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Alexander Herbertsson

Department of Economics, January 2019

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UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

CDS INDEX OPTIONS IN MARKOV CHAIN MODELS

ALEXANDER HERBERTSSON

Abstract. We study CDS index options in a credit risk model where the defaults times have intensities which are driven by a finite-state Markov chain representing the underlying economy. In this setting we derive compact computationally tractable formulas for the CDS index spread and the price of a CDS index option. In particular, the evaluation of the CDS index option is handled by translating the Cox-framework into a bivariate Markov chain. Due to the potentially very large, but extremely sparse matrices obtained in this reformulating, special treatment is needed to efficiently compute the matrix exponential arising from the Kolmogorov Equation. We provide details of these computational methods as well as numerical results. The finite-state Markov chain model is calibrated to data with perfect fits, and several numerical studies are performed. In particular we show that under same exogenous circumstances, the CDS index options prices in the Markov chain framework can be close to or sometimes larger than prices in models which assume that the CDS index spreads follows a log-normal process. We also study the different default risk components in the option prices generated by the Markov model, an investigation which is difficult to do in models where the CDS index spreads follows a log-normal process.

Keywords: Credit risk; CDS index; CDS index options; intensity-based models; dependence modelling; Markov chains; matrix-analytical methods, numerical methods

JEL Classification: G33; G13; C02; C63; G32.

1. Introduction

The development of liquid markets for synthetic credit index products such as CDS index swaps has led to the creation of derivatives on these products, most notably credit index options, sometimes also denoted CDS index options. Essentially the owner of such an option has the right to enter at the maturity date of the option into a protection buyer position in a swap on the underlying CDS index at a prespecified spread; moreover, upon exercise he obtains the cumulative loss of the index portfolio up to the maturity of the option. Credit index options have gained a lot interest the last turbulent years since they allow investors to hedge themselves against broad movements of CDS index spreads or to trade credit volatility.

To date the pricing and the hedging of these options is largely an unresolved problem. In practice this contract is priced by a fairly ad hoc approach: it is assumed that the loss-adjusted spread of the CDS index at the maturity of the option is lognormally distributed under a martingale measure corresponding to a suitable numeraire, and the price of the option is then computed via the Black formula. Details are described for instance in Morini & Brigo (2011) or Rutkowski & Armstrong (2009). However, beyond convenience there is

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no justification for the lognormality assumption in the literature. In particular, it is unclear if a dynamic model for the evolution of spreads and credit losses can be constructed that supports the lognormality assumption and the use of the Black formula, and there is no empirical justification for this assumption either.

In this paper we study CDS index options in a credit risk model where the defaults times have intensities that are functions of a finite-state Markov chain representing the underlying economy. Such models have previously been studied in e.g. Graziano & Rogers (2009) where the authors consider CDOs and CDSs. However, when pricing CDS index options other probabilistic and numerical methods must be used than those in Graziano & Rogers (2009). The methods proposed in this paper are for some sections close to the corresponding methods in Herbertsson & Frey (2018) where the authors apply nonlinear filtering techniques of Frey & Schmidt (2012). More specific, Frey & Schmidt (2012) uses the innovations approach to nonlinear filtering and derive the Kushner-Stratonovich SDE describing the dynamics of the filtering probabilities. The approach in Herbertsson & Frey (2018) creates CDS index spreads that allow for diffusion, drift and jumps which is important for mimicking realistic pricing. The benefit of Herbertsson & Frey (2018) is that this model allow for diffusion, with very few states of the underlying economy. The drawback of Herbertsson & Frey (2018) is that we have to solve for the filtering probabilities by numerical simulations of the Kushner-Stratonovich SDE in order to find Monte Carlo approximations for the price of CDS index options.

In this paper, on contrary to Herbertsson & Frey (2018), the true state of the economy is observable without noise to the market participants and we are thus back in a standard intensity based credit risk model where the default intensities are driven by a Cox-process just as in Lando (1998). In this setting we derive compact computational tractal formulas for the CDS index spreads and CDS index options. Due to the very large, but extremely sparse matrices obtained in this reformulating, special treatment is needed to efficiently compute the matrix exponential arising from the Kolmogorov Equation. We provide details of these computational methods as well as numerical results. The finite-state Markov chain model is calibrated to data with perfect fits, and several numerical studies are performed. In particular we show that under same exogenous circumstances, the CDS index options prices in the Markov chain framework can be close to or sometimes larger than prices in models which assume that the CDS index spreads follows a log-normal process. We also study the different default risk components in the option prices generated by the Markov model, an investigation which is difficult to do in models where the CDS index spreads follows a log-normal process.

Options on a CDS index have been studied in for example Pedersen (2003), Jackson (2005), Liu & Jäckel (2005), Doctor & Goulden (2007), Rutkowski & Armstrong (2009), Morini & Brigo (2011), Flesaker, Nayakkankuppam & Shkurko (2011) and Martin (2012). In all of these papers it is assumed that either the CDS index spread or the so called loss-adjusted CDS index spread at the maturity of the option is lognormally distributed under a martingale measure corresponding to a suitable numeraire, and the price of the option is then computed via the Black formula. For a nice and compact overview of some of the above mentioned papers, see pp.577-579 in Morini & Brigo (2011).

The rest of the paper is organized as follows. First, in Section 2 we give a brief introduction to how a CDS index works and then present a model independent expression for the so called CDS index spread. Section 2 also introduces options on the CDS index and provides a formula for the payoff such an option which holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. Then, in Section 3 we briefly describe the model used in this paper, originally presented in Graziano & Rogers (2009) and we provide the main building blocks that will be necessary to find formulas for portfolio credit derivatives such as e.g. the CDS index as well as credit index options. Examples of such building blocks are the conditional survival distribution, the conditional number of defaults and the conditional loss distribution. In Section 4 we use the results from Section 3 to derive computational tractable formulas for the CDS index in the model presented in Section 3. This will be done in a homogeneous portfolio. Continuing, in Section 5 we derive practical formula for the price of a CDS index option in the Markovian modell.

Finally, in Section 6 we discuss how to estimate or calibrate the parameters in the Markovian model introduced in Section 3 and also calibrate our model and present different numerical results for prices of options on a CDS index. More specific, the Markov model is calibrated to data with perfect fits, and several numerical studies are performed. For example, we show that under same exogenous circumstances, the CDS index options prices in the finite-state Markov chain setting can be several hundred percent bigger compared with models which assume that the CDS index spreads follows a log-normal process. We also compare the Markovian prices with the corresponding prices in the nonlinear filtering model used in Herbertsson & Frey (2018).

2. The CDS index and credit index options

In this section we will discuss the CDS index and options on this index. First, Subsection 2.1 gives a brief introduction to how a CDS index works. Then, in Subsection 2.2 we outline model independent expression for the CDS index spread. Finally, Subsection 2.3 introduces options on the CDS index, sometimes denoted by credit index options, and uses the result form Subsection 2.2 to provide a formula for the payoff such an option which holds for any framework modelling the dynamics of the default times in the underlying credit portfolio.

2.1. **Structure of a CDS index.** Consider a portfolio consisting of m equally weighted obligors. An index Credit Default Swap (often denoted *CDS index* or *index CDS*) for a portfolio of m obligors, entered at time t with maturity T, is a financial contract between a protection buyer **A** and protection seller **B** with the following structure. The CDS index gives **A** protection against all credit losses among the m obligors in the portfolio up to time T where $t < T$. Typically, $T = t + \bar{T}$ for $\bar{T} = 3, 5, 7, 10$ years. More specific, at each default in the portfolio during the period $[t, T]$, **B** pays **A** the credit suffered loss due to the default. Thus, the accumulated value payed by **B** to **A** in the period $[t, T]$ is the total credit loss in the portfolio during the period from t to time T. As a compensation for this **A** pays **B** a fixed fee $S(t, T)$ multiplied what is left in the portfolio at each payment time which are done quarterly in the period $[t, T]$. The fee $S(t, T)$ is set so expected discounted cash-flows between **A** and **B** is equal at time t and $S(t, T)$ is called the *CDS index spread* with maturity $T - t$. For $t = 0$ (i.e. "today") so that $T = \bar{T}$ we sometimes denote $S(0, T)$ by $S(T)$ and the quantity $S(T)$ can be observed on a daily basis for standard CDS indexes such as iTraxx Europe and the CDX.NA.IG index, for maturities $T = 3, 5, 7, 10$ years. The quarterly payments from **B** to **A** are done on the IMM dates 20th of March, 20th of June, 20th of September and 20th of December. Standardized indices such as iTraxx are updated twice a year on so called "index-rolls" which takes place on the two IMM dates 20th of March and 20th of September. The most recent rolled CDS index is referred to the *"on-the-run-index"*. Indices rolled on previous dates are refereed to as *"off-the-run-indices"*. A $\bar{T}$-year on-the-run index issued on 20th of March a given year will mature on 20th of June $\bar{T}$ years later. Similarly, a $\bar{T}$-year on-the-run index issued on 20th of September a given year will mature on 20th of December

$\bar{T}$ years later. Thus, the effective protection period will be somewhere between $\bar{T} - 0.25$ and $\bar{T} - 0.25$ years. For example, a 5-year on-the-run CDS index entered on 20th of March will have a maturity of 5.25 years but if it is entered on the 16th of September the same year it will have a maturity of around 4.75 years. As we will see later, these maturity details will play an important role when pricing options on CDS indices. For more on practical details regarding the CDS index, see e.g Markit (2016) or O'Kane (2008).

In order to give a more explicit description of the CDS index spread $S(t,T)$ we need to introduce some further notations and concepts which is done in the next subsection.

2.2. **The CDS index spread.** In this subsection we give a quantitative description of the CDS index spread. First we need to introduce some notation. Let $(\Omega, \mathcal{G}, \mathbb{Q})$ be the underlying probability space assumed in the rest of this paper. We set $\mathbb{Q}$ to be a risk neutral probability measure which exist (under rather mild condition) if arbitrage possibilities are ruled out. Furthermore, let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration representing the full market information at each time point t. Consider a portfolio consisting of m equally weighted obligors with default times $\tau_1, \tau_2 \dots, \tau_m$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and let $\ell_1, \ell_2, \dots, \ell_m$ be the corresponding individual credit losses at each default time. Typically $\ell_i = (1-\phi_i)/m$ where ϕ_i is a constant representing the recovery rate for obligor i. The credit loss for this portfolio at time t is then defined as $\sum_{i=1}^m \ell_i 1_{\{\tau_i \leq t\}}$. Similarly, the number of defaults in the portfolio up to time t, denoted by N_t, is $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. Note that if the individual loss is constant and identical for all obligors so that $\ell = \ell_1 = \ell_2 = \dots = \ell_m$ then the normalized credit loss L_t is given by $L_t = \frac{\ell}{m} N_t$. In the rest of this paper we will assume that the individual loss is constant and identical for all obligors where $1 - \phi = \ell = \ell_1 = \ell_2 = \dots = \ell_m$ and we therefore have that

$$L_t = \frac{1-\phi}{m} N_t \quad \text{where} \quad N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}. \tag{2.2.1}$$

Finally, for $t < u$ we let $B(t,u)$ denote the discount factor between t and u, that is $B(t,u) = \frac{B_t}{B_u}$ where B_t is the risk free savings account. Unless explicitly stated, we will assume that the risk free interest rate is constant and given by r so that $B_t = e^{rt}$ and $B(t,u) = e^{-r(u-t)}$.

Let $T > t$ and consider an CDS index entered at time t with maturity T on the portfolio with loss process L_t. In view of the above notation we can now define the (stochastic) discounted payments $V_D(t,T)$ from **A** to **B** during the period $[t,T]$, and $V_P(t,T)$ from **B** to **A** in the timespan $[t,T]$, as follows

$$V_D(t,T) = \int_t^T B(t,s)dL_s \quad \text{and} \quad V_P(t,T) = \frac{1}{4}\sum_{n=n_t}^{\lceil 4T \rceil} B(t,t_n)\left(1 - \frac{N_{t_n}}{m}\right) \tag{2.2.2}$$

where n_t denotes $n_t = \lceil 4t \rceil + 1$ and $t_n = \frac{n}{4}$. Recall that it typically holds $T = t + \bar{T}$ for $\bar{T} = 3, 5, 7, 10$ years. We here emphasize that we have dropped the accrued term in $V_P(t,T)$ and also ignored the accrued premium up to the first payment date in $V_P(t,T)$. The expected value of the default and premium legs, conditional on the market information $\mathcal{F}_t$ are given by

$$DL(t,T) = \mathbb{E}\left[V_D(t,T) \,|\, \mathcal{F}_t\right] \quad \text{and} \quad PV(t,T) = \mathbb{E}\left[V_P(t,T) \,|\, \mathcal{F}_t\right] \tag{2.2.3}$$

that is

$$DL(t,T) = \mathbb{E}\left[\int_t^T B(t,s)dL_s \,\middle|\, \mathcal{F}_t\right] \tag{2.2.4}$$

and

$$PV(t,T) = \frac{1}{4}\sum_{n=n_t}^{\lceil 4T \rceil} B(t,t_n)\left(1 - \frac{1}{m}\mathbb{E}\left[N_{t_n} \mid \mathcal{F}_t\right]\right). \tag{2.2.5}$$

In view of structure of a CDS index described in Subsection 2.1, the CDS index spread $S(t,T)$ at time t with maturity T is defined as

$$S(t,T) = \frac{DL(t,T)}{PV(t,T)} \tag{2.2.6}$$

or more explicit, using (2.2.4) and (2.2.5)

$$S(t,T) = \frac{\mathbb{E}\left[\int_t^T B(t,s)dL_s \,\middle|\, \mathcal{F}_t\right]}{\frac{1}{4}\sum_{n=n_t}^{\lceil 4T \rceil} B(t,t_n)\left(1 - \frac{1}{m}\mathbb{E}\left[N_{t_n} \mid \mathcal{F}_t\right]\right)}. \tag{2.2.7}$$

The definition of $S(t,T)$ in (2.2.6) is done assuming that not all obligors have defaulted in the portfolio at time t, that is $S(t,T)$ is defined on the event $\{N_t < m\}$. In the event of a so-called armageddon scenario at time t where $N_t = m$ (i.e. all obligors in the portfolio have defaulted up to time t), we see that the premium leg $V_P(t,T)$ in (2.2.2) is zero at time t, which obviously makes the definition of the spread $S(t,T)$ invalid. Note that for $t = 0$ (i.e. today) the quantity $S(0,T)$ can be observed on a daily basis for standard CDS indexes such as iTraxx Europe and the CDX.NA.IG index, for maturities $T = 3, 5, 7, 10$ years.

We here remark that the outline for the CDS index spread presented in this subsection holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. Consequently, the filtration $\mathcal{F}_t$ used in this subsection can be generated by any credit portfolio model.

2.3. **The CDS index option.** In this subsection we introduce options on the CDS index and discuss how they work. Then we use the result form Subsection 2 in order to provide a formula for the payoff of such an option, which holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. First, let us give the definition of a *payer CDS index option*, which is the same as Definition 2.3 in Morini & Brigo (2011) and Definition 2.4 in Rutkowski & Armstrong (2009).

Definition 2.1. A ***payer CDS index option*** (sometimes called a ***put CDS index option***) with strike κ and exercise date t written on a CDS index with maturity T is a financial derivative which gives the protection buyer **A** the right but not the obligation to enter the CDS index with the protection seller **B** at time t with a fixed spread κ and protection period $T - t$. Moreover, at the exercise date t, the protection seller **B** also pays **A** the accumulated credit loss occurred during the period from the inception time of the option (at time 0, i.e. "today") to the exercise date t, that is **B** pays **A** the loss L_t at time t, which is referred to as the front end protection.

The payoff $\Pi(t,T;\kappa)$ at the exercise time t for a payer CDS index option seen from the protection buyer **A**'s point of view, is given by

$$\Pi(t,T;\kappa) = \left(PV(t,T)\left(S(t,T) - \kappa\right)1_{\{N_t<m\}} + L_t\right)^+ \tag{2.3.1}$$

where $PV(t,T)$ is defined as in (2.2.5). For an analogues expression of (2.3.1), see e.g. Equation (2.18) on p.1045 in Rutkowski & Armstrong (2009) or Equation (2.3) on p.577 in Morini & Brigo (2011). Note that the CDS index at time t is entered only if there are any nondefaulted obligors left in the portfolio at time t, which explains the presence of the indicator function of the event $\{N_t < m\}$ in the expression for the payoff $\Pi(t, T; \kappa)$ in (2.3.1). However, the front end protection L_t will be paid out by $\mathbf{A}$ at time t even if the event $\{N_t = m\}$ occurs. From (2.2.6) we have that

$$PV(t,T)\left(S(t,T)-\kappa\right)1_{\{N_t<m\}} = DL(t,T)1_{\{N_t<m\}} - \kappa PV(t,T)1_{\{N_t<m\}}. \tag{2.3.2}$$

However, since N_t is a non-decreasing process where $N_t \le m$ almost surely for all $t \ge 0$, we have from the definitions in (2.2.4) and (2.2.5) that

$$DL(t,T)1_{\{N_t=m\}} = \mathbb{E}\left[\int_t^T B(t,s)dL_s \,\middle|\, \mathcal{F}_t\right]1_{\{N_t=m\}} = 0 \quad \text{and} \quad PV(t,T)1_{\{N_t=m\}} = 0 \tag{2.3.3}$$

so we can use (2.3.3) to simplify (2.3.2) according to

$$PV(t,T)\left(S(t,T)-\kappa\right)1_{\{N_t<m\}} = DL(t,T) - \kappa PV(t,T). \tag{2.3.4}$$

We here remark that the observations (2.3.3) and (2.3.4) has also been done in Rutkowski & Armstrong (2009) and Morini & Brigo (2011), see e.g Equation (2.6) on p. 1040 in Rutkowski & Armstrong (2009) and Proposition 3.7 on p. 582 in Morini & Brigo (2011). By using (2.3.4) we can rewrite the payoff $\Pi(t, T; \kappa)$ in (2.3.1) as

$$\Pi(t,T;\kappa) = \left(DL(t,T) - \kappa PV(t,T) + L_t\right)^+. \tag{2.3.5}$$

The model outline for payer CDS index option presented in this subsection holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. Consequently, the filtration $\mathcal{F}_t$ used in this subsection can be generated by any credit portfolio model.

Before ending this section we briefly discuss some properties of CDS index options that are not shared with e.g. standard equity options. First, we note that (2.3.1) or (2.3.5) implies that

$$\lim_{\kappa\to\infty} \Pi(t,T;\kappa)1_{\{N_t<m\}} = 0. \tag{2.3.6}$$

Secondly, since the individual loss $1-\phi$ is constant and identical for all obligors and since $L_t = \frac{(1-\phi)N_t}{m}$, we have $L_t 1_{\{N_t=m\}} = (1-\phi)1_{\{N_t=m\}}$ which in (2.3.5) together with (2.3.3) implies that

$$\Pi(t,T;\kappa)1_{\{N_t=m\}} = L_t 1_{\{N_t=m\}} = (1-\phi)1_{\{N_t=m\}} \text{ for all } \kappa \tag{2.3.7}$$

(see also Equation (2.24) on p.1047 in Rutkowski & Armstrong (2009)) and consequently

$$\lim_{\kappa\to\infty} \Pi(t,T;\kappa)1_{\{N_t=m\}} = L_t 1_{\{N_t=m\}} = (1-\phi)1_{\{N_t=m\}}. \tag{2.3.8}$$

So combining (2.3.6) and (2.3.8) renders

$$\lim_{\kappa\to\infty} \Pi(t,T;\kappa) = (1-\phi)1_{\{N_t=m\}} \quad \text{a.s.} \tag{2.3.9}$$

For $s \le t$, the price $C_s(t, T; \kappa)$ of a payer CDS index option at time s with strike κ and exercise date t written on a CDS index with maturity T, is due to standard risk neutral pricing theory given by

$$C_s(t,T;\kappa) = e^{-r(t-s)}\mathbb{E}\left[\Pi(t,T;\kappa)\,|\,\mathcal{F}_s\right]. \tag{2.3.10}$$

Furthermore, since

$$\Pi(t,T;\kappa) = \Pi(t,T;\kappa)1_{\{N_t<m\}} + \Pi(t,T;\kappa)1_{\{N_t=m\}} = \Pi(t,T;\kappa)1_{\{N_t<m\}} + (1-\phi)1_{\{N_t=m\}}$$

then for $s \leq t$, the price $C_s(t,T;\kappa)$ can be expressed as

$$C_s(t,T;\kappa) = e^{-r(t-s)}\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t<m\}} \,\middle|\, \mathcal{F}_s\right] + (1-\phi)e^{-r(t-s)}\mathbb{Q}\left[N_t = m \,|\, \mathcal{F}_s\right]. \quad (2.3.11)$$

From (2.3.6) and (2.3.8) together with the dominated convergence theorem, we conclude that if $s \leq t$ then

$$\lim_{\kappa\to\infty} C_s(t,T;\kappa) = (1-\phi)e^{-r(t-s)}\mathbb{Q}\left[N_t = m \,|\, \mathcal{F}_s\right] \quad (2.3.12)$$

which is in line with the results in (2.3.9). Also note that the results in this section holds for any framework modelling the dynamics of the default times in the underlying credit portfolio. In this paper our numerical examples will be performed for $s = 0$ which in (2.3.12) implies that

$$\lim_{\kappa\to\infty} C_0(t,T;\kappa) = (1-\phi)e^{-rt}\mathbb{Q}\left[N_t = m\right] \quad (2.3.13)$$

Recall that in the standard Black-Scholes model the call option price converges to zero as the strike price converges to infinity but due to the front end protection this will not hold for payer CDS index option, as is clearly seen in Equation (2.3.11), (2.3.12) and (2.3.13).

2.4. **Some previous models for the CDS index option.** In this subsection we will discuss some previously studied models and one of these models will be used as a benchmark to the framework developed in this paper.

Options on a CDS index have been studied in for example Pedersen (2003), Jackson (2005), Liu & Jäckel (2005), Doctor & Goulden (2007), Rutkowski & Armstrong (2009), Morini & Brigo (2011), Flesaker et al. (2011) and Martin (2012). In all of these papers it is assumed that either the CDS index spread or the so called loss-adjusted CDS index spread at the maturity of the option is lognormally distributed under a martingale measure corresponding to a suitable numeraire, and the price of the option is then computed via the Black formula. For a nice and compact overview of some of the above mentioned papers, see pp.577-579 in Morini & Brigo (2011).

