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Value at Risk and Expected Shortfall risk measures using Extreme Value Theory

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Executive Summary

Title	Value at Risk and Expected Shortfall risk measures using Extreme Value Theory
Subject	Master Thesis in Financial Economics (15 credits) at the School of Business, Economics and Law, Gothenburg University
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Key Words	Extreme Value Theory, Generalized Pareto Distribution, Point-Over-Threshold method, risk measures, Value at Risk, Expected Shortfall
Problem	Calculating risk measures as Value at Risk (VaR) and Expected Shortfall (ES) has become popular for institutions and agents in financial markets. A main drawback with these risk measures is that they traditionally assume a specific distribution, as the Normal distribution or the Student's t distribution. When using Extreme Value Theory (EVT) no assumption of the underlying distribution is necessary as the extreme tails can approximately be described by the Generalized Pareto Distribution. How can EVT be used to calculate VaR and ES for a market index?
Purpose	The purpose of this study is to calculate VaR and ES risk measures for 10 market indices. The indices are the Stockholm stock exchange index (OMX30S), the Copenhagen stock exchange (OMXC20), the Helsinki stock exchange (OMXH25), the Deutscher Aktienindex (DAX), the Financial Times Stock Exchange (FTSE-100), the Dow Jones Industrial index (DJI), the Standard and Poor's 500 index (SPX), the NASDAQ-100 index (NDX), the Nikkei-225 stock average index (NKY) and the Bombay stock exchange sensitive index (SENSEX). The purpose is also to find which of these indices are exposed to most extreme losses.
Method	Historical data consisting of daily closing prices were collected from Bloomberg for 10 market indices. These data were then processed in Matlab 7.7.0 (R2008b) using Extreme Value Theory to find VaR and ES risk measures. The risk measures were compared to find out which of the indices was exposed to most extreme loss.
Results	This study has examined the VaR and ES risk measures on 10 market indices. The results show that in terms of VaR and ES, NASDAQ is most exposed to extreme losses. VaR and ES equals 5.340% and 7.002% respectively for the left tail and 5.128% and 7.091 respectively for the right tail.

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1 Introduction

The paragraphs in the introduction section give a brief description of the background, problem definition, delimitations and purpose of this study. The disposition of this report is also presented.

1.1 Background

Managing financial risk for institutions and agents in financial markets has not only become increasingly popular during the past decades but has also been a necessity due to regulatory authorities (Basel Committee in Banking Supervision, 1996).

Financial instruments are today more exposed to volatility when global markets are connected to each other and tend to shift rapidly. Financial instruments are also getting more complex and harder to understand. Therefore institutions and agents in financial markets are paying more interest to the actual risk involved and not only the payoff (Simons, 2000). According to Jorion (2001) risk is defined as "the volatility of unexpected outcomes". Thus, using a model for these outcomes is both valuable and necessary for risk management.

In the field of risk management the focus is starting to shift from not only model the volatility but also to model the extreme events of losses that might occur. Such events are rare but when they happen they often lead to financial stress and high bankruptcy costs. There are some classic examples of catastrophic incidents concerning extreme events (Smith, 2000).

- Barings bank in February 1995 lost about $1.3 billion due to illegal trading activity from a single trader, Nick Leeson. The bank eventually went bankrupt and was sold for one pound.

- Orange County in 1994 lost approximately $1.1 billion due to investment strategies from the treasurer, Robert Citron. Investments were made in a series of derivative instruments tied to interest rates. Interest rates eventually rose, executing the loss.

- Daiwa Bank in July 1995 lost roughly $1.1 billion due to a single trader, Toshihide Iguchi, during 11 years of trading. Losses became clear when Iguchi confessed to his managers.

- Long Term Capital Management nearly collapsed in September 1998. They were trading a complex mixture of derivatives which gave an exposure to market risk of about $200 million.

Risk measures such as Value at Risk (VaR) and Expected Shortfall (ES) have been developed to deal with financial risk. But at times these risk measures, when used traditionally, lack in efficiency and confidence as they are based on the assumption of a specific distribution. These distributions are traditionally the Normal distribution or the Student's t distribution.

A statistical branch has been developed during the years called Extreme Value Theory (EVT). The mathematical foundation was first derived by Fisher (1928), Gnedenko (1943) and further processed by Gumbel (1958). Its application started in hydrology where the flooding of rivers was a concern (Embrechts, Klüppelberg and Mikosch, 1997) but spread into the field of insurance and later to financial risk management. The central result in EVT states that the distribution of the data doesn't need to be assumed. Basically the extreme tails of a wide range of distributions (including the Normal distribution and the Student's t distribution) can approximately be described by the Generalized Pareto Distribution (GPD) (Christoffersen, 2005).

EVT has also been proven useful, not only for the ease to which VaR and ES can be estimated from the GPD parameters, but also because important risk management questions can be answered. These questions are basically:

- What is the expected loss of an instrument?
- If there is a loss, how much will be lost?
- Are there still worse losses to come according to the data?

1.2 Purpose

There are three purposes in this study. Firstly, the purpose of this study is to model the extreme losses from returns on 10 different stock indices using Extreme Value Theory (EVT). The studied indices are OMXS30, OMXC20, OMXH25, DAX, FTSE 100, Dow Jones, S&P500, NASDAQ-100, Nikkei-225 and SENSEX. Secondly, the aim is to use the extreme losses found to calculate Value at Risk (VaR) and Expected Shortfall (ES) risk measures used for risk management purposes. Finally, the third purpose is to see which of these indices are most exposed to extreme losses.

1.3 Problem definition

There are numerous ways to calculate Value at Risk (VaR) and Expected Shortfall (ES) risk measures. How can these risk measures be calculated using a statistical branch called Extreme Value Theory (EVT)? Which of the stock indices in this study are most exposed to extreme losses according to VaR and ES?

1.4 Delimitations

The study presented in this report is limited to calculations of Value at Risk (VaR) and Expected Shortfall (ES) risk measures on the Stockholm stock exchange index (OMX30S), the Copenhagen stock exchange (OMXC20), the Helsinki stock exchange (OMXH25), the Deutscher Aktienindex (DAX), the Financial Times Stock Exchange (FTSE-100), the Dow Jones Industrial index (DJI), the Standard and Poor's 500 index (SPX), the NASDAQ-100 index (NDX), the Nikkei-225 stock average index (NKY) and the Bombay stock exchange sensitive index (SENSEX). Each risk measure is calculated using Extreme Value Theory (EVT) with the Point-Over-Threshold (POT) method. A confidence interval for the risk measures and EVT parameters are also presented.

The time period used for the data from the stock indices range from 5st of January 1970 to 20th of March 2009. Some indices however have shorter time spans. See Table 1 for further details.

1.5 Disposition

The report starts with chapter 1 introducing the subject, the purpose of the study, the problem definitions and the delimitations. It then continues with chapter 2 explaining the methodology and approach to the study. In chapter 3 the theoretical framework goes through the theories used to retrieve the results. Chapter 4 describes the data and the data processing methodology. The results found are then presented in chapter 5 followed by an analysis and discussion in chapter 6. Chapter 7 gives some final thoughts and summarises the study. Further research suggestions of subjects not covered in this study are presented in chapter 8. Finally, the report ends with a list of references used when writing this report.

2 Methodology

The paragraphs in the methodology section describe the type of study that is performed in this report. The approach to the study is also described.

2.1 Choice of methodology

There are traditionally two methods used when investigating problems similar to problems in this report. They are the qualitative and quantitative studies. When using a qualitative study a limited number of units are investigated to gain a deeper understanding of these units. A qualitative study most likely leads to a situation where the researcher's comprehension or interpretation of the information found serves as a basis for the results found in the study.

When using a quantitative study a vast number of units are studied with the purpose of gaining knowledge on a limited number of factors for each unit. Statistical analysis is most likely to take place of the data of interest such that different phenomenon can be explained using a selection of a certain population. A quantitative study can also be used to generalize and to represent other units in a similar population (Holme and Solvang, 1996).

The presented problems and the data used to find the results in this study use a quantitative methodology. It is possible to generalize the studied problems to similar populations (stock indices) and the results are based on statistical analysis of the collected data. A quantitative methodology to the study thus makes sense.

2.2 Approach to the study

In general there are two main approaches to data in a study. These approaches are the deductive approach and the inductive approach. When using a deductive approach the conclusions made for specific events are based on general principles and existing theories. Based on these known theories and principles hypothesis are derived and then further empirically tested for the cases studied. The choice and interpretation of data used are also influenced by the known theories and principles.

When using an inductive approach specific events are not derived from hypothesis derived from existing theories. The events are here studied without prior knowledge or influence of theories. A new theory is compiled and formulated using the results of the study (Davidsson, Patel, 1994).

The study presented in this report uses a deductive approach. Academic research in the field has been performed during several years investigating similar problems. Practices based on these investigations have been prepared and made accessible for the general public. A deductive approach in this case thus makes sense.

3 Theoretical framework

The paragraphs in the theoretical framework describe the theories used to form the results in this study.

3.1 Asset returns, volatility and standard deviation

The return of an asset today is defined as the difference between the closing price of the asset today and its closing price yesterday divided by yesterday's closing price.

$$R_{t+1} = \frac{S_{t+1} - S_t}{S_t} \tag{1}$$

The log return is used widely as the output of the calculation is unit free. Log returns are thus suitable for comparison with other unit free returns. The return or daily geometric return, also called "log" return can be defined as the change in the logarithm of daily closing prices of an asset.

$$R_{t+1} = \ln(S_{t+1}) - \ln(S_t) \tag{2}$$

The volatility is a measure of fluctuations in asset returns. An asset has a high volatility when the return fluctuates over a wide rage. When the return fluctuates over a small range the asset has a low volatility. Volatility can be seen as a risk measure or an uncertainty of asset return movements faced by participants in financial markets. The volatility is measured by the variance or the standard deviation and is a measure of the asset returns dispersion over a specified time period.

$$\sigma^2 = \frac{1}{N-1}\sum_{t=1}^{N}(R_t - \bar{R})^2 \tag{3}$$

The standard deviation is simply the square root of the variance.

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}(R_t - \bar{R})^2} \tag{4}$$

where:

σ^2	is the variance
σ	is the standard deviation
R_t	is the asset return at time t

$\bar{R}$ is the average return over the specified time period
N is the number of days for the specified time period

3.2 Value at Risk

Value at Risk (VaR) as a term was created by Till Guldimann, head of global research at J.P. Morgan in the late 1980s. In short the bank made a decision to aim for higher returns while still keeping control of the corresponding risks. By definition VaR summarizes the worst loss over a target horizon with a given level of confidence level (Jorion, 2001). A common target horizon used is a 1-day horizon and confidence levels range from 1 > q ≥ 0.95. The Basel II committee promotes a confidence level of 0.99 (Basel Committee in Banking Supervision, 1996). Thus a 0.99 confidence level is used for the VaR and ES risk measures in this study.

$$VaR_{t+1}^{q} = \mu_{t+1} + \sigma_{t+1} D_q^{-1} \tag{5}$$

where:

VaR_{t+1}^{q} is the VaR at time t+1 for a confidence level q
μ_{t+1} is the mean at time t+1
σ_{t+1} is the volatility at time t+1
D_q^{-1} is the quantile from the distribution D
q is the confidence level

A 1-day VaR with a 99% confidence level then states that there is a 1% chance of losing more than the VaR number itself during the next trading day under normal market conditions.

