UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

Optimal financial resources for Central Counterparties

Introducing default dependence of clearing members: a mixed binomial approach

Author
Leonardo Di Geronimo

Supervisor
Prof. Alexander Herbertsson

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School of Business, Economics and Law
University of Gothenburg
Sweden

Abstract

Central counterparties (CCPs) are financial intermediaries consisting of clearing members trading financial derivatives between each other. In a financial network, CCPs become the *buyer to every seller and the seller to every buyer*. After the 2007-2008 financial crisis, so called central counterparties have become fundamental financial institutions worldwide. Nahai-Williamson et al. (2013) develop an expected loss function for clearing members to investigate and find the optimal quantities of central counterparties financial resources, i.e. initial margin and default fund, which are safety contributions to the CCP to absorb potential future losses in case of one or several member's defaults. Nahai-Williamson et al. (2013) assume exogenous and independent individual default probabilities, which are uncorrelated with the underlying prices of assets cleared through the CCP. In this thesis, we extend the Nahai-Williamson et al. (2013) model by using a Merton mixed binomial model, which allows for realistic dependencies among default probabilities and lets the prices be correlated with default probabilities themselves. We define a new expected loss function for clearing members, which is minimized with respect to initial margin and default fund and obtains new optimal quantities for CCP's financial resources in our extended model. The new framework with default and price dependencies will change the optimal quantities of sources: initial margin and default fund contributions will be different and higher than previous optimal quantities in Nahai-Williamson et al. (2013). In some cases, our default fund contributions will be 200%, 300% and even 1500% larger than optimal contributions found by Nahai-Williamson et al. (2013). Moreover, the balance between CCP's initial margin and default fund will tend more to the default fund rather than any other financial source. Although it does not concern optimal financial resources, we also find that in the Merton-extended version the expected loss function itself is sometimes 22% and 55% higher than the one defined by Nahai-Williamson et al. (2013) in the same conditions. The economic interpretation of this result is that higher default dependence leads to higher losses, which should be better covered by higher mutualization between clearing members.

Keywords: *Central Counterparties,Risk Management, Merton Model, Mixed Binomial Model, Merton Mixed Binomial Model, Initial Margin, Default Fund.*

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List of Abbreviations

CCP	Central counterparty
CCPs	Central counterparties
CPSS	Committee on Payment and Settlement System
DF	Default fund
EMIR	European Market Infrastructure Regulation
ESMA	European Securities and Market Authority
EU	European Union
G-20	Group of Twenty
IM	Initial margin
IOSCO	International Organization of Securtities Commissions
ITM	In-the-money
OECD	Organization for Economic Cooperation and Development
OTC	Over the counter
OTM	Out-of-the-money
US	United States of America
QCCP	Qualified central counterparty
VaR	Value-at-risk

Chapter 1

Introduction

In the last century, financial markets have become the central place to exchange, sell and buy commodities, stocks, bonds and any other financial contract. Across time, financial markets have become more structured and complex, starting with derivatives at the end of '70s up to very articulated contracts still renewing everyday on the market. However, the baseline of any financial network is still the same: different agents (individuals, companies, institutions, etc.) meet to trade contracts in order to invest, profit, make insurances and so on. These agents have different accesses to information, they are free to choose where to spend their money and they all respect a finer and finer group of laws and regulations stated by national and international institutions.

Financial derivatives are very structured financial contracts whose result is affected by an underlying asset (commodities, indexes, prices and so on). Derivatives can be traded *in-the-counter* (ITC), on officially controlled stock exchanges, or *over-the-counter* (OTC), in a bilateral relation between two private agents. As stated in Ghamami (2015), OTC derivatives played a crucial role in 2007-2008 financial crisis and this is the reason why the 2009 G-20 mandate gave central counterparties a major role in modern derivatives trading. Usually, all financial network agents used to trade derivatives between themselves, often over-the-counter: this means that every agent has an open position with every other agent in the market, facing the possibility of other members' incapability to pay back their obligations. A central counterparty (CCP) is an institution that intermediates between all these agents in the network: the CCP becomes the *buyer to every seller and the seller to every buyer*. In this new framework, agents have just one open position on the market, the one dealing with the CCP, and the clearing house checks the condition of every member's position, it pays off when their positions are positive, it requires to be paid when these positions are negative.

CCPs play a crucial role in contemporary financial network, this is why it has become so important to implement a model to define its financial sources paid by the members. The CCP is a central institution that clears all agents' positions. However clearing houses are not public institutions: Pirrong (2011) claims that they are profit companies, they have to manage their fundings and they can experiment bankruptcy, which would be extremely dangerous given their delicate qualification. Among many models provided to find the optimal sources in a CCP, this thesis will focus on and extend the one implemented by Nahai-Williamson et al. (2013), where the authors provide the framework in which the CCP operates assuming some circumstances on the environment, the network and the administration. Then, Nahai-Williamson et al. (2013) define an expected loss function for CCP's members, considering all the possibilities of multiple defaults and also the CCP's default. This expected loss function for each clearing members contains all the parameters affecting the clearing activity, so it embeds also the financial resources of the CCP. Once the expected loss function is found, the authors minimize this function in terms of CCP's financial resources: they find the optimal quantities of financial sources that clearing houses have to conserve to be able to reduce the expected loss as much as possible. Essentially, Nahai-Williamson et al. (2013) provide a model to know the numerical quantities of financial resources in order to make the CCP work as efficiently as possible.

In this thesis, we replicate and then extend the model by Nahai-Williamson et al. (2013), so that it includes default dependencies among clearing members and the prices of underlying assets to be correlated with the default probabilities. In Nahai-Williamson et al. (2013), the authors make two important assumptions:

- The individual default probabilities of each agent clearing through the CCP is exogenous and independent from anything else; it means that the probability of default is a number given and assumed by the authors;
- The underlying price that is responsible for the changes in every position (because price variations are reflected by derivatives variations) is uncorrelated with default probabilities.

Individual default probabilities are not independent and exogenous: the probability to default depends on many factors, like eventual losses, other agents' defaults, the possibility of losses to spread in the trading network. Moreover, prices are highly affected by the whole economic environment in multiple ways, they are not untied from the rest of the world. In this thesis, we relax the above unrealistic assumptions and by using static credit risk modelling, more specifically a Merton mixed binomial model, we build a modelling framework in which individual default probabilities and underlying prices are dependent and influenced by economic background factors. With this new and more realistic framework with dependent default probabilities and prices, we repeat the same procedure as in Nahai-Williamson et al. (2013): there is an expected loss function (which will be different) and it will be minimized with respect to CCP's financial sources. The optimal quantities of CCP's financial resources in our extended framework will differ from Nahai-Williamson et al. (2013) previous results and will be generally higher than in Nahai-Williamson et al. (2013). Moreover, the balance between different types of CCP's financial resources will be diverse: the optimal quantities of sources will tend on one kind of CCP's financial resource rather than how it was predicted by Nahai-Williamson et al. (2013). More specifically, in our framework, default fund contributions are up to 200%, 300% and even 1500% larger than the ones found by Nahai-Williamson et al. (2013) in similar parameter settings. Hence, CCPs need even fifteen times the amount of default fund resources stated by previous authors. Although the expected loss function itself does not have direct relevance here (because we investigate the optimal quantities of initial margin and default fund that minimize the function), we believe it is worth to mention that in our extended version the expected loss function itself is 22% and even 55% larger than the one found by Nahai-Williamson et al. (2013) in the same setting. As soon as we have dependencies both in default probabilities and underlying prices, the initial margin becomes an inefficient resource to respond to members' losses: higher dependencies bring to higher predicted losses, which are better covered through the sharing mechanism of default fund rather than individual collateral, i.e. initial margin.

The rest of the thesis will be structured as follows: Chapter 2 provides a general and extensive introduction to central clearing and central counterparties, what are the CCPs, which are their financial resources and the main international regulation on the theme. Chapter 3 will present a literature review on the topic of central counterparties and their risk management. Chapter 4 will explain in detail our implementation of the model developed by Nahai-Williamson et al. (2013) and results. Chapter 5 will provide a clear outline of the static credit risk modelling, more specifically, of a Merton mixed binomial model, which allows to construct a framework in which default probabilities and prices are dependent and more realistic. Finally, Chapter 6 will embed the new Merton framework with dependent probabilities and prices in the expected loss for CCPs' members and it will present the new results.

Chapter 2

Central counterparties

Central counterparties are intermediate institutions in financial network grouping together all agents' positions, to be able to net their gains and losses more efficiently and to make the whole market structure more transparent. This chapter contains an overall review of all the basic concepts and issues about central counterparties and their networks: these themes are necessary to be able to understand the model by Nahai-Williamson et al. (2013) and our extension of their model. Hence, the Section 1 begins with the definition of risk, credit risk and its components; Section 2 explains what are central counterparties; Section 3 raises all the issues with CCPs resources and their risk management; Section 4 provides a brief sum of al the international regulations regarding central clearing and CCPs.

2.1 Financial risk

In this section very general concepts as financial risk and risk management are explained: financial risk is what drives the whole modelling around financial markets, to be able to analyse, forecast and prevent losses. This general definition of risk is usually divided in many components, in attempt to optimize risk management and resources. These definitions mainly come from the writings by Herbertsson (2018) and Farago (2018).

2.1.1 Financial risk components

Financial risk is the general risk that occurs whenever managing a portfolio or an investment in any financial market: it is the uncertainty linked to decisions. It is of extreme importance to manage portfolios and investments to meet certain risk criteria, whether decided internally in the institution or externally by regulatory agencies. According to Farago (2018), financial risk can be divided in multiple components to better understand its nature and management:

- *Market risk* is the one arising by changes in market prices: by holding any type of financial contract, everyone suffers continuous movements in prices of assets, interest rates, exchange rates and so on;

- *Operational risk* is the risk of losses resulting from failure or errors in internal processes, people and systems surrounding the whole financial institution; it is observed whenever any external event has a negative impact on financial sources management;

- *Liquidity risk* results from any lack of marketability of any investment; it may happen financial contracts cannot be bought or sold quickly enough to prevent losses or to respect payments and so on; factors like quantity of goods and investment size can make a product very illiquid;

- *Model risk* is the one faced by researchers and analysts and is the risk of using an improper model: the model could be wrong, not sufficiently tested or not statistically significant.

2.1.2 Credit risk

Credit risk is the risk of losses whenever a debtor does not honour its payments to a creditor. Debtors can be of any kind: companies that borrow money from banks in the form of loans, companies that issue bonds, companies or individuals that open a mortgage, anyone who is obliged to pay back someone or some institution can be a debtor. Naturally, many events could happen in between the life of these bilateral contracts: at the end, it is possible that the debtor is not willing to pay or cannot pay because of shortage of financial sources. A *default* occurs anytime the debtor cannot honour his payments: the debtor declares bankruptcy and then the administration and liquidation of its remains are practised following bankruptcy laws, in order to satisfy each creditor. Defaults are extreme by definition and credit risk modelling is the theory that tries to model these events and their probabilities.

According to Schönbucher (2003), credit risk can be itself decomposed in different components:

- *Arrival risk* is the risk connected to whether or not a default will happen in a limited time period;

- *Timing risk* is the uncertainty connected to the precise moment in time in which arrival risk will occur;

- *Recovery risk* describes the uncertainty of the exact amount of losses to face if default really occurs;

- *Default dependency risk* invokes several obligors to jointly default during one specific time period; this concept fades in the definition of systemic risk in the next subsection.

Credit risk is a crucial component of financial risk and it includes the so called counterparty risk, which is the risk that a counterparty in a derivatives transaction will default and therefore make no required payment. Acharya and Richardson (2009) studied the role of counterparty risk in 2007-2008 financial crisis and they found that trading derivatives OTC without any public regulation was the reason for a complex network of risk exposures that imploded in 2008. Counterparty risk has become more regulated since then and it is one of the main reasons for the existence of central clearing and the importance of their financial resources (see Section 2.3).

2.1.3 Systemic risk

In financial markets, *systemic risk* happens when one or several financial investors default and create a domino effect among the entire financial network. One default makes one network member insolvent, then it is possible that those who had to receive the payment become insolvent themselves and this type of events spreads throughout the whole system, eventually causing a threat for local, national or even global financial system. Systemic risk could arise from a loss in one company or institution that spreads through a sort of chain reaction, this is called a *contagion*. More specifically, Pirrong (2011) divides between a so called *distress contagion* and a *default contagion*: the former describes the spread of some limited losses in the network, while the latter defines a domino effect of bankruptcies.

Central counterparties are crucial when preventing systemic risk and default contagion. On one side, the main purpose of CCPs is decreasing counterparty risk thanks to a new and more efficient allocation of risk, but on the other hand, another role of central clearing is avoiding and contain systemic risk. CCPs will hopefully ensure that contagion and default chain reactions do not occur in a financial network. In order to prevent contagion, CCPs have to manage their own risk and therefore be decisive on their financial resources: this thesis is one way to define the optimal quantities of these resources to make the CCP stick to its duty.

2.2 What is a CCP?

The current section gives a more extensive description of a central counterparty as well as discussing its functions and methods to reallocate counterparty risk and minimize financial contagion. Large part of our description of CCPs comes from Pirrong (2011). Central counterparties are organizations that are intended to reduce counterparty risk and systemic risk. Practically, CCPs operate to make it more likely that promised payments will be made. In a general financial network without CCPs, every investor has an open position with many other network members: it means that every one is responsible for his own positions, payments and settlements. Central clearing means that one single organization becomes *seller to every buyer and buyer to every seller*: every single investor has only one relation and one open position, the one with the CCP, already reducing uncertainty and risk exposure. The CCP clears every position daily, or even intra-daily, taking in consideration all positions for one client. What was a complex multilateral network where you had to constantly control every financial contract has now become an easier bilateral relation between one investor and the CCP.

There is a possibility that a debtor is not able to meet his obligations: in this case, the CCP is still obliged to pay the creditor as nothing happened to the obligor and then it has to deal with the loss in alternative ways, that will be examined in Section 2.3. Duffie and Zhu (2011) investigate whether the CCP reduces counterparty risk or not: it may be that counterparty risk reduces thanks to the more solid structure of a central part, but the CCP coul also be the channel for contagion. However, as stated in Pirrong (2011), risk is never eliminated, because it is not possible, it is just reallocated more efficiently. Every single company/investor that was a *node* in the old financial network is now a *clearing member* with one bilateral relation with the CCP. Thereby, central counterparties have two main purposes: reducing counterparty risk and avoiding cascades of losses throughout the financial network. To do that, CCPs affect and reallocate default losses in different methods and these include netting, collateralization, mutualization, equity and eventual insurances.

Netting

Clearing members enter the CCP network with positions on assets and derivatives and all these contracts compensate with equal but opposite positions for some other member. Replacing the buyer to every seller and the seller to every buyer in a process called *novation* (Pirrong, 2011), the CCP knows all these positions and can net out all these offsetting transactions, as it is shown by Figure 2.1. For example, A sells a contract, B buys this contract and sells it again to C and C simply buys this contract. In a multilateral network, if B fails and its position remains open, the other two could be harmed by this default, because both of them would not see their contract exercised. But if all is cleared by a CCP, B's contract would be netted out, because one "positive" position nets out a "negative" one and B's obligations would be extinguished. In general, gains are netted with losses for every single member, this is the first way to limit risk exposures.

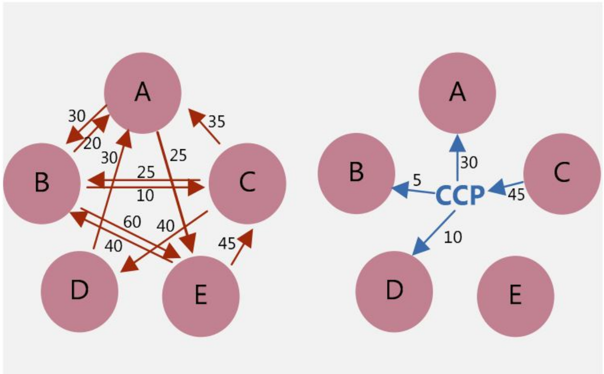


Figure 2.1: Netting without and with a CCP, taken from Domanski et al. (2015)

Collateralization

Value of derivatives and assets varies continuously with market conditions, which make each financial contract an asset for one part and a liability for the other. Hence, in case of default, one of the parties will face the risk of a loss. Parties can reduce this risk by posting *collateral*, which means that the party suffering losses can partially recover with what was initially posted. Pirrong (2011) states that the CCP always requires collateral to all members, more specifically it requires two types of collateral: the *initial margin* (IM) is an amount of money asked at the beginning of the contract, as soon as a node becomes a clearing member; CCPs also observe continuously all the variations of prices and derivatives: whenever they observe a change they ask to compensate with collateral, i.e. *margin call*, and they charge a *variation margin*. Thus, central counterparties compute gains and losses of every portfolio: those whose contracts are now liabilities must pay the CCP for that change in value, and those whose contracts are now gains are always paid by the CCP. One of the main elements of risk management is how to fix the initial margin IM: this is one of the resources Nahai-Williamson et al. (2013) and our thesis investigate. We will find the optimal quantities of collateral to minimize the expected loss function for CCP's members.

Mutualization

CCPs always require members to make an initial contribution to a common fund, called *default fund* (DF) (Pirrong, 2011). Variation margin and initial margin are the first resources to absorb default loss, but in case these are not enough, then the common default fund can be recalled. In this way, even if the defaulter's collateral is not enough, every single member still remains untouched, because the common default fund can absorb the excess losses: this is a form of *loss mutualization*.

Equity

As pointed out by Pirrong (2011), central counterparties are not public companies or non-profit agencies, they are standard profit companies, so they have shareholders who founded the organization and they have equity. Equity can be used to absorb default losses.

In their networks, central counterparties ensure financial stability: they facilitate a more efficient and coordinated replacement of default positions and they reduce the counterparty risk by reallocating it among members and communities. However, Duffie and Zhu (2011) stated that CCPs can either create or reduce systemic risk: collateral and margin calls protect against default, but they can also have a destabilizing effect among traders, firms that have to respect huge margin calls can face liquidity problems and exacerbate their positions. Moreover, severe conditions and defaults can threaten CCPs solvency and make them default as well.

2.3 Risk management

This section describes all the financial resources available for central counterparties and their management. Given the function of the CCP, absorbency of losses, reallocation of counterparty risk and so on, its sources become crucial to be able to fulfill its obligations: decisions and organization of these sources are called risk management. This section will provide a clear and extensive explanation about CCP's resources and how they work.

First of all, CCPs must commit resources to engage a variety of risk measures, because these sources are the primary instrument to implement their functions. Pirrong (2011) notes that CCP risk management interests include:

- Initial margin IM setting: CCPs must decide and periodically review the initial margins levels, that are usually fixed considering the nature of cleared products (riskiness, volatility, liquidity and so on); monitoring market data and backtesting their performance are the most common methodologies;
- Default fund DF calculation: similarly, CCPs must fix and periodically review individual default fund contributions;
- Monitoring members' positions traded through the CCP.

In reality, central counterparties financial resources are more complex than just margins, default fund and equity: Pirrong (2011) describes also *additional calls*, i.e. capped additional contributions that CCPs ask to their members, and insurances against default losses. However, these extra financial resources are not considered both in Nahai-Williamson et al. (2013) and in our model.

2.3.1 CCP's financial resources

Central counterparties can count on different financial resources, respectively: variation margin, initial margin IM, default fund DF and equity.

Variation margin

Central counterparties operate daily and intra-daily valuations of every position, to identify any variation in prices and values and to be able to define who has to pay and who has to be paid in that specific moment: an operation known as *mark-to-market*. *Variation margin* is the amount asked to every member whenever there is a change in value on their positions. Our work about minimizing the expected loss for clearing members to find the optimal quantities of collateral will not consider the variation margin: this is a payment asked in that moment as soon as position value changes, it is not an amount that the CCP has to decide *a priori*.

Thus, if CCPs ask for a variation margin as soon as there is a change in prices and members pay this amount, how is it possible that CCPs incur in losses? Clearing members pay variation margins anytime it is asked by the CCP, but when one clearing member defaults, there is a time gap between the last variation margin payment and the close-out moment of that position: in this time interval, in case of severe market conditions, prices could continue changing, which is why the CCP has to cover additional losses with respect to the obtained variation margin. The real losses CCPs have to cover are the ones arising from price movements between the default time and the close-out, which is called *replacement cost* and is one of the main purposes of the CCP.

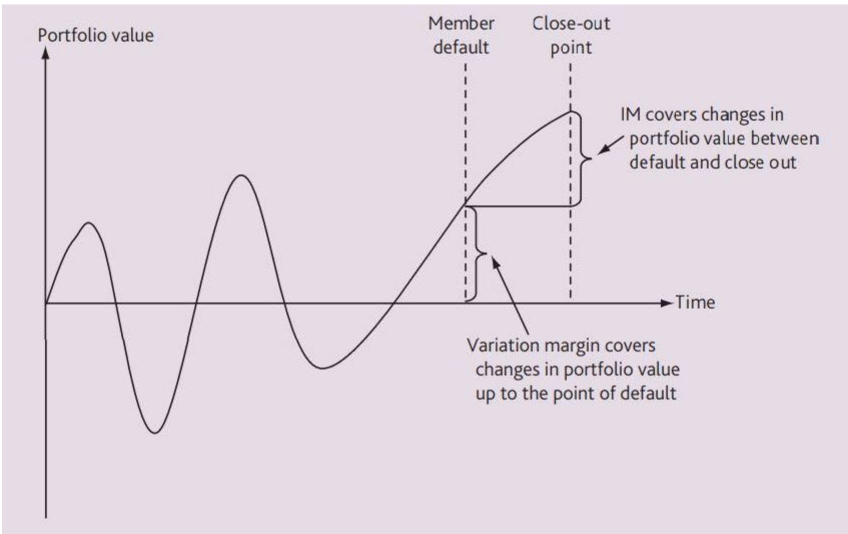


Figure 2.2: How variation margin, initial margin and other sources are used to cover losses, taken from Nahai-Williamson et al. (2013)

Figure 2.2 by Nahai-Williamson et al. (2013) explains this process. For example, a clearing member has an open position, then the CCP asks for variation margin as soon as its position becomes a liability; at time t, this clearing member pays its last variation margin based on the last price movement in t and then defaults, but the CCP closes its position only in $t+1$. Up to time t, any loss is covered by the variation margin only, but further movements in price could happen between t and $t+1$, these are further losses the CCP has to cover with other financial resources.

Initial margin

Central counteparties always collect an *initial margin* (IM), which is comparable to an initial fee clearing members have to pay when their relation with the CCP starts. Pirrong (2011) notes that CCPs do not vary their initial margin based on creditworthiness and credit quality, because it would be too costly to monitor members' nature, indeed they decide initial margins based on riskiness and volatility of cleared products. Initial margins are conventionally calculated so that the probability that prices will move enough to generate losses is sufficiently small. This methodology computes the likelihood that variation margin is exhausted and sets the IM amount to make this event happen with a small probability. In other words, initial margin is computed as a Value-at-Risk (VaR). CCPs choose the amount that, given the variation margin up to the estimated default time, can cover additional losses with a confidence interval of 95%, 99% and so on. In this thesis, we will not provide an analytical model with a VaR to compute the initial margin contribution each clearing member has to post. Instead, the individual initial margin contribution (IM) will be one of the optimal quantities determined by the minimization of the expected loss function with respect to initial margin (IM) and default fund (DF) contributions.

Default fund

Clearing members are also obliged to pay a *default fund* contribution (DF) that will converge in one common fund managed by the CCP. The default fund contribution is the instrument that allows more efficient reallocation of counterparty risk and loss mutualization. If variation and initial margins are not sufficient to cover losses, the CCP starts eroding the default fund. First, the CCP uses the defaulter's contribution to default fund and then the rest of the common account. The default fund is the real characteristic of a CCP, because losses are shared in attempt to avoid contagion and default chain reactions that would lead to systemic risk situations.

Methodologies to compute default fund contribution are various and complex. This thesis will provide one of these methods, which is to minimize the expected loss function for clearing members to determine the optimal pledgeable quantities of initial margin and default fund contributions. Other common models to quantify default fund contributions DF are often based on *stress tests*, a sort of worst case scenario analysis to check how the CCP responds to extreme, but not unlikely, market conditions.

Equity

Recall that CCPs are profit companies with shareholders that invested an initial capital in the firm which constitutes the CCP. It means that the CCP has also its own equity to count on as a financial resource.

2.3.2 Default waterfall

In previous subsections, financial resources were consciously ordered, first variation margin, then the initial margin IM, after the default fund DF and finally the CCP's equity: this is because the claim on one source rather than another is wisely decided sticking to the so called *default waterfall*. The waterfall orders CCP's financial resources and decides which ones have to be used before others. Generally, CCPs mark to market positions with variation margin obtained by clearing members; if default happens, they first rely on defaulter's initial margin IM and then defaulter's contribution DF to default fund; if this is not enough, they also use the common default fund; if losses are still exceeding, then they claim additional contributions to all clearing members; then, further losses can also be absorbed by CCP's equity; if all these resources are still insufficient, then the CCP itself defaults. Figure 2.3 inspired by Nahai-Williamson et al. (2013) gives a general example of financial waterfall for a central counteparty.

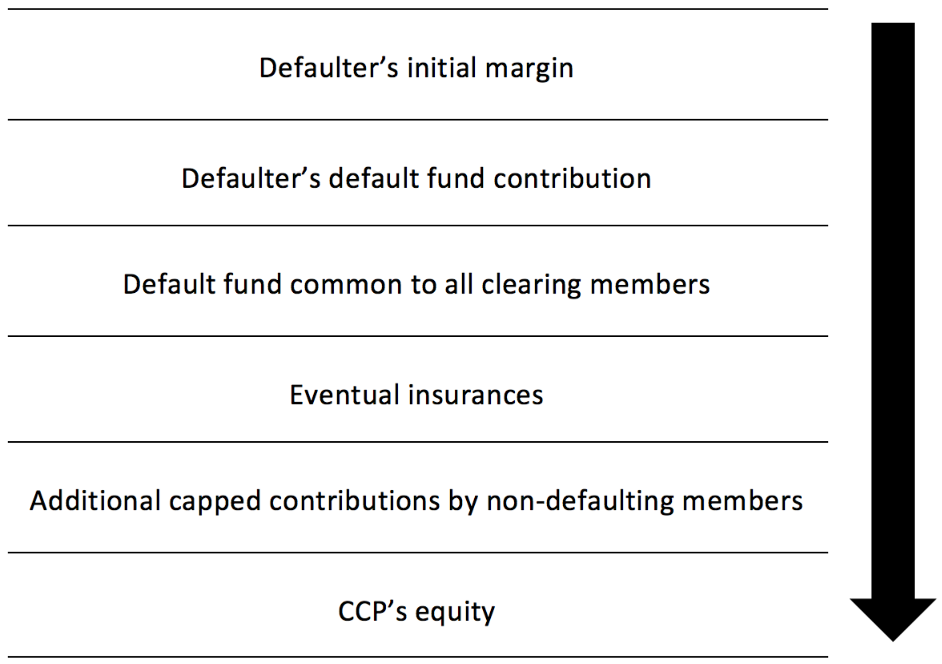


Figure 2.3: Typical default waterfall for CCPs, inspired by Nahai-Williamson et al. (2013)

The elements ins the waterfall can be ordered in a variety of ways depending on the CCP's policy and this order will profoundly affect CCP's risk management. For example, if the CCP does not use any additional call except for the default fund contribution and decides to use equity as the third resource, then the governance will have more incentives to control risk and not to under-collateralize positions, because CCP's own capital could be eroded earlier.

