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ACTA UNIVERSITATIS GOTHOBURGENSIS

The concept *concept* in mathematics education:

A concept analysis

Lotta Wedman

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Abstract

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The notion *concept* is used in different ways within the field of mathematics education. The aim of this study is to carry out a concept analysis of the notion *concept*, within some frequently used frameworks describing conceptual understanding. Building on a philosophical literature review resulting in distinctions that can be used for interpreting views on *concept*, the study addresses the question: Which views on *concept* may be found in texts using the chosen frameworks, from the perspective of the distinctions mental versus non-mental, intersubjective versus subjective and molecular versus holistic? The design involves a literature review in mathematics education, resulting in a selection of texts. Views on *concept*, and to some extent on *concept image*, *conception*, and *schema*, are then interpreted with the help of indicators, and represented in 3D matrices.

There are two categories of views on *concept* within the texts: a mental and intersubjective category, and a non-mental and intersubjective category. One difference between the views is whether conceptual structures have molecular or holistic features. Concerning the notions *concept image*, *conception*, and *schema*, there are generally three different views: an individual view and two culturally dependent views. The different views are sometimes combined. One result is findings regarding how language is used within the texts, where non-mental and mental arenas, and terms and meanings of terms, are not always distinguished. The main contribution of the study is to deepen the understanding of views on the notion *concept* and how terminology is used in mathematics education. This opens the way for a discussion of how the terminology mentioned above may be used coherently within the field of mathematics education.

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Abbreviations

APOS is short for action, process, object and schema, which are the four stages in the development of the cognitive structure in the APOS framework (Dubinsky, 1991; Asiala et al., 1996)

ASI is in Chapter 6 used as a name for the article written by Asiala et al. (1996)

BIMO is in Chapter 5 used as a name for the article written by Bingolbali and Monaghan (2008)

CICD is short for the concept image-concept definition framework (Vinner & Hershkowitz, 1980; Tall & Vinner, 1981; Semadeni, 2008; Bingolbali & Monaghan, 2008)

$CICD^0$ is short for the version of the CICD framework that is used in Vinner and Hershkowitz (1980) and Tall and Vinner (1981). This version is also referred to as the basic version of the CICD framework.

$CICD^+$ is short for the version of the CICD framework that is used in Semadeni (2008).

CICD* is short for the version of the CICD framework that is used in Bingolbali and Monaghan (2008).

DUB is used in Chapter 6 as a name for the article written by Dubinsky (1991).

GT is used in Chapter 6 as a name for the article written by Gray and Tall (1994).

OS is short for the operational-structural framework (Sfard, 1991).

PO is short for the process to object frameworks, which are involving the idea that a cognitive structure develops from including the idea of processes to including the idea of an object that can be used in other processes. The PO frameworks analysed in the study are the OS framework, the procept framework and the APOS framework.

SEM is used in Chapter 5 as a name for the article written by Semadeni (2008).

SF is used in Chapter 6 as a name for the article written by Sfard (1991).

TV is used in Chapter 5 as a name for the article written by Tall and Vinner (1981).

VIHE is used in Chapter 5 as a name for the paper written by Vinner and Hershkowitz (1980).

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1 Introduction

The starting point for this thesis is an interest in how concepts are described, what they are and how they are structured, in the context of mathematics education. This interest has arisen from noting that there are various opposing views on the nature of *concept*[1] in mathematics education research. The study is of a theoretical nature and aims at comparing these views, with the goal of increasing the understanding of the concept *concept*. Today, discussions about concepts may be found in different fields, such as general educational science, philosophy, psychology, and cognitive science. An assumption in the present study is that these discussions can offer new perspectives in the field of mathematics education as well. In recent years, the number of theoretical studies in mathematics education has decreased (Inglis & Foster, 2018; Niss, 2018). However, there are issues that require theoretical approaches. Vagueness, ambiguity and incoherence in conceptual frameworks are examples of such issues. Today, there is a need for concept analyses in mathematics education, with the goal of contributing to theoretical development.

In this opening chapter, I first describe the research interest and the overall design of the project. The first sections offer a background concerning incoherencies when it comes to the notion *concept*, and the methodology of the concept analysis. This methodology is later further described in Chapters 2, 3, and 4. Next, the study is contextualised in the field of mathematics education and argued for on the basis of some trends in this field. The overall design is based on an analytic philosophical approach, which is explained from different perspectives. After the aim and research question are stated, the chapter ends with a description of the overall structure of the thesis.

[1] When a word is italicised in the thesis it refers to a concept. Here, *'concept'* is short for 'the concept *concept*'.

1.1 Incoherencies in the notion *concept* within the field of mathematics education

Independently of how they regard the nature of science, most people agree that scientific research cannot be conducted without some kind of preunderstanding. This preunderstanding consists of experiences and common facts about the object of study. Further, it uses a common vocabulary for describing this object, methods for conducting research, and common norms and beliefs. This is what some people call a paradigm or a research programme. It can also be called a theory or conceptual framework. This preunderstanding can be observed at different levels. It can be seen generally in science, but it can also be seen in different fields and subfields.

In order to offer knowledge, it is important to seek coherence, and try to avoid contrapositions claiming both '*p*' and 'not *p*'. To claim that light is particles and that light is not particles, at the same time, has consequences for people's trust in science. Another form of incoherence is disagreement about the meanings of terms. Our communication is full of ambiguity. The following sentence can be taken as an example of that: 'Petter looked at the cat with one eye'. When Kristina reads this sentence, she may wonder whether Petter used one eye for looking at the cat, or whether the cat had one eye. The ambiguity appears since the sentence has two different interpretations. The same phenomenon appears in texts in mathematics education, and it is not unusual that a term has different meanings in different texts. The two quotes below are chosen to exemplify the fact that the term 'concept' is used with different meanings in the field. In the first, a concept is a theoretical, non-mental, construct, and in the second it is mental. Without any deeper reflection, it seems as if these views[2] on *concept* are incoherent.

> the word "concept" (sometimes replaced by "notion") will be mentioned whenever a mathematical idea is concerned in its "official" form - as a theoretical construct within "the formal universe of ideal knowledge" (Sfard, 1991, p. 3)

> These multiple mental links are necessary for the biological brain to construct the concept of number (Tall, 2013, p. 40)

[2] When I speak about the view on *concept* I mean the characteristics of the concept *concept* that I find in the text irrespective of which views the author or authors might have.

It may be asked why *concept* is an important notion in education and especially in mathematics education. In order to answer that, we can look at the nature of different types of knowledge. All learning is about someone, Nils for example, developing some kind of knowledge. When Nils was a child and learned how to walk, he probably did not need to develop conceptual knowledge. Instead, the knowledge of how to walk was tacit and hard to express explicitly. Compare this learning with learning how to drive a car, which involves concepts that Nils, 15 years later, needs to understand in order to learn the traffic rules. In school, I would say that all subjects contain some concepts that the students must learn. While some school subjects are based on both practical and conceptual knowledge, such as music, other subjects concern the building of theoretical models, involving concepts, which are used for describing the world and the concrete objects in that world. Economics may be taken as an example of that. Mathematics differs from other subjects in that all mathematical objects are abstract, and concepts in mathematics are perhaps even more important than in other fields. Therefore, the notion *concept* is important in the field of mathematics education.

The quality of views on *concept* in mathematics education research differs between different contexts. Often, the meaning of the term ‘concept’ is underspecified and not problematized (Simon, 2017). In some research settings, the meaning of the term ‘concept’ may be vague, imprecise and non-explicated, and it is hard to interpret what is meant. For example, in Tall and Vinner (1981), there is no explication of *concept* and, as will be seen in Chapter 5, two different views are seen in the way terms are used. In other texts, the explication of *concept* and the usage of the notion are not coherent. For example, the usage of terms such as ‘concept acquisition’ in Sfard (1991) seems to refer to a view where concepts are mental[3], which is not coherent with the explication above of *concept* as a theoretical construct. In this case, there is ambiguity in the same text.

