THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Spectral properties of elliptic operators in singular settings and applications

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Abstract

The present thesis is focused on the investigation of the spectral properties of the linear elliptic operators in the presence of singularities. It is divided into three chapters.

In the first chapter, we consider geometric singularities. We construct the heat kernel on surfaces with corners for Dirichlet, Neumann, and Robin boundary conditions as well as mixed problems. We compute the short time asymptotic expansion of the heat trace and apply this expansion to demonstrate a collection of results showing that corners are spectral invariants.

The second chapter deals with linear elliptic second-order partial differential operators with bounded real-valued measurable coefficients. We emphasize that no smoothness assumptions are made on the coefficients. In the first half of this chapter, we study a time-harmonic electromagnetic and acoustic waveguide, modeled by an infinite cylinder with a non-smooth cross section. We introduce an infinitesimal generator for the wave evolution along the cylinder and prove estimates of the functional calculi of these first order non-self adjoint differential operators with non-smooth coefficients. Applying our new functional calculus, we obtain a one-to-one correspondence between polynomially bounded time-harmonic waves and functions in appropriate spectral subspaces. In the second half, we derive Weyl's law for the weighted Laplace equation on Riemannian manifolds with rough metric. Key ingredients in the proofs were demonstrated by Birman and Solomyak nearly fifty years ago in their seminal work on eigenvalue asymptotics.

In the last chapter, we investigate spectral properties of Sturm-Liouville operators with singular potentials. We consider different types of singularities. We find asymptotic formulas for the eigenvalues of the Sturm-Liouville operator on the finite interval, with potentials having a strong negative singularity at one endpoint. We establish that, unlike the case of an infinite interval, the asymptotics for positive eigenvalues does not depend on the potential, and it is the same as in the regular case. The asymptotics of the negative eigenvalues may depend on the potential quite strongly. Next, we study the perturbation of the generalized anharmonic oscillator. We consider a piecewise Hölder continuous perturbation and investigate how the Hölder constant can affect the eigenvalues. Finally, for the the Sturm-Liouville operator with δ-interactions, two-sided estimates of the distribution function of the eigenvalues and a criterion for the discreteness of the spectrum in terms of the Otelbaev function are obtained.

Keywords: Elliptic operators, spectrum, heat kernel, Sturm-Liouville operators, asymptotic of eigenvalues.

List of appended papers

Paper I. Medet Nursultanov, Julie Rowlett, David A. Sher, How to hear the corners of a drum, 2017 MATRIX annals, MATRIX Book Series 2, 2019, 243-278.

Paper II. Medet Nursultanov, Julie Rowlett, David A. Sher, The heat kernel on curvilinear polygonal domains in surfaces (Preprint).

Paper III. Medet Nursultanov, Andreas Rosén, Evolution of Time-Harmonic Electromagnetic and Acoustic Waves Along Waveguides, Integral Equations and Operator Theory, (2018) 90: 53.

Paper IV. Lashi Bandara, Medet Nursultanov, Julie Rowlett, Eigenvalue asymptotics for weighted Laplace equations on rough Riemannian manifolds with boundary (Preprint).

Paper V. Medet Nursultanov, Grigori Rozenblum, Eigenvalue asymptotics for the Sturm-Liouville operator with potential having a strong local negative singularity, Opuscula Math. 37, no. 1 (2017), 109-139.

Paper VI. Ksenia Fedosova, Medet Nursultanov, The asymptotic expansion for the spectrum of a generalized anharmonic oscillator (Preprint).

Paper VII. Medet Nursultanov, Spectral properties of the Schrödinger operator with δ-distribution, Mathematical Notes, Volume 100(2016), Issue 1, 263-275.

Contributions to the appended papers

I made an essential contribution to the development in all the included papers, as well as to writing.

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Contents

Introduction

Due to numerous applications, the linear elliptic partial differential equations of second order form important class of equations in mathematical physics. A second order linear elliptic partial differential equation can be written in the form

$$\mathcal{L}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f, \quad x \in \Omega, \tag{1}$$

where Ω is a geometric object (domain in Euclidean space or manifold) and $a_{ij}(x)$, $b_i(x)$, $c(x)$ are given functions such that $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \neq 0$ for all $x \in \Omega$ and $\xi \in \mathrm{C}^n$. An operator $\mathcal{L}$ is called elliptic.

If Ω is bounded with smooth boundary and all coefficients (a_{ji}, b_j, c) are smooth and bounded, the investigation of such operators becomes comparably easy. However, for an actual real world problem the smoothness and boundedness are not necessarily guarantied, so that one has to consider different singularities, which produce some difficulties.

The present thesis is focused on the investigation of the spectral properties of the elliptic operators in the presence of singularities. It is divided into three chapters. In the first chapter, we consider geometric singularities. More precisely, we consider the Laplace operator on surfaces with non-smooth boundary, in particular with corners. We are interested in heat kernels for Dirichlet, Neumann, Robin, and mixed boundary conditions. The second chapter deals with linear elliptic second-order partial differential operators with bounded real-valued measurable coefficients. We emphasize that no smoothness assumptions are made on the coefficients. We will consider two physical models where such operators arise. We also generalize the Weyl's law of the Laplacian to compact Riemannian manifolds with rough metric. In the last chapter, we investigate the spectral properties of the Sturm-Liouville operators with singular potentials.