3.3 Expected Shortfall

A major drawback with VaR is that extreme losses are ignored. The VaR number only states the probability of losing more than the VaR itself but not the magnitude of the loss. A risk measure used to overcome this drawback is the Expected Shortfall (ES).

$$ES_{t+1}^{q} = -E_t\left[R_{t+1} \mid R_{t+1} < -VaR_{t+1}^{q}\right] \tag{6}$$

where:

ES_{t+1}^{q} is the ES at time t+1 for a confidence level q
VaR_{t+1}^{q} is the VaR at time t+1 for a confidence level q
R_{t+1} is the asset return at time t+1

A 1-day ES with a 99% confidence level then states what the magnitude of the loss is if VaR is exceeded during the next trading day under normal market conditions.

3.4 Extreme Value Theory

Extreme Value Theory (EVT) deals with modelling the extremes from i.e. a financial time series under an unknown distribution. EVT is also used in other scientific fields such as hydrology or insurance. By definition the extremes are present in the left and the right tail of the distribution. There are two kinds of models used to find these extremes called the Block Maxima (BM) model and the Peaks-Over-Threshold (POT) model. When these models are applied to the data the limiting distribution can be found. Consider identically and independently distributed observations $X_1, X_2, \ldots, X_n$ with an unknown underlying distribution function representing daily losses or returns. The left panel in Figure 1 considers the maximum value taken during a specified time period such as monthly or yearly. Variables X_2, X_5, X_7 and X_{11} correspond to the extreme events or block maxima in each period. The right panel in Figure 1 considers the observations exceeding a given threshold u. Variables X_1, X_2, X_7, X_8, X_9 and X_{11} corresponds to the extreme events over the selected threshold.

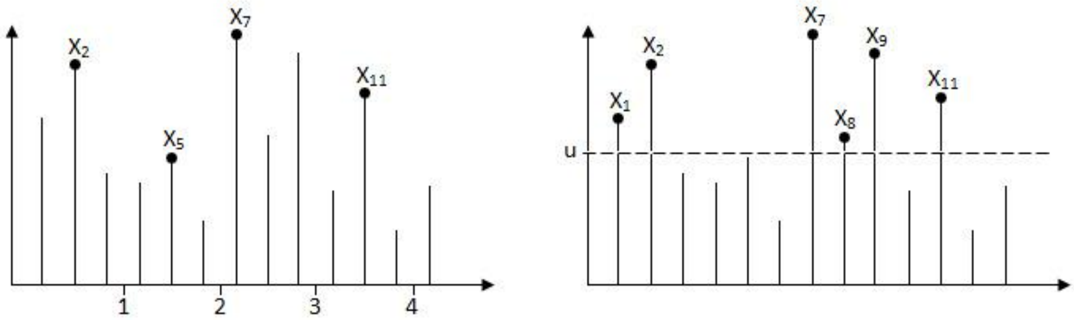

Figure 1 – Block-maxima (left panel) and excess over a threshold u (right panel).

The BM model is traditionally used to analyse data with seasonality i.e. hydrological data and is seen as less efficient for financial data compared to the POT model (Gilli and Këllezi, 2006). The POT model is therefore used in this study and described in more detail.

3.4.1 The Point-Over-Threshold model

The Point-Over-Threshold (POT) models basically consist of two types of analysis. There are the semi-parametric models using the Hill estimator and the fully parametric models based on the Generalized Pareto Distribution (GDP). Following the work of Nyström and Skoglund (2002) this study will be based on the fully parametric model as the semi-parametric model using the Hill estimator was found to be less efficient.

An excess distribution function F_u can be defined when considering the distribution of exceedances x over a certain threshold u (McNeil, Frey and Embrechts, 2005).

$$F_u(y) = P(X - u \leq y | X > u) \tag{7}$$

with $0 \leq y \leq x_F - u$

where:

$F_u(y)$	is the excess distribution function
X	is a random variable
u	is the threshold
y	are the exceedances $x - u$
x_F	is the right endpoint of F

F_u can be expressed in terms of F and the general definition of condition probability in equation (7).

$$F_u(y) = \frac{F(u+y) - F(u)}{1 - F_u} = \frac{F(x) - F(u)}{1 - F(u)} \tag{8}$$

The Pickland-Dalkema-de Haan (1974, 1975) theorem states that when the threshold u gets large under most distribution assumptions the limiting distribution F_u converges to the Generalized Pareto Distribution (GPD).

$$F_u(y) \approx G_{\xi,\beta}(y), \;\; u \to \infty \tag{9}$$

The GPD is general in the sense that it incorporates many distributions including the normal distribution and the Student-t distribution.

$$G_{\xi,\beta}(y) = \begin{cases} 1 - \; (1 + \xi y/\beta)^{-1/\xi} & if\; \xi \neq 0 \\ 1 - exp^{-y/\sigma} & if\; \xi = 0 \end{cases} \tag{10}$$

with β > 0, $y \geq u$ if $\xi \geq 0$ and $(u \leq y \leq u - \beta/\xi)$ if $\xi < 0$.

where:

$G_{\xi,\beta}(y)$	is the generalized Pareto distribution
y	are the exceedances $x - u$
ξ	is the shape parameter of the distribution
β	is the scaling parameter of the distribution

When the shape parameter ξ of the GPD is positive the distribution is heavy tailed as is also the case with the Student-t distribution. The normal distribution within the GPD has a shape parameter equal to zero.

3.4.2 Estimating VaR and ES

Instead of using equation (5) and (6) the VaR and ES risk measures can be derived directly from the GPD parameters (McNeil, Frey and Embrechts, 2005). The expression for VaR is found first by expressing the GPD as a function of x, $x = u + y$.

$$G_{\xi,\beta}(x) = 1 - (1 + \xi(x - u)/\beta)^{-1/\xi} \tag{11}$$

Second by using equation (8), replacing F_u with equation (11) and replacing $F(u)$ by an estimate $(n - N_u/n)$.

$$\hat{F}(x) = 1 - \frac{N_u}{n}\left(1 + \frac{\hat{\xi}}{\hat{\beta}}(x - u)\right)^{-1/\xi} \tag{12}$$

Third by inverting equation (12) for a given probability p.

$$\widehat{VaR}_p = u + \frac{\hat{\beta}}{\hat{\xi}}\left(\left(\frac{n}{N_u}p\right)^{-\xi} - 1\right) \tag{13}$$

where:

VaR_p	is the Value at Risk for a given probability p
u	is the threshold
$\hat{\xi}$	is the estimated shape parameter of the distribution
$\hat{\beta}$	is the estimated scaling parameter of the distribution
n	is the total number of observations
N_u	is the number of observations above the threshold u

The expression for the ES can be rewritten as in (14) where the second term to the right is the expected value of the exceedances over the threshold $\widehat{VaR}_p$ (Gilli and Këllezi, 2006).

$$\widehat{ES}_p = \widehat{VaR}_p + E\big(X - \widehat{VaR}_p | X > \widehat{VaR}_p\big) \qquad (14)$$

If the GPD tail parameter ξ < 1 the mean excess function can be expressed as in (15).

$$e(z) = E(X - z|X > z) = \frac{\beta + \xi z}{1 - \xi}, \quad \beta + \xi z > 0 \qquad (15)$$

The ES can then be derived from the GPD parameters using the definition of ES in (6), expression (14) for $z = \widehat{VaR}_p - u$ and letting X represent the excess y over the threshold u.

$$\widehat{ES}_p = \frac{\widehat{VaR}_p}{1 - \hat{\xi}} + \frac{\hat{\beta} - \hat{\xi} u}{1 - \hat{\xi}} \qquad (16)$$

where:

$\widehat{ES}_p$	is the Expected Shortfall for a given probability p
VaR_p	is the Value at Risk for a given probability p
u	is the threshold
$\hat{\xi}$	is the estimated shape parameter of the distribution
$\hat{\beta}$	is the estimated scaling parameter of the distribution

3.4.3 Selection of the threshold u

Selection of the threshold u can be a difficult task. If the threshold is set to high then there are very few parameters left in the tail making the estimation of the GPD parameters uncertain. On the other hand if the threshold is set to low the EVT theory may not hold meaning that observations above the threshold don’t conform to the GPD (Christoffersen, 2003). As to date there is no algorithm available for an automatic and satisfactory selection of the threshold (Gilli and Këllezi, 2006). The threshold selection process can be performed with a graphical tool called the sample mean excess plot shown in Figure 2 and Figure 3. The sample points are defined in (17) and (18).

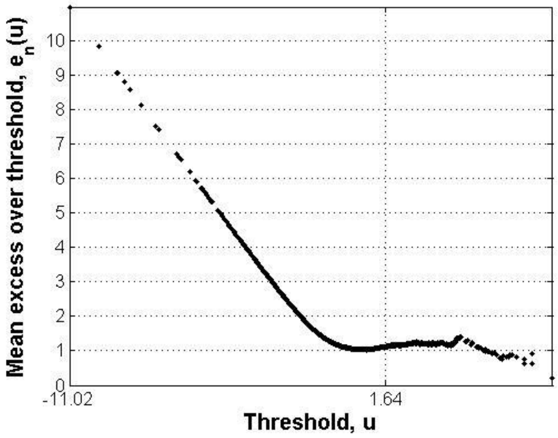

Figure 2 – Sample mean excess plot for the left tail of the OMXS30 index

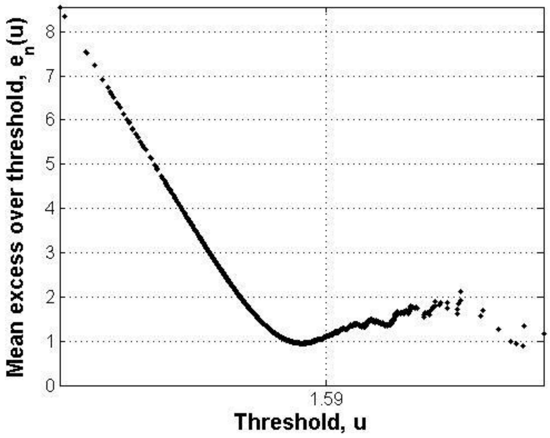

Figure 3 – Sample mean excess plot for the right tail of the OMXS30 index

$$\big(u, e_n(u)\big), \quad x_1^n < x_n^n \tag{17}$$

$$e_n(u) = \frac{\sum_{i=k}^{n}(x_i^n - u)}{n-k+1}, \quad k = min\{i|x_i^n > n\} \tag{18}$$

where:

u	is the threshold
x	is the exceedance observation
$e_n(u)$	is the sample mean excess function
$n-k+1$	is the number of observations exceeding the threshold u

If the data support a GPD model the mean excess function $e_n(u)$ should become increasingly linear for higher values of u. A linear upward trend indicates a GPD model with positive shape parameter ξ, which is the case for heavy tailed distributions. If the plot trends to the horizontal the GPD shape parameter is roughly zero, which is the case for the normal distribution. A downward trend in the GPD shape parameter indicates a negative shape parameter, which is the case for short tailed distributions (McNeil, Frey and Embrechts, 2005). The threshold should thus be set where the plot is linear. Interpreting the sample mean excess plot can however be a subjective task as the mean excess plot can have several linear parts. Another way is to simply set the threshold to a fixed percentage of the distribution.

