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Abstract

A health-capital model is contemplated which accounts for the consumption of many goods, a stock of health and investment in it, as well as an agent's random lifetime and accumulation of wealth. It is shown that if an agent maximizes the expected discounted value of lifetime utility, or if an agent maximizes the expected value of their lifetime, then an agent does not follow the health-investment policy that minimizes the conditional probability of dying at each point in time, in general. What is more, simple and intuitive sufficient, and necessary and sufficient, conditions are identified whereby such agents investment more or less in their health than said policy.

Keywords: health capital; health investment; optimal control; random lifetime

JEL Codes: C61; D11; I12

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1. Introduction

The notion that people make consumption and investment choices with at least some recognition of their health goes back to Grossman (1972). Indeed, two core features of his health-capital model are that health is conceptualized as a stock and that individuals produce increments to it by combining market goods and services with their time. Even though Grossman's (1972) initial formulation focused on medical care goods and services, later extensions allowed for a broader variety of goods, services, and health-related behaviors, as in, e.g., Muurinen (1982), Ehrlich and Chuma (1990), and Bolin and Lindgren (2016). Theoretical treatments that follow in this tradition and also incorporate the influence of individual health-related decisions on the conditional probability of survival include Levy (2002), Yaniv (2002), and Dragone (2009). A common feature of the aforesaid papers is that an individual is assumed to maximize lifetime utility by allocating resources between competing purposes, of which good health is but one.

Lifetime utility depends, of course, on both instantaneous utility and length of life. In fact, for a given time-path of instantaneous utility, lifetime utility is at a maximum when length of life is too. Despite this connection, most theoretical treatments of individual health-investment choices emphasize the trade-off between health-investment and consumption decisions without paying explicit attention to their influence on an agent's lifetime. This is not accurate, needless to say. Moreover, there is substantial medical evidence, some of which is presented below, that supports the existence of a U-shaped relationship between all-cause mortality and various health-related rates of consumption and investment. Consequently, without accounting for the fact that the conditional probability of dying depends on current and accrued health-related decisions, health-capital models remain incomplete.

Perhaps the first paper to analyze the demand for longevity is that by Ehrlich and Chuma (1990). They addressed the matter in a deterministic setting by assuming that an agent maximizes lifetime utility and chooses, among other things, the time of death. But by making the time of death a decision variable, their model only applies to cases in which an agent chooses when to die, to wit, suicide. As a result, the manner in which Ehrlich and Chuma (1990) treat length of life cannot address its inherent stochastic nature nor the fact that its probability distribution can be influenced by certain actions that an agent might take. Similarly, Dalgaard and Strulik (2014) performed a numerical analysis of optimal ageing and death in a deterministic life-cycle model, and they too treated the time of death as a decision variable.

Ehrlich (2000) addressed the above two limitations by treating the time of death as a random variable and permitting its hazard rate to be influenced by health investment. But as in Levy (2002) and Dragone (2009), Ehrlich (2000) did not formally account for the fact that the probability of dying by any time is a state variable and therefore should be included as such in an optimal control representation of a health-capital model. The present work corrects that omission, and as in Yaniv (2002) and Caputo and Levy (2012), includes the said probability as a state variable in the optimal control formulation of the contemplated health-capital model.

As noted above, the medical literature indicates that several health-related consumption goods and investment activities have a non-monotonic effect on the probability of dying. Indeed, for each good and activity, there is medical evidence that a physiological rate that maximizes the positive effect on health, or minimizes the conditional probability of dying, or both, exists. For instance, a U-shaped association between all-cause mortality and alcohol consumption was found by Grönbaeck (2009) and Fuller (2011), and between all-cause mortality and the energy and nutritional content of food by Steinhausen (2002) and Steinhausen and Weber (2009). And Schnohr et al. (2015) and Lee et al. (2015) documented a U-shaped association between all-cause mortality and the amount of jogging. Finally, Morton et al. (2009) and Schnohr et al. (2015) presented evidence that too little or too much exercise increases the probability of dying. So, for many goods and activities, there exists a rate which minimizes the conditional probability of dying, known as the physiological optimal rate. Presumably, nutritional and lifestyle guidelines issued by various health authorities in many countries are based on such epidemiological findings.