2.4 Regulation

This section clarifies the overall regulation that was developed towards central counterparties, more specifically the regulation before and after 2007-2008 financial crisis and the one in force nowadays.

Prior to 1988, regulation was different among nations, this obstacle and the rise of derivatives markets brought to Basel I in 1988. The main focus was to make it mandatory for banks to keep a certain percentage of capital according to its risky assets. For example, cash had a risk weight of zero, investments in other OECD banks weighted 20%, mortgages without any collateral weighted 50% and so on. This first agreement had one weakness: it could not discriminate conforming to credit ratings and credit quality. Enrichments to this common international regulation brought to Basel II, proposed in 1999, but implemented only after 2007. As stated by Hull (2012), regulations started considering also credit quality as evaluated by rating companies. Again, Basel II became obsolete as soon as the 2007-2008 financial crisis happened, which led to new regulation. After the crisis, Basel III was proposed in 2009 and implemented in 2010. This regulatory statements imposed dramatically higher capital requirements and liquidity requirements. Up to 2014, new rules were uploaded in this regulation, and the concept of Qualified CCP (QCCP) was born: a QCCP is one that keeps itself updated with the regulations and publish all information that clearing members need to be able to align with capital requirements. The incentive behind this new mechanisms is the different risk weight that investors suffer if their positions are cleared by a QCCP: when financial institutions deal with a QCCP, they will receive preferential treatment on capital requirements. These last interventions were implemented after December 2017 and some referred to them as "Basel IV", but then new modifications were operated last year, with an implementation date expected for 1st January 2022. In accordance to Poppensieker et al. (2018) for McKinsey & Co., these new parts include: revised approaches for credit risk, rating-based floors, operational risk and market risk and many others.

In the US, the Dodd-Frank Act was adopted by President Obama in 2010 in response to the necessity of central clearing and financial crisis and it's still in force. Following Ejvegård and Romaniello (2016), this regulation required all the OTC derivatives to be cleared by central counterparties. Risk managements standards were established for all intermediaries including CCPs, regulatory agencies were given power to oversight CCPs and US Federal Reserve became a lender of last resort for all these organizations.

The CPSS-IOSCO Principles for Financail Market Infrastructure were published in 2012 by the Committee on Payment and Settlement Systems in the Bank for International Settlements and by the International Organization of Securities Commission. Principle 4 requires that CCPs set up collateral taking into account specific risks inherent to their cleared products. Then, Principle 6 requires that the initial margin should meet a single-tailed confidence level of 99% with respect to the loss exposure, in order to cover losses in the interval between the last variation margin and the position close-out (see Figure 2.2).

European Market Infrastructures Regulation (EMIR) is a body of legislation enacted in 2013 by the European Commission in response to 2007-2008 financial crisis. These laws are based on three main duties for financial intermediaries:

- Obligation to clear: for old OTC derivatives, it became mandatory to take part in a network with a CCP and centrally clear every transaction, at least if a certain threshold in the invested amount is reached;
- Obligation to mitigate risk: some measure to be able to better manage systemic risk became inevitable; counterparties had to confirm the acceptance of any financial contract, the marking to market activity with daily and intra-daily updates of open positions was subscribed, resolution of limited disputes was up to the intermediary and so on;
- Obligation to report: all data regarding derivatives must be collected and reported to *trade repositories* respecting some information standards.

The European Securities and Markets Authority (2019) sent a public statement in March 2019: some modifications has been made to the original text since 2013 and this statement declares a new regime to determine when financial and non-financial counterparties are subject to clearing obligations.

Chapter 3

Related Literature

This chapter provides a short review of the previous CCP literature. Studies about general financial networks and their advantages started at the end of the '90s, but the broad and extensive literature that is observable today is mainly due to 2007-2008 worldwide financial crisis: a stream of studies and researches followed since then. Section 1 gathers all about central clearing, CCPs and contagion; Section 2 pools together all the papers on CCP's risk management and their optimal resources.

3.1 Contagion and central clearing

Several papers developed mathematical models to describe, analyse and prevent contagion in financial networks. In this area, the first pivotal work was made by Eisenberg and Noe (2001): this paper aims to compute how an initial loss cascades through the system. Afterwards, many contagion models were just additions or extensions of Eisenberg and Noe (2001). Glasserman and Young (2015), for example, find that individual quantities like asset size, leverage ad financial connectivity can be used as factors to measure the magnitude of contagion. For contagion models, it is also worth mentioning Battiston et al. (2012), Bardoscia et al. (2017) and Watts (2002).

As it regards the relation between CCPs and contagion, a more specific part of the literature tries to discern to what extent the presence of a CCP can reduce contagion and loss spread. In this sense, another primary work is Duffie and Zhu (2011). Two main results follow their research: first, introducing a CCP for a particular set of derivatives reduces the average counterparty risk if and only if the number of clearing participants is sufficiently large; secondly, netting benefits in general exist only if a clearing house nets across different asset classes, while counterparty credit risk may arise if the clearing process is fragmented across multiple CCPs for different assets. Subsequent works by Cont and Kokholm (2014) and Amini et al. (2016) also underline factors like number of members and netting across multiple types of assets as central to the effectiveness of clearing. Another group of papers analyse the disadvantages of central clearing and all the impacts of inefficiency of CCPs in financial networks, with the intention of providing possible solutions. Some of these works are Koeppl and Monnet (2010), Koeppl et al. (2011), Biais et al. (2012) and Pirrong (2014).

3.2 CCP's risk management

The importance of central counterparties has made the risk management of their financial sources crucial for healthy financial networks. This field of studies about CCP's risk management can be divided into three main areas:

- Some authors study the implications of *transparency* on CCP's risk management, as it was required by the new regulation: this aspect is read in Acharya and Bisin (2014), Oehmke and Zawadowski (2015) and Antinolfi et al. (2016);
- An entire field of CCP's risk management papers concentrates on the real balance between CCP's sources: they provide methods to find the equilibrium between default fund DF, initial margin IM and equity, taking into account what is required by international regulation and laws on financial intermediation. Here we mention some of those after the 2007-2008 financial crisis, like Amini et al. (2015), Capponi et al. (2017), Ghamami and Glasserman (2017) and Menkveld (2017);
- This thesis is a contribution to the third field of risk management, the one investigating CCPs' resources properties and values. There is a whole field of literature that models the expected loss for clearing members and minimize it to find the optimal balance of initial margin, default fund and equity. Differently from the previous authors, international regulatory requirements are not considered in this thesis, to be able to reveal intrinsic properties and potentialities of these sources. Some remarkable works in this sense are Ghamami (2015) and Haene et al. (2009). Ghamami (2015) finds that estabilishing a defaul fund is always optimal, and in some cases a sufficiently large default fund is even all it takes. The use of margin requirements is recommended only if the opportunity cost of collateral is lower than the probability of a particular member's default. Then, Haene et al. (2009) find that the optimal default waterfall is composed by variation margin, initial margin IM and default fund DF. Default fund is defined based on the credit loss distribution of the CCP's portfolio of clearing members' portfolios.

Both the work by Nahai-Williamson et al. (2013) and the present thesis fall in the last path. Nahai-Williamson et al. (2013) is the starting point of this thesis and we will give a detailed explanation in the next chapter. However, it can be considered a sort of upgrade from Haene et al. (2009). Nahai-Williamson et al. (2013) minimize the expected loss function for CCPs' members numerically to find the optimal quantities of individual default fund and initial margin contributions, but the expected loss is now way more sophisticated: it considers parameters like cost of capital, liquidation cost, capital charges and many others, it separates in-the-money and out-of-the-money members in case of one's default and so on. The present thesis wants to be a further enrichment to this last field of studies, improving Nahai-Williamson et al. (2013) with dependent individual default probabilities and realistic prices, to minimize again the expected loss function for CCPs' members and see if these dependencies introduce large differences in the results.

Chapter 4

Nahai-Williamson et al. (2013) model

The current chapter will present the model by Nahai-Williamson et al. (2013), where the authors determine the optimal amount of initial margins and default fund contributions. Nahai-Williamson et al. (2013) do not care about the real size of CCP's financial resources, because these are largely affected by regulatory laws. Instead, the aim of the paper is to study whether the CCPs should have some discretion over the balance between these two sources and how they can find an optimal amount. Nahai-Williamson et al. (2013) create a model that studies the impact of numerous factors on clearing members' expected losses to find the optimal composition between CCP's financial resources. They model an expected loss function for a general surviving member in the CCP's network and minimize this expected loss with respect to initial margin and default fund contributions. In practice, this means a minimization of a two-dimensional function to find IM and DF optimal quantities simultaneously.

Section 1 will provide the theoretical framework of the model and all the considerations behind it. In Nahai-Williamson et al. (2013), the authors perform comparative studies on CCP's financial rsources, i.e. define optimal quantities, and in Section 2 we replicate their findings as well as some additional considerations. This model is more widely explained in the work by Nahai-Williamson et al. (2013): this thesis wants to be an extension of their work and so this chapter represents a respectful summary of their findings.

4.1 The model

In this section, we present the general outline of the model by Nahai-Williamson et al. (2013), such as the assumptions, the rules, the optimization problem and theoretical framework behind the final results about the optimal balance between initial margin IM and default fund DF contributions.

4.1.1 The CCP, its members and network

In Nahai-Williamson et al. (2013), the authors introduce a hypothetical CCP and its clearing network and then apply a mathematical model for the CCP's members expected loss function. The network around the central counterparty is made by n clearing members, which are also the owners of the CCP, i.e. they are shareholders. Every member contribute with an equal amount of equity k to the total amount of capital K possessed by the clearing house. In Nahai-Williamson et al. (2013), the CCP has only *direct members*, which means that only CCP's members clear their position and there is no central clearing for agents that do not have the membership. In reality, there are indirect members, institutions or companies that trade with clearing members through the CCP, but are not clearing members themselves.

Each clearing member has a default probability p that is independent and exogenous. The assumption of independent default probabilities is unrealistic and one of the main purposes of this thesis is to extend Nahai-Williamson et al. (2013) model to allow for default dependencies among clearing members. Our model will be discussed in Chapter 6.

Clearing members have evenly distributed long and short positions of equal size on an imaginary portfolio with initial value of 1. Their positions change as soon as price H in the underlying market changes: in this underlying market, prices ΔH are Normally distributed with mean zero and variance σ, that is $\Delta H \sim N(0, \sigma)$. Note that the initial price is assumed to be zero, so we can either talk of prices H or price changes ΔH. The variance σ can be interpreted as market volatility. Now, evenly distributed market positions of equal size on a portfolio with initial value of 1 imply that, at any market move, $\frac{n}{2}$ members will be *in-the-money* (ITM) and the other $\frac{n}{2}$ members will be *out-of-the-money* (OTM). In-the-money means that you actually gained something on your positions and you are due to be paid, while out-of-the-money means the opposite (see also pages 20-22 in Nahai-Williamson et al. (2013)).

Every member posts a collateral amount y as initial margin at the beginning of the contract with the CCP and an individual default fund contribution z, that will flow into the mutual default fund. Posting collateral comes with an opportunity cost $c > 0$, that can be interpreted as the lost return of that amount of collateral if invested somewhere else. Thus, each clearing member contributes to the CCP's capital with an amount k, they post initial margin for y and take part in the default fund with z, but there are still additional costs: if the CCP cannot cover default losses with its primary resources in the default waterfall, the CCP defaults itself and there is an additional loss due to systemic risk, which will be called s.

Finally, extra costs will affect clearing members in the form of regulatory capital charges: these are capital charges applied to members' IM and DF by regulators, there will be one capital charge for initial margin, d_{IM}, and one capital charge for default fund contributions, d_{DF}. Moreover, collateral does not only have opportunity cost and capital charges, but it also has a further cost that will be called c_c, representing the cost to banks of holding capital in general.

4.1.2 Defaulting process

The model in Nahai-Williamson et al. (2013) defines a very simple default waterfall (see Subsection 2.3.2). As soon as there is a default loss, CCPs use defaulter's initial margin IM and default fund contribution DF. If these funds do not cover all losses, then the CCP proceeds using the mutual default fund made by all contributions. When default fund is exhausted, CCP's equity comes next and if this source is still not sufficient, then the central counterparty goes bankruptcy.

Once the CCP defaults, a liquidation process starts, according to bankruptcy local laws, however in Nahai-Williamson et al. (2013) this process is simplified. In the liquidation process, managers transfer funds from surviving OTM members to ITM members; it is assumed that all surviving OTM members fulfil their obligations, but liquidation has an administrative cost $a \geq 0$. As soon as funds are transferred from surviving OTM members to ITM members, bankruptcy administrators will subtract this liquidation cost from those funds: this is the reason why, in case of CCP's liquidation, only ITM members will suffer this cost.

4.1.3 Clearing member's expected loss function

The centre of the Nahai-Williamson et al. (2013) model is the surviving member's expected loss function that is minimized with respect to y and z, IM and DF contributions. The expected loss function is constructed as follows:

1. We build the individual loss function for both surviving OTM and ITM members;
2. We compute the expected value to find the surviving member's expected loss function for both OTM and ITM members;
3. We aggregate the two expected loss functions for surviving OTM and ITM members and form a final expected loss function by adding a linear part for the cost of collateral (opportunity cost c, capital charges d_{IM} and d_{DF}, etc.). Then, we minimize this function with respect to individual IM and DF contributions, y and z.

The final expected loss function will depend on both IM and DF contributions, making it possible to find the optimal quantities of financial resources that the CCP has to ask to its members.

Individual clearing member's loss function

As soon as there is a market movement, given a certain volatility σ, $\frac{n}{2}$ members will be ITM and $\frac{n}{2}$ will be OTM. If one OTM member cannot pay its position to the CCP, this clearing member goes bankruptcy. So, it is clear that the defaulter will always be an OTM member and that there will be $\frac{n}{2} - 1$ surviving OTM members and still $\frac{n}{2}$ ITM members. If multiple OTM members default at the same time, i members for example, then there will be $\frac{n}{2} - i$ surviving OTM members and still $\frac{n}{2}$ ITM members and so on. Figure 4.1 taken from Nahai-Williamson et al. (2013) explains how OTM and ITM members face losses and how CCP's financial resources absorb these losses.

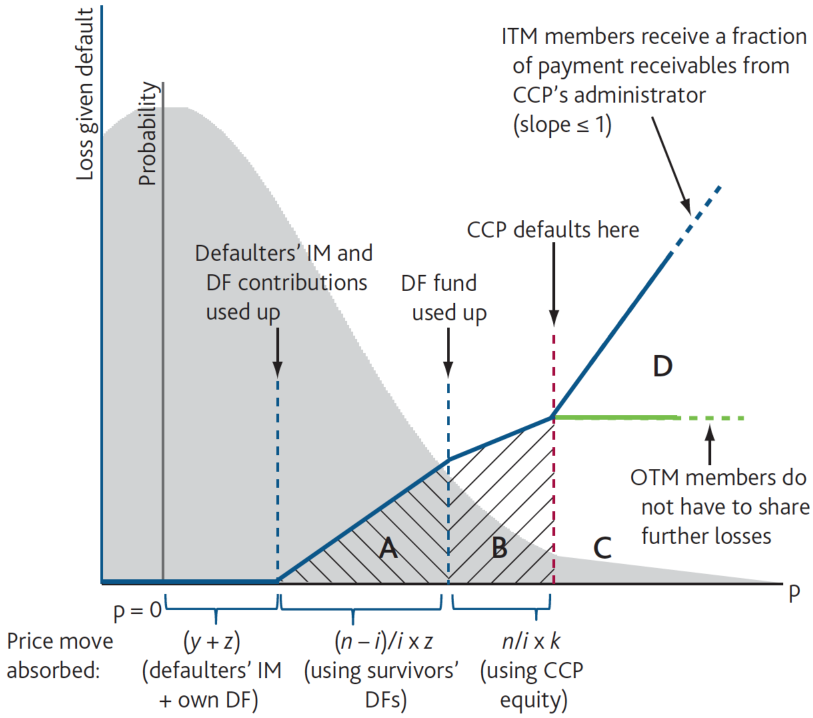


Figure 4.1: Potential loss of ITM and OTM members for different price moves ΔH (called p in Nahai-Williamson et al. (2013) notation) and i defaulting members, taken from Nahai-Williamson et al. (2013)

Note that in Figure 4.1 the prices are denoted by p, which is the notation used in Nahai-Williamson et al. (2013) model, while we will denote prices by H. We start explaining the individual loss function $loss_{OTM}(h, i)$ for OTM surviving members. The reading of Figure 4.1 will proceed from the left to the right:

1. If there is no price move, $\Delta h = 0$, all the evenly distributed positions of members stay the same and no default can occur;

2. If i OTM members default and the price changes up to $y + z$, the defaulter's initial margin and default fund contribution are able to absorb the loss, so all surviving members remain untouched and do not face any loss. Hence, if losses are smaller than defaulter's initial margin and default fund individual contribution, then all surviving members face no loss. This situation represents the first flat region in Figure 4.1, so the individual loss function for surviving OTM member is given by:

$$loss_{OTM}(h, i) = 0 \quad \text{if} \quad h \leq y + z. \tag{4.1}$$

The intervals that define the loss function $loss_{OTM}(h, i)$ are based on the price realization h. Prices cannot be negative, i.e. $h \geq 0$, that is why Figure 4.1 considers only the positive part of the Normal distribution $H \sim N(0, \sigma)$. In our extended model in Chapter 6, the underlying factor that creates default dependence will have both positive and negative realizations.

3. When the price change generates a loss that is larger than $y + z$, then the CCP starts using the common default fund: there are i defaulting members and $n - i$ surviving members, so the fraction of default fund the CCP uses is exactly $\frac{i}{n-i}$ multiplied by the residual loss $h - y - z$ because individual y and z were already consumed. Note that this default fund is eroded more quickly whenever the number of defaulting members i increases: if there is just one default, $i = 1$, then the CCP can use the whole default fund to cover up that loss, it is easier, but whenever i increases, obviously this resources run out faster. This is observed in Region A in Figure 4.1 and means that the loss function $loss_{OTM}(h, i)$ is given by:

$$loss_{OTM}(h,i) = \frac{i}{n-i}(h - y - z) \quad \text{if} \quad y + z < h \leq y + z + \frac{n-i}{i}z. \tag{4.2}$$

4. Once the default fund is exhausted, the clearing house starts using its equity $n \cdot k$: defaulting members are always i, but, as it regards the capital, all members gave this contribution (recall that it was assumed that all clearing members are also the owners of the CCP, they are all shareholders), so the fraction of this source for each member is now $\frac{i}{n}$. This further absorption splits the loss $h - y - z - \frac{n-i}{i}z$ in $\frac{i}{n}$ parts, which explain Region B in Figure 4.1 showing $loss_{OTM}(h, i)$ as:

$$loss_{OTM}(h,i) = \frac{i}{n}\left(h - y - z - \frac{n-i}{i}z\right) \quad \text{if} \quad y+z+\frac{n-i}{i}z < h \leq y+z+\frac{n-i}{i}z+\frac{n}{i}k. \tag{4.3}$$

We believe that this part of the individual loss function $loss_{OTM}(h, i)$, as stated in Equation (1) on page 23 in Nahai-Williamson et al. (2013), is not natural nor in line with the one displayed in Figure 4.1, as it will be discussed later (see the end of this subsection).

5. In case default losses are not absorbed even by equity, then the CCP goes bankrupt: here we observe two different situations for surviving OTM and ITM members. Surviving OTM members have lost both their individual default contribution z and their holding in capital k, because everything has been eroded; moreover, they face further losses s due to contagion and systemic risk. This is described by the flat function $loss_{OTM}(h, i)$ in Region C in Figure 4.1 given by:

$$loss_{OTM}(h,i) = (z + k + s) \quad \text{if} \quad y + z + \frac{n-i}{i}z + \frac{n}{i}k < h. \tag{4.4}$$

Besides, surviving ITM members also lose their default fund contribution z, their holding in capital k and they face systemic cost s, but they lose more. ITM members are the ones that gained from their positions, the larger the price movement h the higher their gain, but if the CCP defaults, this gain becomes a loss, because they are not going to receive their payment: they lose an extra amount $h - y - z - \frac{n-i}{i}z - \frac{n}{i}k$ if $h > y + z + \frac{n-i}{i}z + \frac{n}{i}k$. However, surviving ITM members don't lose all this amount, they have to write down a fraction of their gains. In other words, the liquidation cost a works as a recovery rate ϕ (see pages 61-62 on Herbertsson (2018)). The recovery rate is $\phi = a\frac{\frac{n}{2}-1}{\frac{n}{2}}$ and if $a = 0$, then the recovery rate $\phi = 0$, then surviving ITM members lose all of their promised payment. Here, the loss function $loss_{ITM}(h, i)$ is the increasing straight line in Region C in Figure 4.1, given by:

$$
\begin{aligned}
loss_{ITM}(h,i) = &\left(1 - a\frac{\frac{n}{2}-i}{\frac{n}{2}}\right)\left(h - y - z - \frac{n-i}{i}z - \frac{n}{i}k\right) \quad \text{if} \quad y + z + \frac{n-i}{i}z + \frac{n}{i}k < h. \\
&+ z + k + s
\end{aligned}
\tag{4.5}
$$

Figure 4.1 displays the loss function for one surviving member in case i members default. The plot shows both OTM and ITM surviving member' loss functions, which are different only in the last region in case of CCP's default. As Figure 4.1 displays, the individual loss function is continuous in price h. However, this continuity is not observed in the equations built by Nahai-Williamson et al. (2013). Looking at Equation (4.2), we have:

$$
loss_{OTM}(h,i) = \frac{i}{n-i}\left(h - y - z\right) \quad \text{if} \quad y + z < h \leq y + z + \frac{n-i}{i}z.
$$

In the point $h^* = y + z + \frac{n-i}{i}z$, this function gives:

$$
\begin{aligned}
loss_{OTM}(h^*,i) &= \frac{i}{n-i}\left(h - y - z\right) \\
&= \frac{i}{n-i}\left(y + z + \frac{n-i}{i}z - y - z\right) \\
&= z.
\end{aligned}
$$

This is in line with our description of the individual loss function because in the turning point from Region A to Region B in Figure 4.1 the CCP eroded the whole default fund, so the single clearing members has lost his individual DF contribution z. On the other hand, Equation (4.3) states:

$$
loss_{OTM}(h,i) = \frac{i}{n}\left(h - y - z - \frac{n-i}{i}z\right) \quad \text{if} \quad y + z + \frac{n-i}{i}z < h \leq y + z + \frac{n-i}{i}z + \frac{n}{i}k.
$$

But, in the point $h^* = y + z + \frac{n-i}{i}z$, this function gives:

$$
\begin{aligned}
loss_{OTM}(h^*,i) &= \frac{i}{n}\left(h - y - z - \frac{n-i}{i}z\right) \\
&= \frac{i}{n}\left(y + z + \frac{n-i}{i}z - y - z - \frac{n-i}{i}z\right) \\
&= 0.
\end{aligned}
$$

The loss function jumps down to 0 and this is not displayed in Figure 4.1. In other words, the individual loss function has a discontinuity in h^*, which is present in the equations but not in Figure 4.1. This discontinuity is not intuitive and to avoid this issue we assume that Equation (4.3) should be:

$$loss_{OTM}(h,i) = \frac{i}{n}\left(h - y - z - \frac{n-i}{i}z\right) + z \quad \text{if} \quad y + z + \frac{n-i}{i}z < h \leq y + z + \frac{n-i}{i}z + \frac{n}{i}k. \tag{4.6}$$

Similarly, Nahai-Williamson et al. (2013) assume a constant systemic cost s in case the CCP itself defaults. This means that the individual loss function in Figure 4.1 should have a jump equal to s in the turning point from Region B to Region C, which is not observed in the plot. We assume that Nahai-Williamson et al. (2013) consider a constant systemic cost s in the equations, but not in Figure 4.1. However, this does not change neither the interpretation of Figure 4.1 or the results.

We repeated the implementations by Nahai-Williamson et al. (2013) in Section 4.2 and our optimizations in Section 6.2 by replacing Equation (4.3) with Equation (4.6) and there is not notable numerical difference in the results. However, we will not proceed with the individual loss function in Equation (4.6), although it looks more natural and intuitive. We continue using the one identified by Nahai-Williamson et al. (2013) from Equation (4.1) to (4.5) to be able to compare the optimal CCP's financial resources and discuss the balance between IM and DF contributions.