Following the work of Nyström and Skoglund (2002) the threshold is set to be at the lowest (highest for the right tail) 10% of the distribution. But a threshold of 5% or 15% could equally have been chosen. The impact on the risk measures using different thresholds are shown in Table 2.

3.4.4 EVT Maximum-Likelihood parameter estimation

The risk measures described in previous sections contain unknown parameters if derived from the GPD. These parameters must be estimated in order to fit the GPD to the tails of the distributions from the 10 indices in this study. The method used to find the unknown parameters is based on Maximum Likelihood Estimation (MLE).

For a sample $y = \{y_1, \dots, y_n\}$ the log likelihood function for the GPD is the logarithm of the joint density of the n observations (Gilli and Këllezi, 2006).

$$L(\xi, \beta | y) = \begin{cases} -n \log \beta - \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{n} \log\left(1 + \frac{1}{\xi} y_i\right) & if\ \xi \neq 0 \\ -n \log \beta - \frac{1}{\beta} \sum_{i=1}^{n} y_i & if\ \xi = 0 \end{cases} \tag{19}$$

4 Data and data processing methodology

The paragraphs in the data and data processing methodology section describe the data and the methods used in this study. Data can be divided in primary data and secondary data when used for research purposes. Primary data is data that researchers gather them self's through interviews or observations during the work of the study. Secondary data is data that have already been found by others and that can be extracted from databases, books or journals etc. (Holme and Solvang, 1996). This report only uses secondary data.

4.1 Description of the data

The data consists of daily prices from 10 different stock indices shown in Table 1. Each index lists the most valued companies, which are different in number depending on the index. As an example the OMXS30 index list the 30 most valued companies on the Stockholm stock exchange and the FTSE-100 index consists of the 100 most valued companies on the Financial Times stock exchange. The time period for each index together with the number of sample points are shown in Table 1. There is a difference between the number of sample points (observations) for each index due to different number of holidays and different start and end dates. The data have been retrieved using the Bloomberg databases.

Symbol	Index name	Start	End	Observations
OMXS30	OMX Stockholm 30 Index	02-01-87	20-03-09	5573
OMXC20	OMX Copenhagen 20 Index	02-01-90	20-03-09	4817
OMXH25	OMX Helsinki 25 Index	05-01-87	20-03-09	5571
DAX	Deutscher 30 Aktienindex	05-01-70	20-03-09	9853
UKX	FTSE-100 Index	04-01-84	20-03-09	6378
DJI	Dow Jones Industrial Index	05-01-70	20-03-09	9914
SPX	Standard and Poor's 500 Index	05-01-70	20-03-09	9898
NDX	NASDAQ-100 Index	05-01-87	20-03-09	5602
NKY	Nikkei-225 Stock Average Index	06-01-70	19-03-09	9678
SENSEX	The Bombay Stock Exchange Sensitive Index	04-01-84	20-03-09	5803

Table 1 – Data for the 10 indices analysed in this study

4.2 Criticism of the sources

The quality of the gathered data is very important in order to prevent biased results. It is therefore sound to treat the data with some scepticism. Typical questions to consider are the purpose of the data, when and where the data was gathered or even why the data exists. Further questions to consider are the circumstances when the data was collected, who the originator of the data is and its relation to the data (Davidson and Patel, 1994).

The data used for this study is gathered by Bloomberg and used by many players in the financial markets. There is no reason to think that this data is not correct or biased in any way for purposes only known by Bloomberg.

4.3 Validity and reliability

When researching a topic it is important that the result reflects the questions asked about the topic. Validity thus ensures that this is fulfilled. Reliability deals with the consistency of a number of measurements and is determined based on the processing of the data. Reliability can be increased if the data is treated carefully to avoid errors. Validity implies reliability, but reliability does not imply validity (Holme and Solvang, 1996).

Bloomberg provides data to professional players in financial markets all over the world. They have processes and rules that ensure correct data in their databases. If this were not the case they could not charge users for their services. The data is thus seen as reliable. Methods and calculations used in this study are presented in such a way that they could be replicated. A description and definition of what this study shall find is also provided. The study can then be seen as valid.

4.4 Data processing

The daily closing prices from the 10 indices were used to calculate other necessary data in order to further study the VaR and ES risk measures. All closing prices were transformed to returns using equation (2). From these returns two new return series for each index were created consisting of the returns corresponding to the lower (left) and upper (right) 10% of the distribution. Although for the lower tail of the distribution the returns are multiplied by -1 to make them positive. The excess distribution function F_u defined in (7) could then be created and the GPD parameters could be estimated. For each tail a sample mean excess plot was calculated and the GPD was fitted to the exceedances above the selected threshold. Finaly, the VaR and ES risk measures were calculaed with the estimated GPD parameters using equation (13) and (16). All calculations were performed in the Matlab 7.7.0 (R2008b) environment from MathWorks. To summarise, the steps performed to find the risk measures involved the following:

- Make returns of the closing prices for each index
- Define the threshold for each tail
- Make new return series with exeedances above the threshold for each tail
- Estimate GPD parameters using maximum likelihood estimation
- Calculate VaR and ES risk measures using the estimated GPD parameters

5 Results

The data in the result paragraphs describe the results found on empirical data. First, a comparison of the impact on GPD parameters and risk measures using different thresholds are analysed for the OMXS30 index. Then the point estimates of the GPD parameters and the risk measures are presented for all 10 indices.

5.1 Comparison of parameters using different thresholds

Table 2 reports the impact on the GPD parameters and the VaR and ES risk measures using different thresholds. The thresholds are set at the lowest (highest) 5%, 10% and 15% of the distribution for the OMXS30 index. Similar results were found for the other indices and are thus not reported in this study.

Comparison of parameters under 5%, 10% and 15% exceedences for the OMXS30 index			
Left tail	5%	10%	15%
u	2.390	1.640	1.244
N_u	279	558	835
$\hat{\xi}$	0.013	0.052	0.075
$\hat{\beta}$	1.154	1.060	0.993
$\widehat{VaR}_{0.01}$	4.268	4.235	4.222
$\widehat{ES}_{0.01}$	5.461	5.496	5.535
Right tail	5%	10%	15%
u	2.242	1.594	1.260
N_u	278	559	836
$\hat{\xi}$	0.095	0.164	0.185
$\hat{\beta}$	1.166	0.930	0.832
$\widehat{VaR}_{0.01}$	4.267	4.199	4.185
$\widehat{ES}_{0.01}$	5.769	5.821	5.871

Table 2 – Comparison of EVT parameters and risk measures under 5%, 10% and 15% exceedences for the OMXS30 index

5.2 Mean excess plot and GPD fit

For each tail and each index a mean excess plot was calculated. A plot showing how the GPD fit to exceedances on each index was also calculated. All plots are presented in Appendix 1 through Appendix 10. The VaR and ES risk measures can be read directly from the GPD fit plot or calculated with equations (13) and (16).

5.3 Point estimates of GPD parameters and risk measures

For both tails the VaR and ES risk measures along with the GPD parameters were evaluated. The results are reported in Table 3.

Point estimates using the POT method for the 10 stock indices					
Left tail	OMXS30	OMXC20	OMXH25	DAX	FTSE-100
u	1.640	1.282	2.637	1.332	1.174
N_u	558	482	278	985	637
$\hat{\xi}$	0.052	0.127	0.131	0.205	0.248
$\hat{\beta}$	1.060	0.814	1.349	0.768	0.639
$\widehat{VaR}_{0.01}$	4.235	3.459	5.051	3.592	3.159
$\widehat{ES}_{0.01}$	5.496	4.707	6.966	5.142	4.666
Right tail	OMXS30	OMXC20	OMXH25	DAX	FTSE-100
u	1.594	1.274	1.765	1.359	1.167
N_u	559	480	557	985	637
$\hat{\xi}$	0.164	0.154	0.085	0.271	0.232
$\hat{\beta}$	0.930	0.645	1.218	0.599	0.562
$\widehat{VaR}_{0.01}$	4.199	3.054	4.864	3.274	2.877
$\widehat{ES}_{0.01}$	5.821	4.140	6.483	4.805	4.124
Left tail	Dow Jones	S&P500	NASDAQ	Nikkei-225	SENSEX
u	1.118	1.096	1.997	1.329	1.960
N_u	991	989	560	967	580
$\hat{\xi}$	0.207	0.198	0.062	0.125	0.125
$\hat{\beta}$	0.597	0.631	1.350	0.921	1.127
$\widehat{VaR}_{0.01}$	2.877	2.937	5.340	3.786	4.965
$\widehat{ES}_{0.01}$	4.088	4.180	7.002	5.191	6.679
Right tail	Dow Jones	S&P500	NASDAQ	Nikkei-225	SENSEX
u	1.153	1.128	1.960	1.329	2.068
N_u	991	989	560	967	580
$\hat{\xi}$	0.162	0.167	0.161	0.166	0.096
$\hat{\beta}$	0.605	0.615	1.137	0.755	1.100
$\widehat{VaR}_{0.01}$	2.842	2.854	5.128	3.445	4.904
$\widehat{ES}_{0.01}$	3.891	3.938	7.091	4.772	6.425

Table 3 – Point estimates of GPD parameters and risk measures for the 10 stock indices

6 Analysis and discussion

The sample mean excess plot is presented in Appendix 1 through Appendix 10 along with the selected threshold. When comparing the impact of different thresholds in Table 2 it is clear that the risk measures are fairly unaffected. But the GPD parameters are more dependent on the threshold. As mentioned in paragraph 3.4.3 the data might not fit to the GPD distribution if the threshold is set to low. When analyzing the GPD fit presented in Appendix 1 through Appendix 10 the EVT approach seems to do a fairly good job with the selected thresholds. Most sample points from the exceedances above the selected threshold lies on the distribution curve.

The shape parameter $\hat{\xi}$ has a higher value in the left tail for the OMXS30, OMXH25, FTSE-100, Dow Jones, S&P500 and SENSEX indices indicating heavier tails and thus more extreme losses compared to the right tail. For the OMXC20, DAX, NASDAQ and Nikkei-225 indices the relationship is the opposite. The left tail is important for institutions or agents with short market positions and the right tail with long market positions.

When considering the OMXS30 index the results indicate that tomorrow's loss on a long position will exceed the value 4.235%. The average loss where the loss exceeds 4.235% is 5.496%. If an institution or agent has a short position the tomorrow's loss will exceed the value 4.199% and the average loss where the loss exceeds 4.199% is 5.821%.

From the results in Table 3 it is also clear that it is important to investigate the ES risk measure. For the OMXS30 and NASDAQ-100 indices the right tail has a lower VaR but a higher ES compared to the left tail. Thus these indices seem less risky in terms of VaR but are more risky in terms of ES.

The results also show that in terms of VaR and ES, NASDAQ is most exposed to extreme losses. VaR and ES equals 5.340% and 7.002% respectively for the left tail and 5.128% and 7.091 respectively for the right tail. OMXH25 takes second place for the left tail (5.051% and 6.966%) followed by SENSEX with a second place for the right tail (4.904% and 6.425%).