The aforementioned medical evidence is taken seriously in what follows. In particular, the paper aims to develop a general model of consumption and health investment that, among other things, (i) contemplates the consumption of many goods unrelated to health, (iii) includes a health-investment choice and a stock of health, (iv) allows for a nonmonotonic health production function, and (v) treats the time of death as a random variable whose conditional probability distribution is influenced by certain decisions that an agent makes. In this general setting it is proven that agents do not, in general, follow publicly-issued nutritional and lifestyle guidelines of governmental agencies or of their physician, and instead typically choose to overinvest or underinvest in their health.

2. The Health Capital Model and Assumptions

A crisp development of the generalized health-capital model is presented below. Certain canonical assumptions are noted along the way, but those critical to the main result are stated formally and discussed explicitly after the optimal control problem is developed.

To begin, let $\mathbf{C}(\tau) \in \mathbb{R}_+^M$ denote the consumption vector of nondurable goods that does not affect an agent's stock of health $H(\tau) \in \mathbb{R}_+$ at time τ, and let $I(\tau) \in \mathbb{R}_+$ denote an agent's health-investment rate at time τ. Instantaneous preferences are represented by a felicity function $U(\cdot)$, and agents are assumed to care about the consumption of nondurable goods, health, and investment in health. Accordingly, the value of $U(\cdot)$ at time τ is $U\big(\mathbf{C}(\tau), H(\tau), I(\tau)\big)$.

The state equation for the health stock is $\dot{H}(\tau) = F\big(I(\tau)\big) - \delta H(\tau)$, where $\delta \in \mathbb{R}_{++}$ is an exogenous rate of depreciation of the stock of health, and $F(\cdot)$ is a health production function, mapping the rate of investment in health to the rate-of-change of the health stock with respect to time. This specification is general and realistic, in that, as discussed below, it allows for a nonlinear and nonmonotonic effect of investment on the health stock. What is more, health investment is interpreted broadly, and could include the consumption rates of numerous goods that influence the stock of health, or physical or medical activities that have the same general quality. Such an interpretation is also fully consistent with the medical research cited in §1.

Let $\tau = t$ be an arbitrary but fixed base time of an optimal control problem, that is, time $\tau = t$ is the initial date of the planning horizon and hence the date at which the optimization decision is made. Given this convention, Caputo (2017) has shown that the archetypal lifetime budget constraint from the perspective of base time $\tau = t$ may be written in the form of an ordinary differential equation, viz., $\dot{A}(\tau) = rA(\tau) + Y_0 + Y\big(H(\tau)\big) - \mathbf{p}_C'\mathbf{C}(\tau) - p_I I(\tau)$, and an initial condition, namely, $A(t) = A_t$, where $\mathbf{p}_C \in \mathbb{R}^M_{++}$ is the price vector of the consumption goods, $p_I \in \mathbb{R}_{++}$ is the price of health investment, $r \in \mathbb{R}_{++}$ is an interest rate, $A(\tau)$ is wealth at time τ, $A_t \in \mathbb{R}$ is a given value of wealth at base time t, $Y(\cdot) \in C^{(2)}$ is a health-dependent income flow function, $Y_0 \in \mathbb{R}_{++}$ is an exogenous flow of income, and " $'$ " denotes transposition.

The final aspect of the model addresses the fact that time of death is a random variable. Therefore, let the probability of dying by time τ be given by a cumulative density function $P(\cdot)$, where $P(\tau) \in [0,1]$ for all $\tau \in [0,T]$, $T \in \mathbb{R}_{++}$, $P(0) = 0$, and $P(T) = 1$. It is assumed that the probability of dying at time τ given survival until time τ —known as the hazard function—is a function of the stock of health and health investment, i.e., $\dot{P}(\tau)/[1 - P(\tau)] = \Phi\big(H(\tau), I(\tau)\big)$. This specification reflects the aforementioned medical evidence in that both the rate of health-investment and its accumulated effects, as reflected in the health stock, jointly influence the conditional probability of dying. A simple rearrangement of the above hazard equation yields $\dot{P}(\tau) = \Phi\big(H(\tau), I(\tau)\big)[1 - P(\tau)]$ as a state equation, with initial condition $P(0) = 0$ and terminal condition $P(T) = 1$.