Expected loss function for OTM member

The loss function has to reflect the eventual loss for OTM members that is already described in Figure 4.1 by Nahai-Williamson et al. (2013). We already defined the loss for a surviving OTM member in all the different regions of Figure 4.1 in Equations (4.1), (4.2), (4.3) and (4.4), so combining these equations gives the complete loss function for the single OTM member:

$$loss_{OTM}(h) = \begin{cases} 0 & \text{if } h \leq y + z \\ \frac{i}{n-i}(h - y - z) & \text{if } y + z < h \leq y + z + \frac{n-i}{i}z \\ \frac{i}{n}\left(h - y - z - \frac{n-i}{i}z\right) & \text{if } y + z + \frac{n-i}{i}z < h \leq y + z + \frac{n-i}{i}z + \frac{n}{i}k \\ (z + k + s) & \text{if } y + z + \frac{n-i}{i}z + \frac{n}{i}k < h. \end{cases} \tag{4.7}$$

Let N be the number of defaults among the $\frac{n}{2}-1$ OTM members. Then note that $\mathbb{E}[loss_{OTM}(H,N)]$ is given by:

$$\begin{aligned} \mathbb{E}[loss_{OTM}(H,N)] &= \sum_{i=0}^{\frac{n}{2}-1} \mathbb{E}\left[loss_{OTM}(H,i) \cdot \mathbb{I}_{\{N=i\}}\right] \\ &= \sum_{i=0}^{\frac{n}{2}-1} \mathbb{E}[loss_{OTM}(H,i)]\mathbb{P}[N=i] \end{aligned} \tag{4.8}$$

where the second equality is due to the fact that N and H are independent in the Nahai-Williamson et al. (2013) model (this will be relaxed in our extended model in Chapter 6). Furthermore, since in

Nahai-Williamson et al. (2013) all defaults are independent, we have that N is Binomially distributed within $\frac{n}{2}-1$ OTM members with default probability p, so:

$$\mathbb{P}[N=i]=\binom{\frac{n}{2}-1}{i}p^i(1-p)^{\frac{n}{2}-1-i}. \tag{4.9}$$

Recall that Nahai-Williamson et al. (2013) assume that only OTM members can default and that at least one OTM member survives, that is why the maximum of N is equal to $\frac{n}{2}-1$. Also note that the expected loss function $\mathbb{E}[loss_{OTM}(H,i)]$ s given by:

$$\begin{aligned}\mathbb{E}[loss_{OTM}(H,i)] &= \int_{-\infty}^{+\infty} loss_{OTM}(h,i)f_H(h)dh \\ &= \int_{0}^{+\infty} loss_{OTM}(h,i)f_H(h)dh\end{aligned} \tag{4.10}$$

where the second equality is possible because Nahai-Williamson et al. (2013) consider only the positive part of the Normally distributed price $\Delta H \sim N(0,\sigma)$. In other words, $loss_{OTM}(h,i)=0$ for $h<0$, which is the reason for the integral lower bound. So, by using Equation (4.7) in Equation (4.10) and then in Equation (4.8) together with Equation (4.9), we get that $\mathbb{E}[loss_{OTM}(H,N)]$ is given by:

$$\begin{aligned}\mathbb{E}[loss_{OTM}(H,N;y,z)] = \sum_{i=0}^{\frac{n}{2}-1}\Bigg\{&\binom{\frac{n}{2}-1}{i}p^i(1-p)^{\frac{n}{2}-1-i}\cdot\Bigg[\int_0^{y+z}0\cdot f_H(h)dh \\ &+\int_{y+z}^{y+z+\frac{n-i}{i}z}\frac{i}{n-i}(h-y-z)f_H(h)dh \\ &+\int_{y+z+\frac{n-i}{i}z}^{y+z+\frac{n-i}{i}z+\frac{n}{i}k}\frac{i}{n}\left(h-y-z-\frac{n-i}{i}z\right)f_H(h)dh \\ &+\int_{y+z+\frac{n-i}{i}z+\frac{n}{i}k}^{+\infty}(z+k+s)f_H(h)dh\Bigg]\Bigg\}.\end{aligned} \tag{4.11}$$

In Equation (4.11), the sum of integrals is the expected value of the loss function, where every loss realization is weighted for the probability of price h to fall in that region and make that kind of loss happen. Always in Equation (4.11), the first term outside the brackets, computed as a Binomial probability, considers the event of i members defaulting, which has a strong impact on how the financial sources are used. Note that we have $\mathbb{E}[loss_{OTM}(H,N)]=\mathbb{E}[loss_{OTM}(H,N;y,z)]$ to emphasize that the expected loss can be read as a function of y and z, which will be useful in the optimization.

Expected loss function for ITM member

The loss function for the ITM surviving member is almost identical to the one of the OTM member. Recall that the individual loss function for the surviving ITM member in Figure 4.1 differs from OTM members only for the last region. So, combining Equation (4.1), (4.2), (4.3) and (4.5), $loss_{ITM}(h,i)$ is given by:

$$loss_{ITM}(h,i) = \begin{cases} 0 \\ \dfrac{i}{n-i}(h-y-z) \\ \dfrac{i}{n}\left(h-y-z-\dfrac{n-i}{i}z\right) \\ \left(1-a\dfrac{\frac{n}{2}-i}{\frac{n}{2}}\right)\left(h-y-z-\dfrac{n-i}{i}z-\dfrac{n}{i}k\right)+z+k+s \end{cases} \tag{4.12}$$

where the intervals are defined as in Equation (4.7) and h is a realization of price random variable H.

Note that also in this case we observe the same discontinuity in h^* that we observed for $loss_{OTM}(h,i)$ in Equation (4.3) above. For $h^* = y+z+\frac{n-i}{i}z$, Equation (4.2) gives us $loss_{ITM}(h^*,i) = z$, while Equation (4.3) gives us $loss_{ITM}(h^*,i) = 0$, which is not intuitive and not represented in Figure 4.1. Again, we think that the natural loss function for Region B in Figure 4.1 should be equal to:

$$loss_{OTM}(h,i) = \frac{i}{n}\left(h-y-z-\frac{n-i}{i}z\right)+z \quad \text{if} \quad y+z+\frac{n-i}{i}z < h \leq y+z+\frac{n-i}{i}z+\frac{n}{i}k$$

as already pointed out in Equation (4.6). However, we will continue using the same loss function developed by Nahai-Williamson et al. (2013) both here and in our extended version in Chapter 6, in order to compare results and discuss the balance between IM and DF contributions to CCPs.

The expected value of loss function $loss_{ITM}(h,i)$, i.e. $\mathbb{E}[loss_{ITM}(H,N)]$, follows the same steps showed for $\mathbb{E}[loss_{OTM}(H,N)]$ in Equation (4.11), so:

$$\begin{aligned} \mathbb{E}[loss_{ITM}(H,N;y,z)] = \sum_{i=0}^{\frac{n}{2}} \Bigg\{ \binom{\frac{n}{2}}{i} p^i(1-p)^{\frac{n}{2}-i} \cdot \Bigg[\int_0^{y+z} 0 \cdot f_H(h)dh \\ + \int_{y+z}^{y+z+\frac{n-i}{i}z} \frac{i}{n-i}(h-y-z)f_H(h)dh \\ + \int_{y+z+\frac{n-i}{i}z}^{y+z+\frac{n-i}{i}z+\frac{n}{i}k} \frac{i}{n}\left(h-y-z-\frac{n-i}{i}z\right)f_H(h)dh \\ + \int_{y+z+\frac{n-i}{i}z+\frac{n}{i}k}^{+\infty} \left[\left(1-a\frac{\frac{n}{2}-i}{\frac{n}{2}}\right)\left(h-y-z-\frac{n-i}{i}z-\frac{n}{i}k\right)+z+k+s\right] f_H(h)dh \Bigg] \Bigg\}. \end{aligned} \tag{4.13}$$

There are two differences with respect to the OTM member expected loss function: in case the CCP defaults, i.e. $h > y+z+\frac{n-i}{i}z+\frac{n}{i}k$, the loss and the expected loss are different as it was described in Equation (4.5); secondly, the maximum number of surviving ITM members is now $\frac{n}{2}$, because Nahai-Williamson et al. (2013) assume that only OTM members can survive, so all ITM can survive. Again, note that we have $\mathbb{E}[loss_{ITM}(H,N)] = \mathbb{E}[loss_{ITM}(H,N;y,z)]$ to emphasize that the expected loss can be read as a function of y and z.

Total surviving member's expected loss

The next step is to define the general surviving member's expected loss. This expected loss will be minimized wit respect to initial margin y and default fund contribution z, to find the optimal quantities of sources the CCP should require. The final expected loss function for a general surviving member (despite OTM or ITM) is the sum of the two previous ones plus all the costs of pledging collateral in terms of y and z (opportunity cost c, capital charges d_{IM} and d_{DF} and cost of holding capital decided by banks c_c). Hence, the final function to minimize with respect to y and z is given by:

$$\mathbb{E}[loss_{OTM}(H, N; y, z)] + \mathbb{E}[loss_{ITM}(H, N; y, z)] + (c + d_{IM} \cdot c_c)y + (c + d_{DF} \cdot c_c)z \tag{4.14}$$

and the optimization problem replicated from Nahai-Williamson et al. (2013) paper will be:

$$\min_{y,z} \{\mathbb{E}[loss_{OTM}(H, N; y, z)] + \mathbb{E}[loss_{ITM}(H, N; y, z)] + (c + d_{IM} \cdot c_c)y + (c + d_{DF} \cdot c_c)z\} \tag{4.15}$$

where $\mathbb{E}[loss_{OTM}(H, N; y, z)]$ is given by Equation (4.11) and $\mathbb{E}[loss_{ITM}(H, N; y, z)]$ by Equation (4.13). This optimization is not straightforward to solve analytically. The optimization will be solved numerically and from a starting guess, the algorithm will compute the gradient and it will proceed from that point in the direction suggested by the first derivative to find a minimum with gradient equal or very close to zero. This function will be treated as a two-dimensional function whose minimum characterizes the optimal amount of IM and DF contributions.

4.2 Optimizations and results

This section presents the numerical results and the main findings in our implementation of the studies in Nahai-Williamson et al. (2013). These will be compared with the corresponding numerical studies in our extended version of Nahai-Williamson et al. (2013) model in Chapter 6. The expected loss function in Equation (4.14) is minimized with respect to IM contribution y and DF contribution z for different model parameters such as p, c, d_{IM} and so on. More specifically, Nahai-Williamson et al. (2013) perform *comparative statistics*, where there is a baseline of fixed parameters and then the authors minimize the expected loss varying one parameter and keeping all the others fixed, finding the optimal quantities of IM and DF contributions in different settings (see Table 4.1). We will use the CCP's members expected loss in Equation (4.14) as in Nahai-Williamson et al. (2013) and replicate the same optimizations with the same fixed and changing parameters.

	Variable	Value when fixed	Variation range when changing
p	Individual default probability	5% and 25%	1% to 95%
n	Number of members	20	-
σ	Underlying price volatility	20%	-
c	Opportunity cost	50 bp	25 to 350 bp
k	Equity contribution	0.1%	-
a	Liquidation cost	10%	-
d_{IM}	IM Capital charge	0% and 0.16%	0% to 100%
d_{DF}	DF Capital charge	0% and 0.16%	0% to 100%
c_c	Bank cost of holding capital	10%	-
s	Systemic cost	0	0% to 100%

Table 4.1: Summary of variables used in our numerical studies for the fixed and the changing cases, taken from Table A1 on page 25 in Nahai-Williamson et al. (2013)

The column "Value when fixed" in Table 4.1 displays the standard case of the numerical studies in the Nahai-Williamson et al. (2013) model, that is the baseline values used for the variables when "fixed". The column "Variation range when changing" in Table 4.1 shows the variation range of the same variables in the changing case, which are the same intervals used in Nahai-Williamson et al. (2013). The optimizations in Nahai-Williamson et al. (2013) work as follows: for example, the first minimization with respect to y and z is repeated for all the values of default probability p from 0.01 to 0.95, keeping all the other parameters fixed as in the first column in Table 4.1, and so on. Note that in the first column of Table 4.1 the individual default probability p, as well as capital charges d_{IM} and d_{DF}, has two values. Following Nahai-Williamson et al. (2013), we repeat several optimizations for the two values $p = 5\%$ and $p = 25\%$, because it is considered fundamental to understand the relation between the member's default probability and CCP's risk management.

Just as in Nahai-Williamson et al. (2013), the number of clearing members n, the cost paid to banks to hold capital c_c, the initial individual equity contribution k and the underlying price volatility σ are always constant. Note also that both the research by Nahai-Williamson et al. (2013) and this thesis do not report all the performed optimizations.

4.2.1 Varying default probability p

The first optimization studies the optimal IM and DF contributions as function of the individual clearing member's default probability p, which in Nahai-Williamson et al. (2013) is exogenous and independent between members. We replicate the minimization of Nahai-Williamson et al. (2013) expected loss in Equation (4.14) with respect to y and z for different values of default probability p. Recall that, both in Nahai-Williamson et al. (2013) and in our replications, the nominal value of each member portfolio was initially equal to 1, so this is how to read the results. For example, an optimal quantity equal to 0.5 for the default fund contribution DF can be generalized as an amount equal to the 50% of the original value of portfolio.

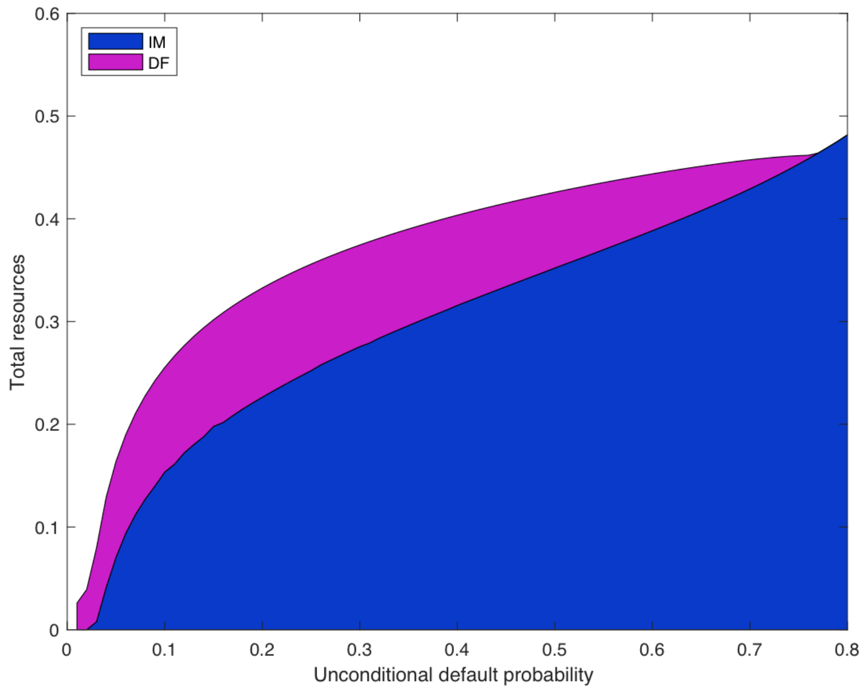


Figure 4.2: The optimal IM contributions y and DF contributions z obtained via Equation (4.15) as functions of the default probability p

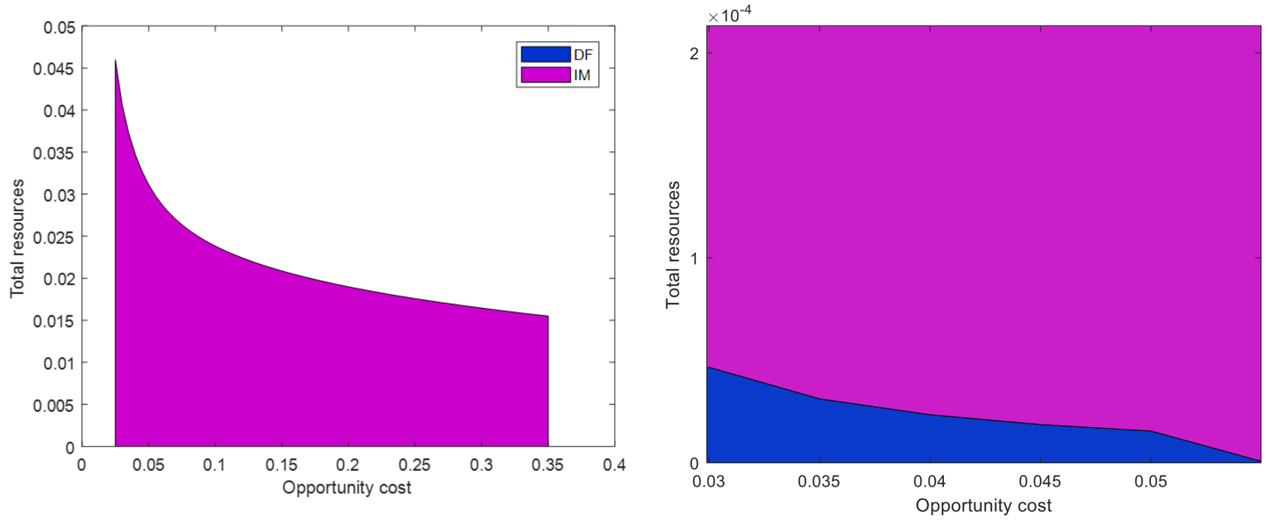
Figure 4.2 displays the results from our optimization in Equation (4.15) in the p range specification in Table 4.1, i.e. 1% to 95%. Nahai-Williamson et al. (2013) perform the same optimization in Figure 5 on page 11 in their paper. The numerical results are different in some regions: in Nahai-Williamson et al. (2013), the optimal DF contribution is equal to zero, $z = 0$, as soon as $p > 40\%$, while in our optimization in Figure 4.2 the DF contribution remains a solid part of CCP's sources balance up to $p \approx 75\%$. However, in our optimization the relation between default probability p and optimal resources is in line with the one in Fgure 5 in Nahai-Williamson et al. (2013). As the probability of default increases, CCPs prefer to manage their risk and cover their losses via initial margin. More specific, when the probability of default is really low, it is better to give a smaller contribution to a common mutual fund than to pay a larger initial margin. As soon as the default probability increases, the result is reversed, because there is a higher cost of mutualization: defaults are very likely and clearing members do not know who will default as everyone has the same probability to default at the beginning, so they prefer to pay a larger initial margin than contribute to a common fund, because with the initial margin their collateral is safe, while the same collateral would be fully consumed in a default fund.

Moreover, both Figure 5 on page 11 in Nahai-Williamson et al. (2013) and our Figure 4.2 show that the optimal level of resources becomes more flat when p is sufficiently large, which reveals that the marginal benefit of holding more collateral for the CCP is decreasing in p. Nahai-Williamson et al. (2013) also observe that when the default probability is zero, $p = 0$, members are risk-free entities, then the perfect amount of resources is zero, because the CCP could offer the clearing service without incurring in losses.

4.2.2 Varying opportunity cost c

The opportunity cost c will be interpreted in this model as the lost gain of the amount of money spent in collateral, since clearing members have to post collateral, which is a cost. Intuitively, as the opportunity cost c increases, it becomes more expensive for clearing members to post any kind of collateral, including IM and DF contributions.

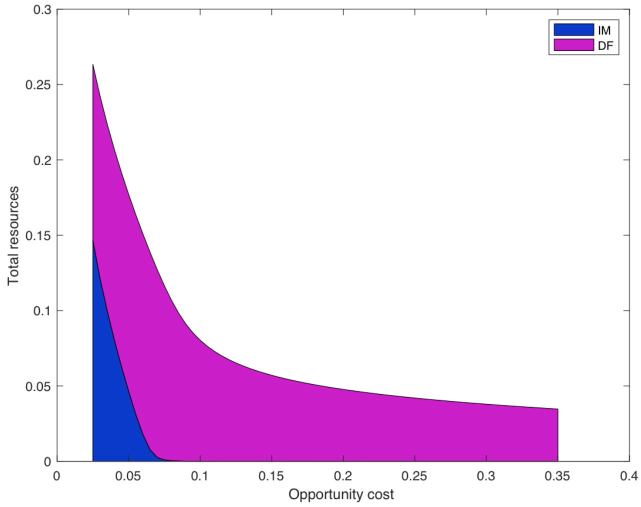
Figure 4.3 displays the results from our optimization in Equation (4.15) in the c variation range in Table 6.1. In this subsection, the optimization of the expected loss in Equation 4.15 with varying values of c will be done twice: once with an individual default probability p equal to 5% and once with $p = 25\%$.



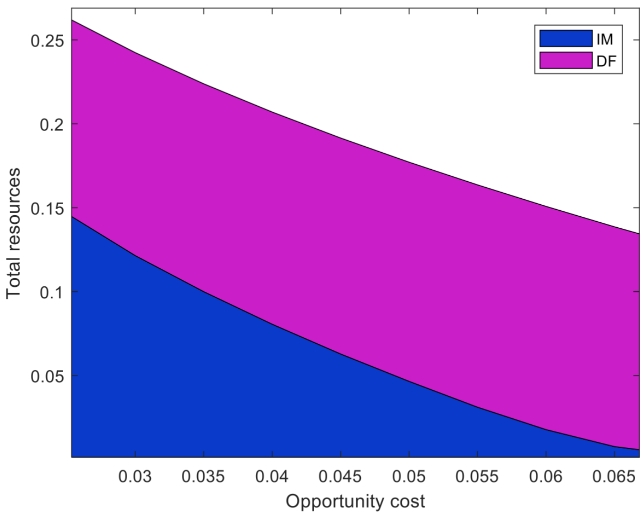
(a) $c \in [0.025, 0.35]$ (b) $c \in [0.03, 0.055]$

Figure 4.3: The optimal IM contributions y and DF contributions z obtained via Equation (4.15) as functions of the opportunity cost c, with $p = 5\%$

The optimization in Figure 4.3 corresponds to our implemented version of Figure 8a on page 13 in Nahai-Williamson et al. (2013). The numerical results are different: in our Figure 4.3a the optimal amounts of IM contributions are much smaller than the ones identified by Nahai-Williamson et al. (2013), which are larger and more persistent as the opportunity cost c increases (in Nahai-Williamson et al. (2013), IM contributions are optimal up to $c = 0.15$). However, the scaled Figure 4.3b shows that the relation between optimal resources and opportunity cost is the same found in Nahai-Williamson et al. (2013). Intuitively, when the opportunity cost c increases, the amount of total resources decreases as pledging collateral becomes generally more costly. However, also the total financial sources composition changes. Both Figure 8a on page 13 in Nahai-Williamson et al. (2013) and our Figure 4.3b show that when c overcomes a certain threshold, default fund DF becomes the only optimal source for the CCP. The default fund covers more losses than a single initial margin, so if every parameter is constant and the opportunity cost c alone is increasing, total resources decrease and the default fund becomes the best choice to cover default losses.



(a) $c \in [0.025, 0.35]$



(b) $c \in [0.025, 0.065]$

Figure 4.4: The optimal IM contributions y and DF contributions z obtained via Equation (4.15) as functions of the opportunity cost c, with $p = 25\%$

The optimization in Figure 4.4 corresponds to the one displayed in Figure 8b on page 13 in Nahai-Williamson et al. (2013), which repeats the optimizations for varying opportunity cost c, but with $p = 25\%$. Again, the optimal quantities are different: in Nahai-Williamson et al. (2013) the balance between IM and DF contributions tends more on IM contributions than reported in our optimization in Figure 4.4b. However, the trend behind the optimal financial resources is the same. In our Figure 4.4a, as collateral becomes more costly, the amount of total resources decreases, but not as much as in the first case (see Figures 8a and 8b on page 13 in Nahai-Williamson et al. (2013) and our Figure 4.3a). Now, it is more likely to observe a default ($p = 25\%$), so even if collateral is costly, it is necessary to have a coverage. Even if initial margin decreases also here, Figure 4.4a shows the pressure of a larger default probability: it was shown from Subsection 4.2.1 that higher default probabilities bring to more dependence on initial margins, this is the reason why here optimal IM contributions are more persistent even if collateral becomes more expensive.

From the outline of Section 4.2, we know that almost every optimization is repeated for the two values $p = 0.05$ and $p = 0.25$, but we will not report all the cases for $p = 0.25$. Figures 8a and 8b on page 13 in Nahai-Williamson et al. (2013) and our Figures 4.3a and 4.4a explain what happens when default probability p is higher. First, it is more likely that clearing members default, so there is a general increase in total resources: IM and DF contributions are higher because CCPs need larger collateral. Secondly, with a higher default probability p, IM contributions always increase: Subsection 4.2.1 showed that if defaults are more likely to happen, there is a higher cost of mutualization and clearing members prefer to pay a larger initial margin than contributing to the common default fund.

4.2.3 Varying capital charges d_{IM} and d_{DF}

Opportunity cost c is not the only cost of pledging collateral in the CCP: posting collateral is expensive also because of initial margin capital charge d_{IM}, default fund capital charge d_{DF} and cost of holding capital assigned by banks c_c. In this subsection, we perform an optimization over IM and DF contributions with changing values of capital charges d_{IM} and d_{DF}. Capital charges d_{IM} and d_{DF} are costs of capital applied to both initial margin and default fund, so again we intuitively expect the optimal total resources to decrease as these charges increase.

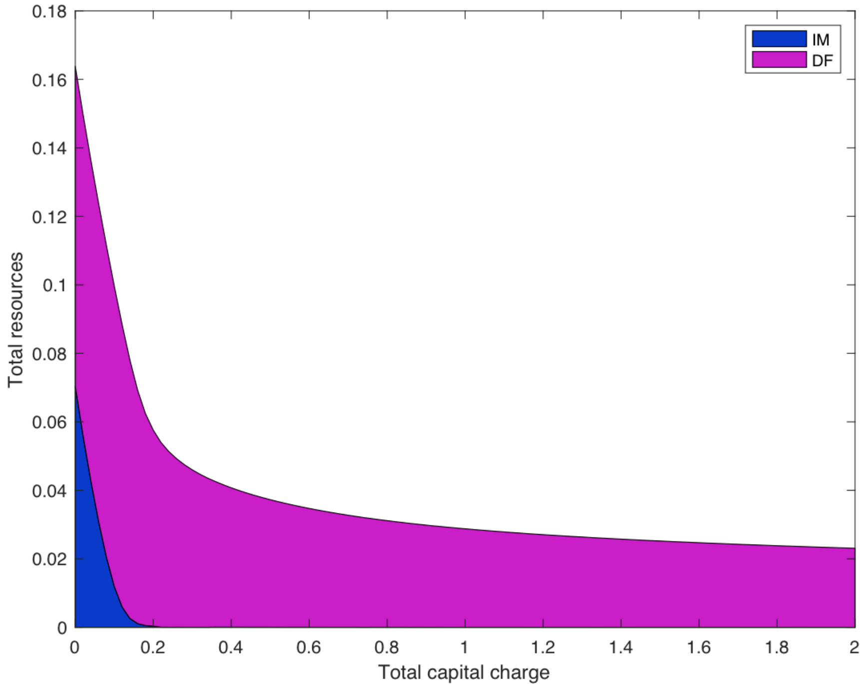
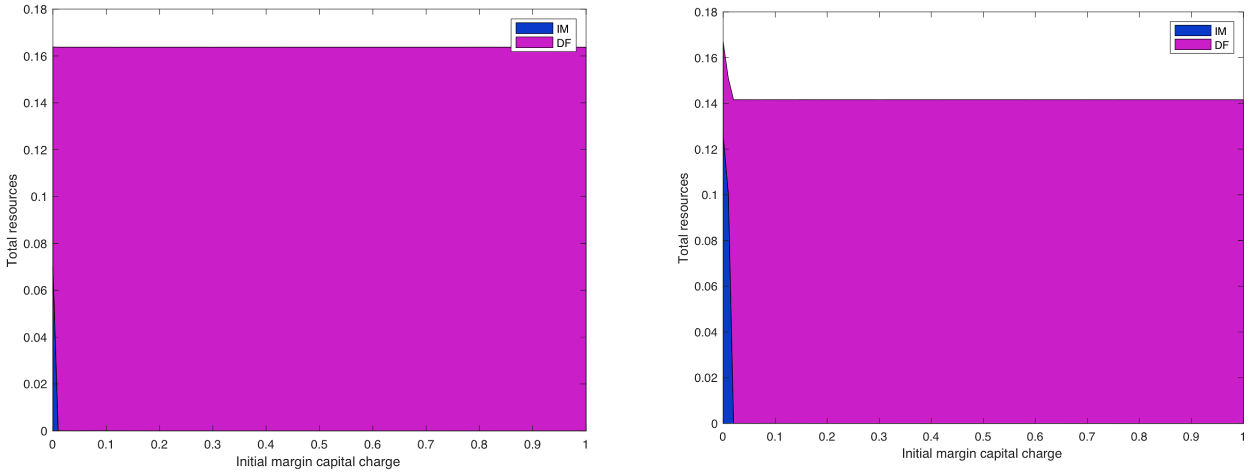


Figure 4.5: The optimal IM contributions y and DF contributions z obtained via Equation (4.15) as functions of the total capital charge $d_{IM} + d_{DF}$, with $p = 5\%$

Figure 4.5 displays the optimal IM and DF contributions via Equation (4.15) on the changing amount of total capital charge, i.e. $d_{IM} + d_{DF}$. Our Figure 4.5 corresponds to the optimization shown in Figure 9a on page 13 in Nahai-Williamson et al. (2013). In our replication, the numerical results are roughly the same as in Nahai-Williamson et al. (2013). In our Figure 4.5, the optimal IM contributions disappear as soon as $d_{IM} + d_{DF} \approx 0.2$, while in Nahai-Williamson et al. (2013) they are persistent up to $d_{IM} + d_{DF} \approx 0.6$. On the other hand, when d_{IM} and d_{DF} both increase, the effect of reducing the amount of total resources is the same. As pledging more collateral becomes more expensive, clearing members just post less collateral. Moreover, sources are only made by DF contributions as soon as total capital charge $d_{IM} + d_{DF}$ is sufficiently high.