7 Summary

This study has examined how the risk measures Value at Risk (VaR) and Expected Shortfall (ES) can be determined using Extreme Value Theory (EVT). The risk measures were found for the Stockholm stock exchange index (OMX30S), the Copenhagen stock exchange (OMXC20), the Helsinki stock exchange (OMXH25), the Deutscher Aktienindex (DAX), the Financial Times Stock Exchange (FTSE-100), the Dow Jones Industrial index (DJI), the Standard and Poor's 500 index (SPX), the NASDAQ-100 index (NDX), the Nikkei-225 stock average index (NKY) and the Bombay stock exchange sensitive index (SENSEX). According to the risk measures NASDAQ is most exposed to extreme losses followed by OMXH25 and SENSEX taking second place. The study also showed that it is important to go beyond VaR as ES was larger than VaR for the Dow Jones, S&P500, Nikkei-225 and SENSEX indices indicating a false relationship to risk to an institution or agent if only VaR is considered.

8 Further research suggestions

Extreme Value Theory (EVT) has been used in this study to calculate the VaR and ES risk measures. Instead of using EVT one could focus on volatility modeling and the quantiles (as in equation (5) and (6)) from the normal distribution or the Student-t distribution to see if results differ.

This study has used an unconditional (no volatility modeling) approach for the evaluation of long horizon risk measures. If considering risk measures for shorter horizons, as daily or weekly, a conditional (volatility modeling) approach might be interesting to evaluate using the GARCH framework (Bollerslev, 2008) on the exceedence data from the tails of the distribution before estimating the GPD parameters.

The Peak Over Threshold (POT) method have been used in this study. However, the method of block maxima can also be useful for GPD parameter estimation. A comparison of these two methods might be interesting to evaluate.

There are two approaches to find the GPD parameters under the POT method. They are the fully parametric maximum likelihood estimation method and the semi-parametric Hill estimator method. It could be interesting to investigate and compare the efficiency of the two methods and compare the results when computing the VaR and ES risk measures.

For further readings in the field of EVT it's recommended to read the book Elements of Financial Risk Management by Christoffersen (2003). Quantitative Risk Management by McNeil, Frey and Embrechts (2005), especially chapter seven, also gives a thorough explanation of EVT concepts. The Gloria Mundi (2009) website contains many interesting articles and research papers free to download. There is also a very interesting webinar from Mathworks (2009) called Market Risk Using GARCH, Extreme Value Theory and Copulas with MATLAB, which gives an understanding of EVT within the Matlab environment. Finally, the research papers mentioned in the references, especially from McNiel, Gilli and Këllezi are strongly recommended for further readings.

9 References

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Appendix 1 – Daily returns and EVT statistics for the OMXS30 index