We will here give a very brief review of the results in some of these papers since these will introduce formulas that we will use as a comparison when benchmarking with our model presented in Section 5.

As discussed in Morini & Brigo (2011), in the initial market approach for pricing CDS index options, the price $C_s^{IM}(t,T;\kappa)$ at time $s \leq t$ of a payer CDS index option with strike κ and exercise date t written on a CDS index with maturity T, is modelled as (see also e.g. Equation (2.4) in Morini & Brigo (2011)))

$$C_s^{IM}(t,T;\kappa) = e^{-r(t-s)}\mathbb{E}\left[V_P(t,T) \,|\, \mathcal{F}_s\right] C^B\left(S(s,T),\kappa,t,\sigma\right) + e^{-r(t-s)}\mathbb{E}\left[L_t \,|\, \mathcal{F}_s\right] \quad (2.4.1)$$

where we have used the same notation as in Subsection 2.3 and where $C^B\left(S,K,T,\sigma\right)$ is the Black-formula, i.e.

$$\begin{aligned} C^B\left(S,K,T,\sigma\right) &= SN(d_1) - KN(d_2) \\ d_1 &= \frac{\ln(S/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} \end{aligned} \quad (2.4.2)$$

and $N(x)$ is the distribution function for a standard normal random variable. As pointed out by Pedersen (2003), and also emphasized in Morini & Brigo (2011), the formula (2.4.1) does not incorporate the front end protection in a correct way given the payoff expression in Equation (2.3.1). To overcome the problem of a wrong inclusions of the front end protection in the option formula, several papers proposed an improvement of the Black-framework, see for example Doctor & Goulden (2007). The idea is to introduce a so called loss-adjusted market index spread defined, see e.g. Equation (2.6) in Morini & Brigo (2011)). More specific, let t be the exercise date for a CDS index option and for $u < t < T$ let $DL_t(u,T)$ and $PV_t(u,T)$ denote

$$DL_t(u,T) = \mathbb{E}\left[B(u,t)V_D(t,T) \,|\, \mathcal{F}_u\right] \;\text{ and }\; PV_t(u,T) = \mathbb{E}\left[B(u,t)V_P(t,T) \,|\, \mathcal{F}_u\right] \qquad (2.4.3)$$

where $V_D(t,T)$ and $V_P(t,T)$ are given by (2.2.2). Next, define loss-adjusted market index spread $\tilde{S}_t(u,T)$ for $u \leq t \leq T$ as

$$\tilde{S}_t(u,T) = \frac{DL_t(u,T) + \mathbb{E}\left[B(u,t)L_t \,|\, \mathcal{F}_u\right]}{PV_t(u,T)}. \qquad (2.4.4)$$

Note that if $u = t$ then $B(t,t) = 1$, $PV_t(t,T) = PV(t,T)$ due to (2.2.3) and since L_t is $\mathcal{F}_t$-measurable $\tilde{S}_t(t,T)$ in (2.4.4) then reduces to

$$\tilde{S}_t(t,T) = \frac{DL(t,T) + L_t}{PV(t,T)} = S(t,T) + \frac{L_t}{PV(t,T)} \qquad (2.4.5)$$

where $S(t,T)$ is defined as in (2.2.6). Also, if $t = 0$ then $L_0 = 0$ a.s. so (2.4.5) then gives

$$\tilde{S}_0(0,T) = S(0,T) \qquad (2.4.6)$$

which makes perfect sense. The benefit with using the loss-adjusted market index spread $\tilde{S}_t(u,T)$ in (2.4.4) is that payoff $\Pi(t,T;\kappa)$ at the exercise time $t > 0$ for a payer CDS index option as given in (2.3.5) can via (2.4.5) be rewritten as

$$\Pi(t,T;\kappa) = PV(t,T)\left(\tilde{S}_t(t,T) - \kappa\right)^+. \qquad (2.4.7)$$

Hence, by using $PV_t(u,T)$ as a numeraire for $u \leq t \leq T$ and assuming that $\tilde{S}_t(u,T)$ is lognormally distributed under a martingale measure corresponding to the chosen numeraire, one can for $s \leq t$, price a payer CDS index option with exercise time t via (2.4.7) and the Black formula according to

$$\widetilde{C}_s(t,T;\kappa) = e^{-r(t-s)}\mathbb{E}\left[V_P(t,T) \,|\, \mathcal{F}_s\right] C^B\left(\tilde{S}_t(s,T), \kappa, t, \tilde{\sigma}\right) \qquad (2.4.8)$$

where we assumed a constant interest rate r. Furthermore, $\tilde{\sigma}$ is the constant volatility of the loss-adjusted market index spread $\tilde{S}_t(u,T)$ and the quantity $C^B(S,K,T,\sigma)$ is the same as in (2.4.2), see also e.g. Equation (2.8) on p.578 in Morini & Brigo (2011).

Remark 2.2. As pointed out on pp.578-579 in Morini & Brigo (2011), there are three main problems with the formula (2.4.8) and the definition of the loss-adjusted market index spread in (2.4.4). The first problem is that loss-adjusted market index spread $\tilde{S}_t(u,T)$ in (2.4.4) is not defined when $PV_t(u,T) = 0$, i.e. when $N_u = m$. The second problem is that when $PV_t(u,T) = 0$, the formula (2.4.8) is undefined and will not be consistent with the expression in (2.3.12) which must holds for any framework modelling the dynamics of the default times in the underlying credit portfolio for the CDS index. The third problem with (2.4.4) is that since $PV_t(u,T) = 0$ on $\{N_u = m\}$ and if $\mathbb{Q}\left[N_u = m\right] > 0$ (which is true for most standard portfolio credit models when $u > 0$), then $PV_t(u,T)$ will not be strictly positive a.s. and will therefore as a numeraire not lead to a pricing measure that is equivalent with the risk-neutral pricing measure $\mathbb{Q}$.

Rutkowski & Armstrong (2009) and Morini & Brigo (2011) have independently developed an approach which overcomes the three problems stated in Remark 2.2 connected to the the loss-adjusted market index spread in (2.4.4) and the pricing formula (2.4.8). The main ideas in Rutkowski & Armstrong (2009) and Morini & Brigo (2011) work as follows (following mainly the notation of Morini & Brigo (2011)). Let $\tau^{(1)} \leq \tau^{(2)} \leq \ldots \leq \tau^{(m)}$ be the ordering of the default times $\tau_1, \tau_2 \ldots, \tau_m$ in the underlying credit portfolio that creates the CDS index. For example, $\tau^{(m)}$ is the maximum of $\{\tau_i\}$, that is

$$\hat{\tau} := \tau^{(m)} = \max\left(\tau_1, \tau_2 \ldots, \tau_m\right) \tag{2.4.9}$$

where we for notational convenience denote $\tau^{(m)}$ by $\hat{\tau}$. So with N_t defined as in previous sections, i.e. $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ we immediately see that

$$\{\hat{\tau} > t\} = \{N_t < m\} \quad \text{and} \quad \{\hat{\tau} \leq t\} = \{N_t = m\}. \tag{2.4.10}$$

Next, both Rutkowski & Armstrong (2009) and Morini & Brigo (2011) assumes the existence of an auxiliary filtration $\hat{\mathcal{H}}_t$ such that underlying full market information $\mathcal{F}_t$ can be decomposed as

$$\mathcal{F}_t = \hat{\mathcal{J}}_t \vee \hat{\mathcal{H}}_t \tag{2.4.11}$$

$$\hat{\mathcal{J}}_t = \sigma\left(\hat{\tau} \leq s; s \leq t\right) \tag{2.4.12}$$

where $\hat{\tau}$ is not a $\hat{\mathcal{H}}_t$-stopping time. Rutkowski & Armstrong (2009) and Morini & Brigo (2011) remarks that one possible construction of (2.4.11)-(2.4.12) is to let $\hat{\mathcal{H}}_t$ be given by

$$\hat{\mathcal{H}}_t = \mathcal{G}_t \vee_{k=1}^{m-1} \mathcal{J}_t^{(k)} \tag{2.4.13}$$

where for each k the filtration $\mathcal{J}_t^{(k)}$ is defined as

$$\mathcal{J}_t^{(k)} = \sigma\left(\tau^{(k)} \leq s; s \leq t\right) \tag{2.4.14}$$

and $\mathcal{G}_t$ in (2.4.13) is a filtration excluding default information, i.e $\mathcal{G}_t$ is the "default free" information. Typically $\mathcal{G}_t$ is a sigma-algebra generated by a d-dimensional stochastic process $(X_t)_{t\geq 0}$ so $\mathcal{G}_t^X = \sigma(X_s; s \leq t)$ where $X_t = (X_{t,1}, X_{t,2}, \ldots, X_{t,d})$ do not contain the random variables $\tau_1, \tau_2 \ldots, \tau_m$ in their dynamics. Such constructions are standard in conditional independent dynamic portfolio credit models, see e.g in Lando (2004) or McNeil, Frey & Embrechts (2005). From the construction in (2.4.11)-(2.4.13) it is clear that $\hat{\tau}$ is not a $\hat{\mathcal{H}}_t$-stopping time. In Remark 3.5 on p.580 in Morini & Brigo (2011) the authors point out that the construction in (2.4.11)-(2.4.12) may under certain, not unreasonable model assumptions, not be possible to construct. Now, for $u < t < T$ let $\widehat{DL}_t(u,T)$ and $\widehat{PV}_t(u,T)$ denote

$$\widehat{DL}_t(u,T) = \mathbb{E}\left[B(u,t)V_D(t,T) \,\middle|\, \hat{\mathcal{H}}_u\right] \quad \text{and} \quad \widehat{PV}_t(u,T) = \mathbb{E}\left[B(u,t)V_P(t,T) \,\middle|\, \hat{\mathcal{H}}_u\right] \tag{2.4.15}$$

where $V_D(t,T)$ and $V_P(t,T)$ are given by (2.2.2). Next, define $\hat{S}_t(u,T)$ as (see Definition 3.8 on p.583 in Morini & Brigo (2011) or in Rutkowski & Armstrong (2009))

$$\hat{S}_t(u,T) = \frac{\widehat{DL}_t(u,T) + \mathbb{E}\left[1_{\{\hat{\tau}>t\}}B(u,t)L_t \,\middle|\, \hat{\mathcal{H}}_u\right]}{\widehat{PV}_t(u,T)} \tag{2.4.16}$$

where t typically is the exercise date for a CDS index option. Furthermore, Morini & Brigo (2011) assumes that

$$\mathbb{Q}\left[\hat{\tau} > s \,|\, \hat{\mathcal{H}}_s\right] > 0 \quad \text{a.s. for any} \quad s > 0 \tag{2.4.17}$$

and Rutkowski & Armstrong (2009) makes a similar assumption but on a bounded interval for s. The reason for the assumption (2.4.17) is that in the derivations of the formulas for the CDS-index spreads presented in Morini & Brigo (2011) and Rutkowski & Armstrong (2009) the quantity $\mathbb{Q}\left[\hat{\tau} > s \,|\, \hat{\mathcal{H}}_s\right]$ will emerge in the denominator of several expressions. More specific, the choice (2.4.11)-(2.4.12) together with (2.4.17) will for $s \leq t$ make the quantity $\widehat{PV}_t(u,T) = \mathbb{E}\left[B(u,t)V_P(t,T) \,|\, \hat{\mathcal{H}}_u\right]$ to be strictly positive a.s. (see e.g. p.581 in Morini & Brigo (2011)) and can thus be used as a numeraire, which was observed both in Rutkowski & Armstrong (2009) and Morini & Brigo (2011) independently of each other. Furthermore, Morini & Brigo (2011) and Rutkowski & Armstrong (2009) also shows that under the condition (2.4.17) the spread $\hat{S}_t(u,T)$ in (2.4.16) is well defined which thus solves the first and third problem specified in Remark 2.2. By using assumption (2.4.17) together with the assumption that $\hat{S}(u,T)$ in (2.4.16) follows a lognormal distribution under a measure defined via $\widehat{PV}_t(u,T)$, Morini & Brigo (2011) and Rutkowski & Armstrong (2009) prove that for $s \leq t$ the price for a payer CDS index option at time s with exercise date t via (2.4.7) is given by

$$\begin{aligned}\widehat{C}_s(t,T;\kappa) &= 1_{\{\hat{\tau}>s\}} e^{-r(t-s)} \mathbb{E}\left[V_P(t,T) \,|\, \mathcal{F}_s\right] C^B\left(\hat{S}_t(s,T), \kappa, t, \hat{\sigma}\right) \\ &+ \frac{1_{\{\hat{\tau}>s\}}}{\mathbb{Q}\left[\hat{\tau} > s \,|\, \hat{\mathcal{H}}_s\right]} \mathbb{E}\left[1_{\{s<\hat{\tau}\leq t\}} e^{r(t-s)}(1-\phi) \,\middle|\, \hat{\mathcal{H}}_s\right] + 1_{\{\hat{\tau}\leq s\}}(1-\phi)e^{-r(t-s)}\end{aligned} \tag{2.4.18}$$

where $\hat{\sigma}$ is the volatility of $\hat{S}_t(u,T)$ under a suitable measure (see e.g. Proposition 4.1, Theorem 4.2 and Corollary 4.3 in Morini & Brigo (2011)). The quantity $C^B(S,K,T,\sigma)$ in (2.4.18) is the same as in (2.4.2). We assumed a constant interest rate r while Morini & Brigo (2011) and Rutkowski & Armstrong (2009) allows for a stochastic discount factor in (2.4.18), see e.g. Equation (2.29) in Rutkowski & Armstrong (2009) and Equation (4.1) and (4.4) in Morini & Brigo (2011). We note that if $s > 0$, then the second term in (2.4.18) is nontrivial to compute in practice. However, an important practical case is to compute $\widehat{C}_s(t,T;\kappa)$ when $s = 0$, i.e. $\widehat{C}_0(t,T;\kappa)$ (the numerical examples in Morini & Brigo (2011) are only done for the case $s = 0$ while Rutkowski & Armstrong (2009) do not provide any numerical examples of their formulas). So letting $s = 0$ in (2.4.18) implies that $\widehat{C}_0(t,T;\kappa)$ is given by the following expression

$$\widehat{C}_0(t,T;\kappa) = e^{-rt}\mathbb{E}\left[V_P(t,T)\right] C^B\left(\hat{S}_t(0,T), \kappa, t, \hat{\sigma}\right) + e^{-rt}(1-\phi)\mathbb{Q}\left[N_t = m\right] \tag{2.4.19}$$

where we used that $\{\hat{\tau} \leq t\} = \{N_t = m\}$. So we clearly see that formula (2.4.19) is consistent with (2.3.13) which must holds for any framework modelling the dynamics of the default times in the underlying credit portfolio for the CDS index. Hence, this solves the second problem pointed out in Remark 2.2. Also note that $\hat{S}_t(0,T)$ will via (2.4.16) simplify to

$$
\begin{aligned}
\hat{S}_t(0,T) &= \frac{\widehat{DL}_t(0,T) + \mathbb{E}\left[1_{\{\hat{\tau}>t\}}B(0,t)L_t\right]}{\widehat{PV}_t(0,T)} \\
&= \frac{DL_t(0,T) + \mathbb{E}\left[1_{\{\hat{\tau}>t\}}B(0,t)L_t\right]}{PV_t(0,T)} \\
&= \frac{DL_t(0,T) + \mathbb{E}\left[B(0,t)L_t\right] - \mathbb{E}\left[1_{\{\hat{\tau}\leq t\}}B(0,t)L_t\right]}{PV_t(0,T)} \\
&= \frac{DL_t(0,T) + \mathbb{E}\left[B(0,t)L_t\right]}{PV_t(0,T)} - \frac{\mathbb{E}\left[1_{\{\hat{\tau}\leq t\}}B(0,t)L_t\right]}{PV_t(0,T)} \\
&= \tilde{S}_t(0,T) - \frac{(1-\phi)\mathbb{E}\left[B(0,t)1_{\{N_t=m\}}\right]}{PV_t(0,T)}
\end{aligned} \tag{2.4.20}
$$

where the second equality follows from (2.4.3) and (2.4.15) with $u = 0$ and last equality is due to the definition of $\tilde{S}_t(u,T)$ in (2.4.4) and the fact that $1_{\{\hat{\tau}\leq t\}}L_t = (1-\phi)1_{\{N_t=m\}}$. Also note that if $t = 0$ then $1_{\{N_0=m\}} = 0$ a.s. which together with (2.4.5) gives

$$
\hat{S}_t(0,T) = \tilde{S}_0(0,T) = S(0,T) \tag{2.4.21}
$$

which makes perfect sense. Furthermore, if we assume that the interest rate is deterministic we can rewrite (2.4.20) as

$$
\hat{S}_t(0,T) = \tilde{S}_t(0,T) - \frac{(1-\phi)\mathbb{Q}\left[N_t = m\right]}{\mathbb{E}\left[V_P(t,T)\right]} \tag{2.4.22}
$$

where $V_P(t,T)$ is defined in (2.2.2).

There are several numerical issues to be considered in (2.4.19). First, as pointed out on p.1051 in Rutkowski & Armstrong (2009), since the loss adjusted spread $\hat{S}_t(u,T)$ is not directly observable on the market at any time point $u \geq 0$, it is quite challenging to estimate the volatility $\hat{\sigma}$ of $\hat{S}_t(u,T)$ where $\hat{\sigma}$ is used in the Black-formula present in (2.4.19). Secondly, computing the quantity $\mathbb{Q}\left[N_t = m\right]$ for large m (for example, $m = 125$ both in the iTraxx Europe and CDX NAG index) is numerically nontrivial and requires special attention even in simple standard portfolio credit models such as the one-factor Gaussian copula model. Note that if the interest rate is deterministic, then $\mathbb{Q}\left[N_t = m\right]$ emerges both in the second term of (2.4.19) aswell as in $\hat{S}_t(0,T)$ used in the Black-formula given by (2.4.19), as seen in (2.4.22). While Rutkowski & Armstrong (2009) do not provide any numerical examples, Morini & Brigo (2011) use a one-factor Gaussian copula model but do not specify which numerical method they use to compute $\mathbb{Q}\left[N_t = m\right]$. In conditional independent models such as copula models, there exists many methods for computing $\mathbb{Q}\left[N_t = k\right]$, $0 \leq k \leq m$, see for example in Gregory & Laurent (2003) and Gregory & Laurent (2005).

In order to numerically benchmark the CDS index model presented in Section 3-5 against Morini & Brigo (2011), we will also implement the model in Morini & Brigo (2011) using a one-factor Gaussian copula model just as Morini & Brigo (2011) do. Our choice of numerical method when computing $\mathbb{Q}\left[N_t = m\right]$ in (2.4.19) and (2.4.22) will be based on the normal approximation of the mixed binomial distribution, similar to the method in Frey, Popp & Weber (2008). To be more specific, for any integer $1 \leq k \leq m$ we use the following approximation for $\mathbb{Q}[N_t \leq k]$ in the one-factor Gaussian copula model

$$\mathbb{Q}[N_t \leq k] \approx \int_{-\infty}^{\infty} N\left(\frac{k+0.5-mp_t(z)}{\sqrt{mp_t(z)(1-p_t(z))}}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \quad \text{for } k \leq m \tag{2.4.23}$$

where $p_t(z)$ is given by

$$p_t(z) = N\left(\frac{N^{-1}(\mathbb{Q}[\tau \leq t]) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \tag{2.4.24}$$

and $N(x)$ is the distribution function for a standard normal random variable, ρ is the correlation parameters and τ has the same distribution as the exchangeable default times $\{\tau_i\}$ in the underlying credit portfolio, see e.g. Corollary 2.5 in Frey et al. (2008). The term 0.5 in (2.4.23) is a so-called "half-correction" which seem to produce better approximations that the ordinary normal approximation of a binomial distribution. Next, since

$$\mathbb{Q}[N_t = m] = \mathbb{Q}[N_t \leq m] - \mathbb{Q}[N_t \leq m-1] \tag{2.4.25}$$

we use (2.4.23) with $k = m-1$ and $k = m$ in the right hand side of (2.4.25) to retrieve an approximation to the quantity $\mathbb{Q}[N_t = m]$ in (2.4.19) and (2.4.22). Next we need to find an expression for $\mathbb{Q}[\tau \leq t]$ used in (2.4.23) via (2.4.24). A standard assumption made in the homogeneous portfolio credit risk one-factor Gaussian copula model is that the default times $\{\tau_i\}$ have constant default intensity λ, that is they are exponentially distributed with parameter λ, i.e. if τ has the same distribution as $\{\tau_i\}$ then

$$\mathbb{Q}[\tau \leq t] = 1 - e^{-\lambda t} \tag{2.4.26}$$

where λ is given by

$$\lambda = \frac{S_{\mathcal{M}}(\bar{T})}{1-\phi} \tag{2.4.27}$$

and $S_{\mathcal{M}}(\bar{T})$ is the market quote for the $\bar{T}$-year CDS-index spread today and ϕ is the recovery rate. The relation (2.4.27) is the so-called credit triangle, frequently used among market practitioners assuming a "flat" CDS term structure, i.e. assuming that the default intensity will be constant for all time points after t.

A derivation of the relation (2.4.27) in the case with quarterly payments is given in Proposition B.1 in Appendix B, since the existing proofs of (2.4.27) found in the litterature are only done in the unrealistic case when the CDS index premium is paid continuously, see e.g pp.70-71 in Brigo, Morini & Pallavicini (2013). In practice the CDS premiums are paid quarterly.

