Rating Based Bond Pricing

Pricing risky bonds under the martingale measure mean we cannot use empirical probabilities of default for this purpose to assess how much we are expected to have left of a portfolio after a period of uncertainty. An illustration of this remark given by Lando (2004), shows that the real spread of a corporate bond could be much greater than the implied spreads, Figure 2. Since we use real probabilities as inputs to simulate the rating transitions in the Markov chain, the elements need to account for a risk-premium, set by market prices, if used in an equation to price bonds.

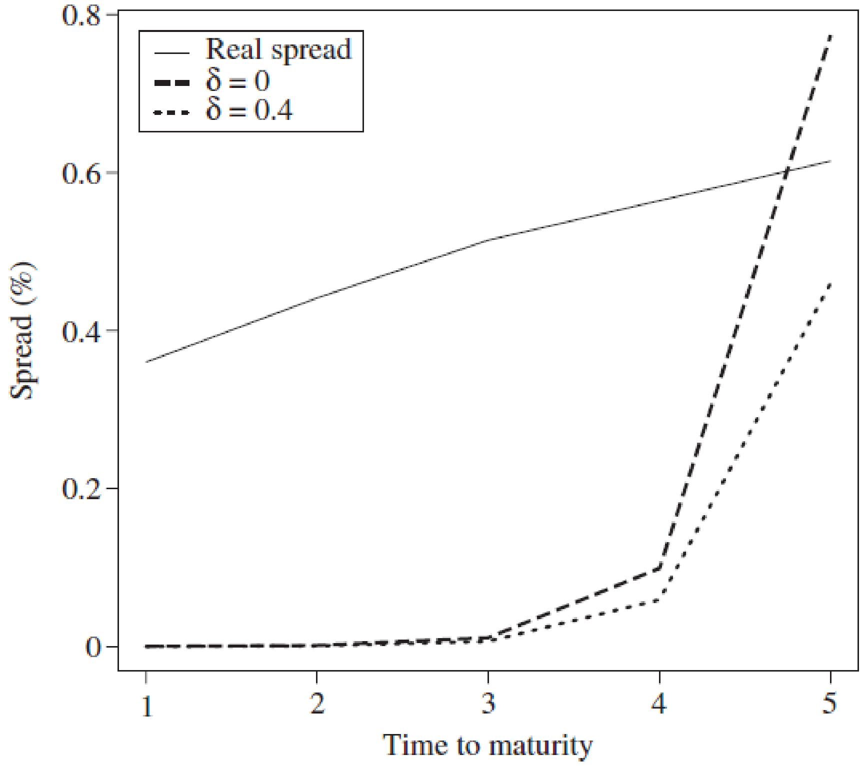


Figure 2: Comparison of Implied and Real Credit Spreads. Illustration of the difference between implied spread using two loss given defaults, δ, and the real spread for a single investment-graded bond issue (from Lando, 2004, p.147).

Jarrow and Turnbull (1995) provide the pricing equation of risky zero-coupon bond (zcb), maturing at time T, by discounting the expected payoff given the risk of default at time τ^* using the risk-free zcb $p(t,T)$. For a loss given default δ and a martingale measure denoted $\tilde{Q}$, which we assume is independent of the term-structure for risk-free bonds[2], the bond pricing equation becomes:

$$v(t,T) = p(t,T)(\delta + 1(1-\delta)\tilde{Q}_t(\tau^* > T)). \tag{6}$$

Jarrow, Turnbull and Lando (JLT) (1997) were the first to provide a model for calibrating the rating Markov chain process to prices, by extending Equation (6) into a Markov model for the term-structure. We assume that there exist estimates of real transition probabilities, *e.g.* published by a rating agency, under probability measure

[2]This assumption is more realistic for investment-graded bonds than for speculative rated issues (Jarrow, Turnbull and Lando, 1997).

Q, and that the $K \times K$ transition matrix fufills:

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1K} \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{K-1,1} & q_{K-1,2} & q_{K-1,3} & \cdots & q_{K-1,K} \\ 0 & 0 & \ldots & \ldots & 1 \end{pmatrix}, \tag{7}$$

where $q_{ij} \geq 0 \ \forall i, j, \ j \neq i$ and $q_{ii} = 1 - \sum_{j=1}^{K} q_{ij}, \ i = 1, ..., K-1$. In the JLT-pricing model (1997), Equation (6) depends on credit rating i, and the survival probability is such that:

$$\tilde{Q}_t^i(\tau^* > T) = \sum_{j \neq K} \tilde{q}_{ij}(t, T) = 1 - \tilde{q}_{iK}, \tag{8}$$

where $\tilde{q}_{ij}(t, T)$ are the elements of a transition matrix under risk-neutral probabilities, $\tilde{Q}$. As Figure 2 show, the market implied default probabilities are greater than real default probabilities[3]. In Equation (8), we should have probabilities of default for all ratings under the $\tilde{Q}$ measure to be able to price the portfolio of bonds correctly in the stress-test, where we will be using Equation (6). In general, it is not a problem that we simulate the Markov chain under real probabilities since the pricing Equation (6) is independent of the path of ratings or the transition probabilities other than those into default. Therefore it is possible to retrieve market implied probabilities of default from CDS spreads and use this implied PD as $\tilde{q}_{iK}$ in Equation (8).

This way the portfolio of bonds will be able to transition between ratings under the real probabilities, and we are consistent with pricing under the risk-neutral measure. We are aware that using risk-neutral measures for forecasting future losses on loans for a bank is not appropriate for a stress-test, nor an industry standard, for a bank since they likely want to analyze real losses and not those measured under a risk-neutral setting. Hence it is a limitation in our model that the losses in the stress-test are under the risk-neutral measures. However, as we use empirical distributions of rating transitions, the dynamics of the rating changes will be under real-world probabilities which we believe adds to the robustness of combining a Value-at-Risk simulation with risk-neutral pricing approach in Equations (6) and (8).

2.4.2 Limitations of Using Ratings Based Term-Structures

> ”Credit ratings are often viewed by the market as lagging indicators. For these bonds, the ratings are difficult to reconcile with the default probabil-

[3] In Appendix D we show the sensitivity of our stress-test to different risk premiums.

ities as assessed by the markets. It might be argued that it is not sufficient to focus only on credit ratings when assessing whether assets are ”low risk”, according to CDS spreads.” (Ernst & Young, 2014)

Rating agencies have a reputation for being slow, compared to *e.g.* Moody’s KMV-database, when it comes to re-evaluating their rated issues. There is also some raised criticism on regulation on Basel II, which further emphasized a score-based credit risk model for institutions acting on the financial markets. McNeil, Frey, and Embrechts (2010) explain this criticism in that such regulation might be ”eating its own tail”. The risks are often much more complicated than merely a rating could ever describe. Further, this lack of public data on companies creditworthiness means that statistical models that account for credit ratings are limited to trust the agencies opinion. Another limitation is the default dependence, meaning that if we price a portfolio of bonds according to the benchmark for a similarly rated corporate bond, we do not consider the domino effect of firms’ default rate as they are likely correlated. As recessions start to happen, one company’s default will affect others through the linkage of borrowing. The result of such a limitation of external ratings is that once the recession starts, companies dependence in the slow external rating’s model leads to a large Value-at-Risk (VaR) for a portfolio, defined for a fixed percentage as the certain losses. The effect might, in reality, be a lot stronger than what the underlying model shows (McNeil, Frey and Embrechts, 2010).

3 Models

One is likely to encounter problems if future credit loss-models were only using the historically estimated transition matrix. To begin with, some events, *e.g.* default or change into a speculative rating, for investment graded bonds have zero probability since the sample does not include any of those cases. As Lando (2004) explains, this is something you want to avoid to have in models of the credit risk for a portfolio of bonds, as it does not reflect real-world scenarios. Of course, historical estimates do not capture all events, and there is always an inherent default-risk, even for a top investment-grade bond. The dataset that is used to estimate the TM might not include any of those scenarios. However, once the default happens, it could lead to massive unexpected losses for carelessly calibrated credit risk models, see for example the previous discussion about correlated defaults in Subsection 2.4.2. Therefore, an exclusion of the probability of unlikely events in credit risk models might be enough to ensure that capital is not available to cushion for unexpected losses once they occur.

Secondly, there is the problem with using empirically estimated default-probabilities in pricing a portfolio, as we explained in Subsection 2.4.1, as risk-neutral probabilities are the way to go.

In what follows in this section, the aim is to explain our models used for rating processes, by accounting for systematic risk, and how the mixture of Markov chains (MMC)-model work by mixing the process for credit ratings with the process for future stochastic cycles in the economy. We begin in Subsection 3.1, by outlining the continuous-time Markov chain assumption and then show how the Markov chain is simulated with a constant rate and with a rate conditional on background information from the macroeconomic environment. Then, in Subsection 3.2 we show how to calculate the transition matrix in discrete-time from a Markov chain, several years forward. We show the discrete-time calculations in the time-homogeneous case, using both an assumption about consecutive ”expansions” and ”contraction” and in a Mixture of Markov chains-model, which as mentioned mixes these with the process for macroeconomic cycles.

3.1 Markov Chain for Ratings’ Processes

In this subsection, we introduce continuous-time Markov chains. We begin by first defining the essential concepts in a continuous-time Markov chain, which are the Markov property, the generator matrix, and the Chapman-Kolmogorov differential equation of choice. In Subsection 3.1.1 we explain the estimation procedure for the intensities, which uses the duration spent in ratings. In Subsection 3.1.2 we show how to simulate a Markov chain with constant intensities. Then in Subsection 3.1.3 we state the simulation technique for a Markov chain conditional on background information.

The arguably most important assumption, used in many models for credit rating dynamics, is that ratings follow a continuous-time Markov chain process. This assumption is much appreciated for its broad applications when modeling transition probabilities, or merely corporate default. In simple terms, the Markov property means that a future state can be predicted by its present state alone, thus not requiring knowledge about the history of how the process moved between states. The Markov chain property is also the foundation in our of construction of a conditional TM for credit ratings. Using survival analysis, we analyze how well the Markov property is fulfilled historically, by looking at the past distribution of credit rating upgrades and downgrades.[4] The assumption applied in the analysis of how rating changes

[4]See Appendix B for this analysis.

are distributed differently in economic cycles from what the empirical results of the Markov property show, lead to a benchmark for the distribution of rating upgrade and downgrades occurring at a constant rate. For a set of, K, scores: $S = \{1, ..., K\}$, the Markov property can be formulated as follows (Schönbucher, 2003):

Markov property. *A stochastic process for the credit rating, is said to be a Markov process if the probability of the rating being $r \in S$ at a later time $T > t$ only depends on the current rating, R_t:*

$$P(R_T = r|\Omega_t) = P(R_T = r|R_t),\ \forall r \in S,$$

where Ω_t is the information set until time t.

Data on credit score changes comes in discrete time intervals, whereas the rate of a credit event (*e.g.* upgrade or downgrade) can be assumed to be the probability of the credit event occurring at an arbitrary small times after an initial rating's issue. The Markov assumption in continuous-time lets you construct a generator matrix, which is well-suited in modeling the dynamics of the rating for a group of homogeneous obligors, using the infinite-small probabilities for moving from state i to state j during Δt conditional on being in state i at time t.

The intensity of a credit event is the probability of the event occuring after a short period, Δt (Trueck and Rachev, 2009). To get a formulation for the intensities for ratings, assume that the probabilities of a transition are proportional to time when the time, Δt, is small. Using the specification for the rating process as specified in the Markov property, we have (Trueck and Rachev, 2009):

$$P(R_{t+\Delta t} = j|R_t = i) = \lambda_{ij}\Delta t, \text{ for } i \neq j. \tag{9}$$

As shown in Schönbucher (2003), a matrix where the off-diagonal elements are represented by each intensities such that: $\lambda_{ij} \geq 0\ \forall i, j \in S$, $i \neq j$, the diagonal elements are defined as: $\lambda_i = -\sum_{i \neq j} \lambda_{ij}$, $\forall i \in S$ and the matrix is the logarithm of the transition probabilities for a fixed time t, is called a generator matrix and has the form:

$$\Lambda = \begin{pmatrix} -\lambda_1 & \lambda_{12} & \lambda_{13} & ... & \lambda_{1K} \\ \lambda_{21} & -\lambda_2 & \lambda_{23} & ... & \lambda_{2K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{K,1} & \lambda_{K,2} & ... & ... & -\lambda_K \end{pmatrix}. \tag{10}$$

The generator matrix (Λ) is neatly connected to the TM through the Chapman-Kolgomorov differential equations. We will use a time-homogeneous model, which is a simplifying assumption for solving this forward differential equation:

$$\frac{\partial}{\partial T}Q(t,T) = Q(t,T)\Lambda(T),\ T > t, \tag{11}$$

with the solution:

$$Q(t,T) = \exp((T-t)\Lambda) := \sum_{n=0}^{\inf} \frac{1}{n!}((T-t)\Lambda)^n. \tag{12}$$

Note that $Q(t,T)_{ij} = P(R_T = j|R_t = i)$. For more on Chapman-Kolmogorov differential equations related to credit ratings and the generator matrix in the non-timehomogeneous case, see Appendix 2.3 in Lando (2004). In the Chapman-Kolmogorov equations, transitions consist of intermediate stages between each movement for ratings, and the probability of moving from one grade to another means going through all the middle hazards. The generator matrix, $\Lambda(t)$, which has the instantaneous probabilities at t, and the TM, $Q(t,T)$, that spans a longer period from t to T, is connected through the forward differential equations, by the assumption of intermediate steps, in the Equation (12). We will use the assumption of a time-homogeneous model in every calculation of the transition matrix. Given that we have a generator matrix, it is a straightforward task to solve Equation (12) in MATLAB, using the function EXPM, which calculates the matrix exponential (MATLAB Release 2018a, The MathWorks, Inc.).

3.1.1 Estimation of Intensities Using Duration

The intensities that make up the generator matrix in Equation (10) is calculated using a maximum likelihood formula, which we will not go into detail about since the solution to the maximum likelihood estimation in this case is a widely popular equation. A yearly λ_{ij} of transitioning from rating, i, to, j, is given by the formula (Schönbuscher, 2003):

$$\lambda_{ij} = \frac{n_{ij}}{\int_0^t Y_i(s)ds}, \tag{13}$$

where n_{ij} are transitions from, i, to, j, in the interval $[0,t]$ and $Y_i(s)$ is the number of firms rated, i, in a year, s. The denominator in Equation (13) represents the total duration spent for all the obligors in rating, i, over the period $[0,t]$ and has the unit of time. It is therefore readily recognized that the intensities, λ, are the average migrations per unit of time (Lando, 2004). In the time-varying case, the interpretation of intensities merely means that the instantaneous likelihood of changing from one state to another changes continuously. From Equation (9) we know that under a short period, the probability of moving from rating i to j is equal to the corresponding transition intensity. The sum of the rows in the generator matrix is zero because λ_{ij} are the probabilities for moving away and the diagonal is the negative sum of that (Lando, 2004). The modeling technique of using continuous-time Markov chains for rating processes are also widely used in the empirical literature, by *e.g.* Bangia *et al.* (2002) for their calculation of transition matrices conditional on specific economic cycles.

3.1.2 How to Simulate a Markov Chain with Constant Intensities

The arrival times spent in each rating, S, are random and only determined by the given state of the process, R_t. Assuming constant intensities is the most basic kind of Markov process, where S, are exponentially distributed. In computational terms, it means that each S can be simulated from a uniformly distributed random variable, U, using the formula (Kay, 2006):

$$S = \frac{1}{\lambda} \ln \frac{1}{1-U}, \tag{14}$$

where $U \sim U(0,1)$ and λ is the negative of the diagonal element in the generator matrix in Equation (10) that corresponds to the rating the process R_t is currently in. An illustration of simulated rating changes for a Markov chain using Equation (14) is shown in Figure 3. Each time spent in the ratings, S, in Figure 3 are exponentially distributed with an intensity equal to the negative of the diagonal element corresponding to the current rating R_t in the generator matrix in Equation (13).