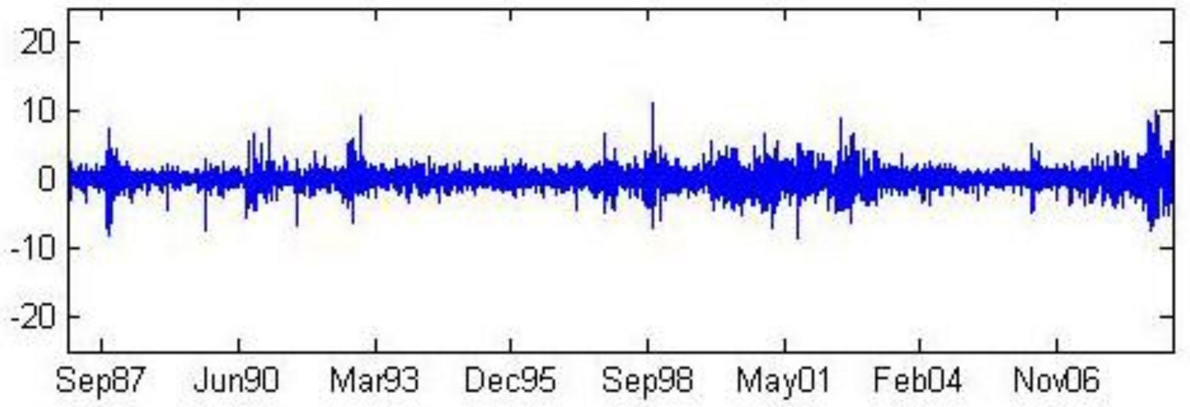

Figure 4 – Daily returns of the OMXS30 index

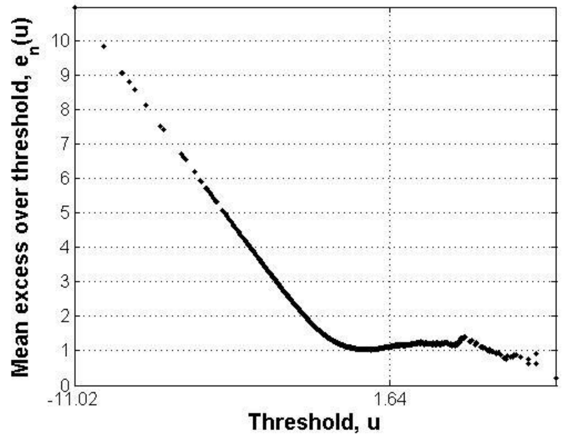

Figure 5 – Sample mean excess plot for the left tail of the OMXS30 index

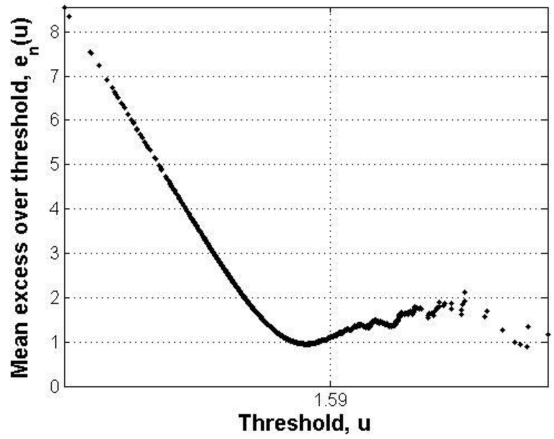

Figure 6 – Sample mean excess plot for the right tail of the OMXS30 index

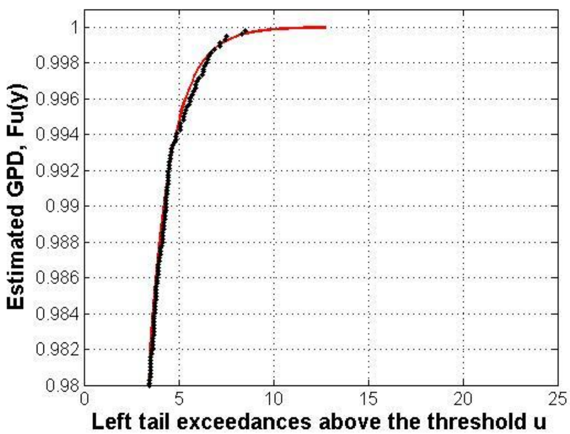

Figure 7 – GPD fitted to the left tail exceedances above the threshold u = 1.64

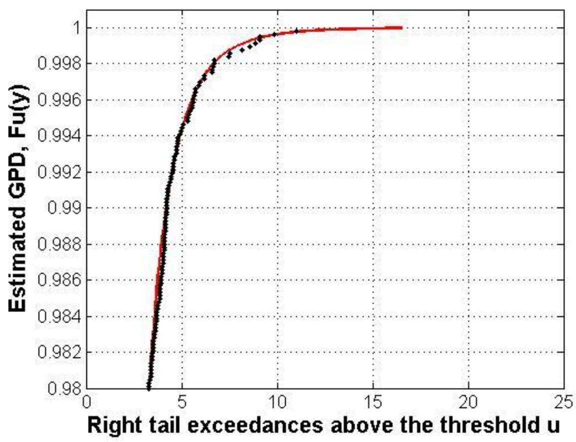

Figure 8 – GPD fitted to the right tail exceedances above the threshold u = 1.59

Appendix 2 – Daily returns and EVT statistics for the OMXC20 index

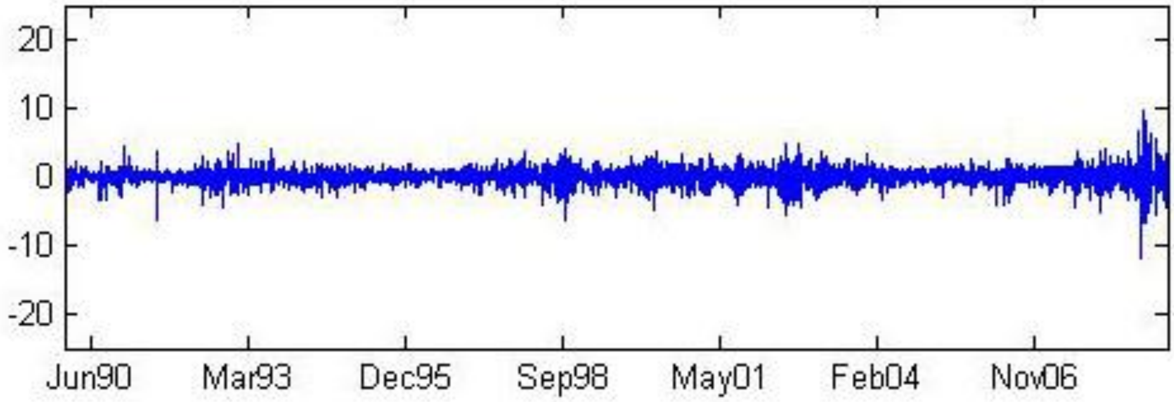

Figure 9 – Daily returns of the OMXC20 index

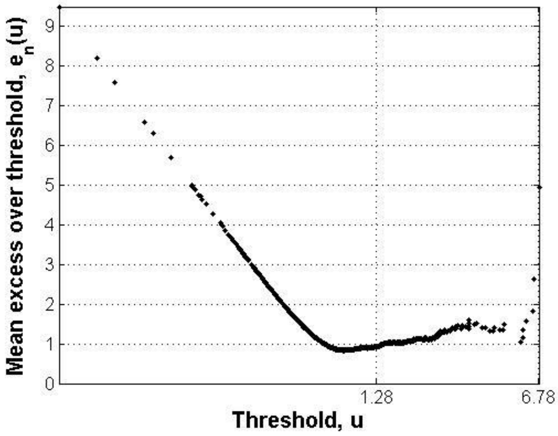

Figure 10 – Sample mean excess plot for the left tail of the OMXC20 index

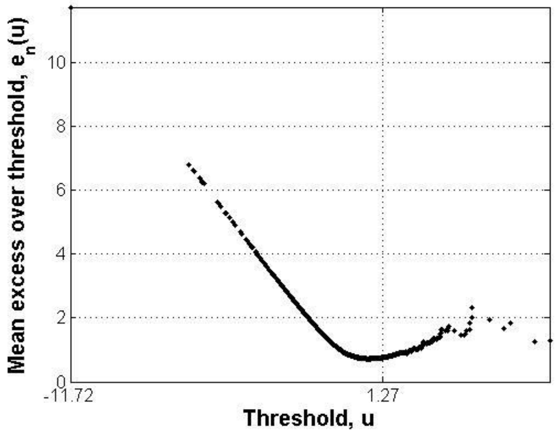

Figure 11 – Sample mean excess plot for the right tail of the OMXC20 index

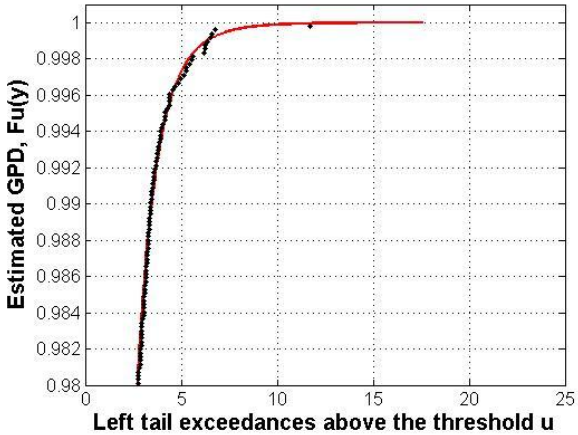

Figure 12 – GPD fitted to the left tail exceedances above the threshold u = 1.28

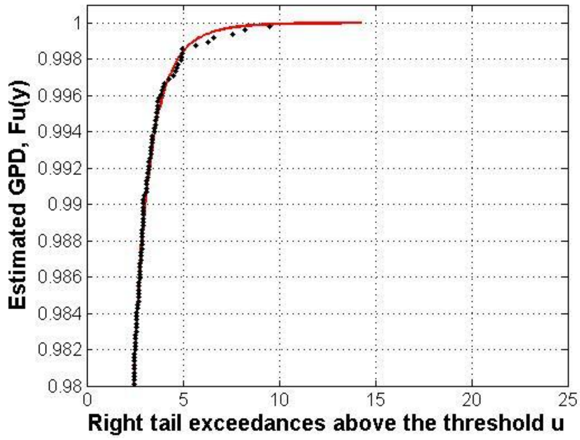

Figure 13 – GPD fitted to the right tail exceedances above the threshold u = 1.27

Appendix 3 – Daily returns and EVT statistics for the OMXH25 index

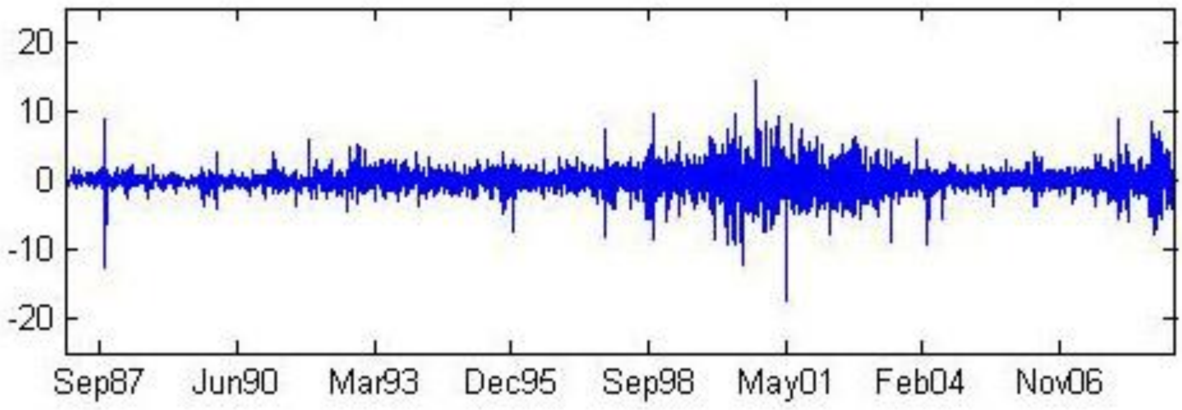

Figure 14 – Daily returns of the OMXH25 index

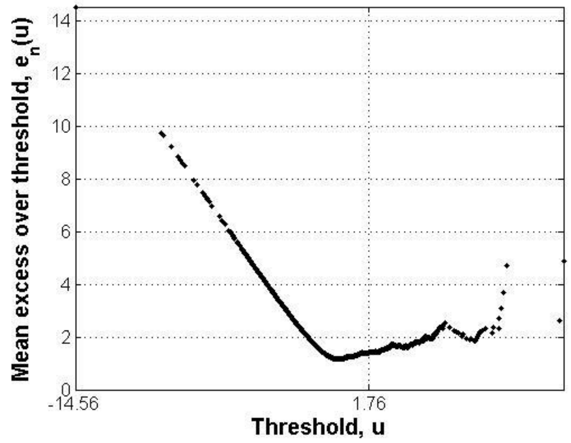

Figure 15 – Sample mean excess plot for the left tail of the OMXH25 index

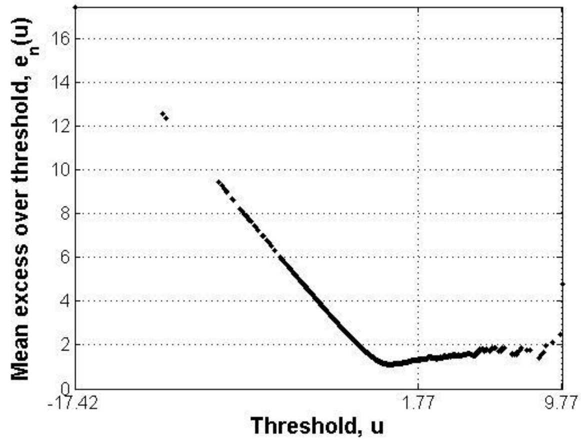

Figure 16 – Sample mean excess plot for the right tail of the OMXH25 index

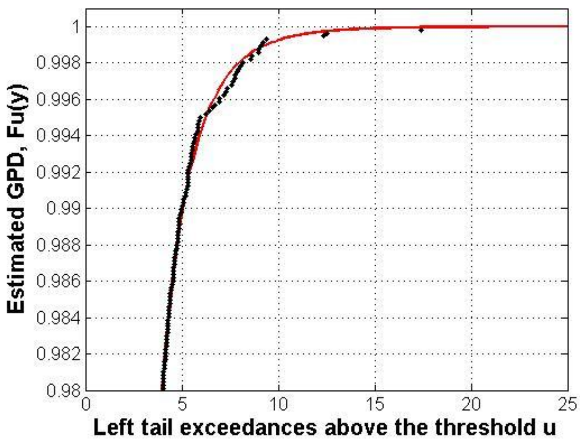

Figure 17 – GPD fitted to the left tail exceedances above the threshold u = 1.76

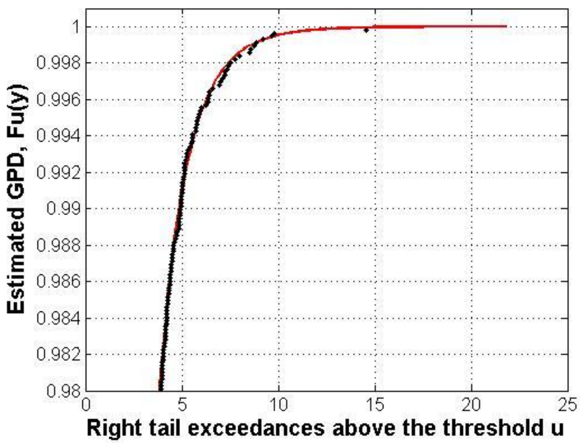

Figure 18 – GPD fitted to the right tail exceedances above the threshold u = 1.77

Appendix 4 – Daily returns and EVT statistics for the DAX index

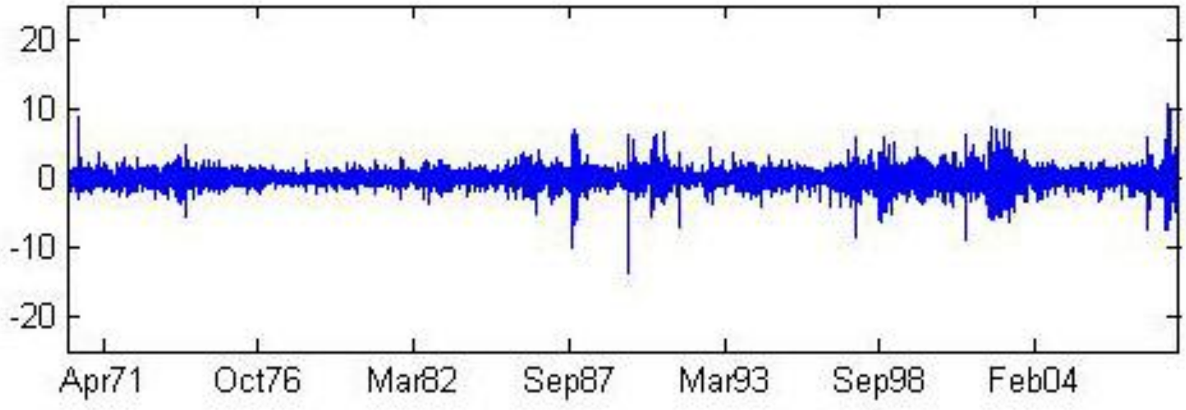

Figure 19 – Daily returns of the DAX index

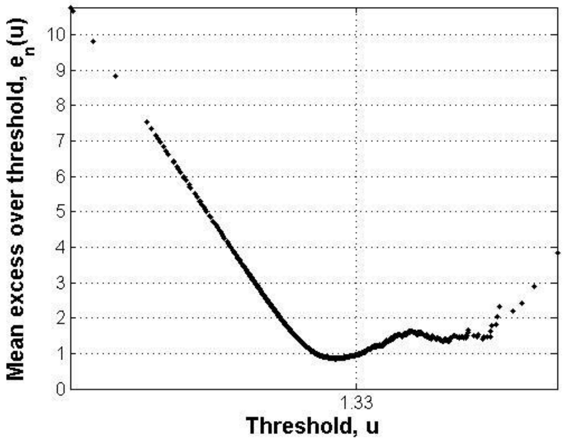

Figure 20 – Sample mean excess plot for the left tail of the DAX index

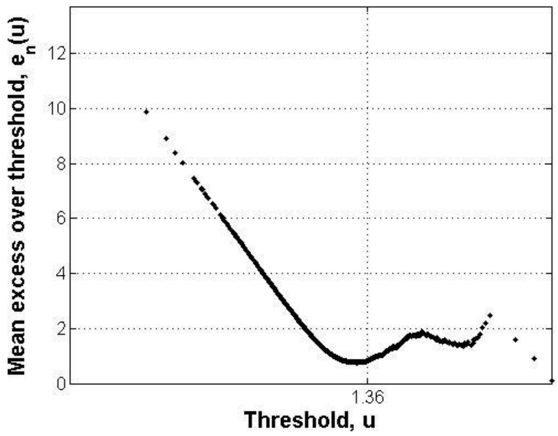

Figure 21 – Sample mean excess plot for the right tail of the DAX index

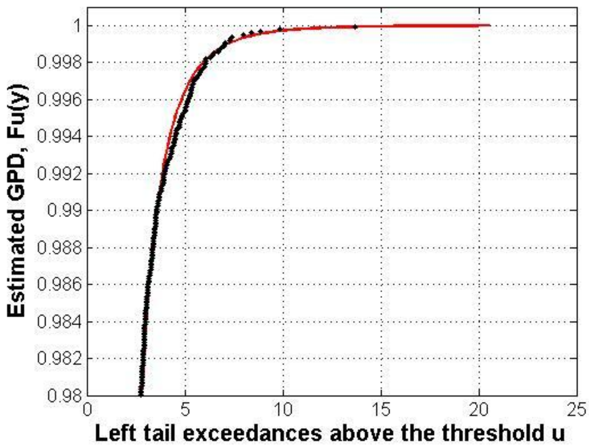

Figure 22 – GPD fitted to the left tail exceedances above the threshold u = 1.33

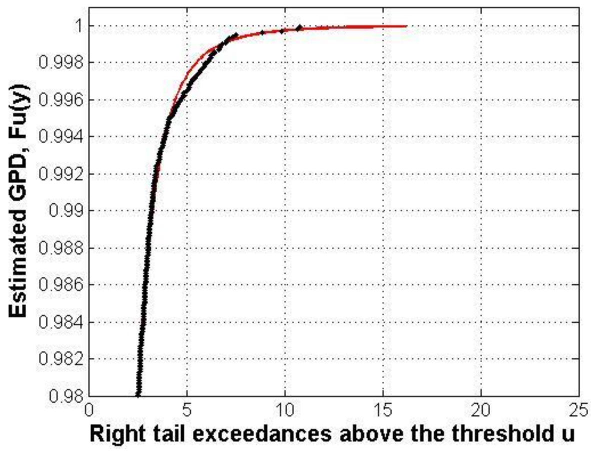

Figure 23 – GPD fitted to the right tail exceedances above the threshold u = 1.36