In order to formulate the objective functional of an expected discounted lifetime-utility-maximizing agent, the stochastic nature of an agent's lifetime must be taken in to account. To this end, note that the discounted value of lifetime utility at base time $t \in [0,T)$, from base time t until time $\tau > t$ in the future, is

$$\int_t^\tau U\big(\mathbf{C}(s), H(s), I(s)\big) e^{-\rho[s-t]}\, ds\,, \tag{1}$$

where $\rho \in \mathbb{R}_{++}$ is the rate of time preference. By definition, expected discounted lifetime utility is found by multiplying the discounted value of lifetime utility between base time t and a possible time of death τ given in Eq. (1), by the probability of dying at time τ, namely, $\dot{P}(\tau)$, and then integrating the aforesaid product over all possible times of death $\tau \in [t,T]$, thereby resulting in the expression

$$\int_t^T \dot{P}(\tau)\left[\int_t^\tau U\big(\mathbf{C}(s),H(s),I(s)\big)e^{-\rho[s-t]}\,ds\right]d\tau\,. \tag{2}$$

The final step is to render Eq. (2) suitable for analysis by the methods of optimal control. To this end, let $u(\tau)\overset{\text{def}}{=}\int_t^\tau U\big(\mathbf{C}(s),H(s),I(s)\big)e^{-\rho[s-t]}\,ds$ and $dv(\tau)\overset{\text{def}}{=}\dot{P}(\tau)d\tau$, and integrate Eq. (2) by parts to arrive at

$$\begin{aligned}\int_t^T \dot{P}(\tau)\left[\int_t^\tau U\big(\mathbf{C}(s),H(s),I(s)\big)e^{-\rho[s-t]}\,ds\right]d\tau &= \left[P(\tau)\int_t^\tau U\big(\mathbf{C}(s),H(s),I(s)\big)e^{-\rho[s-t]}\,ds\right]_{\tau=t}^{\tau=T}\\ &\quad-\int_t^T P(\tau)U\big(\mathbf{C}(\tau),H(\tau),I(\tau)\big)e^{-\rho[\tau-t]}\,d\tau \\ &=\int_t^T U\big(\mathbf{C}(\tau),H(\tau),I(\tau)\big)[1-P(\tau)]e^{-\rho[\tau-t]}\,d\tau,\end{aligned} \tag{3}$$

seeing as the dummy variable of integration is immaterial, $P(t)\in[0,1]$, and $P(T)=1$.

Pulling all of the above information together, an agent is asserted to behave as if solving the optimal control problem

$$\begin{aligned}&\max_{\mathbf{C}(\cdot),I(\cdot)}\int_t^T U\big(\mathbf{C}(\tau),H(\tau),I(\tau)\big)\big[1-P(\tau)\big]e^{-\rho[\tau-t]}\,d\tau\\ &\text{s.t. }\ \dot{A}(\tau)=rA(\tau)+Y_0+Y\big(H(\tau)\big)-\mathbf{p}_C'\mathbf{C}(\tau)-p_I I(\tau),\ A(t)=A_t\,,\\ &\qquad \dot{H}(\tau)=F\big(I(\tau)\big)-\delta H(\tau),\ H(t)=H_t\,,\\ &\qquad \dot{P}(\tau)=\Phi\big(H(\tau),I(\tau)\big)\big[1-P(\tau)\big],\ P(t)=P_t\,,\ P(T)=1\,,\end{aligned} \tag{4}$$

where $H_t\in\mathbb{R}_{++}$ is a given value of the health stock at the base time and $P_t\in[0,1]$ is a given value of the cumulative density function at the base time. In order to ease the notation in what follows, define $\boldsymbol{\theta}\overset{\text{def}}{=}(t,A_t,H_t,P_t,T,\mathbf{p}_C,p_I,Y_0,\delta,\rho,r)\in\mathbb{R}^{M+10}$ as the parameter vector.

The following assumptions are imposed on problem (4) and discussed subsequently.

(A1) For all $(\mathbf{C},H,I)\in\mathbb{R}_{++}^M\times\mathbb{R}_{++}\times\mathbb{R}_{++}$, $U(\cdot)\in C^{(2)}$, $U_{C_i}(\mathbf{C},H,I)\geq 0$, $i=1,2,\ldots,M$, and $U_H(\mathbf{C},H,I)>0$.

