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CHILD HUMAN CAPITAL—THE IMPORTANCE OF PARENTING STYLE

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Abstract

Investments in the human capital of children during their upbringing determine the opportunities available in adulthood. Recognizing that the parent-child interaction plays a significant role in the accumulation of child human capital, we develop a differential game in which the parent may invest directly in child human capital and the child consumes goods that influence the accumulation of their human capital. We compare the accumulation of child human capital between three different parenting styles, formalized as three different solution concepts to the differential game: (i) the parent and the child maximizes a joint utility function (cooperative solution), (ii) the parent announces a strategy dependent on time only (open-loop Stackelberg), (iii) the parent's strategy depends on the accrued amount of human capital (feedback Stackelberg). We show that under rather general assumptions the open-loop Stackelberg equilibrium is time consistent, and coincides with a feedback Stackelberg equilibrium. Using cooperative parenting as a benchmark, we find that less or more child human capital may be accumulated over the family's planning horizon under "open-loop Stackelberg" parenting, depending on parental and child preferences for human capital and wealth at the terminal time of the family's planning horizon, and on the extent to which child consumption influences the accumulation of their human capital. In particular, if the child's preference for terminal time wealth is strong enough, more human capital will be accumulated under "open-loop Stackelberg" parenting.

Introduction

The two most important components of human capital are education and health. Child and adolescent (hereafter child) human capital is important largely because it affects that person's future prospects for health, education, labour-market, and other essential lifetime outcomes. Ample evidence from both the epidemiological and economic literature supports this statement, see, e.g., Campbell et al. (2014), Currie (2009) and Currie and Almond (2011). Investments in child human capital are made both through public efforts, e.g., vaccination programs and compulsory schooling, and intra-family allocation of time and resources. Public and private investments may be offset or reinforced by activities and consumption choices made by children. For instance, alcohol consumption at an early age, illegal drug abuse, and risky sexual behaviour adversely affect the formation of human capital either directly, through physiological mechanisms, or indirectly, by rendering the production of human capital less effective. Other activities and consumption choices may have the opposite effect, examples of which include following stipulated educational curricula and a wholesome diet, exerting appropriate amounts of physical activities, and taking part in non-trivial cultural activities.

Several intra-family decision-making mechanisms have been proposed in the literature, and typically involve cooperative or strategic interaction between the family members. In a dynamic context, the mechanisms suggested in the literature may be further classified as to whether or not future allocations are made contingent on accruing key outcomes. The interaction between parents and their children is typically characterised by asymmetries regarding, for instance, financial resources and farsightedness, and by parents being the main holders of the decision-making initiative. In the literature on the economics of parenting styles, as in Doepke and Zilibotti (2017), no clear distinction is made between decision-making mechanisms per se and the ultimate objectives that parents and their children strive to attain. A parent is typically assumed to be both altruistic and paternalistic, and unless purely altruistic, there is a conflict between child and parental preferences. This conflict is recognised and the different parenting styles considered are simply different ways to achieve its resolution. We adopt a different set of definitions of parenting styles by departing from the typical decision-making mechanism used to allocate resources. Thus, our main objective is to examine the importance of different intra-family decision-making mechanisms for the accumulation of child human capital. To this end, we formulate a differential game that accounts for the parent-child interaction and apply three different solution concepts to it, reflecting three different "parenting styles".

One of the pioneers in the development of economic theories of family decision-making and human capital was Gary Becker (1981, 1993). The focus of his analyses was on gains to family formation assuming cooperation on the part of the family members. Even though there may be strong incentives for honouring cooperatively made decisions about intra-family conduct, few, if any, domestic actions can be controlled by enforceable agreements. Consequently, incentives for strategic behaviour, generally entailing allocations that differ from cooperatively made Pareto efficient allocations, cannot be perfectly curbed. Thus, several authors argue that strategic behaviour needs to be taken into account when analysing intra-family decision making—see, for instance, King (1982), Konrad and Lommerud (1995), Dufwenberg (2002), and Oosterbeek et al., (2003). The survey of economic theories of the family by Bergström (1997) focused on consensus models of the family, and strategic considerations were only briefly discussed. No doubt, this owes to the fact that, at the time, non-cooperative game theory was largely unexplored as a means of studying the family. More recent contributions to this field include parent-child interaction, as in Chang (2009) and Chang et al. (2005), but from the perspectives of children competing for parental resources.

A family is exposed to strategic behaviour, but parental decisions regarding investments in the human capital of young children may be less so, as (a) young children make no intentional investments on their own and, hence, cannot strategically offset parental investment by lowering their own, and (b) parents of young children are able to monitor and enforce good habits—at least to some degree. On the other hand, when children reach the age when *some* autonomy is an essential part of eventually becoming an independent adult, but still are dependent on the family for survival, they will be able to behave strategically in relation to the "freedom" offered by parents.

Given that children can only be governed imperfectly, parents must decide how resources are allocated *given* strategic, selfish and relatively short-sighted child behaviour. This is reminiscent of Becker's famous Rotten-kid theorem, which implies that a selfish child will cooperate in maximising family welfare if financial incentives are properly tuned. Thus, a parent who faces a related, but more complex, situation of making optimal investments in child human capital during the child's upbringing may design financial incentives not only as a means of fulfilling altruistic objectives, but also as a tool for influencing child behaviour.

The differential game contemplated refines some specific aspects of the multifaceted parent-child relationship. General properties of the game are first established. Employing some simplifying assumptions it is then explicitly solved using three different solution concepts, each of which illustrates a different parenting style. Much of the focus is on the amount of child human capital at the end of the planning horizon. By employing the aforementioned assumptions, the said game belongs to the class of differential games referred to as "tractable games" by Dockner et al. (2000), their defining property being explicit solvability. Dockner et al. (2000) argued that "analytical tractability is stressed, since analytical solutions have the advantage of shedding light on the qualitative properties of equilibria in a general way". Tractable differential games have been applied to various areas in economics, such as drugs, corruption and terrorism, the environment and natural resources, industrial organisation and political economy—see, for instance, Dockner et al. (2000), Grass et al. (2008), Hori and Shibata (2010), Long (2010, 2011), and Novak et al. (2010). The theoretical modelling of dynamic and strategic interactions within the family has even been investigated by mathematicians and biologists, as in Ewald et al. (2007).