Appendix 5 – Daily returns and EVT statistics for the FTSE-100 index

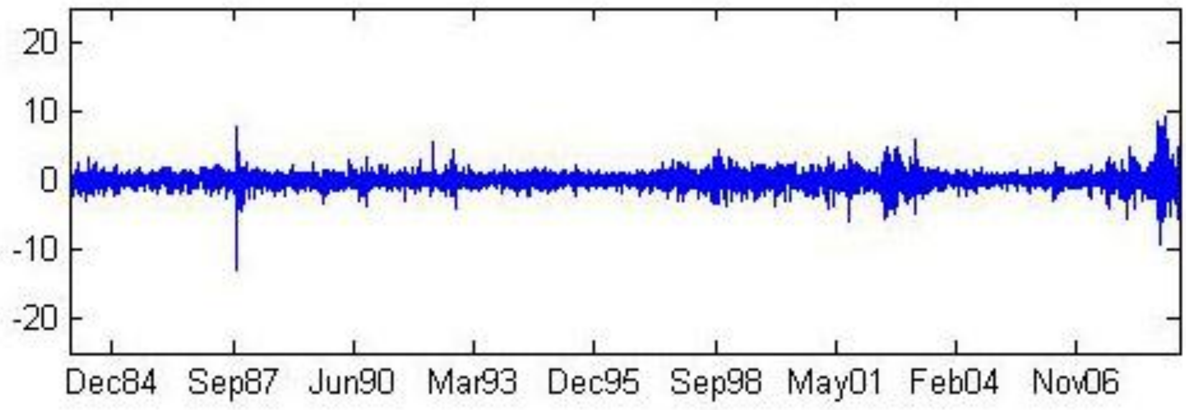

Figure 24 – Daily returns of the FTSE-100 index

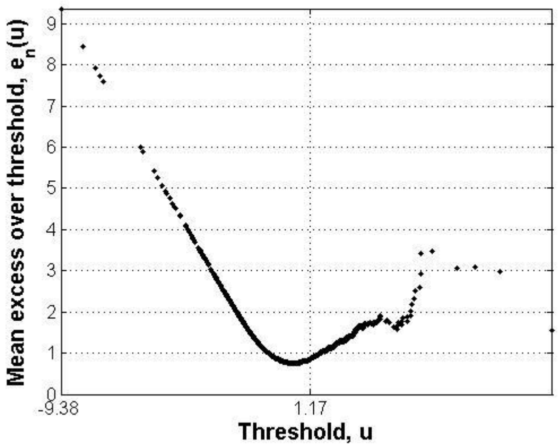

Figure 25 – Sample mean excess plot for the left tail of the FTSE-100 index

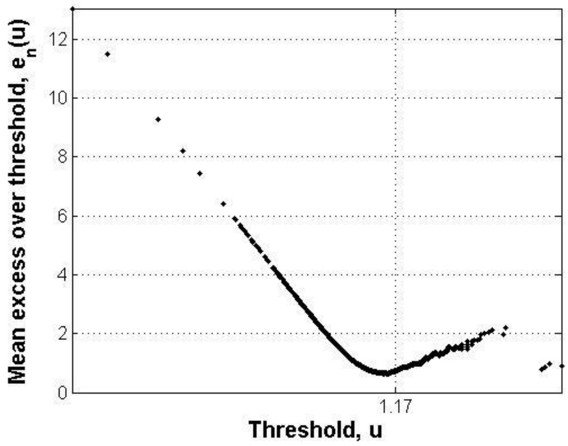

Figure 26 – Sample mean excess plot for the right tail of the FTSE-100 index

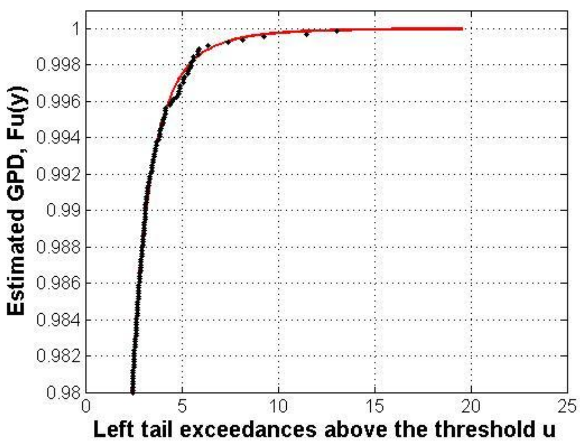

Figure 27 – GPD fitted to the left tail exceedances above the threshold u = 1.17

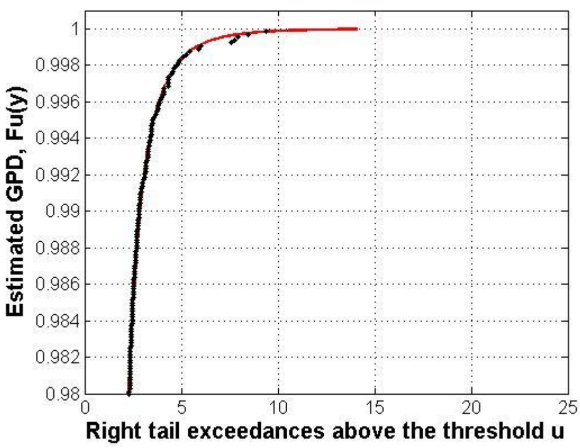

Figure 28 – GPD fitted to the right tail exceedances above the threshold u = 1.17

Appendix 6 – Daily returns and EVT statistics for the Dow Jones index

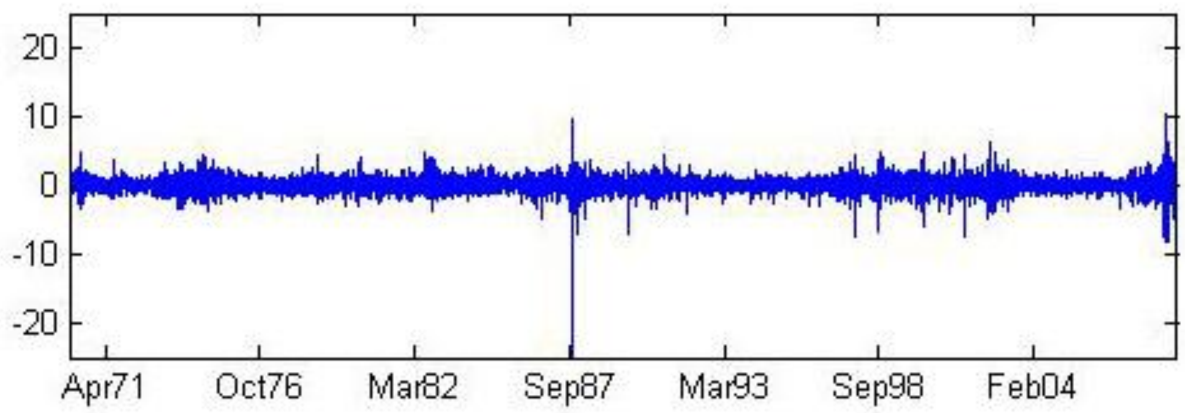

Figure 29 – Daily returns of the Dow Jones index

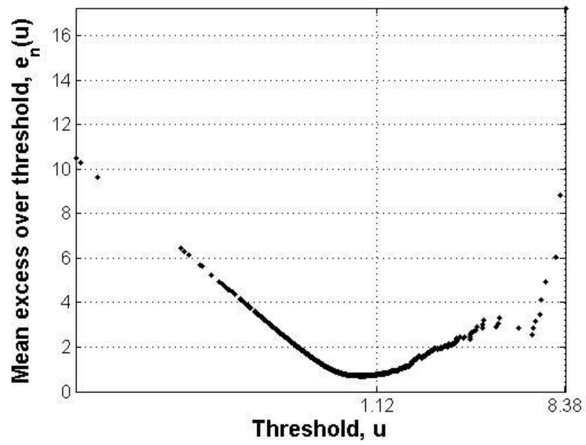

Figure 30 – Sample mean excess plot for the left tail of the Dow Jones index

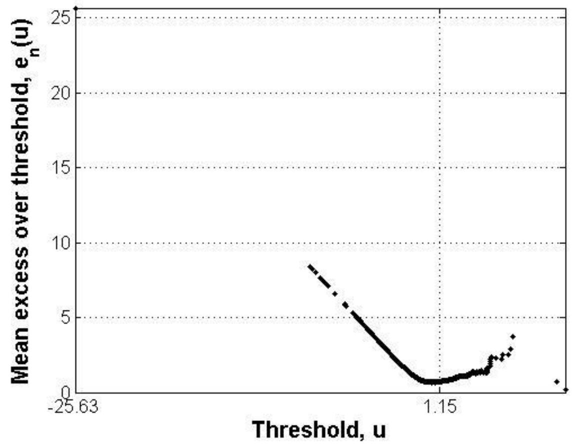

Figure 31 – Sample mean excess plot for the right tail of the Dow Jones index

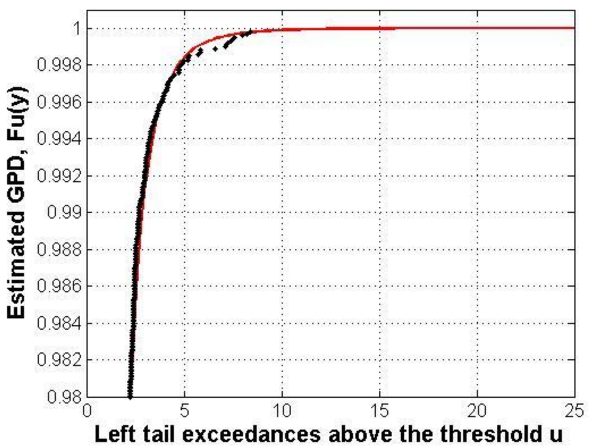

Figure 32 – GPD fitted to the left tail exceedances above the threshold u = 1.12

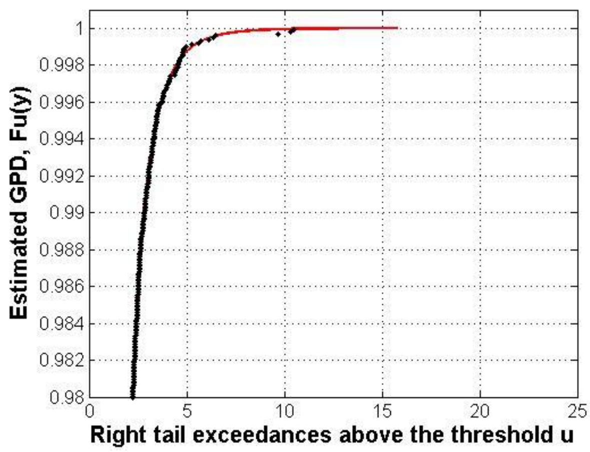

Figure 33 – GPD fitted to the right tail exceedances above the threshold u = 1.15