(A2) For all $H\in\mathbb{R}_{++}$, $Y(\cdot)\in C^{(2)}$ and $Y'(H)>0$.

(A3) For all $I\in\mathbb{R}_{++}$, $F(\cdot)\in C^{(2)}$.

(A4) For all $(H,I) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$, $\Phi(\cdot) \in C^{(2)}$, strongly convex, and $\Phi_H(H,I) < 0$, while for all $H \in \mathbb{R}_{++}$ there exists an $I = \bar{I}(H) \in \mathbb{R}_{++}$ such that $\Phi_I\left(H, \bar{I}(H)\right) \equiv 0$.

(A5) For all $\boldsymbol{\theta} \in B(\boldsymbol{\theta}^0;\varepsilon)$, there exists an interior and finite optimal control vector for problem (4), denoted by $\left(\mathbf{C}^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right)$, with corresponding states $\left(A^*(\tau;\boldsymbol{\theta}), H^*(\tau;\boldsymbol{\theta}), P^*(\tau;\boldsymbol{\theta})\right)$ and costates $\left(\lambda_A(\tau;\boldsymbol{\theta}), \lambda_H(\tau;\boldsymbol{\theta}), \lambda_P(\tau;\boldsymbol{\theta})\right)$, where $\boldsymbol{\theta}^0$ is a given value of $\boldsymbol{\theta}$, and $B(\boldsymbol{\theta}^0;\varepsilon)$ is an open $(M+10)$-ball centered at $\boldsymbol{\theta}^0$ of radius $\varepsilon > 0$.

Assumption (A1) asserts that instantaneous preferences are monotonic in the goods unrelated to health, but strongly monotonic in health, meaning that health is always a good as far as instantaneous preferences are concerned. On the other hand, as no stipulation on the sign of the marginal utility of health investment has been made, health investment could be a good, neutral, or a bad with regard to instantaneous preferences. This allows for a wider range of behavior than the prototypical assumption, scilicet, $U_I(\mathbf{C},H,I) \equiv 0$, as shown in the ensuing section. Note too that concavity of the felicity function, a ubiquitous assumption in the literature extant, has not been assumed.

Supposition (A2) says that the health-dependent income flow function is strictly increasing in the health stock, i.e., health is a good with respect to income, which is the usual assumption. Note too that no concavity assumption has been placed on the health production function.

Assumption (A3) allows for health investment to have a nonmonotonic effect on the rate of change of the health stock with respect to time, seeing as the marginal product of health investment may be positive, zero, or negative. As will be demonstrated in the ensuing section, the fact that $F'(I) \gtreqless 0$ permits the model to capture a wider range of behavior than if the archetypal assumption were made, to wit, $F'(I) > 0$. Observe again that no concavity assumption has been placed on the health production function either.

By a well-known theorem from convex optimization, assumption (A4) implies that investing in health at the rate $I = \bar{I}(H)$ globally and uniquely minimizes the conditional probability of dying at each point in time for any given positive value of the stock of health. This implies a U-shaped relationship between health investment and the conditional probability of dying for any positive value of the health stock. It is important to note that this assumption and the properties that it implies reflect the medical evidence presented in §1 regarding a U-shaped relationship between all-cause mortality and various forms of health-related behaviors.

And finally, due to the lack of assumptions on the functions employed in formulating problem (4), supposition (A5) is essential, as without it one cannot guarantee that a solution of the necessary conditions exists. Moreover, there is not a simple set of conditions that could be placed on the felicity, health-dependent income, and health production functions, such as concavity, that would permit one to invoke Mangasarian- or Arrow-type sufficient conditions, thereby pointing to the essential nature of assumption (A5).

3. The Main Theorem and Extensions

Before the main theorem can be established, the following technical result must be introduced. Its proof and that of Theorem 1 are given in the Appendix.

Lemma 1. *Let assumptions (A1)–(A5) hold. If* $\lambda_P(\tau;\boldsymbol{\theta}) < 0 \, \forall (\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$ *then* $\lambda_H(\tau;\boldsymbol{\theta}) > 0 \, \forall (\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$.