A differential-game approach to investments in child human capital

The core features of the model involve one parent who has a first-mover advantage in the allocation of wealth between investment in child human capital and their own consumption. Given the parent's allocation and a lifetime budget constraint, the child chooses how much to consume of each of two goods, of which one has a detrimental effect on human capital while the other has a beneficial effect. It is assumed that the child is old enough to make autonomous decisions regarding how to allocate the available resources between the two types of consumptions.

We derive solutions under three different informational structures governing the parent's and child's decision making. The first structure assumes that the parent and child make decisions hierarchically under open-loop information, referred to as the *non-evaluating parenting style*, thereby yielding an open-loop Stackelberg equilibrium. This means that the leader, i.e., parent, announces their strategy for the entire planning period at the initial time and thus posses a first-mover advantage in this sense, which is fully committed and conditioned only on the clock and exogenous variables. The second assumes that the parent and child make decisions cooperatively under open-loop information, referred to as the *cooperative parenting style*, thus resulting in a cooperative equilibrium. The cooperative case reflects the situation in which the parent has implemented joint, as opposed to strategic, decision making. And the third assumes that the parent and child make decisions hierarchically under feedback information, referred to as the *evaluating parenting style*, hence yielding a feedback Stackelberg equilibrium. This is a situation in which a parent allocates resources contingent on time, the accrued stock of human capital, wealth and exogenous variables, and in which they posses a first-mover advantage at each point in time.

General structure

The formal structure of the model is as follows. At each point in time, $\tau \in [0,T]$, the parent derives utility from investments, $I(\tau) \geq 0$, in child human capital, $H(\tau) \geq 0$, and their own consumption, $Q(\tau) \geq 0$, and the child derives utility from two types of consumption, $Q_1(\tau) \geq 0$ and $Q_2(\tau) \geq 0$. Consumption of Q_1 adds to the change in the child's stock of human capital over time, while consumption of Q_2 subtracts from it. The human-capital state equation is

$$\dot{H}(\tau) = f(I(\tau), Q_1(\tau), Q_2(\tau)) - \delta H(\tau),\ H(0) = H_0, \tag{1}$$

where $f(\cdot)$ is a human-capital production function such that $f(\cdot) \in C^{(2)}$, with partial derivatives $f_I > 0$, $f_{Q_1} > 0$, and $f_{Q_2} < 0$, and H_0 is the initial stock of child human capital.

The following additional assumptions are then adopted:

(A1) Parental (p) and child (c) preferences are represented by the utility functions

$$U^p(Q(\tau), I(\tau)) \in C^{(2)}, \tag{2}$$

$$U^c(Q_1(\tau), Q_2(\tau)) \in C^{(2)}, \tag{3}$$

where $U_j^i > 0$, $i = p, c;\ j = 1, 2$.

(A2) Parental preferences for the stocks of human capital and financial assets at $\tau = T$, are represented by the utility function $\alpha_H H(T) + \alpha_A A(T)$, where $\alpha_H \geq 0$ and $\alpha_A \geq 0$.

(A3) Child preferences for the stocks of human capital and financial assets at $\tau = T$, are represented by the utility function $\beta_H H(T) + \beta_A A(T)$, where $\beta_H \geq 0$ and $\beta_A \geq 0$.

(A4) There are perfect financial markets, which allow lending and borrowing at the continuously compounded interest rate $r > 0$.

(A5) Under each solution concept there exists a unique optimal control vector, such that all control variables are strictly positive for $\tau \in [0, T)$.

Assumption (A1) places prototypical restrictions on the instantaneous utility functions, that is, (i) they are twice continuously differentiable and (ii) Q, I and Q_1, Q_2 are goods. The formulation of the scrap-value functions in stipulations (A2) and (A3) permits the derivation of explicit solutions to the differential game. Assumption (A4) is a standard assumption. It means that financial resources can be transferred between time periods and thus that a lifetime budget constraint applies, as may be seen in what follows. On the other hand, in the absence of financial markets, borrowing and lending is not possible, in which case a different budget constraint would bind at each point in time.

Budget constraint

Given perfect financial markets, the budget constraint faced by both agents asserts that the present value of expenses must be less than or equal to the present value of available resources. More formally, the budget constraint is

$$\int_0^T [I(\tau)+Q(\tau)+Q_1(\tau)+Q_2(\tau)]e^{-r\tau}\,d\tau \leq A_0 + \int_0^T ye^{-r\tau}d\tau\,, \tag{4}$$

where $A_0 \geq 0$ is the initial stock of assets, and $y \geq 0$ is a fixed flow of income. By defining

$$A(\tau) \overset{\text{def}}{=} A_0 e^{r\tau} + \int_0^\tau [y - I(t) - Q(t) - Q_1(t) - Q_2(t)]\mathrm{e}^{-r(t-\tau)}\,dt\,, \tag{5}$$

and then using Leibniz's rule and the lifetime budget constraint, it follows that

$$\dot{A}(\tau) = rA(\tau) + y - I(\tau) - Q(\tau) - Q_1(\tau) - Q_2(\tau)\,,\; A(0) = A_0\,,\; A(T) \geq 0\,. \tag{6}$$

The optimal control problems

In the non-evaluating (open-loop) and evaluating (feedback) cases, the parent and the child act as if solving certain optimal control problems. Letting $\rho_p \geq 0$ be the parental rate of time preference, the parent acts as if solving

$$V^p(A_0,H_0) \overset{\text{def}}{=} \max_{Q(\cdot),I(\cdot)} \int_0^T U^p[Q(\tau),I(\tau)]e^{-\rho_p\tau}d\tau + [\alpha_H H(T) + \alpha_A A(T)]e^{-\rho_p T}, \tag{7}$$

under different informational structures, and subject to the state equation (1), the budget constraint (6), and the non-negativity constraints on the control variables. Similarly, letting ρ_c be the child rate of time preferences, the child acts as if solving

$$V^c(A_0,H_0) \overset{\text{def}}{=} \max_{Q_1(\cdot),Q_2(\cdot)} \int_0^T U^c[Q_1(\tau),Q_2(\tau)]e^{-\rho_c\tau}d\tau + [\beta_H H(T) + \beta_A A(T)]e^{-\rho_c T}, \tag{8}$$

subject to the state equation (1), the budget constraint (6), and the said non-negativity constraints.