The heat kernel and geometric spectral invariants on surfaces with corners (Papers I and II)

Let M be a smooth, n-dimensional topological manifold with smooth boundary, ∂M, such that the closure, $\overline{M} = M \cup \partial M$, is compact. If M is equipped with a smooth Riemannian metric, g, then there is a naturally associated Laplace operator, which in local coordinates is

$$\Delta_g = -\frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det(g)} \frac{\partial}{\partial x_j} \right). \tag{2}$$

This is a second order elliptic operator with smooth coefficients, inherited from the smoothness of the Riemannian metric. It is well known in this setting that the Laplacian, Δ_g, has a discrete, non-negative set of eigenvalues which accumulate only at ∞. A natural question arise: if two compact Riemannian manifolds (M, g) and (M', g') have the same Laplace spectrum, then are they isometric? No, they are not. However, isospectrality does imply that M and M' are of the same dimension, n. Moreover, they must also have the same n-dimensional volume. Thus, both dimension and volume are spectral invariants, in the sense that they are determined by the spectrum. This fact follows from Weyl's law, see [60],

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\omega_n \mathrm{Vol}(M)}{(2\pi)^n}. \tag{3}$$

Above, $N(\lambda)$ is the number of eigenvalues of the Laplacian, counted with multiplicity, which do not exceed λ, ω_n is the volume of the unit ball in $\mathbb{R}^n$, and $\mathrm{Vol}(M)$ is the volume of M. It is natural to ask, what other geometric features are spectral invariants?

The next geometric spectral invariant was discovered by Pleijel [51] some forty years after Weyl's law. For an n-dimensional manifold with smooth boundary, the $n-1$ dimensional volume of the boundary is a spectral invariant. About ten years later, McKean and Singer [45] proved that certain curvature integrals are also spectral invariants. For smooth surfaces and smoothly bounded planar domains, McKean & Singer [45] and independently M. Kac [31] proved that the Euler characteristic is a spectral invariant. In both approaches, they used the existence of a short time asymptotic expansion for the heat trace, together with the calculation of the coefficients in this expansion.

Here, we are interested in the heat kernel on surfaces with non-smooth boundary, in particular, with corners. This includes curvilinear polygonal domains in the plane, as well as more exotic non-planar examples. We are interested in the heat kernel for such surfaces because it may allow us to determine new geometric spectral invariants. Indeed, we show that in general, the presence or lack of corners is a spectral invariant for Dirichlet, Neumann, Robin, and mixed boundary conditions. Moreover, we shall see that a jump in boundary condition from Dirichlet to Neumann is also a spectral invariant.

Let us specify what is meant by a *surface with corners.*

Definition 0.1. We say that Ω is a *curvilinear polygonal domain* if it is a subdomain of a smooth Riemannian surface (M, g) with piecewise smooth boundary and a vertex at each non-smooth point of $\partial\Omega$. A *vertex* is a point p on the boundary of Ω at which the following are satisfied.

(1) The boundary in a neighborhood of p is defined by a continuous curve $\gamma(t)$: $(-a,a) \to M$ for $a > 0$ with $\gamma(0) = p$. We require that γ is smooth on $(-a,0]$ and $[0,a)$, with $||\dot{\gamma}(t)|| = 1$ for all $t \in (-a,a)$, and such that

$$\lim_{t\uparrow 0} \dot{\gamma}(t) = v_1, \quad \lim_{t\downarrow 0} \dot{\gamma}(t) = v_2,$$

for some vectors $v_1, v_2 \in T_pM$, with $-v_1 \neq v_2$.

(2) The *interior angle* at the point p is the *interior angle* at that corner, which is the angle between the vectors $-v_1$ and v_2.

Note that requiring $-v_1$ and v_2 to be distinct means that the interior angle will be an element of $(0, 2\pi)$, which rules out inward and outward pointing cusps. An angle of π, corresponding to a phantom vertex, is allowed.

There is some substantial work in the literature on heat trace expansions for certain surfaces with corners. The heat trace expansion for a polygonal domain in the plane, with the Dirichlet boundary condition, has been known since at least the 1960s; see for example [24], [23], [31]. Its most simplified form and calculation can be found in a paper of van den Berg and Srisatkunarajah [58], although the expression there is originally due to unpublished work of Ray. However, this result applies only to exact polygons. Although it has been widely assumed that an analogous result holds for curvilinear polygons, a rigorous proof was not given until [42]. Similar results hold for Neumann boundary conditions, see [43]. Although Robin conditions have been studied on manifolds with boundary [61], to our knowledge there is no work in the literature about heat trace expansions with Robin conditions in the presence of corners, even in the plane. For certain corner angles, however, we refer to the physical approach of [7]. Outside the planar case, or even in the planar case with mixed boundary conditions, less is known.

Our result allows us to handle the general case of compact surfaces with corners, with any combination of Dirichlet, Neumann, and/or Robin boundary conditions on the various smooth boundary components. Throughout, we consider the Laplacian on such a surface defined as in (2) with $n = 2$. Our convention for the Robin boundary condition on any portion of the boundary is:

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \kappa u|_{\partial\Omega}\,.$$

Here, the derivative on the left is the *inward* pointing normal derivative, and therefore, on the right, κ is a *positive* function. Under this condition the spectrum is non-negative. We assume throughout, for simplicity, that κ is smooth.

Our main result is:

Theorem 0.2. *Let Ω be a curvilinear polygonal domain in a smooth surface with finitely many vertices $V_1, \ldots, V_n$ of angles $\alpha_1, \ldots, \alpha_n$. Define its edges $E_1, \ldots, E_n$ by letting E_j be the segment of the boundary between V_{j-1} and V_j, with subscripts taken mod n. Let $\mathcal{E}_D$, $\mathcal{E}_N$, and $\mathcal{E}_R$ be three disjoint sets whose union is $\{1, \ldots, n\}$. For each $j \in \mathcal{E}_D$, $\mathcal{E}_N$, and $\mathcal{E}_R$, we impose Dirichlet, Neumann, and Robin conditions with parameter $\kappa_j(x)$, respectively, along E_j. Assume that all functions $\kappa_j(x)$ are positive and smooth.*

Let $\mathcal{V}_=$ be the set of j for which vertex V_j has either zero or two Dirichlet edges adjacent to it, i.e. either both j and $j+1 \in \mathcal{E}_D$ or neither are. Conversely, let $\mathcal{V}_{\neq}$ be *the set of j for which V_j has exactly one adjacent Dirichlet edge. Also let $K(z)$ and $k_g(x)$ be the Gauss curvature and geodesic/mean curvature of Ω and $\partial\Omega$ respectively.*