As another example, in Sfard (2008) the notion *concept* is defined as a symbol together with its uses. In that explication, the symbol is combined with its meaning, seen as usage in communication, with a reference to Wittgenstein (1953/1992).[4] Naturally, it is possible to combine elements of different nature, but it may have the effect that this combined object becomes hard to

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[3] Expressions such as ‘concept acquisition’ seem to refer to the ideas in Piaget’s perspective where a concept is a mental representation (Furth, 1969).

[4] In Subsection 3.2.3.2, it is described that Wittgenstein (1953/1992) has another view on *concept*, where concepts are mental representations.

understand. Further, as the explication is not in line with other explications, where concepts are often seen as meanings of symbols, the nature of concepts in Sfard (2008) is not similar to the nature of concepts in other views.

Here, it may be noted that if an explication of the notion *concept* is made which does not fit with how *concept* is used in other settings in the field, then it can be hard to manage to use one single view throughout a single text. This occurs since conceptual frameworks are often based on the findings of others. That is why it is important to evaluate how a framework relates to other frameworks in the field, and how a view on *concept* relates to other views on *concept*, as well. Additionally, if a view on *concept* is specified within a framework, then one can ask how the results based on that framework can be combined with results based on other frameworks, using other views on *concept*.

In Yoon (2006), views on the related notion *conceptual system*, in the Models and Modelling literature, are categorised. In this analysis, three distinct views are found. In the first view, which is the most common one, a conceptual system is a "mental framework that students use to interpret the world around them" (Yoon, 2006, p. 33). Here, conceptual systems are internal objects likened to cognitive structures, which function as cognitive lenses through which someone may view a situation. In the second view, a conceptual system is a "system that underlies a student's mathematical or scientific model, and is made up of elements, operations, relations, and patterns" (Yoon, 2006, p. 34). These elements are mathematical components. Finally, in the third view, a conceptual system "underlies a student's understanding of mathematical or scientific ideas" (Yoon, 2006, p. 35).

It is not clear how an analysis of the notion *conceptual system* has a bearing on the notion *concept*, even though it is reasonable to adopt the idea that these notions are related. However, the citations above, from Sfard (1991; 2008) and Tall (2013), show that there are different meanings of 'concept' within the field of mathematics education. This may lead to unclear theoretical frameworks, if it is not apparent whether concepts are mental or non-mental. This in turn makes scientific results difficult to interpret and to compare. One cannot easily combine the results from a study using concepts as mental representations with a study using concepts as non-mental theoretical constructs. In the first case, concepts develop in the mind of an individual. In the latter case, concepts develop from a cultural perspective through communication between individuals. It may be more rule than exception that different researchers have different opinions about the meaning of a term. In order to compare results from different studies and generalise findings, it is necessary to specify meanings in explications and try to achieve an agreement. The goal of obtaining a unique meaning for every term may be utopian, but fewer specified meanings is something to strive towards.

It might be argued that the main reason for conducting research in mathematics education is to improve teaching about mathematics. Recently, more and more demands for practice-based studies have been raised. At the same time, it is necessary for mathematics education to develop as a field. In order to be able to claim knowledge of how to improve school practice, mathematics education needs a common mature research programme, with common orientations. In such a field, vagueness, ambiguities and incoherencies are fought against. This is why research has to balance between offering results aiming at developing school practice, developing new methodologies, and being theoretical. A contribution of theoretical research can be to offer new models for understanding mathematical learning.

Several ways of tackling conceptual incoherencies may be found in the field of mathematics education. One way is to stick to a single framework and use the notions[5] within it. To exemplify, Simon (2017) specifies the notion *mathematical concept* in a constructivist approach. As another example, Gray and Tall (2007) introduce the notion *thinkable concept*, because of the many meanings of 'concept' and a wish to emphasise the specific meaning in the three world framework. This strategy of inventing new notions when the existing ones are considered ambiguous or dependent on theoretical frameworks is not unique. For example, Tall (2013) uses the notion *knowledge structure* instead of *concept image* (Tall & Vinner, 1981), *conception* (Sfard, 1991) or *schema* (Asiala et al., 1996). These notions may be similar. However, I would say that lack of precise explications in these frameworks makes it difficult to compare them; they are formed within different contexts, but they all claim to describe or explain cognitive structures.

Another way of tackling incoherencies is to carry out a concept analysis to clarify the meanings of terms in an area, both in explications and in the way terms are used, while evaluating the consistency of underlying assumptions (Machado & Silva, 2007). In such a study, it is presupposed that different views

[5] It is a tongue twister to say that the study is a concept analysis of the concept *concept*. To distinguish between *concept* as the object of the study, and the usage of 'concept' at a meta-level, where *concept*, *conception*, *concept image* etcetera are analysed, I have chosen to use the term 'notion' at the meta-level of concepts. Consequently, the expressions 'the notion *concept*' and 'the notion *conception*' are used.

on *concept* could, and should, be compared, in order to broaden the understandings coming from certain frameworks. The reason for doing a concept analysis of the notion *concept* in this thesis is that it can offer a clearer understanding of different views on *concept* in the field. Such an understanding can build a foundation for developing conceptual frameworks.

1.2 Concept analyses in mathematics education

The purpose of concept analyses may be to solve conceptual problems, to create new concepts, or to contribute to theory development. These analyses can be used either for clarifying a conceptual framework for a research project, or as studies in themselves. In the first approach, scientific method includes three essential clusters of activities: experimentation, mathematisation and conceptual, or theoretical, analysis (Machado & Silva, 2007). In educational research, Petocz and Newbery (2010) argue that concept analysis should be seen as a first step in scientific inquiry, but also as something that continues during the whole research process. The second approach is taken by Jenkins (2008; 2014) and Yoon (2006), who conduct concept analysis as research in itself. This is the approach taken in the current thesis, with the purpose of clarifying and comparing views on the notion *concept*, categorising different views, and contributing to theory development. As an example, the study can be used as a first step in the development of a conceptual framework for analysing concepts. (Braun & Clarke, 2006; Nuopponen, 2010a; Nuopponen, 2011)

There are few methodological texts concerning concept analyses in mathematics education, and such analyses are not always explicitly described. One common approach is to start with a literature review and then make a text analysis, where different views on a concept are interpreted, compared and categorised (Nuopponen, 2010b). The object of my concept analysis is the notion *concept* itself. It would be an overly wide project to try to find all usages of *concept* in the field of mathematics education. Instead, the focus is on some current frameworks describing conceptual understanding. How the frameworks and texts are selected, and a more concrete description of the methods used in the analysis are to be found in Chapter 4.

A possible approach would have been to read the texts, interpret the views on *concept*, and sort them into categories. However, in an early phase of the study it became clear that such an approach did not offer sufficient understanding for comparing views on *concept* in different frameworks, since it was hard to relate different views to each other. Based on that insight, I realised that some kind of preunderstanding was needed.

Since views on *concept* are frequently discussed in philosophy of language, a philosophical literature review is conducted. In addition, in philosophical texts discussing concepts, references are often made to texts from concept research within cognitive science. Consequently, the literature review is conducted within the common field of philosophy of language and concept research within cognitive science.

A result of this review is a philosophically flavoured tool for analysing views on *concept* in texts in mathematics education. This tool involves three distinctions, described in Chapter 3, between mental versus non-mental, intersubjective versus subjective, and molecular versus holistic views on *concept*. A reason for this approach is that views on *concept* are often more clearly described in philosophy than in mathematics education, and there are reasons to believe that these have affected views in mathematics education as well, since texts in mathematics education occasionally refer to philosophical texts. For example, this can be seen in Sfard (1991), who refers to Frege (1952/1960), in Tall (2001), who refers to the logicist ideas of Hilbert, Russell and Frege, and in Semadeni (2008), who refers to both logicist and empiricist ideas.