In the cooperative case, the parent and the child behave as if solving

$$V(A_0,H_0) \overset{\text{def}}{=} \max_{I(\cdot),Q(\cdot)Q_1(\cdot),Q_2(\cdot)} \left\{ \begin{array}{l} \int_0^T \left[U^p(Q(\tau),I(\tau)) + U^c(Q_1(\tau),Q_2(\tau))\right]e^{-\rho\tau}d\tau \\ + \left[\alpha_H H(T) + \alpha_A A(T) + \beta_H H(T) + \beta_A A(T)\right]e^{-\rho T} \end{array} \right\}, \tag{9}$$

subject to the aforementioned equations.

In the subsequent sections we, first, demonstrate that (i) the open-loop Stackelberg equilibrium is time consistent, (ii) an open-loop Nash equilibrium is a degenerate feedback Nash equilibrium, (iii) an open-loop Stackelberg equilibrium is identical to an open-loop Nash equilibrium, and (iv) a feedback Stackeberg equilibrium is identical to a feedback Nash equilibrium, and, second, derive an optimal control vector for each parenting style. Function arguments are omitted where appropriate.

General properties of noncooperative equilibria

Following Dockner et al. (2000) or Grass et al. (2008), an open-loop Stackelberg equilibrium is defined as follows: (a) the parent makes a binding commitment, at $\tau = 0$, to investment and consumption time paths, $I(\tau)$ and $Q(\tau)$, (b) given $I(\tau)$ and $Q(\tau)$, the child chooses consumption time paths, $Q_1(\tau)$ and $Q_2(\tau)$ according to (8), and (c) the parent chooses investments in child human capital and consumption according to (7), using the child's reaction functions.

An open-loop Stackelberg equilibrium is not in general time consistent, as shown by Xie (1997) and Dockner et al. (2000). However, the aforesaid authors demonstrated that if the initial values of the follower's costate variables are not controllable by the leader, then an open-loop Stackelberg equilibrium is time consistent. We now demonstrate that this is the case in the model outlined above.

To that end, define the child's, i.e., the follower's, current-value Hamiltonian function by

$$\Psi^c(A,H,Q_1,Q_2,\lambda_A^c,\lambda_H^c) \stackrel{\text{def}}{=} U^c(Q_1,Q_2)+\lambda_H^c[f(I,Q_1,Q_2)-\delta H]+\lambda_A^c[rA+y-I-Q-Q_1-Q_2]. \quad (10)$$

where λ_A^c and λ_H^c are the child's costate variables corresponding to the state equations for assets and human capital, respectively. Given assumption (A5), the maximum principle gives the following necessary and sufficient conditions for optimality:

$$\begin{aligned} U_{Q_1}^c(Q_1,Q_2)+\lambda_H^c f_{Q_1}(I,Q_1,Q_2)-\lambda_A^c=0, \\ U_{Q_2}^c(Q_1,Q_2)+\lambda_H^c f_{Q_2}(I,Q_1,Q_2)-\lambda_A^c=0, \\ \dot{\lambda}_H^c=[\rho_c+\delta]\lambda_H^c,\ \lambda_H^c(T)=\beta_H, \\ \dot{\lambda}_A^c=[\rho_c-r]\lambda_A^c,\ \lambda_A^c(T)=\beta_A. \end{aligned} \quad (11)$$

By the implicit function theorem the follower's reaction functions, say $Q_1=R^1(I,\lambda_A^c,\lambda_H^c)$ and $Q_2=R^2(I,\lambda_A^c,\lambda_H^c)$, exist and are locally $C^{(1)}$.

Observe that the two costate equations are wholly independent of the parent's control variables, and thus so are the initial values of said costate variables, thereby establishing the ensuing proposition.

Proposition 1: The open-loop Stackelberg equilibrium in the parenting game described by Eqs. (7) and (8) is time consistent.

The preceding means that the announced consumption and investment strategies are indeed credible in the eyes of the child. Next, observe that (i) the instantaneous utility functions of the parent and the child are independent of the state variables (A,H), (ii) the state equations (1) and (6) are linear in (A,H), and (iii) the salvage value functions are linear in (A,H). Consequently, the differential game under consideration meets the definition of a linear state game by Dockner et al. (2000, p. 188). As discussed by Dockner et al. (2000, p. 189), the following proposition is an immediate consequence of this fact.

Proposition 2: If an open-loop Nash equilibrium exists in the differential game defined by Eqs. (1), (6), (7) and (8), then it is a degenerate feedback Nash equilibrium and, hence, subgame perfect.

Define the parent's, i.e., the leader's, current value Hamiltonian function by

$$\begin{aligned}\Psi^{p}(A,H,Q,I,\lambda_A^p,\lambda_H^p,\mu_A^p,\mu_H^p) \overset{\text{def}}{=} {}& U^p(Q,I)+\lambda_H^p\Big[f\left(I,R^1(I,\lambda_A^c,\lambda_H^c),R^2(I,\lambda_A^c,\lambda_H^c)\right)-\delta H\Big] \\ &+\lambda_A^p\Big[rA+y-I-Q-R^1(I,\lambda_A^c,\lambda_H^c)-R^2(I,\lambda_A^c,\lambda_H^c)\Big] \\ &+\mu_H^p[\rho_c+\delta]\lambda_H^c+\mu_A^p[\rho_c-r]\lambda_A^c,\end{aligned} \tag{12}$$

where λ_A^p and λ_H^p are the parent's costate variables corresponding to the state equations for assets and human capital, respectively, and μ_A^p and μ_H^p are the parent's costate variables corresponding to the costate equations of the child for assets and human capital. Given assumption (A5), the necessary and sufficient conditions are

$$\begin{aligned}U_I^p(I,Q)+\lambda_H^p\Big[&f_I\left(I,R^1(I,\lambda_A^c,\lambda_H^c),R^2(I,\lambda_A^c,\lambda_H^c)\right) \\ &+f_{Q_1}\left(I,R^1(I,\lambda_A^c,\lambda_H^c),R^2(I,\lambda_A^c,\lambda_H^c)\right)R_I^1(I,\lambda_A^c,\lambda_H^c) \\ &+f_{Q_2}\left(I,R^1(I,\lambda_A^c,\lambda_H^c),R^2(I,\lambda_A^c,\lambda_H^c)\right)R_I^2(I,\lambda_A^c,\lambda_H^c)\Big] \\ &+\lambda_A^p\Big[-1-R_I^1(I,\lambda_A^c,\lambda_H^c)-R_I^2(I,\lambda_A^c,\lambda_H^c)\Big]=0,\end{aligned}$$