Then the heat trace $\mathrm{Tr}H^{\Omega}(t)$ for the Laplacian with those boundary conditions has a complete polyhomogeneous conormal expansion in t as $t \to 0$. Moreover, the first few terms of this expansion have the form

$$\mathrm{Tr}H^{\Omega}(t) = a_{-1}t^{-1} + a_{-1/2}t^{-1/2} + a_0 + O(t^{1/2}\log t),$$

where:

$$a_{-1} = \frac{A(\Omega)}{4\pi}; \tag{4}$$

$$a_{-1/2} = \frac{1}{8\sqrt{\pi}}\Big(\sum_{j\notin\mathcal{E}_D} \ell(E_j) - \sum_{j\in\mathcal{E}_D} \ell(E_j)\Big); \tag{5}$$

$$a_0 = \frac{1}{12\pi}\int_\Omega K(z)\,dz + \frac{1}{12\pi}\int_{\partial\Omega} k_g(x)\,dx + \frac{1}{2\pi}\sum_{j\in\mathcal{E}_R}\int_{E_j} \kappa_j(x)\,dx \tag{6}$$

$$+ \sum_{j\in V_=} \frac{\pi^2-\alpha_j^2}{24\pi\alpha_j} + \sum_{j\in V_{\neq}} \frac{-\pi^2-2\alpha_j^2}{48\pi\alpha_j}. \tag{7}$$

The proof of this result contains several ingredients which may be of independent interest. The main strategy is to use geometric microlocal analysis to construct the heat kernel on a heat space created by blowing up $\Omega \times \Omega \times [0,\infty)$ along various p-submanifolds. On this heat space we show that the heat kernel has a polyhomogeneous conormal expansion at every boundary hypersurface, and indeed the heat kernel is constructed by solving suitable model problems at the various boundary hypersurfaces. This gives a full description of the heat kernel on a surface with corners in all asymptotic regimes, and as such is useful for any application in which fine structure information about the heat kernel near $t = 0$ is needed.

A major advantage of this method is that a complete asymptotic description of the heat *kernel,* rather than just its trace, is obtained. This allows precise asymptotic analysis for expressions such as the gradient of the heat kernel and is likely of interest for future work.

Elliptic second-order PDEs with bounded real-valued measurable coefficients (Papers III and IV)

In this chapter, we consider elliptic second-order partial differential equations with bounded real-valued measurable coefficients. Such operators naturally arise in physics. In the first part, we investigate two physical models, namely, we explore time-harmonic electromagnetic and acoustic waves along waveguides. Another reason to study operators with bounded measurable coefficients is that a pullback of a smooth metric by lipeomorphism is only guaranteed to have such regularity. Such a transformation allows for objects with singularities to be studied more simply. This leads one to consider the notions rough metric and rough Riemannian manifold defined by Bandara in [14]. In the second part, we establish eigenvalue asymptotics for weighted Laplace equation on rough Riemannian manifolds.

2.1. Here we study time-harmonic electromagnetic and acoustic waves along waveguides. We begin by considering the classical boundary value problems for a divergence form second order elliptic equation

$$\operatorname{div}_{(t,x)} A(x) \nabla_{(t,x)} u(t,x) = 0 \tag{8}$$

for functions on the upper half space $\mathrm{R}^{1+n} := \{(t,x) \in \mathrm{R} \times \mathrm{R}^2;\ t > 0\}$, with boundary data in $L_2(\mathrm{R}^n)$. Here A is t-independent bounded and accretive in the sense that $A \in L_\infty(\mathrm{R}^n; \mathcal{L}(\mathrm{C}^{1+n}))$ and there exists $C > 0$ such that

$$\operatorname{Re} \int_{\mathrm{R}^n} (A(x)f(x), f(x))dx \geq C \int_{\mathrm{R}^n} |f(x)|^2 dx$$

for all $f \in L_2(\mathrm{R}^n; \mathrm{C}^{1+n})$. In [12], Auscher, Axelsson and McIntosh present an interesting approach to investigate such equations. They express equation (8) as a vector-valued ordinary differential equation. For more detailed explanation write $v \in \mathrm{C}^{1+n}$ as $v = (v_\perp, v_\parallel)$, where $v_\perp \in \mathrm{C}$ and $v_\parallel \in \mathrm{C}^n$. They obtain the equivalence between equation (8) and the following equation

$$\partial_t f + DBf = 0 \tag{9}$$

with constraint $\operatorname{curl}_x f = 0$, where B is a bounded uniformly accretive multiplication operator formed pointwise from A, and D is the self-adjoint differential operator

$$D := \begin{bmatrix} 0 & \operatorname{div} \\ -\nabla & 0 \end{bmatrix}$$

in $L_2(\mathrm{R}^n; \mathrm{C}^{1+n})$. By equivalence, it is meant that the equation (8) for u implies that $f := ((A\nabla_x u)_\perp, \nabla_x u)$ solves (9), and conversely, if f solves (9), then there exists a unique solution (up to a constant) u to (8) such that $f = ((A\nabla_x u)_\perp, \nabla_x u)$.

It was proved that $T := DB$ is an ω-bisectorial operator on $\mathcal{H} := \overline{\mathbf{R}(D)}$, i.e.

$$\sigma(T) \subset S_\omega := \{\zeta \in \mathrm{C} :\ |\arg \zeta| < \omega \text{ or } |\arg(-\zeta)| < \omega\}$$

and for all $\mu \in (\omega, \frac{\pi}{2})$, there exists $C_\mu > 0$ such that $\|(\lambda - T)^{-1}\| \leq C_\mu / |\lambda|$ for all $\lambda \notin S_\omega$. Moreover, it is proved by Axelsson, Keith and McIntosh [13], that T satisfies the quadratic estimate

$$\int_0^\infty \|tT(I + t^2T^2)^{-1}u\| \frac{dt}{t} \leq C\|u\|^2, \quad u \in \mathcal{H}. \tag{10}$$