Furthermore, note that we have used the CDS index spread $S_{\mathcal{M}}(\bar{T})$ in (2.4.27) because this spread will in a homogeneous credit portfolio be identical to the the individual CDS spread for an obligor in the reference portfolio, see e.g. Proposition Lemma 6.1 in Herbertsson, Jang & Schmidt (2011). This ends the specification of how we compute $\mathbb{Q}[N_t = m]$. In Figure 1 we plot $\mathbb{Q}[N_t = m]$ for $t = 9$ months and $m = 125$ as function of the correlation parameter ρ where we used (2.4.23)-(2.4.27) to compute $\mathbb{Q}[N_t = m]$ with $\phi = 40\%$ and $S_{\mathcal{M}}(5) = 200$ bps. As can be seen in Figure 1, the effect of ρ on $\mathbb{Q}[N_t = m]$ will only come in to play when ρ is bigger than 85% and for smaller ρ, the armageddon probability $\mathbb{Q}[N_t = m]$ will in practice be neglible, see also Figure 5.1 in Morini & Brigo (2011)

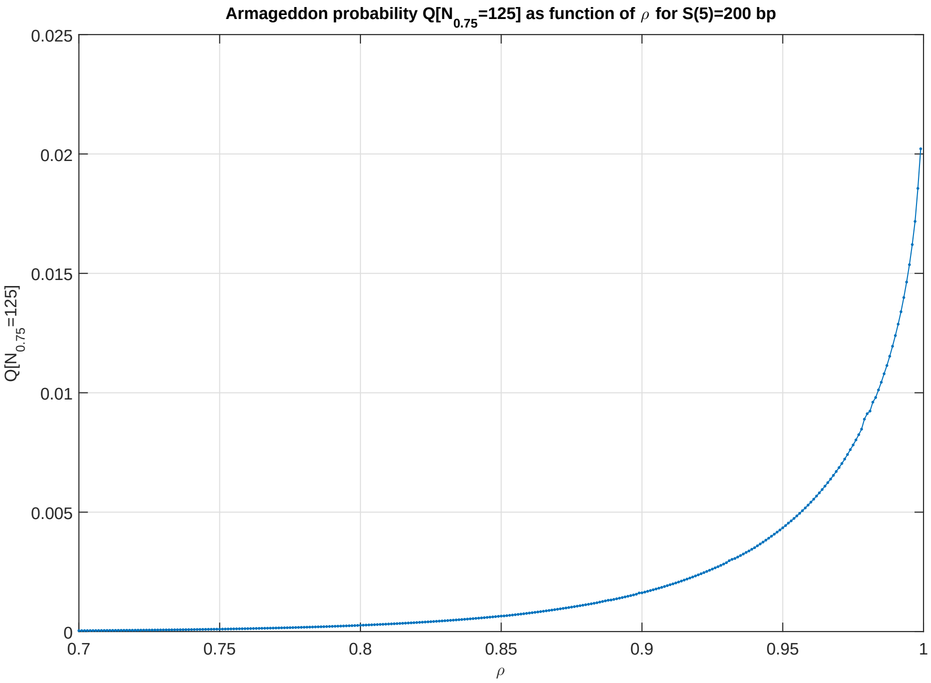

Figure 1. The Armageddon probability $\mathbb{Q}\left[N_{0.75}=125\right]$ as function of the correlation $\rho =$ where $S(0,5)=200$ and $\phi = 40\%$ bp.

So what is left to compute in (2.4.19) is $\hat{S}_t(0,T)$. This is done in the following proposition.

Proposition 2.3. *Consider a CDS index with maturity T on a homogeneous credit portfolio where the obligors have constant default intensity λ. Then, with notation as above*

$$\hat{S}_t(0,T) = 4(1-\phi)e^{-rt}\left(1-e^{-\frac{(r+\lambda)}{4}}\right)\frac{\left(\frac{\lambda}{\lambda+r}e^{rt}\left(e^{-(r+\lambda)t}-e^{-(r+\lambda)T}\right)+1-e^{-\lambda t}-\mathbb{Q}\left[N_t=m\right]\right)}{e^{-\frac{(r+\lambda)n_t}{4}}-e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}}} \tag{2.4.28}$$

where $n_t = \lceil 4t \rceil + 1$.

Proof. From (2.4.22) we have

$$\hat{S}_t(0,T) = \tilde{S}_t(0,T) - \frac{(1-\phi)\mathbb{Q}\left[N_t=m\right]}{\mathbb{E}\left[V_P(t,T)\right]} \tag{2.4.29}$$

so we need explicit expressions for the quantities $\mathbb{E}\left[V_P(t,T)\right]$ and $\tilde{S}_t(0,T)$. First, to find $\mathbb{E}\left[V_P(t,T)\right]$ we use the exchangeability of the default times $\{\tau_i\}$ all having the same distribution as in (2.4.26), which in the definition of $V_P(t,T)$ given by (2.2.2) with properties for geometric series and some computations yields

$$\mathbb{E}\left[V_P(t,T)\right] = \frac{e^{rt}\left(e^{-\frac{(r+\lambda)n_t}{4}}-e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}}\right)}{4\left(1-e^{-\frac{(r+\lambda)}{4}}\right)} \tag{2.4.30}$$

where n_t denotes $n_t = \lceil 4t \rceil + 1$ as in (2.2.2). Next, we provide an explicit expression for $\tilde{S}_t(0,T)$ given by (2.4.4) with $u = 0$ and constant interest rate r, that is

$$
\begin{aligned}
\tilde{S}_t(0,T) &= \frac{DL_t(0,T) + e^{-rt}\mathbb{E}\left[L_t\right]}{PV_t(0,T)} \\
&= \frac{DL_t(0,T) + e^{-rt}(1-\phi)\mathbb{Q}\left[\tau \leq t\right]}{PV_t(0,T)} \\
&= \frac{\mathbb{E}\left[V_D(t,T)\right]}{\mathbb{E}\left[V_P(t,T)\right]} + \frac{(1-\phi)\mathbb{Q}\left[\tau \leq t\right]}{\mathbb{E}\left[V_P(t,T)\right]} \\
&= \frac{e^{rt}\mathbb{E}\left[\int_t^T e^{-rs} dL_s\right]}{\mathbb{E}\left[V_P(t,T)\right]} + \frac{(1-\phi)\mathbb{Q}\left[\tau \leq t\right]}{\mathbb{E}\left[V_P(t,T)\right]} \\
&= \frac{(1-\phi)e^{rt}\int_t^T e^{-rs} f_\tau(s) ds}{\mathbb{E}\left[V_P(t,T)\right]} + \frac{(1-\phi)\mathbb{Q}\left[\tau \leq t\right]}{\mathbb{E}\left[V_P(t,T)\right]}
\end{aligned}
\tag{2.4.31}
$$

where the second equality follows the definition of the loss L_t in (2.2.1) together with the exchangeability of the default times $\{\tau_i\}$ all having the same distribution as τ and the third equality comes from the definition of $DL_t(u,T)$ and $PV_t(u,T)$ in (2.4.3) with $u = 0$ using that the interest rate is constant, given by r. The fourth equality is due to the expected value of $V_D(t,T)$ in (2.2.3) together with (2.2.4) and that $B(t,s) = e^{r(s-t)}$ since the interest rate is constant. The last equality in (2.4.31) follows from Equation (6.3.3) in Lemma 6.1, p.1203 in Herbertsson et al. (2011) where $f_\tau(s)$ is the density of the default time τ. So plugging (2.4.31) into (2.4.29) we get that $\hat{S}_t(0,T)$ can be rewritten as

$$
\hat{S}_t(0,T) = \frac{1-\phi}{\mathbb{E}\left[V_P(t,T)\right]}\left(e^{rt}\int_t^T e^{-rs} f_\tau(s) ds + \mathbb{Q}\left[\tau \leq t\right] - \mathbb{Q}\left[N_t = m\right]\right). \tag{2.4.32}
$$

Note that (2.4.32) holds for any distribution of τ, and to make $\hat{S}_t(0,T)$ more explicit we use that τ in this paper (as in most articles treating homogeneous one-factor Gaussian copula models applied to portfolio credit risk) has constant default intensity λ, i.e. τ is exponentially distributed with parameter λ as in (2.4.26) which implies

$$
\int_t^T e^{-rs} f_\tau(s) ds = \int_t^T \lambda e^{-(r+\lambda)s} ds = \frac{\lambda}{\lambda + r}\left(e^{-(r+\lambda)t} - e^{-(r+\lambda)T}\right). \tag{2.4.33}
$$

So (2.4.26), (2.4.30) and (2.4.33) in (2.4.32) renders an explicit formula for $\hat{S}_t(0,T)$ given by

$$
\hat{S}_t(0,T) = 4(1-\phi)e^{-rt}\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)\frac{\left(\frac{\lambda}{\lambda+r}e^{rt}\left(e^{-(r+\lambda)t} - e^{-(r+\lambda)T}\right) + 1 - e^{-\lambda t} - \mathbb{Q}\left[N_t = m\right]\right)}{e^{-\frac{(r+\lambda)n_t}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T \rceil + 1)}{4}}}
$$

which concludes the proposition. □

The quantity $\mathbb{Q}\left[N_t = m\right]$ used in $\hat{S}_t(0,T)$ given by (2.4.28) will in this paper be computed via the equations (2.4.23)-(2.4.27) where λ is given by (2.4.27).

In Subsection 6.2 we will use $\widehat{C}_0(t,T;\kappa)$ given by (2.4.19), $\hat{S}_t(0,T)$ in (2.4.28) and the method (2.4.23)-(2.4.27) for computing $\mathbb{Q}\left[N_t = m\right]$, as a benchmark against the model developed in the next sections.

We here remark that Morini & Brigo (2011) do not provide any explicit expression of $\hat{S}_t(0,T)$ given on the form (2.4.28), see e.g. the equation under Table 5.1 on p.589 in Morini

& Brigo (2011). But as will be seen in Subsection 6.2, our numerical values for (2.4.19), roughly coincide with those presented in Table 5.1-5.2 in Morini & Brigo (2011). We have not done any numerical benchmark against Rutkowski & Armstrong (2009) since there are no numerical results presented in Rutkowski & Armstrong (2009).

Furthermore, we will also show that the finite-state Markov chain modell presented in this paper will for the same CDS index spread $S(0,T)$ create CDS index option prices that can be several hundred percent, or even several thousands percent bigger (depending on the value of ρ and t and the strike κ) than those given by (2.4.19) with the same CDS index spread $S(0,T)$, and at the same time it will hold that $\mathbb{Q}\left[N_t = m\right] = 0$ in the finite-state Markov chain model while $\mathbb{Q}\left[N_t = m\right] > 0$ in the one-factor Gaussian copula as used in Morini & Brigo (2011).

3. Credit portfolio models using Markov chains

In this section we shortly recapitulate the model of Graziano & Rogers (2009) and also introduce some notation needed for later on. Then we describe the main building blocks that will be necessary to find formulas for portfolio credit derivatives such as e.g. the CDS index and CDS index options. Examples of such building blocks are the conditional survival distribution, the conditional number of defaults and the conditional loss distribution.

3.1. **The main building blocks.** Let $(\Omega, \mathcal{G}, \mathbb{P})$ be the underlying probability space assumed in the rest of this paper.

Let X_t be a finite state continuous time Markov chain on the state space $S^X = \{1, 2, \dots, K\}$ with generator $\boldsymbol{Q}$. Let $\mathcal{F}_t^X = \sigma(X_s; s \leq t)$ be the filtration generated by the factor process X. Consider m obligors with default times $\tau_1, \tau_2 \dots, \tau_m$ and let the mappings $\lambda_1, \lambda_2 \dots, \lambda_m$ be the corresponding $\mathcal{F}_t^X$ default intensities, where $\lambda_i : S^X \mapsto \mathbb{R}^+$ for each obligor i. This means that each default time τ_i is modeled as the first jump of a Cox-process, with intensity $\lambda_i(X_t)$. It is well known (see e.g. Lando (1998)) that given an i.i.d sequence $\{E_i\}$ where E_i is exponentially distributed with parameter one, such that all $\{E_i\}$ are independent of $\mathcal{F}_\infty^X$, then

$$\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_i(X_s) ds \geq E_i \right\}. \tag{3.1.1}$$

Hence, for any $T \geq t$ we have

$$\mathbb{Q}\left[\tau_i > t \,\middle|\, \mathcal{F}_T^X\right] = \exp\left(-\int_0^t \lambda_i(X_s) ds\right) \tag{3.1.2}$$

and thus

$$\mathbb{Q}\left[\tau_i > t\right] = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_i(X_s) ds\right)\right]. \tag{3.1.3}$$

Note that the default times are conditionally independent, given $\mathcal{F}_\infty^X$.

The states in $S^X = \{1, 2, \dots, K\}$ are ordered so that state 1 represents the best state and K represents the worst state of the economy. Consequently, the mappings $\lambda_i(\cdot)$ are chosen to be strictly increasing in $k \in \{1, 2, \dots, K\}$, that is $\lambda_i(k) < \lambda_i(k+1)$ for all $k \in \{1, 2, \dots, K-1\}$ and for every obligor in the portfolio.

Let $Y_{t,i}$ denote the random variable $Y_{t,i} = 1_{\{\tau_i \leq t\}}$ and Y_t be the vector $Y_t = (Y_{t,1}, \dots, Y_{t,m})$. The filtration $\mathcal{F}_t^Y = \sigma(Y_s; s \leq t)$ represents the default portfolio information at time t, generated by the process $(Y_s)_{s \geq 0}$.

We set the full information $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ to be the biggest filtration containing all other filtrations with $\mathcal{G} = \mathcal{F}_\infty$. We can for example let $\mathcal{F}_t$ be given by

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^Y. \tag{3.1.4}$$

Next, recall from (2.2.1) that N_t and L_t are given by

$$N_t = \sum_{i=1}^{m} Y_{t,i} = \sum_{i=1}^{m} 1_{\{\tau_i \leq t\}} \quad \text{and} \quad L_t = \frac{1-\phi}{m} N_t \tag{3.1.5}$$

where ϕ_i is the recovery rate for obligor i.

Figure 2 and Figure **??** visualizes a simulated path of X_t and N_t in an example where $K = 5$ and $m = 125$ in a homogeneous model where $\lambda_i(X_t) = \lambda(X_t)$, using fictive parameters for $\boldsymbol{Q}$ and $\boldsymbol{\lambda}$. The first and second subfigures in Figure 2 - **??** shows the corresponding trajectories for X_t and N_t. Note how the defaults presented by N_t cluster as X_t switches to higher states, representing the worse economic state among $\{1, 2, \ldots, 5\}$ since $\lambda(k) < \lambda(k+1)$ for all $k \in \{1, 2, \ldots, 4\}$ and for every obligor in the portfolio.

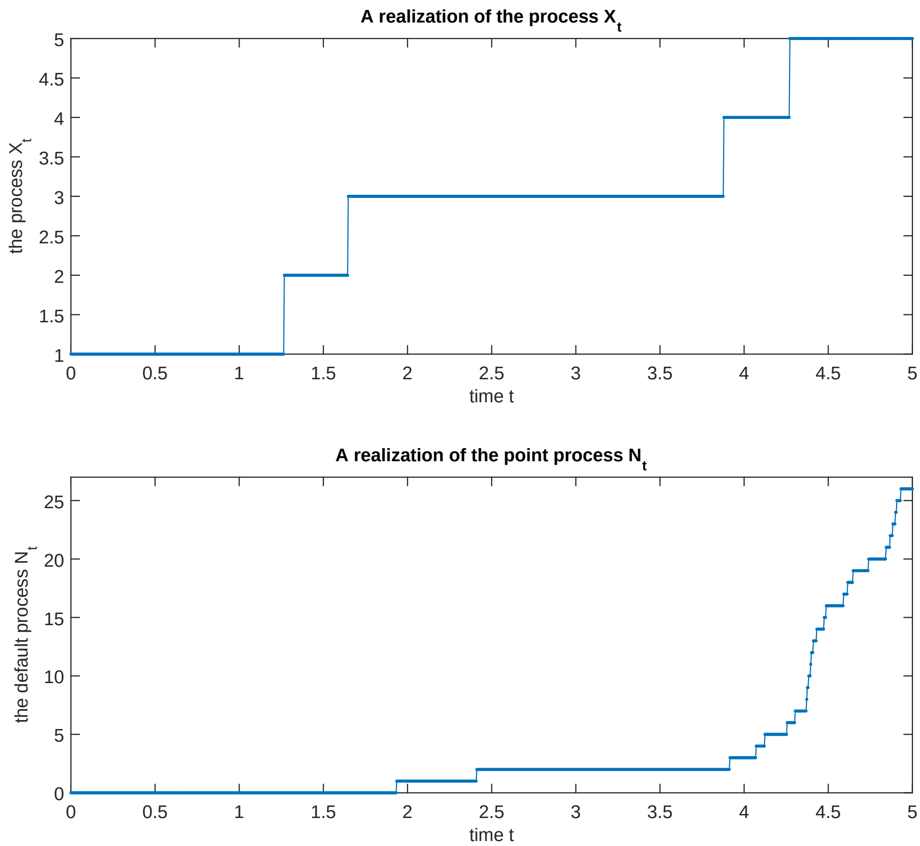

Figure 2. A simulated trajectory of X_t and N_t where $K = 5$ and $m = 125$.

Our main task in the rest of this section is to find the following quantities

$$\mathbb{Q}\left[\tau_i > T \,|\, \mathcal{F}_t\right], \quad \mathbb{E}\left[N_T \,|\, \mathcal{F}_t\right] \quad \text{and} \quad \mathbb{E}\left[L_T \,|\, \mathcal{F}_t\right]$$

where $T > t$. These expressions will be useful when deriving formulas for the CDS index spread $S(t,T)$ as well as the CDS index option discussed in Section 4.

3.2. **The conditional survival distribution.** In this subsection we study the conditional survival distribution $\mathbb{Q}\left[\tau_i > T \,|\, \mathcal{F}_t\right]$ for $T > t$ in the finite state Markov chain model. To do this we need to introduce some notation. If X_t is a finite state Markov jump process on $S^X = \{1, 2, \ldots, K\}$ with generator $\boldsymbol{Q}$, then, for a function $\lambda(x) : S^X \mapsto \mathbb{R}$ we denote the matrix $\boldsymbol{Q}_\lambda = \boldsymbol{Q} - \boldsymbol{I}_\lambda$ where $\boldsymbol{I}_\lambda$ is a diagonal-matrix such that $(\boldsymbol{I}_\lambda)_{k,k} = \lambda(k)$.

Furthermore, let $\boldsymbol{e}_k \in \mathbb{R}^m$ be a row vector where the entry at position k is 1 and the other entries are zero and let $\mathbf{1}$ be a column vector in $\mathbb{R}^K$ where all entries are 1. The following proposition is an important result, which also can be found in modified version in Graziano & Rogers (2009) originally coming from a result on pp.273-274 in Rogers & Williams (2000). We state the result here since it introduces notation needed in the rest of this paper and also uses a slightly different version than Graziano & Rogers (2009).

Proposition 3.1. *Consider a credit portfolio specified as in Section 3 and let $\lambda_i(X_t)$ be the $\mathcal{F}_t^X$-intensity for obligor i. If $T \geq t$ then, with notation as above*

$$\mathbb{Q}\left[\tau_i > T \,|\, \mathcal{F}_t\right] = 1_{\{\tau_i > t\}} \boldsymbol{e}_{X_t} e^{\boldsymbol{Q}_\lambda (T-t)} \mathbf{1} \tag{3.2.1}$$

where $\boldsymbol{e}_{X_t} = \sum_{k=1}^K 1_{\{X_t = k\}} \boldsymbol{e}_k = \left(1_{\{X_t=1\}}, \ldots, 1_{\{X_t=K\}}\right)$ is a row vector in $\mathbb{R}^m$ and where the matrix $\boldsymbol{Q}_{\lambda_i} = \boldsymbol{Q} - \boldsymbol{I}_{\lambda_i}$ is defined as above.

Proof. Since $T > t$, then

$$\mathbb{E}\left[1_{\{\tau_i > T\}} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[1_{\{\tau_i > T\}} \,\middle|\, \mathcal{F}_t^X \vee \mathcal{F}_t^{Y_i}\right] = 1_{\{\tau_i > t\}} \mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s) ds} \,\middle|\, \mathcal{F}_t^X\right] \tag{3.2.2}$$

where the first equality is due to the fact that conditionally on X, then τ_i is independent of τ_j for $j \neq i$. The second equality follows from a standard result for the first jump time of a Cox-process, see e.g. p.102 in Lando (1998), Corollary 9.1 in McNeil et al. (2005) or Corollary 6.4.2 in Bielecki & Rutkowski (2001). Since $T > t$ and due to the Markov property of X we can rewrite the quantity $\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s) ds} \,\middle|\, \mathcal{F}_t^X\right]$ as

$$\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s) ds} \,\middle|\, \mathcal{F}_t^X\right] = \mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s) ds} \,\middle|\, X_t\right] = \sum_{k=1}^K \mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s) ds} \,\middle|\, X_t = k\right] 1_{\{X_t = k\}} \tag{3.2.3}$$

and by using Theorem A.1 in Appendix A we have that

$$\mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s) ds} \,\middle|\, X_t = k\right] = \boldsymbol{e}_k e^{\boldsymbol{Q}_{\lambda_i}(T-t)} \mathbf{1} \tag{3.2.4}$$

where the matrix $\boldsymbol{Q}_{\lambda_i}$ is defined as previously. So (3.2.4) in (3.2.3) and (3.2.2) yields

$$\mathbb{Q}\left[\tau_i > T \,|\, \mathcal{F}_t\right] = 1_{\{\tau_i \geq t\}} \sum_{k=1}^K 1_{\{X_t = k\}} \boldsymbol{e}_k e^{\boldsymbol{Q}_\lambda (T-t)} \mathbf{1} = 1_{\{\tau_i \geq t\}} \boldsymbol{e}_{X_t} e^{\boldsymbol{Q}_\lambda (T-t)} \mathbf{1} \tag{3.2.5}$$

where $\boldsymbol{e}_{X_t} = \sum_{k=1}^K 1_{\{X_t = k\}} \boldsymbol{e}_k = \left(1_{\{X_t=1\}}, \ldots, 1_{\{X_t=K\}}\right)$ is a row vector in $\mathbb{R}^m$. Inserting (3.2.5) into XX proves the theorem.

$\square$

Theorem 3.1 allows us to state credit related derivatives quantizes in very compact and computational convenient formulas, as will seen later in this paper. We also remark that a version Theorem 3.1 for a nonlinear filtering model can also be found in Herbertsson & Frey (2018) and this filtering version has previously also been successfully used in Herbertsson & Frey (2014).

3.3. **The conditional number of defaults.** In this subsection we derive practical expressions for $\mathbb{E}\left[N_t \,\middle|\, \mathcal{F}_t^M\right]$. We consider an homogeneous credit portfolios where $\lambda_i(X_t) = \lambda(X_t)$ so that $\boldsymbol{Q}_{\lambda_i} = \boldsymbol{Q}_\lambda$ for each obligor i. Recall that $N_t = \sum_{i=1}^m 1_{\{\tau_i \le t\}}$. The main message of this subsection is the following proposition.