Lemma 1 states that if the expected lifetime marginal utility of the probability of dying by time τ is negative at each point in time of an agent's lifetime but the last, then the expected lifetime marginal utility of health is positive at all points in time but the last. This is a desirable property stemming from the intuitive stipulation that an increase in the probability of dying by time τ makes an agent worse off. Given the assumptions $U_H(\mathbf{C},H,I) > 0$, $Y'(H) > 0$, and $\Phi_H(H,I) < 0$, i.e., health is a good in all manners possible, it is not surprising that the expected lifetime marginal utility of health is positive at all points in time but the last. Indeed, any other conclusion would be implausible under the foregoing assumptions. With Lemma 1 in place, the main result can be stated.

Theorem 1. *Let assumptions (A1)–(A5) hold and assume* $\lambda_P(\tau;\boldsymbol{\theta}) < 0 \, \forall (\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$.

(i) *If for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$, $U_I\left(\mathbf{C}^*(\tau;\boldsymbol{\theta}), H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right) = 0$ *and* $F'\left(I^*(\tau;\boldsymbol{\theta})\right) = 0$, *then* $I^*(\tau;\boldsymbol{\theta}) = \bar{I}\left(H^*(\tau;\boldsymbol{\theta})\right)$ *for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$.

(ii) *If for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$, $U_I\left(\mathbf{C}^*(\tau;\boldsymbol{\theta}), H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right) \geq 0$, $F'\left(I^*(\tau;\boldsymbol{\theta})\right) \geq 0$, *and either derivative is positive, then* $I^*(\tau;\boldsymbol{\theta}) > \bar{I}\left(H^*(\tau;\boldsymbol{\theta})\right)$ *for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$.

(iii) *If for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$, $U_I\left(\mathbf{C}^*(\tau;\boldsymbol{\theta}), H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right) \leq 0$, $F'\left(I^*(\tau;\boldsymbol{\theta})\right) \leq 0$, *and either derivative is negative, then* $I^*(\tau;\boldsymbol{\theta}) < \bar{I}\left(H^*(\tau;\boldsymbol{\theta})\right)$ *for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$.

Part (i) of Theorem 1 gives simple sufficient conditions under which the health-investment rate that maximizes an agent's expected lifetime utility also minimizes the conditional probability of dying at each point in time but the last. In particular, it shows that if the instantaneous marginal utility and marginal product of health investment vanish at each point in time of an agent's lifetime but the last, then the expected lifetime utility maximizing rate of health investment also minimizes the conditional probability of dying at each point in time but the last. Note too that this pair of sufficient conditions is a knife's-edge case—a probability zero event that almost never happens—and thus forms a dividing line between the overinvestment and underinvestment cases given in parts (ii) and (iii), respectively.

If $I = \bar{I}(H)$ is interpreted as the health-investment recommendation of physicians, or that of a health agency, then Theorem 1 can be interpreted colloquially. In this case, it asserts that rational agents do not generally follow the advice of their physician or health authority regarding health-investment decisions. Said differently, if one's physician or a health agency asserts that certain health-investment rates are best, rational agents will almost always choose a different rate of health investment than recommended.

Note too that Theorem 1 is a general result, as it applies to an optimal control problem with M consumption goods that are unrelated to heath, health investment, and three state variables, namely, health, wealth, and the cumulative density function of an agent's lifetime. What is more, it only relies on assumptions (A1)–(A5) and Lemma 1. Indeed, there are no strong or unusual assumptions placed on instantaneous preferences, or on the income flow, health production, or conditional probability of dying, functions, nor are the said functions assumed to take a particular functional form. What is more, $U(\cdot)$, $Y(\cdot)$, and $F(\cdot)$ are not assumed to be concave.

Parts (ii) and (iii) of Theorem 1 give simple and intuitive sufficient conditions for "overinvestment" and "underinvestment" over an agent's lifetime, respectively, relative to a policy that minimizes the conditional probability of dying at each point in time. Take part (ii), for example. It asserts that if the instantaneous marginal utility and marginal product of health investment are nonnegative all along the solution, and either is positive, then rational agents overinvest in their health relative to a policy that minimizes the conditional probability of dying at each point in time. In this case, health investment provides a marginal benefit via the felicity function and another by way of the health production function. With health investment having two marginal benefits, it is plausible that rational agents would overinvest in health relative to a policy that minimizes the conditional probability of dying at each point in time. Note too that this case subsumes the archetypal assumptions $F'(I) > 0$ and $U_I(\mathbf{C}, H, I) \equiv 0$. Indeed, the prototype stipulations imply that rational agents *always* overinvest in their health at each point in time.