More realistically, the optimization is repeated with only one capital charge among d_{IM} and d_{DF} as the changing variable.



(a) With $d_{DF} = 0$

(b) With $d_{DF} = 0.16\%$

Figure 4.6: The optimal IM contributions y and DF contributions z obtained via Equation (4.15) as functions of the IM capital charge d_{IM}, with $p = 5\%$

The optimization in Equation (4.15) on the changing amount of IM capital charge d_{IM} is repeated for two values of DF capital charge, $d_{DF} = 0$ and $d_{DF} = 0.16\%$, with $p = 5\%$. The optimizations in Figure 4.6 correspond to the same optimizations in Figures 10a and 10b on page 14 in Nahai-Williamson et al. (2013). Note that Figures 10a and 10b on page 14 in Nahai-Williamson et al. (2013) are zoomed to focus on the small amount of IM contributions, but the results are roughly the same in our Figure 4.6. Consistently with Nahai-Williamson et al. (2013) and both Figures 4.6a and 4.6b, we can say that when IM capital charge d_{IM} increases, the only resource to manage risk consists almost only of DF contributions. Collateral is necessary to the normal functioning of every CCP and it cannot be eliminated, so applying an asymmetric modification varying IM capital charge d_{IM} makes all the clearing members shift to the cheaper form of collateral. Initial margin is preferred just because of a smaller cost of mutualization, otherwise it is less effective than default fund in terms of loss-absorption. So, if initial margin is also more expensive, then it becomes the least attractive source of all.

In reverse, the next optimization is done via Equation (4.15) on the changing amount of DF capital charge d_{DF}, with $d_{IM} = 0$ and $p = 5\%$.

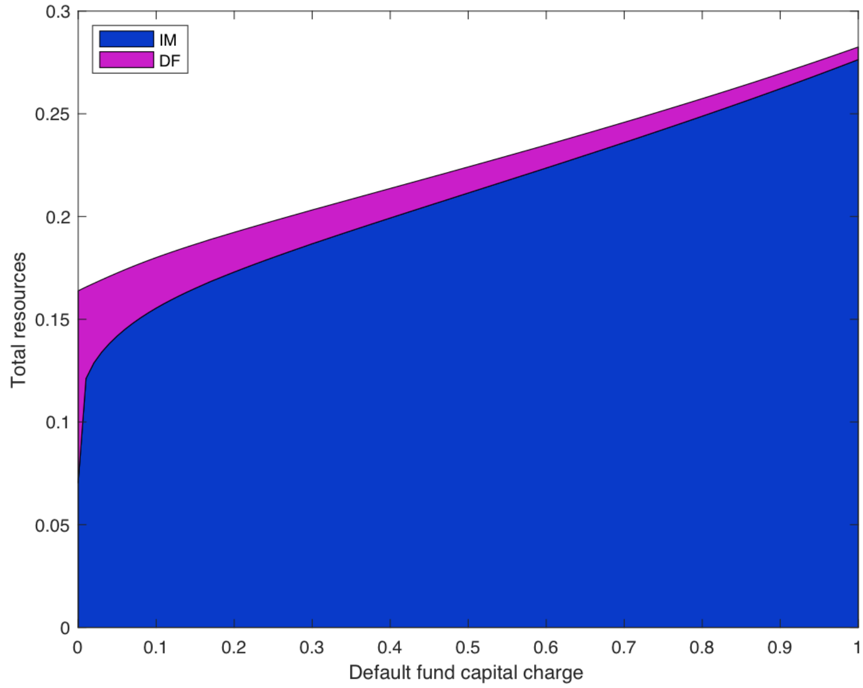


Figure 4.7: The optimal IM contributions y and DF contributions z obtained via Equation (4.15) as functions of the DF capital charge d_{DF}, with $d_{IM} = 0$ and $p = 5\%$

Figure 4.7 displays the result of the optimal quantities for IM and DF contributions on the changing DF capital charge d_{DF}. This optimization is also shown in Figure 11a on page 14 in Nahai-Williamson et al. (2013) and the results are very similar to our Figure 4.7: the total amount of resources in Nahai-Williamson et al. (2013) is ≈ 0.32, while in our optimization is ≈ 0.28, but the relation between financial resources and d_{DF} as well as the balance of sources are the same. According to both Figure 11a on page 14 in Nahai-Williamson et al. (2013) and our Figure 4.7, even with a very large DF capital charge d_{DF}, clearing members do not renounce giving DF contributions, because even if it bears a mutualization cost, it can absorb more default losses, so it takes a much higher capital charge to erase the benefit of DF contributions. The case with $d_{IM} = 0.16\%$ (not reported here), corresponding to Figure 11b on page 14 in Nahai-Williamson et al. (2013), shows a similar equilibrium of sources: the only difference is that optimal amounts of IM contributions are smaller because also IM contributions are expensive.

The last optimizations in Figures 4.5, 4.6 and 4.7 show that capital charges have an extreme impact on CCP's resources: they can change the CCPs resources composition almost completely. Slight differences and modifications on capital charges d_{IM} and d_{DF} give totally different numerical results. Even if in reality CCPs do not choose their own capital charges (as they are mainly decided by international laws), these forms of costs of collateral must be heavily considered when dealing with CCPs risk management.

4.2.4 Varying systemic cost s

The last relation to investigate is the one between optimal DF and IM contributions and the systemic cost s, which represents a constant further loss occurring after CCP's default due to contagion effects. This is a cost occurring only in case of CCP's default, so it is plausible to think that increasing s will make total resources shift towards DF contributions, because default fund postpones defaulting events more than any other collateral.

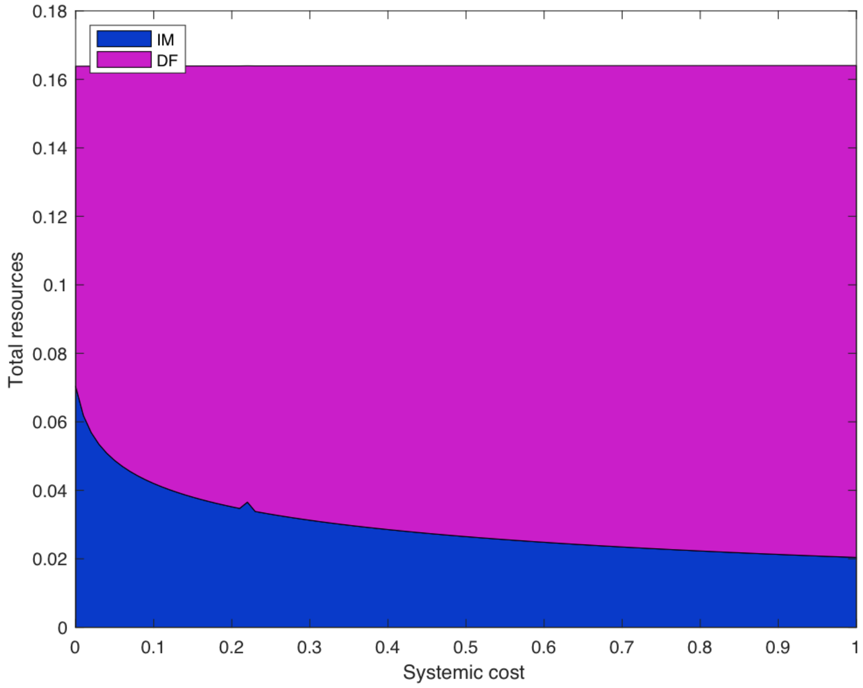


Figure 4.8: The optimal IM contributions y and DF contributions z obtained via Equation (4.15) as functions of the systemic cost s, with $p = 5\%$

Figure 4.8 portraits the results of the optimal quantities for IM and DF contributions on the changing systemic cost s. The same optimization is displayed in Figure 13a on page 15 in Nahai-Williamson et al. (2013): in Nahai-Williamson et al. (2013), the IM contribution equals zero as soon as $s \approx 0.11$, while in our Figure 4.8 IM contributions y decrease, but they never disappear; moreover, Nahai-Williamson et al. (2013) optimal DF contribution is ≈ 0.25, while our optimal z is ≈ 0.16. However, both in Figure 13a on page 15 in Nahai-Williamson et al. (2013) and in our Figure 4.8, the optimal DF contribution is constant and the IM contribution decreases as the systemic cost s increases. An increasing post-default cost makes clearing members want to postpone CCP's defaulting events, so there is an incentive towards DF contributions, always due to its loss mutualization and absorbency. Consistently with Figure 13b on page 15 in Nahai-Williamson et al. (2013), this optimization is repeated for $p = 25\%$, showing that a larger individual default probability makes IM contributions more persistent.

4.3 Unrealistic assumptions

The model in Nahai-Williamson et al. (2013) makes some unrealistic assumptions about clearing members, which will be now briefly discussed:

1. In Nahai-Williamson et al. (2013), the individual probabilities of default are exogenous and independent. In other words, the number of defaults N in Nahai-Williamson et al. (2013) model follows a Binomial distribution, i.e. $N \sim \text{Bin}(\frac{n}{2} - 1, p)$ for $loss_{OTM}(h, i)$ in Equation (4.7) and $N \sim \text{Bin}(\frac{n}{2}, p)$ for $loss_{ITM}(h, i)$ in Equation (4.12). Such models have very light tails, i.e. $\mathbb{P}[N > i]$ is very small for larger i, even if $i \leq \frac{n}{2} - 1$ or $i \leq \frac{n}{2}$, and this is not realistic compared to what is observed in reality, especially in crises times. Moreover, several authors like McNeil et al. (2015) and Lando (2004) agree that the assumption of independent default probabilities is unrealistic;

2. In Nahai-Williamson et al. (2013), the underlying prices H and clearing members default probabilities are uncorrelated.

The authors in Nahai-Williamson et al. (2013) minimize the expected loss function for different settings and they find optimal balances of default fund and initial margin individual contributions. In this thesis, we will relax the two assumptions above introducing dependent default probabilities and asset prices which are correlated with the default probabilities themselves. To be able to create a framework in which default probabilities and prices are more realistic, we will use a static credit risk modelling framework, which means a different distribution for the number of defaulting members N with a higher probability to observe a large number of defaults i, i.e. fatter tails. More specific, we will use a Merton mixed binomial model, which will be presented in the next chapter.

Chapter 5

Static credit risk modelling

In this chapter, we will build a framework in which individual default probabilities and prices become dependent and thus more realistic. So, in this extended model, we create a new expected loss function for CCP's members and minimize it to find the optimal quantities of IM and DF contributions. Schönbucher (2003) points out that credit risk can be divided in multiple components such as arrival risk, timing risk, recovery risk and dependency risk or risk of contagion. Studying and modelling credit risk can be done in multiple ways: for example, it is possible to study both the arrival risk and timing risk, i.e. whether the default occurs or not and in which exact moment in time. Other analyses only consider arrival risk and contagion risk, i.e. whether the default happens or not in a given time period $t \in [0, T]$ and whether or not it affects other counterparties. Models that do not consider timing risk are often called *static credit risk models*. To make probabilities and prices dependent, we will use one of the most common models in static portfolio credit risk, the Merton mixed binomial model (see Frey and McNeil (2001), Frey and McNeil (2003) and McNeil et al. (2015)). To be able to use the Merton mixed binomial model, Section 1 will introduce the baseline of Merton model, Section 2 will explain the mixed binomial model and Section 3 will mix these to have the Merton mixed binomial model, which will be used to make default and price dependencies.

5.1 Merton model

This section explains the basic concepts and the functioning of Merton model, the most common and largely used static credit risk model in the literature. The baseline follows previous reviews of Merton model, mostly by Frey and McNeil (2001), Frey and McNeil (2003), Lando (2004) and Herbertsson (2018).

5.1.1 Assumptions and setup

The main assumption in the Merton model is the Black-Scholes setting: it means that trading has no effects on prices, there are no transaction costs, short selling is possible and borrowing and lending are done at a risk-free rate r. Moreover, in the Black-Scholes setting, both equity and debt are priced like derivatives with the stock price as the underlying asset. Consider a company C. The value of C's assets at time t, denoted by V_t, follows a geometric Brownian motion according to:

$$dV_t = \mu V_t dt + \sigma V_t dW_t \tag{5.1}$$

where W_t is a standard Brownian motion, which is a stochastic process continuous in time with the following properties:

- $W_0 = 0$;
- W_t has a continuous path;
- $W_t - W_s \sim N(0, t-s)$, so $W_t - W_s$ is Normally distributed with zero mean and variance $t-s$;
- W_t has independent increments, it means that, if $t_1 < t_2 < t_3 < t_4$, then $W_{t_2} - W_{t_1}$ is independent from $W_{t_4} - W_{t_3}$.

By using Ito's lemma on Equation (5.1), one can show that the asset value V_t is given by:

$$V_t = V_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \tag{5.2}$$

where $(\mu - \frac{1}{2}\sigma^2)t$ is a deterministic term and σW_t is the stochastic term, depending on the standard Brownian motion.

5.1.2 Total assets value, equity and debt claims

The company C issues two types of claims: debt and equity. The debt is issued in terms of zero coupon bonds and, at the maturity T, C pays the amount D to debtholders. The value of the zero coupon bond at any time t is B_t, while the value of equity at any time t is S_t. If these are the only claims issued by the company, then the company total assets is defined as:

$$V_t = B_t + S_t \tag{5.3}$$

Total asset value is only made up by debt claims and equity claims, so, at any time $t \in [0, T]$, it is equal to the sum of zero coupon bond value B_t and the equity value S_t.

Now, two possible situations may arise at maturity T: either the total assets value V_T exceeds the amount D which will be paid to debtholders, or it does not. More specifically, at maturity T, these two conditions lead to different consequences. Whenever the total assets value is larger than the amount D, i.e. $V_T > D$, then the amount D is paid to debtoholders and the remaining $V_T - D$ is earned by the equity owners. On the contrary, if the total assets value is lower than the amount D, i.e. $V_T < D$, then we can say that the company faces a default in T and the whole amount V_T is paid to debtholders. So the debthoders lose $D - V_T$ and the equity owners get nothing.

In the Merton model a default is only possible at maturity T and not before T. We need this outline in order to merge it with the mixed binomial model to obtain a Merton mixed binomial model and introduce dependencies between default probabilities and prices in the final expected loss function for CCP's members.

5.2 Mixed binomial model

This section reviews the so called *mixed binomial model* (see Frey and McNeil (2001), Frey and McNeil (2003) and Lando (2004)). In Section 4.3, we pointed out that Nahai-Williamson et al. (2013) use a *binomial model* to find the probability of multiple defaults. Hence, in Nahai-Williamson et al. (2013) the default probabilities are independent, while the mixed binomial model we are going to introduce randomizes the individual default probabilities and allows for a stronger dependence between them. Again, the main source of this material comes from Lando (2004) and Herbertsson (2018).

5.2.1 Model outline

Consider a portfolio/network with m obligors, where each obligor can default up to time T. Let X_j be a random variable such that $X_j = 1$ if obligor j defaults up to time T, otherwise $X_j = 0$. It is assumed that X_1, X_2,..., X_m are all independent with identical distribution. Furthermore, let the unconditional default probability be $\mathbb{P}[X_j = 1] = \overline{p}$ and the unconditional survival probability be $\mathbb{P}[X_j = 0] = 1 - \overline{p}$.

The mixed binomial model randomizes the individual default probabilities. More specific, the individual default probability is a function of one *common factor* W that represents some common economic background variable that affects all the obligors in the network at the same time. The variable W can be interpreted differently for different market models, but in this model W is a random variable modelling the economic background, affecting all the individuals and creating dependence and correlation. So, W is a random variable with density $f_W(w)$ and the default probability is a function of W, namely $p(W)$, given by:

$$p(W) = \mathbb{P}[X_j = 1 \mid W]. \tag{5.4}$$

It is clear that also $p(W)$ is a random variable and it represents the conditional default probability $\mathbb{P}[X_j = 1 \mid W]$. More specifically, let $\overline{p}$ be given by:

$$\overline{p} = \mathbb{E}[p(W)] \tag{5.5}$$

and note also that:

$$\mathbb{P}[X_j = 1] = \mathbb{E}[X_j] = \mathbb{E}[\mathbb{E}[X_j \mid W]] = \mathbb{E}[p(W)] = \overline{p} \tag{5.6}$$

$$\mathrm{Var}(X_j) = \overline{p}(1 - \overline{p}) \tag{5.7}$$

$$\mathrm{Cov}(X_{j_1}, X_{j_2}) = \mathrm{Var}(p(W)). \tag{5.8}$$

For more details on Equations (5.6), (5.7) and (5.8) see Chapter 9.2 on page 216 in Lando (2004).

5.2.2 Default probability

Let N be the number of defaults among m obligors. Then, N is given by:

$$N = \sum_{j=1}^{m} X_j.$$

Hence, we next need to study

$$\mathbb{P}[N = i]. \tag{5.9}$$

which represents the probability to observe multiple defaults, more precisely, i simultaneous defaults.

The key point of the mixed binomial model is that individual variables describing the default X_1, X_2,..., X_m are not independent anymore, they are only *conditionally* independent, as seen in Equations (5.4) and (5.8). Having only conditional independence is what allows for dependence between the agents, in our case, the clearing members: this common factor and the conditional probability as a function of this common factor is what we will use to build dependencies between clearing members. If these variables are only conditional independent, then the conditional probability of observing i defaults can be computed through a Binomial:

$$\mathbb{P}[N = i \mid W] = \binom{m}{i} p(W)^i (1 - p(W))^{m-i} \tag{5.10}$$

where i can be interpreted as the number of "successes" and $m - i$ the number of failures in the Binomial distribution: in this case, the "success case" is the default $X_j = 1$. Note that:

$$\mathbb{P}[N = i] = \mathbb{E}[\mathbb{P}[N = i \mid W]]. \tag{5.11}$$

So combining Equation (5.10) with Equation (5.11):

$$\mathbb{P}[N = i] = \mathbb{E}\left[\binom{m}{i} p(W)^i (1 - p(W))^{m-i}\right]. \tag{5.12}$$

Recall the general rule for the expected value of any function of random variables, $\mathbb{E}[A(X)] = \int_{-\infty}^{+\infty} A(x) f_X(x) dx$, which implies that Equation (5.12) is given by:

$$\mathbb{P}[N = i] = \int_{-\infty}^{+\infty} \binom{m}{i} p(W)^i (1 - p(W))^{m-i} f_W(w) dw. \tag{5.13}$$

In a mixed binomial model, this is the analytical formula for the probability of observing exactly i defaults. Mixed binomial models consist in a very general framework, the conditional default probability $p(W)$ can now be derived according to different models: we will use the Merton model, but a Beta distribution and a Logit-Normal distribution could also be used to find a closed formula for $p(W)$ (see McNeil et al. (2015) and Lando (2004)).

5.3 Merton mixed binomial model

This section merges the Merton model and the mixed binomial model. In our case, the mixed binomial model is inserted in a Merton model framework, with all Merton assumptions and features seen in Section 5.1. The baseline of our presentation in this section is taken from Frey and McNeil (2001), Frey and McNeil (2003), Lando (2004) and Herbertsson (2018). The Merton mixed binomial model leads to individual default probabilities, which are dependent as well as related with asset prices.

5.3.1 Conditional default probability

Consider a portfolio/network with m obligors, each obligor is described by a Bernoullian random variable X_j, which is either $X_j = 1$ whenever this individual defaults or $X_j = 0$ otherwise. X_1, X_2,..., X_m will be only conditionally independent and identically distributed, as in every mixed binomial model. But now, each obligor stays in a Merton framework. Hence, the j-obligor total assets follow the dynamics:

$$dV_{t,j} = \mu_j V_{t,j} dt + \sigma_j V_{t,j} dB_{t,j}$$

where $B_{t,j}$ is a stochastic process defined by:

$$B_{t,j} = \sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,j} \tag{5.14}$$

and where $W_{t,0}$, $W_{t,1}$, $W_{t,2}$,..., $W_{t,m}$ are independent standard Brownian motions. Even if we have an additional transformation from the standard Brownian motions, i.e. through $B_{t,j}$, the total assets value $V_{t,j}$ is still a geometric Brownian motion. Applying Ito's lemma, it is possible to write:

$$V_{t,j} = V_{0,j} e^{(\mu_j - \frac{1}{2}\sigma_j^2)t + \sigma_j B_{t,j}}.$$

As shown in Equation (5.14), total assets value for each obligor is driven by a common process $W_{t,0}$, that is not j-indexed and affects every single obligor at the same time: this is the economic environment. Then, $B_{t,j}$, and therefore $V_{t,j}$, is affected by an individual process $W_{t,j}$ that is unique for each obligor j. The assets will depend both on a systemic process and on an idiosyncratic process: the common process will create dependence among individuals. Note that Equation (5.14) displays a new variable ρ, the Merton correlation, which describes the correlation between members' total assets returns. The derivation and the meaning of this variable are explained in Appendix A.

As stated in Subsection 5.1.2, in the Merton model, default can occur only at time T, depending on whether the total assets of the company are enough to pay debtholders or not. Now, let D_j be the individual amount that each j-obligor has to pay to its debtholders at time T, the bankruptcy occurs if anf only if $V_{T,j} < D_j$. Then default happens whenever:

$$V_{0,j} e^{(\mu_j - \frac{1}{2}\sigma_j^2)T + \sigma_j B_{T,j}} < D_j.$$

Applying the logarithmic transformation $f(x) = \ln(x)$, the inequality still holds because it is a strictly increasing function, so:

$$\ln(V_{0,j}) - \ln(D_j) + \left(\mu_j - \frac{1}{2}\sigma_j^2\right) T + \sigma_j B_{T,j} < 0.$$

Substituting $B_{T,j}$ with what stated in Equation (5.14):

$$\ln(V_{0,j}) - \ln(D_j) + \left(\mu_j - \frac{1}{2}\sigma_j^2\right) T + \sigma_j(\sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,j}) < 0. \tag{5.15}$$

In the standard Brownian motion, increments between t_2 and t_1 are independent and Normally distributed with mean zero and variance $(t_2 - t_1)$ and $W_0 = 0$. So, in this case at time T:

$$W_{T,j} - W_{0,j} \sim N(0, T - 0)$$

$$\Rightarrow W_{T,j} \sim N(0,T).$$

Now, if $Y_j \sim N(0,1)$, i.e. Y_j is a standard Normal, then $W_{T,j}$ has the same distribution as $\sqrt{T}Y_j$, where Y_1, Y_2,..., Y_m are also independent. After this transformation, the common background variable $W_{T,0}$ is now turned into $\sqrt{T}Y_0$: this is where the fusion between Merton model and mixed binomial model happens, because we define this new common variable Y_0 as the common background factor W in the mixed binomial model, that drives conditional default probabilities. Thus, $Y_0 = W$, so the event in Equation (5.15) has the same conditional probability (and thus same probability) as the event:

$$\ln(V_{0,j}) - \ln(D_j) + \left(\mu_j - \frac{1}{2}\sigma_j^2\right) T + \sigma_j(\sqrt{\rho}\sqrt{T}W + \sqrt{1-\rho}\sqrt{T}Y_j) < 0. \tag{5.16}$$

By dividing with $\sigma_j\sqrt{T}$ in Equation (5.16), we get:

$$\frac{\ln(V_{0,j}) - \ln(D_j) + \left(\mu_j - \frac{1}{2}\sigma_j^2\right) T}{\sigma_j\sqrt{T}} + \sqrt{\rho}W + \sqrt{1-\rho}Y_j < 0$$

and making the substitution

$$C_j = \frac{\ln(V_{0,j}) - \ln(D_j) + \left(\mu_j - \frac{1}{2}\sigma_j^2\right) T}{\sigma_j\sqrt{T}}$$

then Equation (5.16) is equivalent with the event:

$$C_j + \sqrt{\rho}W + \sqrt{1-\rho}Y_j < 0, \tag{5.17}$$

that is

$$Y_j < \frac{-(C_j + \sqrt{\rho}W)}{\sqrt{1-\rho}}. \tag{5.18}$$

The result in Equation (5.18), which is equivalent with (5.15), says that the individual component of risk, Y_j has to be smaller than a certain amount to observe a default at time T. Default occurs whenever individual riskiness falls before a certain threshold. This is very useful, because the above calculations showed that the default event $V_{T,j} < D_j$ has the same probability (and conditional probability) as $Y_j < \frac{-(C_j+\sqrt{\rho}W)}{\sqrt{1-\rho}}$. Next, define X_j as $X_j = 1$ when the j-obligor defaults and zero otherwise. Then, the mixed binomial model then leads to:

$$\begin{aligned} p(W) &= \mathbb{P}[X_j = 1 \mid W] \\ &= \mathbb{P}\left[Y_j < \frac{-(C_j + \sqrt{\rho}W)}{\sqrt{1-\rho}} \mid W\right] \\ &= N\left(\frac{-(C_j + \sqrt{\rho}W)}{\sqrt{1-\rho}}\right) \end{aligned} \tag{5.19}$$

where last equality in Equation (5.19) is due to the fact that the Y_is are independent and standard Normal. So, conditional default probability becomes a realization of a standard Normal cumulative distribution.