$$U_Q^p(I,Q)-\lambda_A^p=0\,,$$

$$\dot{\lambda}_H^p = [\rho_p + \delta]\lambda_H^p,\ \lambda_H^p(T) = \alpha_H,$$

$$\dot{\lambda}_A^p = [\rho_p - r]\lambda_A^p,\ \lambda_A^p(T) = \alpha_A,$$

$$\begin{aligned}
\dot{\mu}_H^p &= [\rho_p - \rho_c - \delta]\mu_H^p \\
&\quad - \Big[f_{Q_1}\left(I, R^1(I,\lambda_A^c,\lambda_H^c), R^2(I,\lambda_A^c,\lambda_H^c)\right) R^1_{\lambda_H^c}(I,\lambda_A^c,\lambda_H^c) \\
&\quad + f_{Q_2}\left(I, R^1(I,\lambda_A^c,\lambda_H^c), R^2(I,\lambda_A^c,\lambda_H^c)\right) R^2_{\lambda_H^c}(I,\lambda_A^c,\lambda_H^c) \Big] \lambda_H^p \\
&\quad + \Big[R^1_{\lambda_H^c}(I,\lambda_A^c,\lambda_H^c) + R^2_{\lambda_H^c}(I,\lambda_A^c,\lambda_H^c) \Big] \lambda_A^p, \\
\dot{\mu}_A^p &= [\rho_p - \rho_c + r]\mu_A^p \\
&\quad - \Big[f_{Q_1}\left(I, R^1(I,\lambda_A^c,\lambda_H^c), R^2(I,\lambda_A^c,\lambda_H^c)\right) R^1_{\lambda_A^c}(I,\lambda_A^c,\lambda_H^c) \\
&\quad + f_{Q_2}\left(I, R^1(I,\lambda_A^c,\lambda_H^c), R^2(I,\lambda_A^c,\lambda_H^c)\right) R^2_{\lambda_A^c}(I,\lambda_A^c,\lambda_H^c) \Big] \lambda_H^p \\
&\quad + \Big[R^1_{\lambda_A^c}(I,\lambda_A^c,\lambda_H^c) + R^2_{\lambda_A^c}(I,\lambda_A^c,\lambda_H^c) \Big] \lambda_A^p.
\end{aligned}$$

(13)

Now assume that $f(\cdot)$ is jointly linear in all its arguments. This assumption implies the follower's reaction functions, and that the necessary and sufficient conditions determining the leader's open-loop Stackelberg equilibrium are identical to the necessary and sufficient conditions that determine an open-loop Nash equilibrium, thereby establishing the next proposition.

Proposition 3: If $f(\cdot)$ is jointly linear in all its arguments, then an open-loop Stackelberg equilibrium coincides with an open-loop Nash equilibrium.

The linearity of $f(\cdot)$ therefore implies that the leadership by a parent has no effect on the equilibrium of the game given open-loop information. This section is brought to a close by deducing one more general result. First, observe that $U_Q^p(Q,I)$ and $U_I^p(Q,I)$ are independent of (Q_1,Q_2), and that $U_{Q_1}^c(Q_1,Q_2)$ and $U_{Q_2}^p(Q_1,Q_2)$ are independent of (Q,I). Second, note that the two state equations are linear functions of the four control variables (Q,I,Q_1,Q_2). It therefore follows from Definition 2.4 of Rubio (2006) that the reaction functions of the differential game are orthogonal, assuming that

that the information structure is of the feedback variety. The veracity of the ensuing proposition then follows immediately from footnote 3 and Proposition 2.3 of Rubio (2006).

Proposition 4: If $f(\cdot)$ is jointly linear in all its arguments and the information structure of the game defined by Eqs. (1), (6), (7), and (8) is of the feedback variety, then a feedback Stackelberg equilibrium coincides with a feedback Nash equilibrium.

Taken together, Proposition 3 and 4 show that the first-mover advantage of a parent vanishes when $f(\cdot)$ is linear, whether the information structure is open-loop or feedback. In what follows, we maintain the assumption that $f(\cdot)$ is jointly linear in all its arguments.

Next, we derive explicit solutions associated with the three different parenting styles. To that end, we begin by placing additional stipulations on the differential game.

Additional assumptions regarding the model specification

The following additional assumptions are made:

(A1') Parental and child preferences are represented by quadratic instantaneous utility functions

$$U^p(Q(\tau), I(\tau)) = Q(\tau) - \tfrac{1}{2}Q^2(\tau) + I(\tau) - \tfrac{1}{2}I^2(\tau), \tag{14}$$

$$U^c(Q_1(\tau), Q_2(\tau)) = \theta\left[Q_1(\tau) - \tfrac{1}{2}Q_1^{\,2}(\tau)\right] + Q_2(\tau) - \tfrac{1}{2}Q_2^{\,2}(\tau), \tag{15}$$

where $\theta \geq 0$ is a parameter reflecting the child's relative preferences for the two consumption goods.

(A5') The optimal control vector, under each solution concept, is strictly positive, i.e., all control variables are strictly positive for $\tau \in [0, T)$.

(A6) Consumption of Q_1 adds $\varphi_1 Q_1$ $(\varphi_1 \geq 0)$ to, while consumption of Q_2 deducts $\varphi_2 Q_2$ $(\varphi_2 \geq 0)$ from, the stock of human capital.

(A7) The stock of human capital does not depreciate during the part of the child's lifecycle considered here.

(A8) The rate of time discounting is zero.

Assumptions (A1') and (A6) are made in order to derive closed-form solutions, whereas assumption (A7) is made for convenience. By assumption (A5') complications involving boundary solutions are avoided. Due to the linearity of $f(\cdot)$ and assumptions (A1') and (A6), the Mangasarian sufficiency theorem can be invoked and, hence, the optimal control vector is unique. A more general specification, allowing for proportional deterioration of the stock of human capital, would not influence the qualitative content of our results. Likewise, assumption (A8) does not qualitatively influence the equilibria either, as it merely introduces a scaling of the equilibrium amount of child human capital at $\tau = T$ compared to the case with a strictly positive rate of time discounting.

Given the linearity of $f(\cdot)$ and assumption (A6), the human-capital state equation becomes

$$\dot{H}(\tau) = I(\tau) + \varphi_1 Q_1(\tau) - \varphi_2 Q_2(\tau) . \tag{16}$$

We start by considering the cooperative family, in which case the allocation derived is by construction Pareto efficient. This solution serves as a benchmark for the two non-cooperative solutions considered.