Proposition 3.2. *Consider an exchangeable credit portfolio with m obligors in a model specified as in Section 3. Then, for $T \ge t$ and with notation as above*

$$\mathbb{E}\left[N_T \,\middle|\, \mathcal{F}_t\right] = m - (m - N_t)\, \boldsymbol{e}_{X_t} e^{\boldsymbol{Q}_\lambda (T-t)} \boldsymbol{1}. \tag{3.3.1}$$

Proof. Let $T > t$ and first note that

$$\mathbb{E}\left[N_T \,\middle|\, \mathcal{F}_t\right] = m - \sum_{i=1}^m \mathbb{E}\left[1_{\{\tau_i > T\}} \,\middle|\, \mathcal{F}_t\right] = m - \sum_{i=1}^m 1_{\{\tau_i > t\}} \mathbb{E}\left[e^{-\int_t^T \lambda_i(X_s) ds} \,\middle|\, \mathcal{F}_t^X\right] \tag{3.3.2}$$

where the last equality is due to Equation (3.2.2) in Theorem 3.1. Furthermore, in a homogeneous portfolio we have $\lambda_i(X_s) = \lambda(X_s)$ for all obligors i and this in (3.3.2) implies that $\mathbb{E}\left[N_T \,\middle|\, \mathcal{F}_t\right] = m - (m - N_t)\, \mathbb{E}\left[e^{-\int_t^T \lambda(X_s) ds} \,\middle|\, \mathcal{F}_t^X\right]$. Thus, by using (3.2.3) and (3.2.4) and the notation $\boldsymbol{e}_{X_t} = \sum_{k=1}^K 1_{\{X_t = k\}} \boldsymbol{e}_k$ in Theorem 3.1 with $\lambda_i(X_s) = \lambda(X_s)$ for all obligors, we conclude that $\mathbb{E}\left[N_T \,\middle|\, \mathcal{F}_t\right] = m - (m - N_t)\, \boldsymbol{e}_{X_t} e^{\boldsymbol{Q}_\lambda (T-t)} \boldsymbol{1}$ which proves the proposition. $\square$

A similar proof can be found for inhomogeneous portfolios.

3.4. **The conditional portfolio loss: The case with constant recovery.** This is trivial for homogeneous portfolios, given the results from Subsection 3.3. To see this, recall that $N_t = \sum_{i=1}^m 1_{\{\tau_i \le t\}}$ and $L_t = \frac{1}{m} \sum_{i=1}^m (1 - \phi_i) 1_{\{\tau_i \le t\}}$ where ϕ_i are constants and in a homogeneous portfolio we have $\phi_1 = \phi_2 = \ldots = \phi_m = \phi$ so that $L_t = \frac{(1-\phi)}{m} N_t$. Thus,

$$\mathbb{E}\left[L_T \,\middle|\, \mathcal{F}_t\right] = \frac{(1 - \phi)}{m} \mathbb{E}\left[N_T \,\middle|\, \mathcal{F}_t\right] \tag{3.4.1}$$

where $\mathbb{E}\left[N_T \,\middle|\, \mathcal{F}_t\right]$ is explicitly given in Subsection 3.3 for homogeneous portfolios. To be more specific, (3.4.1) with Proposition 3.2 yields

$$\mathbb{E}\left[L_T \,\middle|\, \mathcal{F}_t\right] = (1 - \phi)\left(1 - \left(1 - \frac{N_t}{m}\right) \boldsymbol{e}_{X_t} e^{\boldsymbol{Q}_\lambda (T-t)} \boldsymbol{1}\right). \tag{3.4.2}$$

Similar results can also be obtained in an inhomogeneous portfolio both with identical or different recoveries.

3.5. **Auxiliary computational tools.** In this subsection we outline some auxiliary tools that will be utilized when pricing CDS index options in the Markovian model specified in the Subsection 3.1 - 3.4. To be more specific, the pricing of CDS index options in the Markovian model needs the probabilities $\mathbb{Q}\left[X_t = k, N_t = j\right]$ and $\mathbb{Q}\left[N_t = j\right]$ in the previous subsections. We will also use these tools in our numerical studies in later sections of this paper.

Consider a bivariate Markov process H_t on a state space S^H defined as

$$S^H = \{1, \dots, K\} \times \{0, 1, \dots, m\} \tag{3.5.1}$$

where $|S^H| = K(m+1)$. So each state $\boldsymbol{j} \in S^H$ can be written as a pair $\boldsymbol{j} = (k, j)$ where k and j are integers such that $1 \leq k \leq K$ and $0 \leq j \leq m$. The first component of H_t belongs to $\{1, \dots, K\}$ while the second component of H_t is defined on $\{1, \dots, m\}$. The intuitive idea behind the bivariate Markov process H_t is of course that the first component of H_t should "mimic" the factor process X_t defined in Subsection 3.1 while the second component of H_t should represent N_t, i.e. the number of defaulted obligors in the portfolio at time t, as defined in previous sections. More specific, for any pair $(k, j) \in S^H$ and for any time point $t \geq 0$, we want that the events $\{H_t = (k, j)\}$ and $\{X_t = k, N_t = j\}$ should have the same probability under the risk-neutral measure $\mathbb{Q}$, that is

$$\mathbb{Q}\left[H_t = (k, j)\right] = \mathbb{Q}\left[X_t = k, N_t = j\right] \quad \text{where} \quad (k, j) \in S^H \quad \text{and} \quad t \geq 0. \tag{3.5.2}$$

In view of the above description of the bivariate Markov process H_t we now specify the generator $\boldsymbol{Q}_H$ for H_t on S^H. For a fixed value k of the first component of H_t we can treat the second component of H_t as a pure death process on $\{0, 1, \dots, m\}$, i.e. a process which counts the number of defaulted obligors in the portfolio given that the underlying economy is in state k, that is $X_t = k$. Therefore, for any $j = 0, 1, \dots, m-1$ the process H_t can jump from (k, j) to $(k, j+1)$ with intensity $(m-j)\lambda(k)$ where the mapping $\lambda(\cdot)$ is the default intensity same for all obligors, see also in Subsection 3.3. Recall that $\lambda(k)$ is the individual default intensity when the factor process is in state k, i.e. $X_t = k$. Next, for a fixed value j of the second component of H_t (i.e. the number of defaulted obligors at time t are j) consider two distinct states k and k' in $\{1, \dots, K\}$. Then, inspired by the construction of the underlying factor process X_t with generator $\boldsymbol{Q}$, we let the bivariate process H_t jump from (k, j) to (k', j) with intensity $\boldsymbol{Q}_{k,k'}$ where $k \neq k'$. These are the only allowed transitions for H_t. Hence, the generator $\boldsymbol{Q}_H$ for H_t is then given by

$$\begin{aligned} (\boldsymbol{Q}_H)_{(k,j),(k,j+1)} &= (m-j)\lambda(k) \quad && 0 \leq j \leq m-1, \quad 1 \leq k \leq K \\ (\boldsymbol{Q}_H)_{(k,j),(k',j)} &= \boldsymbol{Q}_{k,k'} && 0 \leq j \leq m, \quad 1 \leq k, k' \leq K \quad k \neq k' \end{aligned} \tag{3.5.3}$$

and for each pair k, j we also have that

$$(\boldsymbol{Q}_H)_{(k,j),(k,j)} = -\sum_{(k',j') \in S^H, k' \neq k, j' \neq j} (\boldsymbol{Q}_H)_{(k,j),(k',j')}\,. \tag{3.5.4}$$

where the other entries in $\boldsymbol{Q}_H$ are zero. In view of this construction one can show that, see e.g. Proposition 2.3 in Mandjes & Spreij (2016),

$$\mathbb{Q}\left[H_t = (k, j)\right] = \mathbb{Q}\left[X_t = k, N_t = j\right] \quad \text{where} \quad (k, j) \in S^H \quad \text{and} \quad t \geq 0. \tag{3.5.5}$$

Let $\boldsymbol{\alpha}_H \in \mathbb{R}^{K(m+1)}$ be the initial distribution of the Markov process H_t on the state space S^H with generator $\boldsymbol{Q}_H$ and consider $\boldsymbol{j} \in S^H$. From Markov theory we know that

$$\mathbb{Q}\left[H_t = \boldsymbol{j}\right] = \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t} \boldsymbol{e}_{\boldsymbol{j}}, \tag{3.5.6}$$

where $\boldsymbol{e}_{\boldsymbol{j}} \in \mathbb{R}^{K(m+1)}$ is a column vector where the entry at position $\boldsymbol{j}$ is 1 and the other entries are zero. Furthermore, $e^{\boldsymbol{Q}_H t}$ is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of $\boldsymbol{Q}_H$. Thus, in view of (3.5.5) and (3.5.6) we have for any $\boldsymbol{j} = (k, j) \in S^H$ and $t \geq 0$ that

$$\mathbb{Q}\left[X_t = k, N_t = j\right] = \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t} \boldsymbol{e}_{(k,j)}. \tag{3.5.7}$$

So (3.5.7) provides us with an efficient way to compute the probabilities $\mathbb{Q}[X_t = k, N_t = j]$ for any $t \geq 0$ and any pair $\boldsymbol{j} = (k, j)$ where k and j are integers such that $1 \leq k \leq K$ and $0 \leq j \leq m$. Note that there exist over 20 different ways to compute the matrix exponential, for more on this see e.g in Moeler & Loan (1978) and Moeler & Loan (2003).

Since $\mathbb{Q}[N_t = j] = \sum_{k=1}^{K} \mathbb{Q}[X_t = k, N_t = j]$ we retrieve that

$$\mathbb{Q}[N_t = j] = \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t} \sum_{k=1}^{K} \boldsymbol{e}_{(k,j)} = \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t} \boldsymbol{e}_{(\cdot,j)} \tag{3.5.8}$$

where $\boldsymbol{e}_{(\cdot,j)} \in \mathbb{R}^{K(m+1)}$ is a column vector defined as $\boldsymbol{e}_{(\cdot,j)} = \sum_{k=1}^{K} \boldsymbol{e}_{(k,j)}$. Finally, let us specify the the initial distribution $\boldsymbol{\alpha}_H \in \mathbb{R}^{K(m+1)}$ of the Markov process H_t on the state space S^H, defined as in (3.5.1). First, let $\boldsymbol{\alpha}$ be the initial distribution of the process X_t defined in Subsection 3.1. Then $\boldsymbol{\alpha}_k = \mathbb{Q}[X_0 = k]$ and given the row vector $\boldsymbol{\alpha} \in \mathbb{R}^K$ we now specify the initial distribution $\boldsymbol{\alpha}_H \in \mathbb{R}^{K(m+1)}$. We assume that all obligors in the portfolio are "alive" (non-defaulted) at time $t = 0$, i.e. today, which implies that the second component must be zero for all states of the economy background process modelled by the first component of the bivariate Markov process. Hence, it must hold that

$$\sum_{k=1}^{K} (\boldsymbol{\alpha}_H)_{(k,0)} = 1 \quad \text{and} \quad (\boldsymbol{\alpha}_H)_{(k,j)} = 0 \quad \text{for} \quad j = 1, 2, \ldots, m \tag{3.5.9}$$

which in turn guarantees that the sum of the entries in $\boldsymbol{\alpha}_H$ are one.

As we will see later, by using the formulas (3.5.7), (5.12) and (3.5.9) we can efficiently compute numerical values for CDS index options in a Markovian model specified in the previous Subsections 3.1 - 3.4.

Since typically m and K are allowed to be large, especially m, we will in general deal with very high dimensional state spaces of size $(m+1) \times K$, which requires special treatment when numerically dealing with the matrix exponential of the generator for H_t. Just computing the matrix exponential with standard algorithms will make the implementation slow and also inaccurate. Instead we will rely on the so-called uniformization method which has successfully been utilized in high-dimensional state space applications of portfolio credit risk, see e.g. in Herbertsson (2007), Herbertsson & Rootzén (2008), Herbertsson (2011), Bielecki, Crépey & Herbertsson (2011) and Lando (2004). In our case we will also exploit the sparseness of the transition matrices for H_t which makes the running times even quicker. With the help of H_t we will also display the loss distribution $\mathbb{Q}[N_t = k]$ for $k = 0, 1, \ldots, m$ and in particular the armageddon probabilities $\mathbb{Q}[N_t = m]$ for some calibrated examples in the Markov chain model outlined in Section 3 - 5. Finally, we have also performed robustness testes in order to increase the reliability of the implemented code. For example, we have checked that $\mathbb{Q}[X_t = k]$ is the same via $\boldsymbol{Q}$ and $\boldsymbol{Q}_H$.

Figure 3 displays the probabilities $\mathbb{Q}[X_t = k, N_t = j]$ for all states (k, j) computed via (3.5.6) with $\boldsymbol{Q}_H$ constructed from $\boldsymbol{Q}$ given by a birth-death process X_t with $K = 100$. The number of obligors are $m = 125$. The lower subplot in Figure 3 is in logscale. Furthermore, Figure 4 shows the nonzero entries in the matrix $\boldsymbol{Q}_H$ used in Figure 3.

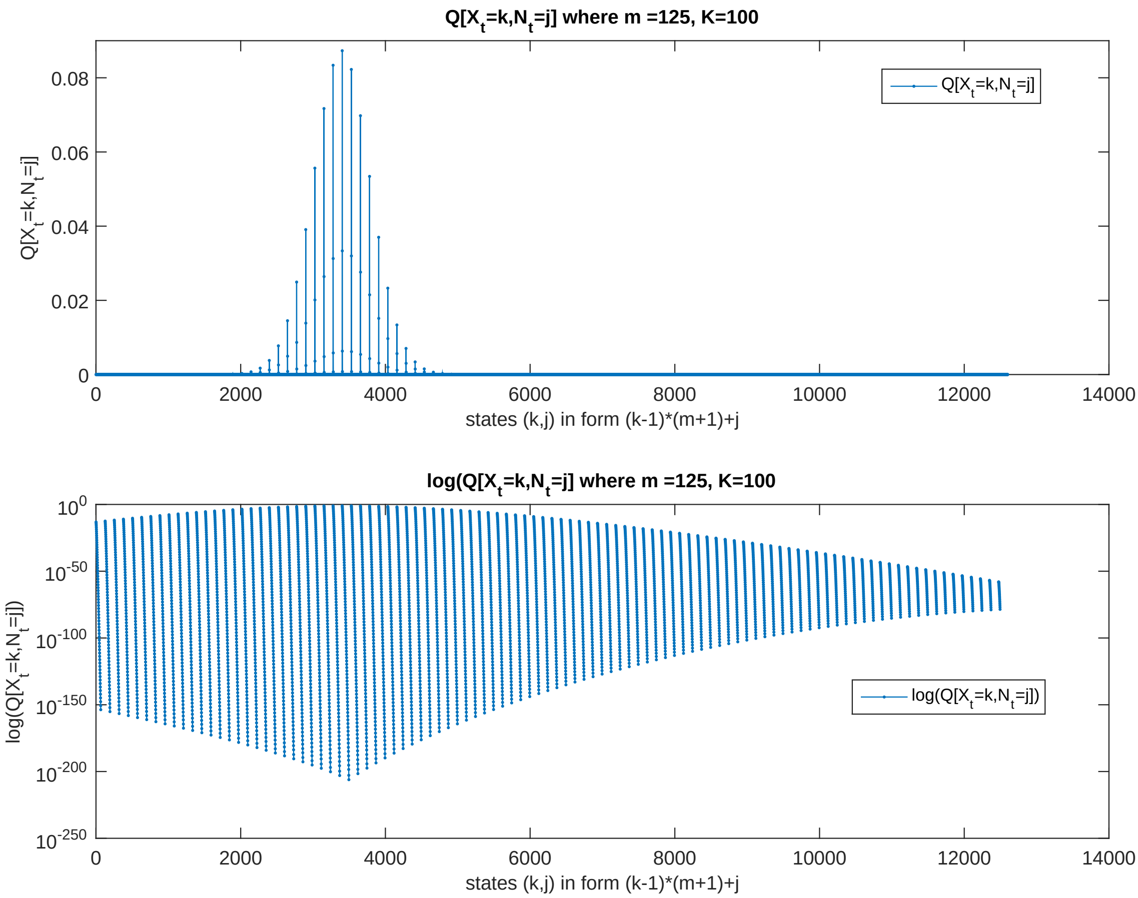

Figure 3. The probabilities $\mathbb{Q}\left[X_t = k, N_t = j\right]$ for all states (k, j) computed via (3.5.6) with $\boldsymbol{Q}_H$ constructed from $\boldsymbol{Q}$ given by a birth-death process X_t with $K = 100$. The number of obligors are $m = 125$. The lower subplot is in logscale.

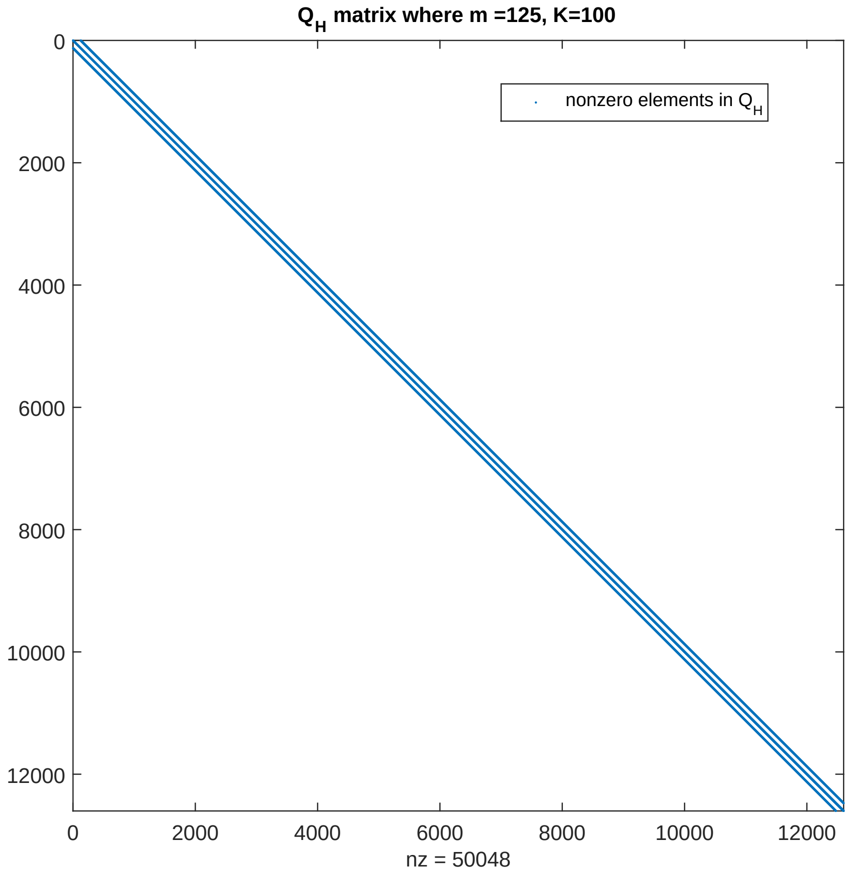

Figure 4. The nonzero entries in the matrix $\boldsymbol{Q}_H$ used in Figure 3. $\boldsymbol{Q}_H$ is constructed from $\boldsymbol{Q}$ given by a birth-death process X_t with $K = 100$. The number of obligors are $m = 125$.

4. The CDS index in the Markov chain model

In this section we apply the results from Section 3 together with Subsection 2 to find formulas for the CDS index spreads in the models introduced in Section 3. This will be done in a homogeneous portfolio. We will assume that the risk free interest rate is constant and given by r and for $t < s$ we let $B(t, s)$ denote $B(t, s) = e^{-r(s-t)}$. We can now state the following theorem.