Since the standard Brownian motions $W_{t,0}$, W_{t,j_1} and W_{t,j_2} are independent, the $W = Y_0$, Y_1, Y_2,..., Y_m are also independent. Now, assume that all the obligors in the model are identical, so $V_{0,j} = V_0$, $D_j = D$, $\mu_j = \mu$, $\sigma_j = \sigma$ and $C_j = C$, then:

$$p(W) = \mathbb{P}[X_j = 1 \mid W] = N\left(\frac{-(C + \sqrt{\rho}W)}{\sqrt{1-\rho}}\right). \tag{5.20}$$

The function $p(W)$ gives us the conditional probability of default as function of the factor W. The mixed binomial model considers the conditional default probability as a function of factor W and by applying the Merton Model this function $p(W)$ becomes explicit, as seen in Equation (5.19) or (5.20).

Now, going back to the unconditional default probability and using Equation (5.17), we get that:

$$\begin{aligned} \mathbb{P}[X_j = 1] &= \mathbb{P}[V_{T,j} < D] \\ &= \mathbb{P}[C + \sqrt{\rho}W + \sqrt{1-\rho}Y_j < 0] \\ &= \mathbb{P}[\sqrt{\rho}W + \sqrt{1-\rho}Y_j < -C]. \end{aligned} \tag{5.21}$$

Furthermore, since W and Y_j are both standard Normal, also a linear transformation like $\sqrt{\rho}W + \sqrt{1-\rho}Y_j$ remains a standard Normal, which in Equation (5.21) implies that:

$$\mathbb{P}[X_j = 1] = \mathbb{P}[\sqrt{\rho}W + \sqrt{1-\rho}Y_j < -C] = N(-C)$$
$$\Rightarrow \mathbb{P}[X_j = 1] = N(-C). \tag{5.22}$$

From the outline of mixed binomial model, in Equation (5.6) we have $\mathbb{P}[X_j = 1] = \mathbb{E}[p(W)] = \overline{p}$, so Equation (5.22) implies:

$$\overline{p} = N(-C) \Rightarrow N^{-1}(\overline{p}) = -C \Rightarrow C = -N^{-1}(\overline{p}). \tag{5.23}$$

Merging this result with Equation (5.20) we obtain a final closed formula for the individual conditional default probability given by:

$$p(W) = \mathbb{P}[X_j = 1 \mid W] = N\left(\frac{N^{-1}(\overline{p}) - \sqrt{\rho}W}{\sqrt{1-\rho}}\right) \tag{5.24}$$

where $\overline{p}$ represent the unconditional default probability of clearing members and ρ is the Merton correlation between members asset returns, which is illustrated in Appendix A.

5.3.2 The probability of i defaults

In a mixed binomial model inspired by the Merton model, the number of defaults N among m obligors is given by:

$$N = \sum_{j=1}^{m} X_j.$$

Next, we derive an expression for $\mathbb{P}[N = i]$ and, according to Equation (5.10), $\mathbb{P}[N = i]$ is equal to:

$$
\begin{aligned}
\mathbb{P}[N=i] &= \mathbb{E}[\mathbb{P}[N=i] \mid W] \\
&= \mathbb{E}\left[\binom{m}{i} p(W)^i (1-p(W))^{m-i}\right] \\
&= \int_{-\infty}^{+\infty} \binom{m}{i} p(w)^i (1-p(w))^{m-i} f_W(w) dw.
\end{aligned} \tag{5.25}
$$

The Merton model made the function $p(W)$ explicit in Equation (5.24), Equation (5.25) then yields to:

$$
\mathbb{P}[N=i] = \int_{-\infty}^{+\infty} \binom{m}{i} N\left(\frac{N^{-1}(\overline{p}) - \sqrt{\rho}W}{\sqrt{1-\rho}}\right)^i \left(1 - N\left(\frac{N^{-1}(\overline{p}) - \sqrt{\rho}W}{\sqrt{1-\rho}}\right)\right)^{m-i} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw. \tag{5.26}
$$

So, we have derived an analytical formula to compute the probability to have i multiple members defaulting.

Now we have all the instruments to extend the model by Nahai-Williamson et al. (2013) and realize a more realistic framework. Thanks to the common background factor defined in this chapter in Merton mixed binomial model, we will make individual default probabilities unconditionally independent and we will build underlying prices as a function of the same background factor. This will be the key point to be able to construct a new expected loss function for CCP's members and minimize it to find the new optimal quantities of IM and DF contributions.

Chapter 6

Introducing default and price dependencies in Nahai-Williamson et al. (2013)

This chapter presents the core of our thesis as well as our new contribution to central clearing risk management literature. We implement an extension of Nahai-Williamson et al. (2013) model to a more realistic setting, which allows for dependent default probabilities and relates the underlying prices with the factor driving the individual defaults. The main purpose of this extension is to see if this new framework introduces large differences with Nahai-Williamson et al. (2013) in terms of optimal quantities of IM and DF contributions. The Merton mixed binomial model in Chapter 5 will be used to model the new individual default probability and the same background factor will be used to model prices.

The possibility of dependent default probabilities and more realistic prices changes the CCP's framework, in particular the expected loss function for surviving clearing members. First, Section 1 builds the new expected loss function for CCP's surviving member. Secondly, Section 2 repeats all the original optimizations (as done in Nahai-Williamson et al. (2013)) with the same fixed and varying parameters, minimizing this new function to find the new optimal quantities for individual IM and DF contributions.

6.1 Model and method

In the following section, we explain which assumptions in Nahai-Williamson et al. (2013) are relaxed and how the new expected loss function for CCP's surviving member is constructed. We also discuss how to integrate dependent default probabilities and realistic prices in the loss minimization and then repeat the optimizations as done in Nahai-Williamson et al. (2013).

6.1.1 Relaxing assumptions

The model in Nahai-Williamson et al. (2013) is based on several assumptions, for example regarding the financial network in which the CCP operates and the expected loss function of surviving members, as discussed in Section 4.1. Among these assumptions, two are crucial: the individual default probability of each clearing member is exogenous and assumed to be independent from other members' default and the economic environment. The second assumption is the price structure: underlying prices in market are the values deciding which positions are in-the-money and out-of-the-money. Nahai-Williamson et al. (2013) assume an initial price equal to zero and then use equivalently price H or price variation ΔH, assuming that $\Delta H \sim N(0, \sigma)$.

It is unrealistic to assume that clearing members' individual default probabilities are exogenous and independent. Default probability changes due to two main drivers: first, default probability is affected by price movements in the market through many channels (companies' positions, their investments value, their revenues and sales can be drastically different and so on); second, default probability also reflects other members' losses and defaults. On the other hand, it is possible to assume that price changes are Normally distributed with a variance equal to the estimated market volatility σ. Again, the problem is that price H is not correlated with anything else, while obviously prices are affected by the whole economic background as well as by defaults.

In this thesis, we will extend the model in Nahai-Williamson et al. (2013) relaxing the two assumptions above. Individual default probabilities will be modelled with a Merton mixed binomial model, as presented in Chapter 5, and prices will become a function of the same underlying factor driving the Merton mixed binomial model. The new parameters introduced in the extended model will consequently affect the surviving members' expected loss function and optimal IM and DF quantities for CCPs resources.

6.1.2 Underlying factor W

As discussed in Chapter 5, the Merton mixed binomial model uses a common underlying factor W to define both conditional default probability and, in this case, the market prices. It is not fundamental to find the specific nature of this factor: it is open to interpretation, it could be the variation of any microeconomic or macroeconomic value, index, commodity and so on. What is important is that the factor W is distributed as a standard Normal, $W \sim N(0, 1)$, which is very plausible for any single random variable whose realizations are continuous in time. Hence, the density to the factor W will be:

$$f_W(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}.$$

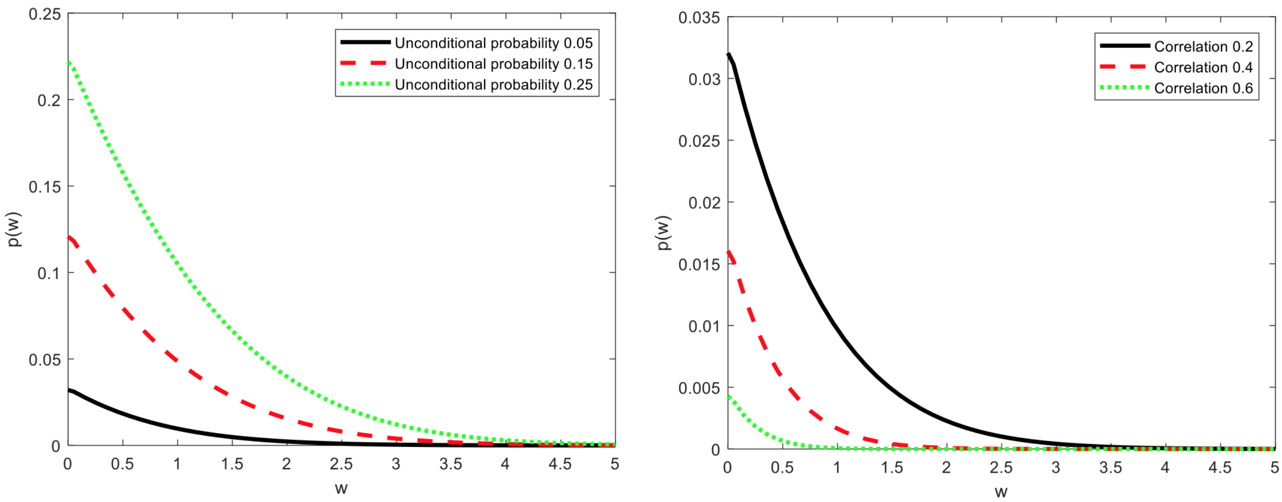
As will be seen later, in the extended model the factor W drives both the individual conditional default probability and the market price, which means both the probability and the price are functions of W.

6.1.3 Individual conditional default probability $p(W)$

We will extend the version by Nahai-Williamson et al. (2013) with an adaptation to Merton mixed binomial model to have dependent default probabilities. The default dependencies in our framework are created via the common background factor W as specified by Equation (5.24). The correlation parameter ρ in the Merton model allows dependence between clearing members, whose total assets value moves in relation to this parameter. Furthermore, dependence is also defined through default correlation ρ_X, which ignites also a relation through clearing members' default random variables. See Appendix A for more detail on Merton correlation ρ and default correlation ρ_X. Recall from Equation (5.24) in Chapter 5 that:

$$p(W) = \mathbb{P}[X_j = 1 \mid W] = N\left(\frac{N^{-1}(\overline{p}) - \sqrt{\rho}W)}{\sqrt{1-\rho}}\right) \tag{6.1}$$

where $\overline{p}$ is the unconditional default probability and ρ is the Merton correlation (see Appendix A). Both unconditional probability $\overline{p}$ and Merton correlation ρ are the inputs of our extended model, while in Nahai-Williamson et al. (2013) the exogenous input was the independent default probability p (see Subsection 4.1.1). Here, Equation (6.1) states the relation between the common underlying factor W and the conditional default probability $p(W)$. Also note that $\mathbb{E}[p(W)] = \mathbb{E}[\mathbb{P}[X_j = 1 \mid W]] = \overline{p}$ (see Equations (5.4) and (5.5)).



(a) With $\overline{p} = 0.05$, $\overline{p} = 0.15$ and $\overline{p} = 0.25$

(b) With $\rho = 0.2$, $\rho = 0.4$ and $\rho = 0.6$

Figure 6.1: Conditional default probability $p(w)$ as function of factor w

Figure 6.1 shows that the individual conditional default probability $p(W)$ decreases as the underlying factor W realizes in larger values. Hence, the Merton underlying factor W and the new conditional default probability $p(W)$ are inversely proportional. Moreover, Figure 6.1a displays an intuitive result: when the unconditional and exogenous default probability $\overline{p}$ is larger, then $p(W)$ will also be larger. On the other hand, when Merton correlation ρ is larger, i.e. clearing members' asset move more similarly in the market, the lower unpredictability makes $p(W)$ decrease.

From now on, we will use the conditional default probability in Equation (6.1) for clearing members and their expected loss function. Recall that $\overline{p}$ represents the possibility of one company to default as if it was alone and completely detached from its environment. The conditional probability $p(W)$ in Equation (6.1) contains both the relation between default probability and the economic background, because it is a function of factor W, but it also contains a link with other clearing members, due to Merton correlation ρ (see Appendix A for more details).

In our extended model, Nahai-Williamson et al. (2013) individual default probability p is replaced with Merton conditional probability $p(W)$. This brings new parameters on the table: there are now the common background factor W and the Merton correlation ρ, where the former explains the link between default probabilities and the environment, while the latter shows how default probabilities are linked to members' assets conditions.

6.1.4 Market price $g(W)$

In Nahai-Williamson et al. (2013), prices H are not correlated with the economic environment and they are Normally distributed $H \sim N(0, \sigma)$. To be able to connect prices with the economical background, we let prices be a function $g(W)$ of the same environmental factor W that drives the conditional default probabilities. In other words, compared to prices H in Nahai-Williamson et al. (2013), now prices are defined by $H = g(W)$. The next step is defining how this background factor W affects prices $g(W)$. We let $g(w)$ be given by:

$$g(w) = w^2 \cdot \mathbb{I}_{\{w>0\}} = \begin{cases} w^2 & \text{if } w > 0 \\ 0 & \text{if } w < 0. \end{cases} \tag{6.2}$$

Recall that we need a function $g(\cdot)$ that links the background factor W to prices $g(W)$, in order to have more realistic prices that move with the economic environment. In the choice of this function $g(\cdot)$, some conditions must be respected: prices cannot be negative by definition, so there cannot be a realized value w that gives us a realization $g(w)$ smaller than zero ($g(w) \geq 0$ should always hold). Moreover, prices either increase or decrease as the factor W increases, this is again up to interpretation: it depends on what is identified as the factor W and which type of asset the price models. Our analysis only wants to study the consequences of dependent prices on the final CCPs' financial resources, so we can indistinctly use an increasing or decreasing function. In addition, the function $g(\cdot)$ relating the common factor to prices should not be symmetrical: prices change with different magnitude when other factors increase rather than decreasing.

Note that if we set the Merton correlation equal to zero in Equation (6.1), i.e. $\rho = 0$, and let the price be the positive linear function $h = g(w) = w \cdot \mathbb{I}_{\{w>0\}}$, then we are back in the model by Nahai-Williamson et al. (2013), where the number of defaults N is Binomially distributed with parameters p, $\frac{n}{2} - 1$ and $\frac{n}{2}$, i.e. $N \sim \text{Bin}(\frac{n}{2} - 1, p)$ for OTM members and $N \sim \text{Bin}(\frac{n}{2}, p)$ for ITM members. In this case, Equation (6.1) shows that the unconditional default probability $\overline{p}$ would be equal to the conditional one $p(W)$, $\overline{p} = p(W)$. So, the individual default probability would be exogenous and there would not be any background factor W: it is exactly as modelled in Nahai-Williamson et al. (2013).

The function we chose in our thesis for $g(w)$ in Equation (6.2) is plotted in Figure 6.2:

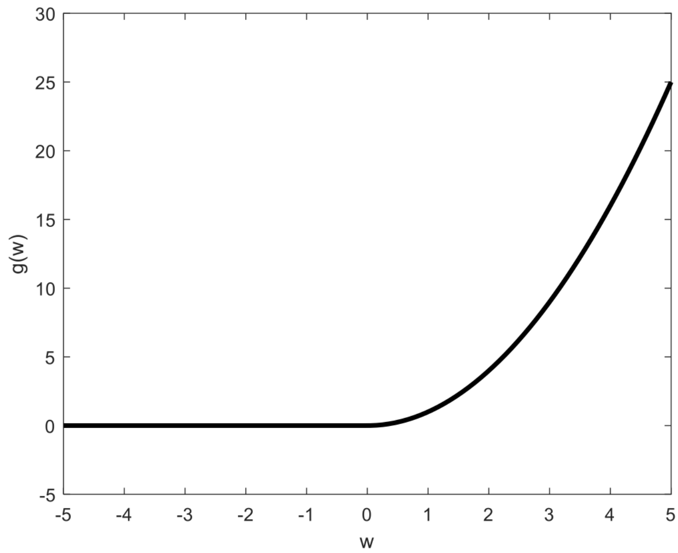


Figure 6.2: Price $g(w)$ as function of the background factor w

From Figure 6.2 we see that $g(w)$ behaves like a parabola through the origin $[0, 0]$, when $w > 0$, and it is equal to zero otherwise. Note that every other function that is positive, asymmetrical and strictly increasing or decreasing could have been used in place of $g(w)$ in Equation (6.2). For example, possible functions could have been both the exponentials $g(w) = e^w$ and $g(w) = e^{-w}$, but also positive linear functions like $g(w) = \max(|w|, 0)$ and $g(w) = \max(w, 0)$.

6.1.5 New expected loss function

The surviving members' expected loss function is different: in our extended version of Nahai-Williamson et al. (2013), the individual default probability p is substituted by conditional default probability $p(W)$, while prices H are now a function $g(W)$ of the background factor W. The next step is proceeding with the new expected loss function for the single surviving OTM member, to see how the model is changed. The loss function L^{OTM} for the single OTM members can be now written as a function of both number of defaults N and factor W, that is:

$$L^{OTM} = L^{OTM}(N, W)$$

Next, note that:

$$\mathbb{E}[L^{OTM}(N, W)] = \mathbb{E}[\mathbb{E}[L^{OTM}(N, W) \mid W]] \tag{6.3}$$

where the last equation is due to law of iterated expectation. So, we now focus on $L^{OTM}(N, W)$ and note that:

$$L^{OTM}(N, W) = \sum_{i=0}^{\frac{n}{2}-1} L^{OTM}(i, W) \cdot \mathbb{I}_{\{N=i\}} \tag{6.4}$$

and Equation (6.4) then implies that

$$
\begin{aligned}
\mathbb{E}[L^{OTM}(N,W) \mid W] &= \mathbb{E}\left[\sum_{i=0}^{\frac{n}{2}-1} L^{OTM}(i,W) \cdot \mathbb{I}_{\{N=i\}} \mid W\right] \\
&= \sum_{i=0}^{\frac{n}{2}-1} \mathbb{E}[L^{OTM}(i,W) \cdot \mathbb{I}_{\{N=i\}} \mid W].
\end{aligned} \tag{6.5}
$$

The conditional expected value inside the sum can be rewritten as:

$$
\begin{aligned}
\mathbb{E}[L^{OTM}(i,W) \cdot \mathbb{I}_{\{N=i\}} \mid W] &= \mathbb{E}[L^{OTM}(i,W) \mid W] \cdot \mathbb{E}[\mathbb{I}_{\{N=i\}} \mid W] \\
&= \mathbb{E}[L^{OTM}(i,W) \mid W] \cdot \mathbb{P}[N=i \mid W]
\end{aligned} \tag{6.6}
$$

where the first equality in Equation (6.6) is due to the fact that W and N are independent *conditionally* on W. The second equality in Equation (6.6) shows that the expected value of an indicator function $\mathbb{I}_{\{A\}}$ is equal to the probability of the indicator function event A to happen. Recall that the probability $\mathbb{P}[N=i \mid W]$ in Equation (6.6) is the realization of a Binomially distributed random variable that counts a number i of "successes" with probability $p(W)$, so:

$$
\mathbb{P}[N=i \mid W] = \binom{\frac{n}{2}-1}{i} p(W)^i (1-p(W))^{\frac{n}{2}-1-i}. \tag{6.7}
$$

Next, the first term in Equation (6.6) can be rewritten as:

$$
\mathbb{E}[L^{OTM}(i,W) \mid W] = L_i^{OTM}(W) \tag{6.8}
$$

where $L_i^{OTM}(w)$ is given by

$$
L_i^{OTM}(w) = \begin{cases}
0 & \text{if } g(w) \leq y+z \\
\dfrac{i}{n-i}\,(g(w)-y-z) & \text{if } y+z < g(w) \leq y+z+\dfrac{n-i}{i}z \\
\dfrac{i}{n}\left(g(w)-y-z-\dfrac{n-i}{i}z\right) & \text{if } y+z+\dfrac{n-i}{i}z < g(w) \leq y+z+\dfrac{n-i}{i}z+\dfrac{n}{i}k \\
(z+k+s) & \text{if } y+z+\dfrac{n-i}{i}z+\dfrac{n}{i}k < g(w).
\end{cases} \tag{6.9}
$$

Equation (6.9) displays the loss function for any surviving OTM member. This is equal to the loss function displayed by Equation (4.7) describing the same regions in Figure 4.1 in Chapter 4. The only difference is that price realization is not defined as h, but as a realization $g(w)$ of Equation (6.2). Again, as long as default losses are absorbed by defaulter's IM and DF contributions, the loss is equal to zero. When the defaulter's individual IM and DF contributions are not enough, each surviving OTM member faces a fraction $\frac{i}{n-i}$ of the extra losses. When default fund is also eroded, CCP starts using equity (which was given by all clearing members), so extra losses are paid by each surviving OTM member through a fraction equal to $\frac{i}{n}$. The, in case the CCP defaults, surviving OTM members lose their default fund contribution z, their holding in capital k and systemic cost s. For more details on these regions, see Subsection 4.1.3. Note that in Equation (6.9) we took the same structure as in Equation (4.3) for $y + z + \frac{n-i}{i} z < g(w) \leq y + z + \frac{n-i}{i} z + \frac{n}{i} k$, in order to use the same structure as in Nahai-Williamson et al. (2013) and not to lose comparison with their results. However, Equation (4.3), as well the corresponding region in Equation (6.9), has the same discontinuity problem discussed at the end of Subsection 4.1.3.

Now, combining Equation (6.5) and Equation (6.6), we get:

$$\begin{aligned} \mathbb{E}[L^{OTM}(N, W) \mid W] &= \sum_{i=0}^{\frac{n}{2}-1} \mathbb{E}[L^{OTM}(i, W) \mid W] \cdot \mathbb{P}[N = i \mid W] \\ &= \sum_{i=0}^{\frac{n}{2}-1} L_i^{OTM}(W) \cdot \mathbb{P}[N = i \mid W] \end{aligned} \tag{6.10}$$

and this result with Equation (6.3) stating the law of iterated expectation gives:

$$\begin{aligned} \mathbb{E}[L^{OTM}(N, W)] &= \mathbb{E}\left[\sum_{i=0}^{\frac{n}{2}-1} L_i^{OTM}(W) \cdot \mathbb{P}[N = i \mid W]\right] \\ &= \sum_{i=0}^{\frac{n}{2}-1} \mathbb{E}\left[L_i^{OTM}(W) \cdot \mathbb{P}[N = i \mid W]\right]. \end{aligned} \tag{6.11}$$

Recall that for any function $A(X)$ of a continuous random variable X with density $f_X(x)$ we have that $\mathbb{E}[A(X)] = \int_{-\infty}^{+\infty} A(x) f_X(x) dx$, so:

$$\mathbb{E}\left[L_i^{OTM}(W) \cdot \mathbb{P}[N = i \mid W]\right] = \int_{-\infty}^{+\infty} L_i^{OTM}(w) \cdot \mathbb{P}[N = i \mid w] f_W(w) dw$$

and thus we have:

$$\mathbb{E}[L^{OTM}(N, W)] = \sum_{i=0}^{\frac{n}{2}-1} \int_{-\infty}^{+\infty} L_i^{OTM}(w) \cdot \mathbb{P}[N = i \mid w] f_W(w) dw. \tag{6.12}$$

Combining Equation (6.7) and Equation (6.12), we get:

$$\mathbb{E}[L^{OTM}(W; y, z)] = \sum_{i=0}^{\frac{n}{2}-1} \int_{-\infty}^{+\infty} L_i^{OTM}(w) \cdot \binom{\frac{n}{2}-1}{i} p(W)^i (1 - p(W))^{\frac{n}{2}-1-i} f_W(w) dw \tag{6.13}$$

where $L_i^{OTM}(w)$ is given by Equation (6.9) and $f_W(w)$ is the density of a standard Normal distribution. Note that $\mathbb{E}[L^{OTM}(N, W)] = \mathbb{E}[L^{OTM}(W; y, z)]$, where we have emphasized the dependence on y and z, which will become useful in the optimization. Thus, Equation (6.13) gives the new unconditional expected loss for the single surviving OTM member in the new model. This is the first part of the new expected loss function that will be minimized to find the optimal quantities of IM and DF contributions in the next optimizations. The difference with Nahai-Williamson et al. (2013) expected loss function is straightforward: it is not possible to separate the expected losses and the probability of observing i defaults, because these default probabilities are not independent any more.

We also need to define the unconditional expected loss for any surviving ITM member. By using the same arguments as above from Equation (6.3) to Equation (6.8), we see that the new expected loss function for a single ITM survivor is given by:

$$\mathbb{E}[L^{ITM}(W;y,z)] = \sum_{i=0}^{\frac{n}{2}} \int_{-\infty}^{+\infty} L_i^{ITM}(w) \cdot \binom{\frac{n}{2}}{i} p(W)^i (1-p(W))^{\frac{n}{2}-i} f_W(w) dw \tag{6.14}$$

where $L_i^{ITM}(w)$ is defined as:

$$L_i^{ITM}(w) = \begin{cases} 0 \quad \text{if } g(w) \leq y+z \\ \dfrac{i}{n-i}\left(g(w)-y-z\right) \quad \text{if } y+z < g(w) \leq y+z+\dfrac{n-i}{i}z \\ \dfrac{i}{n}\left(g(w)-y-z-\dfrac{n-i}{i}z\right) \quad \text{if } y+z+\dfrac{n-i}{i}z < g(w) \leq y+z+\dfrac{n-i}{i}z+\dfrac{n}{i}k \\ \left(1-a\dfrac{\frac{n}{2}-i}{\frac{n}{2}}\right)\left(g(w)-y-z-\dfrac{n-i}{i}z-\dfrac{n}{i}k\right)+z+k+s \quad \text{if } y+z+\dfrac{n-i}{i}z+\dfrac{n}{i}k < g(w). \end{cases} \tag{6.15}$$

The composition and structure are identical to the loss function described in Equation (4.12) for Figure 4.1 in Chapter 4. As stated in Subsection 4.1.4, the only difference with OTM members is what happens in case of CCP's default: ITM members lose all the payments they should have received from OTM members, but not the total amount, just a fraction considering the liquidation cost a paid to administrators. Again, note that in Equation (6.14), we wrote $\mathbb{E}[L^{ITM}(W;y,z)]$ to emphasize that the expected loss function can be read as a function of y and z. In Equation (6.9) we took the same structure as in Equation (4.3) for $y+z+\frac{n-i}{i}z < g(w) \leq y+z+\frac{n-i}{i}z+\frac{n}{i}k$, but also here we notice the discontinuity problem described in Subsection 4.1.3. We repeated the implementations by Nahai-Williamson et al. (2013) in Section 4.2 and our optimizations in Section 6.2 by replacing Equation (4.3) with the more intuitive Equation (4.6) and there is not notable numerical difference in the results. However, we use the same expected loss composition as in Nahai-Williamson et al. (2013) to be able to discuss our results.