Cooperative parenting

The parent and the child act as if solving optimization problem (9), subject to Eqs. (6) and (16). The Hamiltonian function is:

$$\Psi(A,H,Q,I,Q_1,Q_2,\lambda_A,\lambda_H) \overset{\text{def}}{=} Q-\tfrac{1}{2}Q^2+I-\tfrac{1}{2}I^2+\theta\left(Q_1-\tfrac{1}{2}Q_1^2\right)+Q_2-\tfrac{1}{2}Q_2^2$$
$$+\lambda_H\left(I+\varphi_1 Q_1-\varphi_2 Q_2\right)+\lambda_A\left(rA+y-Q-I-Q_1-Q_2\right), \tag{17}$$

where λ_A and λ_H are the costate variables corresponding to the asset and human capital state equations, respectively. The necessary and sufficient optimality conditions for a strictly positive control vector are:

$$\begin{gathered}1-Q-\lambda_A=0\,,\\ 1-I+\lambda_H-\lambda_A=0\,,\\ \theta(1-Q_1)+\lambda_H\varphi_1-\lambda_A=0\,,\\ 1-Q_2-\lambda_H\varphi_2-\lambda_A=0\,,\\ \dot{\lambda}_H=0\,,\ \lambda_H(T)=\alpha_H+\beta_H\,,\\ \dot{\lambda}_A=-r\lambda_A\,,\lambda_A(T)=\alpha_A+\beta_A\,.\end{gathered} \tag{18}$$

Seeing as $\lambda_H(\tau)=\alpha_H+\beta_H\geq 0$ and $\lambda_A(\tau)=(\alpha_A+\beta_A)e^{r(T-\tau)}\geq 0$, it follows that the cooperative equilibrium is given by

$$\begin{gathered}I^*(\tau)=1+(\alpha_H+\beta_H)-(\alpha_A+\beta_A)e^{r(T-\tau)}\,,\\ Q^*(\tau)=1-(\alpha_A+\beta_A)e^{r(T-\tau)}\,,\\ Q_1^*(\tau)=1+\theta^{-1}\varphi_1(\alpha_H+\beta_H)-\theta^{-1}(\alpha_A+\beta_A)e^{r(T-\tau)}\,,\\ Q_2^*(\tau)=1-\varphi_2(\alpha_H+\beta_H)-(\alpha_A+\beta_A)e^{r(T-\tau)}\,.\end{gathered} \tag{19}$$

Child human-capital investment, and the consumption of all three goods, are increasing over time at a rate proportional to the joint preference for the terminal stock of financial assets, $\alpha_A+\beta_A$. This is so since the shadow price of wealth decreases over time and, hence, reduces the marginal cost of investment and the consumption goods. The magnitude of each control variable at each point in time relative to unity—the satiation rate of investment and consumption—is partly determined by the same preferences. For example, human-capital investment is below 1 for all $\tau\in[0,T]$ if joint preferences for the terminal stock of financial assets are stronger than joint preferences for the terminal stock of human capital, i.e., if $(\alpha_H+\beta_H)-(\alpha_A+\beta_A)<0$. A similar observation can be made for $Q_1^*(\tau)$. In contrast, parental consumption and child consumption of the good that is harmful to human capital are always less than the satiation rate.

The terminal value of child human capital is endogenous, and so must be found by solving the state equation for human capital using the cooperative equilibrium. Its derivation is given in the appendix and yields the proof of the following proposition.

Proposition 5: *Given Eq. (19) and the human-capital state equation, the optimal amount of child human capital at* $\tau = T$ *in the cooperative family is given by:*

$$H^*(T) = H_0 - r^{-1}(\alpha_A + \beta_A)\left(\theta^{-1}\varphi_1 - \varphi_2 + 1\right)\left(e^{rT} - 1\right) + \left[1 + \varphi_1 - \varphi_2 + (\theta^{-1}\varphi_1^2 + \varphi_2^2 + 1)(\alpha_H + \beta_H)\right]T \,. \quad (20)$$

A few observations about the optimal amount of child human capital for the cooperative equilibrium at the end of the planning horizon are in order. First, note that the strength of the parent's and the child's preferences for the child's human capital at $\tau = T$ is positively related to the size of the human capital stock at that time. This is so because the stronger those preferences are, more is invested and spent on the good with a beneficial effect on human capital, and less is spent on the good with a detrimental effect. Second, the parent's and the child's preferences for the stock of financial wealth at $\tau = T$ might be positively or negatively related to the stock of human capital at the end of the planning horizon. The reason is that the stronger the parent's or child's preferences for terminal wealth the less is spent on *all* goods that influence the stock, and that the beneficial effect of reducing Q_2 may or may not outweigh the negative effect of reducing I and Q_1. Third, the extent to which the detrimental effect that good two has on the child's stock of human capital (φ_2) at the end of the planning horizon is given by

$$\frac{\partial H^*(T)}{\partial \varphi_2} = r^{-1}(\alpha_A + \beta_A)(e^{rT} - 1) + \left[2\varphi_2(\alpha_H + \beta_H) - 1\right]T > 0$$

if $2\varphi_2(\alpha_H + \beta_H) > 1$. The intuition behind this somewhat counterintuitive result is that the effect on the stock of an increase in φ_2 works through Q_2 only, and that the positive indirect effect, $-\varphi_2 \, \partial Q_2^* / \partial \varphi_2$, outweighs the negative direct effect, $-Q_2^*$.