Appendix 7 – Daily returns and EVT statistics for the S&P500 index

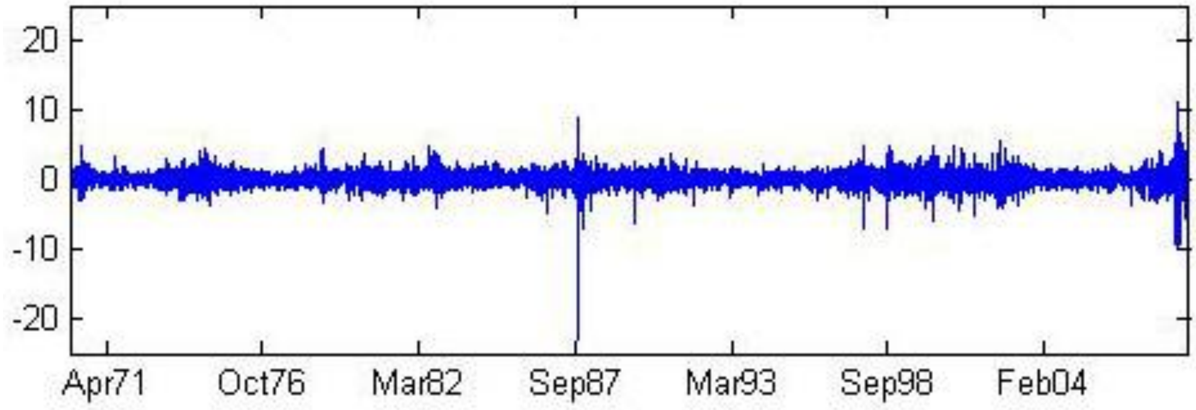

Figure 34 – Daily returns of the S&P500 index

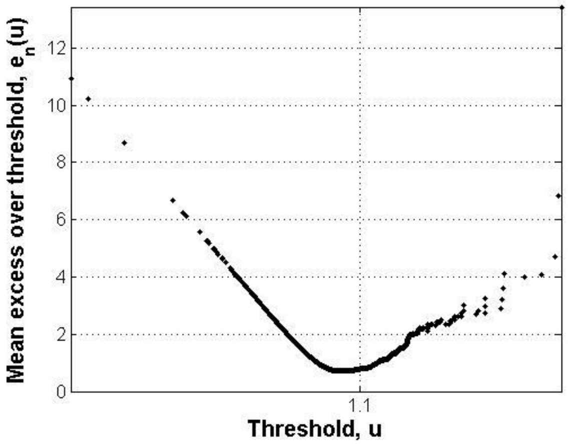

Figure 35 – Sample mean excess plot for the left tail of the S&P500 index

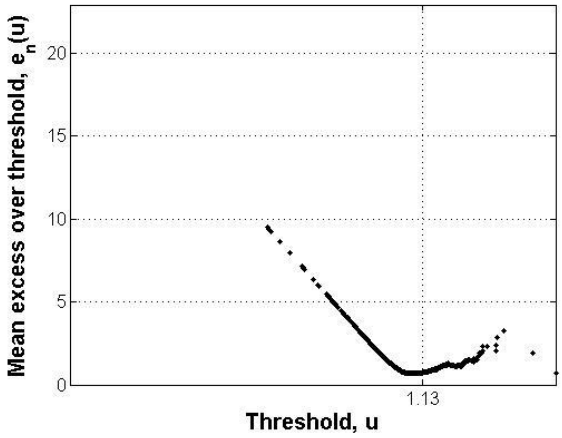

Figure 36 – Sample mean excess plot for the right tail of the S&P500 index

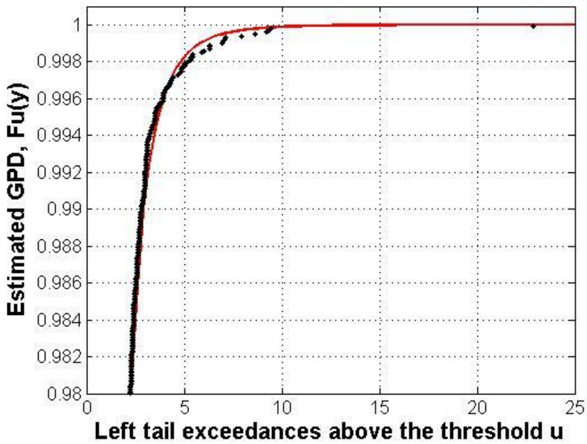

Figure 37 – GPD fitted to the left tail exceedances above the threshold u = 1.10

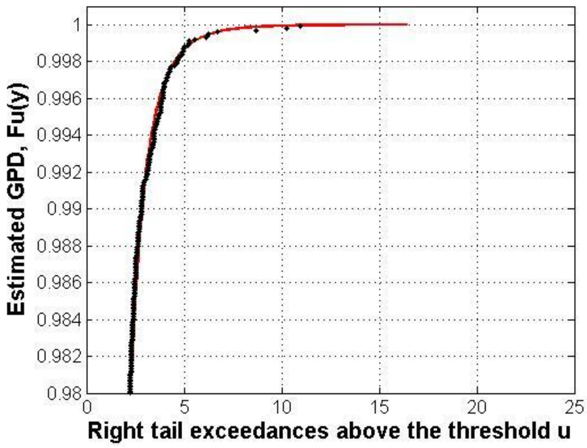

Figure 38 – GPD fitted to the right tail exceedances above the threshold u = 1.13

Appendix 8 – Daily returns and EVT statistics for NASDAQ index

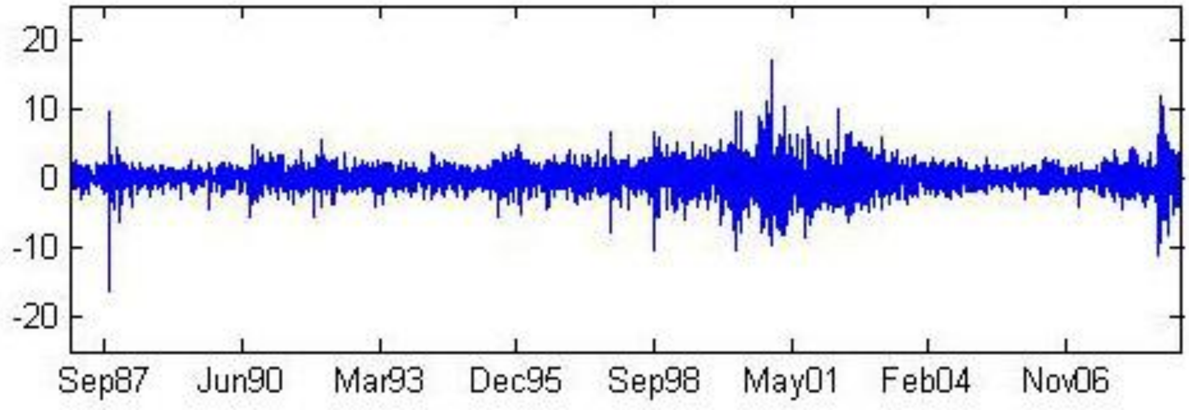

Figure 39 – Daily returns of the NASDAQ index

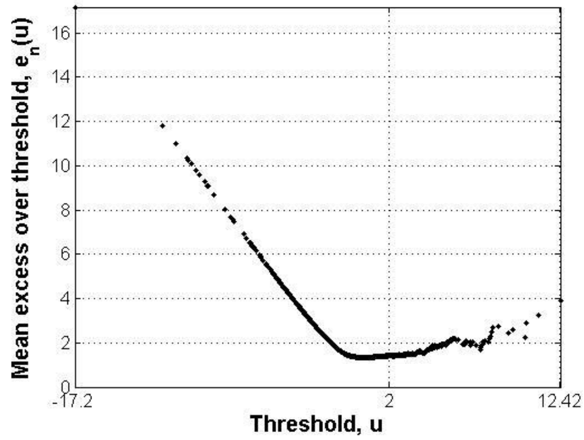

Figure 40 – Sample mean excess plot for the left tail of the NASDAQ index

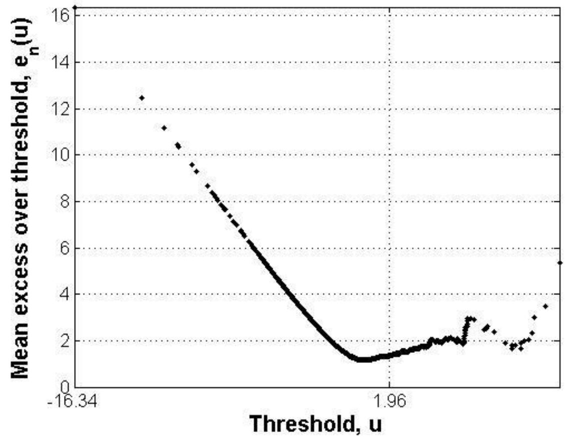

Figure 41 – Sample mean excess plot for the right tail of the NASDAQ index

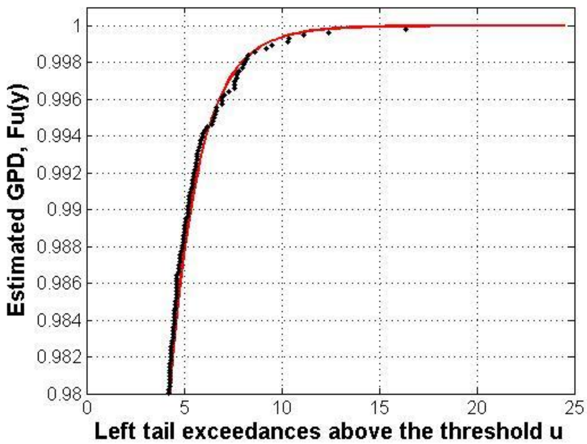

Figure 42 – GPD fitted to the left tail exceedances above the threshold u = 2.00

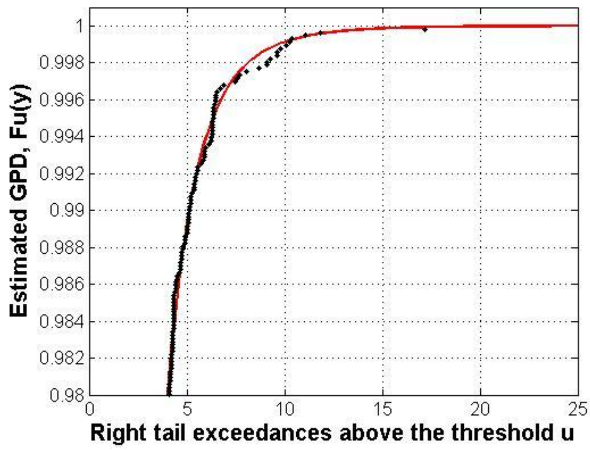

Figure 43 – GPD fitted to the right tail exceedances above the threshold u = 1.96