3.1.3 How to Simulate a Markov Chain with Time-Varying Intensities

McNeil, Frey, and Embrechts (2010), as well as Lando (2004), gives an explicit algorithm for simulating arrival times in a generalized Poisson process, in a simple 2-step process and we attribute the model to Lando's (1998) work on Cox processes for credit ratings. The requirement is that the time-dependent $\lambda(s)$, is continuous and non-negative. Note that $\lambda(s)$ does not need to be deterministic. In fact, it would maybe make more sense that it depends on a process for the short-rate, X_s. However, our contribution is to utilize a deterministic function that is achieved by calibrating the empirical distributions of transition rates. The algorithm for simulating the first jump of a continuous-time Markov chain process conditional on background information is

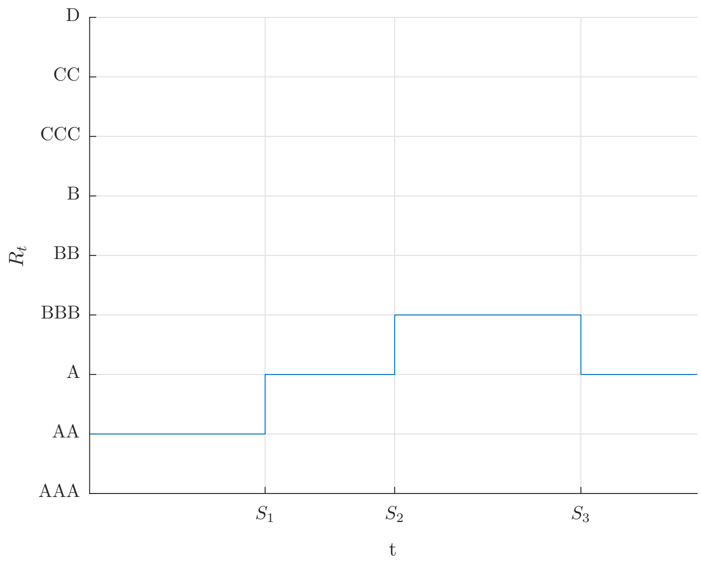


Figure 3: Illustration of Simulated Markov Chain. Illustration of a simulation of the rating Markov chain, R_t, for a single company that is starting in rating AA.

as follows:

1. Simulate a standard exponentially distributed random variable, E, that is independent of the intensity function $\lambda(t)\ \forall t \geq 0$. The random variable, E, acts as the "threshold" for accepting the time, S.
2. McNeil, Frey and Embrechts (2010) and Lando (2004), then show that for a strictly increasing and finite cumulative hazard function $h(t) = \int_0^t \lambda(s)ds$, $t > 0$, the conditional first stochastic arrival time will have the distribution:

$$S = h^{-1}(E) = \inf\{t \geq 0 : h(t) \geq E\}, \tag{15}$$

where, $E \sim \text{Exp}(1)$. Equation (15) is an important result, that can be used to analyze the effect of systematic risk (or macroeconomic factors) on credit rating transitions. From the results of Equation (15), it would then be possible to specify a time-dependent intensity, $\lambda(t)$, and have an exponentially distributed random variable, E, with unit intensity simulated to get the first time spent in the initial rating. By iterating Equation (15) for a company until finally $S > T$, can give us multiple times spent in each rating, S, for a company during T years conditional on a time-changing intensity. This will prove very useful in our analysis of the macroeconomic effect on transition probabilities, as the hazard rate distribution of downgrade and upgrades can first be estimated empirically and then be deterministic and depend on time as shown in Figure 16 in Appendix A (Lando, 1998).

Low Cycles		
Transition	Contraction	Expansion
Downgrade	1.2	0.8
Upgrades	0.8	1.2
Medium Cycles		
Transition	Contraction	Expansion
Downgrades	1.3	0.7
Upgrades	0.7	1.3
Huge Cycles		
Transition	Contraction	Expansion
Downgrades	1.5	0.5
Upgrades	0.5	1.5

Table 2: Weibull α-Shape Parameter Used in Markov Chain Simulation in Business Cycles. This table shows the values for the shape parameter, α, in the Weibull probability density function $f(t) = \lambda\alpha t^{\alpha-1}$ for the time spent in a rating until a credit event, *downgrade* or *upgrade* happens. An $\alpha > 1$ means that the rate is increasing in time, and an $\alpha < 1$ means that the rate is decreasing in time. We distinguish three cycles of varying parameter for, α, named in the order from least effect of time-variation to most time-dependent: ”low”, ”medium” and ”huge”.

We will adjust the intensities conditional on a 2-state economy, downturn, and expansion, by adjusting the shape distribution parameter, α, in a Weibull-distribution for the transition rates, to correspond to cycles in the economy, depending on if it is an upgrade or downgrade.[5] In total, 3 magnitude cycles will be accounted for with different α:s that determine the magnitudes of time-variation (”low”, ”medium”, ”huge”). The distributions for the transition rates that are used in our simulations are summarized in Table 2. Under the distribution assumption in Table 2, we can calculate the discrete-time transition matrix for each Markov chain using Equations (10), (12) and (13), such that the transition matrices Q_C will represent the business cycle of economic contraction, where upgrades are less common, and downgrades are more common and transition matrices Q_E will represent the business cycle of economic expansion, where upgrades are more common, and downgrades are less common.

3.2 Calculating Forward Transition Matrix in Discrete-Time

The Cohort model is used by rating agencies when they publish one-year average TMs for all rating classes that belong to the corresponding cohort. The model works well for presenting ordinary TMs, when using an extensive database of rating changes, such as the one available to significant rating agency companies (Schönbucher, 2003).

In the Cohort model of transition probabilities, the probability of going from rating, i, to, j, is usually calculated over a year and the formula for the probabilities are (Schönbucher, 2003):

$$Q(t, t+1)_{ij} = \frac{n_{ij}}{n_i}, \tag{16}$$

where, n_{ij}, are the number of transitions from, i, to, j, and, n_i, are the number of

[5]See Appendix A for the survival probability of Weibull-distribution.

obligors in rating i in the beginning (previous year).

In discrete-time, the transition matrix can be calculated n-years into the future, if the model for the transition matrix is time-homogeneous and in discrete-time, by simply having a yearly TM $Q(t, t+1)$ and raising it to power, n (Perederiy, 2017):

$$Q(t, t+n) = Q(t, t+1)^n. \tag{17}$$

Equation (17) therefore corresponds to the case where the same TM is expected to hold consecutively for, n-years, with non-switching business cycles. One way to obtain a forward estimate for the credit rating TM that is conditional on switching cycles in the economy is to use a regime switching Markov chain model for business cycles. A stochastic process for the business cycle combined with the Markov chain for ratings leads to a Mixture of Markov Chains-model, which conditions the TM on the uncertainty of the future states as well. These models have been used to study credit rating data by *e.g.* Bangia, *et al.* (2002), Frydman and Schuermann (2008) and Fei *et al.* (2012).

In the regime switching model, the future state is still unknown, but institutional changes between macroeconomic cycles are modeled as a Markov chain, which means that changes to business cycles only depends on the current cycle that we are in. Each of the cycles has a rating Markov chain for companies characterized by the cycle. The uncertain future state of the economy, after one year, is represented by a transition probability matrix of two business cycles, expansion (E) and contraction (C) as:

$$Q_{BC}(t, t+1) = \begin{pmatrix} p_{E,E} & (1 - p_{E,E}) \\ (1 - p_{C,C}) & p_{C,C} \end{pmatrix}. \tag{18}$$

By making use of the assumption of time-homogeneity for credit ratings, the resulting future rating transition matrix, calculated for some years forward, becomes an average of the TMs conditional on economic contraction and expansion, weighted by the probabilities of the different states occurring in the current time in Equation (18) (Trueck and Rachev, 2009). A way to implement a Mixture of Markov Chains-model for the rating process in practice is to simulate Markov chain conditional on cycles such as those explained in the previous section, with 1-year transition matrices $Q_E(t, t+1)$, and, $Q_C(t, t+1)$, by using the simulation technique explained in Subsection 3.1.3. Then, estimate the business cycle transition matrix in Equation (18) from published research on business cycles. Of course, only relying on published business cycle reports from *e.g.* National Bureau of Economic Research (NBER), means that we also need to know if there exists a relationship between the stochastic process which determines the business cycle probabilities and credit rating events. Fortunately, the MMC-model, in this implementation has already been used by Bangia *et al.* (2002) and Fei *et al.* (2012). Bangia *et al.* (2002), use quarterly data on U.S. business cycles from NBER to estimate state transition probabilities for one year ahead. With these past studies in mind, we deem the method to be justifiable to use in our case as well.

The methods used by Bangia *et al.* (2002) and Fei *et al.* (2012) of calculating the rating transition probabilities corresponding to different economic cycles are the same in both their articles. Bangia *et al.* (2002) and Fei *et al.* (2012) use a method to divide the sample of rating changes into periods of different economic activity and use the data in these periods to get conditional transition probabilities corresponding to each state. However, Fei *et al.* (2012), generalizes the regime-switching models, to be able to account for the evolution of transition matrices, n-years into the future, whereas Bangia *et al.* (2002) use a model only for one forward year. Since we want to look at losses for the full life-time the n-years method will be used. A n-year forward estimate of the TM conditional on cycles is obtained by following the method of Fei *et al.* (2012). We write out the matrix dimensions for easier reading the formulas. A block matrix, consisting of the TMs conditional on the two states and the corresponding transition probabilities for the each state is first defined as:

$$Q_M(t,t+1) = \underset{2K\times 2K}{\begin{pmatrix} p_{E,E}Q_E(t,t+1) & (1-p_{E,E})Q_C(t,t+1) \\ (1-p_{C,C})Q_E(t,t+1) & p_{C,C}Q_C(t,t+1) \end{pmatrix}}, \tag{19}$$

where the number of ratings (discrete state-space) in the Markov chain is K. It is then possible to obtain the TM conditional of business cycles after, n-years, assuming time-homogeneity. The formulas for the conditional TMs then become (Fei *et. al*, 2012):

$$Q(t,t+n) = FQ_M^{n-2}L', \tag{20}$$

where in Equation (20), the variables, L' and F are defined as:

$$L' = \underset{2K\times K}{\begin{pmatrix} p_{E,E}Q_E(t,t+1) + (1-p_{E,E})Q_C(t,t+1) \\ (1-p_{C,C})Q_E(t,t+1) + p_{C,C}Q_C(t,t+1) \end{pmatrix}}, \tag{21}$$

and if the current regime is expansion (E):

$$F = \underset{K\times 2K}{\begin{pmatrix} p_{E,E}Q_E(t,t+1) & (1-p_{E,E})Q_C(t,t+1) \end{pmatrix}}, \tag{22}$$

or if the current state is contraction (C):

$$F = \underbrace{\Big((1-p_{C,C})Q_E(t,t+1) \quad p_{C,C}Q_C(t,t+1)\Big)}_{K\times 2K}. \tag{23}$$

In our results we will assume that the current state is expansion. From Table 2 we have in our mixture of Markov chains three block matrices, Equation (19), since the magnitude of the conditional Markov chains defined by the transition matrices, $Q_E(t,t+1)$ and $Q_C(t,t+1)$ are either ”low”, ”medium” or ”huge”.

4 Portfolio Application for Stress-Testing on Losses

CreditMetrics^{TM} is a portfolio-based approach to credit risk, introduced by JP Morgan's risk management team in 1997. The approach in CreditMetrics^{TM} is to weigh prices of claims with its probabilities, assuming that bonds prices should depend on all transition probabilities of a rating, and obtain a value for the full portfolio that way (Gupton, Finger and Bhatia, 2007). As we have seen in Subsection 2.4.1, this is not how you price bonds. The pricing formulas in Gupton *et al.* (2007) are not industry standards in bond pricing, nor do they provide an accurate method of pricing a portfolio of bonds, as only the risk-neutral default-probabilities are relevant. For the sake of applying our transition matrix in a stress-test to analyze expected and unexpected losses, we use the standards in the CreditMetrics^{TM}-method for simulating asset values that determine the new ratings, which we explained in Subsection 2.1. However, for pricing the portfolio under a ratings based model, we will not use the formulas in CreditMetrics^{TM}, since they would be wrong for pricing bonds trading on the market. Thus we price the portfolio of bonds using Equation (6) which we explained in Subsection 2.4.1.

A Value-at-Risk (VaR)-analysis is used to assess what losses the bank is certain to incur on a portfolio of bonds from default. We are interested in the VaR due to credit, as it is an essential measure for a bank to account for losses on credit, which are required by IFRS 9 for financial instruments (IASB, 2014). CreditMetrics^{TM} uses a three-step method for analyzing the credit-VaR of a portfolio, which we modify to bond pricing formulas:

1. **Simulate several evolutions of the underlying assets of the portfolio of bonds and assign new credit ratings, R_T, based on the upper-thresholds for each simulation**

In Subsection 2.1, we explained how to do this using thresholds, where a formula for the default threshold was given in Equation (5). The industry correlation matrix, Table 3, will be used to simulate correlated asset returns. Correlation in asset returns is a realistic assumption, and in CreditMetrics™ it is motivated for analyzing diversification benefits between different fixed-income investments. However, our analysis is on the marginal effect on the losses from having different probabilities of default and not a study on diversification. To be consistent with CreditMetrics™, we also assume assets to be correlated by industry. For the sake of examining marginal effect due to the transition matrix as well as comparing the results between the mixture of Markov chain model and the model that assumes consecutive years of contraction or expansion, we set the asset distribution the same in each models.

	Energy	**Basic materials**	**Industrials**	**Cyclical consumer goods services**	**Non cyclical consumer goods services**	**Financials**	**Healthcare**	**Technology**	**Tele-com**	**Utilities**
Energy	1.00	0.80	0.72	0.63	0.55	0.62	0.49	0.61	0.55	0.43
Basic materials	0.80	1.00	0.80	0.76	0.57	0.68	0.62	0.74	0.56	0.41
Industrials	0.72	0.80	1.00	0.92	0.76	0.89	0.74	0.84	0.61	0.47
Cyclical consumer goods services	0.63	0.76	0.92	1.00	0.76	0.85	0.72	0.86	0.60	0.41
Non-cyclical consumer goods services	0.55	0.57	0.76	0.76	1.00	0.71	0.75	0.65	0.69	0.62
Financials	0.62	0.68	0.89	0.85	0.71	1.00	0.70	0.75	0.49	0.34
Healthcare	0.49	0.62	0.74	0.72	0.75	0.70	1.00	0.67	0.49	0.47
Technology	0.61	0.74	0.84	0.86	0.65	0.75	0.67	1.00	0.52	0.42
Tele-com	0.55	0.56	0.61	0.60	0.69	0.49	0.49	0.52	1.00	0.57
Utilities	0.43	0.41	0.47	0.41	0.62	0.34	0.47	0.42	0.57	1.00

Table 3: Industry Asset Return Correlations. This table shows the correlation matrix for the ten major industry classifications. The correlation matrix was calculated using 10-years index return data of the *Thomson Reuters North America Indicies*, retrieved 2018 – 04 – 03. In CreditMetrics™ a correlation matrix is used when simulating the asset returns of the portfolio of bonds.

2. **Calculate the portfolio loss**

   In our risk-neutral pricing formulas, the loss, L, is the sum of the current value of the bonds that defaulted after simulation of the asset returns V_T which determines the new rating, R_T. The loss function is:

$$L = \sum_l v_l(0, T), \tag{24}$$

   where $v_l = \{v^i(0, T) : R_T = K, i = 1, ..., K-1 | R_0 = i\}$ are the defaulted bonds. A defaulted zero-recovery bond in the future will be worth nothing and hence be a loss to us. This means that we do not account for expected losses on credit from anything other than the default, which comes from having, $\tilde{q}_{iK}$ in the valuation formula for a risky bond in Equation (8). The maximum loss is equal to the portfolio value today, which corresponds to the case where all bonds default in the future. If no bonds defaulted in the future but got downgraded to the lowest rating, the loss would still be zero. An assumption we make in the stress-tests are that the loss is under the risk-neutral pricing formulas, given that the transition matrix for the future changes conditional on cycles is used to determine the new ratings.