An assumption in the study is that distinctions made within the common field of philosophy of language and concept research within cognitive science can contribute in the field of mathematics education. The approach taken here will broaden, and at the same time delimit, what may be found in the analysis. On the one hand, it will facilitate seeing philosophical aspects in the views on *concept*, whether concepts are mental or non-mental, intersubjective or subjective, and molecular or holistic. On the other hand, it can obstruct the discovery of aspects that are not included in the analysing tool.

One procedure for finding views on *concept* in the educational texts consists of interpreting explications. However, the most common procedure involves interpreting the usage of the term 'concept'. In order to ensure that different texts are interpreted in similar ways, indicators are developed from the characteristics of the analysing tool, and involve certain formulations indicating the different views. The method then consists of searching for these indicators, both in explications and in the way terms are used. This method used for the concept analysis, including the indicators, is described in Chapter 4.

Having described the focus of the study and the design at an overall level, I now want to make some reflections before specifying aim and research question. First, the study is placed in the field of mathematics education, from the background of some trends that can be seen during the last few decades. These are described in Section 1.3. Second, concept analyses have their roots in analytic philosophy, where such studies are common. Consequently, this study takes an analytic philosophical approach. This raises some questions about the role of analytic philosophy in the field of mathematics education, which I consider from three different directions, in Section 1.4.

1.3 The nature of mathematics education research

Around 1970, the first journals for mathematics education research appeared and the first conferences within this field were held. With those events, one can claim that the modern field of mathematics education was established. Until then, mathematics education had been ruled by behaviourism, where the researcher took the role of a neutral observer. Over the next few decades, conceptual frameworks, which could provide as an understanding of mathematics education and build a foundation for research, were mostly developed in a constructivist tradition. With that change, the researcher often took the role of an interpreter, trying to understand the meaning that teaching and learning of mathematics had for the participants. (Niss, 2018; Inglis & Foster, 2018; Kilpatrick, 1993)

From the 1970s, one can distinguish some trends within the field of mathematics education. A first trend concerns the subject matter of research studies. New topics that appeared during this time were, among others, gender-related issues, cultural issues, technology-related issues, mathematical modelling, beliefs and affects, assessment in mathematics, and professional development of mathematics teachers. With these new topics came new kinds of methodological approaches. A second trend in the development is a decrease in the use of experimental methods and a shift from quantitative to qualitative methods. Discussions of interpretation in qualitative research resulted in a need for new conceptual frameworks, and a third trend is an increased theoretical diversity. In the 1980s, constructivism was the dominant approach in the field, developed from the perspective of Piaget. The focus was on children's construction of personal schemas, and research was often based on interviews in which participants were asked to reflect on their thought processes. In the 1990s, there was a change towards more sociocultural approaches. With this change, the focus of research became observations of interaction. Since then, a multitude of approaches and frameworks have appeared. Nowadays, there are constructivist frameworks, sociocultural frameworks, and many other approaches as well. Further, it is not unusual to use different approaches for different kinds of studies, or to attempt to find connections between frameworks, in order to synthesise ideas. (Inglis & Foster, 2018; Kilpatrick, 2014; Niss, 2018)

Consequently, the area of mathematics education research constitutes a diverse field, when it comes to subject, methods and conceptual frameworks. The contemporary field of mathematics education seeks inspiration from directions such as mathematics, psychology, cognitive science, neuroscience, philosophy, linguistics, semiotics, history, and so on (Niss, 2018). Furthermore, the number of studies of learning in mathematics has decreased. At the same time, the number of studies of learning in mathematics in the field of psychology has increased, and it seems as if these kinds of studies have partly migrated from mathematics education to the psychological field. (Inglis & Foster, 2018)

The usage of the term 'concept' is more frequent in articles using constructivist frameworks, even if the term can be seen in other frameworks as well (Inglis & Foster, 2018). It is reasonable to assume that with the lower focus on mathematical learning, and on constructivist frameworks, the usage of 'concept' in mathematics education research has declined[6]. Simultaneously, in concept research within the field of cognitive science, the discussion of the notion *concept* is vital (see for example Margolis and Laurence (2015)).

The term 'concept' is still an important one in the context of mathematics education, both in curricula and in teacher education. Hence, curriculum developers, teacher educators, and students interested in concepts are mainly directed either towards constructivist frameworks, or to the field of cognitive science. Here, one may draw the conclusion that if we want curricula and teacher education to build on current scientific research, it may not be enough to focus on articles in mathematics education. We may also need knowledge about findings in psychology and other adjacent fields. That is why Inglis and Foster (2018) argue for more between-programme engagement and interaction between experimental psychology and mathematics education research. In

[6] This conclusion is made from a study presented in Inglis and Foster (2018), based on an analysis of articles in *Educational Studies in Mathematics* and the *Journal for Research in Mathematics Education*.

addition, based on this discussion I would say that there is a need for new frameworks in mathematics education, which offer an understanding of concepts, and which take modern concept research into account.

A fourth trend in the development of the field of mathematics education is "a gradually reduced spectrum of variation in the nature of 'mainstream' published research" Niss (2018, p. 35). From the 1970s to the late 1980s and onward, there has been a decrease in theoretical discussions and reflections. Formal analyses of educational issues, position or opinion papers, and papers with an exposition of mathematical content in an educational context have become more unusual. (Inglis & Foster, 2018; Niss, 2018)

It may be asked whether this last trend is something that is desirable in the field of mathematics education. Naturally, one can have different opinions on that, but I will argue here that the answer is no. First, on the website for the *Journal for Research in Mathematics Education* (JRME) it is claimed that

> JRME publishes a wide variety of research reports that move the field of mathematics education forward. These include, but are not limited to, various genres and designs of empirical research; philosophical, methodological, and historical studies in mathematics education; and literature reviews, syntheses, and theoretical analyses of research in mathematics education. (NCTM, 2020)

This quote shows a willingness to publish other kinds of research, including theoretical studies in JRME, opposing the trend in recent years. Such an attitude is in line with the position in Niss (2010) that even though a majority of papers and PhD theses present empirical studies, mathematics education research should not be limited to being empirical. There are important issues and problems in the field that require theoretical investigations. Such studies can propose new distinctions or concepts. Further, they can be used for analysing, comparing and linking conceptual frameworks, or presenting and analysing research methods. (Niss, 2010)

A concept analysis of the notion *concept*, can be seen as an example of the kind of theoretical research that has decreased within the field in the last few decades. However, there seems to be a need for more such work. As seen in Section 1.1, there are ambiguities within the field and notions may be vague and are not always used coherently. With the trend of combining frameworks, it may be more important than ever to be cautious since there are different views on *concept* within different frameworks. In addition, when studying concepts in mathematics education, it is essential to take concept research in cognitive psychology, and the views and approaches in that field, into consideration. It is very likely that such research can be used in the development of mathematics education.

As expressed in Section 1.2, this thesis takes an analytic philosophical approach within the field of mathematics education, with the purpose of contributing to theory development. The next issue to be addressed is how such an approach can be placed within a philosophical context. Analytic philosophy has played a significant role within the field of mathematics education, during the 20th century. Below, I discuss this role from three different directions, all taking a philosophical stance.

1.4 Using analytic philosophy in mathematics education research

One way of interpreting analytic philosophy within education is by considering that observational studies have often been grounded in a logical positivist view, imitating methods from science. This involves setting up hypotheses and trying to verify or falsify them in order to get justified and robust results.