The non-evaluating parenting style

By Proposition 3, the equilibrium resulting from the non-evaluating parenting style is derived by computing the open-loop Nash equilibrium of the game. Thus, the Hamiltonian functions associated with the child's and the parent's optimization problems are

$$\Psi^c(A, H, Q_1, Q_2, \lambda_A^c, \lambda_H^c) \stackrel{\text{def}}{=} \theta\left[Q_1 - \tfrac{1}{2}Q_1^{\,2}\right] + Q_2 - \tfrac{1}{2}Q_2^{\,2} + \lambda_H^c\left[I + \varphi_1 Q_1 - \varphi_2 Q_2\right] + \lambda_A^c\left[rA + y - I - Q - Q_1 - Q_2\right] \tag{21}$$

$$\Psi^p(A, H, Q, I, Q_1, Q_2, \lambda_A^p, \lambda_H^p, \mu_A^p, \mu_H^p) \stackrel{\text{def}}{=} Q - \tfrac{1}{2}Q^2 + I - \tfrac{1}{2}I^2 + \lambda_H^p\left[I + \varphi_1 Q_1 - \varphi_2 Q_2\right] + \lambda_A^p\left[rA + y - I - Q - Q_1 - Q_2\right]. \tag{22}$$

The necessary and sufficient conditions for optimality for a strictly positive child control vector are

$$\begin{aligned} &\theta(1 - Q_1) + \varphi_1 \lambda_H^c - \lambda_A^c = 0, \\ &1 - Q_2 - \varphi_2 \lambda_H^c - \lambda_A^c = 0, \\ &\dot{\lambda}_H^c = 0, \ \lambda_H^c(T) = \beta_H, \\ &\dot{\lambda}_A^c = -r\lambda_A^c, \ \lambda_A^c(T) = \beta_A, \end{aligned} \tag{23}$$

And those for the parent are

$$\begin{aligned} &1 - I^* - \lambda_A^p + \lambda_H^p = 0, \\ &1 - Q^* - \lambda_A^p = 0, \\ &\dot{\lambda}_H^p = 0, \ \lambda_H^p(T) = \alpha_H, \\ &\dot{\lambda}_A^p = -r\lambda_A^p, \ \lambda_A^p(T) = \alpha_A. \end{aligned} \tag{24}$$

Seeing as $\lambda_A^c(\tau) = \beta_A e^{r[T-\tau]}$, $\lambda_H^c(\tau) = \beta_H$, $\lambda_A^p(\tau) = \alpha_A e^{r[T-\tau]}$, and $\lambda_H^p(\tau) = \alpha_H$, the time-consistent open-loop Stackelberg equilibrium, i.e., the equilibrium allocation in the non-evaluating family is

$$\begin{aligned} I^S(\tau) &= 1 + \alpha_H - \alpha_A e^{r(T-\tau)}, \\ Q^S(\tau) &= 1 - \alpha_A e^{r(T-\tau)}, \end{aligned}$$

$$Q_1^S(\tau) = 1 + \theta^{-1}\varphi_1\beta_H - \theta^{-1}\beta_A e^{r(T-\tau)},$$
$$Q_2^S(\tau) = 1 - \varphi_2\beta_H - \beta_A e^{r(T-\tau)}. \quad (25)$$

As was the case in the cooperative family, investment and all consumption increase over time in the non-evaluating family. In the present case, however, the rate of increase is proportional to each agent's marginal valuation of the terminal stock of financial assets, rather than to the family's marginal valuation of the terminal stock, implying that the rate of increase is smaller in the non-evaluating family than in the cooperative family. Deductions concerning the position of each control trajectory in relation to unity are analogous to those made in the cooperative case, with the difference that in this case the relative strength of each agent's preferences for the terminal stocks determines the position of each trajectory. So, for instance, if $\alpha_A > \alpha_H$, then $I^*(\tau) < 1$ for all $\tau \in [0, T]$.

Again, the stock of child human capital at $\tau = T$ in the non-evaluating family is obtained by solving the human-capital state equation (16) using the above open-loop Stackelberg equilibrium. The result is given in proposition 6 and proven in the appendix.

Proposition 6: *Given Eq. (25) and the human-capital state equation, the equilibrium amount of child human capital at* $\tau = T$ *in the non-evaluating family is given by:*

$$H^S(T) = H_0 + r^{-1}\left[\beta_A(\varphi_2 - \theta^{-1}\varphi_1) - \alpha_A\right](e^{rT} - 1) + \left[1 + \alpha_H + \varphi_1(1 + \theta^{-1}\varphi_1\beta_H) - \varphi_2(1 - \varphi_2\beta_H)\right]T. \quad (26)$$

The following observations are noteworthy. First, the more the parent prefers child human capital at time $\tau = T$, the larger is the stock of it at that time. This is so, since as α_H increases, more is invested in child human capital during the family's lifetime. The opposite is true regarding the strength of the parent's preferences for terminal wealth $A(T)$. Second, as was the case in the cooperative family, the child's preferences for $A(T)$ might be positively or negatively related to $H^S(T)$, as they are negatively linked to Q_1^* and Q_2^*. If, however, $\varphi_2 > \theta^{-1}\varphi_1$, then $\partial H^S(T)/\partial\beta_A > 0$. Third, the strength of the child's preferences for $H(T)$ is positively related to $H^S(T)$, since β_H is positively related to $Q_1^S(\tau)$ and negatively related to $Q_2^S(\tau)$. And fourth, the effect of an increase in j_2 on the child's terminal human capital is given by $\partial H^S(T)/\partial\varphi_2 = r^{-1}\beta_A(e^{rT}-1)+(2\varphi_2\beta_H-1)T$, which may be positive or negative, depending on the magnitude of β_A and β_H. But if $2\varphi_2\beta_H \geq 1$, then an increase in the marginal damage to human capital as a result of the child's consumption of good two leads to a larger terminal stock of human capital.

The accumulation of child human capital differs between the two preceding parenting styles according to the following proposition, as may be seen by comparing Eqs. (20) and (26).

Proposition 7: *The difference* $H^*(T)-H^S(T)$ *is given by*

$$H^*(T)-H^S(T) = r^{-1}\left[\alpha_A(\varphi_2-\theta^{-1}\varphi_1)-\beta_A\right]\left[e^{rT}-1\right]+\left[\alpha_H(\theta^{-1}\varphi_1^2+\varphi_2^2)+\beta_H\right]T . \qquad (27)$$

The amount of child human capital accumulated at the end of the family's lifetime may be larger or smaller under cooperative parenting compared to non-evaluating parenting, depending on the preferences of the parent and the child, and the magnitude of the effects of the child's consumption on the evolution of their capital stock. For example, the larger is the magnitude of the detrimental human-capital effect of Q_2, the larger is the difference $H^*(T)-H^S(T)$. So, cooperative parenting will lead to more child human capital than non-evaluating parenting when φ_2 is (sufficiently) large. The one situation in which the non-evaluating parenting style leads to more child human capital than the cooperative parenting style involves sufficiently strong child preferences for the stock of financial assets at $\tau = T$, which, in the cooperative case, will be incorporated directly in the family's decision making, and thereby leads to smaller investments in human capital.