The final expected loss function for a general surviving member has the same structure as in Nahai-Williamson et al. (2013). The final expected loss to minimize with respect to IM and DF contribution is:

$$\mathbb{E}[L^{OTM}(W;y,z)] + \mathbb{E}[L^{ITM}(W;y,z)] + (c + d_{IM} \cdot c_c)y + (c + d_{DF} \cdot c_c)z. \tag{6.16}$$

Again, the minimization problem faced in every optimization will be:

$$\min_{y,z}\{\mathbb{E}[L^{OTM}(W;y,z)] + \mathbb{E}[L^{ITM}(W;y,z)] + (c + d_{IM} \cdot c_c)y + (c + d_{DF} \cdot c_c)z\} \tag{6.17}$$

where:

$$\mathbb{E}[L^{OTM}(W;y,z)] = \sum_{i=0}^{\frac{n}{2}-1} \int_{-\infty}^{+\infty} L_i^{OTM}(w) \cdot \binom{\frac{n}{2}-1}{i} p(W)^i (1-p(W))^{\frac{n}{2}-1-i} f_W(w) dw$$

$$\mathbb{E}[L^{ITM}(W;y,z)] = \sum_{i=0}^{\frac{n}{2}} \int_{-\infty}^{+\infty} L_i^{ITM}(w) \cdot \binom{\frac{n}{2}}{i} p(W)^i (1-p(W))^{\frac{n}{2}-i} f_W(w) dw.$$

The minimization of this new function will lead to the optimal quantities of IM and DF contributions taking into account dependent individual default probabilities and prices.

6.2 New optimizations and results

This section is dedicated to all the numerical results coming from the new expected loss function minimizations. In Chapter 4, we replicated the same optimizations by Nahai-Williamson et al. (2013), now we will repeat the optimizations with the new expected loss function, using the same values for fixed and changing model parameters.

	Variable	Value when fixed	Variation range when changing
$\overline{p}$	Unconditional default probability	5% and 25%	1% to 95%
n	Number of members	20	-
c	Opportunity cost	50 bp	25 to 350 bp
k	Equity contribution	0.1%	-
a	Liquidation cost	10%	-
d_{IM}	IM Capital charge	0% and 0.16%	0% to 100%
d_{DF}	DF Capital charge	0% and 0.16%	0% to 100%
c_c	Bank cost of holding capital	10%	-
s	Systemic cost	0	0% to 100%
ρ	Merton correlation	20%	10% to 100%

Table 6.1: New summary of variables used in our numerical studies for the fixed and the changing cases, inspired by Table A1 on page 25 in Nahai-Williamson et al. (2013)

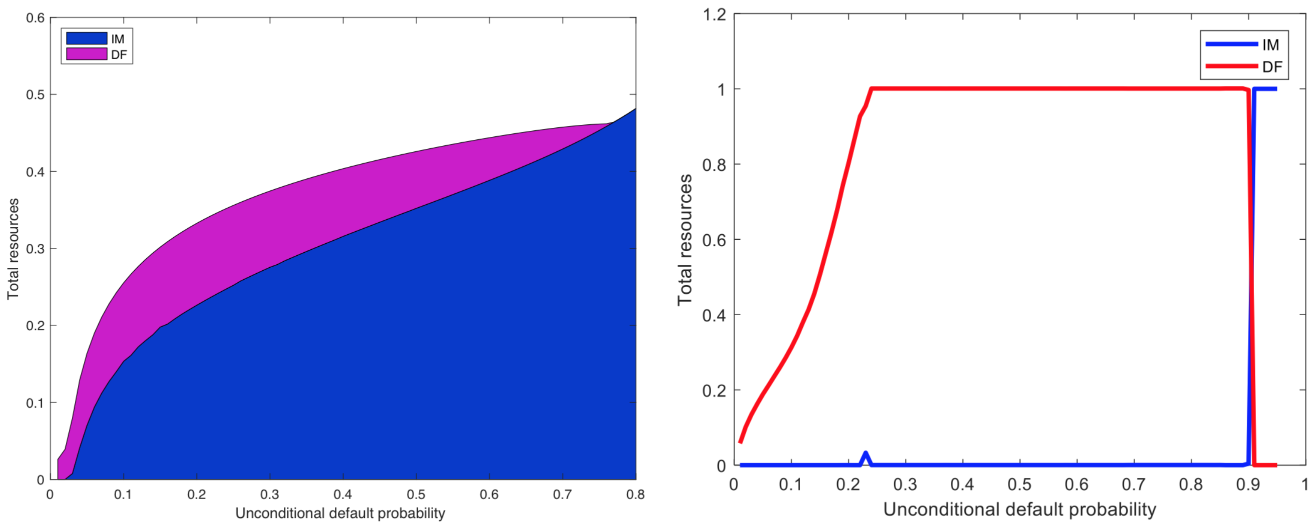
The column "Value when fixed" in Table 6.1 displays the baseline of new optimizations: these are the default values when they are fixed. Then, the column "Variation range when changing" in Table 6.1 shows the variation range of parameters when they are changing, which are the same intervals used in Nahai-Williamson et al. (2013). So, the new optimizations work like this: for example, the first minimization is repeated for all the values of unconditional default probability $\overline{p}$ from 0.01 to 0.95, keeping all the other parameters fixed as in the first column in Table 6.1, and so on. Note that in the first column of Table 6.1 the unconditional default probability $\overline{p}$, as well as capital charges d_{IM} and d_{DF}, has two values. Following Nahai-Williamson et al. (2013), we repeat several optimizations twice for two different default values of $\overline{p}$, $\overline{p} = 5\%$ and $\overline{p} = 25\%$.

Table 6.1 shows that some parameters are constant: the number of members in the financial network n, the initial equity contribution k, the administration cost a and the cost of banks to hold capital c_c.

Moreover, the Merton mixed binomial model allows for a new parameter: Merton correlation ρ. This correlation is discussed in the Appendix A and it will be the center of a new optimization displayed in Subsection 6.2.5.

6.2.1 Varying unconditional default probability $\overline{p}$

The first optimization studies the optimal IM and DF contributions as functions of the individual unconditional default probability $\overline{p}$. More numerical details on the optimization are shown in the Appendix C.



(a) Via Equation (4.15)

(b) Via Equation (6.17)

Figure 6.3: The optimal IM contributions y and DF contributions z as functions of the unconditional default probability $\overline{p}$

In Figure 6.3a, replicating the optimization by Nahai-Williamson et al. (2013) via Equation (4.15), when default probability is extremely low, CCP's resources are made completely of DF contributions (if no one can default, it is the most efficient way to collect resources), but as soon as default probability increases, initial margin IM becomes the only source CCP must collect. As stated in Subsection 4.2.1, Nahai-Williamson et al. (2013) find that when the default probability is very high, i.e. it is likely that someone defaults, there is a higher cost of mutualization, then clearing members prefer to post initial margin IM, so that their sources will not be eroded in common funds to pay default losses. In our optimization via Equation (6.17) in Figure 6.3b, note that the optimization has a shift around $\overline{p} \approx 0.9$, where optimal IM and DF contributions switch values. However, the same result from Nahai-Williamson et al. (2013) remains valid: the higher the individual unconditional default probability $\overline{p}$, the higher the optimal IM contribution.

To see the difference between the optimal quantities of IM and DF contributions in the Merton mixed binomial framework via Equation (6.17) and the ones found by the minimization in Equation (4.15), we compute a percentage relative difference RF between the IM and DF optimal contributions:

$$RF_{IM} = \frac{IM_{new} - IM_{nahai}}{IM_{nahai}} \cdot 100 \tag{6.18}$$

$$RF_{DF} = \frac{DF_{new} - DF_{nahai}}{DF_{nahai}} \cdot 100. \tag{6.19}$$

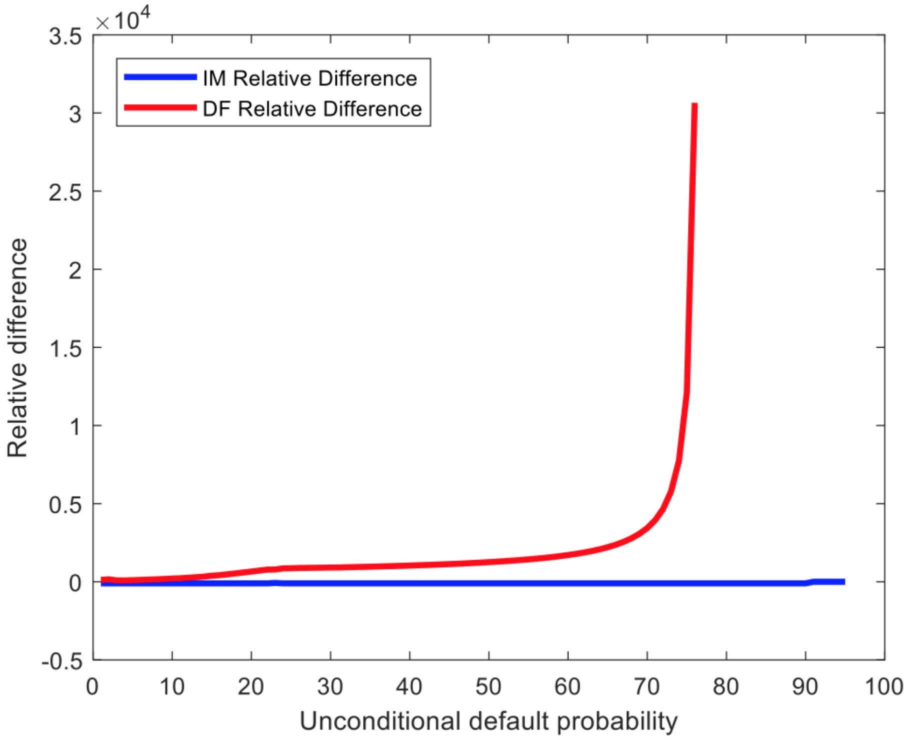


Figure 6.4: The IM relative difference RF_{IM} and the DF relative difference RF_{DF} as functions of the unconditional default probability $\overline{p}$

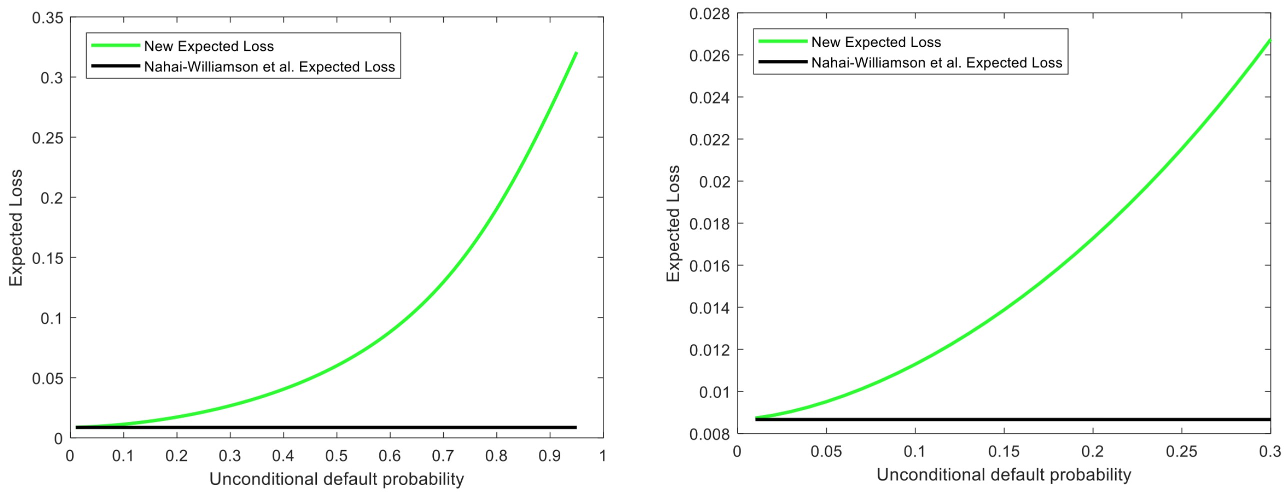
The optimization in Merton framework in Figure 6.3b and the optimal contributions relative difference in Figure 6.4 show two main results:

1. The total amount of financial resources, i.e. the sum of IM and DF optimal contributions, is generally higher: Figure 6.3a inspired by Nahai-Williamson et al. (2013) displays a maximum contribution of ≈ 0.5, while the new sum of total resources in Figure 6.3b almost reaches 1, i.e. the total value of the cleared portfolio;

2. Figure 6.4 displays the relative differences RF_{IM} and RF_{DF} as stated in Equations (6.18) and (6.19). The optimal DF contribution relative difference RF_{DF} in Figure 6.4 is very large and increasing. For smaller values of unconditional default probabilities, i.e. $\overline{p} \in [0, 30\%]$, which are the most observed in reality, Figure (6.4) shows that the new optimal DF contribution is almost 1500% bigger, that is more than fifteen times bigger than the one found via Equation (4.15) in Figure 6.3a. For larger values of $\overline{p}$, the difference gets even larger. Moreover, the balance between

IM and DF contributions is different: in Figure 6.3b DF contributions are more persistent (up to $\overline{p} = 90\%$). Note that in our implemented version of the optimization by Nahai-Williamson et al. (2013) in Figure 6.3a, the optimal IM and DF contributions are added on top of each other (for $\overline{p} \approx 30\%$, the optimal DF contribution is ≈ 0.06).

Hence, in our framework with dependent default probabilities and prices, when the unconditional default probability $\overline{p}$ increases, the CCP needs more resources than how it was prevented by Nahai-Williamson et al. (2013). Furthermore, central counterparties need to relate more on default fund than initial margins, due to its loss-absorption capacity. However, for extreme default probabilities, $\overline{p} > 90\%$, initial margin remains the best source, because cost of mutualization is too high to use a common fund.



(a) $\overline{p} \in [0.01, 0.95]$

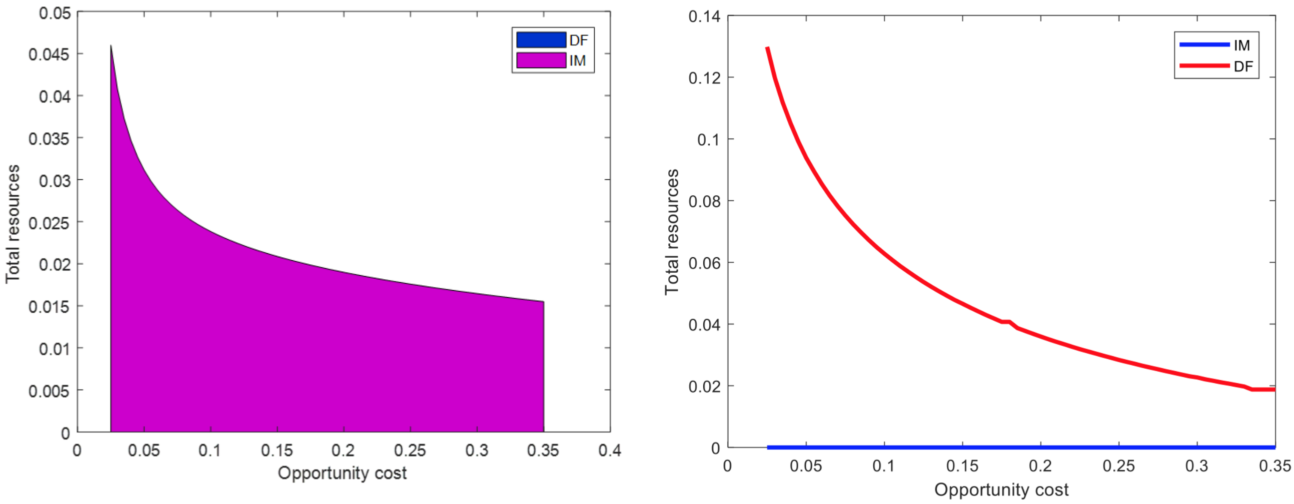
(b) $\overline{p} \in [0.01, 0.30]$

Figure 6.5: The Nahai-Williamson et al. (2013) expected loss via Equation (4.14) and the expected loss via Equation (6.16) as functions of the unconditional default probability $\overline{p}$ in different regions

In Figure 6.5 the IM and DF contributions are fixed as the average values of the optimal quantities coming from optimization in Figure 6.3b via Equation (6.17). Then, we compute the values of the surviving member's expected loss function both in Nahai-Williamson et al. (2013) case in Equation (4.14) and in our case in Equation (6.16). Figure 6.5 shows that, while Nahai-Williamson et al. (2013) function in Equation (4.14) slightly increases, the new expected loss function in Equation (6.16) has a much more steep increase: our expected loss function will in general be much higher and this will be true for all our optimizations. Although it is not relevant up to the IM and DF optimal quantities that minimize the expected loss function, we believe that it is worth to mention that the expected loss function in our Merton-extended model is even 22% and 55% larger than the expected loss value in Nahai-Williamson et al. (2013): in Figure 6.5b with $\overline{p} \in [0.01, 0.30]$, ≈ 0.011 and ≈ 0.014 against ≈ 0.009. As soon as we allow for dependence between default probabilities and assets prices, the expected loss function for surviving clearing members increases: this is the reason behind our results. If the expected loss function is higher, then we expect higher optimal quantities for both IM and DF contributions as well as more default fund to cover wider losses.

6.2.2 Varying opportunity cost c

The second optimization studies the optimal IM and DF contributions as functions of the opportunity cost of collateral c (see Subsection 4.1.1). Further details on the optimization are displayed in the Appendix C.



(a) Via Equation (4.15)

(b) Via Equation (6.17)

Figure 6.6: The optimal IM contributions y and DF contributions z as functions of the opportunity cost c, with $\overline{p} = 5\%$

Figure 6.6a replicating the optimization by Nahai-Williamson et al. (2013) via Equation (4.15) shows that the optimal amount of total resources decreases, as the opportunity cost c increases, i.e. collateral becomes more expensive (see Subsection 4.2.2). In the minimization via Equation (6.17) in the Merton framework, Figure 6.6b displays the same result: when the opportunity cost c increases, collateral (both IM and DF contributions) becomes more costly and thereby decreases. In both cases, there is always an amount of optimal DF contribution, even when collateral is extremely costly, because of default fund's property of mutualization. However, our optimal quantities in Figure 6.6b are larger: optimal DF contribution has a maximum of ≈ 0.13, instead of ≈ 0.045 in Figure 6.6a.

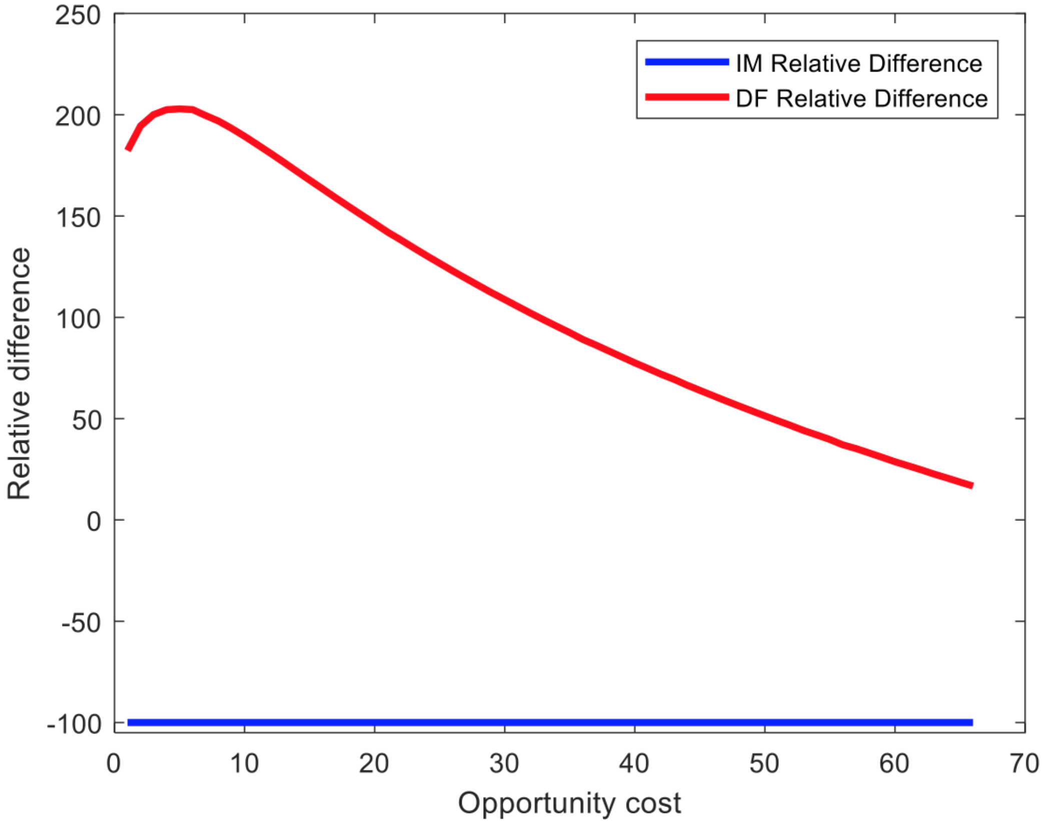


Figure 6.7: The IM relative difference RF_{IM} and the DF relative difference RF_{DF} as functions of the opportunity cost c, with $\overline{p} = 5\%$

Figure 6.7 shows the relative differences for the optimal quantities in Figure 6.6 as stated in Equations (6.18) and (6.19) as functions of the opportunity cost c. Note that the optimal quantities of IM contributions in the extended model in Figure 6.6b are almost equal to zero, so relative difference RF_{IM} in Equation (6.18) gives a standard value of $\approx -100\%$ (see Appendix B). On the contrary, the relative difference RF_{DF} in Figure 6.7 has a positive and decreasing path, which means that the new DF optimal quantities in Figure 6.6b come closer and closer to the ones found by replicating Nahai-Williamson et al. (2013) in Figure 6.6a. However, Figure 6.7 shows that in our extended model optimal DF contributions are often 200%, 150% and 100% larger than in Nahai-Williamson et al. (2013) model. The optimal quantities of DF contribution CCPs need to minimize surviving members' expected loss are larger than the forecast made by Nahai-Williamson et al. (2013).

Recall that every optimization in Nahai-Williamson et al. (2013) is repeated for two independent default probabilities, $p = 5\%$ and $p = 25\%$, so in the extended model we will repeat the minimizations for the two different values $\overline{p} = 5\%$ and $\overline{p} = 25\%$. For brevity, we will not report the optimization results for the cases with $\overline{p} = 25\%$: we already know from Subsections 4.2.1, 4.2.2 and 6.2.1 that increasing the individual unconditional default probability $\overline{p}$ has the only effect of increasing the need for IM contributions. For example, in this case, repeating the optimization in Figure 6.6b via Equation (6.17) we obtain optimal IM and DF contributions with the same path, but with slightly larger IM contributions.

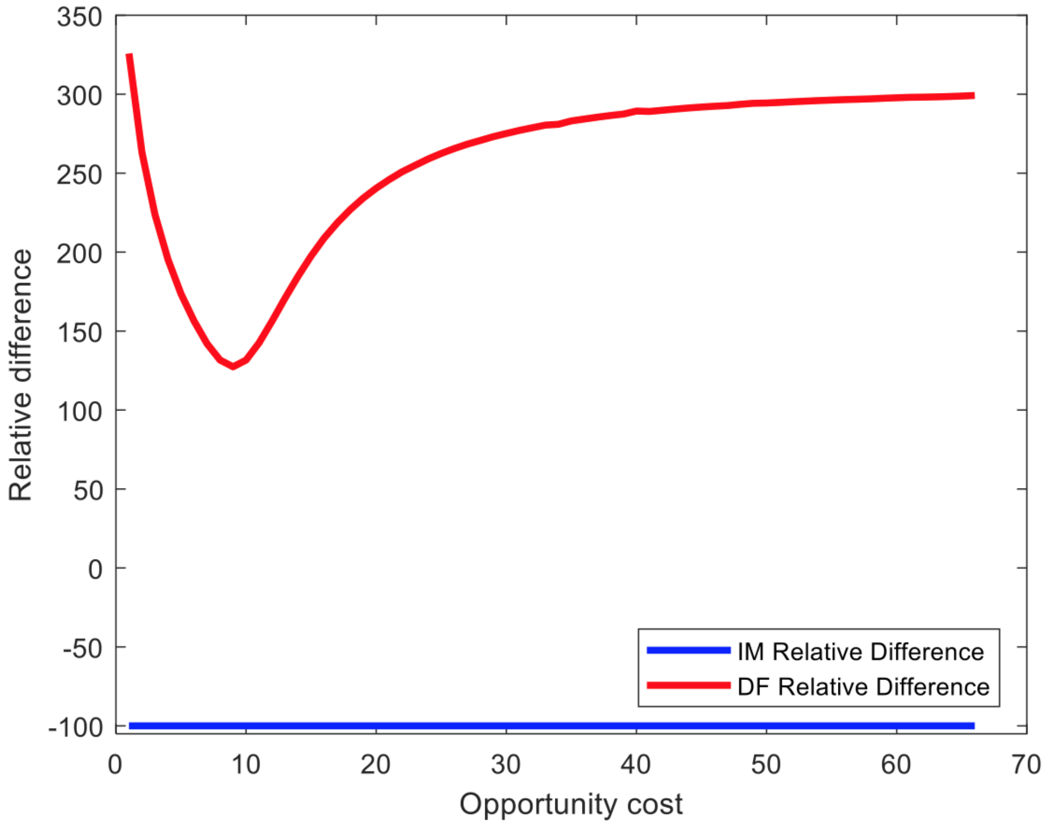
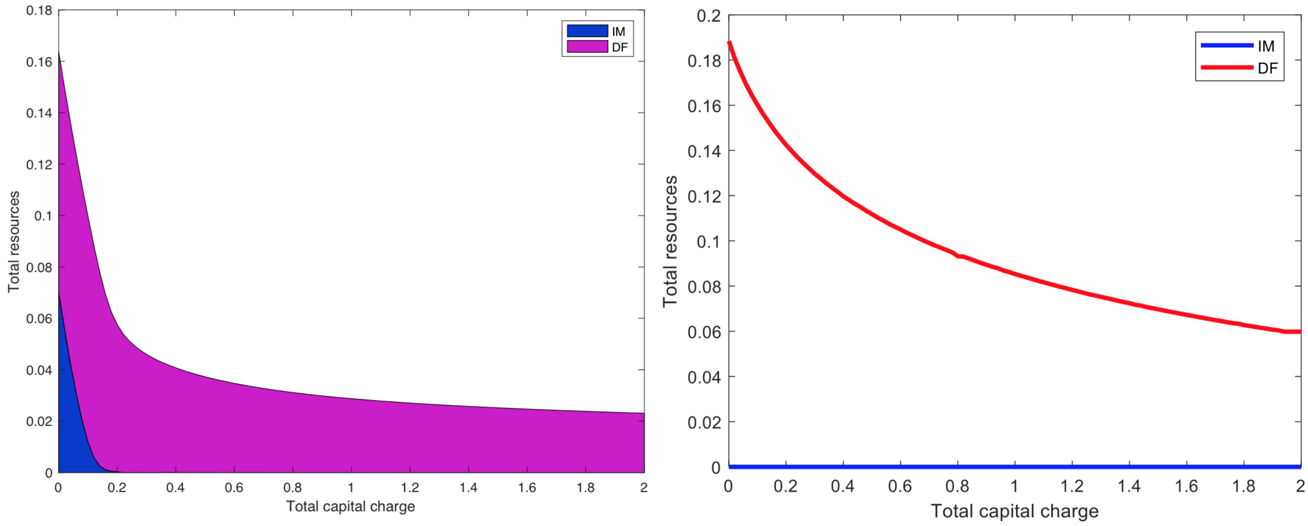


Figure 6.8: The IM relative difference RF_{IM} and the DF relative difference RF_{DF} as functions of the opportunity cost c, with $\overline{p} = 25\%$

Figure 6.8 shows the percentage relative differences between the optimal IM and DF quantities computed by Equation (4.15) as Nahai-Williamson et al. (2013) and the corresponding optimal quantities in the extended Merton mixed binomial model with $\overline{p} = 25\%$. The optimal IM contributions are still too close to zero to be displayed by RF_{IM} in Equation (6.18) in Figure 6.8. However, the optimal DF quantities in the Merton framework are larger than the original ones displayed in Figure 4.4 in Chapter 4: in our case, optimal DF contributions gets 300%, 200% and 250% larger than the quantities found replicating Nahai-Williamson et al. (2013) in Subsection 4.2.2. When Merton framework allows for dependencies between default probabilities and asset prices, default fund's ability to mutualize and share the losses becomes crucial and more efficient.