Another alternative is to regard analytic philosophy as a direction within philosophy of education, which can be considered a branch of applied philosophy concerned with the nature of education. In that branch, the problems arise from educational theory and practice. Analytic philosophy was the most influential in philosophy of education from about 1950 to 1980, when philosophers applied concept analysis to the field of education. The common belief was that some problems in education arose from a misuse of language. For a number of reasons, for example the scarcity of results with an impact on school practice, the influence of analytic philosophy in education declined. Today, I would say, analytic philosophy and concept analyses are more common in other fields, than they are in educational studies. This claim is affirmed by the fourth trend seen in Section 1.3, that there has been a decrease in theoretical discussions and formal analyses in mathematics education in the last few decades. However, there may today be a slight shift and an incipient trend towards more concept analyses in educational studies as well. (Portelli, 1987; Siegel & Callan, 2018)

Analytic philosophy within mathematics education can also be seen from the perspective of a group of researchers currently working in philosophy of mathematics education. That topic is based on the intersection of the fields of mathematics education and philosophy, including philosophical, anthropological, historical and ethno-mathematical perspectives. Questions asked in this group concern the nature of mathematics, how mathematics relates to society, what learning and teaching mathematics is, and so on. Concept analyses and analytic philosophical approaches in mathematics education can be seen under this umbrella, but, in my experience, this group seems to be more concerned with social issues. (Ernest, 2016)

The current study can be regarded from either one of the two last directions. However, where to position the study from a philosophical viewpoint is not of primary importance. Instead, what is of importance is that the analytic philosophical approach brings with it certain assumptions. As one example, it brings with it the idea that different views on *concept* could, and should, be compared, in order to broaden the understanding coming from a certain framework. Further, it brings to mind the idea of reductionism that simpler theories, including fewer concepts, are to be preferred. This idea is in the spirit of Ockham's razor, meaning that you should not assume more phenomena or concepts than are needed to explain things. As stated above, you can find several terms with similar meanings, such as 'conception', 'concept image', 'schema', and 'knowledge structure'. The reason for this may be that the meanings of the terms are dependent on the framework in which they are found. There is therefore a need to specify a meaning in a certain context. However, this makes texts from different frameworks hard to compare. Further, in many texts different terms are used in parallel, especially when relating results to findings of other studies, using other frameworks. The consequence of this is that several terms with similar meanings are used in the same text, resulting in blurred views. This is why I have been governed by the principle of using a small number of general, text-independent, terms rather than terms that are used in the texts referred to.

1.5 Aim and research question

The aim of the study is to carry out a concept analysis of the notion *concept*, within some frequently used frameworks describing conceptual understanding in mathematics education. The purpose is to influence mathematics education research in several ways. The study makes both a theoretical and a methodological contribution. Theoretically, it will give clearer understanding of views on the notion *concept*, within the studied frameworks, examining the coherence of the texts. Further, it will also offer categories that can be used for describing concepts. Methodologically, the design will contribute to the method of making concept analyses in mathematics education. The study addresses the following research question:

> Which views on *concept* may be found in texts using the chosen frameworks, from the perspective of the distinctions mental versus non-mental, intersubjective versus subjective and molecular versus holistic?

1.6 Structure of the thesis

It has been impossible to write this thesis in chronological order, starting with one chapter followed by others. It began by reading texts in mathematics education. When I realised that I could not grasp the nature of *concept* within the field I started to read texts within philosophy, which led to the field of concept research in cognitive science. The purpose of reading philosophical texts was to try to understand views on *concept* in texts in mathematics education. After that, a first version of the analysing tool was developed. When I read the texts in mathematics education again and made a first version of the analysis, applying the analysing tool, I realised that I needed to change the tool, in order to capture the essence of the educational views. After reanalysing the texts, indicators were developed in order to ensure that the different texts were interpreted similarly. Consequently, the indicators were developed in an interplay between the philosophical views and the views found in the analysed texts. The final analyses, using the indicators, were conducted in a more rigorous way.

In this study in mathematics education, philosophy is used in two different ways. First, the study takes an analytic philosophical approach as it assumes that a concept analysis can contribute with new perspectives in the field of mathematics education. Second, it uses philosophy as a method, since the analysing tool is developed from philosophical ideas. Consequently, this thesis refers to discussions within the field of philosophy and uses some terminology that may be unfamiliar for researchers in mathematics educations. In order to establish a common philosophical background, involving terminology used for describing views on *concept*, Chapter 2 presents a historically flavoured description of discussions about the nature of *concept*.

Together, Chapters 3 and 4 describe the design of the study. In Chapter 3, the analysing tool is developed from views on *concept* within philosophy of language and concept research within cognitive science, through the lenses of a number of texts. Again, these views have resulted in the three distinctions mental versus non-mental, intersubjective versus subjective, and molecular versus holistic. The chapter examines these distinctions. In Chapter 4, the methodology of the concept analysis is described. This includes selection of texts, how the initial analysis was made, reflections on how the analysing tool is used, indicators used for interpreting texts in mathematics education, and the object of study. Further, at the end of Chapter 4 a summary of the study, including delimitations, philosophical assumptions, and some comments on the design, is collected, based on the content in Chapters 1 to 4.

Chapters 5 and 6 present concept analyses of texts employing some frequently used frameworks describing conceptual understanding in mathematics education. In Chapter 5, texts using different versions of the concept image-concept definition (CICD) framework, a basic version and some developments of this version, are analysed. Chapter 6 contains analyses of the process to object (PO) frameworks. This includes the operational-structural (OS) framework, the procept framework, and the APOS (action, process, object, schema) framework.

Chapter 7 presents a summary of the study and conclusions that can be drawn from it, comparing the results of the analyses. This includes views on *concept*, views on some related notions, and comments on how different views relate to each other. Further, findings of how language is used in the texts are collected. Finally, in Chapter 8 the results and the methodology are discussed. Additionally, the chapter includes comments on how the used philosophy may be interpreted from the field of mathematics education. The thesis ends with an example of how the theoretical results may be used in future empirical research.

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2 Philosophical discussions of *concept*

In contemporary discussions of *concept*, one can discern traces of a long philosophical tradition, from the ideas of Plato and Aristotle, via medieval discussions of the problem of universals, with views that were later developed within different epistemologies. Since these affect the views that are held today, it is worthwhile to devote some pages to this history. Mostly, such discussions have been held within a philosophical context, where there are many positions and approaches relevant to contemporary views. This notwithstanding, there is a risk that an overly detailed description here of this background removes focus from the study itself. This is the reason that this chapter focuses only on major features. The story told is a simplified, rather rough, version aiming at setting a context for the study[7]. An aim is to establish a terminology that will be used further on for describing views on *concept* in mathematics education.

When conducting a study about concepts in mathematics education, the question of the differences between mathematical concepts and other types of concepts arises. A stance in the thesis is that the meaning of the term 'concept' in different fields should basically be the same, but that there are different types of concepts, with different features. Features of different types of concepts affect interpretation of texts within different fields. In mathematics education texts, most often the view on *concept* is to be interpreted as describing mathematical concepts. This justifies the partial focus on the nature of mathematics in this chapter. Another reason for writing about philosophy of mathematics is that there is a mutual exchange between mathematics education and philosophy of mathematics, where mathematics education uses philosophy and vice versa. Examples where philosophy is used in mathematics education may be seen in Sfard (2008), who refers to Wittgenstein (1953/1992), in Tall (2001), who refers to the logicist ideas of Hilbert, Russell and Frege, and in Semadeni (2008), who refers to both logicist and empiricist ideas. In addition, a fairly recent example of mathematics education influencing the philosophy of

[7] For more extensive explanations and formal definitions, I can recommend the *Stanford Encyclopedia of Philosophy*: https://plato.stanford.edu/

mathematics is Cole (2008) discussing social constructivism as a philosophy of mathematics.

When using views and approaches from philosophy in mathematics education, there is a risk that the philosophical context is less known to the reader. The texts referred to use philosophical vocabulary that is often taken for granted. I have consciously tried to decrease the density of philosophical terms, and different positions are exemplified rather than elaborated on in more detail, in order to increase readability.

As the content can, to a large extent, be seen as being in the public domain in philosophy, this lowers the requirements for referencing. I have preferred to refer to encyclopedias with good reputations (e.g. the *Stanford Encyclopedia of Philosophy*[8]) and compared the descriptions there with other sources. The content of the rest of the thesis has guided the selection of ideas, which should be seen as examples from the philosophical discussions. Note that all classification and generalisation of ideas means simplifications of the positions: at the same time as it offers possibilities for comparison, it restricts accuracy. Hence, the content in this chapter, as well as in Chapter 3, should be seen as somewhat superficial.