3. **Summarize the result as a distribution of the losses**
   Expected losses are the average value of the losses. The upper tail of the loss-distribution, correspond to the bank's credit-VaR, such that for a loss-density $f_L(s)$:
$$F_L(x) = P(L \leq x) = \int_{-\inf}^{x} f_L(s) ds, \tag{25}$$
$$VaR_\alpha(L) = F_L^{-1}(\alpha). \tag{26}$$
   Hence, VaR_α is the α-quantile of L. The expected shortfall is:
$$ES_\alpha = E[L|L > VaR_\alpha] \tag{27}$$

5 Model Implementation

In our analysis, we choose to analyze credit rating changes from S&P and restrict the obligors to U.S. firms, to account for some otherwise unwanted heterogeneity of the data. We also do not distinguish between firms on watchlists, unsolicited and solicited ratings, as this information was not possible to tell from our dataset. In the portfolio stress-test, our restriction to U.S. companies facilitates pricing of risky bonds, since we treat all firms within a rating homogeneously in Equation (6). In as many manners as was possible, our implementation of the models for the rating process, calculation of transition matrix and portfolio stress-test will coincide with practices explained in the empirical literature on credit ratings, or suggested by financial institutions. We begin by mentioning some of the assumptions in our model, then in Subsection 5.1 we describe the rating migration data. In Subsection 5.2 we give a data description of the macroeconomic index we used for the Markov chain for business cycles. Finally, in Subsection 5.3 we detail our simulation techniques.

We restrict the scale by removing all relative strength modifiers, (+) and (−). An assumption we make in the stress-tests are that industry, as a variable, provides correlation to asset returns. The use of 95%-percentile for the VaR is also used in other studies as a standard risk measure. Lastly, we chose to have a portfolio that mimics a real index (Frydman and Schuermann, 2008; Bangia *et al.* 2002; Gupton, Finger and Bhatia, 2007). However, an unusual assumption we make is to exclude claim seniority and hence, any recovery for the bonds in the portfolio. Fortunately, we use the same bonds in each test, such that the marginal effect, of having different transition matrices, on the losses still allows for analysis. The Markov chains are simulated for 10-years, which matches the maturity of the bonds in the portfolio. This analysis of discrete-time forward transition matrices coincides with Perederiy (2017) study on rating transitions, in light of IFRS 9 emphasis on long-term horizons, since it also simulates transition rates 10-years forward. It also coincides with the loss distribution standard of 10-years provided in Basel III (The Basel Committee on Banking Supervision, 2017).

5.1 Rating Data Description

In this subsection, we explain our data on rating migrations. Further in Subsection 5.1.1 we detail the sample selection bias of defaults, which affected the data, and how we accounted for the bias in the analysis. In Subsection 5.1.2 we detail the discrete state-space of ratings that are used in our analysis.

The rating data consists of changes to long-term issuer credit ratings from Standard and Poor's and spans a total of 32 years, with the first observed rating change occurring at $1986-01-05$ until the last observed rating in $2018-01-13$. There are a total of 934 unique companies in the dataset. The companies are those that had an assigned long-term issuer-rating changed by S&P as well as survived, and was a part of the constituents of the indices: S&P500, Nasdaq Composite, and S&P400 MidCap in $2018-01-13$. Figure 4 shows the number of companies represented by each industry in the full dataset. As can be seen, the most common industries in the dataset are consumer discretionary, financials and technology.

5.1.1 Sample Selection Bias

A consequence of only modeling the surviving companies is that there will be no transitions into default. However, as Table 5 shows, there are transitions from default to a non-default rating. This is because, in our initial dataset, a default was not an absorbing state, which means a company can be rated as, D, and then restructure and survive. Compared to *e.g.* Lando's (2004) definitions for generator matrices, K, is an absorbing state. A total of 29 observations where a company gets out of default are present in our data. To account for default-risk in our dataset, which has a selection bias for defaulted companies, therefore, we must find out the default probability externally, and the rating data is mainly used for analyzing the transitions

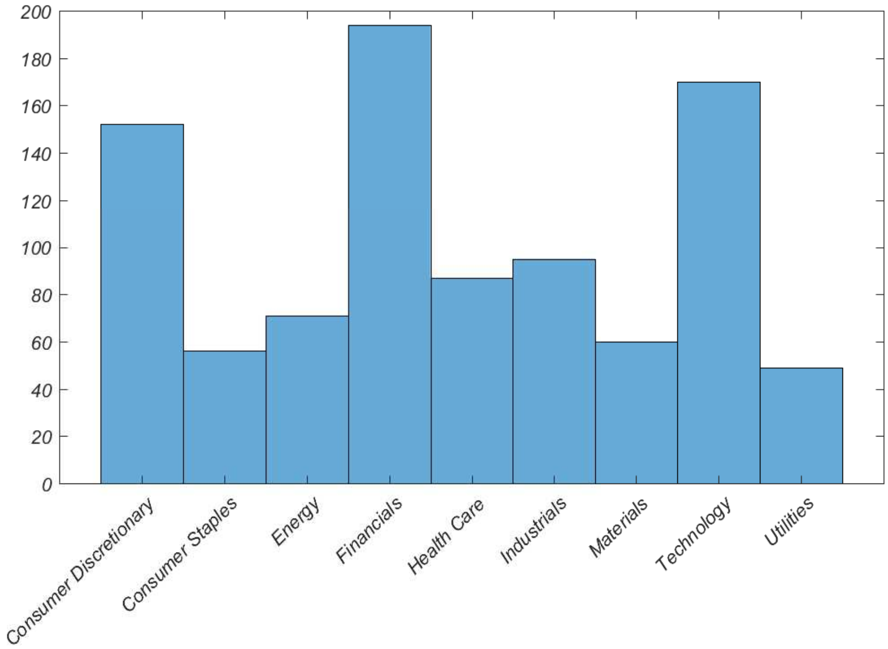


Figure 4: Industries of the Companies in the Dataset. This histogram shows which industries the companies in our dataset are classified into.

between non-default ratings. Our choice was to use the market-implied probability of default from CDS-spreads for this purpose, and thus we approximated the yearly default-intensities, λ_{iK}, $i = 1, ..., K-1$ in the generator matrix, from the 1-year implied default probability. The implied probabilities of default from the CDS spreads, for each rating class in the U.S. corporate benchmark retrieved from Thomson Reuters Eikon as of $2018-04-03$ are summarized in Table 4. The CDS spreads implies that

Rating	$\tilde{Q}_0^i(\tau \leq 1)$
AAA	0.41
AA	0.18
A	0.28
BBB	0.44
BB	0.72
B	2.96
CCC	16.61

Table 4: 1-Year Implied Probabilities of Default. This table shows the retrieved implied probabilities of default in percentage for 1 year U.S. corporations retrieved from Thomson Reuters Eikon.

the risk-neutral PD is larger for, AAA, rated bonds than for, AA, or, A, and almost equal to the PD of, BBB as shown in Table 4. Although somewhat unexpected, it reflects the market's credit risk premium and means that the risk premium for AAA is very large over a year.

Table 5 shows the total number of changes observed in the data. Excluding the diagonal, we have that the upper-right half corresponds to downgrades and the lower-left half corresponds to upgrades in ratings. The total observed upgrades are 604 and total observed downgrades are 727. The reason for there being more downgrades, than upgrades, begs the questions if the companies can well-represent a diverse set of company ratings for a proper analysis of the transition dynamics. However, as we show in the survival analysis of rating upgrades and downgrades in Appendix B, rating upgrades were greatly time-decreasing with a Weibull-α of 0.57, due to the economy during the time we observe the ratings. Downgrades had a Weibull-α of 1.14. The result of this survival analysis is that downgrades dominated over upgrades. Thus it might be tricky to find a dataset of credit ratings, during this period, that is not characterized by this same behavior as well as a full dataset with companies that defaulted.

5.1.2 Available Ratings

We only consider rating changes into letters, to reduce the set of available ratings. It means that we convert the scores that are either of the modifiers: $(+)$ or $(-)$, into their corresponding letter grade. Rating changes into, NR, are completely omitted since it does not indicate a rating. Ultimately, the nine ratings in the data are: *AAA, AA, A, BBB, BB, B, CCC, CC* and *D*. Typically analysts use a scale of eight ratings, omitting, CC. Because we choose not to merge, CCC, into, CC, we are consistent with each letter rating in the migration analysis. An unusual property of the migrations in Table 5, which happens to be an effect of the conversions, is that the diagonal is non-zero. This means ratings have moved from one rating to the same. As explained above, this is due to the joining of grades that have been added or removed a $(+)$ or $(-)$ by S&P. Regarding the analysis, it will merely be treated as any change to another rating and the only difference is the definition of the generator matrix. The diagonal of the generator matrix is defined as the negative sum of all transition probabilities away from a rating over the duration of the rating. When performing simulations, we want to check how well the simulations compared to the real data, and therefore we treat these as a rating transition on its own. By substituting the diagonal entries that are traditionally in the generator matrix, the method for simulation through intensities work in all cases. To summarize, we allow rating changes to occur into the same rating, and at the same rate as observed in the real data.

Table 6 shows the complete, average and standard deviation of the firm-duration, in years, of the ratings. The number corresponds to the total amount of years that all the companies in total have spent in each classification. The grade with the most extended span is, BBB. The longer the duration is for a rating, the smaller the intensity becomes. Therefore, λ_{ij} away from a score i with long duration $e.g.$ BBB becomes smaller and the transition less likely.

		Terminal Rating								
		AAA	AA	A	BBB	BB	B	CCC	CC	D
Initial Rating	AAA	2	13	0	0	0	0	0	0	0
	AA	3	139	78	4	0	0	1	0	0
	A	0	34	842	218	1	1	0	0	0
	BBB	0	0	130	1443	153	4	0	2	0
	BB	0	1	2	180	1146	139	1	0	2
	B	0	0	1	3	176	608	56	6	2
	CCC	0	0	0	0	2	37	63	18	9
	CC	0	0	0	0	0	2	4	4	19
	D	0	0	0	1	3	11	14	0	2

Table 5: Credit Rating Changes in the Dataset. Data description of the credit rating changes for long-term issuer data by S&P for 934 U.S. companies during the period 1986 – 01 – 05 to 2018 – 01 – 13. Observations where the final rating is the same as the initial rating are also shown. The dataset contains observations from one rating to the same (diagonal elements) because the scores got modified by (+) or (−), which S&P defines as a sign of relative strength and weakness respectively within the rating category.

	AAA	AA	A	BBB	BB	B	CCC	CC	D
Total Duration (Years)	96.3	773.2	3568.9	5356.4	3332.2	1675.7	143.9	21.3	37.7
Average Duration (Years)	6.42	7.98	11.06	10.42	7.59	5.72	2.57	1.01	1.35
Standard Deviation (Years)	9.38	6.97	7.54	7.00	5.86	4.95	3.49	3.52	1.86

Table 6: Credit Rating Duration of the Dataset. This table summarizes the duration in years for each rating, where rating histories for a total of 934 U.S. companies, observed in the period 1986 – 01 – 05 to 2018 – 01 – 13. The dataset does not contain ratings prior to 1986 – 01 – 05, thus the first observed rating for each company occurred sometime during the 32 years period. All durations equals 15,005.6 years, which means the dataset on average observe each company's rating history for 16 years.

5.2 Macroeconomic Data Description

In this subsection, we give a description of the data from National Bureau of Economic Research (NBER) used for stochastic business cycles in the Mixture of Markov chain (MMC)-model of the transition matrix. In the MMC-model, the same method as used by Bangia *et al.* (2002) is used, where data on business cycles is used for calculating the cycle TM. Business cycle data on the U.S. economy is available from the website for NBER.[6] The data is monthly and spans from December 1854 to June 2009. Business cycles are classified as either "peak" or "trough". For the total period, peak years have a total duration of 48 years and trough has a total duration of 106.5 years. The TM for business cycles are estimated using the duration method. The transition matrix for the business cycle can be annualized, by assuming time-homogeneity and calculating the exponential power of the matrix equal to the number of periods, which is 12 for monthly business cycle data.

5.3 Simulation Technique

In total, two different kinds of simulations are performed. The first simulation is of the continuous-time Markov chains conditional on cycles, as we explain in Subsection 5.3.1. From this, we calculate transition matrix conditional on cycles in discrete-time.

[6]NBER
http://www.nber.org/cycles/cyclesmain.html

The second simulation is the portfolio returns in the stress-test, which we detail in Subsection 5.3.2.

5.3.1 Simulating the Markov Chains

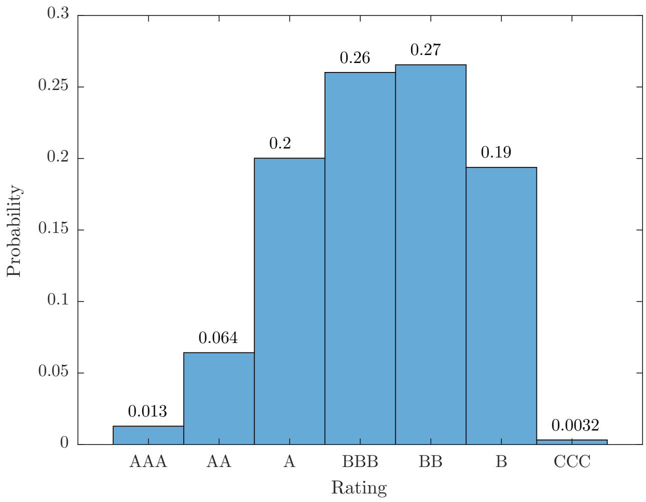


Figure 5: Distribution of the Initial Ratings for the Companies in the Dataset. Histogram showing the initial long-term issuer ratings from S&P for the 934 U.S. companies in the dataset. The histogram is not explanatory for whether it is the first rating ever for the company since it shows the first rating change for each company in the period 1986 – 01 – 05 to 2018 – 01 – 13. This initial distribution is a useful information when an attempt is made at trying to simulate the same rating history, as observed during the real 32 year period, to see how well the model can be described by a data generating process.

At first, the constant intensities are calculated using Equation (13) on all credit rating migration data. Then we simulate Markov chains for each company with the rates equal to the computed intensities. The initial rating of each company, as shown in Figure 5, are then observed for the same amount of time as in the real data, to see how well the constant intensities can be used to simulate the real ratings for a total period of 32 years. After we establish that the data generating process works for the estimated constant intensities from Equation (13), it is time to calibrate the TM for cycles in the economy. A set of uniformly distributed ratings (including D) for $1,000$ companies will be used to simulate the rating changes during a 10-years period. The rating distribution that is used for the simulations are shown in Figure 6. The conditions for the cycles are shown in Table 2. To further consider the probabilities of ending up in a business cycle, a regime switching Markov model is used. The 1-year TM for the business cycles is calculated from data on U.S. economic cycles determined by NBER during the period December 1854 – June 2009. The conditional forward TM for credit ratings are then calculated, using the equation with consecutive years of expansions and contractions given in Equation (17), and also in the Mixture of Markov chains in Equation (20). In all stress-tests, we set the forward time, n, to 10 years.