In contrast to the archetypal assumptions, the present assumptions permit the model to explain why individuals might choose to underinvest in their health. Part (iii) shows that underinvestment in health is optimal when all along the optimal path the marginal utility of investment is negative and the marginal product of investment is not positive, or vice versa. With two marginal costs associated with health investment it is no wonder that agents underinvest in their health. The said archetypal assumptions yield a model incapable of explaining such behavior.

Implicit in Theorem 1 are weaker statements about the optimal rate of health investment too. To see this, observe that Theorem 1 is predicated on the signs of certain derivatives holding over an agent's lifetime. But what if instead the marginal utility and marginal product of health investment were positive only over an subinterval of an agent's lifetime, or even just at a particular point in time? In this case, overinvest in health occurs over the said subinterval or point in time, rather than over an agent's entire lifetime. Consequently, the model is capable of delivering a rational explanation as to why some individuals might underinvest in their health over some periods of time but overinvest in others, something the archetypal assumptions rule out.

Theorem 1 also holds if a more general assumption is made about a subset of the functions and parameters, as summarized by the following.

Theorem 2. *Let assumptions (A1)–(A5) hold. If* $\lambda_P(\tau;\boldsymbol{\theta})<0\,\forall(\tau,\boldsymbol{\theta})\in[t,T)\times B(\boldsymbol{\theta}^0;\varepsilon)$*, and* $(\mathbf{p}_C,p_I,Y_0,\delta,r)$ *and* $\left(F(\cdot),\Phi(\cdot),Y(\cdot)\right)$ *are explicit functions of time, then the conclusions of Theorem 1 et seq. continue to hold.*

Theorem 2 asserts that if all the parameters but the rate of time preference are functions of time, and the income flow, health production, and conditional probability of dying, functions are too, then rational agents continue to generally ignore the advice of their physician or health authority regarding health-investment decisions. Its veracity follows from the fact that the necessary conditions under Theorem 2 are identical in form to those given by Eqs. (7)–(14) in the Appendix, as all that differs is the explicit time-dependence of the so-noted functions and parameters. Consequently, Lemma 1 continues to hold, as do the conclusions of Theorem 1.

Now consider an agent who maximizes a different objective functional, namely, their expected lifetime from the perspective of base time t, given by $\int_t^T \tau\dot{P}(\tau)d\tau$. By employing the same integration-by-parts approach that was used in Eq. (3), it can be shown that

$$\int_t^T \tau\dot{P}(\tau)d\tau=\int_t^T[1-P(\tau)]d\tau+t[1-P(t)]. \tag{5}$$

Consequently, maximizing an agent's expected lifetime, viz., $\int_t^T \tau\dot{P}(\tau)d\tau$, is equivalent to maximizing $\int_t^T[1-P(\tau)]d\tau$, as $t\in[0,T]$ is an arbitrary but fixed base time and $P(t)=P_t$ is given. With this latter objective functional, it is straightforward to verify that the necessary conditions associated with it are a special case of those given in Eqs. (7)–(14) in the Appendix, found by setting $U(\mathbf{C},H,I)\equiv 1$ and $\rho\equiv 0$. Hence the following corollary is immediate.

Corollary 1. *Let assumptions (A2)–(A5) hold and replace the objective functional of problem (4) with* $\int_t^T \tau \dot{P}(\tau)d\tau$ *. If* $\lambda_P(\tau;\boldsymbol{\theta}) < 0 \forall (\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$ *, then*

(i) $F'\left(I^*(\tau;\boldsymbol{\theta})\right) = 0$ *for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$ *if and only if* $I^*(\tau;\boldsymbol{\theta}) = \bar{I}\left(H^*(\tau;\boldsymbol{\theta})\right)$ *for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$*,*

(ii) $F'\left(I^*(\tau;\boldsymbol{\theta})\right) > 0$ *for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$ *if and only if* $I^*(\tau;\boldsymbol{\theta}) > \bar{I}\left(H^*(\tau;\boldsymbol{\theta})\right)$ *for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$*, and*