2.1 A story about concepts

This story begins in Athens in a dialogue between Plato and his disciple Aristotle around 300 BCE. The discussion concerns roses, and how we can know that roses of different kinds and colours all share the same property of being a rose. What does it mean to be a rose? Where is this meaning situated? And how can we have knowledge about it? One way of discussing these questions is by looking at the nature of the concept *rose* and other types of concepts.

Plato and Aristotle had different ideas about what concepts are. In fact, one might make the mistake of thinking that these ideas are completely opposite to each other. However, they agree that there is an abstract object called a 'universal', or a concept, that contains the essence of what it is to be a rose. This means that we as human beings are not just conscious of specific roses with different kinds of colours and appearances, but if we have enough experiences and if we are sufficiently intelligent we also have knowledge about the nature

[8] https://plato.stanford.edu/

of the concept *rose*. The reason that concepts are important is that if we did not have concepts, then all concrete objects would be new to us and we would not know anything about roses. (Klima, 2017; Russell, 1917/1959)

Further, according to Plato and Aristotle concepts must have some kind of existence. The argumentation between Plato and Aristotle concerns where these concepts exist, or are situated, and the relationship between the concept *rose* and physical roses. According to Plato, concepts are ideas that exist in a non-mental and abstract world. This world contains concepts that have references including physical objects, such as the concept *rose*, and also mathematical concepts, such as *circle*. The roses in the physical world are imperfect copies of the concept *rose* and different round things are imperfect copies of perfect circles. Aristotle disagrees with his master and refutes the idea of an ideal world. Instead, in his view the concept *rose* exists as possibilities in the physical roses and arises in the human mind as an abstraction, where the concept *rose* becomes a mental image of an ideal rose. (Klima, 2017)

One can see two underlying distinctions in this debate. First, there is a distinction between concrete and abstract objects. While concrete objects are the objects of everyday life, such as roses and books, abstract objects do not exist in space-time. Instead, they can be thoughts, either thoughts of a single person or thoughts that are common to a group of people. An example is redness, where redness as such does not exist in the real world, but is the common property of red objects. Other examples are mathematical objects such as numbers or functions, which do not exist in the physical world.

In the two positions above, there is an agreement that concepts are abstract objects. What distinguishes the two views is whether concepts are mental, as in Aristotle's view, or non-mental, as in Plato's view. Consequently, the discussion acts on three different arenas. The first one is concrete, containing physical objects. The second is a non-mental and abstract arena, where concepts exist from a Platonic perspective. Finally, the third arena is mental and abstract, and this is the place for concepts from an Aristotelian perspective. In brief, these can be referred to as a concrete, a non-mental and a mental arena (Figure 2.1).

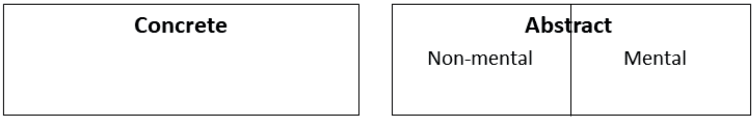

Figure 2.1 Three different arenas

There are different views on the nature of these arenas. For example, in an Aristotelian view there are no such things as abstract and non-mental objects. Therefore, this arena does not exist in such a view. Further, a philosopher claiming a physicalist position, which appeared in the 19th century, would say that the only existing substance is concrete. In his or her view, concepts may be neural structures that are part of, or use, the sensory-motor system of our brains. (Stoljar, 2017)

However, going back to the two positions considering concepts as abstract objects, the discussion continued over the years, and the views of Plato and Aristotle had great impact. In a medieval and Christian version of the Platonic position, the so called neoplatonist view, concepts were principal, immutable and eternal forms of things that were contained in a divine understanding of the world. One result of the interest in Greek philosophy at this time was that from the middle of the 12th century into the 13th century, over 42 of Aristotle's books were copied, translated and applied to Christian thought. Albertus Magnus, just to mention one example, interpreted and systematised most of the works of Aristotle in accordance with the Church's doctrine, and his disciple, Thomas Aquinas, wrote a dozen commentaries. This resulted in a Christian reception of Aristotelianism in Western Europe. (Wildberg, 2019; McInerny & O'Callaghan, 2018; Führer, 2019)

Simultaneously with this development, the realist positions were challenged by those claiming that such views were incoherent. The argument for this was that an abstract object could not exist in many physical instances at the same time, and constitute the substance of these physical objects. Unlike the two realist views described above, two non-realist positions appeared that denied that concepts existed independently of the human mind. First, a nominalist position appeared at the beginning of the 12th century, advocated by for example William of Ockham and Peter Abelard. In this view, universals were mere terms and the only things existing in the world were physical objects. Consequently, there was nothing in the individual rose but the rose itself.

According to this view, universality was not an ontological feature of the world, but a semantic feature of language. (King & Arlig, 2018; Spade & Panaccio, 2019)

When discussing language, a distinction between syntax and semantics is useful. Syntax, on the one hand, is the rules governing grammar, and a syntactical issue is how we may construct sentences. Semantics, on the other hand, concerns the meanings of terms and how terms refer. This leads us to another distinction, the one between terms, classes of objects[9] that a term can refer to (in some views called 'references' or 'extensions') and the meaning of terms (in some views called 'senses' or 'intensions').

In classical philosophy of language, different types of expressions get their meanings in different ways. The noun phrase 'this is a rose' gets its meaning from what it means to be a rose, referring to the class of roses. The verb phrase 'Alma runs', on the other hand, gets its meaning from what an individual called Alma is doing right now, referring to situations where there is a person called Alma who is running.

Ockham and Abelard counted the terms 'rose' and 'run' as universals, since they referred to more than one object or situation. However, they did not think that terms had abstract meaning, and that there were such things as senses. Instead, in their view, the term 'rose' signified the roses themselves, and there was no meaning apart from the actual roses. Consequently, all we need to assume is that two roses are more similar to each other than either of them to, for example, a peony, and that all flowers are more similar to each other than any of them are to an animal, and so on. (Klima, 2017; King & Arlig, 2018; Spade & Panaccio, 2019)

The second non-realist position accepted the existence of concepts, but not an existence independent of the human mind. This position is called conceptualism, or psychologism, and arose as an intermediate between realism and nominalism. In this view, the mind has the ability to create concepts, which are abstractions in the mind. During the 17th century, it was embraced by early modern thinkers, such as John Locke, and until the 20th century, this was the most common view on *concept*. (Balaguer, 2016)

[9] I will use 'class' throughout the thesis in an everyday way in order to avoid a context of set theory or category theory, which would not be relevant to the study.

Summarising the discussion of the problem of universals, we end up with four approaches: *Platonic realism*, *Aristotelian realism*, *nominalism* and *conceptualism*. As with all categorisations and generalisations, these are simplified versions of the different positions and a single specific view can in fact combine different approaches.

On entering the 17th century and the current era, where the dominant approach was conceptualism, the discussion of concepts can be regarded in the light of an epistemological discussion between rationalists and empiricists. The question concerned to what extent we can obtain knowledge without sense experience, that is if we can develop mental concepts without experience from the physical world. According to rationalists such as René Descartes, Baruch Spinoza and Gottfried Wilhelm Leibniz the answer is that we can. From their perspective, knowledge exceeds the information provided by our senses and we can create some concepts as part of our rational nature; reason can provide additional information about the world and we develop some knowledge by means of our intuition, or by deduction from already known facts. (Markie, 2017)

Empiricists such as John Locke, George Berkeley and David Hume defended the position that sense experience is the only source for knowledge. In their view, we do not have innate concepts and reason alone cannot give us any new ones. The total conceptual structure can be described as a mirror representing facts about the real world, and their relations. This mirroring occurs because our concepts are grounded in experience of the world; concepts either have references including physical objects, or are composed of other, more basic, concepts with references including physical objects. This in turn ensures that by exploring this mental structure, we can obtain knowledge of the world. (Markie, 2017; Jenkins, 2008)

In a rational view, mathematics has a special position, offering a model of rigour for rational inquiry, giving tools and foundation for other sciences, such as physics. Mathematics differ from science in that mathematical objects are all abstract. Numbers and circles do not exist in the physical world. In physics, on the other hand, some concepts have references that include concrete objects and other concepts have references that include abstract objects[10]. This relates to the distinction between analytic propositions, which are true based on the meanings of the terms involved and on how these are related, and synthetic

[10] While *gold* is an example of the first kind, *acceleration* is an example of the second kind.

propositions, which are true based on how the terms involved refer to the physical world and facts about the world. Mathematics is often seen as analytic, where mathematical knowledge is obtained by deduction from already known facts. Consequently, it is perhaps natural to have a rationalist view on mathematics rather than an empiricist one. A relevant question is then what the nature of mathematics is, from an empiricist perspective. Further, how can we develop mathematical knowledge, if all knowledge depends on sense experience?