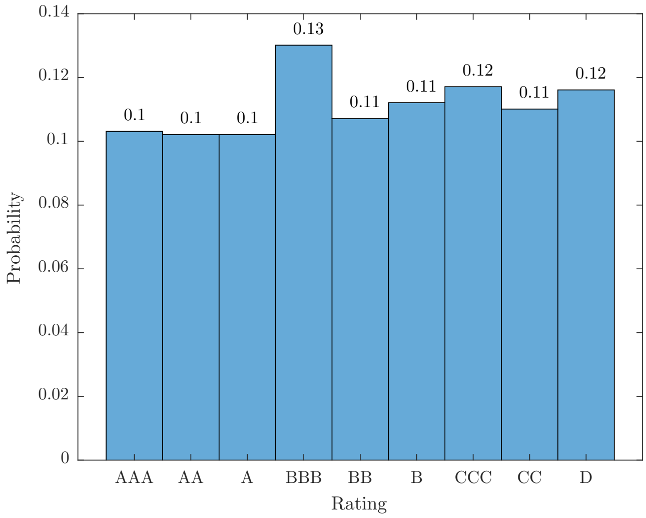


Figure 6: Distribution of Ratings for 1000 Companies Before Simulations. This figure shows the rating distribution for $1,000$ companies, which will be used to simulate the rating changes in a generalized Poisson process with time-varying intensities. All ratings are included (even default) as these ratings are not used to represent a real portfolio of companies, but rather used for analyzing the dynamics of the ratings. Since the state, D, is non-absorbing it poses no problem to also include it and allow it to switch as well.

5.3.2 Simulations for the Portfolio Stress-Tests

Lastly, the future TMs are used as an input in CreditMetrics™ to analyze changes in the distribution of losses, arising from the differences in probabilities of default in the transition matrices due to macroeconomic cycles. Here the asset return that is subject to correlation, which is multivariate normally distributed, due to its industry. The correlation matrix is shown in Table 3. In Matlab, the function mvnrnd is used to simulate the return distributions for each asset (Matlab Release 2018a, The MathWorks, Inc.).

Consistent with CreditMetrics™, a factor loading parameter, ω, for the asset returns weight, which is due to the industry it is operating in, must be specified. We choose to arbitrarily set the factor loading parameter to 0.65 in all companies. As long as we are consistent with the factor loading parameter, the results will still be comparable since the same companies are used in each stress-test. The expected return of the portfolio is, therefore, the same in each of the tests. The weight of the idiosyncratic return, the non-correlated univariate part of the return, is then solved from Equation (3): $\sqrt{(1-0.65^2)} = 0.76$ (Gupton, Finger and Bhatia, 2007).

The portfolio will consists of zero-coupon bonds, with the principals uniformly distributed between $10,000$ and $100,000$ USD in increments of $10,000$ USD. The portfolio is constructed such that the initial rating distribution mimics that of the companies in the S&P500 Index that we have long-term issuer rating data from S&P on as of $2018-01-13$. In total it means 441 companies. The bonds are set to mature after 10 years, which will make the duration of the portfolio of bonds coincide with the simulation period for Markov chains in cycles. In the initial time, all bonds in the portfolio is valued by Equation (6) and (8) corresponding to each rating. The risk-neutral default probabilities are the implied default-probabilities from CDS spreads summarized in Table 7. A simplifying assumption that we make is that, LGD,

Rating	$\tilde{Q}_0^i(\tau \leq 10)$
AAA	7.94
AA	10.23
A	13.8
BBB	20.2
BB	35.67
B	61.12
CCC	85.71

Table 7: 10-Year Implied Probabilities of Default. This table shows the retrieved implied probabilities of default in percentage for 10 year U.S. corporations retrieved from Thomson Reuters Eikon.

is 1 for each credit rating in the portfolio stress-test. As stated in CreditMetrics^{TM}, recovery rates are random and unpredictable unless advanced models are used.

6 Results

We present estimated empirical intensities in Subsection 6.1 and empirical transition matrices in Subsection 6.2. The result of the calculated 1-years transition matrices from simulated Markov chains using exponentially and Weibull-distributed time in ratings are presented in Subsection 6.3. The resulting 10-years transition matrices of the discrete-time Mixture of Markov chains-model is detailed in Subsection 6.4. Finally, the result of the portfolio stress-test is shown in Subsection 6.5.

6.1 Empirically Estimated Intensities

		Terminal Rating								
		AAA	AA	A	BBB	BB	B	CCC	CC	D
Initial Rating	AAA	-0.135	0.135	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	AA	0.004	-0.111	0.101	0.005	0.000	0.000	0.001	0.000	0.000
	A	0.000	0.010	-0.071	0.061	0.000	0.000	0.000	0.000	0.000
	BBB	0.000	0.000	0.024	-0.054	0.029	0.001	0.000	0.000	0.000
	BB	0.000	0.000	0.001	0.054	-0.098	0.042	0.000	0.000	0.001
	B	0.000	0.000	0.001	0.002	0.105	-0.146	0.033	0.004	0.001
	CCC	0.000	0.000	0.000	0.000	0.014	0.257	-0.459	0.125	0.063
	CC	0.000	0.000	0.000	0.000	0.000	0.094	0.188	-1.175	0.893
	D	0.000	0.000	0.000	0.027	0.080	0.292	0.372	0.000	-0.770

Table 8: Empirical Generator Matrix. Results showing the estimated generator matrix (Λ) from rating transition data. Each row sums up to 0, and all off-diagonal elements are non-negative which makes it a proper generator matrix. The off-diagonal elements should be interpreted as the distribution parameter for the arrival time of a type of transition. For example, the element corresponding to an initial rating, AAA, and terminal rating, AA, means that the average time until that migration occurs for an obligor with rating AAA, is: $\mu_{exp} = 1/\lambda = 1/0.135 = 7.41$ years.

Table 8 shows the generator matrix, estimated using Equation (13) in Subsection 3.1.1. The intensities are very low, due to the large number of right-censored

AAA	AA	A	BBB	BB	B	CCC	CC	D
0.02	0.18	0.24	0.27	0.34	0.36	0.44	0.19	0.05

Table 9: Empirical Intensity of an Obligor Being Rated the Same. Result of estimates for the constant rate of getting assigned the same rating as before. These rates counts as right-censored observations, as they are not a rating transition and exist because S&P modifiers for relative strength were converted into its letter-grade equivalent in our analysis. These rates are still used in the simulations, when trying to simulate the real data of transitions that was observed. The rating with the highest rate of this kind is CCC, corresponding to an average of $1/\lambda = 1/0.44 = 2.3$ years until a CCC got modified by either (+) or (−) by the dataset from S&P.

observations, when an obligor migrates to the same rating. The rating data initially had 21 ratings, where there were (+) and (−)-modifiers for the S&P long-term issuer ratings from AA to CCC (no C rating was ever present in the sample). Therefore, an explanation for the low intensities is due to obligors staying long in their initial ratings until a change occurs. The duration, shown in Table 6 is therefore significant for many of the scores, and the transition count is low, which further lowers the intensities. In Table 9, the intensity for the right-censored arrival time in each rating has been calculated, which is the intensity by which the new rating will be the same as it already is. It is estimated by dividing the diagonal of Table 5, which are the right-censored observations, with the durations in Table 6. As we see in Table 6, the rate at which the same rating is being assigned is higher than all other possible migrations combined for ratings AA down to B, which is the negative of the diagonal in the generator matrix. The scores where there is a higher intensity of migrating than staying in the same rating is AAA, CCC, CC, and because of the sample selection bias, D. Hence it is not relevant to compare the intensity of default in the empirical tables, since not all defaulted companies were not included from the beginning in the dataset.

6.2 Empirically Estimated Transition Matrices

		Terminal Rating								
		AAA	AA	A	BBB	BB	B	CCC	CC	D
Initial Rating	AAA	94.792	5.208	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	AA	0.393	89.253	9.567	0.655	0.000	0.000	0.131	0.000	0.000
	A	0.000	0.946	93.009	5.903	0.086	0.057	0.000	0.000	0.000
	BBB	0.000	0.000	2.432	94.614	2.741	0.135	0.000	0.039	0.039
	BB	0.000	0.031	0.062	5.362	90.555	3.616	0.218	0.031	0.125
	B	0.000	0.000	0.062	0.373	10.441	86.389	2.051	0.311	0.373
	CCC	0.000	0.000	0.000	0.000	1.493	23.881	73.134	0.746	0.746
	CC	0.000	0.000	0.000	0.000	0.000	13.043	13.043	65.217	8.696
	D	0.000	0.000	0.000	2.778	8.333	19.444	2.778	0.000	66.667

Table 10: Empirical Transition Matrix - Cohort Method. Results showing the 1-year TM in percentage, estimated using the Cohort method. The dataset included ratings by S&P for 934 U.S. companies during the time 1986 − 01 − 05 to 2018 − 01 − 13. We see that rating downgrades for the initial investment graded ratings: AA to BBB, are far more common than upgrades for the same ratings. For the speculative ratings: BB to CC, rating upgrades were more common. Another characteristic of this Cohort estimated TM is that all rating cohorts have a high probability of staying in the same rating after one year. The table also shows that migrations to neighboring ratings, for both upgrades and downgrades, are dominating for all grades except for the lowest, CC.

Table 10 shows the yearly TM estimated using the cohort method. Other than for the lowest speculative rating classes, CCC and CC, the probability of a rating migration to any different rating than a neighboring one is smaller than 1%. The estimated yearly TM using the duration method is displayed in Table 11. By

		Terminal Rating								
		AAA	AA	A	BBB	BB	B	CCC	CC	D
Initial Rating	AAA	87.399	11.936	0.613	0.044	0.001	0.001	0.007	0.000	0.000
	AA	0.343	89.541	9.219	0.762	0.012	0.015	0.099	0.005	0.004
	A	0.002	0.870	93.244	5.745	0.108	0.029	0.001	0.001	0.000
	BBB	0.000	0.011	2.282	94.890	2.656	0.125	0.005	0.022	0.010
	BB	0.000	0.028	0.119	5.020	90.977	3.716	0.087	0.009	0.046
	B	0.000	0.002	0.061	0.425	9.355	87.030	2.551	0.314	0.263
	CCC	0.000	0.000	0.008	0.158	2.435	20.518	65.139	5.740	6.001
	CC	0.000	0.000	0.011	0.696	2.655	14.000	16.325	31.656	34.658
	D	0.000	0.001	0.035	2.015	6.719	22.144	20.723	1.102	47.262

Table 11: Empirical Transition Matrix - Duration Method. Results showing the 1-year TM in percentage, estimated using the duration method. The dataset included rating changes by S&P for 934 U.S. companies during the time $1986-01-05$ to $2018-01-13$. We see that rating downgrades for the initial investment graded ratings: AA to BBB, are far more common than upgrades for the same ratings. For the speculative ratings: BB to CC, rating upgrades were more common. All ratings have a high probability of staying in the same grade after one year. The table also shows that migrations to neighboring ratings are dominating.

comparing the two empirical TMs, we see that they are similarly distributed for the transition probabilities for ratings AA to B, and in fact differ less than 1% in each probability. There is a higher probability of 7 percentage units of being downgraded from the best rating, AAA, in the duration TM. Downgrade probabilities for the junk ratings: CCC to C or default and CC into default, are also much more common in the duration matrix.

The duration matrix manages to ”smooth” out some of the probabilities away from the diagonal of only migrating to the neighboring rating. This effect is small, but it shows that the duration matrix is a better choice than the cohort matrix when it comes to modeling some transitions that are less likely. Both historically estimated TMs are, however, downgrade-biased which was caused by the increase in downgrades over the full period and the decreasing rate of upgrades, lasting from the very first rating in 1986 until the last upgrade was assigned at 18th December 2018[7]. Upgrades also mainly occurred after 1–2 years after the previous rating change, which is a contributing factor why some upgrade probabilities are low, due to their low occurrence in the rating changes data.

Regarding the probability to default, these empirically estimated transition matrices lack default data, which means it should not be interpreted as the real default probabilities.

6.3 1-Year Transition Matrices From Simulations

First, in Subsection 6.3.1, the exponentially distributed simulation of the TM is presented, then the simulated Weibull-distributed transition matrices in Subsection 6.3.2. For transition matrices from simulated Markov chains, we use heatmaps to visual-

[7]See Appendix B for a survival analysis of the empirical upgrades and downgrades in the dataset.

ize the transition matrices graphically and to distinguish them from the empirically estimated transition matrices.

6.3.1 Exponentially Distributed Rating Changes

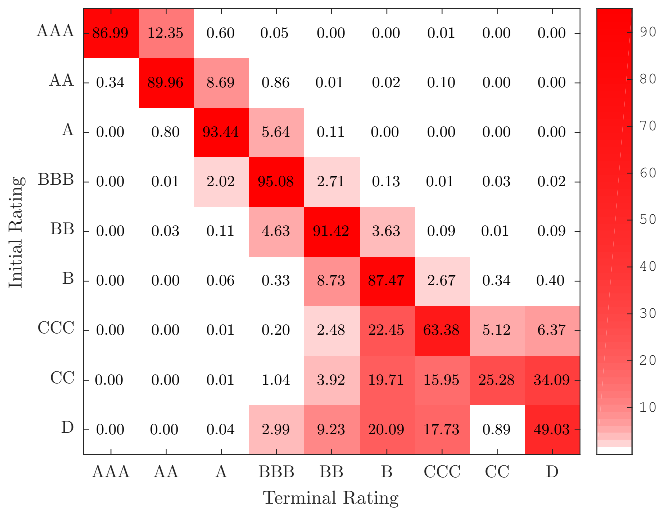


Figure 7: 1-Year Transition Matrix From Exponentially Distributed Rating Changes. Result of a 1-year transition matrix in percentage from simulated Markov chains with constant rates for the initial ratings in Figure 5, displayed as a heatmap. By comparing the transition probabilities to those estimated from the real data in Table 11, we see that constant intensities are able to provide a reasonably good data generating process.

		Terminal Rating								
		AAA	AA	A	BBB	BB	B	CCC	CC	D
Initial Rating	AAA	0	-0.57	0	0	0	0	0	0	0
	AA	-0.08	-1.37	-0.11	-0.11	0	0	-0.05	0	0
	A	0	-0.38	-22.4	-6.45	0.17	-0.05	0	0	0
	BBB	0	0	-0.7	6.87	2.18	0.32	0	0.24	0
	BB	0	-0.09	0.33	3.95	25.73	3.32	-0.08	0	0.1
	B	0	0	0.03	0.24	3.47	-1.39	0.94	0.02	0.22
	CCC	0	0	0	0	-0.05	1.69	0.62	0.12	0.37
	CC	0	0	0	0	0	-0.02	-0.2	0.42	0.09
	D	0	0	0	0.01	0.05	0.35	0.14	0	-0.08

Table 12: Mean Error in the Constant Intensity Model. Result of calculated means of the difference between simulated rating transitions and observed rating transitions for 100 simulations of rating changes with constant intensities. Each initial rating in the dataset was used to simulate the rating changes. The companies were observed for the same amount of time as in the real data. A low (large) absolute value in any cell means that the rating transition is (not) approximated well with a constant intensity.

In Figure 7, the initial ratings for the companies in the dataset were used in the simulation and we assumed that all times in ratings are exponentially distributed. The intensities were calculated using the duration method from the real data. The companies were observed for as long as they were observed in the real data. Figure 7 show that, for a basic model, exponentially distributed arrival time will do a decent job of modeling the TM, as long as the constant intensities come from the duration formula. The rating transitions which fits the best are speculative ratings, AAA, down to, B. The elements in Figure 7, show that for these ratings the probability of staying in the same rating over a year can be well-estimated and likewise each of their transition probabilities. It is a beneficial feature to have in a model since it will show with what probabilities the scores won't change, *e.g.* in the case of a portfolio manager looking to keep a certain proportion of investment grades each year. The transition probabilities for, AAA, down to B, all differ by less than 1% in the simulated data compared to the real data in Table 11. The junk ratings, CCC, CC and default, D, in the TM are, therefore, most sensitive to changing rates over time and constant intensities functions worse for these ratings. Transitions are more likely to happen for these and there is a lot less data, which explains the difference. The mean error of the simulations seems to be affected by the duration and rating changes observations. Therefore, as one uses a larger dataset, it is clear that a constant intensity better approximates the TM.