Theorem 4.1. *Consider a CDS index in the finite state Markov chain model outlined in Section 3 to* **??**. *Then, with notation as above*

$$DL(t, T) = \mathbb{E}\left[\int_t^T B(t, s) dL_s \,\middle|\, \mathcal{F}_t\right] = \left(1 - \frac{N_t}{m}\right) \boldsymbol{e}_{X_t} \boldsymbol{A}(t, T)\mathbf{1} \tag{4.1}$$

and

$$PV(t, T) = \mathbb{E}\left[V_P(t, T) \,|\, \mathcal{F}_t\right] = \left(1 - \frac{N_t}{m}\right) \boldsymbol{e}_{X_t} \boldsymbol{B}(t, T)\mathbf{1} \tag{4.2}$$

where $\boldsymbol{A}(t,T)$ *and* $\boldsymbol{B}(t,T)$ *are defined as*

$$\boldsymbol{A}(t,T) = (1-\phi)\left[\boldsymbol{I} - e^{\boldsymbol{Q}_\lambda(T-t)}\left(\boldsymbol{I} + r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right)e^{-r(T-t)} + r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right] \tag{4.3}$$

$$\boldsymbol{B}(t,T) = \frac{1}{4}\sum_{n=n_t}^{\lceil 4T\rceil} e^{\boldsymbol{Q}_\lambda(t_n-t)}e^{-r(t_n-t)} \tag{4.4}$$

and if $\boldsymbol{Q}_\lambda - r\boldsymbol{I}$ *has distinct eigenvalues or is symmetric then*

$$\boldsymbol{B}(t,T) = \frac{1}{4}\left(e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})\frac{1}{4}} - \boldsymbol{I}\right)^{-1}\left(e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})(\frac{\lceil 4T\rceil - 4t+1}{4})} - e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})(\frac{n_t}{4}-t)}\right). \tag{4.5}$$

Furthermore, if $N_t < m$ *we have*

$$S(t,T) = \frac{\boldsymbol{e}_{X_t}\boldsymbol{A}(t,T)\boldsymbol{1}}{\boldsymbol{e}_{X_t}\boldsymbol{B}(t,T)\boldsymbol{1}} = \sum_{k=1}^{K} 1_{\{X_t=k\}}\frac{\boldsymbol{e}_k\boldsymbol{A}(t,T)\boldsymbol{1}}{\boldsymbol{e}_k\boldsymbol{B}(t,T)\boldsymbol{1}} \tag{4.6}$$

Proof. First we recall the definitions of $DL(t,T)$, $PV(t,T)$ and $S(t,T)$ from (2.2.3), (2.2.4), (2.2.5) and (2.2.6). Next, the term $\int_t^T B(t,s)dL_s$ used in $DL(t,T)$ can be rewritten in a more practical form using integration by parts (see e.g. Theorem 3.36, p.107 in Folland (1999)), so that $\int_t^T B(t,s)dL_s = B(t,T)L_T - L_t + \int_t^T rB(t,s)L_s ds$ and by applying Fubini-Tonelli on this expressions then renders

$$\mathbb{E}\left[\int_t^T B(t,s)dL_s \,\middle|\, \mathcal{F}_t\right] = B(t,T)\mathbb{E}\left[L_T \,|\, \mathcal{F}_t\right] - L_t + \int_t^T rB(t,s)\mathbb{E}\left[L_s \,|\, \mathcal{F}_t\right]ds. \tag{4.7}$$

Furthermore, if $s > t$ then (3.4.2) gives

$$\mathbb{E}\left[L_s \,|\, \mathcal{F}_t\right] = (1-\phi)\left(1 - \left(1 - \frac{N_t}{m}\right)\boldsymbol{e}_{X_t}e^{\boldsymbol{Q}_\lambda(s-t)}\boldsymbol{1}\right)$$

so using this in (4.7) and recalling that $B(t,s) = e^{-r(s-t)}$ for $s > t$, we get

$$\begin{aligned}\mathbb{E}\left[\int_t^T B(t,s)dL_s \,\middle|\, \mathcal{F}_t\right] &= B(t,T)\mathbb{E}\left[L_T \,|\, \mathcal{F}_t\right] - L_t + \int_t^T rB(t,s)\mathbb{E}\left[L_s \,|\, \mathcal{F}_t\right]ds \\ &= e^{-r(T-t)}(1-\phi)\left(1 - \left(1 - \frac{N_t}{m}\right)\boldsymbol{e}_{X_t}e^{\boldsymbol{Q}_\lambda(T-t)}\boldsymbol{1}\right) - \frac{(1-\phi)}{m}N_t \\ &\quad + \int_t^T re^{-r(s-t)}(1-\phi)\left(1 - \left(1 - \frac{N_t}{m}\right)\boldsymbol{e}_{X_t}e^{\boldsymbol{Q}_\lambda(s-t)}\boldsymbol{1}\right)ds.\end{aligned} \tag{4.8}$$

The integral in the RHS of (4.8) can be simplified according to

$$\begin{aligned}&\int_t^T re^{-r(s-t)}(1-\phi)\left(1 - \left(1 - \frac{N_t}{m}\right)\boldsymbol{e}_{X_t}e^{\boldsymbol{Q}_\lambda(s-t)}\boldsymbol{1}\right)ds \\ &= (1-\phi)\left(1 - e^{-r(T-t)}\right) \\ &\quad - r(1-\phi)\left(1 - \frac{N_t}{m}\right)\boldsymbol{e}_{X_t}\left(e^{\boldsymbol{Q}_\lambda(T-t)}e^{-r(T-t)} - \boldsymbol{I}\right)\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\boldsymbol{1}\end{aligned} \tag{4.9}$$

where the last equality in (4.9) is due to the fact that

$$\int_t^T e^{-r(s-t)}e^{\boldsymbol{Q}_\lambda(s-t)}ds = \int_t^T e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})(s-t)}ds = \left(e^{\boldsymbol{Q}_\lambda(T-t)}e^{-r(T-t)} - \boldsymbol{I}\right)\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}.$$

Note that $(\boldsymbol{Q}_\lambda - r\boldsymbol{I})^{-1}$ exists since $\boldsymbol{Q}_\lambda - r\boldsymbol{I}$ by construction is a diagonal dominant matrix, implying that $\det(\boldsymbol{Q}_\lambda - r\boldsymbol{I}) \neq 0$ by the Levy-Desplanques Theorem. By plugging (4.9) into (4.8) and performing some trivial but tedious computations we get

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^T B(t,s)dL_s \,\middle|\, \mathcal{F}_t\right] \\
&= (1-\phi)\left(1-\frac{N_t}{m}\right)\left(1-\boldsymbol{e}_{X_t}\left(e^{\boldsymbol{Q}_\lambda(T-t)}\left(\boldsymbol{I}+r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right)e^{-r(T-t)} - r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right)\boldsymbol{1}\right) \\
&= (1-\phi)\left(1-\frac{N_t}{m}\right)\boldsymbol{e}_{X_t}\left[\boldsymbol{I} - e^{\boldsymbol{Q}_\lambda(T-t)}\left(\boldsymbol{I}+r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right)e^{-r(T-t)} + r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right]\boldsymbol{1} \\
&= \left(1-\frac{N_t}{m}\right)\boldsymbol{e}_{X_t}\boldsymbol{A}(t,T)\boldsymbol{1}
\end{aligned}
$$

where we in the second equality used that $1 = \boldsymbol{e}_{X_t}\boldsymbol{1} = \boldsymbol{e}_{X_t}\boldsymbol{I}\boldsymbol{1}$ and where $\boldsymbol{A}(t,T)$ in the final equality is given by

$$
\boldsymbol{A}(t,T) = (1-\phi)\left[\boldsymbol{I} - e^{\boldsymbol{Q}_\lambda(T-t)}\left(\boldsymbol{I}+r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right)e^{-r(T-t)} + r\left(\boldsymbol{Q}_\lambda - r\boldsymbol{I}\right)^{-1}\right]
$$

which proves (4.1) and (4.3). To derive the expression for the premium leg we use (3.3.1) in Proposition 3.2 with $s > t$ and obtain $1 - \frac{1}{m}\mathbb{E}\left[N_s \,|\, \mathcal{F}_t\right] = \left(1-\frac{N_t}{m}\right)\boldsymbol{e}_{X_t}e^{\boldsymbol{Q}_\lambda(s-t)}\boldsymbol{1}$ which in Equation (2.2.5) then renders that

$$
\begin{aligned}
PV(t,T) = \frac{1}{4}\sum_{n=n_t}^{\lceil 4T\rceil} B(t,t_n)\left(1-\frac{1}{m}\mathbb{E}\left[N_{t_n} \,|\, \mathcal{F}_t\right]\right) &= \frac{1}{4}\left(1-\frac{N_t}{m}\right)\sum_{n=n_t}^{\lceil 4T\rceil}\boldsymbol{e}_{X_t}e^{\boldsymbol{Q}_\lambda(t_n-t)}\boldsymbol{1}e^{-r(t_n-t)} \\
&= \left(1-\frac{N_t}{m}\right)\boldsymbol{e}_{X_t}\boldsymbol{B}(t,T)\boldsymbol{1}
\end{aligned}
$$

where $\boldsymbol{B}(t,T) = \frac{1}{4}\sum_{n=n_t}^{\lceil 4T\rceil} e^{\boldsymbol{Q}_\lambda(t_n-t)}e^{-r(t_n-t)}$ and this proves (4.2) and (4.4).

Next, some elementary computations with the fact $t_n = \frac{n}{4}$ gives us

$$
\begin{aligned}
\frac{1}{4}\sum_{n=n_t}^{\lceil 4T\rceil} e^{\boldsymbol{Q}_\lambda(t_n-t)}e^{-r(t_n-t)} &= \frac{1}{4}\left(\sum_{n=n_t}^{\lceil 4T\rceil} e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})t_n}\right)e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})t} \\
&= \frac{1}{4}\left(\sum_{n=n_t}^{\lceil 4T\rceil} e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})\frac{n}{4}}\right)e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})t} \\
&= \frac{1}{4}\left(\sum_{n=0}^{\lceil 4T\rceil - n_t} e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})\frac{n}{4}}\right)e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})\left(\frac{n_t}{4}-t\right)}.
\end{aligned}
\tag{4.10}
$$

If assume that the matrix $(\boldsymbol{Q}_\lambda - r\boldsymbol{I})\frac{1}{4}$ has distinct eigenvalues or is symmetric then from Lemma B2 in Appendix of Herbertsson (2017) we know that

$$
\sum_{n=0}^{\lceil 4T\rceil - n_t} e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})\frac{n}{4}} = \left(e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})\frac{1}{4}} - \boldsymbol{I}\right)^{-1}\left(e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})\left(\frac{\lceil 4T\rceil - n_t + 1}{4}\right)} - \boldsymbol{I}\right) \tag{4.11}
$$

so combining (4.10) and (4.11) with some simple computations finally implies that if $\boldsymbol{Q}_\lambda - r\boldsymbol{I}$ has distinct eigenvalues or is symmetric, then

$$\boldsymbol{B}(t,T) = \frac{1}{4}\left(e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})\frac{1}{4}} - \boldsymbol{I}\right)^{-1}\left(e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})(\frac{\lceil 4T\rceil - 4t+1}{4})} - e^{(\boldsymbol{Q}_\lambda - r\boldsymbol{I})(\frac{n_t}{4}-t)}\right).$$

which proves (4.5).

Finally, (4.6) follows from the definition in (2.2.6) together with the expressions for the default leg and premium leg in (4.1) and (4.2). □

Note that Equation (4.5) in Theorem 4.1 is very useful from a computational point of view since the sum in the right hand side of (4.4) requires the computation of $\lceil 4T\rceil - n_t + 1$ different matrix exponentials while the right hand side in (4.5) only requires the computation of *three* different matrix exponentials and one matrix inversion. For large $\lceil 4T\rceil - n_t + 1$ or/and large matrices $\boldsymbol{Q}_\lambda$ this will substantially reduce the computational time when finding the sum in in the right hand side of Equation (4.4). Recall that computations of the matrix exponential $e^{\boldsymbol{T}}$ for large matrices $\boldsymbol{T}$ can be very time consuming and is often also numerically challenging, see e.g. in Moeler & Loan (1978), Moeler & Loan (2003), Sidje & Stewart (1999) and for credit risk applications see also e.g. in Herbertsson (2007), Herbertsson & Rootzén (2008), Herbertsson (2008*b*), Herbertsson (2008*a*), Herbertsson (2011), Bielecki et al. (2011) and Lando (2004).

Remark 4.2. At $t = 0$ one can assume that X_0 is not observable. This is the case in Graziano & Rogers (2009), see Remark 2.3 on p.49 in Graziano & Rogers (2009). Since X_0 is not observable then $\mathcal{F}_0$ is not the trivial sigma algebra but rather $\sigma(X_0)$. Thus, since $N_0 = 0$, then for $t = 0$ the relations (4.1), (4.2) and (4.6) reduces to

$$DL(0,T) = \mathbb{E}\left[\int_0^T B(0,s)dL_s \,\middle|\, X_0\right] = \boldsymbol{e}_{X_0}\boldsymbol{A}(0,T)\boldsymbol{1} \tag{4.12}$$

and

$$PV(0,T) = \mathbb{E}\left[V_P(0,T) \,|\, X_0\right] = \boldsymbol{e}_{X_0}\boldsymbol{B}(0,T)\boldsymbol{1} \tag{4.13}$$

so that

$$S(0,T) = \sum_{k=1}^{K} 1_{\{X_0 = k\}} \frac{\boldsymbol{e}_k\boldsymbol{A}(0,T)\boldsymbol{1}}{\boldsymbol{e}_k\boldsymbol{B}(0,T)\boldsymbol{1}}. \tag{4.14}$$

Furthermore, if $\boldsymbol{\alpha}$ is the initial distribution of the process X_t so that $\boldsymbol{\alpha}_k = \mathbb{Q}[X_0 = k]$ then $\mathbb{E}[S(0,T)]$ will be our proxy to the observed CDS index spread on the market at time $t = 0$ where $\mathbb{E}[S(0,T)]$ thus is given by

$$\mathbb{E}[S(0,T)] = \sum_{k=1}^{K} \boldsymbol{\alpha}_k \frac{\boldsymbol{e}_k\boldsymbol{A}(0,T)\boldsymbol{1}}{\boldsymbol{e}_k\boldsymbol{B}(0,T)\boldsymbol{1}}. \tag{4.15}$$

So when calibration the Markov model CDS index spread at time $t = 0$ against the corresponding observed market spread we will use the formula (4.15).

Note that in the case when we don't now the state X_0, the model can be seen as special case of a hidden Markov model.

Remark 4.3. Sometimes it is from a financial point of view more convenient to assume that X_0 is known with 100% so that the model spread $S(0,T)$ also will be known for sure at time

$t = 0$, i.e. "today" which is the calibration date. Compare for example with the Black-Scholes model for stock prices where the spot stock price S_0 in the model will coincide with the observed market price.

Thus, in the Markov model this means that $\boldsymbol{\alpha} = \boldsymbol{e}_{k^*}$ for some $k^* = X_0$. In this paper the exact state will be obtained/relieved after the calibration. Let us briefly discuss this how k^* then are found. From Equation (4.6) imply that we can write the CDS index spread as

$$S(t,T) = \sum_{k=1}^{K} 1_{\{X_t=k\}} S_k(t,T) \tag{4.16}$$

where

$$S_k(t,T) = \frac{\boldsymbol{e}_k \boldsymbol{A}(t,T)\mathbf{1}}{\boldsymbol{e}_k \boldsymbol{B}(t,T)\mathbf{1}} \quad \text{for } k = 1,\ldots,K \tag{4.17}$$

In particular, for $t = 0$ we will for any transition matrix $\boldsymbol{Q}$ and intensity vector $\boldsymbol{\lambda}$ be able to compute the "state-space" spreads $S_1(0,T), S_2(0,T)\ldots, S_K(0,T)$ as in (4.17). For a given observed market spread $S_{\mathcal{M}}(T)$ we could then calibrate the model parameters for $\boldsymbol{Q}$ and $\boldsymbol{\lambda}$ so that one of the values $S_k(0,T)$, say $S_{k^*}(0,T)$, is as close as possible to the corresponding market spread $S_{\mathcal{M}}(T)$. We will come back to this discussion in Section 6.1.

Remark 4.4. Our numerical studies in Subsection 6.1 for the CDS index options will be based on examples where X_t is a birth-death process. Note that if a Markov chain X_t is a birth-death process with same up and down transition intensities, then the generator $\boldsymbol{Q}$ to X_t will be a symmetric matrix, and thus $\boldsymbol{Q}_\lambda - r\boldsymbol{I}$ will also be symmetric. Hence, if X_t is a birth-death process then Equation (4.5) in Theorem 4.1 can be used.

Note that the term $1 - N_t/m$ in the right hand side of both (4.1) and (4.2) implies that the conditional expectations of the default and premium legs will be zero for the armageddon event $N_t = m$. This fact is in line with the conclusion in (2.3.3) which holds for any model of the default times $\tau_1,\ldots,\tau_m$. Furthermore, note that the right hand side in (4.6) is still well defined when $N_t = m$.

From Theorem 4.1 we conclude that given the vector $\boldsymbol{e}_{X_t}$, then the formulas for the default and premium leg in the Markov model as well as the CDS index spread $S(t,T)$ are compact and computationally tractable closed-form expressions in terms of $\boldsymbol{e}_{X_t}$ and $\boldsymbol{Q}_\lambda$. Furthermore, Theorem 4.1 will also help us to find tractable formulas for the payoff of more exotic derivatives with the CDS index as a underlyer. Example of such derivatives are call options on the CDS index, which we will treat in the next section.

5. CDS index options in the Markovian model

In this section we apply the results from Section 4 and Subsection 2.3 to present a highly computationally tractable formula for the payoff of a so called CDS index option in the model presented in Section 3.

By inserting the explicit expressions for the default and premium legs for the index-CDS spread given by (4.1) and (4.2) in Theorem 4.1 into the expression of the payoff $\Pi(t,T;\kappa)$ for the CDS index option in Equation (2.3.5), that is

$$\Pi(t,T;\kappa) = \left(DL(t,T) - \kappa PV(t,T) + L_t\right)^+.$$

we immediately make the payoff $\Pi(t,T;\kappa)$ very explicit in terms of $\boldsymbol{e}_{X_t}$, N_t, $\boldsymbol{A}(t,T)$ and $\boldsymbol{B}(t,T)$, as summarized in the following lemma.

Lemma 5.1. *Consider a CDS index portfolio in the Markov chain model. Then, the payoff $\Pi(t,T;\kappa)$ for an CDS index option with strike κ, exercise date t and maturity T for the underlying CDS index, is given by*

$$\Pi(t,T;\kappa) = \left(\boldsymbol{e}_{X_t} \Big[\boldsymbol{A}(t,T) - \kappa \boldsymbol{B}(t,T) \Big] \boldsymbol{1} \left(1 - \frac{N_t}{m}\right) + \frac{(1-\phi)\,N_t}{m} \right)^{+} \tag{5.1}$$

where $\boldsymbol{A}(t,T)$ and $\boldsymbol{B}(t,T)$ are defined as in Theorem 4.1.

Note that on the event $\{N_t = m\}$, the right-hand side in (5.1) reduces to the random variable $(1-\phi)1_{\{N_t=m\}}$ for any strike spread κ, which is consistent with Equation (2.3.7).

In view of Lemma 5.1 and since the price of the CDS index option $C_0(t,T;\kappa)$ at time 0 (i.e. today) is given by $C_0(t,T;\kappa) = \mathbb{E}\left[e^{-rt}\Pi(t,T;\kappa)\right]$ we therefore get

$$C_0(t,T;\kappa) = e^{-rt}\mathbb{E}\left[\left(\boldsymbol{e}_{X_t} \Big[\boldsymbol{A}(t,T) - \kappa \boldsymbol{B}(t,T) \Big] \boldsymbol{1} \left(1 - \frac{N_t}{m}\right) + \frac{(1-\phi)\,N_t}{m} \right)^{+} \right]. \tag{5.2}$$

Next we derive analytical expressions for the formulas in the RHS of Equation (5.2).

Proposition 5.2. *Let $C_0(t,T;\kappa)$ be the price today of an CDS index option with strike κ, exercise date t and maturity T. Then, with notation as above,*

$$C_0(t,T;\kappa) = e^{-rt}\boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t}\boldsymbol{h}^{(\Pi)}(t,T;\kappa) \tag{5.3}$$

where

$$\boldsymbol{h}^{(\Pi)}(t,T;\kappa) = \sum_{j=0}^{m-1} \boldsymbol{h}^{(\Pi)}(t,T;\kappa,j) + (1-\phi)\boldsymbol{e}_{(\cdot,m)} \tag{5.4}$$

with

$$\boldsymbol{h}^{(\Pi)}(t,T;\kappa,j) = \sum_{k=1}^{K} \left(p_k(t,T;\kappa)\left(1 - \frac{j}{m}\right) + \frac{(1-\phi)\,j}{m} \right)^{+} \boldsymbol{e}_{(k,j)}. \tag{5.5}$$

and

$$p_k(t,T;\kappa) = \boldsymbol{e}_k \big[\boldsymbol{A}(t,T) - \kappa \boldsymbol{B}(t,T) \big] \boldsymbol{1} \tag{5.6}$$

for $\boldsymbol{A}(t,T)$ and $\boldsymbol{B}(t,T)$ defined as in Theorem 4.1. Furthermore, $\boldsymbol{Q}_H$ is the generator to a bivariate Markov chain H_t defined on $S^H = \{1,\dots,K\} \times \{0,1,\dots,m\}$ as in Subsection 3.5 and $\boldsymbol{\alpha}_H$ is the initial distribution of H_t. The column vectors $\boldsymbol{e}_{(k,j)} \in \mathbb{R}^{K(m+1)}$ and $\boldsymbol{e}_{(\cdot,m)} \in \mathbb{R}^{K(m+1)}$ are defined as in Subsection 3.5.

Proof. From Equation (2.3.11) we have

$$C_0(t,T;\kappa) = e^{-rt}\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t<m\}}\right] + (1-\phi)e^{-rt}\mathbb{Q}\left[N_t = m\right] \tag{5.7}$$

and note that $\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t<m\}}\right]$ can be rewritten as

$$\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t<m\}}\right] = \sum_{j=0}^{m-1} \mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t=j\}}\right]. \tag{5.8}$$

We will now derive an exact expression for the quantity $\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t=j\}}\right]$ and for this we need some more notation. For each state k in the state space of the underlying process

X_t defined in Section 3, let $p_k(t,T;\kappa)$ denote the k-th component in the vector $\big[\boldsymbol{A}(t,T) - \kappa\boldsymbol{B}(t,T)\big]\mathbf{1}$, that is

$$p_k(t,T;\kappa) = \boldsymbol{e}_k\big[\boldsymbol{A}(t,T) - \kappa\boldsymbol{B}(t,T)\big]\mathbf{1}. \tag{5.9}$$

Hence, this observation together with Equation (5.1) then implies that we can rewrite the quantity $\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t=j\}}\right]$ as follows

$$\begin{aligned}
\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t=j\}}\right] &= \mathbb{E}\left[\left(\boldsymbol{e}_{X_t}\Big[\boldsymbol{A}(t,T) - \kappa\boldsymbol{B}(t,T)\Big]\mathbf{1}\left(1-\frac{j}{m}\right) + \frac{(1-\phi)\,j}{m}\right)^+ 1_{\{N_t=j\}}\right] \\
&= \sum_{k=1}^{K}\left(\boldsymbol{e}_k\Big[\boldsymbol{A}(t,T) - \kappa\boldsymbol{B}(t,T)\Big]\mathbf{1}\left(1-\frac{j}{m}\right) + \frac{(1-\phi)\,j}{m}\right)^+ \mathbb{E}\left[1_{\{X_t=k\}}1_{\{N_t=j\}}\right] \\
&= \sum_{k=1}^{K}\left(p_k(t,T;\kappa)\left(1-\frac{j}{m}\right) + \frac{(1-\phi)\,j}{m}\right)^+ \mathbb{Q}\left[X_t=k, N_t=j\right] \\
&= \sum_{k=1}^{K}\left(p_k(t,T;\kappa)\left(1-\frac{j}{m}\right) + \frac{(1-\phi)\,j}{m}\right)^+ \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t}\boldsymbol{e}_{(k,j)} \\
&= \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t}\sum_{k=1}^{K}\left(p_k(t,T;\kappa)\left(1-\frac{j}{m}\right) + \frac{(1-\phi)\,j}{m}\right)^+ \boldsymbol{e}_{(k,j)} \\
&= \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t}\boldsymbol{h}^{(\Pi)}(t,T;\kappa,j)
\end{aligned} \tag{5.10}$$

where the third equality in (5.10) follows from (5.9). The fourth equality in (5.10) is due to relation (3.5.7) and the construction of a bivariate Markov chain H_t defined on $S^H = \{1,\ldots,K\}\times\{0,1,\ldots,m\}$ with generator $\boldsymbol{Q}_H$ given by (3.5.3)-(3.5.4) and $\boldsymbol{\alpha}_H$ is the initial distribution of H_t. Here, $\boldsymbol{e}_{(k,j)} \in \mathbb{R}^{K(m+1)}$ is a column vector where the entry at position $(k,j)\in S^H$ is 1 and the other entries are zero. Furthermore, $\boldsymbol{h}^{(\Pi)}(t,T;\kappa,j) \in \mathbb{R}^{K(m+1)}$ is a column vector defined as

$$\boldsymbol{h}^{(\Pi)}(t,T;\kappa,j) = \sum_{k=1}^{K}\left(p_k(t,T;\kappa)\left(1-\frac{j}{m}\right) + \frac{(1-\phi)\,j}{m}\right)^+ \boldsymbol{e}_{(k,j)}. \tag{5.11}$$

From (5.12) we have that

$$\mathbb{Q}\left[N_t = m\right] = \boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t}\boldsymbol{e}_{(\cdot,m)} \tag{5.12}$$

where $\boldsymbol{e}_{(\cdot,m)} \in \mathbb{R}^{K(m+1)}$ is a column vector defined as $\boldsymbol{e}_{(\cdot,m)} = \sum_{k=1}^{K}\boldsymbol{e}_{(k,m)}$. So combing (5.10) and (5.12) we get

$$\begin{aligned}
C_0(t,T;\kappa) &= e^{-rt}\boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t}\sum_{j=0}^{m-1}\boldsymbol{h}^{(\Pi)}(t,T;\kappa,j) + (1-\phi)e^{-rt}\boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t}\boldsymbol{e}_{(\cdot,m)} \\
&= e^{-rt}\boldsymbol{\alpha}_H e^{\boldsymbol{Q}_H t}\boldsymbol{h}^{(\Pi)}(t,T;\kappa)
\end{aligned}$$

where

$$\boldsymbol{h}^{(\Pi)}(t,T;\kappa) = \sum_{j=0}^{m-1}\boldsymbol{h}^{(\Pi)}(t,T;\kappa,j) + (1-\phi)\boldsymbol{e}_{(\cdot,m)}$$

which proves (5.3)-(5.5) and concludes the proposition.