Bisectoriality of operator T and the quadratic estimate (10) allow one to construct the $H^\infty(S^0_\mu)$ (where $S^0_\mu := S_\mu \setminus \{0\}$) functional calculus designed by McIntosh in [44], i.e. there is an algebra homomorphism

$$\Phi_T : H^\infty(S^0_\mu) := \{f : S^0_\mu \to \mathrm{C} : f \text{ holomorphic }, \|f\|_\infty < C\} \to \mathcal{L}(\mathcal{H}),$$

which satisfies the following conditions

(1) There exists $C > 0$ such that $\Phi_T(f) \leq C\|f\|_\infty$ for all $f \in H^\infty(S^0_\mu)$.
(2) If $g(z) = 1$ for all $z \in S^0_\mu$, then $\Phi_T(g) = I$ on $\mathcal{H}$.
(3) If $\lambda \notin S_\mu$ and $f(z) = (\lambda - z)^{-1}$ for all $z \in S^0_\mu$, then $\Phi_T(f) = (\lambda - T)^{-1}$.
(4) If $\{f_n\}_{n=1}^\infty$ is a sequence in $H^\infty(S^0_\mu)$ that converges uniformly on compact subsets of S^0_μ to $f \in H^\infty(S^0_\mu)$, then $\Phi_T(f_n)$ converges to $\Phi_T(f)$ in $\mathcal{H}$.

These properties are useful tools to investigate (9), and hence (8). The goal of this section is to derive such tools to investigate Helmholtz's equation and the Maxwell's system of equations.

Let us look at a waveguide along straight pipe $\mathrm{R}\times\Omega$. For simplicity of explanation we assume that the cross section $\Omega \subset \mathrm{R}^n$ is bounded and has smooth boundary $\partial\Omega$ with outward normal n. However, in our work Ω is a bounded Lipschitz domain.

Suppose we consider an electromagnetic waveguide ($n = 2$) with permittivity ε, permeability μ and conductivity σ. Define $\varepsilon_* := \varepsilon + i\sigma/\omega$, where ω is the frequency. Then the time-harmonic electric and magnetic fields, $\widetilde{E}(s, x, t) = E(s, x)e^{-i\omega t}$ and $\widetilde{H}(s, x, t) = H(s, x)e^{-i\omega t}$, satisfy Maxwell's equations

$$\begin{cases} \mathrm{div}_{(s,x)}\mu H = 0, \\ -i\omega H + \mathrm{curl}_{(s,x)}E = 0, \\ -i\omega\varepsilon_* E - \mathrm{curl}_{(s,x)}H = 0, \\ \mathrm{div}_{(s,x)}\varepsilon_* H = 0, \end{cases} \tag{11}$$

with boundary condition $E \times \mathrm{n} = 0$ and $\mu H \cdot \mathrm{n} = 0$ on $\partial\Omega$, where $(s, x) \in \mathrm{R} \times \Omega$ and t is time. (Note that in Paper III we do not use a time variable and we use t as a special variable along the waveguide instead of s. Also we skip notation $*$ for ε_*, so that ε is not permittivity but rather the combination of permittivity and conductivity.) We assume that μ, $\varepsilon_* \in L_\infty(\Omega; \mathcal{L}(\mathrm{C}^3))$ are uniformly strictly accretive and s-independent, i.e. the properties of material do not change along the waveguide. By uniformly strictly accretivety, we mean that there exist $C > 0$ such that $\mathrm{Re}(\mu(x)\zeta, \zeta) > C|\zeta|^2$ for all $x \in \Omega$ and all $\zeta \in \mathrm{C}^3$.

Further on, suppose we study an acoustic waveguide($n \geq 1$) with wave number k. Then we need to consider the Helmholtz equation, or reduced wave equation

$$\begin{bmatrix}\mathrm{div}_{(s,x)} & k\end{bmatrix} A(x) \begin{bmatrix}\nabla_{(s,x)} \\ k\end{bmatrix} u = 0 \tag{12}$$

with Dirichlet boundary condition $u(s, \cdot) = 0$ on $\partial\Omega$ for all $s \in \mathrm{R}$. We also assume that $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathrm{C}^{n+2}))$ is s-independent uniformly strictly accretive, so that properties of matter do not change along the waveguide.

In both cases, we express the Helmholtz equation (12) and the Maxwell's system of equations (11) as a vector valued ordinary differential equation

$$\partial_s f + DBf = 0 \tag{13}$$

as in [12], where B is a bounded uniformly accretive matrix formed from μ, ε_* or A, and D is an operator, which is equal to

$$D_H := \begin{bmatrix} 0 & \mathrm{div} & k \\ -\nabla_0 & 0 & 0 \\ -k & 0 & 0 \end{bmatrix}$$

in Helmholtz's case and

$$D_M := \begin{bmatrix} 0 & \mathrm{div}_0 & 0 & 0 \\ -\nabla & 0 & 0 & i\omega J \\ 0 & 0 & 0 & \mathrm{div} \\ 0 & -i\omega J & -\nabla_0 & 0 \end{bmatrix}, \quad \text{where} \quad J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

in Maxwell's case. Here ∇, ∇_0, div and div_0 denote gradient and divergence operators on $H^1(\Omega)$, $H^1_0(\Omega)$, $H_{\mathrm{div}}(\Omega; \mathrm{C}^n)$ and $H^0_{\mathrm{div}}(\Omega; \mathrm{C}^n)$ respectively.

The main difference from divergence form equations, we mentioned before, is that D is not self-adjoint, and hence $T := DB|_{\mathbf{R}(D)}$ is not necessary a bisectorial operator. This leads us to modify the functional calculus we discussed above. Observing that D is the perturbation of a self-adjoint operator by a bounded operator, we prove that T is "close" to the bisectorial operator in the sense that there exist $\tau > 0$ and $\omega \in (0, \frac{\pi}{2})$ such that

$$\sigma(T) \subset S_{\omega,\tau} := \{x + iy \in \mathrm{C} : |y| < |x| \tan \omega + \tau\}$$

and there exists a constant $C > 0$ such that for any $\lambda \notin S_{\omega,\tau}$,

$$\|(\lambda - DB)^{-1}\| \leq \frac{C}{\mathrm{dist}(\lambda, S_{\omega,0})}, \quad u \in \mathcal{H} := \mathbf{R}(D). \tag{14}$$

Anther difference is the boundedness of Ω. This implies that operator T has purely discrete spectrum with only accumulation point at infinity and each eigenvalue has finite algebraic multiplicity.