6.2.3 Varying capital charges d_{IM} and d_{DF}

In this subsection, we display the third optimization that studies the optimal IM and DF contributions as functions of the initial margin and default fund capital charges d_{IM} and d_{DF} (see Subsection 4.1.1). More details about the optimization are explained in the Appendix C.



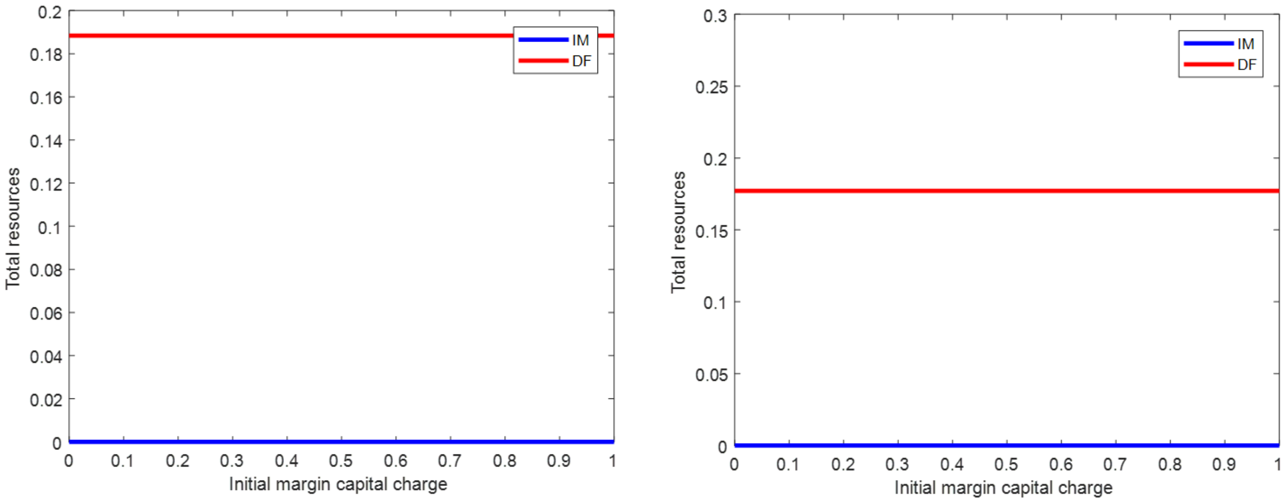
(a) Via Equation (4.15)

(b) Via Equation (6.17)

Figure 6.9: The optimal IM contributions y and DF contributions z as functions of the total capital charge $d_{IM} + d_{DF}$, with $\overline{p} = 5\%$

Figure 6.9a displays the optimization via Equation (4.15) replicating Nahai-Williamson et al. (2013) to see how optimal IM and DF contributions change when the total capital charge on collateral, i.e. $d_{IM} + d_{DF}$, changes, with $\overline{p} = 5\%$. In Figure 6.9a the optimal quantities decrease as collateral becomes more expensive. However, there is still a part of initial margin up to $d_{IM} + d_{DF} \approx 0.2$ and the maximum value of total resources is ≈ 0.16. In the extended model with Merton mixed binomial model, Figure 6.9b displays the same optimization but via Equation (6.17). Figure 6.9b shows that also in our extended model optimal quantities decrease as collateral becomes more costly. Nevertheless, the maximum value of total resources is ≈ 0.19 and Figure B.1 in Appendix B shows that optimal DF contributions are most of the time 200%/150% bigger than the ones pictured in Figure 6.9a. The same optimization in Figure 6.9b is repeated for the case $\overline{p} = 25\%$: there is just a slight increase in optimal IM contributions.

Following the same structure in Nahai-Williamson et al. (2013) and in Subsection 4.2.3, we proceed with an optimization that studies the optimal IM and DF contributions as functions of the initial margin capital charges d_{IM}, fixing DF capital charge to $d_{DF} = 0$ and $d_{DF} = 0.16\%$.



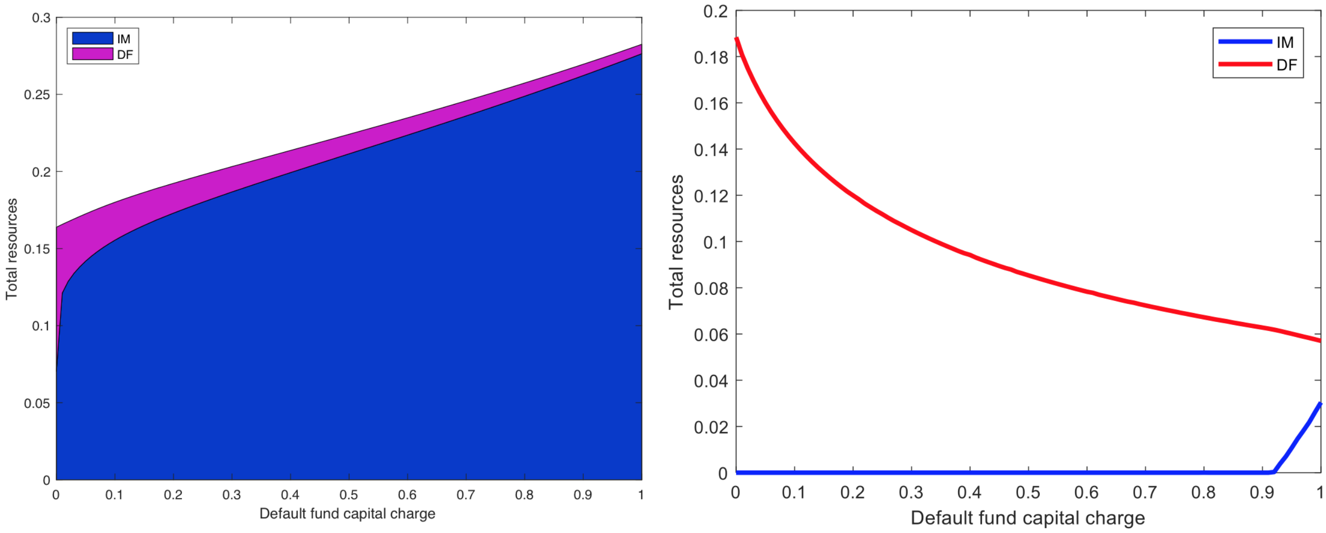
(a) With $d_{DF} = 0$

(b) With $d_{DF} = 0.16\%$

Figure 6.10: The optimal IM contributions y and DF contributions z via Equation (6.17) as functions of the IM capital charge d_{IM}, with $\overline{p} = 5\%$

Figure 6.10 shows our implementation via Equation (4.15) of the same optimization via Equation (4.15) in Figure 4.6, which is the same as in Figures 10a and 10b on page 14 in Nahai-Williamson et al. (2013). In both cases for $d_{DF} = 0$ and $d_{DF} = 0.16\%$, the optimal resources were mainly made of a constant optimal DF contribution, except for a slight portion of initial margin when d_{IM} is close to zero. In the extended model with dependencies between default probabilities and prices, Figure 6.10 shows the same optimization, but via Equation (6.17). As soon as IM capital charge increases, clearing members shift to the cheaper form of collateral, DF contributions. In the case with $d_{DF} = 0$, Figure 6.10a shows that IM contributions are almost equal to zero, however the constant optimal DF contribution is 20% higher than the one in Figure 4.6a in Subsection 4.2.3 (≈ 0.16 against ≈ 0.19, see Figure B.2 in Appendix B for more details on RF_{DF}). In our more realistic environment with Merton mixed binomial model, the expected loss function is larger, and it needs a wider amount of default fund to be able to cover default losses. The same optimization is repeated in our extended model via Equation (6.17) for the case $d_{DF} = 0.16\%$ and $\overline{p} = 5\%$, so with an increment in DF capital charge. Figure 6.10b shows that IM contributions are always avoided because they are more expensive, but the constant optimal DF contribution is now lower, i.e. ≈ 0.17, due to the higher cost of DF. Nevertheless, the optimal DF contribution in Figure 6.10b is still 25%/30% higher than the one found in Figure 4.6a in Subsection 4.2.3 via Equation (4.15) (see Figure B.3 in Appendix B for more details).

Contrary to above, the next optimization studies the optimal IM and DF contributions as functions of the default fund capital charges d_{DF}, fixing IM capital charge to $d_{IM} = 0$.



(a) Via Equation (4.15)

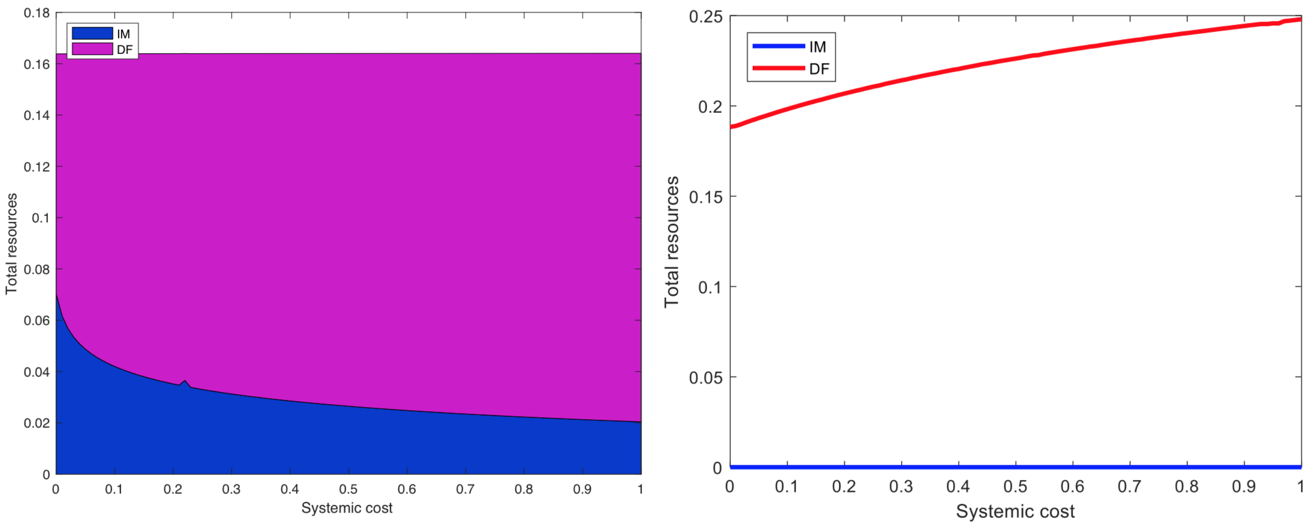
(b) Via Equation (6.17)

Figure 6.11: The optimal IM contributions y and DF contributions z as functions of the DF capital charge d_{DF}, with $d_{IM} = 0$ and $\overline{p} = 5\%$

Figure 6.11a shows the results from the optimization via Equation (4.15) replicating the one by Nahai-Williamson et al. (2013) in Figure 11a on page 14 in their paper. Figure 6.11a displays that when default fund becomes more expensive, IM contributions increase, but DF contributions never disappear. Even with the highest cost $d_{DF} = 90\%$, DF contributions are still present in the balance of sources, due to its efficiency. Figure 6.11b describes the same optimization in our extended Merton framework via Equation (6.17). Figure 6.11b shows that DF contributions decrease when default fund becomes more costly. However, even if the cost d_{DF} increases, the majority of financial resources is made by DF contribution, which play a crucial role in the final balance for CCPs. When the default fund capital charge is extreme, $d_{DF} \approx 90\%$, then IM contributions start increasing again even in our Figure 6.11b, but DF contributions are always higher. The optimal DF contributions in our extended model in Figure 6.11b are even 600%, 700% and 800% bigger than the ones found in the replication of Nahai-Williamson et al. (2013) via Equation (4.15) (see Figure B.4 in Appendix B for more details).

6.2.4 Varying systemic cost s

In this subsection, we display another optimization that studies the optimal IM and DF contributions as functions of the systemic cost s (see Subsection 4.1.1), with $\overline{p} = 5\%$. Appendix C contains more details on the optimization process.



(a) Via Equation (4.15)

(b) Via Equation (6.17)

Figure 6.12: The optimal IM contributions y and DF contributions z as functions of the systemic cost s, $\overline{p} = 5\%$

Figure 6.12a shows the results from the optimization via Equation (4.15) replicating the one by Nahai-Williamson et al. (2013) in Figure 13a on page 15 in their paper. Figure 6.12a displays a constant optimal DF contribution and a decreasing IM contribution. Recall that the systemic cost s happens only in case of CCP's default, so clearing members prefer default fund because it can procrastinate CCP's default more than the initial margin can do. The same optimization is repeated in our Merton-extended model via Equation (6.17). In Figure 6.12b the balance between optimal financial resources is different: optimal IM contributions are very close to zero, while the optimal DF contributions increase when the systemic cost s gets larger. When a post-default cost like s increases, clearing members need to retard CCP's default and DF contributions are the most efficient way to absorb losses. This behaviour is observed also in the optimization in Figure 6.12a via Equation (4.15), however our optimal DF contributions in Figure 6.12b are 60%, 65% and even 70% bigger than the ones found in Figure 6.12a (see Figure B.5 in Appendix B for more details).

6.2.5 Varying Merton correlation ρ

Our extended model introducing dependencies among default probabilities and prices allows us to perform an optimization that is not present either in Nahai-Williamson et al. (2013) or in our replication of their model in Chapter 4. The Merton framework introduces a new parameter via Equation (5.14): the Merton correlation ρ. In Appendix A we elaborate more on the role of ρ. Our last optimization presented in Figure 6.13 studies the optimal IM and DF contributions as functions of the Merton correlation ρ (see Section 5.3), with $\overline{p} = 5\%$.

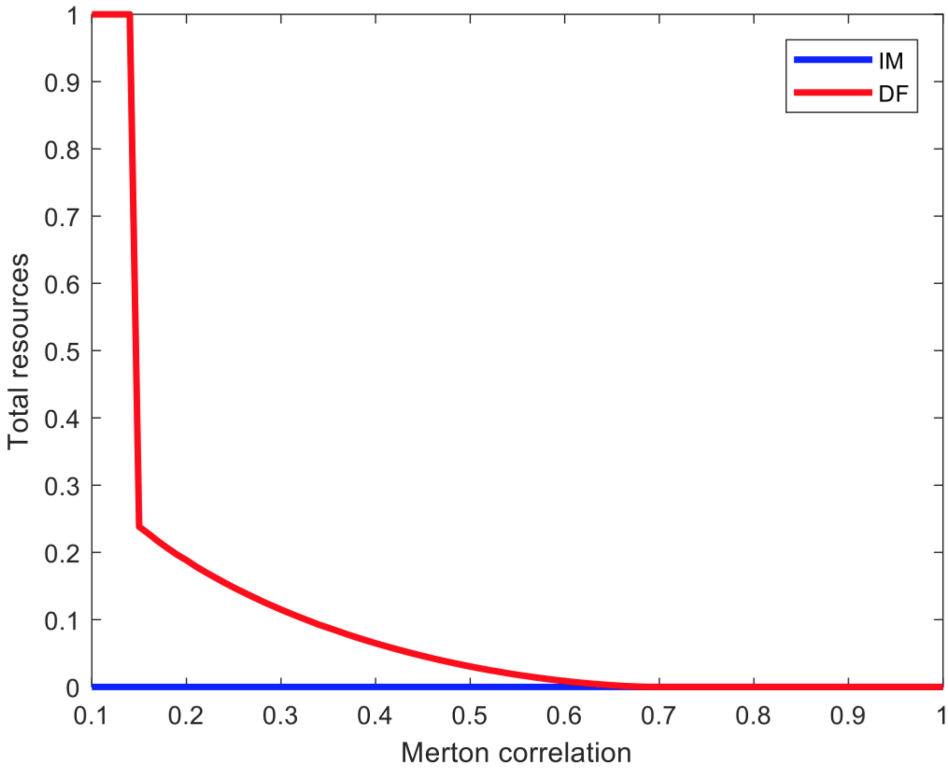


Figure 6.13: The optimal IM contributions y and DF contributions z via Equation (6.17) as functions of the Merton correlation ρ, with $\overline{p} = 5\%$

Figure 6.13 displays the results from the optimization in our Merton framework via Equation (6.17). Recall that Merton correlation ρ describes how much total assets values of clearing members are related to each other and how much they move together on the markets (see Appendix A for more details). In the extreme situation where $\rho = 0$, clearing members' assets values are completely detached and every clearing member is independent from each other. Substituting $\rho = 0$ in Equation (6.1), the conditional default probability $p(W)$ is equal to the exogenous unconditional probability $\overline{p}$. In other words, if $\rho = 0$ in the Merton model, we are back in a standard binomial model with independent defaults just as in the Nahai-Williamson et al. (2013) model. On the other side, if the Merton correlation realized in a theoretical situation like $\rho = 1$, then all clearing members' assets would be the same and move together. In this case, the unpredictability of assets returns is extremely low and also clearing members' default probability is the same and very low (recall that $\overline{p} = 5\%$), which is why Figure 6.13 displays optimal sources that are almost equal to zero in this case. However, for more plausible value of Merton correlation ρ, i.e. $\rho \in [0.2, 0.6]$, optimal DF contributions decrease as Merton correlation becomes higher. Recall that from Equation (6.1) the conditional default probability $p(W)$ decreases as Merton correlation ρ increases. Moreover, if $p(W)$ decreases, also the conditional probability of i defaults, i.e. $\mathbb{P}[N = i \mid W]$, in Equation (6.7) is negatively affected. The negative impact of Merton correlation ρ on both conditional default probability $p(W)$ and probability $\mathbb{P}[N = i \mid W]$ is the reason behind the negative relation between ρ and the optimal contributions in Figure 6.13. As soon as Merton correlation ρ increases, both $p(W)$ and $\mathbb{P}[N = i \mid W]$ decrease, clearing members experience less defaults and need less DF contributions.

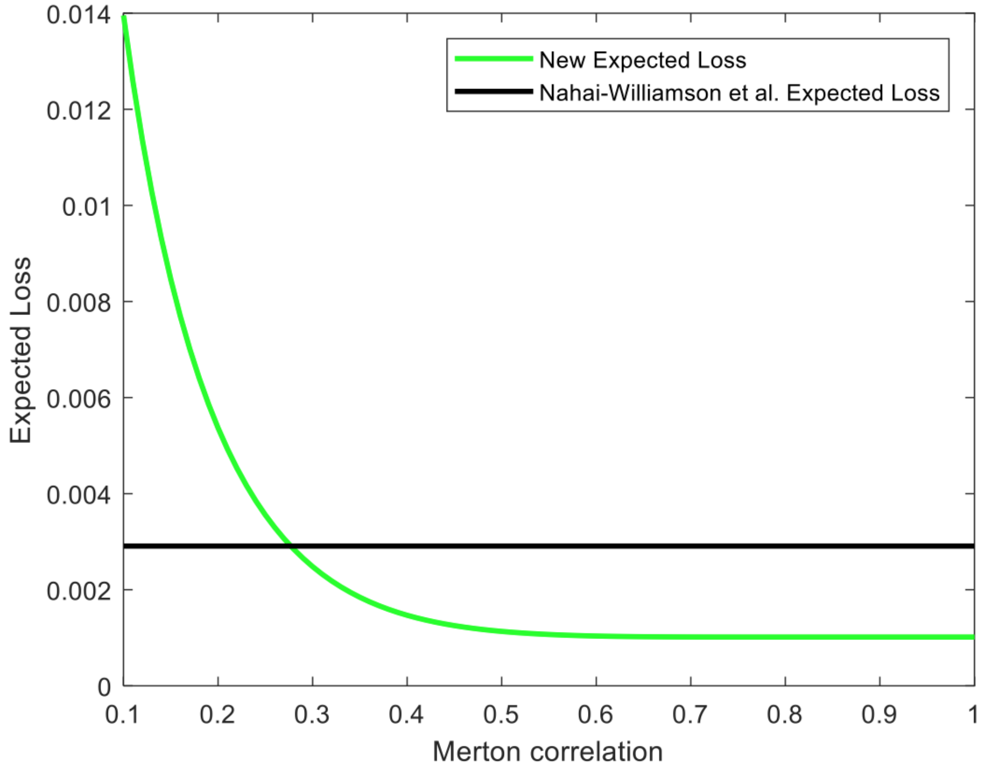


Figure 6.14: The Nahai-Williamson et al. (2013) expected loss via Equation (4.14) and the expected loss via Equation (6.16) as functions of the Merton correlation ρ, with $\overline{p} = 5\%$

In Figure 6.14 the IM and DF contributions are fixed as the average values of the optimal quantities coming from optimization in Figure 6.13 via Equation (6.17). Then, we compute the values of the surviving member's expected loss function both in Nahai-Williamson et al. (2013) case in Equation (4.14) and in our case in Equation (6.16). Figure 6.14 shows that, while Nahai-Williamson et al. (2013) function via Equation (4.14) is constant (there is no ρ in their model), the new expected loss function via Equation (6.16) has a decreasing pattern. For smaller values of Merton correlation ρ, i.e. $\rho \in [0, 0.27]$, Figure 6.14 shows that the expected loss function in our Merton framework is higher than the Nahai-Williamson et al. (2013) one. So, if we take in consideration the correlation between clearing members' total assets, CCPs need a higher amount of financial resources when this correlation is low, i.e. there is more unpredictability on the market. On the other hand, for larger values of Merton correlation ρ, i.e. $\rho \in [0.27, 1]$, Figure 6.14 shows that the expected loss function in the extended model via Equation (6.16) is lower than the one observed via Equation (4.14). When ρ is larger, both individual conditional default probability $p(W)$ and $\mathbb{P}[N = i \mid W]$ are lower, which results in the only case in our extended model where our expected loss function is lower than the one in Nahai-Williamson et al. (2013).

Chapter 7

Conclusion

Nahai-Williamson et al. (2013) develop a model to find the optimal quantities of default fund and initial margin contributions. This is done by finding an expected loss function for surviving CCP's members, which is minimized with respect to members' IM and DF contributions. Nahai-Williamson et al. (2013) assume that clearing members' default probabilities are exogenous and independent and that prices are not related to the background economic environment. We relax these assumptions with the help of the Merton mixed binomial model: the individual default probabilities and prices are related via the common background factor W. Once this framework of more realistic probabilities and prices is defined, we derive the new expected loss function for surviving clearing members and we minimize it to find the new optimal quantities of IM and DF contributions.

Our results are presented in Section 6.2. Recall that we introduce dependencies between default probabilities and prices to investigate if there is a difference in the optimal values of initial margin and default fund contributions. Our results show that there are large differences between the optimal financial resources in our Merton-extended model and the ones coming from our replication of Nahai-Williamson et al. (2013) in Chapter 4.

In the optimizations of the CCP's Merton mixed binomial model, the numerical results for optimal quantities of both IM and DF contributions are different compared to the ones in the model by Nahai-Williamson et al. (2013). Recall that these quantities must be read as percentages of portfolio values asked to clearing members as collateral, so even a slight change can be a decisive amount of money. Moreover, in our Merton-extended model, the value of total resources (IM and DF contributions considered together) is always higher than in Nahai-Williamson et al. (2013) optimizations in Chapter 4.

The optimal default fund contributions are generally higher in the extended model compared to the model in Nahai-Williamson et al. (2013). Once we allow for dependencies among default probabilities and asset prices, we show that the optimal DF contributions are sometimes 200%, 300% or even 1500% larger than the ones identified in Chapter 4 through Nahai-Williamson et al. (2013). Moreover, in the extended version through Merton mixed binomial model, results prove that the balance of CCP's financial resources tends more to the default fund DF rather than initial margin IM. For example, our optimizations in Figures 6.3b and 6.11b show that even when the individual default probability or DF capital charge increase, IM always represents a lower part of the total CCP's resources. On the contrary, the optimizations in Figures 6.10 and 6.12b in the extended CCP's Merton model show that, whenever DF contributions are more persistent, IM contributions almost disappear and default fund becomes the only reliable source in the CCP's balance, due to its sharing mechanism of losses. In Nahai-Williamson et al. (2013), the initial margin is preferable because clearing members at the beginning do not know who will default and every one has the same independent default probability. In a more realistic model where we allow for dependencies between default probabilities and prices, defaults are not detached from the rest of the world and they are hardly covered only by defaulter's initial margin. In the extended version where clearing members are related thanks to a Merton mixed binomial model, the presence of Merton correlation ρ among members' assets and default correlation ρ_X (see Appendix A) makes defaults more likely, which is the reason why the loss-absorption of default fund is more efficient.

In conclusion, as soon as we allow for a common background factor that makes default probabilities and prices related with each other via the economic environment, then the optimal quantity of total resources (IM and DF contributions together) will be generally higher compared to Nahai-Williamson et al. (2013) model. Moreover, the balance of CCP's financial resources will be dominated by the default fund form of collateral. The model developed by Nahai-Williamson et al. (2013) is probably too prudent: in a more realistic framework, CCPs need larger and more unbalanced amounts of collateral.