□

Thus, Proposition 5.2 establish a an exact formula for the option price $C_0(t, T; \kappa)$ as function of the probabilities $\mathbb{Q}[X_t = k, N_t = j]$ and $\mathbb{Q}[N_t = j]$ for each state k and $j = 0, 1, \ldots, m$.

Remark 5.3. Note that CDS index options typically has short exercise maturities between 3 to 6 months, see e.g Exhibit 2, 6 and 8 in, Chapter 2 of Mahadevan, Musfeldt & Naraparaju (2011) , p.6 in Jackson (2005) or p.1043 in Rutkowski & Armstrong (2009). In fact, also very short maturities such as 1, 1.5 and 2 months are common, see Exhibit 2 in Chapter 2 of Mahadevan et al. (2011) or p.7 in Flesaker et al. (2011). Note that a CDS index option on a standardized index such as e.g. iTraxx Europe, CDX IG or iTraxx Xover, with a maturity longer than 6 months would mean that the underlying index for the option would become an off-the-run series. This, since all standardized indices are updated on the roll-dates of March 20th and September 20th every year. Hence, having an CDS index option on a standardized index with a maturity longer than 6 months would therefore in practice mean that the underlying portfolio would change. This also motivates exercise maturities between 1 to 6 months. We will in our numerical studies consider maturities of $1, 3$ and 6 months. However, in the numerical examples of Morini & Brigo (2011) only maturities of 9 months are considered. Thus, in order to benchmark our model to the one in Morini & Brigo (2011) we will also consider maturities of 9 months, even though this in practice would mean that the underlying index would become an off-the-run series.

6. Numerical studies

In this section we perform various numerical studies of the CDS index spread and CDS index option prices presented in Section 4 and 5, which in turn are based on the model outlined in Section 3. The numerical studies are performed by calibration all parameters to market data

First, in Subsection 6.1 we gives a detailed outline of how the matrix $\boldsymbol{Q}$ and vector $\boldsymbol{\lambda}$ are chosen and then discuss how to estimate/calibrate the parameters $\boldsymbol{\theta} = (\boldsymbol{Q}, \boldsymbol{\lambda})$ in the finite-state Markov chain model introduced in Section 3 - 5.

Next, in Subsection 6.2 we use the results of Subsection 6.1 to calibrate, compute and display the CDS index option prices as well as the lower bounds for these prices as functions of varies parameters such as the strike, the maturity and the spot-spread. We also calibrate the benchmark model 2.4.19 to the same spot CDS index spread. After this we compare the Markov chain prices, their lower bound prices and the benchmark prises for different values of ρ and $\hat{\sigma}$. In particular, a numerical study is performed where we show that the lower bound in our model can be several hundred percent bigger compared with models which assume that the CDS-spreads follows a log-normal process. Also a systematic study is done in order to understand the impact of various model parameters on CDS index options (and on the index itself)

6.1. **Specifying the parameters and calibrating the Markov chain model.**

In this section we discuss how to estimate/calibrate the parameters $\boldsymbol{\theta} = (\boldsymbol{Q}, \boldsymbol{\lambda})$ in the Markovian model introduced in Section 3.

There are many ways to estimate/calibrate the parameters $\boldsymbol{\theta} = (\boldsymbol{Q}, \boldsymbol{\lambda}, \boldsymbol{\alpha})$ in the Markov chain model. First of all, we can parameterize $\boldsymbol{Q}, \boldsymbol{\lambda}, \boldsymbol{\alpha}$ in different ways. Secondly, we can use observable values for different credit derivatives, financial instruments and other quantities when calibrating the model.

Regarding parametrization of $\boldsymbol{\theta} = (\boldsymbol{Q}, \boldsymbol{\lambda}, \boldsymbol{\alpha})$ we will here use the setup in Remark 4.3, that is we proceed as follows. For $\boldsymbol{\lambda} = (\lambda(1), \dots, \lambda(K))$ we will use the following piecewise linear parametrization of the mapping $\lambda(k)$ for the individual default intensity,

$$\lambda(k) = b + \beta k \tag{6.1.1}$$

where β and b are constants such that $\beta > 0$ and $b > 0$. We here remind the reader that the states in $S^X = \{1, 2, \dots, K\}$ are ordered so that state 1 represents the best state and K represents the worst state of the economy. Consequently, the mapping $\lambda(\cdot)$ is chosen to be strictly increasing in $k \in \{1, 2, \dots, K\}$.

The parametrization of $\lambda(k)$ in (6.1.1) is convenient in the sense that it describes the function $\lambda(k)$ with only two parameters β and b, regardless of the number of states K.

Next, we will assume that the finite state continuous time Markov chain X_t on the state space $S^X = \{1, 2, \dots, K\}$ is a birth-death process with identical up and down transition intensities given by q. Hence, the generator $\boldsymbol{Q}$ will satisfy

$$\boldsymbol{Q}_{i,j} = \begin{cases} q & \text{if } i = j-1 \text{ or } i = j+1 \\ -2q & \text{if } 2 \leq i = j \leq K-1 \\ -q & \text{if } i = j = 1 \text{ or } i = j = K \\ 0 & \text{otherwise} \end{cases} \tag{6.1.2}$$

where $q > 0$. So (6.1.2) gives us only one parameter describing the generator $\boldsymbol{Q}$ regardless of the number of states K.

Hence, given the parametrization of $\boldsymbol{Q}, \boldsymbol{\lambda}$ in (6.1.1)-(6.1.2), the parameters to be estimated/calibrated are then $\boldsymbol{\theta} = (b, \beta, q)$. In this paper we will estimate $\boldsymbol{\theta} = (b, \beta, q)$ by calibrating the model spot CDS-index spread $S(0, 5)$ towards the corresponding observed market spread by using Equation (4.6) for $t = 0$.

Recall from Remark 4.2 in general the finite-state Markov chain model X_0 is not observable at time $t = 0$ so $S(0, 5)$ is random according to Equation (4.14). In such a case one would calibrate $\mathbb{E}\left[S(0, T)\right]$ given by (4.15) against the corresponding observed market spread and this means that we also need to find the K values of $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_K)$ where $\boldsymbol{\alpha}_k = \mathbb{Q}\left[X_0 = k\right]$. So the parameters to be estimated are then $\boldsymbol{\theta} = (b, \beta, q, \boldsymbol{\alpha})$.

Let $S_{\mathcal{M}}(T)$ be observed market value for T-year CDS-index spread today. Let $S^{(\mathbb{E})}(T; \boldsymbol{\theta})$ denote the T-year model CDS index spread in the Markov model, so

$$S^{(\mathbb{E})}(T; \boldsymbol{\theta}) = \mathbb{E}\left[S(0, T)\right] = \sum_{k=1}^{K} \boldsymbol{\alpha}_k \frac{\boldsymbol{e}_k \boldsymbol{A}(0, T)\boldsymbol{1}}{\boldsymbol{e}_k \boldsymbol{B}(0, T)\boldsymbol{1}}.$$

Let $S_{\mathcal{M}}(T)$ be the observed market quote for the T-year CDS-index spread today, i.e. at time $t = 0$. Furthermore, let $S^{(\mathbb{E})}(T; \boldsymbol{\theta})$ denote the T-year model CDS-index spread in the finite-state Markov model. Then, by using (4.15) with $\boldsymbol{\alpha}$ we have

$$S^{(\mathbb{E})}(T; \boldsymbol{\theta}) = \mathbb{E}\left[S(0, T)\right] = \sum_{k=1}^{K} \boldsymbol{\alpha}_k \frac{\boldsymbol{e}_k \boldsymbol{A}(0, T)\boldsymbol{1}}{\boldsymbol{e}_k \boldsymbol{B}(0, T)\boldsymbol{1}}. \tag{6.1.3}$$

So when calibration the Markov chain model CDS-index spread at time $t = 0$ against the corresponding observed market spread we will use the formula (6.1.3) and $\boldsymbol{\theta} = (b, \beta, q, \boldsymbol{\alpha})$ is then calibrated via the following minimization routine

$$\begin{aligned}
&\min_{\boldsymbol{\theta}} \left(\frac{S^{(\mathbb{E})}(T; \boldsymbol{\theta}) - S_{\mathcal{M}}(T)}{S_{\mathcal{M}}(T} \right)^2 \\
&\qquad \text{subject to} \\
&\qquad\quad \boldsymbol{\alpha}\mathbf{1} = 1 \\
&0 \le \alpha_k \le 1 \quad \text{for} \quad k = 1, \dots, K \\
&\beta > 1, b > 0, q > 0
\end{aligned} \tag{6.1.4}$$

where $S_{\mathcal{M}}(T)$ is the market quote for the T-year CDS-index spread today, i.e. at time $t = 0$ and $S^{(\mathbb{E})}(T; \boldsymbol{\theta})$ is given by (6.1.3). The calibration of the parameters $\boldsymbol{\theta} = (b, \beta, q, \boldsymbol{\alpha})$ is thus a constrained nonlinear optimization problem and such routines are mostly solved numerically using standard mathematical software packages such as e.g. matlab. Numerical optimization routines typically requires the user to provide an initial guess $\boldsymbol{\theta}_0 = \left(b_0, \beta_0, q_0, \boldsymbol{\alpha}^{(0)}\right)$ before running the scheme.

Note that if K is large, and in order to keep the number of parameters in $\boldsymbol{\theta} = (b, \beta, q, \boldsymbol{\alpha})$ small, we can parameterize the entries in the initial distribution $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_K)$ where $\boldsymbol{\alpha}_k = \mathbb{Q}[X_0 = k]$, just as we did for the vector $\boldsymbol{\lambda} = (\lambda(1), \dots, \lambda(K))$. One can consider several different parameterizations of $\boldsymbol{\alpha}_k = \mathbb{Q}[X_0 = k]$. One example (which is a bit unrealstic) is to use a uniform distribution on a subset of $1, 2, \dots, K$ and thus specify $\boldsymbol{\alpha}_k$ as

$$\boldsymbol{\alpha}_k = \begin{cases} \frac{1}{h_\alpha} & \text{if } k = k_L, k_L + 1, \dots, k_L + h_\alpha \\ 0 & \text{otherwise} \end{cases} \tag{6.1.5}$$

Hence, $\boldsymbol{\alpha}_k$ in (6.1.5) is uniformly distributed over the interval $k = k_L, k_L + 1, \dots, k_L + h_\alpha$. The parametrization in (6.1.5) is thus described by the two integers h_α and k_L where k_L is where the uniform distribution start and h_α is the "length" of the uniform distribution. Note that h_α and k_L are integers that must satisfy

$$1 \le k_L + h_\alpha \le K \quad \text{and} \quad k_L, h_\alpha \quad \text{are positive integers.} \tag{6.1.6}$$

Thus, when using the parameterization (6.1.5) we also add the integer constraint (6.1.6) to the optimization problem (6.1.4) used to find the parameters $\boldsymbol{\theta} = (b, \beta, q, k_L, h_\alpha)$. Another options is simply to specify the parameters h_α and k_L and calibrate over $\boldsymbol{\theta} = (b, \beta, q)$ in (6.1.4).

A third option to find is to way to calibrate the

As already mentioned in Remark 4.3, it is sometimes from a financial point of view more convenient to assume that X_0 is known with 100% probability so that the model spread $S(0, T)$ also will be known for sure at time $t = 0$, i.e. "today" which is the calibration date. Compare for example with the Black-Scholes model for stock prices where the spot stock price S_0 in the model will coincide with the observed market price.

Thus, in the Markov model this means that $\boldsymbol{\alpha} = \boldsymbol{e}_{k^*}$ for some $k^* = X_0$. In this paper the exact state will be obtained/relieved after the calibration. Let us briefly discuss this how k^* then are found. From Equation (4.6) imply that we can write the CDS index spread as

$$S(t, T) = \sum_{k=1}^{K} 1_{\{X_t = k\}} S_k(t, T) \tag{6.1.7}$$

where

$$S_k(t,T) = \frac{\boldsymbol{e}_k \boldsymbol{A}(t,T)\mathbf{1}}{\boldsymbol{e}_k \boldsymbol{B}(t,T)\mathbf{1}} \quad \text{for } k = 1, \dots, K \tag{6.1.8}$$

In particular, for $t = 0$ we will for any transition matrix $\boldsymbol{Q}$ and intensity vector $\boldsymbol{\lambda}$ be able to compute the "state-space" spreads $S_1(0,T), S_2(0,T) \dots, S_K(0,T)$ as in (6.1.8). For a given observed market spread $S_{\mathcal{M}}(T)$ we could then calibrate the model parameters for $\boldsymbol{Q}$ and $\boldsymbol{\lambda}$ so that one of the values $S_k(0,T)$, say $S_{k^*}(0,T)$, is as close as possible to the corresponding market spread $S_{\mathcal{M}}(T)$. The parameters $\boldsymbol{\theta} = (b, \beta, q,)$ are thus calibrated via the following minimization routine

$$\begin{aligned} &\min_{\boldsymbol{\theta}} \left(\frac{S_{k^*}(0,T) - S_{\mathcal{M}}(T)}{S_{\mathcal{M}}(T} \right)^2 \\ &\qquad \text{subject to} \\ &S_1(0,T) \leq S_{\min}, \quad S_K(0,T) \geq S_{\max} \\ &\beta > 0, b > 0, q > 0 \text{ for some integer } k^* = 1, 2 \dots, K \end{aligned} \tag{6.1.9}$$

where $S_{\mathcal{M}}(T)$ is the market quote for the T-year CDS-index spread today, i.e. at time $t = 0$ and $S_{\min}$ and $S_{\max}$ are specified bounds. The constraints $S_1(0,T) \leq S_{\min}, S_K(0,T) \geq S_{\max}$ will for exogenously given bounds $S_{\min}$ and $S_{\max}$ make sure that the "distribution" $\{S_k(0,T)\}$ will be spread out properly. Hence, in the estimation (6.1.9) the initial state $X_0 = k^*$ is obtained as a "bi-product" from the calibration.

We here remark that we can extend the above calibration routine by including market CDS-index spreads $\{S_M(\bar{T})\}_{T \in \mathcal{T}}$ for a several maturities such as e.g. $\bar{T} \in \mathcal{T} = \{3, 5, 7, 10\}$. In such a case the objective function in (6.1.4) is then replaced with

$$\sum_{\bar{T} \in \mathcal{T}} \left(\frac{S^{(\mathbb{E})}(\bar{T}; \boldsymbol{\theta}) - S_{\mathcal{M}}(\bar{T})}{S_{\mathcal{M}}(\bar{T})} \right)^2$$

and analogously for (6.1.9). We will in this paper only use one maturity $\bar{T} = 5$ which historically has been the most liquidly quoted CDS-index spread, in particular for the individual entries in the index.

The calibration of the parameters $\boldsymbol{\theta} = (b, \beta, q, \boldsymbol{\alpha})$ is thus a constrained nonlinear optimization problem and such routines are mostly solved numerically using standard mathematical software packages such as e.g. matlab. Numerical optimization routines typically requires the user to provide an initial guess $\boldsymbol{\theta}_0 = \left(b_0, \beta_0, q_0, \boldsymbol{\alpha}^{(0)}\right)$ before running the scheme. In our initial guess $\boldsymbol{\theta}_0$ we let $\boldsymbol{\alpha}_k^{(0)} = \frac{1}{K}$ for each k.

6.2. **Computing prices of options on the CDS index in the Markovian model and comparing with lognormal model prices.** In this subsection we use the results of Subsection 6.1 to calibrate, compute and display the CDS index option prices as functions of varies parameters such as the strike, the maturity and the spot-spread. We also calibrate the benchmark model (2.4.19) to the same spot CDS index spread.

We will use the calibration (6.1.9) with the parametrization for $\boldsymbol{\theta} = (\boldsymbol{Q}, \boldsymbol{\lambda})$ given by (6.1.1) and (6.1.2). Hence, $\boldsymbol{\theta} = (\boldsymbol{Q}, \boldsymbol{\lambda})$ is described by the three variables β, b, q. Furthermore, the initial state X_0 is obtained from (6.1.9).

We will compare this Markov chain model with the benchmark model given by Equation (2.4.19), that is, the model developed in Morini & Brigo (2011), and also in Rutkowski &

Armstrong (2009). Note that the model in Equation (2.4.19) with $\hat{S}_t(0,T)$ given by Proposition (2.3) and $\mathbb{Q}[N_t = m]$ computed in a one-factor Gaussian copula model via the equations (2.4.23), (2.4.24) and (2.4.25). Thus, this model is described by three variables: $\hat{\sigma}, \lambda, \rho$. Also note that $\hat{\sigma}$ is the volatility of a unobservable quantity, and ρ is also nontrivial to estimate. We use iTraxx Europe Main, 5 year, $m = 125$ obligors. The market data was sampled on July 5th, 2018 where market CDS index spread is 73 bp and retrieved via Bloomberg. The Markov model renders perfect calibration against market CDS index spread and we use $K = 100$ states. The parameters are displayed in Table 2.

In the benchmark model (Morini-Brigo model), the λ is perfectly fit to the spread via the credit triangle given by (2.4.27) or via Equation (B.9). Furthermore, since ρ difficult to estimate lacking liquidly traded portfolio instruments such as CDO tranches, we use a conservative value $\rho = 0.45$ (which is quite high!).

Table 1. The calibrated parameters for model (6.1.1)-(6.1.2) where k^* is the obtained/calibrated initial state for X_t at $t = 0$, and $S_{k^*}(0,5)$ is the corresponding model spread. The market spread $S_{\mathcal{M}}(5)$ is $S_{\mathcal{M}}(5) = 73$ bps.

q	b	β	k^*	$S_{k^*}(0,5)$
20	4.09662e-18	0.000436025	28	73.5244

Given the parameters in Table 2 we can also compute the implied 5-year default probability and the 9 months armageddon probabilities in the Markov model. We do the same for the Morini-Brigo model with $\rho = 0.45$ and compare the probabilities against each other, which are displayed in Table

Table 2. The implied 5-year default probability and the 9 months armageddon probabilities in the Markov model and Morini-Brigo model. The parameters in the Markov model are given by Table .

Model	$\mathbb{Q}[\tau_i \leq 5]$	$\mathbb{Q}[N_{0.75} = 125]$
Markov chain	5.920%	0
Morini-Brigo	5.902	2.82414e-07 %

So the 5-year default probabilities are quite close, but not the 9 months armageddon probabilities. Note that for the Morini-Brigo model the above values do not need the volatility $\hat{\sigma}$.

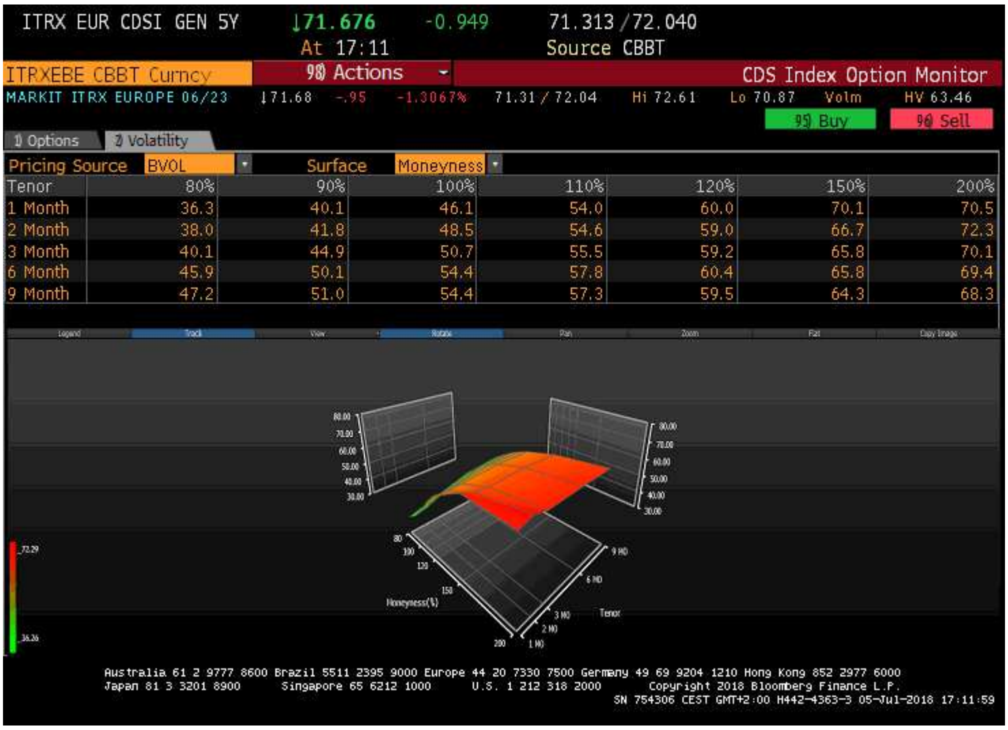

Figure 5. Implied volatilities of the 5 year CDS index Itraxx Europe main, series 29, retrieved from Bloomberg via the Bloomberg function OMON and pricing source: BVOL/CBBT, sampled at July 5th, 2018.