Discreteness of the spectrum allows us to separate the eigenvalues on regions $\{\mathrm{Re}\lambda < 0\}$, $\{\mathrm{Re}\lambda = 0\}$, and $\{\mathrm{Re}\lambda > 0\}$ by appropriate curves. A finiteness of algebraic multiplicities of the eigenvalues, together with resolvent bounds (14) and quadratic estimate (10), give us necessary inequalities to build a functional calculus. Applying our new functional calculus, we prove that all polynomial bounded time-harmonic waves in the semi-infinite or bi-infinite waveguide have representation in $\mathbf{R}(\Pi_0)$ or $\mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_+)$, where Π_-, Π_0 and Π_+ are spectral projections corresponding to the spectrum on the left-half plane, imaginary axis, and on the right-half plane.

2.2 In Chapter I, we briefly discussed Weyl's law. This law has both geometric generalizations, in which the underlying domain or manifold is no longer smooth; as well as analytic generalizations, in which the Laplace operator is replaced by a different, but typically Laplace-like operator. Here we simultaneously consider both a geometric generalization as well as an analytic generalization. We are inspired by the work of the Soviet mathematicians, Birman and Solomyak [19], who made a fundamental contribution to the study of the eigenvalue asymptotics for elliptic operators with non-smooth coefficients nearly fifty years ago.

We consider compact manifolds with a smooth differentiable structure and allow the possibility that such manifolds also carry a smooth boundary. However, the Riemannian-like metric in our setting, known as a rough metric, is only assumed to be measurable. Such a rough metric is only required to be bounded above in an L^∞ sense, and essentially bounded below. A smooth topological manifold, M, equipped with a rough Riemannian metric, g, is henceforth dubbed a *rough Riemannian manifold*.

Definition 0.3 (Rough metric)**.** We say that a symmetric $(2,0)$ measurable tensorfield g is a *rough metric* if it satisfies the following *local comparability condition*: for each $x \in M$, there exists a chart (U_x, ψ_x) containing x and a constant $C(U_x) \geq 1$ such that

$$C(U_x)^{-1}|u|_{\psi_x^*\delta(y)} \leq |u|_{g(y)} \leq C(U_x)|u|_{\psi_x^*\delta(y)}$$

for almost-every $y \in U_x$, for all $u \in T_yM$. Above, $\psi_x^*\delta$ is the pullback to U_x of the $\mathbb{R}^n$ scalar product inside $\psi(U_x)$.

Remark 0.4. As a consequence of the compactness of M, we note that the compatibility condition is equivalent to demanding that there exists a *smooth Riemannian metric, h,* on M such that

$$C(U_x)^{-1}|u|_h \leq |u|_g \leq C(U_x)|u|_h$$

for almost-every $y \in U_x$, where U_x, u, and $C(U_x)$ are as in Definition 0.3.

Due to the regularity of the coefficients of a general rough metric g, it is unclear how to associate a canonical distance structure to g. However, the expression

$$\sqrt{\det g(x)}\ d\psi_x^*\mathcal{L},$$

for almost-every $x \in U_x$ inside a compatible a chart (U_x, ψ_x), can readily be checked to transform consistently under a change of coordinates. This yields a Radon measure that is independent of coordinates, which we denote by μ_g. Therefore we may define $L^k(T^{(p,q)}M,\ d\mu_g)$ spaces in the usual way.

Now, let us state the problem we want to solve. Let M be a compact manifold with a smooth differential structure and smooth boundary. We consider the Laplace operator with admissible boundary condition, that is the operator, $\Delta_{g,\mathcal{W}}$, associated with the form

$$\mathcal{E}_{g,\mathcal{W}}[u,v] = (\nabla u, \nabla v)_{L^2(T^*M,\ d\mu_g)}, \quad u, v \in \mathrm{D}(\mathcal{E}_{g,\mathcal{W}}) := \mathcal{W},$$

where $\mathcal{W}$ is a closed subspace of the Sobolev space $H^1(M)$ containing $H_0^1(M)$. Let $\beta > \frac{n}{2}$ and $\rho \in L^\beta(M,\ d\mu_g)$ is a real-valued function such that $\int_M \rho\ d\mu_g \neq 0$. We aim to investigate the eigenvalue problem for the weighted Laplace equation

$$\Delta_{g,\mathcal{W}} u = \lambda \rho u. \tag{15}$$

We understand this eigenvalue problem in the following way. Consider the subspace:

$$Z(\rho) = \begin{cases} \mathcal{W} & \text{if } \mathcal{E}_{g,\mathcal{W}} \text{ generates the norm in } \mathcal{W}, \\ & \text{which is equivalent to } H^1 \text{ norm }, \\ \left\{u \in \mathcal{W} :\ \int_M \rho u\ d\mu_g = 0\right\} & \text{otherwise.} \end{cases}$$

We show that $Z(\rho) \subset \mathcal{W}$ closed in H^1 norm, and that $\mathcal{E}_{g,\mathcal{W}}[\cdot,\cdot]$ is equivalent to H^1 norm. Therefore, $Z(\rho)$, equipped with the norm $\mathcal{E}_{g,\mathcal{W}}[\cdot,\cdot]$, is a Hilbert space, $(Z(\rho), \mathcal{E}_{g,\mathcal{W}})$. In this space, the form

$$\rho[u,v] := \int_M \rho u \bar{v}\ d\mu_g, \quad \mathrm{D}(\rho) = Z(\rho)$$

is completely continuous, hence the eigenvalue problem

$$\rho[u,v] = \lambda \mathcal{E}_{g,\mathcal{W}}[u,v], \quad u,v \in (Z(\rho), \mathcal{E}_{g,\mathcal{W}}) \tag{16}$$

is well defined. Finally, we note that if λ is solution for (16), the $1/\lambda$ is solution for (15), and visa versa. The main result of this paper is