Appendix 9 – Daily returns and EVT statistics for Nikkei-225 index

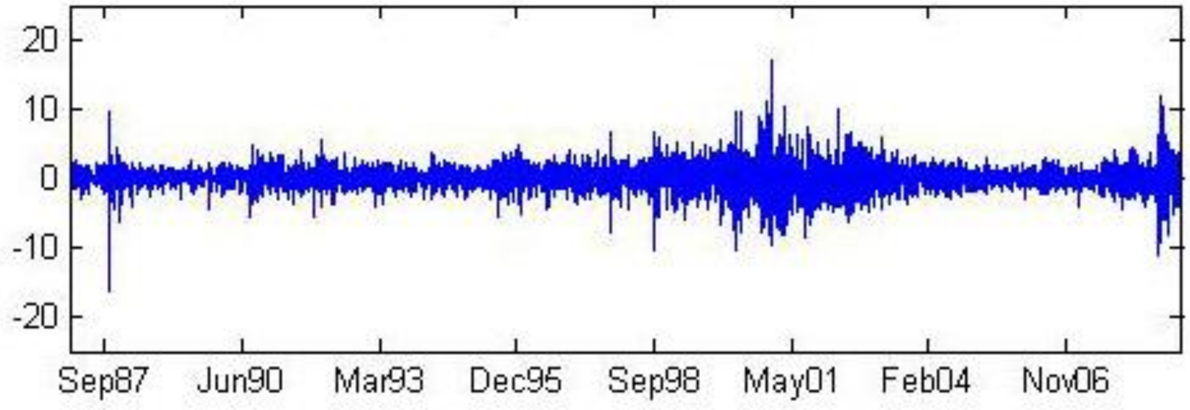

Figure 44 – Daily returns of the Nikkei-225 index

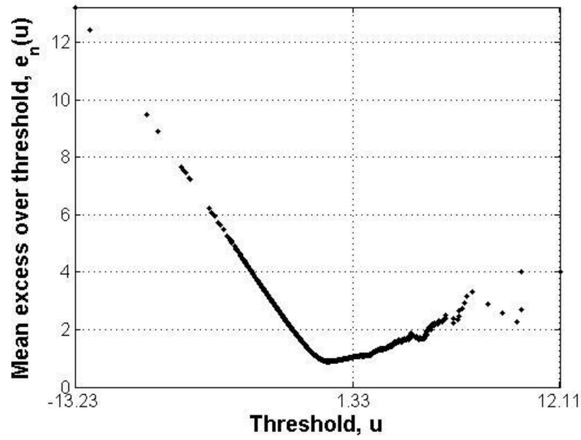

Figure 45 – Sample mean excess plot for the left tail of the Nikkei-225 index

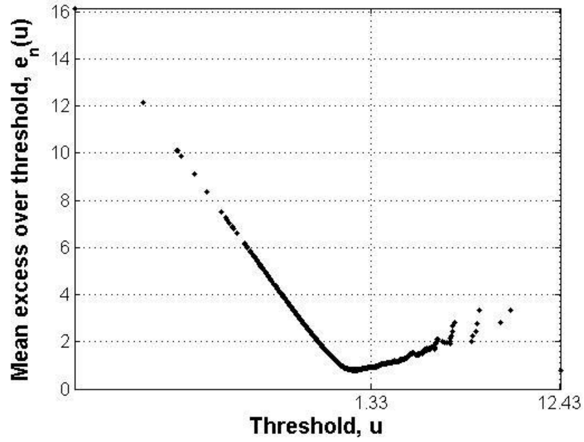

Figure 46 – Sample mean excess plot for the right tail of the Nikkei-225 index

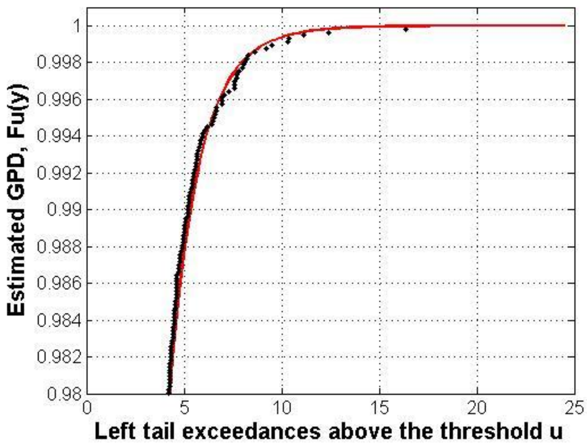

Figure 47 – GPD fitted to the left tail exceedances above the threshold u = 1.33

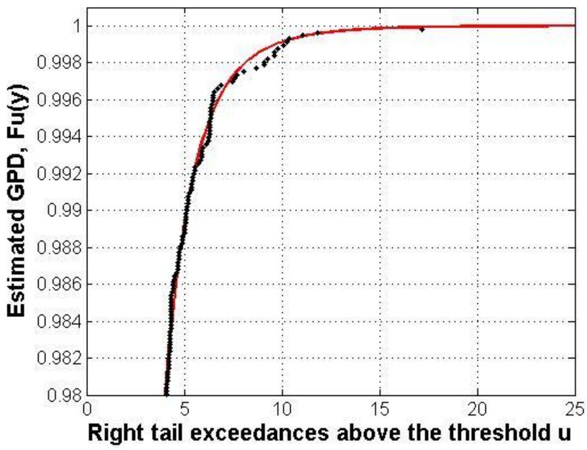

Figure 48 – GPD fitted to the right tail exceedances above the threshold u = 1.33

Appendix 10 – Daily returns and EVT statistics for SENSEX index

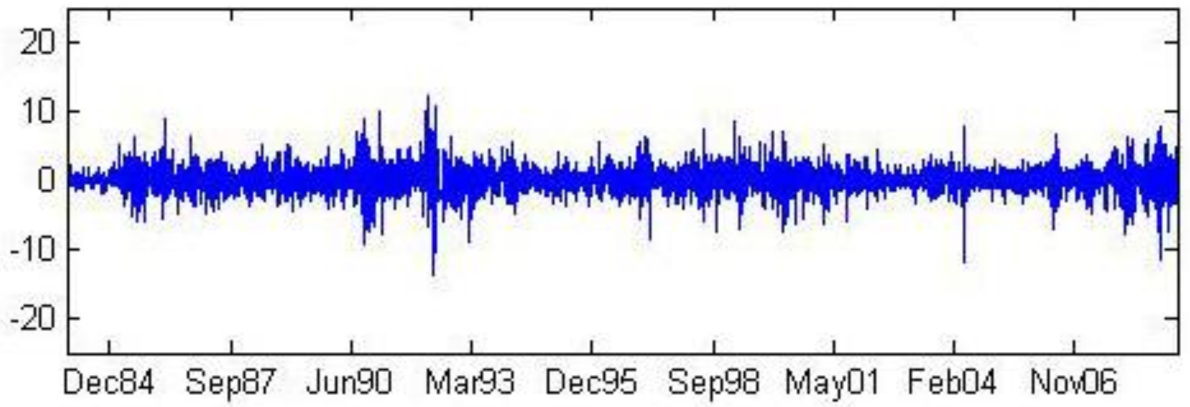

Figure 49 – Daily returns of the SENSEX index

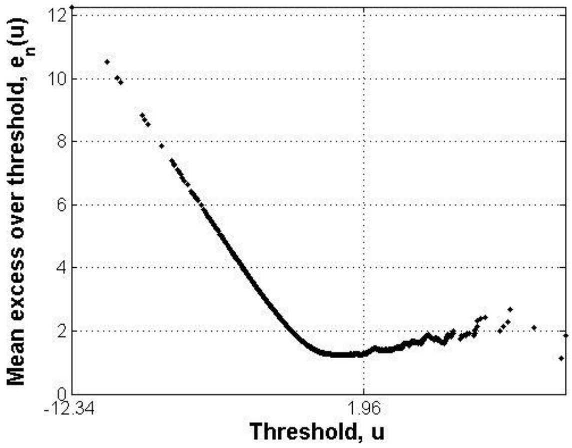

Figure 50 – Sample mean excess plot for the left tail of the SENSEX index

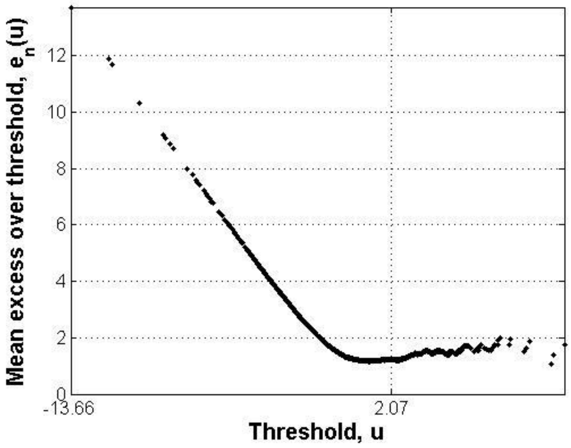

Figure 51 – Sample mean excess plot for the right tail of the SENSEX index

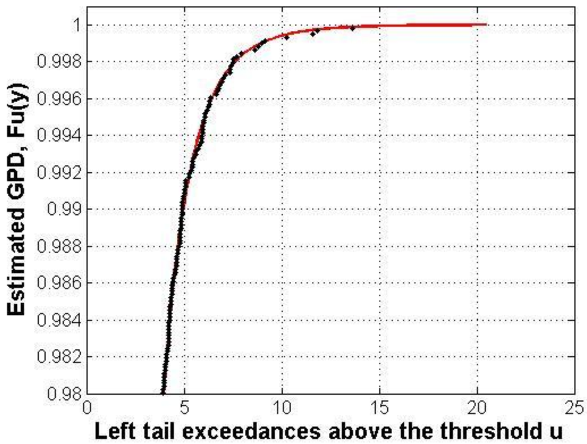

Figure 52 – GPD fitted to the left tail exceedances above the threshold u = 1.96

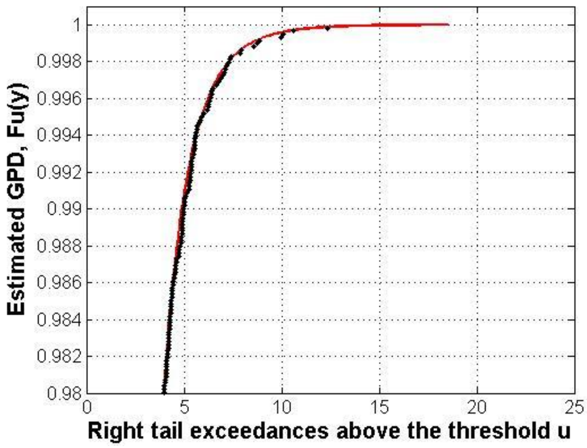

Figure 53 – GPD fitted to the right tail exceedances above the threshold u = 2.07