The parameter $\hat{\sigma}$ is not possible to calibrate directly. Instead we set $\hat{\sigma}$ to be higher than all implied volatilities with strike 110% for the CDS index option on iTraxx Europe Main (5 year), obtained from Bloomberg on July 5th, 2018, via the Bloomberg function OMON and pricing source: BVOL/CBBT, see in Figure 5. We then price CDS index options in both models with the above parameters for different maturities and compare prices and do other numerical studies.

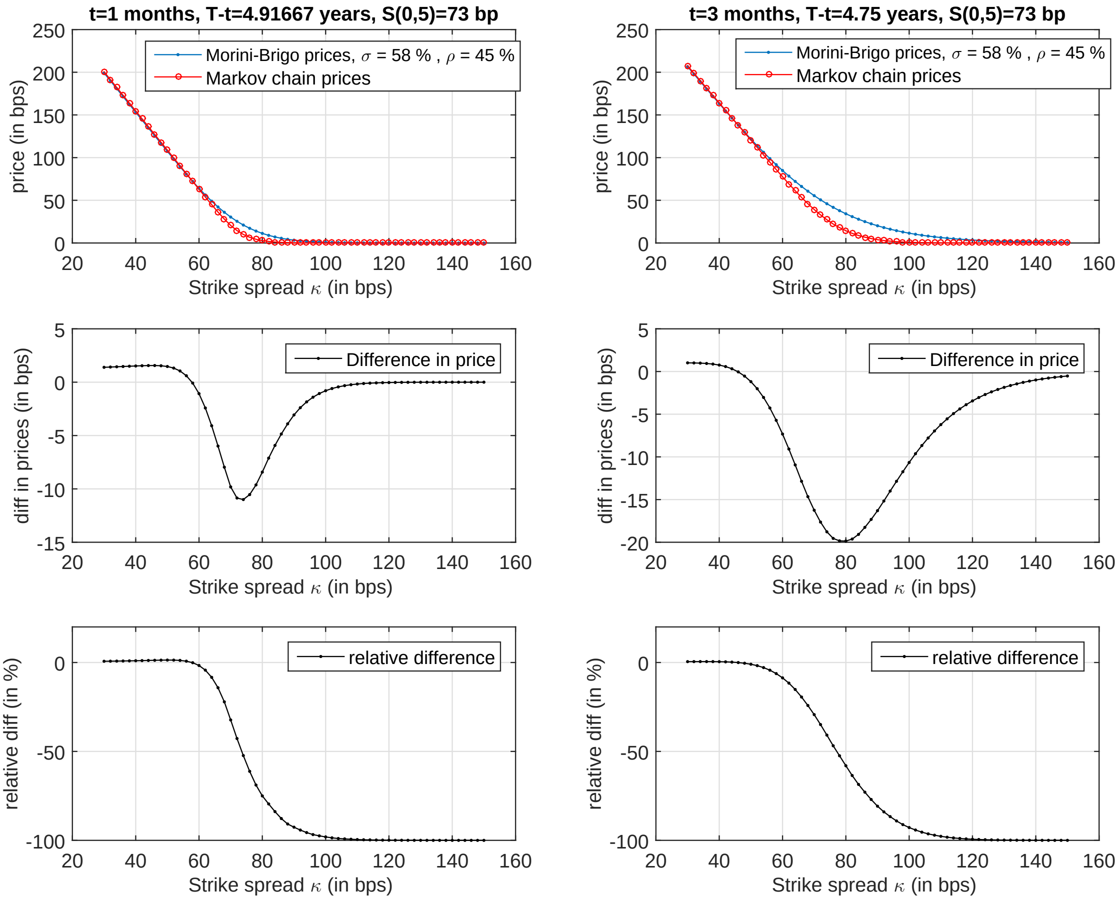

Figure 6. The Markov chain option prices and the Morini-Brigo prices for $t = 1$ month and $t = 3$ months where market spot spread is 73 bps. The parameters in the Markov model are given by Table 2 and $\hat{\sigma} = 58\%$ and $\rho = 45\%$ are used in the Morini-Brigo formula.

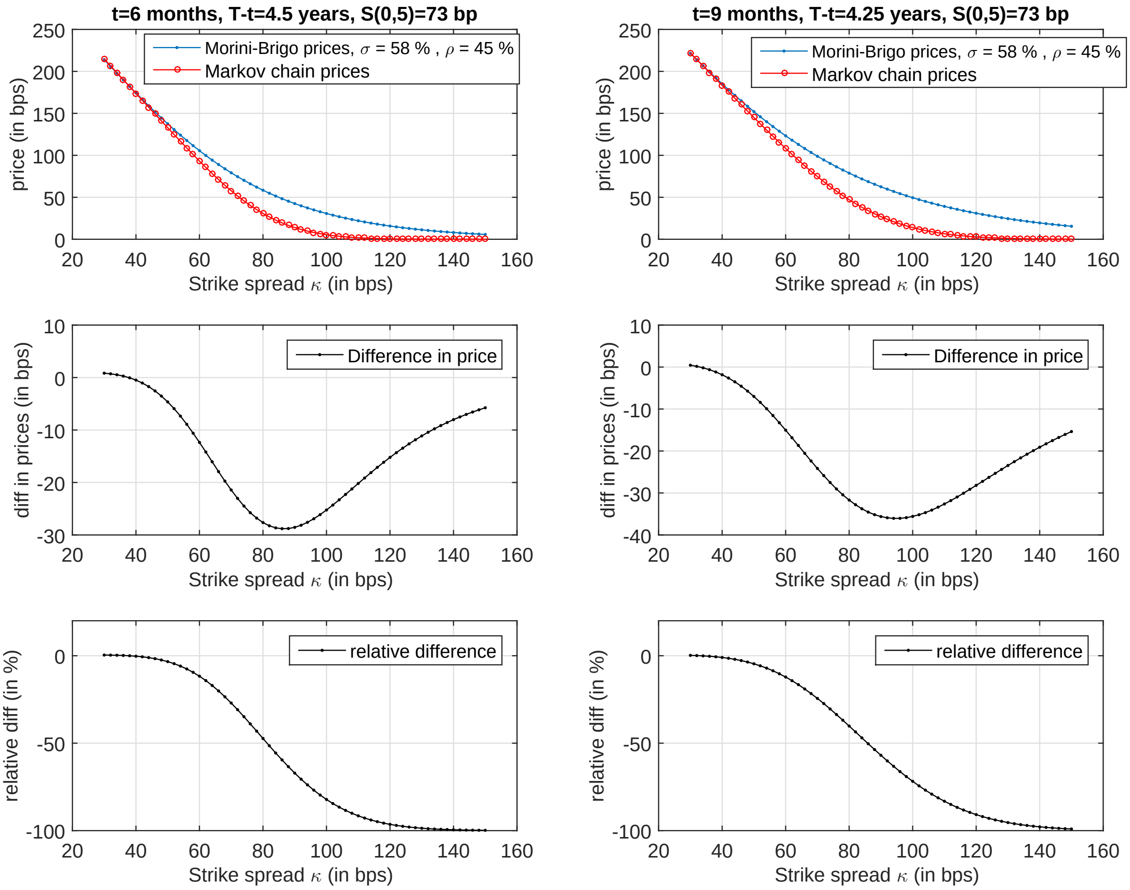

Figure 7. The Markov chain option prices and the Morini-Brigo prices for $t = 6$ month and $t = 9$ months where market spot spread is 73 bps. The parameters in the Markov model are given by Table 2 and $\hat{\sigma} = 58\%$ and $\rho = 45\%$ are used in the Morini-Brigo formula.

From Figure 6 and Figure 7 we see that the CDS-index option prices in the Markovian modell are able to produce prices that sometimes are quite close to the prices obtained in the log-normal model of Morini & Brigo (2011). However, recall that beyond convenience there is no justification for the lognormality assumption for the CDS-index spread. In particular, it is unclear if a dynamic model for the evolution of spreads and credit losses can be constructed that supports the lognormality assumption and the use of the Black formula, and there is no empirical justification for this assumption either.

It is of course interesting to find the implied volatilies of Markov prices via Morini-Brigo formula for the maturities studied in the above Figure 6 and Figure 7. More specific, given all other equal in the Markov chain model and the Morini-Brigo model (for a fixed correlation ρ), what volatility $\hat{\sigma}_{\mathrm{imp}}$ should we plug in the Morini-Brigo formula (2.4.19) in order to make the option prices agree in the two different models. The answer to this question is displayed in

Figure ?? and Figure 9. For example, in the case with $t = 1$ month we see in Figure ?? that the implied volatitiliy $\hat{\sigma}_{\text{imp}}$ should range between 40% to 250% in the Morini-Brigo model to obtain the corresponding option prices in the Markov chain. For $t = 3$ months $\hat{\sigma}_{\text{imp}}$ must lie between 20% to 120%. We also note that if we would decrease ρ all the values of $\hat{\sigma}_{\text{imp}}$ would increase even more. Comparing the values of $\hat{\sigma}_{\text{imp}}$ in Figure ?? and Figure 9 with the historical volatilities (10 days, 30 days ,60 days, and 90 days) on iTraxx Europe main 5 years retrieved from Bloomberg on July 5, 2018 via the Bloomberg built in function for historical volatilities, we see that the implied volatilities $\hat{\sigma}_{\text{imp}}$ are sometimes several factors bigger that the historical volatilities, even for the case $t = 6, t = 9$ months, see e.g. in Figure 10.

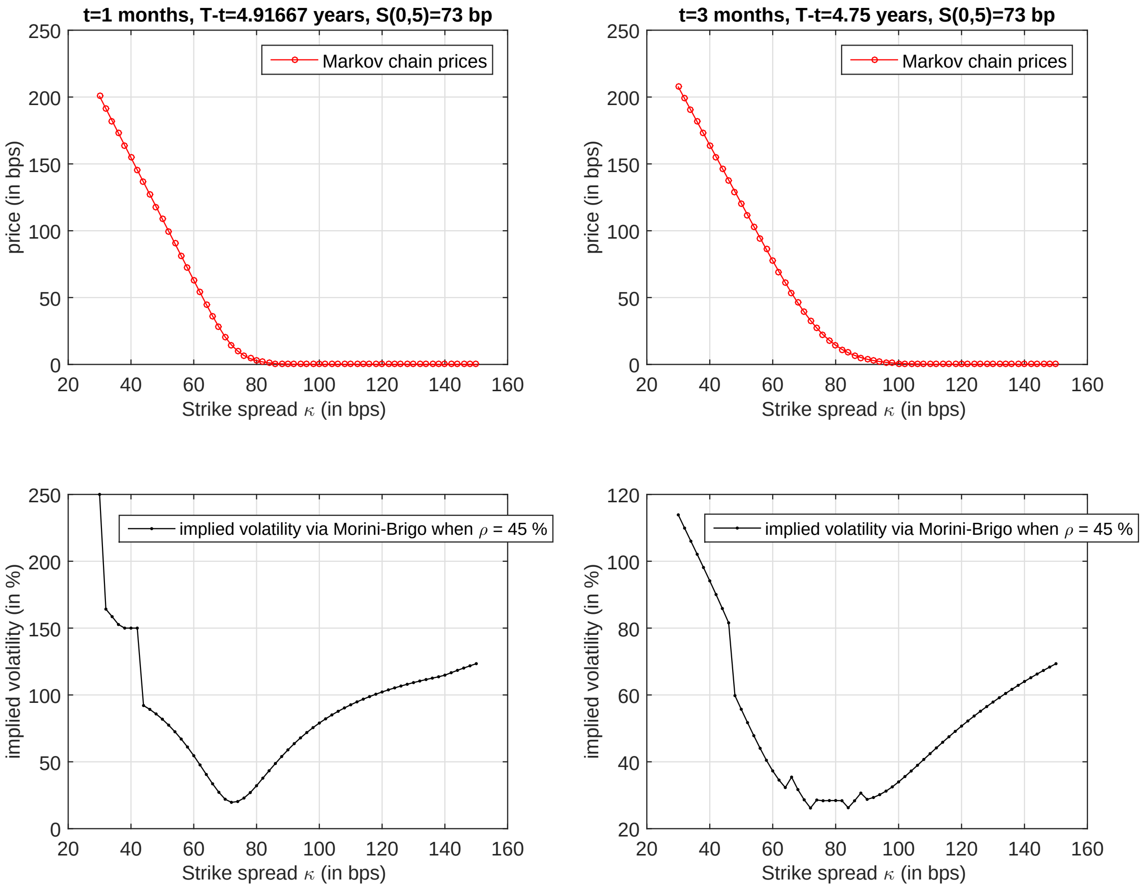

Figure 8. Implied volatilies of Markov prices via Morini-Brigo formula for $t = 1$ month and $t = 3$ months with the same parameters as in Figure 6.

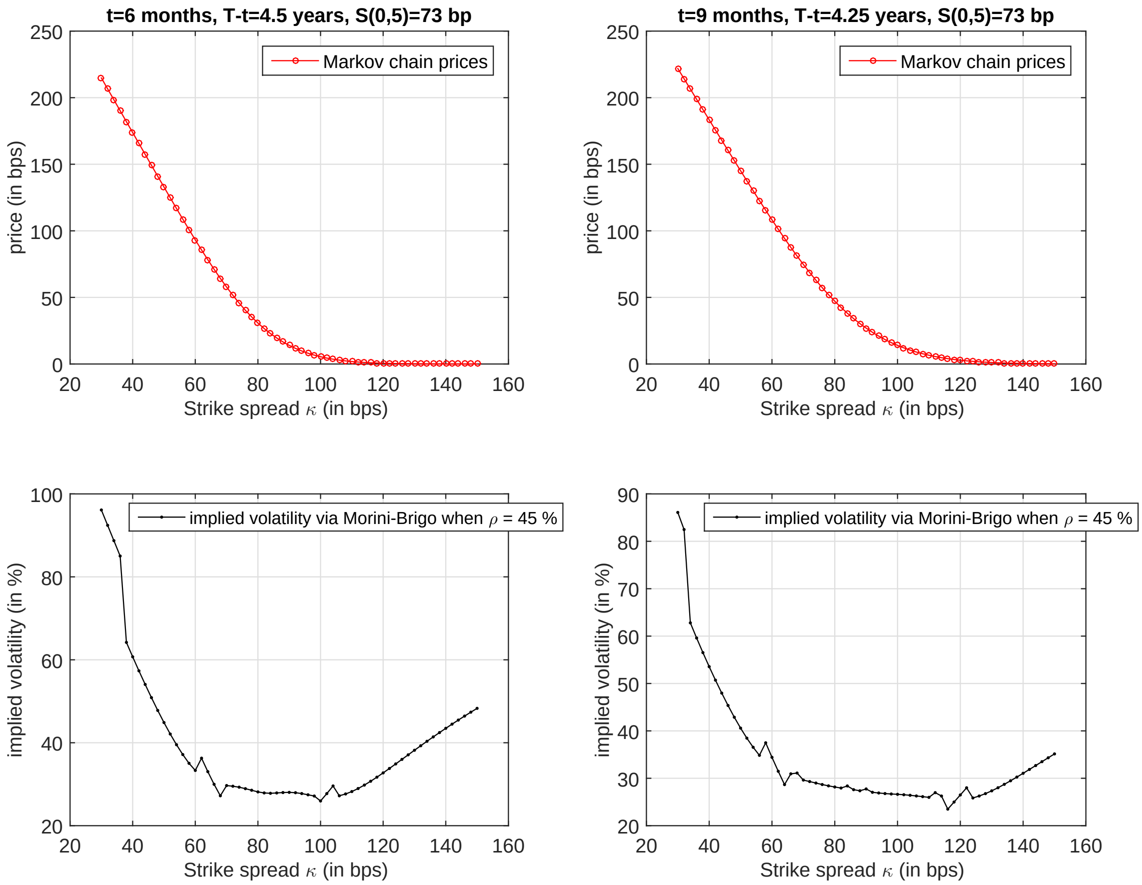

Figure 9. Implied volatilities of Markov prices via Morini-Brigo formula for $t = 6$ and $t = 9$ months with the same parameters as in Figure 6.

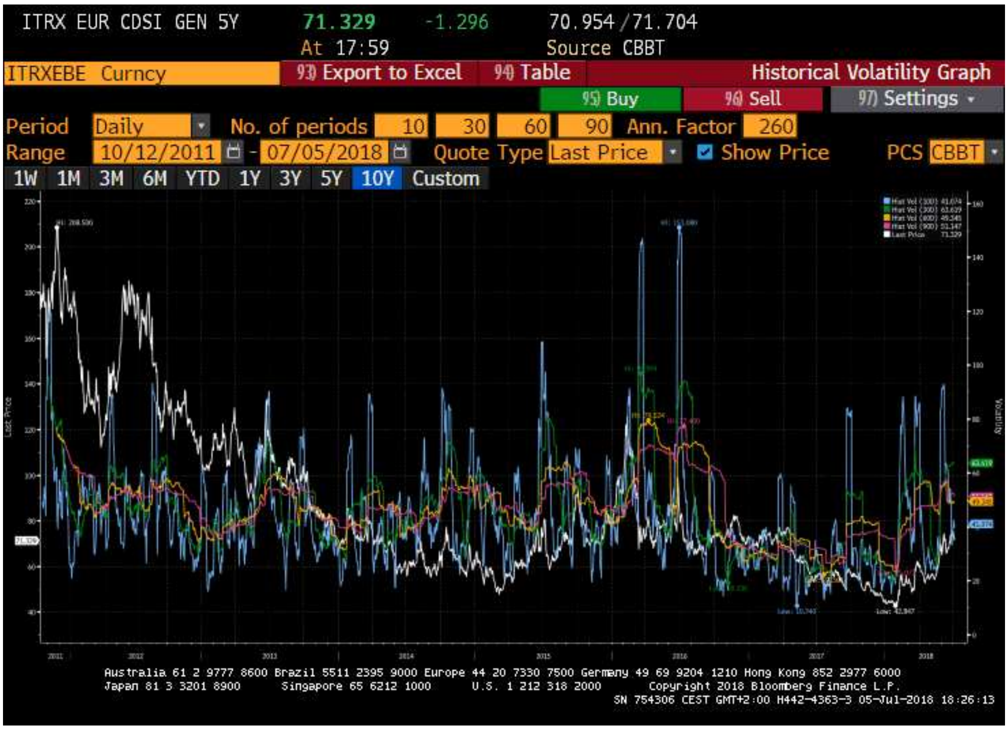

Figure 10. Historical volatilities (10 days, 30 days ,60 days, and 90 days) on iTraxx Europe main 5 years retrieved from Bloomberg on July 5, 2018 via the Bloomberg built in function for historical volatilities and source CBBT.

Recall that

$$C_0(t,T;\kappa) = e^{-rt}\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t<m\}}\right] + (1-\phi)e^{-rt}\mathbb{Q}\left[N_t = m\right]$$

and $\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t<m\}}\right] = \sum_{j=0}^{m-1}\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t=j\}}\right]$. In Figure 11 we plot $e^{-rt}\mathbb{E}\left[\Pi(t,T;\kappa)1_{\{N_t=j\}}\right]$ for $j = 0,1,2,\ldots,7$ as well as the Morini-Brigo prices. The parameters used to generate Figure 11 are given by Table 2 and thus the same as in Figure 6 and 7.

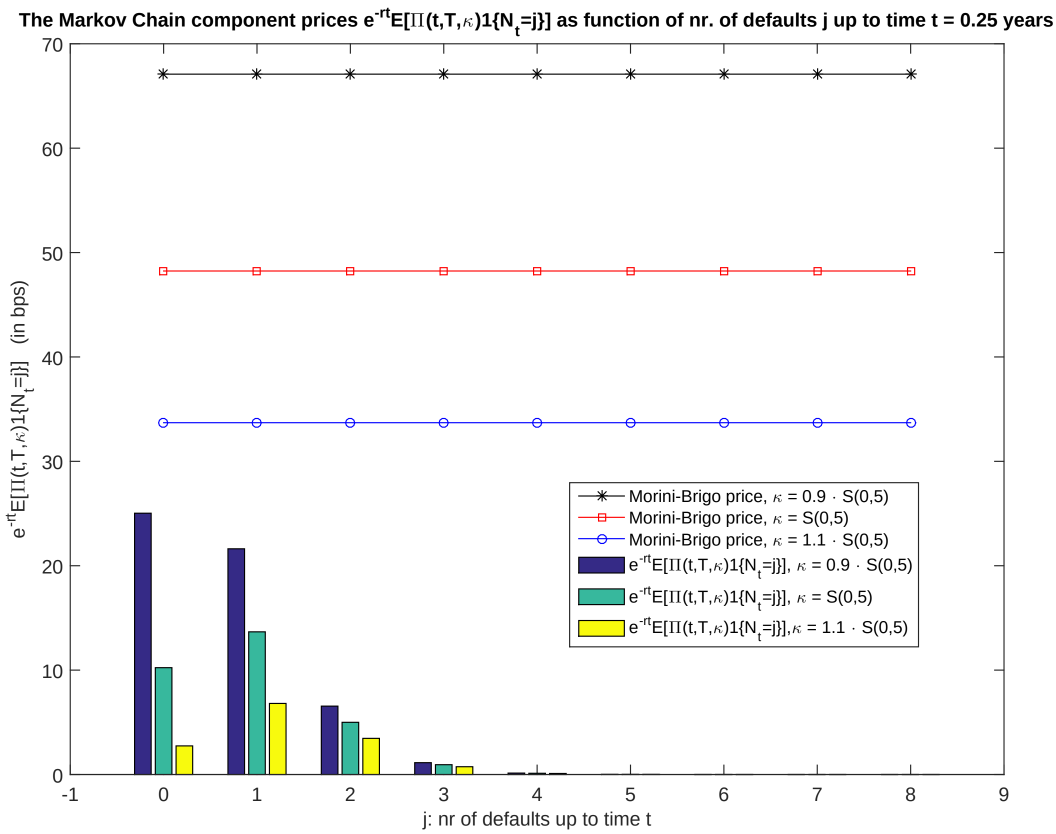

Figure 11. Markov option prices componentwise vs Morini-Brigo price where parameters for the Markov model is given Table 2 and parameters for Morini-Brigo are same as in Figure 6

6.3. **Other quantities in the Markovian model.** Given our calibrated Markov model with parameters specified as in Table 2 we can study the model in more detail. This might help us to understand the option prices etc. displayed in the previous subsection.