Theorem 0.5. *Let M be a smooth compact manifold of dimension ≥ 2 with smooth boundary, and let g be a rough metric on M. Then, the eigenvalues of (16) are discrete with finite dimensional eigenspaces with positive and negative eigenvalues, $\{-\lambda_j^-(\mathcal{W}); \lambda_j^+(\mathcal{W})\}_{j=1}^\infty$, such that*

$$-\lambda_1^-(\mathcal{W}) \leq -\lambda_2^-(\mathcal{W}) \leq \ldots < 0 < \ldots \leq \lambda_2^+(\mathcal{W}) \leq \lambda_1^+(\mathcal{W}).$$

Moreover, they satisfy the Weyl asymptotic formula

$$\lim_{k\to\infty} \lambda_k^\pm(\mathcal{W}) k^{\frac{2}{n}} = \left(\frac{\omega_n}{(2\pi)^n}\right)^{\frac{2}{n}} \left(\int_{M^\pm} |\rho(x)|^{\frac{n}{2}}\ d\mu_g\right)^{\frac{2}{n}} = \left(\frac{\omega_n}{(2\pi)^n}\right)^{\frac{2}{n}} \|\rho\|_{L^{\frac{n}{2}}(M^+,\ d\mu_g)}.$$

Above, $M^\pm := \{x \in M : \pm\rho(x) > 0\}$.

Spectral properties of the Sturm-Liouville operators with singular potentials (Papers V, VI , and VII)

Spectral properties of the Sturm-Liouville operators have been studied for more than a century due to numerous applications. One of the best studied topics in this theory is the eigenvalue asymptotics. There are many publications in this field; we mention only the books [37, 39, 40, 49, 57].

Sturm-Liouville spectral problems are naturally divided into two classes. The problem

$$Hy \equiv -y'' + q(x)y = \lambda y, x \in I = (x_0, x_1), \tag{17}$$

with certain boundary conditions at the endpoints x_0, x_1 was initially called regular if the interval I is finite and the "potential" q is continuous on $\bar{I}$, otherwise the problem used to be called singular. In the regular case, the spectral theory can be usually reduced to some problems in complex analysis and algebra. The singular problems are much more complicated and they require some hard analysis. Here we consider the following three problems:

(1) The problem on $I = (0,1)$ with $q(x)$ behaving like $-x^{-\alpha}$, $\alpha > 2$ near zero;

(2) The problem on $I = \mathbb{R}$ with $q(x) = |x|^\alpha + V(x)$, $\alpha > 0$ and V is τ-Hölder;

(3) The problem on $I = (0,\infty)$ with $q(x) = \sum c_i \delta(x - t_i)$, $c_k > 0$.

These problems are significantly different from each other. Therefore we use different approaches in order to investigate these problems.

3.1 The study of the asymptotic behavior of eigenvalues of the Sturm-Liouville operator on $(0,\infty)$, with $q(x)$ bounded near zero and tending to $-\infty$ at infinity sufficiently fast, so that the limit circle at infinity takes place, started in 1954, see [28]. The specifics of the problem required a new approach. The operator is not semi-bounded, so the variational method, very efficient for semi-bounded operators, could not be applied. Bookkeeping zeroes of solutions, also widely used for semi-bounded problems (the number of the eigenfunction for a regular problem is closely related to the quantity of its zeroes), could not be applied either, since all solutions oscillate rapidly at infinity and have infinitely many zeroes. P. Heywood found in [28] a modification of the zero-counting method. First, the problem on the finite interval $(0, b)$ was considered, with some boundary conditions set at the point b. The corresponding operator is denoted H_b. For fixed $\lambda > 0, -\mu < 0$, the number of eigenvalues on $(0, \lambda)$ and $(-\mu, 0)$, $N(H_b; (0, \lambda))$ and $N(H_b; (-\mu, 0))$, are studied. This is achieved by evaluating the number of zeroes $n(b, s)$ of the solutions of the equation $(H - s)y = 0$ on the interval $(0, b)$ for $s = 0, s = \lambda$ and $s = -\mu$. Although, as $b \to \infty$, each of these quantities grows unboundedly, the differences $n(b, \lambda) - n(b, 0)$ and $n(b, 0) - n(b, -\mu)$ turn out to be bounded, uniformly in b, and, moreover, they admit an explicit expression, not depending on b, with an error term, uniformly bounded in b. This information on the zeroes produces the expressions for $N(H_b; (0, \lambda))$ and $N(H_b; (-\mu, 0))$. The final step consists in proving that these expressions converge, as $b \to \infty$, to the corresponding counting functions for the operator on the semiaxis.

This result in [28] differs drastically from the ones for the semibounded case. Some regularity conditions, to be specified later on, are imposed, the first one being the (eventual) monotonicity of the potential $q(x)$, and the asymptotic formulas, with $h(p(\mu)) = \mu$, are

$$N(H,(0,\lambda)) = \pi^{-1}\int_0^\infty [(\lambda + h(x))^{\frac{1}{2}} - h(x)^{\frac{1}{2}}]dx + O(1),\ \lambda > 0, \tag{18}$$

$$N(H,(-\mu,0)) = \pi^{-1}\int_0^{p(\mu)} h(x)^{\frac{1}{2}}dx + \pi^{-1}\int_{p(\mu)}^\infty [h(x)^{\frac{1}{2}} - (h(x)-\mu)^{\frac{1}{2}}]dx + O(1). \tag{19}$$

Much later, in 1974, without (initially) knowing about [28], the problem of the eigenvalue asymptotics for the case $q(x) \to -\infty$ was considered by Belogrud and Kostuchenko, [16]. Actually, a short note, without proofs, appeared in [16], but a more detailed exposition was published in (now inaccessible) [15], with the final presentation filling chapters 5 and 9 in the book [37].