There are several proposed answers to these questions. One way of answering appears to be what may be called mathematical empiricism, advocated by for example John Stuart Mill in the 19th century, and contemporarily by Carrie Jenkins. Consider 3 + 5 = 8 as an example. In an empiricist view, the signs '3', '5', and '8' refer to sensory data, where the mental number-concept *three* can be developed from the experience of the amount of three things, in different versions. The number-concepts *five* and *eight* can similarly be developed from the amounts of five and eight things. Even the concept *addition* has an empirical ground, developed from the experience of combining different groups of things, and a child can grasp 3 + 5 = 8 by thinking about what it means to combine three fingers with five fingers. The statement is then knowable through examination of our empirically grounded concepts. Consequently, in this view, there seems to be no difference between how we acquire knowledge about roses and how we acquire knowledge about numbers, from the perspective that the knowledge is empirical. (Jenkins, 2008)[11]

Another answer to the question of how we can develop mathematical knowledge distinguishes between different kinds of knowledge, where some knowledge comes from sense experience and some does not. In this way, we can be rationalists in mathematics and empiricists when it comes to science. Immanuel Kant advocated such a solution and distinguished between empirical knowledge, based on experience of the physical world, and knowledge a priori, which was independent of experience and arose in pure reason[12]. Mathematical concepts are developed a priori. Further, Kant distinguishes between geometric

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[11] Jenkins' theory of concept-grounding for mathematics has been criticised by e.g. Sjögren and Bennet (2014). The criticism points out that the theory deals almost exclusively with simple arithmetical concepts and that the conclusions therefore go too far. For instance, the empiricist view cannot be used for extending mathematics to the areas of infinity or set theory.

[12] The terms 'a priori' and 'a posteriori' describe different types of knowledge, where knowledge a priori involves necessary conclusions and knowledge a posteriori involves conclusions based on sense experience.

concepts, which are based on pure intuition of space, and arithmetic concepts, which are based on pure intuition of time. Concepts of numbers are then developed through addition of units in time. (Markie, 2017; Kant, 1783/2004)

Hitherto, three different views on the nature of mathematics have appeared, directly or indirectly, in this historical background. The first one is mathematical Platonism, in which non-mental mathematical concepts exist in an ideal world, which may be discovered by human beings. The second one is mathematical empiricism, in which mental mathematical concepts have an empirical ground and are built on sense experience. The third view is the one advocated by Kant, in which mental mathematical concepts arise in reason, not depending on sense experience, but where mathematics is still about the physical world. For Kant, mathematics is a priori, but synthetic.

During the 17th century, with Leibniz as a central figure, philosophy of mathematics began to focus on the relation between mathematics and logic, which dominated the discussions until the 20th century. In this work, philosophers such as Gottlob Frege and Bertrand Russell had great impact. The common challenge was the psychologism in Mill's empiricism, which threatened the status of mathematics as certain knowledge (Frege, 1884/1950). In order to get rid of psychologism, Frege (1884/1950) tried to reduce mathematics to logic and by that give mathematics a rigorous foundation. According to him, our reason for believing that 3 + 5 = 8 is not that we have found that an amount of 3 things and an amount of 5 things together make an amount of 8 things. Instead, the number 3 is a sum of three ones, 1 + 1 + 1. In line with that '3 + 5' and '8' refer to the same object, 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1. Still, 3 + 5 = 8 contributes to our knowledge, since '3 + 5' and '8' have different senses.

Unfortunately, the logistic idea failed, since the basic conditions turned out to be contradictory, as was shown via Russell's paradox (Russell, 1902/1967). The failure of logicism was one reason for David Hilbert to start a programme for an axiomatic formalisation of mathematics, starting with a rigorous formalisation of geometry. Parallel to the work of Frege and Russell, Richard Dedekind had, during the 19th century, worked on an axiomatization of analysis by reducing analysis to arithmetic, as a reaction to problems within differential calculus. This work was based on the Dedekind-Peano axioms for the natural numbers, which offered a rigorous foundation for arithmetic. Hilbert's programme was a continuation of that work, with the purpose of giving mathematics as a whole a rigorous foundation. (Reck, 2017; Zach, 2016)

Hilbert's formalism offered a view on mathematics that may be seen as an alternative to traditional Platonism, mathematical empiricism, and Kant's perspective. There are also other views regarding the nature of mathematics and what mathematical concepts are (Horsten, 2019). One way of distinguishing mathematical concepts from non-mathematical concepts is by describing them as 'well determined' and 'well defined'. Such views are presented below, after some clarifications.

That a concept is not well determined may depend on vagueness or ambiguity. If a term is vague, this means that the bounds of the extension of the term are indeterminate. An example is the term 'bald'. Some people are definitely bald. Others are definitely not bald. However, one cannot tell how much hair someone needs to have in order not to be bald and there are borderline cases. Concepts that are not well determined can also be ambiguous. This means that a term has different meanings. Take the term 'dress' that may refer to an outfit or to a verb as an example. Most often the meaning of 'dress' can be understood from the context, but not always. In the expression 'Axel saw her dress', it is hard to tell which meaning is intended. Further, as mentioned in Chapter 1, ambiguity in texts in mathematics education can appear when a term has different meanings in different parts of a text.

Explications can be used for pointing out which meaning is intended in a text, and for replacing vague and imprecise concepts with more exact ones. When Carnap (1947) describes explications, he describes the task of making a vague concept more exact, or the task of replacing a concept used in everyday life or in an earlier stage of scientific development with a more modern scientific concept. In defining, one often tries to find an explication that is both sufficient and necessary (Murphy, 2004). If it is sufficient, an object that fulfils the conditions of the explication is a member of the class of objects that is the reference of the explicated concept. If it is necessary, an object that does not fulfil the conditions of the explication is not a member of this class.

Some concepts are well determined, which means that they are unambiguous and not vague. There are those who, such as Priss (2017), suggest that mathematical concepts are well determined. Further, that a concept is well defined means that its definition offers a unique interpretation. This requires that the concept is well determined. In an ideal world, all people who have knowledge about the concepts involved in the definition would interpret the definition in the same way. In physics, the concept *gravitation* has different definitions in Newtonian mechanics and in general relativity, and will perhaps be given another definition in future theories. In mathematics, however, there are those (Sjögren & Bennet, 2014) who argue that mathematical concepts have often unique explications.

The notion *concept* is not a mathematical notion, but a notion from areas such as philosophy, psychology and education. There are also other notions, or concepts, used in mathematics education that are non-mathematical. Concepts such as *discourse* and *student* are examples of that. In fact, concepts in mathematics education may have been taken from the different fields that mathematics education has been developed from. For example, *discourse* comes from linguistics, *reification* is taken from psychology, and *circle* is a mathematical concept. Further, some concepts are developed within the field of mathematics education itself. *Procept* (Gray & Tall, 1994) can be taken as one example of such a concept.