(iii) $F'\left(I^*(\tau;\boldsymbol{\theta})\right) < 0$ *for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$ *if and only if* $I^*(\tau;\boldsymbol{\theta}) < \bar{I}\left(H^*(\tau;\boldsymbol{\theta})\right)$ *for all* $(\tau,\boldsymbol{\theta}) \in [t,T) \times B(\boldsymbol{\theta}^0;\varepsilon)$*.*

Corollary 1 asserts that an agent who maximizes the expected value of their lifetime chooses to invest in their health at a rate that generally differs from that which minimizes the conditional probability of dying at each point in time, just like an expected lifetime utility maximizing agent. But in this case the sufficient conditions are simplified and are also necessary, making for a more powerful conclusion. Despite the similarities between Theorem 1 and Corollary 1, the investment rates of an expected discounted lifetime-utility maximizer and an expected value of lifetime maximizer are not, in general, the same, as their objective functionals differ.

4. Conclusion

Agents who maximize expected lifetime utility or the expected value of their lifetime, choose to invest in their health at a rate that generally differs from than that which minimizes the conditional probability of dying at each point in time. Stated colloquially, rational agents do not follow the advice of their physician or health agency regarding health investment decisions, in general. The traditional assumptions imply that rational agents over invest in their health at every point in time of their lifetime, whereas the more general assumptions employed here permit an intuitive rationale as to why rational agents might overinvest or underinvest in their health, or even, on occasion, follow the recommendation of their physician.

5. Appendix

Define the current-value Hamiltonian by

$$\Psi(A,H,P,\mathbf{C},I) \overset{\text{def}}{=} U(\mathbf{C},H,I)[1-P] + \lambda_A\left[rA + Y_0 + Y(H) - \mathbf{p}_C'\mathbf{C} - p_I I\right] + \lambda_H\left[F(I) - \delta H\right] + \lambda_P \Phi(H,I)[1-P]. \tag{6}$$

By Theorem 10.1 of Caputo (2005), the necessary conditions obeyed by $\left(\mathbf{C}^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right)$ are

$$U_{C_i}(\mathbf{C},H,I)[1-P] - p_{C_i}\lambda_A = 0\ ,\ i = 1,2\ldots,M\ , \tag{7}$$

$$U_I(\mathbf{C},H,I)[1-P] - \lambda_A p_I + \lambda_H F'(I) + \lambda_P \Phi_I(H,I)[1-P] = 0\ , \tag{8}$$

$$\dot{\lambda}_A = [\rho - r]\lambda_A\ ,\ \lambda_A(T) = 0\ , \tag{9}$$

$$\dot{\lambda}_H = [\rho+\delta]\lambda_H - [U_H(\mathbf{C},H,I) + \lambda_P \Phi_H(H,I)][1-P] - \lambda_A Y'(H)\ ,\ \lambda_H(T) = 0\ , \tag{10}$$

$$\dot{\lambda}_P = [\rho + \Phi(H,I)]\lambda_P + U(\mathbf{C},H,I)\ , \tag{11}$$

$$\dot{A} = rA + Y_0 + Y(H) - \mathbf{p}_C'\mathbf{C} - p_I I\ ,\ A(t) = A_t\ , \tag{12}$$

$$\dot{H} = F(I) - \delta H\ ,\ H(t) = H_t\ , \tag{13}$$

$$\dot{P} = \Phi(H,I)\left[1-P\right],\ P(t) = P_t\ ,\ P(T) = 1\ , \tag{14}$$

Note that the solution of the necessary conditions holds for all $\boldsymbol{\theta} \in B(\boldsymbol{\theta}^0;\varepsilon)$, so this stipulation will not be repeatedly spelled out in the following proofs in order to ease the notational burden.