The evaluating parenting style

In this section, we assume that the parent posses a first-mover advantage and has adopted a feedback strategy. This means that the parent's decisions about investment and consumption at each point in time are made based upon the accrued values of the stocks of financial assets and human capital. As demonstrated by Dockner et al. (2000), the feedback Stackelberg equilibrium can be derived using dynamic programming. More specifically, in our case we derive an explicit solution by conjecturing specific functional forms of the value functions of the parent and the child and, then, by employing the method of undetermined coefficients to the two Hamilton-Jacobi-Bellman (HJB) equations. Recall that the feedback Stackelberg and the Nash equilibria coincide, by Proposition 4. We thus derive the feedback Nash equilibrium.

By Theorem 3.1 of Dockner et al. (2000) the HJB equation corresponding to the parent's optimization problem (7) is given by

$$-V_\tau^p(\tau,A,H) = \max_{I,Q}\left\{I - \tfrac{1}{2}I^2 + Q - \tfrac{1}{2}Q^2 + V_H^p\left[I + \varphi_1 Q_1 - \varphi_2 Q_2\right] + V_A^p\left[rA + y - Q - I - Q_1 - Q_2\right]\right\}, \quad (28)$$

with terminal condition $V^p(T, A(T), H(T)) = \alpha_A A(T) + a_H H(T)$. Similarly, the HJB equation corresponding to the child's problem (8) is given by

$$-V_\tau^c(\tau,A,H) = \max_{Q_1,Q_2}\left\{\begin{matrix}\theta\left[Q_1 - \tfrac{1}{2}Q_1{}^2\right] + Q_2 - \tfrac{1}{2}Q_2{}^2 + V_H^c\left[I + \varphi_1 Q_1 - \varphi_2 Q_2\right] \\ +V_A^c\left[rA + y - Q - I - Q_1 - Q_2\right]\end{matrix}\right\}, \quad (29)$$

with terminal condition $V^c(T, A(T), H(T)) = \beta_A A(T) + \beta_H H(T)$.

Given the functional forms in the two optimization problems, reasonable conjectures about the parent's and child's value function are

$$V^p(\tau,A,H) = a(\tau)H + b(\tau)A + c(\tau). \quad (30)$$

$$V^c(\tau,A,H) = d(\tau)H + e(\tau)A + f(\tau), \quad (31)$$

In order for the conjectured value functions to satisfy the corresponding HJB equations, the time dependent coefficients must be determined so that the value functions satisfy the corresponding HJB equations (28) and (29), for all (feasible) values of (τ, A, H). This amounts to (i) maximizing the right-hand sides of the HJB equations with respect to the control variables and solving for the control variables, (ii) substituting the solutions from (i) back into to each HJB equation, (iii) substituting the partial derivatives of the conjectured value functions in Eqs. (30) and (31) in the HJB equations, and (iv) equating the coefficients on $(A, H, 1)$ on the right- and left-hand sides of the HJB equations. This procedure leads to a six-equation system of ordinary differential equations, the solution to which is given in the following proposition—the proof is outlined in the appendix.

Proposition 8: *Let * denote a solution to a differential equation for a coefficient in Eqs. (30) and (31). Then, given the functional forms in Eqs. (14)–(16), Eqs. (30) and (31) are value functions for the corresponding optimal control problems (7) and (8), if*

$$[a^{*}(\tau), b^{*}(\tau)] = [\alpha_H, \alpha_A e^{r(T-\tau)}], \tag{32}$$

$$[d^{*}(\tau), e^{*}(\tau)] = [\beta_H, \beta_A e^{r(T-\tau)}], \tag{33}$$

where $c^{*}(\tau)$ and $f^{*}(\tau)$ satisfy the differential equations $-\dot{c} = h(t, \mathbf{x})$ and $-\dot{f} = g(t, \mathbf{x})$, where $\mathbf{x}$ is the vector of parameters (see the appendix).

The evaluating family equilibrium amount of child human-capital investment and consumption may then be obtained by using the conjectured value functions and (32) and (33). These calculations are outlined in the appendix, and the result is summarized in the following corollary.

Corollary 1: *The feedback Stackelberg equilibrium resulting from the value functions derived in Proposition 8, is identical to the open-loop Stackelberg equilibrium in Eq. (25).*

Corollary 1 shows that the evaluating parenting and non-evaluating parenting styles result in identical decisions. What is more, because the open-loop Stackelberg equilibrium, open-loop Nash equilibrium, feedback Nash Equilibrium, and feedback Stackelberg equilibrium all coincide with each other, there is no first mover advantage for the parent whether the parent commits to a plan of action at the outset of the game or does not commit at all.

Summary and conclusion

We analysed the significance of parenting style for the accumulation of child human capital over a given planning period. Three different parenting styles were defined according to three intra-family decision-making mechanisms, namely cooperative parenting, non-evaluating parenting, and evaluating parenting. Cooperative parenting means that a joint utility function is maximized, while non-evaluating parenting means that intra-family decisions are equated with an open-loop Stackelberg equilibrium, with the parent as the leader. The evaluating parenting style means that decisions are made according to a feedback Stackelberg equilibrium, and that the parent's actions are contingent on time and the accrued stocks of child human capital and financial assets. We showed that the open-loop Stackelberg equilibrium in the parenting game is time consistent, and an open-loop Nash equilibrium, if it exists, is a degenerate feedback Nash equilibrium, and thus subgame perfect, and a feedback Stackelberg equilibrium coincides with a feedback Nash equilibrium, for a broader class of games than for which we derived closed form solutions. Moreover, we demonstrated that given linearity of the human-capital state equation, the open-loop Stackelberg equilibrium coincides with the open-loop Nash equilibrium thereby implying that there is no advantage to be the first-mover, i.e., parent, under open-loop or feedback information structures. Finally, given the specified forms of the functions appearing the parent's and the child's optimal control problems and the state equation governing human capital, we also demonstrated that equilibria for the evaluating parenting style and the non-evaluating parenting style are identical. So, it is immaterial for the terminal stocks of child human capital and financial assets, whether a plan is made and committed to at the outset or if the parent makes decisions at each point in time contingent on the accrued stocks.