Naturally, texts within mathematics education may include mathematical concepts, which according to some views should be well determined, but also non-mathematical concepts, which may be vague and ambiguous rather than well determined. Further, the approach taken and the framework used in the text affect which non-mathematical concepts appear. Hence, different types of non-mathematical terms appear in different texts, something that is claimed by Inglis and Foster (2018):

> we would expect a research paper that reports a constructivist analysis of geometry learning to contain words such as triangle, circle, and angle but also words such as schema, constructivism, and interaction. (Inglis & Foster, 2018, p. 469)

Even when concepts are well determined and have unique explications in mathematics, they may not be well defined. Note that it is not possible to define every concept. As an example of that, there are real numbers that do not have names, since there is a countable number of names, but an uncountable number of real numbers. Again, the number of possible definitions, assuming a finite or countable language, is countable. Naturally, one can give names and definitions to numbers that lack names, but there will always be concepts which do not have names and that are not defined. Furthermore, it can take some time before a definition takes its final form. The concepts *function* and *set* have both had a long and tricky history.

The discussions of the nature of mathematics continued during the 20th century. In the 1930s, Willard Van Orman Quine, published an axiomatic set theory, called New Foundations. Later, in the early 50s, he contributed to the epistemological discussions between rationalists and empiricists by criticising the distinction between analytic and synthetic, described above. This criticism is based on the observation that no satisfactory explanation of analyticity had been given, and that one cannot distinguish between analytic and synthetic propositions. The argument is that in order to distinguish between these two, one must use synonymy, which can be understood as interchangeability. For example, the statement 'all and only bachelors are unmarried men' holds, because 'bachelors' can be interchanged with 'unmarried men'. However, in the statement "bachelors' has fewer than ten letters', 'bachelors' cannot be interchanged with 'unmarried men'. In order to explain this, one needs to use analyticity again, and one ends up in an infinite regress. Consequently, it is impossible to draw a line between synthetic statements, depending on sense experience, and analytic statements. An empiricist answer to this criticism would be to claim the idea that every statement is constructed from terms referring to sense experience. However, Quine shows that this view is as problematic as the distinction between analytic and synthetic, for similar reasons. (Forster, 2018; Quine, 1951/1985)

Quine's solution was to take the holistic stance that all statements are interconnected and that the whole system of science is verified or falsified at the same time. Therefore, it is misleading to consider the meaning of a single term or sentence. A term such as 'rose' is connected to 'flower', which in turn is connected to 'dandelion', which in Sweden is connected to a political party, and so on. By similar chains, every term is related to every other term in the language, and the change of the meaning of one term affects the meanings of related terms. According to Quine, meanings at the edge of the system are more likely to be affected by sense experience than those of the interior, with rose being at its edge and logical concepts at its centre. Further, physical objects and non-physical objects are not of different types; they are all sources for our conceptions, although there is a floating boundary between concrete and abstract. In this view, the meaning of 'rose' is not just determined by the class of all physical roses and how members are pointed out by the sense, but it is determined by how the term 'rose' is used in communication. (Jackman, 2017; Quine, 1951/1985)

Another advocator of a holistic view, active at the same time as Quine, was Ludwig Wittgenstein who claimed that the properties of the objects included in the reference of a concept form a complicated network of similarities, called family resemblances, and that concepts are related to each other in many different ways (Wittgenstein, 1953/1992). For instance, something is called a number because it has a relationship with several other things that have previously been called a number:

> […] this can be said to give it an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres. (Wittgenstein, 1953/1992, p. 32)

The holistic idea opposed, among other things, the idea of compositionality: that the meaning of a complex expression is determined by the meanings of the terms included in it and the rules for combining them (Jackman, 2017). Hence, you cannot explain the content of 'Alma is running' based only on the content of the terms 'Alma', 'is', and 'running', if advocating a holistic view.

With these ideas and other discussions appearing during the second half of the 20th century, the idea of mathematics as certain and immutable knowledge was challenged. With Wittgenstein, among others, a radical constructivist view on mathematics appeared, in which mathematical propositions did not concern mathematical objects (Rodych, 2018). In that view, the human mind invented mathematics and did not discover it. Later, during the early 1980s, the claim that mathematical concepts do not depend on culture was further questioned, and philosophers, mathematicians, and mathematics educators started to discuss mathematical practice from a cultural perspective.

Within philosophy, Imre Lakatos' (1976/1979) view on concept development had an impact on views on mathematics. His position concerned how mathematical concepts develop through history, from an intersubjective perspective, and how new concepts are created. This may be seen as a combination of empirical and rational development. First, the empirical development is based on observations of the physical world. Second, the rational development is based on proofs and refutations. By exploring mathematical theorems and proofs, and stretching the boundaries of the relevant concepts, mathematics can develop. This is a process of guessing about relationships in mathematics followed by giving counterexamples. In Popper

(1934/2002), this is formulated in terms of a context of discovery versus a context of justification. (Lakatos, 1976/1979)

During the 1980s, mathematicians such as D'Ambrosio (1985/2004) and Bishop (1988/2004) started to reconsider what mathematics is, based on an idea that different cultures have different mathematical practices. In such a view, mathematics is not merely an academic activity, given that mathematics is practiced differently among national-tribal societies, labour groups, children of a certain age bracket, professional classes, and so on. Hence, advocators asked for an extended definition of mathematics. In D'Ambrosio (1985/2004), traditional mathematics, what is learnt in school, is called academic mathematics and the term 'ethnomathematics' is used for the phenomenon that mathematics is practiced differently within different cultural groups. A broader view on mathematics includes these aspects[13]. This idea has later been developed further, and today Skovsmose (2016) is one of many who distinguish between different forms of mathematics as a practice, such as everyday mathematics, engineering mathematics, academic mathematics and ethnomathematics.

In recent times, discussions about concepts have taken place within different fields, resulting in a multitude of views on *concept*. First, the discussion has continued within philosophy of language, where concepts are often seen as some kind of senses, which again are non-mental meanings of terms. Second, there are discussions within concept research in cognitive science, where concepts are often seen as some kind of mental representations. Third, there are several frameworks studying concepts within educational research, which are based on different views on *concept*. Murphy (2004) describes the diversity in the research on concepts as ranging from mathematical models tested in artificial category-learning experiments, to anthropological studies in rain forests, to linguistic analyses of word and phrase meaning.

Independent of field, many views are based on the assumption that people organise the world through representation. Representation in a wide sense means that an object in one of the three arenas described above (see Figure 2.1) can point to an object in one of the other. For example, a mathematical circle can be represented by $x^2 + y^2 = 9$, or a round physical object. As another example, a mental image of a circle can represent the non-mental object *circle*.

[13] Note that it is possible to use the term 'mathematics' for school mathematics and the term 'mathematical practice' for the phenomenon that mathematics is practiced differently in different contexts. However, this is not what D'Ambrosio advocates.

However, *representation* is also a concept on which there are various views. In social representation theory (Potter & Edwards, 1999), representations are considered primarily cognitive. In discursive psychology, on the other hand, representations are seen as objects that people construct in speech and text (Potter & Edwards, 1999). These two different meanings of 'representation' can be referred to as 'internal representation'[14] and 'external representation'. While internal representations of a circle are structures that enable people to perceive, interpret and categorise the physical world, external representations are different round objects, pictures, equations and other descriptions of circles. These two types of representations are situated in two different arenas, a mental arena and a concrete arena.

One conclusion that can be drawn from the history of concepts, and its multitude of views, is that it would be naïve to hope that these discussions would end up with one single view where the notion *concept* is well determined. Maybe, this is not even desirable. In addition, when discussing concepts there are some distinctions that are important to consider. One is the distinction between syntax and semantics, between terms and meanings of terms. A similar distinction is the one between terms, classes of objects that the terms refer to and the content of the terms. A third distinction is the one between a concrete, a mental, and a non-mental arena.

Many philosophical contradictions appear because of a misuse of language when these arenas are not distinguished. A simple example where syntax is not distinguished from semantics is seen in the sentences:

> The chapter has 42 pages.
>
> The chapter has ten letters.