Further activities in this topic concentrated on improving the asymptotic estimates. This is impossible to do in the terms of the counting function: since $N(H,(\lambda_1,\lambda_2))$ is an integer, a remainder estimate better than $O(1)$ is impossible. On the other hand, if a formula is found, expressing the eigenvalues themselves in an implicit form as solutions of some equations, such results can give improved asymptotic formulas for the eigenvalues, with a higher order of accuracy. The first result of this kind was obtained by Alenitsyn in [6]. By finding an asymptotic expression, with several terms, of solutions of the equation, using the WKB method, Alenitsyn derived two-term equations (for the positive and for the negative spectrum) determining the eigenvalues in an implicit form. An important feature of this sharpening is that one can trace the dependence of the eigenvalue asymptotics on the parameter fixing the self-adjoint extension by setting the boundary conditions at infinity - which was impossible by the previously used methods.

Some years later, a series of papers by Atkinson and Fulton appeared; see [9–11]. In the seminal paper [9] a new approach to non-semibounded problems is presented, based upon a modified Prüfer transform, reducing the second order linear equation to a system of first order nonlinear equations, for which the asymptotic analysis becomes more feasible. Besides deriving an improved Heywood formula, in Alenitsyn style, and demonstrating a number of interesting examples and consequences, the authors in [9] announce subsequent papers, [8, 10, 11], where the approach would be developed further, in order to give algorithmically arbitrary many higher order terms in the implicit expression for the eigenvalues. The three cases announced are:

(1) The problem on $(0,\infty)$ with $q(x)$ tending to $-\infty$ faster than $-x^2$ at infinity;
(2) The problem on $(0,1)$ with $q(x)$ behaving like $Cx^{-\alpha}$, $\alpha \in [1,2)$ near zero;
(3) The problem on $(0,1)$ with $q(x)$ behaving like $-x^{-\alpha}$, $\alpha > 2$ near zero.

The papers [10,11], containing the analysis of the cases (1) and (2), have appeared. However, the paper [8], although announced several times, was never published.

A rather complete spectral analysis of the singular non-semibounded Sturm-Liouville operator on the semi-axis with singularity at infinity was performed long ago, while for the complementing case, the potential tending rapidly to $-\infty$ at the finite endpoint remains completely unresolved.

This part is devoted to filling this gap. We modify the approach initiated by Heywood and find the asymptotic formulas for the eigenvalue counting functions.

The main part of the paper is devoted to finding the asymptotic formulas for eigenvalues. These formulas show, in particular, the presence of a new effect, not existing for the limit-circle problem at infinity, considered previously. Namely, it turns out that the asymptotics for the positive eigenvalues, according to the formula obtained, is the same, at least in the leading term, for all potentials subject to the regularity conditions, in particular, the same as for the regular problem. On the other hand, the asymptotics for the negative eigenvalues depends essentially on the potential. Moreover, it turns out that, asymptotically, there are fewer negative eigenvalues than positive ones, on intervals of the same length.

3.2. In the second part of this chapter, we consider a perturbation of the generalized anharmonic oscillator

$$H \equiv -\frac{d^2}{dx^2} + |x|^\alpha + V(x), \qquad x \in \mathbb{R}, \qquad \alpha > 0.$$

Due to some important applications in physics, this class of operators are well studied. Especially the harmonic oscillator ($\alpha = 2$ and $V = 0$). Its eigenvalues equal $\lambda_n = 2n - 1$ ($n \in \mathbb{N}$) and the corresponding normalized eigenfunctions are explicitly expressed in terms of the Chebyshev-Hermite polynomials; see for instance [39]. In the general setting, perturbed anharmonic oscillator cannot be solved analytically, and one has resort to the asymptotic solutions. Under rather mild conditions on V, the main term of the spectral asymptotics of such operators are well known; see for instance [57]. It is more difficult to obtain further terms of the asymptotics. We mention works concerning the spectral asymptotics of the perturbed anharmonic oscillator: [1, 20–22, 25–27, 33, 36, 52, 53].

In most works concerning the spectral properties of perturbed anharmonic oscillator, the perturbations are smooth. For an actual real-world potential smoothness is not necessarily guaranteed. For this reason, we want to reduce the smoothness and explore how this will affect the eigenvalues, we require $V(x)$ to be only piecewise Hölder-continuous:

Theorem 0.6. *Let H be the self-adjoint operator in $L^2(\mathbb{R})$, generated by*

$$-\frac{d^2}{dx^2} + |x|^\alpha + V(x), \tag{20}$$

where $\alpha > 0$, and $V(x)$ is a bounded, real-valued, compactly supported, piecewise Hölder continuous function with an exponent $\tau > 0$. Then the sequence of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of H satisfies the following asymptotic formula

$$\begin{aligned}\lambda_n &= C_1^{-\frac{2\alpha}{\alpha+2}}(2n-1)^{\frac{2\alpha}{\alpha+2}} \\ &+ \frac{2\alpha}{\alpha+2} C_0 C_1^{-\frac{\alpha+4}{\alpha+2}}(2n-1)^{-\frac{2}{\alpha+2}} \\ &+ \frac{2\alpha}{\alpha+2}\frac{1}{4\pi} C_1^{-\frac{\alpha+4}{\alpha+2}}(2n-1)^{-\frac{2}{\alpha+2}} \int_{-\infty}^{\infty} V(s)\cos\left(2C_1^{-\frac{\alpha}{\alpha+2}}(2n-1)^{\frac{\alpha}{\alpha+2}} s\right) ds \\ &+ \frac{2\alpha}{\alpha+2} C_2 C_1^{-\frac{\alpha+6}{\alpha+2}}(2n-1)^{-\frac{4}{\alpha+2}} + O\left(n^{-1}\right),\end{aligned} \tag{21}$$