First, Figure 12 displays the outcomes for the "state-space" spreads $\{S_k(t,T)\}$ for $t = 0, 1, 3, 6,$ and $t = 9$ months when computed as in (6.1.8) with parameters given by by Table 2. As seen in Figure 12 $\{S_k(t,T)\}$ is spread out nicely covering a reasonable interval given historical outcomes for CDS index spreads on iTraxx Europe main.

Furthermore, Figure 13 shows the distribution $\mathbb{Q}\left[X_t = k\right]$ for the Markov chain X_t for same t values as used in Figure 6.

Finally, we can combine the probabilities $\mathbb{Q}\left[X_t = k\right]$ and the outcomes $\{S_k(t,T)\}$ in order to get the probability distribution for $\{S_k(t,T)\}$ at $t = 1, 3, 6,$ and $t = 9$ months, which is displayed in Figure 14. Note that the values of the horizontal bars in Figure 14 are the same as in Figure 13 while the values on the x-axes in Figure 14 are given by Figure 12 for proper choices of t. From Figure 14 we clearly see that the probability of having spread values $\{S_k(t,T)\}$ larger than e.g. 110 bps for $t = 1, 3, 6,$ or $t = 9$ months is essentially zero. This might explain the lower option prices for the Markov model (compared with the Morini-Brigo model) which are displayed in Figure 6 and 7. Finally, Figure 15 shows the individual default intensities $\lambda(k)$ as function of the state space variable k (lower plot) as well as the spot CDS index $S_k(0,T)$ (upper plot) as function of k.

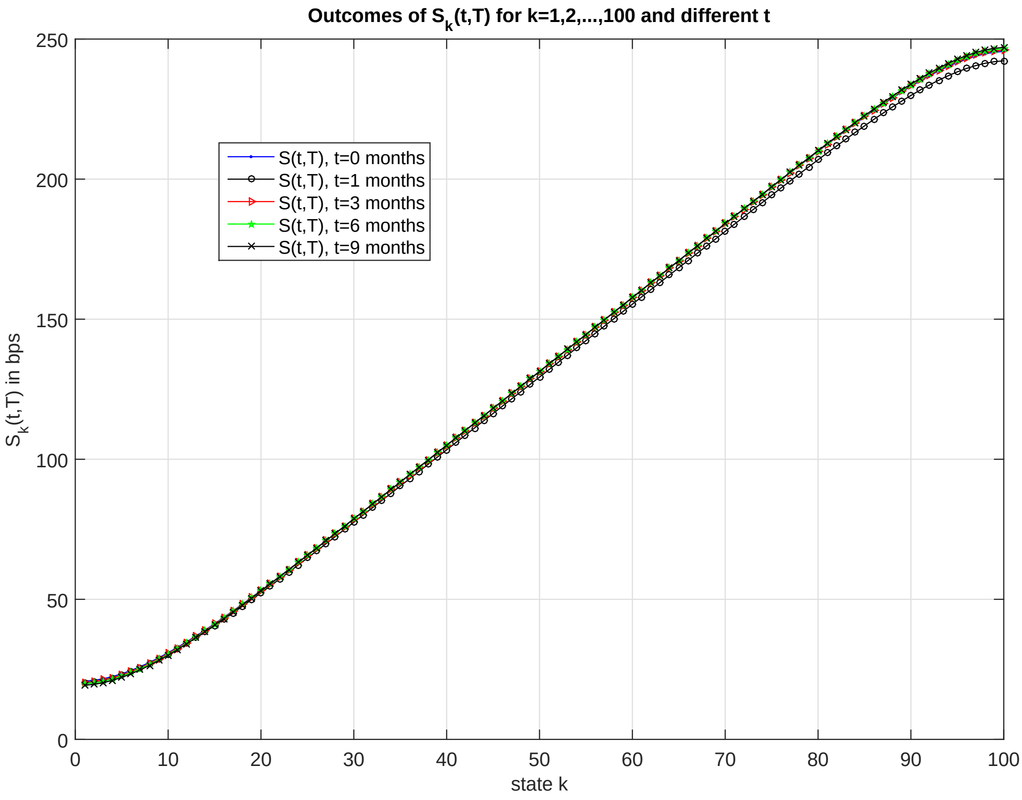

Figure 12. The "state-space" spreads $\{S_k(t,T)\}$ for $t = 0, 1, 3, 6$, and $t = 9$ months for the Markov model with parameters given by Table 2.

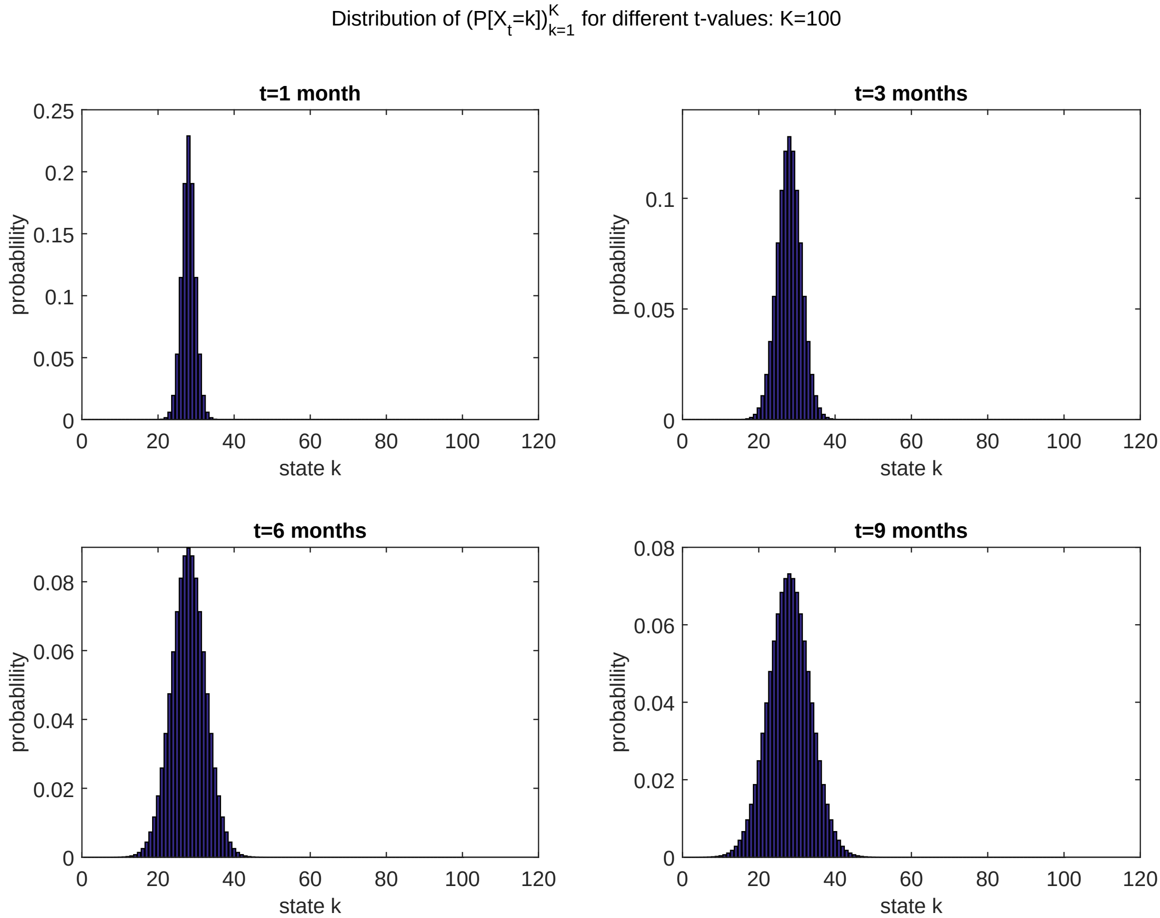

Figure 13. The distribution $\mathbb{Q}[X_t = k]$ for $t = 1, 3, 6$, and $t = 9$ months for the Markov model with parameters given by Table 2.

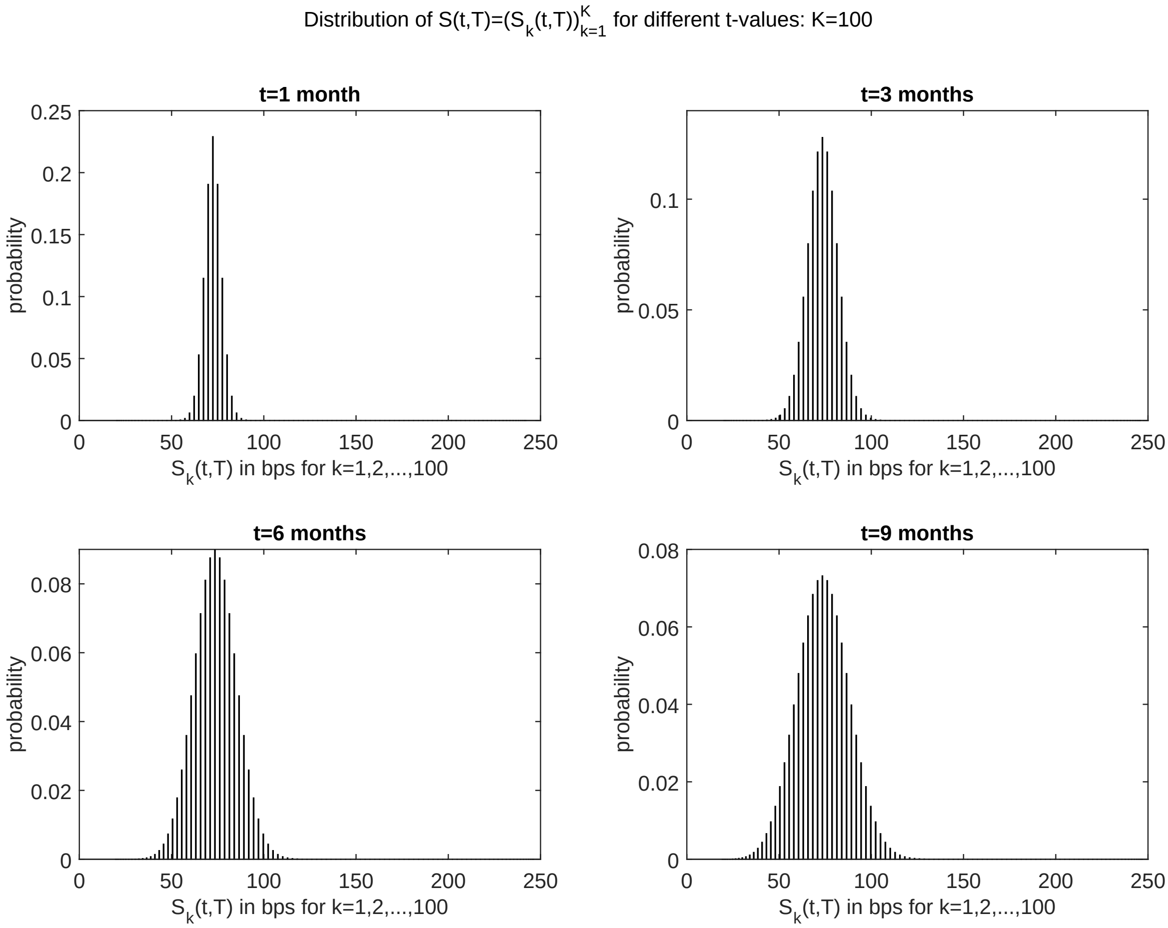

Figure 14. The distribution for the spreads $\{S_k(t,T)\}$ at $t = 1, 3, 6$, and $t = 9$ months for the Markov model with parameters given by Table 2.

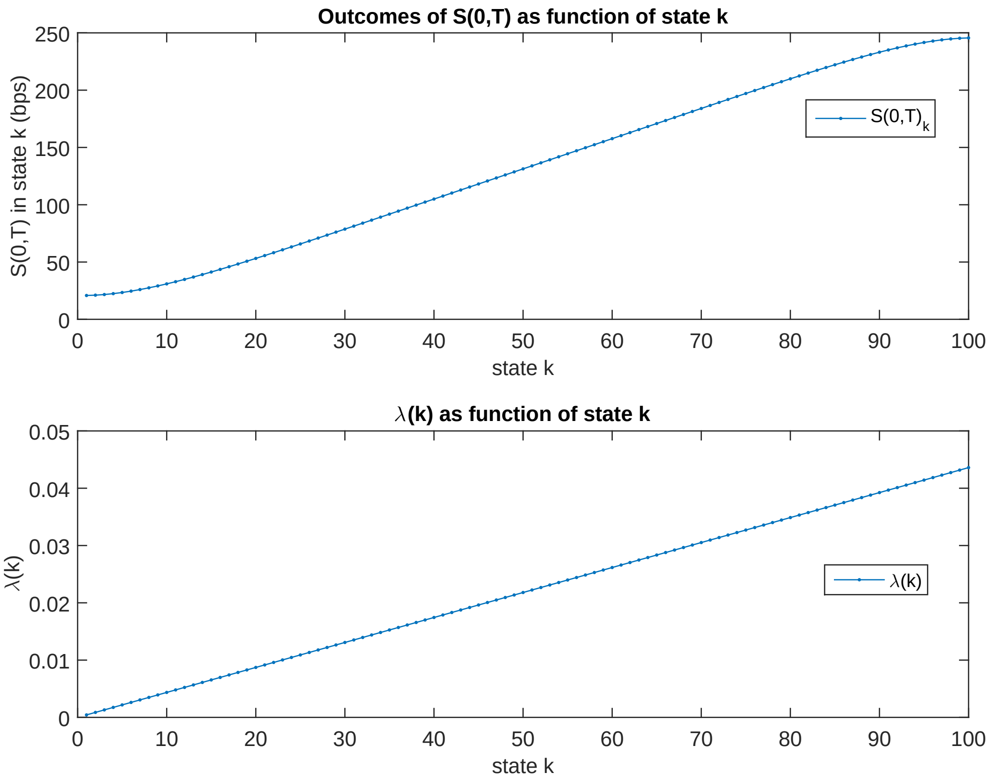

Figure 15. The default intensity $\lambda(k)$ (lower plot) as function of k and the spreads $\{S_k(0,5)\}$ as function of k (upper plot).

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Appendix A. Feyman-Kac formulas for finite-state Markov chains

The purpose of this section is to introduce some useful formulas, that will be used throughout this paper. We first introduce some notation. Let $S^X = \{1, 2, \dots, K\}$ and consider a functions $f(x) : S^X \mapsto \mathbb{R}$, then we denote $\boldsymbol{f}$ a $\mathbb{R}^K$-column vector with $\boldsymbol{f}_j = f(j)$ for $j \in S^X$. Next, let X_t be a finite state Markov jump process on $S^X = \{1, 2, \dots, K\}$ with generator $\boldsymbol{Q}$. Then, for a function $\lambda(x) : S^X \mapsto \mathbb{R}$ we denote the matrix $\boldsymbol{Q}_\lambda = \boldsymbol{Q} - \boldsymbol{I}_\lambda$ where $\boldsymbol{I}_\lambda$ is a diagonal-matrix such that $(\boldsymbol{I}_\lambda)_{k,k} = \lambda(k)$ and $\boldsymbol{e}_x$ is a row vector in $\mathbb{R}^K$ where the entry at position x is 1 and the others entries are zero.

We now state with the following result.

Theorem A.1. *Let X_t be a finite state Markov jump process on $S^X = \{1, 2, \dots, K\}$ with generator $\boldsymbol{Q}$. Consider functions $\lambda(x), f(x) : S^X \mapsto \mathbb{R}$. Then, with notation as above*

$$\mathbb{E}\left[e^{-\int_0^t \lambda(X_s)ds} f(X_t) \,\middle|\, X_0 = x \right] = \boldsymbol{e}_x e^{\boldsymbol{Q}_\lambda (T-t)} \boldsymbol{f}. \tag{A.1}$$

A proof of Proposition A.1 can be found on pp.273-274 in Rogers & Williams (2000). It is easy to extend Theorem A.1 to yield the following equality, for $T \geq t$

$$\mathbb{E}\left[e^{-\int_t^T \lambda(X_T)ds} f(X_T) \,\middle|\, X_t = x \right] = \boldsymbol{e}_x e^{\boldsymbol{Q}_\lambda (T-t)} \boldsymbol{f} \tag{A.2}$$

where the rest of the notation are as in Theorem A.1. The main point in Theorem A.1 is that given the matrix $\boldsymbol{Q}_\lambda$, then the left-hand side in Equation (A.1) (and Equation (A.2) is straightforward to implement using standard mathematical software.

We note that Theorem A.1 does not hold if the functions λ, f also depend on time t, i.e $\lambda(t, x), f(t, x) : [0, \infty) \times S^X \mapsto \mathbb{R}$. In such cases, one generally has to rely on numerical ODE method in order to find the quantity $\mathbb{E}\left[e^{-\int_0^t \lambda(s, X_s)ds} f(t, X_t) \,\middle|\, X_0 = x \right]$.

Appendix B. Derivation of the credit triangle

The purpose of this section is to derive the relation

$$\lambda = \frac{S(\bar{T})}{1 - \phi} \tag{B.1}$$

where $S(\bar{T})$ is the $\bar{T}$-year CDS index spread for a homogeneous credit portfolio where the default times $\{\tau_i\}$ have constant default intensity λ which means that they are exponentially distributed with parameter λ, i.e. if τ has the same distribution as $\{\tau_i\}$ then

$$\mathbb{Q}\left[\tau \leq t\right] = 1 - e^{-\lambda t}. \tag{B.2}$$

The existing proofs of (B.1) found in the credit literature are only done for the unrealistic case when the CDS premium is paid continuously. In practice the CDS premiums are done quarterly. Furthermore, formula (B.1) is used repeatedly in portfolio credit risk, see e.g. Equation (9.11) on p.404 in McNeil et al. (2005). Below, we will for notational convenience write T instead of $\bar{T}$.

Proposition B.1. *Consider a CDS index with maturity T on a homogeneous credit portfolio where the obligors have constant default intensity λ and where the interest rate is r. Then,*

$$S(T) = 4(1-\phi)\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)\frac{\lambda}{\lambda + r}\frac{1 - e^{-(r+\lambda)T}}{e^{-\frac{r+\lambda}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil + 1)}{4}}} \tag{B.3}$$

and if $r + \lambda$ is small it holds that

$$\lambda \approx \frac{S(T)}{1-\phi}. \tag{B.4}$$

Proof. First recall that $S(T)$ is shorthand notation for $S(0,T)$ and using the definition of $S(0,T)$ in Equation (2.2.6) we have that

$$S(T) = S(0,T) = \frac{DL(0,T)}{PV(0,T)} \tag{B.5}$$

where

$$DL(0,T) = \mathbb{E}\left[\int_0^T e^{-rs} dL_s\right] = (1-\phi)\int_0^T e^{-rs} f_\tau(s)ds = \frac{(1-\phi)\lambda}{\lambda + r}\left(1 - e^{-(r+\lambda)T}\right) \tag{B.6}$$

and the last two equations in (B.6) follows from (B.2) and similar computations as in Equation (2.4.31) and (2.4.33) in Propositon 2.3 with $t = 0$. Furthermore, (2.2.5) implies that

$$PV(0,T) = \frac{1}{4}\sum_{n=1}^{\lceil 4T\rceil} e^{-rt_n}\left(1 - \frac{1}{m}\mathbb{E}\left[N_{t_n}\right]\right) \tag{B.7}$$

where $t_n = \frac{n}{4}$ and which after identical computations as in (2.4.30) with $t = 0$, renders that

$$PV(0,T) = \frac{e^{-\frac{(r+\lambda)}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil + 1)}{4}}}{4\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)}. \tag{B.8}$$

Hence, (B.6) and (B.8) in (B.5) then gives

$$S(T) = 4(1-\phi)\left(1 - e^{-\frac{(r+\lambda)}{4}}\right)\frac{\lambda}{\lambda + r}\frac{1 - e^{-(r+\lambda)T}}{e^{-\frac{r+\lambda}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil + 1)}{4}}} \tag{B.9}$$

which proves (B.3). Next, if r and λ are small we can use the following first order Taylor expansion

$$e^{-\frac{r+\lambda}{4}} \approx 1 - \frac{r+\lambda}{4} \tag{B.10}$$

which renders

$$4(1-\phi)\left(1 - e^{-\frac{r+\lambda}{4}}\right)\frac{\lambda}{\lambda + r} \approx (1-\phi)\lambda \tag{B.11}$$

and

$$\frac{1 - e^{-(r+\lambda)T}}{e^{-\frac{r+\lambda}{4}} - e^{-\frac{(r+\lambda)(\lceil 4T\rceil + 1)}{4}}} \approx \frac{1 - e^{-(r+\lambda)T}}{1 - e^{-\frac{(r+\lambda)(\lceil 4T\rceil + 1)}{4}} - \frac{r+\lambda}{4}} \approx 1 \tag{B.12}$$

since $\frac{\lceil 4T\rceil+1}{4} \approx T$ and $\frac{r+\lambda}{4}$ is small compared to $1-e^{-\frac{(r+\lambda)(\lceil 4T\rceil+1)}{4}}$ when T is larger (typically $T=5$ or $T=10$). Hence, under (B.10) the approximations (B.11)-(B.12) inserted in (B.9) then imply that

$$S(T) \approx (1-\phi)\lambda$$

that is

$$\lambda = \frac{S(T)}{1-\phi}$$

which proves (B.4). $\square$

We here remark that if one in the single-name CDS spread assumes that the default payment in the default leg is postponed to the end of the quarter in which the default happens, then, assuming (B.2), one can prove (B.4) for a general interest rate which not necessary have to be constant. By Lemma 6.1, p.1203 in Herbertsson et al. (2011) this will therefore also hold for a CDS-index.

In a perfectly calibrated model we have by definition that $S_{\mathcal{M}}(T) = S(0,T)$ which in (B.4) can be used to find a numerical value for λ given that the recovery ϕ is known.

(Alexander Herbertsson), Centre For Finance, Department of Economics, School of Business, Economics and Law, University of Gothenburg, P.O Box 640, SE-405 30 Göteborg, Sweden

E-mail address: alexander.herbertsson@cff.gu.se