Proof of Lemma 1. By Eq. (9), $\lambda_A(\tau;\boldsymbol{\theta}) \equiv 0$ for all $\tau \in [t,T]$. Consequently, Eq. (10) reduces to $\dot{\lambda}_H = [\rho+\delta]\lambda_H - G^*(\tau;\boldsymbol{\theta})$, $\lambda_H(T) = 0$, where

$$G^*(\tau;\boldsymbol{\theta}) \overset{\text{def}}{=} \left[U_H\left(\mathbf{C}^*(\tau;\boldsymbol{\theta}), H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right) + \lambda_P(\tau;\boldsymbol{\theta})\Phi_H\left(H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right)\right]\left[1 - P^*(\tau;\boldsymbol{\theta})\right],$$

and $G^*(T;\boldsymbol{\theta}) = 0$ as $P(T) = 1$ from Eq. (14). Integration of $\dot{\lambda}_H = [\rho+\delta]\lambda_H - G^*(\tau;\boldsymbol{\theta})$ yields the general solution $\lambda_H(\tau) = -\int_t^\tau e^{-[\rho+\delta][s-\tau]} G^*(s;\boldsymbol{\theta})\,ds + ae^{[\rho+\delta]\tau}$, where a is a constant of integration. The transversality condition $\lambda_H(T) = 0$ gives $a = \int_t^T e^{-[\rho+\delta]s} G^*(s;\boldsymbol{\theta})\,ds$, hence the specific solution is $\lambda_H(\tau;\boldsymbol{\theta}) = \int_\tau^T e^{-[\rho+\delta][s-\tau]} G^*(s;\boldsymbol{\theta})\,ds$. Next, note that if $\lambda_P(\tau;\boldsymbol{\theta}) < 0$ for all $\tau \in [t,T)$, then $G^*(\tau;\boldsymbol{\theta}) > 0$ for all $\tau \in [t,T)$, as $U_H(\mathbf{C},H,I) > 0$ and $\Phi_H(H,I) < 0$ by assumptions (A1) and (A4) respectively. But then it follows that $\lambda_H(\tau;\boldsymbol{\theta}) > 0$ for all $\tau \in [t,T)$. *Q.E.D.*

Proof of Theorem 1. As $\lambda_A(\tau;\boldsymbol{\theta}) \equiv 0$ for all $\tau \in [t,T]$ by Eq. (9), Eq. (8) reduces to

$$U_I\left(\mathbf{C}^*(\tau;\boldsymbol{\theta}), H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right)\left[1 - P^*(\tau;\boldsymbol{\theta})\right] + \lambda_H(\tau;\boldsymbol{\theta})F'\left(I^*(\tau;\boldsymbol{\theta})\right) \equiv -\lambda_P(\tau;\boldsymbol{\theta})\Phi_I\left(H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right)\left[1 - P^*(\tau;\boldsymbol{\theta})\right] \tag{15}$$

for all $\tau \in [t,T)$. By definition, $P^*(\tau;\boldsymbol{\theta}) \in [0,1]$ for all $\tau \in [t,T]$. Furthermore, if $\lambda_P(\tau;\boldsymbol{\theta}) < 0$ for all $\tau \in [t,T)$, then by Lemma 1 $\lambda_H(\tau;\boldsymbol{\theta}) > 0$ for all $\tau \in [t,T)$. Consequently, if $U_I\left(\mathbf{C}^*(\tau;\boldsymbol{\theta}), H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right) \geq 0$, $F'\left(I^*(\tau;\boldsymbol{\theta})\right) \geq 0$, and either is positive for all $\tau \in [t,T)$, then it follows from Eq. (15) that $\Phi_I\left(H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right) > 0$ for all $\tau \in [t,T)$. Moreover, $\Phi_I\left(H^*(\tau;\boldsymbol{\theta}), \bar{I}\left(H^*(\tau;\boldsymbol{\theta})\right)\right) \equiv 0$ for all $\tau \in [t,T)$ by assumption (A2). Putting the two preceding deductions together gives $\Phi_I\left(H^*(\tau;\boldsymbol{\theta}), I^*(\tau;\boldsymbol{\theta})\right) > \Phi_I\left(H^*(\tau;\boldsymbol{\theta}), \bar{I}\left(H^*(\tau;\boldsymbol{\theta})\right)\right)$ for all $\tau \in [t,T)$. But as $\Phi_{II}(H,I) > 0$ for all $(H,I) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$, the preceding inequality implies that $I^*(\tau;\boldsymbol{\theta}) > \bar{I}\left(H^*(\tau;\boldsymbol{\theta})\right)$ for all $\tau \in [t,T)$, which proves part (ii). The proofs of the other parts are essentially identical and so are omitted. *Q.E.D.*

6. References

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