6.3.2 Weibull Distributed Rating Changes

The result of the simulated conditional yearly TMs in Figure 8 show how sensitive the probabilities were to the change in rates. Results are clear, namely that our model for the dynamics of the transition probabilities are in-line with expectations for almost all rating classes. As the absolute value of the shape-parameter increases, the magnitude of cycles, the effect is that rating changes closest to the diagonal will become more frequent. In contractions (expansions) it is the probabilities right (left) of the diagonal that increase corresponding to a downgrade (upgrade).

In huge cycles, AAA bonds have a less than 6% of being downgraded in expansions, whereas that probability is over 22% in contractions. The simulated probabilities of default imply that they are larger for every rating class in huge recessions compared to when the country experiences huge economic growth. In huge cycles of contraction, the recovery of a default, which occurred 29 times in our dataset, got reduced. Historically, the probability of staying defaulted was 47%, which in the simulated worst-case increased to 61%, hence 14 percentage units greater. The simulated TMs thereby capture the increased difficulty of restructuring in bad cycles. The downgrade probability of an AA bond is 17.6% in contractions and only 6.5% in expansions (huge cycles). For both A, and BBB-rated bonds, the downgrade probability in huge contractions is 2.7 times greater than in huge expansions. Even though the proportion diminishes the lower the rating is for the investment graded bond, there is still notable differences in the distribution and rate of downgrades. The upgrade probability is more than double for BBB-rated bonds in vast cycles of expansion compared to contraction. For rating BB, the dynamics in huge cycles means that upgrades are more than double as likely, and downgrades are half as likely in expansions as in contractions.

In conclusion, the calculation of conditional transition matrices using hazard processes conditional on background information proved successful. The dynamics ac-

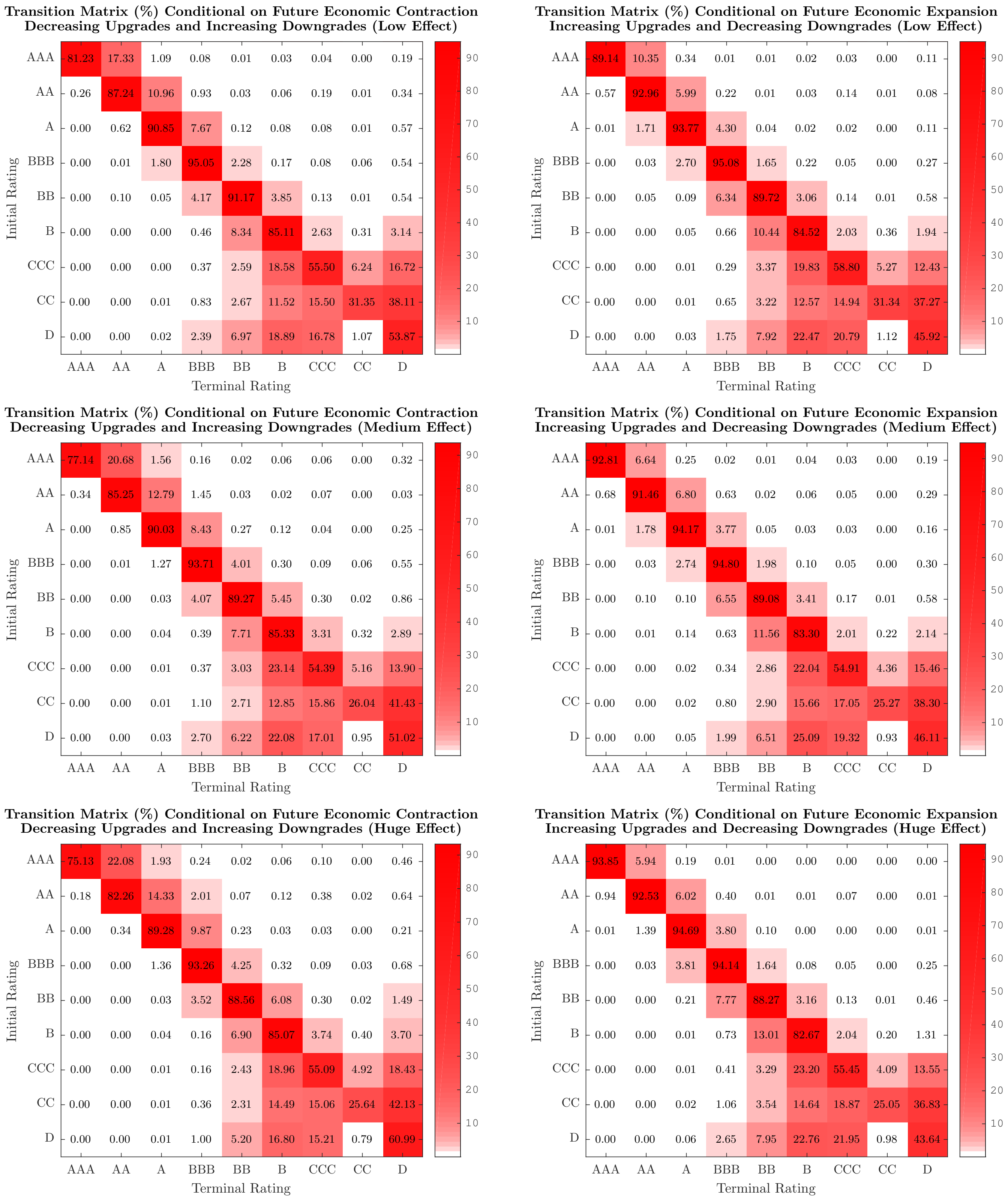


Figure 8: 1-Year Transition Matrices Conditional on Business Cycles. Simulation results of yearly TMs conditional on time-varying intensities displayed as heatmaps. An initial set of 1,000 companies with uniformly distributed ratings, shown in Figure 6, were used to simulate all the Markov chains that makes up the transition matrix. The length that every company was observed for was 10-years. Left-hand panels are the result of economic contraction, with upgrade intensities decreasing and downgrades intensities increasing in time. Right-hand panels are the results of economic expansion, with upgrade intensities increasing and downgrades intensities decreasing in time. The intensity of being rated the same, shown in Table 9, were constant in all cases. The magnitude of time-variation is lowest in the upper panels, medium in the middle panels and strongest in the lower panels. The rating changes are distributed as a Weibull-distribution, with the magnitude of the shape-parameter, α, depending on cycle. The α:s are summarized in Table 2.

counted for the historical distribution, sample-selection bias, and in each rating change, the models managed to develop a realistic probability given the cycles. Speculative grades were trickier to model because of fewer data than dynamics of investment-grade transitions.

6.4 10-Years Transition Matrices From the Mixture of Markov Chains-Model

The estimated yearly TM for U.S. cycles of economic expansion, E, and contraction, C, from NBER's data during the period December 1854 – June 2009 is in percentage:

$$\hat{Q}_{BC}(t,t+1) = \begin{pmatrix} \hat{p}_{E,E} & (1-\hat{p}_{E,E}) \\ (1-\hat{p}_{C,C} & \hat{p}_{C,C} \end{pmatrix} = \begin{pmatrix} 56.5 & 43.5 \\ 19.6 & 80.4 \end{pmatrix}.$$

The resulting credit rating TM for 10-years forward in a MMC-model are shown in

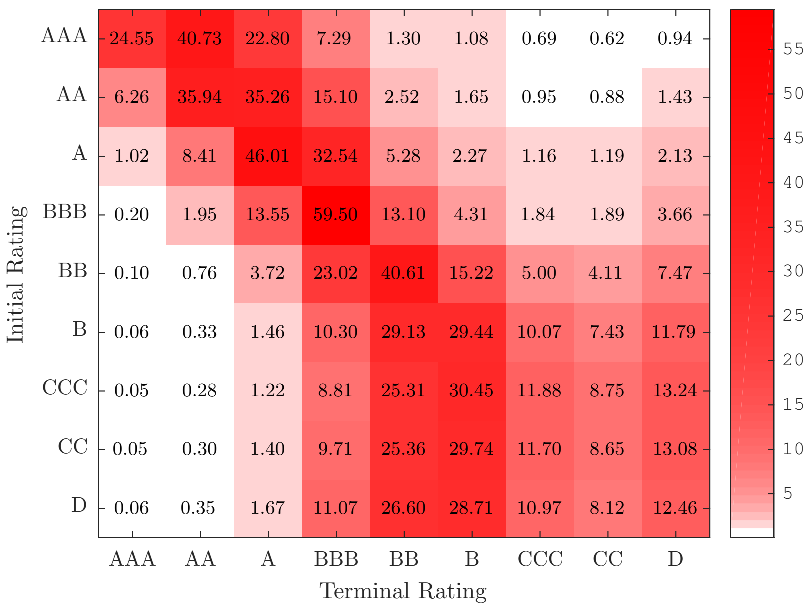


Figure 9: 10-Years Transition Matrix in Low Cycles. Result of MMC-model for credit rating TM (in percentage) assuming low cycles in the next 10-years displayed as a heatmap. The business cycle TM from NBER, $\hat{Q}_{BC}$, combined with 1-year credit rating transition matrices from simulated ratings conditional on cycles, $\hat{Q}_E(t,t+1)$ and $\hat{Q}_C(t,t+1)$, with time-varying rates for rating changes corresponding to low cycles in Table 2, leads to this forward TM.

Figures 9, 10 and 11. The resulting TMs are spread when it comes to the probabilities. Results does not differ particularly much between cycles. Therefore, we can conclude that the MMC-model smooths out the resulting forward TM in all scenarios that we modeled.

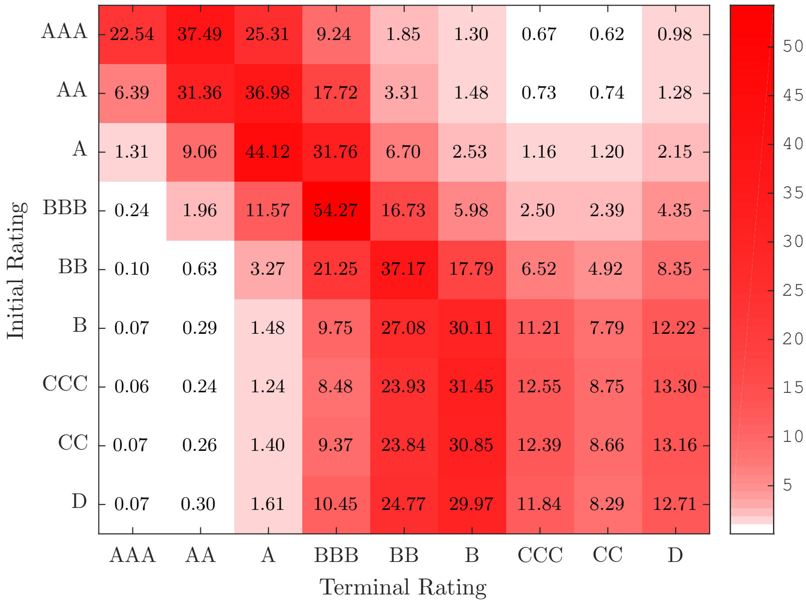


Figure 10: 10-Years Transition Matrix in Medium Cycles. Result of MMC-model for credit rating TM (in percentage) assuming medium cycles in the next 10-years displayed as a heatmap. The business cycle TM from NBER, $\hat{Q}_{BC}$, combined with 1-year credit rating transition matrices from simulated ratings conditional on cycles, $\hat{Q}_E(t,t+1)$ and $\hat{Q}_C(t,t+1)$, with time-varying rates for rating changes corresponding to medium cycles in Table 2, leads to this forward TM.

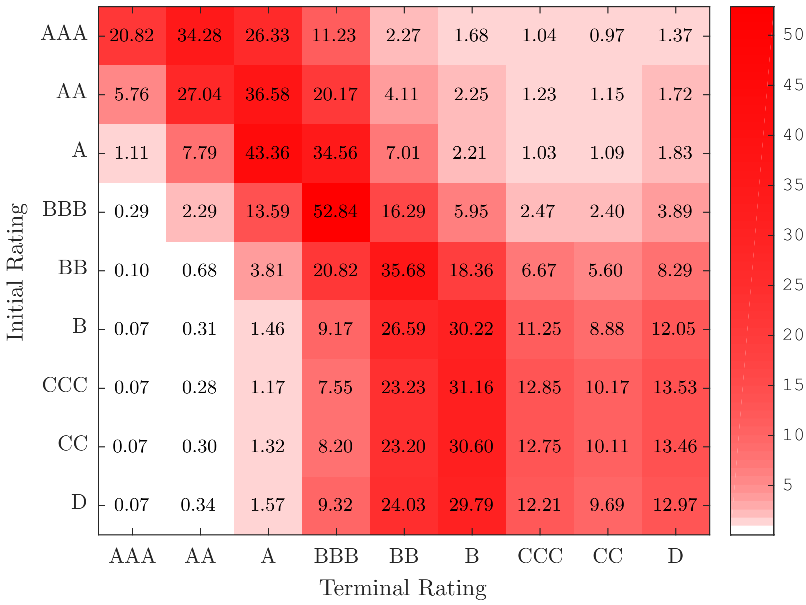


Figure 11: 10-Years Transition Matrix in Huge Cycles. Result of MMC-model for credit rating TM (in percentage) assuming huge cycles in the next 10-years displayed as a heatmap. The business cycle TM from NBER, $\hat{Q}_{BC}$, combined with 1-year credit rating transition matrices from simulated ratings conditional on cycles, $\hat{Q}_E(t,t+1)$ and $\hat{Q}_C(t,t+1)$, with time-varying rates for rating changes corresponding to huge cycles in Table 2, leads to this forward TM.

6.5 Portfolio Stress-Test Application

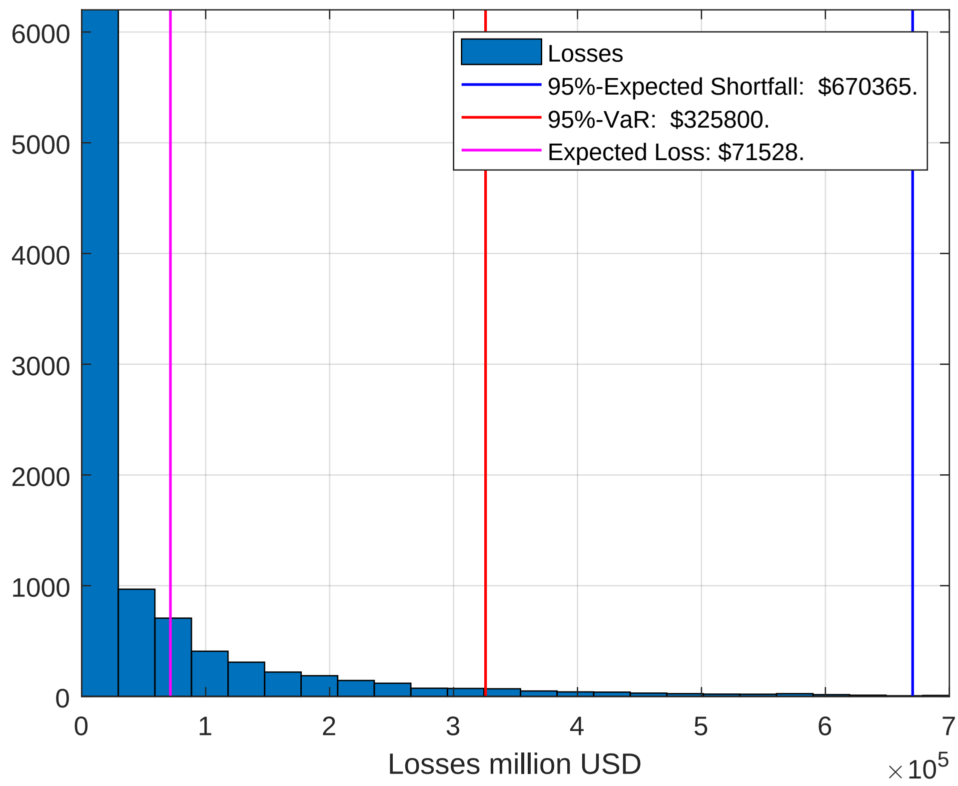


Figure 12: Loss Distribution Unconditional of Cycles. CreditMetrics™ output for 10,000 simulated portfolio losses, using the PDs in the empirical transition matrix for 10-years forward, risk-free rate, $r_f = 0\%$ and $\tilde{Q}_0^i(\tau \leq 10)$ as in Table 7. On the x-axis is losses defined by Equation (24) and on the y-axis is the frequency.