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Appendix

Proof of Proposition 5

Substituting the variables in the human-capital state equation (16) for the optimal control vector given by (19) yields

$$\begin{aligned}\dot{H}(\tau) = {} & 1+(\alpha_H+\beta_H)-(\alpha_A+\beta_A)e^{r(T-\tau)}+\varphi_1\left[1+\theta^{-1}\varphi_1(\alpha_H+\beta_H)-\theta^{-1}(\alpha_A+\beta_A)e^{r(T-\tau)}\right] \\ & -\varphi_2\left[1-\varphi_2(\alpha_H+\beta_H)-(\alpha_A+\beta_A)e^{r(T-\tau)}\right].\end{aligned}$$

This differential equation is of the form $\dot{H}(\tau) = a+be^{r(T-\tau)}$, where

$$a \overset{\text{def}}{=} 1+(\alpha_H+\beta_H)+\varphi_1\left[1+\theta^{-1}\varphi_1(\alpha_H+\beta_H)\right]-\varphi_2\left[1-\varphi_2(\alpha_H+\beta_H)\right],$$

$$b \overset{\text{def}}{=} -(\alpha_A+\beta_A)-\varphi_1\theta^{-1}(\alpha_A+\beta_A)+\varphi_2(\alpha_A+\beta_A).$$

The general solution is $H(\tau)=a\tau-r^{-1}be^{r(T-\tau)}+c$, where c is a constant of integration. The initial condition $H(0)=H_0$ gives $c=H_0+r^{-1}be^{rT}$. Hence $H^*(\tau)=H_0+r^{-1}b(e^{rT}-e^{r(T-\tau)})+a\tau$ and $H^*(T)=H_0+r^{-1}b(e^{rT}-1)+aT$. Inserting the expressions for a and b in $H^*(T)$ and rearranging terms gives Eq. (20).

Proof of Proposition 6

Substituting the open-loop Stackelberg equilibrium values given by (25) in the human-capital state equation (16) yields

$$\dot{H}(\tau) = 1+\alpha_H-\alpha_A e^{r(T-\tau)}+\varphi_1\left[1+\theta^{-1}\varphi_1\beta_H-\theta^{-1}\beta_A e^{r(T-\tau)}\right]-\varphi_2\left[1-\varphi_2\beta_H-\beta_A e^{r(T-\tau)}\right].$$

Proceeding analogously to the proof of Proposition 2 gives Eq. (26).

Proof of Proposition 8

Assuming an interior solution, the necessary and sufficient conditions for a maximum of the right-hand sides of the HJB equations (28) and (29) are

$$1-I+V_H^p-V_A^p=0\,,$$
$$1-Q-V_A^p=0\,,$$
$$\theta[1-Q_1]+\varphi_1 V_H^c-V_A^c=0\,,$$
$$1-Q_2-\varphi_2 V_H^c-V_A^c=0\,.$$

Next, solve for the four control variables from the preceding optimality conditions, substitute them in each HJB equation, and then substitute the partial derivatives of the conjectured value functions in Eqs. (30) and (31) in the HJB equations to arrive at the form of the HJB equations used to find the six coefficients in Eqs. (30) and (31). For the parent, the HJB equation becomes

$$-\dot{a}(\tau)H-\dot{b}(\tau)A-\dot{c}(\tau) \quad = \quad rb(\tau)A+\begin{Bmatrix}[y-4]b(\tau)+1+[1+\varphi_1-\varphi_2]a(\tau)-a(\tau)b(\tau)\\ +b^2(\tau)+\tfrac{1}{2}a^2(\tau)+\left[1+\theta^{-1}\right]e(\tau)b(\tau)+\left[\varphi_2-\theta^{-1}\varphi_1\right]b(\tau)d(\tau)\\ +\left[\varphi_2-\theta^{-1}\varphi_1\right]a(\tau)e(\tau)+\left[\theta^{-1}\varphi_1^2+\varphi_2^2\right]a(\tau)d(\tau)\end{Bmatrix},$$

where
$$-\dot{c}(\tau)=\begin{Bmatrix}[y-4]\alpha_A e^{r(T-\tau)}+1+[1+\varphi_1-\varphi_2]\alpha_H-\alpha_H\alpha_A e^{r(T-\tau)}\\ +\alpha_A^2e^{2r(T-\tau)}+\tfrac{1}{2}\alpha_H^2+\left[1+\theta^{-1}\right]\alpha_A e^{r(T-\tau)}\beta_A e^{r(T-\tau)}+\left[\varphi_2-\theta^{-1}\varphi_1\right]\beta_H\alpha_A e^{r(T-\tau)}\\ +\left[\varphi_2-\theta^{-1}\varphi_1\right]\alpha_H\beta_A e^{r(T-\tau)}+\left[\theta^{-1}\varphi_1^2+\varphi_2^2\right]\alpha_H\beta_H\end{Bmatrix}$$

and for the child the HJB equation is

$$-\dot{d}(\tau)H-\dot{e}(\tau)A-\dot{f}(\tau) \quad = \quad re(\tau)A+\begin{Bmatrix}\tfrac{1}{2}[\theta+1]+[y-4]e(\tau)+[1+\varphi_1-\varphi_2]d(\tau)\\ +\left[\varphi_2-\theta^{-1}\varphi_1\right]d(\tau)e(\tau)+\tfrac{1}{2}\left[1+\theta^{-1}\right]e^2(\tau)+2e(\tau)b(\tau)\\ -a(\tau)e(\tau)-d(\tau)b(\tau)+d(\tau)e(\tau)\end{Bmatrix},$$

where
$$\dot{f}(\tau)=\begin{Bmatrix}\tfrac{1}{2}[\theta+1]+[y-4]\beta_A e^{r(T-\tau)}+[1+\varphi_1-\varphi_2]\beta_H\\ +\left[\varphi_2-\theta^{-1}\varphi_1\right]\beta_H\beta_A e^{r(T-\tau)}+\tfrac{1}{2}\left[1+\theta^{-1}\right]\beta_A e^{2r(T-\tau)}+2\beta_A e^{r(T-\tau)}\alpha_A e^{r(T-\tau)}\\ -\alpha_H\beta_A e^{r(T-\tau)}-\beta_H\alpha_A e^{r(T-\tau)}+\alpha_H\beta_H\end{Bmatrix}$$

In order for the conjectured value functions to satisfy the corresponding HJB equations for all (τ, A, H), the coefficients on $(A, H, 1)$ must be identical on each side of both HJB equations. Identifying the coefficients on the right- and left-hand sides of each HJB equation gives a system of six ordinary differential equations, which can be readily inferred from the preceding two equations, and which can be solved using the boundary conditions $V^{p}(T, A(T), H(T)) = \alpha_A A(T) + a_H H(T)$ and $V^{c}(T, A(T), H(T)) = \beta_A A(T) + \beta_H H(T)$.