Appendix A

Clearing members' correlation

A.1 Default correlation ρ_X in mixed binomial models

Recall the definition of the correlation $\mathrm{Corr}(X,Y)$ between two random variables X and Y:

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}. \tag{A.1}$$

Correlation describes the dependence between two random variables. As stated in Subsection 5.2.1, consider a portfolio/network with m obligors, where each obligor can default up to time T. Let X_j be a random variable such that $X_j = 1$ if obligor j defaults up to time T, otherwise $X_j = 0$. According to Frey and McNeil (2001) and Frey and McNeil (2003), in all mixed binomial models as the one in Section 5.2, it is possible to define a default correlation which describes how members default indicator functions X_{j_1} and X_{j_2} are related. In the mixed binomial model, we are interested in $\rho_X = \mathrm{Corr}(X_{j_1}, X_{j_2})$, which would be the correlation between two individual default indicators. From Equations (5.7) and (5.8), we know that:

$$\mathrm{Var}(X_j) = \overline{p}(1-\overline{p})$$
$$\mathrm{Cov}(X_{j_1}, X_{j_2}) = \mathrm{Var}(p(W))$$

where $\mathbb{E}[p(W)] = \overline{p}$ as in Equation (5.5). Thus, the correlation in a mixed binomial model is given by:

$$\rho_X = \frac{\mathrm{Cov}(X_{j_1}, X_{j_2})}{\sqrt{\mathrm{Var}(X_{j_1})}\sqrt{\mathrm{Var}(X_{j_2})}} = \frac{\mathbb{E}[p(W)^2] - \overline{p}^2}{\overline{p}(1-\overline{p})} = \frac{\mathrm{Var}(p(W))}{\overline{p}(1-\overline{p})} \tag{A.2}$$

where $\overline{p}$ is the unconditional default probability of each member. So, the correlation in (A.2) describes the relation between the random variables representing the default status for each clearing member.

A.2 Merton correlation ρ

Following Herbertsson (2018), the Merton model introduces a new variable, the correlation ρ in Equation (5.14), which weights how much of total assets value depends on the common economic environment $W_{t,0}$ and how much on the individual part $W_{t,j}$ (see Subsection 5.3.1). The clearing members' assets returns, correlated through Equation (5.14), imply that $\mathrm{Corr}(B_{t,j_1}, B_{t,j_2}) = \rho$. To see this, note that:

$$
\begin{aligned}
\mathrm{Cov}(B_{t,j_1}, B_{t,j_2}) &= \mathbb{E}[B_{t,j_1}, B_{t,j_2}] - \mathbb{E}[B_{t,j_1}]\mathbb{E}[B_{t,j_2}] \\
&= \mathbb{E}[(\sqrt{\rho}W_{t,0} + \sqrt{1-\rho}W_{t,j_1})(\sqrt{\rho}W_{t,0} + \sqrt{1-\rho}W_{t,j_2}] \\
&= E[\rho W_{t,0}^2] + \sqrt{\rho}\sqrt{1-\rho}\mathbb{E}[W_{t,0}W_{t,j_1}] + \sqrt{\rho}\sqrt{1-\rho}\mathbb{E}[W_{t,0}W_{t,j_2}] + (1-\rho)\mathbb{E}[W_{t,j_1}W_{t,j_2}] \\
&= \rho\mathbb{E}[W_{t,0}^2] \\
&= \rho t
\end{aligned}
\tag{A.3}
$$

where the first and the second equalities in (A.3) are due to the fact that the expected value of a Brownian motion is zero. In the third equality in (A.3), the other elements disappear because the standard Brownian motions $W_{t,0}$, W_{t,j_1} and W_{t,j_2} are independent. Then the last equality in (A.3) follows because the second moment of a Brownian motion is equal to t, $\mathbb{E}[W_t^2] = t$. Furthermore:

$$
\begin{aligned}
\mathrm{Var}(B_{t,j}) &= \mathrm{Var}(\sqrt{\rho}W_{t,0} + \sqrt{1-\rho}W_{t,j}) \\
&= \rho\mathrm{Var}(W_{t,0}) + (1-\rho)\mathrm{Var}(W_{t,j}) \\
&= \rho t + (1-\rho)t \\
&= t.
\end{aligned}
\tag{A.4}
$$

Hence:

$$
\mathrm{Corr}(B_{t,j_1}, B_{t,j_2}) = \frac{\mathrm{Cov}(B_{t,j_1}, B_{t,j_2})}{\sqrt{\mathrm{Var}(B_{t,j_1})}\sqrt{\mathrm{Var}(B_{t,j_2})}} = \frac{\rho t}{\sqrt{t}\sqrt{t}} = \rho. \tag{A.5}
$$

So, the Merton correlation ρ represents the mutual dependence among obligors assets returns created by the macroeconomic latent common variable $W_{t,0}$. The higher the Merton correlation ρ, the more members are correlated to each other (technically, their total assets values move altogether).

The Merton correlation ρ describes the dependence between the total assets values among agents. Recall from Subsection 5.1.2 that in a Merton framework default happens if and only if $V_{T,j} < D$, so the zero-one random variable X_j describing the default of obligor j before time T can be written as:

$$
X_j = \mathbb{I}_{\{V_{T,j} < D\}}.
$$

Since X_j also depends on total assets value $V_{T,j}$, we know that X_{j_1} and X_{j_2} are dependent whenever $\mathrm{Cov}(B_{t,j_1}, B_{t,j_2})$ in Equation (A.3) is equal to ρt with $\rho \neq 0$. Briefly, if $\rho \neq 0$, it holds that $\mathrm{Cov}(X_{j_1}, X_{j_2}) \neq 0$, while if $\rho = 0$ then $\mathrm{Cov}(X_{j_1}, X_{j_2}) \neq 0$, that is:

$$
\begin{aligned}
&\mathrm{Cov}(X_{j_1}, X_{j_2}) = 0 \quad \text{if } \rho = 0 \\
&\mathrm{Cov}(X_{j_1}, X_{j_2}) \neq 0 \quad \text{if } \rho \neq 0
\end{aligned}
\tag{A.6}
$$

where $\mathrm{Cov}(X_{j_1}, X_{j_2})$ is a measure of default dependence between the zero-one random variables X_j. These are two different concept of correlation. The Merton correlation ρ and $\mathrm{Cov}(B_{t,j_1}, B_{t,j_2})$ describe the dependence between clearing members' total assets values (the higher this correlation, the more total assets values move in a similar path), while $\mathrm{Cov}(X_{j_1}, X_{j_2})$ describes the correlation between the random variables X_{j_1} and X_{j_2} representing the default status for members j_1 and j_2. The latter is the same correlation ρ_X described in Equation (A.2), a default dependence between all network members. Equation (A.6) shows that default correlation ρ_X is affected by Merton correlation ρ via the quantity $\mathbb{E}[p(W)^2]$.

The Merton correlation ρ is a variable which is not present in the work by Nahai-Williamson et al. (2013). This new correlation parameter ρ allows us to perform the optimization displayed in Subsection 6.2.5 to study the optimal IM and DF as functions of correlation ρ. Note that if either correlation ρ or default correlation ρ_X are equal to zero, i.e. $\rho = 0$ or $\rho_X = 0$, the conditions in Section 6.1 that allow for dependencies between default probabilities and asset prices are not valid any more and we are back to a standard binomial model just as in Nahai-Williamson et al. (2013).

Appendix B

Relative differences RF_{IM} and RF_{DF}

The percentage relative difference between the optimal IM and DF contributions in our replication of the Nahai-Williamson et al. (2013) model (see Chapter 4) and our optimal quantities in the Merton-extended version (see Chapter 6) is displayed in Section 6.2 only for the optimizations with varying unconditional default probability $\overline{p}$ (see Figure 6.4) and with varying opportunity cost c (see Figures 6.7 and 6.8). In this appendix, we display all the other relative differences between the optimal quantities from Chapter 4 through Nahai-Williamson et al. (2013) and the ones from our optimizations in the extended version from Chapter 6. Recall from Equations (6.18) and (6.19) that the relative differences RF_{IM} and RF_{DF} are given by:

$$RF_{IM} = \frac{IM_{new} - IM_{nahai}}{IM_{nahai}} \cdot 100$$

$$RF_{DF} = \frac{DF_{new} - DF_{nahai}}{DF_{nahai}} \cdot 100.$$

Note that in the Merton-extended model in Chapter 6 the optimal quantities of IM contributions are generally almost equal to zero. Then, the value IM_{new} in Equation (6.18) is so close to zero that Equation (6.18) gives a standard value approximately equal to -100%. Our results in Chapter 6 state that the optimal total resources (IM and DF together) are higher than the one found by Nahai-Williamson et al. (2013) and that also the optimal DF contributions are larger. However, we never convey conclusions on optimal IM contributions alone, which are actually bigger in Nahai-Williamson et al. (2013) implementation.

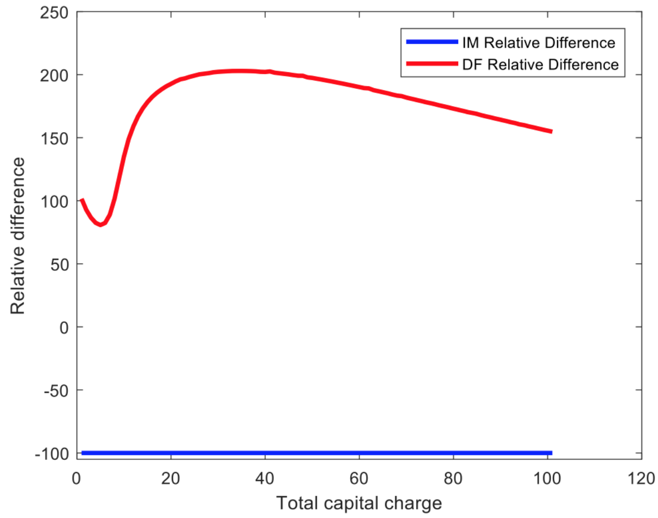


Figure B.1: The IM relative difference RF_{IM} and the DF relative difference RF_{DF} as functions of total capital charge $d_{IM} + d_{DF}$, with $\overline{p} = 5\%$

Figure B.1 shows the relative differences RF_{IM} and RF_{DF} between the optimal IM and DF contributions found by replicating Nahai-Williamson et al. (2013) optimization via Equation (4.15) with varying total capital charge $d_{IM} + d_{DF}$ and the optimal quantities found repeating the same optimization in the Merton-extended model via Equation (6.17) (see Subsection 6.2.3).

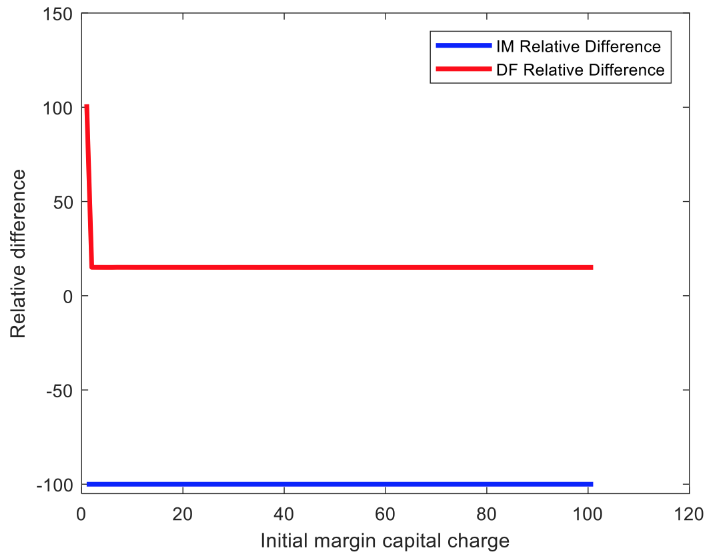


Figure B.2: The IM relative difference RF_{IM} and the DF relative difference RF_{DF} as functions of IM capital charge d_{IM}, with $d_{DF} = 0$ and $\overline{p} = 5\%$

Figure B.2 displays the relative differences RF_{IM} and RF_{DF} between the optimal IM and DF contributions found by replicating Nahai-Williamson et al. (2013) optimization via Equation (4.15) with varying IM capital charge d_{IM} and the optimal quantities found repeating the same optimization in the Merton-extended model via Equation (6.17) (see Subsection 6.2.3). The optimal quantities replicating Nahai-Williamson et al. (2013) in Figure 4.6a display almost a constant optimal value for the DF contribution, except for very small values of d_{IM}. For the interval $d_{IM} \in [0, 0.01]$, we observe an optimal IM contribution and a higher DF contribution. This is the reason for the discontinuity in Figure B.2: in our extended version, the optimal DF contributions are up to 100% and 50% larger than the ones found in Figure 4.6a for $d_{IM} \in [0, 0.01]$.

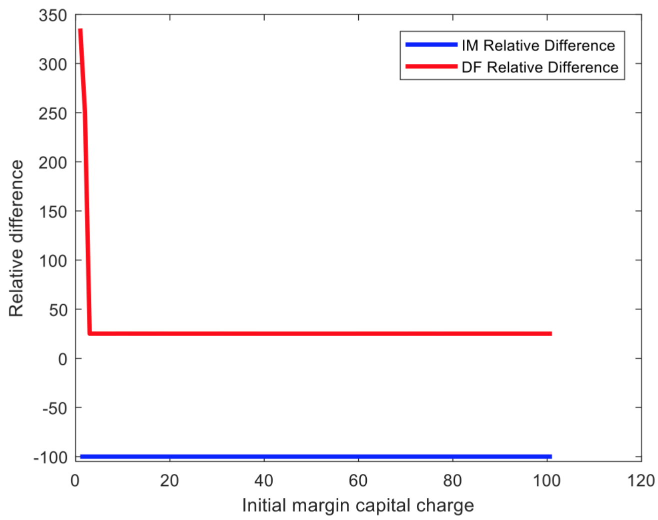


Figure B.3: The IM relative difference RF_{IM} and the DF relative difference RF_{DF} as functions of IM capital charge d_{IM}, with $d_{DF} = 0.16\%$ and $\overline{p} = 5\%$

Figure B.3 shows the relative differences RF_{IM} and RF_{DF} between the optimal IM and DF contributions found by replicating Nahai-Williamson et al. (2013) optimization via Equation (4.15) with varying IM capital charge d_{IM} and the optimal quantities found repeating the same optimization in the Merton-extended model via Equation (6.17) (see Subsection 6.2.3). The optimal quantities replicating Nahai-Williamson et al. (2013) in Figure 4.6b display almost a constant optimal value for the DF contribution, except for very small values of d_{IM}. For the interval $d_{IM} \in [0, 0.251]$, we observe an optimal IM contribution and a higher DF contribution. As in Figure B.3, this is the reason for the discontinuity in Figure B.3: in our extended version, the optimal DF contributions are even 300%, 250% and 200% larger than the ones found in Figure 4.6b for $d_{IM} \in [0, 0.025]$.

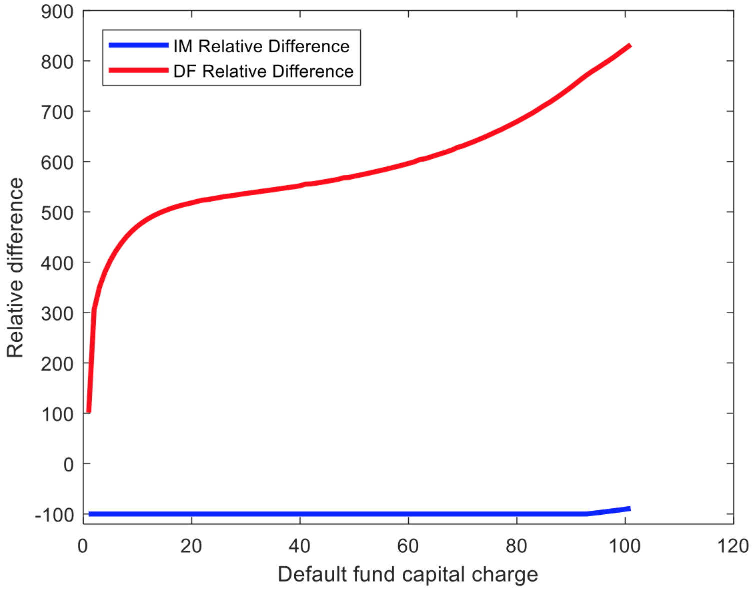


Figure B.4: The IM relative difference RF_{IM} and the DF relative difference RF_{DF} as functions of IM capital charge d_{DF}, with $d_{IM} = 0$ and $\overline{p} = 5\%$

Figure B.4 shows the relative differences RF_{IM} and RF_{DF} between the optimal IM and DF contributions found by replicating Nahai-Williamson et al. (2013) optimization via Equation (4.15) with varying DF capital charge d_{DF} and the optimal quantities found repeating the same optimization in the Merton-extended model via Equation (6.17) (see Subsection 6.2.3). Note that in Figure B.4 there is a slight increase in the optimal IM relative difference RF_{IM} for extremely high values of d_{DF}. This is coherent with our optimizations in Figure 6.11b, where optimal IM contributions start to increase for the largest values of d_{DF}.

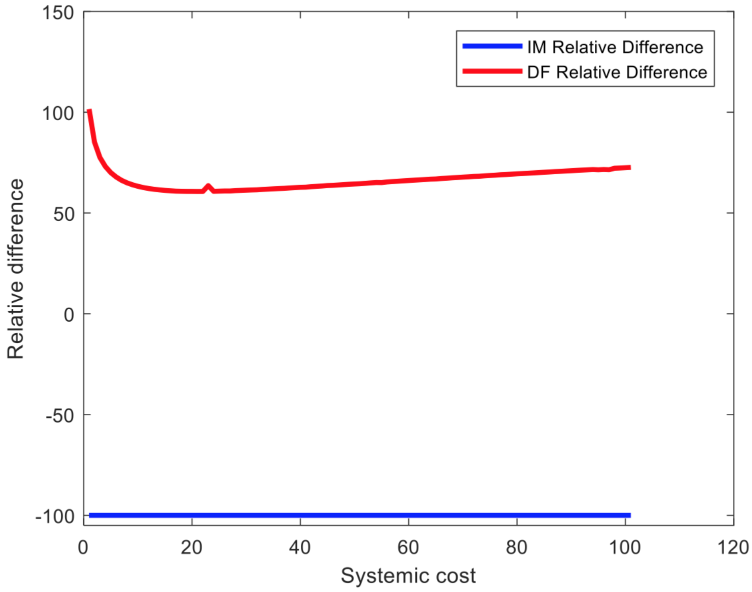


Figure B.5: The IM relative difference RF_{IM} and the DF relative difference RF_{DF} as functions of systemic cost s, with $\overline{p} = 5\%$

Figure B.5 shows the relative differences RF_{IM} and RF_{DF} between the optimal IM and DF contributions found by replicating Nahai-Williamson et al. (2013) optimization via Equation (4.15) with varying systemic cost s and the optimal quantities found repeating the same optimization in the Merton-extended model via Equation (6.17) (see Subsection 6.2.4).

Appendix C

Numerical optimization

The results displayed in Chapter 6 come from a series of optimizations. As described in Table 6.1, in each optimization one parameter is floating while all the others are fixed. Recall the optimization problem given by Equation (6.17), that is:

$$\min_{y,z}\{\mathbb{E}[L^{OTM}(W;y,z)] + \mathbb{E}[L^{ITM}(W;y,z)] + (c + d_{IM}\cdot c_c)y + (c + d_{DF}\cdot c_c)z\}.$$

The optimizations are implemented using the software MATLAB®.

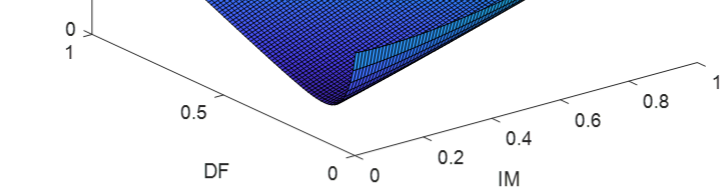
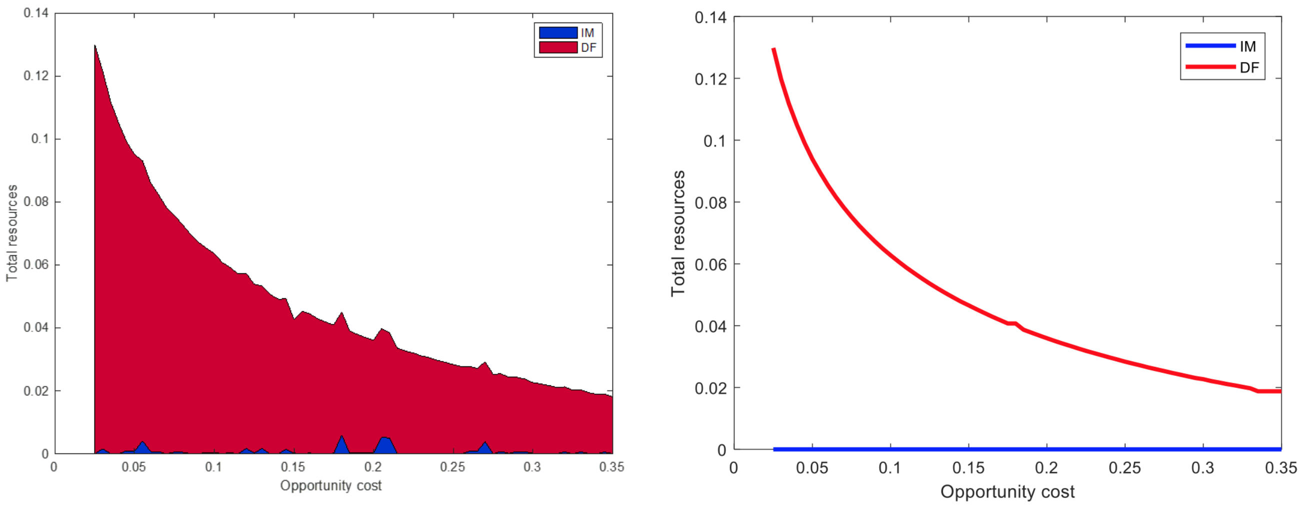


Figure C.1: Expected loss function in Equation (6.16) with parameters fixed as in the last column of Table 6.1 and opportunity cost $c = 0.025$

Figure C.1 plots the surface of the expected loss function in the extended version of the Nahai-Williamson et al. (2013) model adapted to a Merton framework as given in Equation (6.16) with opportunity cost $c = 0.025$ and all the other parameters fixed as displayed in the first column of Table 6.1. So, this is the first expected loss function that we minimize for the optimization results in Subsection 6.2.2 in Figure 6.6b. Figure C.1 shows that this new expected loss function is smooth and has a global minimum, which is clear from Figure C.1. The expected loss function has this smooth behaviour for each value of its parameters: so when we change opportunity cost c, for example, the function can shift up, but it still has a global minimum.

Optimizations are usually done with the command `fmincon`, which uses the Levenberg–Marquardt algorithm. However, when we minimize the expected loss function in Equation (6.16) in Section 6.2 with `fmincon`, the results are not regular, while minima for different parameters should be smooth.

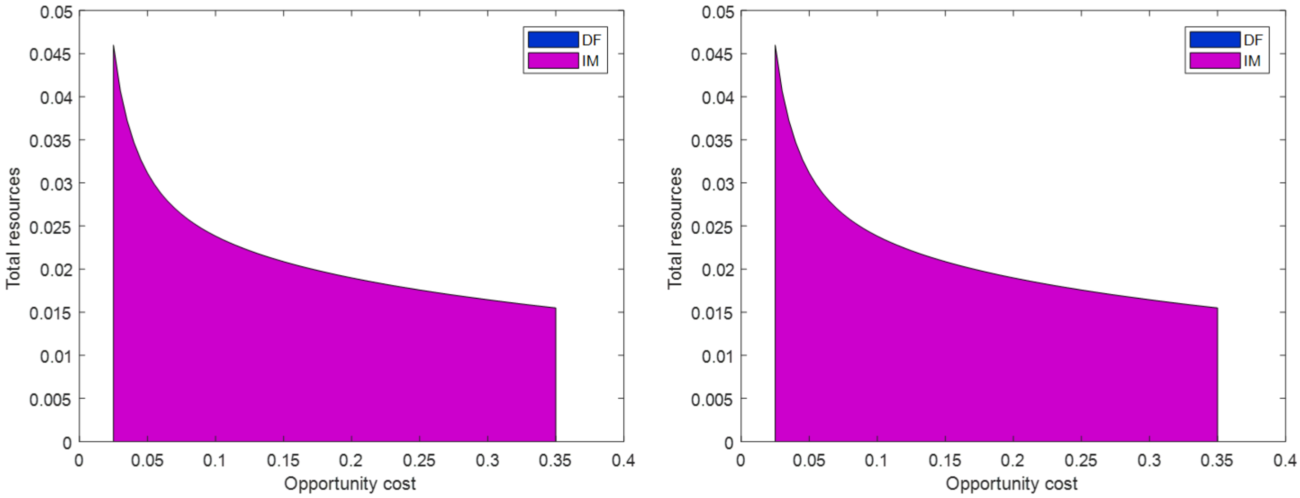


(a) With `fmincon` (b) With `patternsearch`

Figure C.2: Different results for the same optimization in Merton-extended version in Subsection 6.2.2 using command `fmincon` and `patternsearch`

Figure C.2a shows the same optimization results as in Subsection 6.2.2, but using the command `fmincon`. We cannot accept that irregular behaviour with jumps, because from Figure C.1 we know that both the function and the series of minima are smooth. The irregular behaviour in Figure C.2a is mainly due to numerical issues inside the command: the expected loss function is very complex and requires a lot of iterations. Hence, we used a different optimization routine in MATLAB®, called `patternsearch`, to solve Equation (6.16). The command `patternsearch` provides a minimization algorithm with more iterations and precision: Figure C.2b shows the right results for this minimization.

However, it would not be possible to compare our results with our implementations of Nahai-Williamson et al. (2013) if the authors used the classic Levenberg–Marquardt algorithm. Figure C.3 shows that repeating Nahai-Williamson et al. (2013) optimizations with the new command `patternsearch` brings to exactly the same results.



(a) With `fmincon`

(b) With `patternsearch`

Figure C.3: Our implemented versions of Nahai-Williamson et al. (2013) optimization in Subsection 4.2.1 using commands `fmincon` and `patternsearch`

Figure C.3 shows that the results are exactly the same in the implemented Nahai-Williamson et al. (2013) using both commands `fmincon` and `patternsearch`. Thus, the comparison between our results and Nahai-Williamson et al. (2013) remains valid.

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