Figures 12, 13, 14 and 15 shows the portfolio stress-test results using the same portfolio, and asset returns, after 10-years. The figures can be compared between each other to see the marginal effect that the transition matrix have on losses. The portfolio consists of zero-coupon, zero-recovery bonds of companies that have a rating and industry distribution that mimics the S&P500 as of $2018 - 01 - 13$. In total there are 441 companies in the portfolio. The principal for each bonds are chosen from a uniform distribution between $10,000$ to $100,000$ USD in increments of $10,000$ USD. The industry correlation of the companies in the portfolio were accounted for, by having a factor loading parameter for the systematic return of 0.65. In each of the scenarios, the asset return over the period are fixed, so that only the marginal effect on losses due to a change in the forward transition matrix can be analyzed between the different types of macroeconomic cycles. The maturity of the bonds are ten years and the portfolio value today using Equation (6) with the risk-neutral probabilities of default in Table 7 is $19,510,290$ USD.

To begin, we can make a comparison of our simulations conditional on macroeconomic cycles in Figures 13, 14 and 15 against the purely historically-estimated tran-

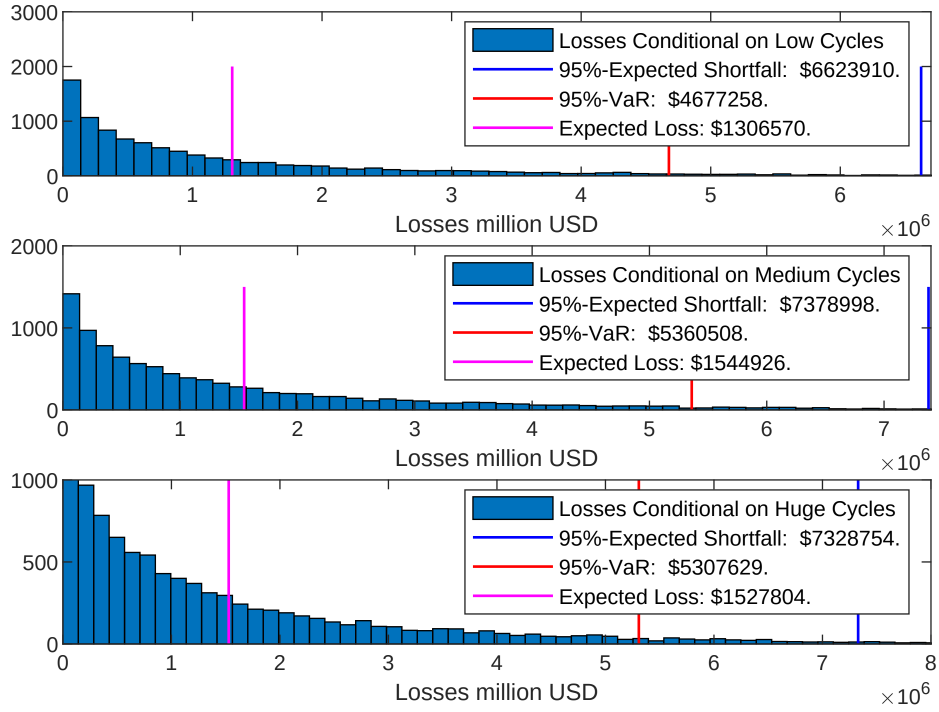


Figure 13: Loss Distribution Conditional on Business Cycles. CreditMetrics™ output for 10, 000 simulated portfolio losses, using the PDs from an MMC-model for the forward transition matrix for 10-years forward, risk-free rate, $r_f = 0\%$ and $\tilde{Q}_0^i(\tau \leq 10)$ as in Table 7. On the x-axis is losses defined by Equation (24) and on the y-axis is the frequency.

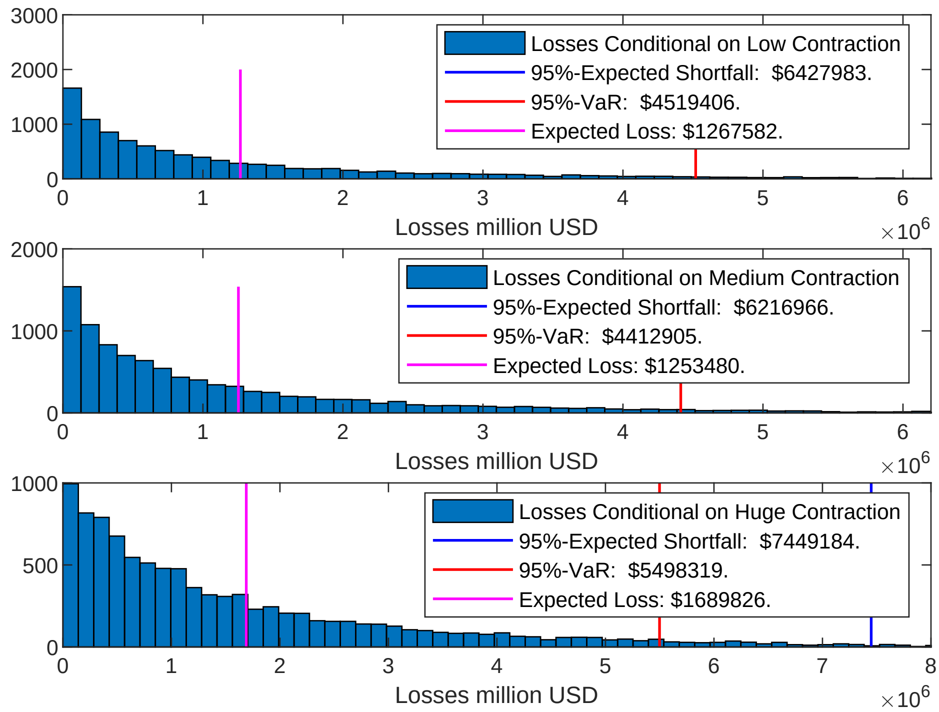


Figure 14: Loss Distribution Conditional on Consecutive Contraction. CreditMetrics™ output for 10, 000 simulated portfolio losses, using the PDs from the transition matrix conditional on consecutive economic contraction for 10-years forward, risk-free rate, $r_f = 0\%$ and $\tilde{Q}_0^i(\tau \leq 10)$ as in Table 7. On the x-axis is losses defined by Equation (24) and on the y-axis is the frequency.

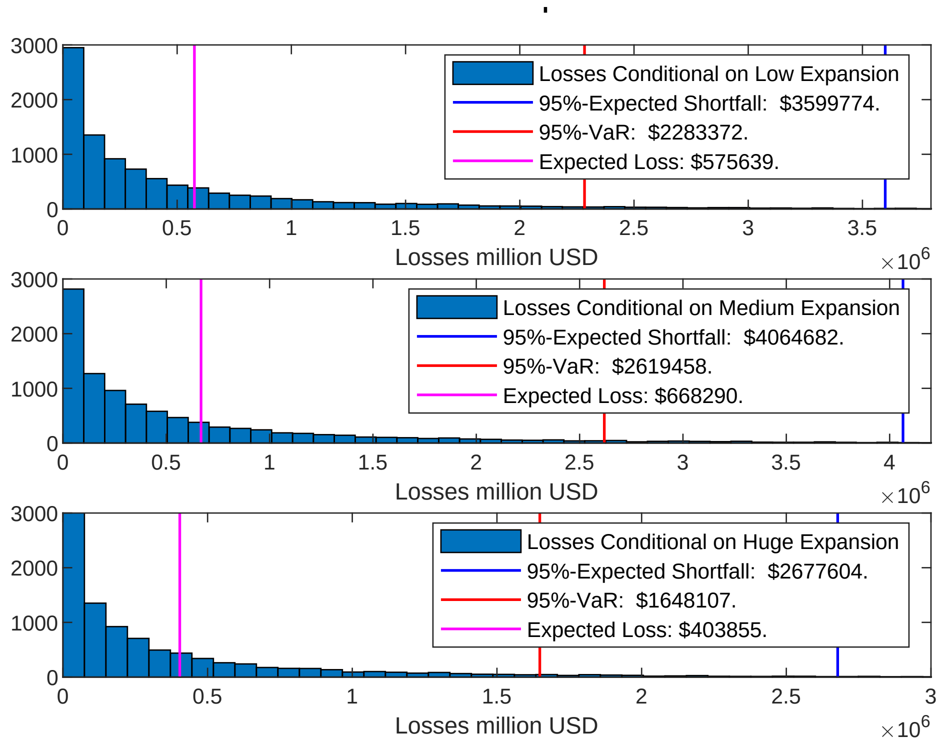


Figure 15: Loss Distribution Conditional on Consecutive Expansion. CreditMetrics™ output for 10,000 simulated portfolio losses, using the PDs from the transition matrix conditional on consecutive economic expansion for 10-years forward, risk-free rate, $r_f = 0\%$ and $\tilde{Q}_0^i(\tau \leq 10)$ as in Table 7. On the x-axis is losses defined by Equation (24) and on the y-axis is the frequency.

sition matrix, with sample selection bias, in Figure 12, where we estimate losses using the empirical default-rates in Table 11. In this benchmark, the losses are the lowest out of all stress-tests since the data did not include cases where companies stayed defaulted, which means there were low default probabilities in the transition matrix. The 95%-VaR in the benchmark is 325, 800 USD. Not so unexpected, the closest out of all stress-tests conditional on cycles, the closest 95%-VaR to this value is the scenario with huge expansions consecutively for ten years, with 1, 648, 107 USD. Hence by only using the empirical transition matrix, with few defaults, losses are also fewer and corresponds to a relatively rare case where economic expansion is expected to happen with large magnitude each of the next 10-years, in our stress-test. The reason behind the large difference to the purely empirical benchmark is that we adjusted the simulations with implied PDs on the market, which the empirical matrix did not include due to the sample selection bias of defaults in the dataset. Obviously Figure 12 shows some issues in credit risk models when you are uncareful about which historical data to use.

The Mixture of Markov chain (MMC)-models, were not practical to use for modeling transition matrices, as the transition probabilities became smoothed and similar in all periods. In Figure 13 we show that the stress-test of the MMC-models resulted in expected losses, 95%-VaR and expected shortfall that changed little in relative terms depending on cycle magnitude. The 95%-VaR change with an increase of 13% from low to huge cycles, in comparison to change in absolute value of 22% for contraction and 39% for expansion going from low to huge. A conclusion arising from this is that such a model smooths out the weighted probabilities between good and distressed times, in no small degree making the outputs less useful. Therefore, the effect on losses becomes marginal when MMC is used to account for probabilities of cycles. The models with time-homogeneous TMs used consequently, with results shown in Figures 14 and 15, is therefore a better alternative for stress-tests as they show greater variation in regards to cycle strength.We see a change in expected losses of a factor 0.7 (a decrease of 30%) when expansions go from low to huge cycles. For contractions, expected losses increase similarly with a factor 1.33 from small to huge business cycles (an increase of 33%). In all three stress-tests of macroeconomic cycles, the medium cycles showed, oddly enough, the opposite effect than what was expected given the direction of intensities. In part, it could be due to the limitations in the stress-test, such as the assumption of zero-recovery and that implied probability of default, in Table 4, used in the simulation of rating changes was higher than other investment-grade bonds. However, as we see from the transition matrices in Figure 8 the speculative grades dynamics were more challenging to model, due to fewer data on those ratings initially. Since speculative ratings account for most of the default, it hence limits our stress-test to some degree. Huge cycles, however, more accurately managed to capture the dynamics of a recession and expansion respectively, of credit rating events when compared to low cycles. By only comparing huge cycles and low cycles, the stress-test show results that were in-line with expectations.

The conditional 10-years forward transition matrices assuming time-homogeneity showed greater changes in losses between expansion and contraction simulations. We see from Figure 15 that 95%-VaR lies between $1,648,107$ and $2,283,372$ USD in expansions and between $4,519,406$ and $5,498,319$ USD for contractions. The results show that modeling the direction of intensities in Markov chains for ratings, in cycles, has pronounced effect on the 95%-VaR for a portfolio of bonds when used in this stress-test. Overall from the CreditMetrics^{TM} output, we see that the simple time-homogeneous model for future TMs is an excellent choice to account for higher variation in transition probabilities due to cycles. Our simulation of conditional Markov chains has therefore shown to have an apparent marginal effect on the losses on a portfolio of assets.

7 Conclusions

Our analysis of intensity models for the rating transitions showed that, given that the empirical distribution of the rating transitions are known, one could account for macroeconomic cycles by changing the shape-parameter, α, in the Weibull-distribution, which incorporate time-variation in rates. In particular, the simulations show that stronger cycles shift the transition probabilities in the transition matrix towards added upgrades and fewer downgrades in economic expansions. For economic contractions, the simulated TM show the opposite effect. Therefore, the implementation of simulated rating changes using Lando's (1998)-model provided results, for the TM, which are in-line with economic theory. To account for a potential progressive decline in credit quality for outstanding loans, banks, and other financial institutions can use this model for immediately recognizing losses on the deteriorated claims in the future, required under IFRS 9 accounting regulation.

Lastly, regarding the portfolio stress-test, we can conclude if our hypothesis stated in the introduction were true:

- The Mixture of Markov Chains-model had the effect of weighing the conditional TMs, which smoothed out the probabilities and we ended up with similar losses in each magnitude of cycles, compared to the other stress-tests. However, the time-homogeneous model of consecutive ”expansion” or ”contraction” years showed a clear difference between low and huge cycles for expansion and contraction respectively.
- By setting constant a portfolio of bonds, asset return, and correlation, we developed a Value-at-Risk-model that worked well in showing the marginal effect of having different probabilities of default from the conditional transition matrices in the future.

A drawback of our model is that we assumed no recovery in the case of default for the portfolio stress-tests and that we had to use implied probabilities of default as the initial default probability when we simulated credit rating Markov chains for companies, which would provide different results if better real-world estimates were chosen instead. Further research on generalized Markov chain processes for analyzing the transition matrix in IFRS 9 context of portfolio stress-test against future losses, seem to point two ways – either a state variable could be used to incorporate for example the current short-term rate in the intensity function. Or, the model of deterministic Weibull-distributed rates which we used can be extended, for example by changing the scaling parameter. Either direction, those models would provide more insights into the macroeconomic effect of cycles on the transition matrix and its effect on credit losses.

A Distributions for Random Events

The goal of our analysis of past distribution of transition rates is to obtain and analyze the parameters for the distributions of credit transition probabilities, by fitting the empirically estimated functions to known probability density functions. In our models, the distributions for rating times, which we use, will be the same as those explained by McNeil, Frey, and Embrechts (2010). Such fitted deterministic functions can illustrate the change in the probability distribution over time, which is useful when a model should be calibrated for time-varying business cycle effects later on. By doing a survival analysis, a comparison between rating change, in different ratings, can from the estimated distribution parameters be illustrated graphically as well.