where

$$C_1 = \frac{4\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\pi\Gamma\left(\frac{3}{2}+\frac{1}{\alpha}\right)}, \quad C_0 = \frac{1}{\pi}\int_{-\infty}^{\infty} V(s)ds, \quad C_2 = \frac{\alpha-1}{12\pi(2+\alpha)}\cot\left(\frac{\pi}{\alpha}\right) C_1^{-1},$$

and $\Gamma(\cdot)$ *is the gamma function.*

Theorem 0.6 shows that the perturbation, $V(x)$, does not affect the first term. However, it appears in the second term, while the regularity, the parameter τ, affects only the third term. Indeed, in case $V(x)$ being smooth and compactly supported, the third term would decay rapidly. When $V(x)$ is Hölder continuous with an exponent $\tau > 0$, we can say only that the third term is $O(n^{-\frac{\alpha\tau+2}{\alpha+2}})$. In order to demonstrate more explicitly the effect of the smoothness, we construct an example. There we consider the operator H from Theorem 0.6 for $\alpha = 2$ and $V(x)$ being the Weierstrass nowhere differentiable function. Then we find the subsequence of the eigenvalues $\{\lambda_{n_k}\}_{k=1}^{\infty}$ such that

$$\lambda_{n_k} = 2n_k - 1 + n_k^{-\frac{1}{2}} \frac{1}{4\sqrt{2}} \int_{-\pi}^{\pi} V(s)ds + n_k^{-\frac{1+\tau}{2}} 2^{-\frac{5+3\tau}{2}} + O(n_k^{-1}).$$

3.3. The last part of this chapter concerns Sturm-Liouville operator with the potential being a sum of delta functions

$$Hy \equiv -y'' + \sum_{t_i \in I} c_i \delta(x - t_i) y = \lambda y \quad x \in I = (x_0, x_1), \tag{22}$$

where c_i is a coupling constant attached to the point source located at t_i, and δ is Dirac δ-function(i.e. the unit measure concentrated at 0).

Models of this type have already been discussed extensively, particularly in the physical literature concerned with problems in atomic, nuclear, and solid state physics. They occur in the literature under various names, like "point interaction models", "zero-range potential models", "delta interaction models", "Fermi pseudopotential models", and "contact interaction potential".

Historically, the first influential paper on this models of (22) was that by Kronig and Penney [38], in 1931, who treated the case $\{t_k\}_{k=-\infty}^{\infty} = \mathbb{Z}$ with $c_k = c$ independent of t_k. This "Kronig-Penney" has become a standard reference model in solid state physics, see for instance [32], [59]. It provides a simple model for a nonrelativistic electron moving in a fixed crystal lattice. A few years later, Bethe and Peierls [18] and Thomas [56] started to discus models of three dimensional generalization $-\Delta + q(x)$ with only one interaction at 0. Such models also arise in the theory of sound and electromagnetic wave prorogation in dielectric media, where the role of the point interactions is replaced by boundary conditions at suitable geometric configurations. Such relations have been pointed out and exploited in the work by Heisenberg, Jost [30], Lieb and Koppe [41], Nussenzveig [50], and others.

Subsequent studies aimed at the clarification of this state of affairs led in particular to the first rigorous mathematical work by Berezin and Faddeev [17] in 1961 on the definition of operator of type (22)(in three dimensional case) as self-adjoint operators in $L_2(\mathrm{R}^3)$. For more details see [4].

A correct definition of the Sturm-Liouville operator whose potential $q(x)$ is a distribution(not only delta potential) of first order is given by Savchuk and Shkalikov in [54]. Also we mention relatively modern papers [2, 3, 5, 29, 34, 35, 46, 47, 55].

In this paper we consider a case, when $I = [0, \infty)$, $\{c_k\}_{k=1}^{\infty}$ is a sequence of positive numbers and $T = \{t_k\}_{k=1}^{\infty} \subset I$ is an increasing sequence, which tends to ∞. We understand the operator H in the following way. Let H_0 be the one-dimensional

Laplace operator with domain

$$\mathbf{D}(H_0) = \Big\{ y \in W_2^1(\mathbb{R}^+) \cap W_2^2(\mathbb{R}^+ \backslash T) : y'(0) = 0,\ \sum_{j=1}^{+\infty} c_j |y(t_j)|^2 < \infty,$$
$$y'(t_j + 0) - y'(t_j - 0) = c_j y(t_j),\ j \in \mathbb{N} \Big\}.$$

Then, by operator H, we mean the self-adjoint extension of H_0. The reason we associate this extension with (22) is that formally, for $u, v \in \mathrm{D}(H_0)$, we compute

$$(H_0 u, v) = \|y'\|^2 + \sum_{k=1}^{\infty} c_k |y(t_k)|^2 = \left\langle -y'' + \sum c_k \delta(x - t_k) y, y \right\rangle.$$

The main result of this work is two-sided estimates of the distribution function of the eigenvalues, which are given in terms of the regularized function q^* of q (averaging of q in some literature)

$$q^*(x) := \inf_{d>0} \left\{ d^{-2} : \sum_{t_k \in \Delta(d,x)} c_k \le d^{-1} \right\}$$

where $\Delta(d, x) = [x - d/2, x + d/2]$ for $x \in I$. Different versions of such functions appear in [37], [48] and later papers by the authors of [48]. The result we mentioned above states

$$\frac{\sqrt{\lambda}}{2\sqrt{\pi^2+1}} \mathrm{mes} \left\{ q^*(x) \le \frac{\lambda}{16(\pi^2+1)} \right\} < N(H, (0, \lambda)) < \pi\sqrt{\lambda} \mathrm{mes}\{q^*(x) \le \pi^2 \lambda\}.$$

To prove the upper bound we use localization method, i.e. we divide I to the intervals $\{\Delta_k\}$ in appropriate way and estimate $N(H, (0, \lambda))$ by sum of $N(H_{\Delta_k}, (0, \lambda))$. To prove the lower bound we use variational methods.

As a consequence, we prove a criteria for discretness of the spectrum in terms of the function q^*. We also establish s criterion for the resolvent to belong to the Schatten class $\mathfrak{S}_p$.

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