An exponential distribution is appropriate in the case where we assume that the times until a downgrade or upgrade occur randomly but at a constant rate, λ. The exponential density function is (McNeil, Frey and Embrechts, 2010):

$$f(u) = a \exp(-\lambda u). \tag{28}$$

The survival function, $S(t)$, for the exponential distribution is calculated by substituting Equation (28) into (Rodríguez, 2007):

$$f(u) = -S'(u), \tag{29}$$

and integrating over t. The resulting survival function only depends on the constant rate, λ, and has the form (McNeil, Frey and Embrechts, 2010):

$$S(t) = \exp(-\lambda t). \tag{30}$$

The mean and variance of the exponential density function are (McNeil, Frey and Embrechts, 2010):

$$\mu_{\text{exp}} = \frac{1}{\lambda},\ \sigma^2_{\text{exp}} = \frac{1}{\lambda^2}$$

The exponentially constant rate for the distribution corresponds to the cases in which the Markov property, explained in Section 3.1 is wholly fulfilled (Frydman and Schuermann, 2008). Another density function for describing the rate of the random times are Weibull-distributions. A Weibull-distribution has a time-varying rate at which the random arrival times occur. The Weibull-probability density function, in the form used by McNeil, Frey and Embrechts (2010) is:

$$f(u) = \lambda \alpha u^{\alpha-1} \exp(-\lambda u^{\alpha}). \tag{31}$$

The surival function in a Weibull distribution gets the form (McNeil, Frey and Embrechts, 2010):

$$S(t) = \exp(-\lambda t^{\alpha}). \tag{32}$$

The mean and variance of the Weibull density function are (McNeil, Frey and Embrechts, 2010):

$$\mu_{WB} = \lambda^{-\frac{1}{\alpha}}\Gamma(1+\frac{1}{\alpha}),\ \sigma^2_{WB} = \lambda^{-\frac{2}{\alpha}}(\Gamma(1+\frac{2}{\alpha}) - (\Gamma(1+\frac{1}{\alpha}))^2),$$

where Γ is the gamma function.

Both distributions are popular choices for fitting the estimated survival functions and can be used to adjust empirical data on rating changes, *e.g.* upgrades, and downgrades, to obtain the parameter values for their respective distributions. Note from the survival functions in Equation (30) and Equation (32), that it is clear that the Weibull-distribution will have a changing hazard-rate, whereas the exponential model has a constant hazard rate. The Weibull-shape parameter, α, in Equation (31) determines the time-variation and is used to control the magnitudes of our cycles shown in Figure 16.

B Survival Analysis of Upgrade and Downgrades

Figure 17 and 18 shows the survival probability using the Kaplan-Meier estimators, and the Weibull-fitted curves (Kaplan and Meier, 1958). We see that the rate at which downgrades occurs, in the beginning, is 0.0383. If this rate had been kept constant throughout 32 years, it would correspond to an average of 26.1 years for any non-defaulted company until a downgrade. We see that the survival function in the left-hand panel of Figure 17 exhibits some of the Markov property, meaning that downgrades appear to occur at a reasonably linear rate and does not seem to be too much affected by how long it has survived up until a point in time. Evident from the estimated shape parameter, determined to be 1.1416, which in a case with full Markov property would be 1, some Markov property holds. From this, we conclude that the Markov property, characterized by arrival times with exponential distribution at a constant rate, to some extent provides a good model of downgrades.

The slope of the downgrade survival function is, however, not entirely linear, particularly in the right-hand panel of Figure 17. The right-hand panel shows the survival function with omitted right-censored observations. In this case, α is 0.8003 and the survival probability of the random time is a lot steeper from the initial time

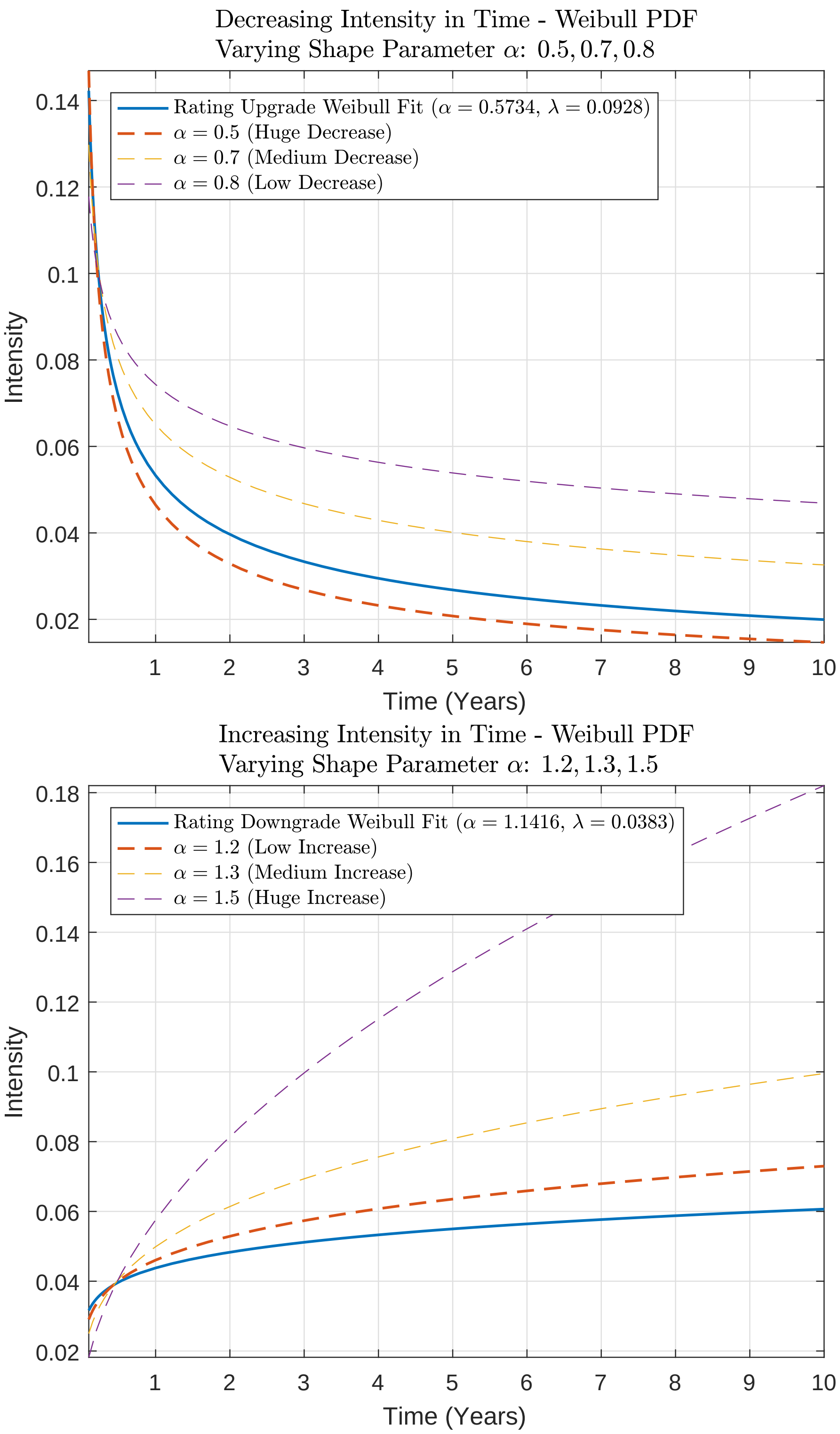


Figure 16: Weibull-Distributed Intensity Functions. The upper graph shows decreasing intensities in time, and the lower graph shows decreasing intensities based for random events with Weibull probability density function (PDF) with different Weibull-shape parameter, α. The functional form in both graphs is: $\lambda(t) = \lambda_0 \alpha t^{\alpha-1}$.

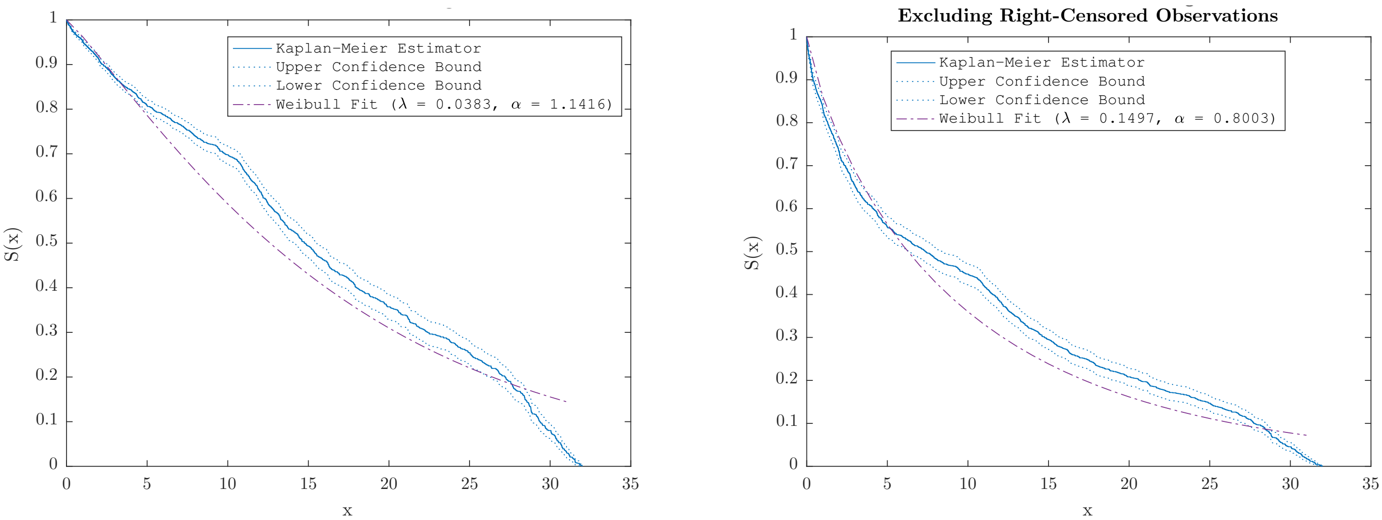


Figure 17: Empirical Survival Function for Rating Downgrades. Results showing empirically estimated survival functions, $S(x) = 1 - P(\text{Downgrade}|\tilde{\tau} > x)$, of any rating downgrade occuring at time $\tilde{\tau} > 0$. The left panel shows the survival probability distribution for any obligor at risk of a downgrade, calculated from the Kaplan-Meier product limit estimator (Kaplan and Meier, 1958). The right panel shows the function when right-censored observations are excluded, such that only the cases where downgrades actually occurred counts, calculated from the Kaplan-Meier product limit estimator. 95%-confidence bounds are also displayed. The Weibull-fitted curve in the left panel has a scale-parameter, $\lambda = 0.0383$ and a shape-parameter, α, of 1.1416. This means that the hazard increases in time. The rate seems to be decreasing from time 0 up until 10 years and then again from 10 to around 27 years but at a different rate. This means that the Markov property is not completely fulfilled for rating downgrades, as the cumulative probability distribution of downgrades depend on how much time has passed before the rating got changed.

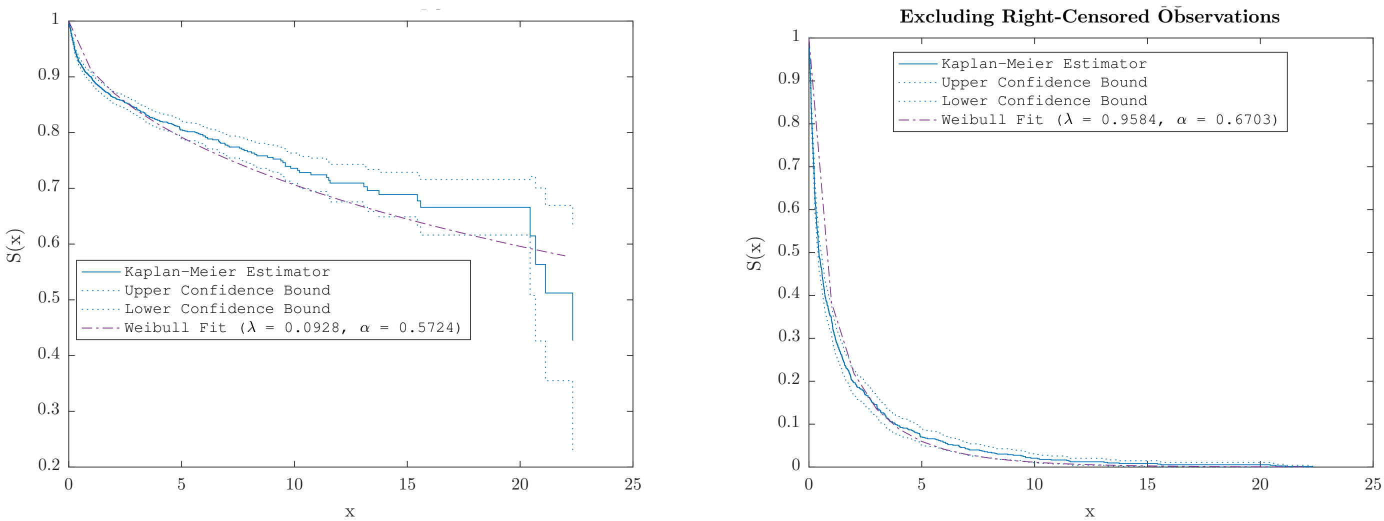


Figure 18: Empirical Survival Function for Rating Upgrades. Results showing empirically estimated survival functions, $S(x) = 1 - P(\text{Upgrade}|\tilde{\tau} > x)$, of any rating upgrade occuring at time $\tilde{\tau} > 0$ The left panel shows the survival probability distribution for any obligor at risk of an upgrade, calculated from the Kaplan-Meier product limit estimator (Kaplan and Meier, 1958). The right panel shows the function when right-censored observations are excluded, such that only the cases where upgrades actually occurred counts, calculated from the Kaplan-Meier product limit estimator. 95%-confidence bounds are also displayed. The Weibull-fitted curve in the left panel has a scale-parameter, $\lambda = 0.0928$ and a shape-parameter, α, of 0.5724. This means that the hazard, or instantaneous probability of an upgrade, decreased a lot during the total period. If the Markov property were fully satisfied, the curves would be sloping down linearly, as upgrades then occurs at a constant rate.

up to 12 years. It then flattens out, making the curve convex, which is a property of having an α below 1, where the rate decreases over time. The right-hand panel of Figure 17, hence indicates both visible and from the shape parameter, that there is an apparent effect of the time on the evaluation of the rating agency. The Weibull mean for the left-hand panel in Figure 17 corresponds to a downgrade time, where the increase in intensity is accounted for, that over the full time is estimated to be an average of 16.6 years. Distinguishing between the averages is essential as it does not mean that this was the average time for the obligors that got a lower rating. To get this average time, we use the survival function which excludes all the cases with obligors being upgraded or assigned the same score, shown in the right-hand panel in Figure 17. The mean time until a downgrade for those obligors that downgraded were instead 12 years.

We also see from the graph that the rate at which the decrease in survival probability occurs are convex in both panels of Figure 17. Looking at the Kaplan-Meier estimated survival function, there seem to be at least four time-intervals characterized by different distribution for the random time of downgrades. The first is the period between 0 and 10 years. Either upgrades or ratings into the same rating are shown to be more common than downgrades in the initial 10 years after a rating is assigned. The second period after 10 years, shows a sharper decline in the empirical survival function, going on for 4 years, such that the survival probability goes from 70% down to 50% in the four years that follows. The increase in the rate for this period compared to the time that precedes it is apparent by the hunched shape at time 10 in the left-panel of Figure 17. The third distribution phase is the goes from 14 to 28 years, and lastly, the sharpest decline occurs in the last four years. It is interesting to note that the shape of the real distribution of the survival function and the one that excludes right-censored data has a somewhat similar distribution towards the second half of the period. What this means is that in the first half, more obligor was rated the same as before, or got upgraded. In the latter half (firms unrated for over 15 years) downgrades are so frequent that they make up almost all rating changes.

In Figure 18 the empirical distribution of the survival function for rating upgrades are shown. Since upgrades that occurred very heavily distributed in the beginning year, the empirical function will not go to zero as there are only right-censored observations after 22 years. When a Weibull fit is used for this function it will, therefore, give a very small α and λ such that a rating upgrade is not expected to occur until 102.1 years into the future. This, of course, does not make sense and as mentioned has to do with trying to fit values that a very skewed towards a low number of years for the rating upgrades that occurred in the data. For example, the mean of the right-hand panel in Figure 18, shows that the average time until the upgrade for the obligors that were upgraded was 1.41 years. Almost no upgrade occurred after 10 years. In the end, we conclude that upgrades in our sample were not approximated well by Markov properties, as indicated in both panels of Figure 18.

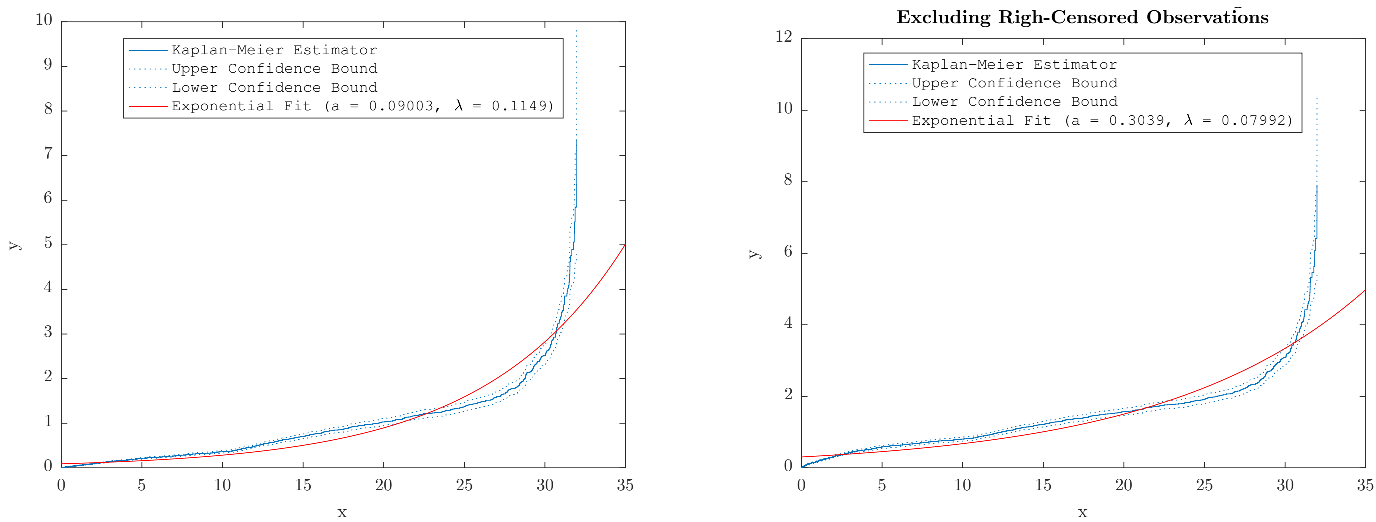
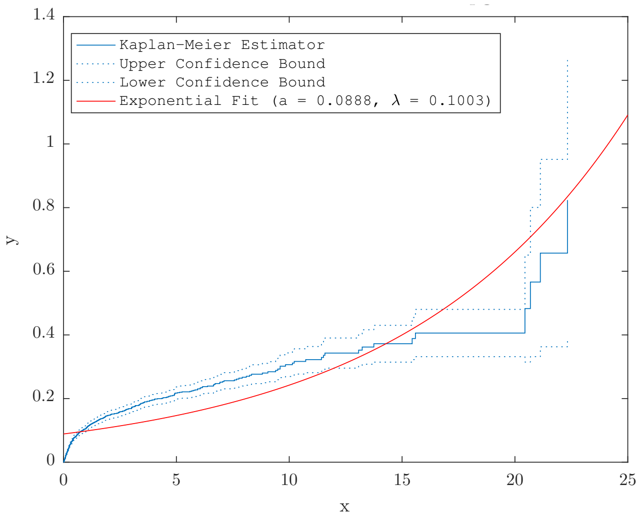


Figure 19: Empirical Cumulative Hazard Function for Rating Downgrades. Result of the empirically estimated cumulative hazard function, $\hat{h}(x)$, for any rating downgrade in the dataset. The left panel shows the cumulative hazard for any obligor at risk of a downgrade, calculated from the Kaplan-Meier product limit estimator (Kaplan and Meier, 1958). The right panel shows the function when right-censored observations are excluded, such that only the cases where downgrades actually occurred counts, calculated from the Kaplan-Meier product limit estimator. 95%-confidence bounds are also displayed. The function is fitted to the exponential function: $y = a\exp(\lambda x)$. If the Markov property were fully satisfied, the curves would be sloping linearly to the upper-right, as the expected transition increase linearly with time at the same rate.

Figure 19 shows the cumulative hazard functions for rating downgrades. The graphs further illustrate the time-varying distribution of downgrades, since the curves would be linear otherwise. The left panel in Figure 19 indicates what the expected number of downgrades is at each time. From the analysis of the survival function for downgrades, we know that the average time until a downgrade for a non-defaulted obligor was 16.1 years. As shown in Figure 19, we hence see that this value on the x-axis corresponds to a cumulative hazard of 1 (the first hitting time). Figure 20 shows the cumulative hazard function for upgrades. Since, α, is below 1, for the fitted curve, it means that the shape of the cumulative hazard is logarithmic. Thus it bends the other way compared to that of the downgrade cumulative distribution. We also see that this corresponds to a much stronger decrease rate over time for upgrades. The right-hand panel in Figure 20 shows that an upgrade is likely to happen only few years from the previous rating. In comparison to the downgrade cumulative hazard, the time until a change is much sooner for upgrades.

From the survival analysis for any upgrade or downgrade, we conclude that the most significant difference between upgrade and downgrade distributions are the estimated, α. Upgrades diverged far more from Markov properties ($\alpha = 1$) than for downgrades. The motivation of using different shape-parameter in cycles, is therefore



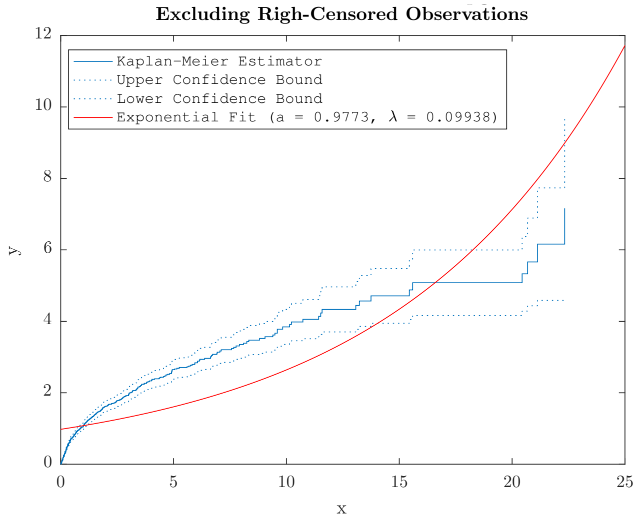


Figure 20: Empirical Cumulative Hazard Function for Rating Upgrades. Result of the empirically estimated cumulative hazard function, $\hat{h}(x)$, for any rating upgrade in the dataset. The left panel shows the cumulative hazard for any obligor that can be upgraded, calculated from the Kaplan-Meier product limit estimator (Kaplan and Meier, 1958). The right panel shows the function when right-censored observations are excluded, such that only the cases where upgrades actually occurred counts, calculated from the Kaplan-Meier product limit estimator. 95%-confidence bounds are also displayed. The function is fitted to the exponential function: $y = a\exp(\lambda x)$. If the Markov property were fully satisfied, the curves would be sloping linearly to the upper-right, as the expected transition increase linearly with time at the same rate.

strong, as it accounts for differences in the expected number of credit events.

C Simulated Final Ratings

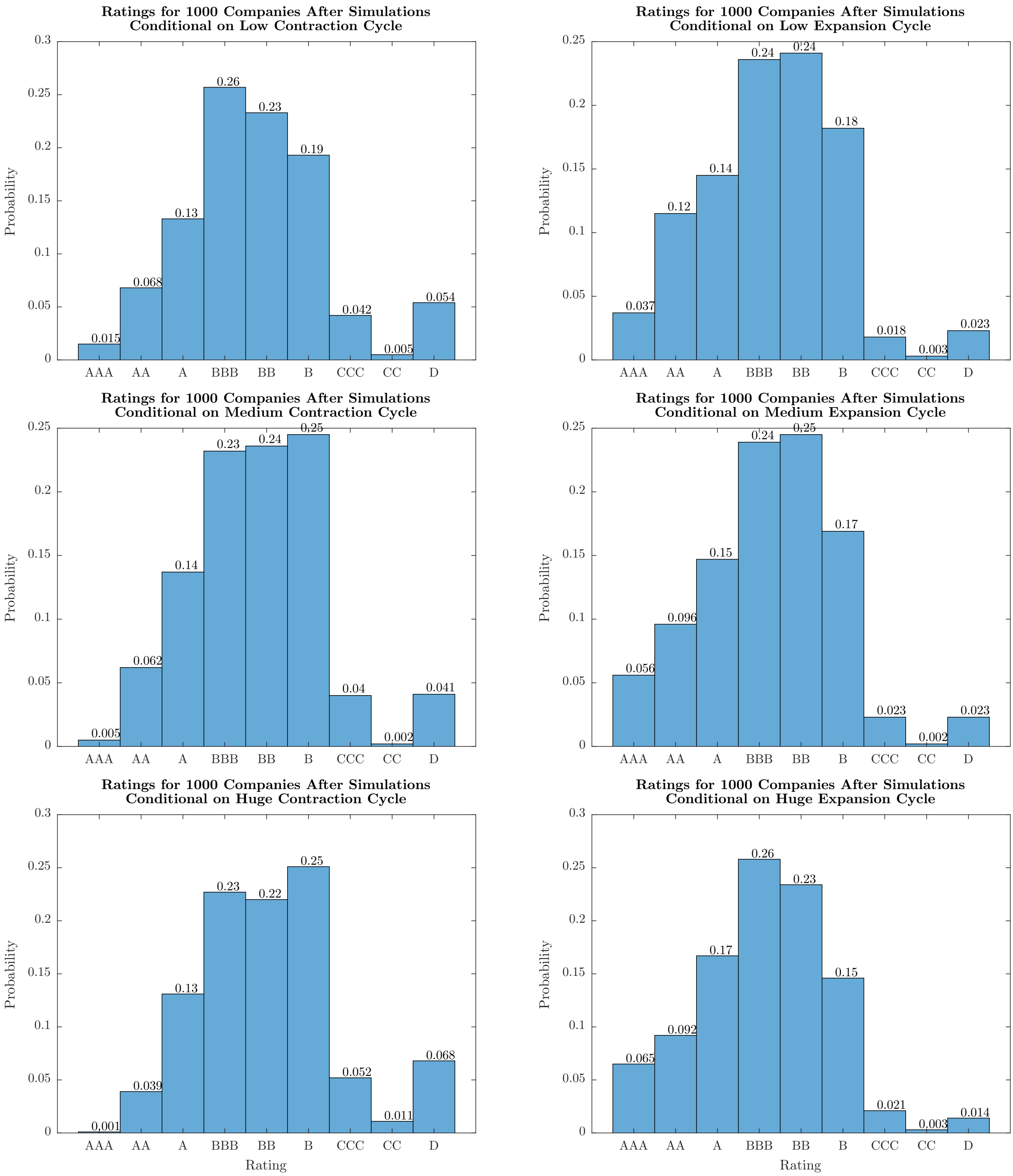


Figure 21: Final Rating Distributions after Simulations. Simulation results, showing how the final ratings of the 1,000 companies in Figure 6, ended up after their rating changes were dependent on different economic cycles. Note that no state was absorbing, so each distribution still consists of 1,000 companies.

D Loss Distributions Sensitivity to Risk Premiums

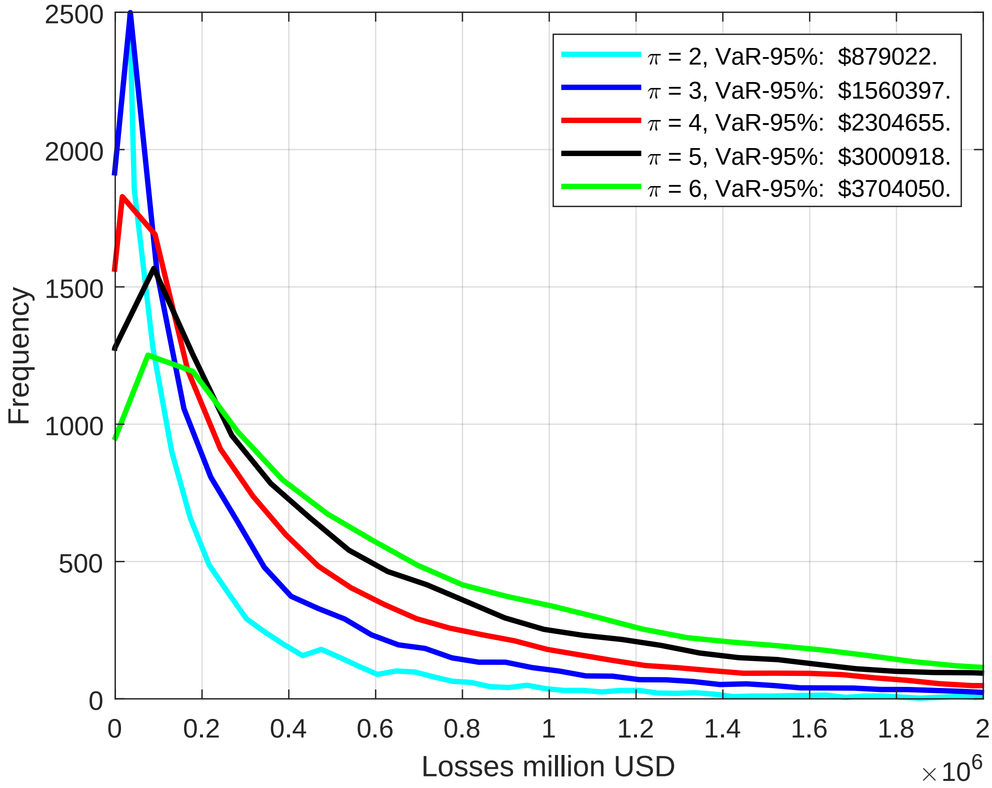


Figure 22: Loss Distribution for Different Risk Premiums. CreditMetrics™ output for 10,000 simulated portfolio losses, using the empirical forward TM for 10-years and setting a constant risk premium, π, for all ratings relating the empirical probabilities of default to the risk-neutral probabilities of default. On the x-axis is losses defined by Equation (24) and on the y-axis is the kernel function values for the loss distribution.

As we discussed in Subsection 2.4.1, the risk-neutral default probabilities are often much more substantial, implying that there is a risk premium for ratings. Figure 22 shows the same stress-test as performed in Figure 12, utilizing the corresponding empirical probabilities of default but using five different risk-premiums, which we denote π. The real probabilities of transitions are related to the risk-neutral probabilities in the formula (Lando, Jarrow and Turnbull, 1997):

$$\tilde{q}_{ij}(t, t+1) = \pi_i(t) q_{ij},\ \pi_i(t) \geq 0, i = 1, ..., K-1 \tag{33}$$

Note in Equation (33) that the risk-premias $\pi_i(t)$ for a rating class i only depends on i and not j, but can still be used to transform an empirical transition matrix Q into a risk-neutral transition matrix $\tilde{Q}$, which are used in Equation (6) to price bonds. By modelling the term-structure from a Markov chain of ratings using the model by Jarrow, Turnbull and Lando (1997) in Equation (6) means we have to know the risk premium at maturity of the portfolio which we want to value, 10-years. In the Jarrow, Turnbull, and Lando (JLT) (1997)-model the risk premium is used to relate risk-neutral transition probabilities with empirical transition probabilities. From Figure

22 it is clear that losses increase if we choose to work with a higher risk premium in the stress-tests.

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