Most likely, the chapter contains more than ten letters. Instead, it is the expression 'the chapter' which contains ten letters. In this example, the mistake may not have consequences for our understanding. We can judge that it is not reasonable for a chapter to have ten letters and we can thus conclude that it should be written that the expression 'the chapter' has ten letters. In other examples, mistakes can affect our understanding. Take the following sentences, from a context of mathematics education, as examples of this:

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[14] In this thesis, internal representation is mostly called mental representation.

> The symbols have properties specified in the axioms.
>
> The term 'concept' refers to geometrical figures, numbers, algebraic expressions etc.

In the first sentence, symbols are part of syntax and should not have properties in the sense of properties specified in axioms. Written symbols can have properties, such as being large or small, but this is not what is meant here. What actually have properties are concepts, but this is a semantical issue. Similarly, in the second sentence, the term 'concept' may include geometrical figures and numbers in its references. However, algebraic expressions are part of syntax. A consequence of these category mistakes is that it is impossible to understand the meaning of these sentences, which complicates the interpretation of the view on *concept* in this part of the text.

As another example, in Section 1.1, it is mentioned that in Sfard (2008), the notion *concept* is defined as a symbol together with its uses. Here, the symbol is combined with its meaning, seen as usage in communication. If elements of different nature are combined, then the nature of this combined object becomes hard to understand. Further, as the explication is not in line with other explications, where concepts are often seen as the meaning of symbols, the nature of concepts in Sfard (2008) is not similar to the nature of concepts in other views.

Regarding the three arenas, the mistakes are of a different kind. We do not ask our children to go and fetch redness, but we ask them to fetch something red. However, sometimes we can mistake the nature of the phenomenon that we experience. When a teacher calls a parallelogram 'a trapezoid', it may be an indication that in her conception, parallelograms and trapezoids are mixed up. She does not realise that this is her own experience of mathematics, but believes that this is mathematical truth. Consequently, mental experiences and facts within mathematics as a field of knowledge are mixed up. As another example, when it comes to views on *concept*, different views may appear in the same text. Sometimes a sentence involving a view where concepts seem to be mental is followed by a sentence where concepts seem to be non-mental. It may even be the case that a mental and a non-mental view appear in the same sentence. In such cases, a mental and a non-mental arena are combined.

3 Building a tool for analysing views on *concept*

> […] although the notion of a concept has been employed in various ways, there are not all that many precise theories of concepts. (Zalta, 2001, p. 335)

The purpose of this chapter is not to provide a philosophically precise theory of the notion *concept*, but to clarify differences and similarities between different contemporary views. These views are interpreted from texts in the common field of philosophy of language and concept research within cognitive science, collected from a configurative[15] literature review. The purpose of this review is to find some key references and to get an overall picture of the contemporary discussion of concepts within philosophy. Further, this review should be seen as separate from the literature review in mathematics education, described in Section 4.1. It can be considered a substudy, with the purpose of elaborating a framework for the concept analysis, facilitating interpretation of views on *concept* in texts in mathematics education.

The texts have been found in different ways. First, articles, discussions and reviews from two journals, *The British Journal for the Philosophy of Science* and *Philosophia Mathematica*, were chosen from the time period between 2000 and 2014. These journals are highly ranked within the field, and together they offer opportunities to compare the discussions regarding concepts in science and concepts in mathematics. In *The British Journal for the Philosophy of Science*, the titles which were selected were those of articles, discussions and reviews including the terms 'mathematics' (but not applied mathematics), 'arithmetic', 'number'[16], 'concept', 'conception', 'indispensability'[17], 'representation', 'meaning', 'objects', 'cognitive' (but not cognitive science) and other titles which I associated with mathematics, arithmetic or concepts. The abstracts of these texts were then read, together with some other abstracts that seemed interesting from the perspective that it offered new insights into the nature of concepts. In

[15] The distinction between configurative and aggregative literature reviews is described in Section 4.1.
[16] The reason for searching for views on *number* is that the number-concept is fundamental in the field of mathematics, underlying a wide area of mathematical domains.
[17] I was looking for views related to the view in Quine (1951/1985; 1960).

Philosophia Mathematica, all abstracts and introductions of reviews were read. When reading, my interest was in ontological views on mathematics, arithmetic or numbers and also views on concepts and cognitive structures. My interest was not in philosophy of arithmetic or philosophy of number, but in philosophy of concepts and number-concepts. Bibliographical articles were avoided. This review resulted in a list of key researchers and also some books, where some of them were anthologies and others not[18]. Reading the abstracts gave an overview of the field of philosophy in these journals, and it also led me to the field of concept research within cognitive science. There, I found a discussion regarding the nature of *concept* which started about 50 years ago. This review, together with *The Stanford Encyclopedia of Philosophy*, Google searches, recommendations from colleagues, and references that were found when reading texts, provided the material from which the views selected in this chapter were chosen.

The views were chosen depending on how often they are referred to, how clear the description of *concept* is and whether they contribute new perspectives to the discussion[19]. While the purpose of Chapter 2 is to give a brief introduction to the basic philosophical terminology used in the thesis, from a historical point of view, the purpose here is to mirror distinctions within contemporary views. However, in order to do that, some views from Chapter 2 have been repeated.

From my understanding of the texts, a philosophically flavoured analysing tool is built for interpreting texts in mathematics education. As seen in Chapter 2, one dimension of the nature of *concept* visible in the discussions concerns two positions, where concepts are considered by some (Plato, 399 BCE/1997a; Frege, 1892/1985; 1892/1951; Quine, 1960; Zalta, 2001) as non-mental and by others (Hume, 1888/1978; Jenkins, 2008; Murphy, 2004; Carey, 2009) as mental. A mental view, on the one hand, may either be useful when describing the development of mathematics from an empiricist position, or when describing how people develop concepts when learning mathematics. A non-mental view, on the other hand, may rather be of use when describing how concepts develop in formal mathematics from a Platonic realist position, or

[18] As an observant reader may notice, none of the articles in these journals are actually referred to in the thesis, since I found other works by the same authors, which were more appropriate for the purpose. Further, most references have been found through the anthologies.

[19] Of course, the intention here is not to give anything close to a complete overview of the philosophical discussions about *concept*.

when describing how concepts develop in a cultural context, from a linguistic perspective.

Another dimension concerns whether concepts are subjective or intersubjective, or even objective. The difference between a subjective and an intersubjective view can be shown using the example of a student, Karin, who has a subjective concept *circle*. The teacher tries to compare Karin's concept *circle* with her own concept *circle* in order to assess Karin's knowledge. In that comparison, an intersubjective common understanding among mathematics teachers can be used. In addition, hopefully the teacher's concept has features that are similar to the features of the non-mental intersubjective concept described in the curriculum.

A third dimension concerns how concepts are structured. In Figure 3.1, two different models for the cognitive structure underlying the current discussions are represented. The left-hand picture represents the molecular model claiming that concepts are hierarchically structured, where some concepts are more complex than others and could be defined from more basic concepts. Such a view may be represented by Katz (1972/1999) and Jenkins (2008). The right-hand picture represents the holistic model that concepts interrelate with others in a web-like structure, in which no concept is more basic than another. This view may be represented by Wittgenstein (1953/1992) and Quine (1960; 1951/1985). The last model is based on the idea that concepts are related in complicated networks, where it is not possible to clearly delimit concepts.

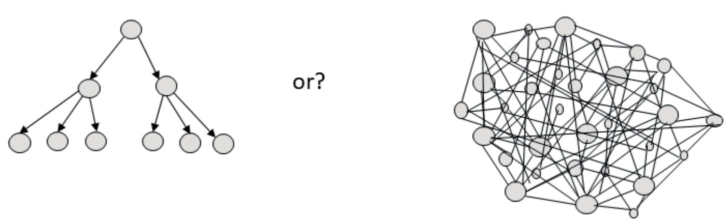

Figure 3.1 Concepts considered as molecular or as holistic

In this chapter, the distinction mental versus non-mental is discussed in Section 3.1, the distinction intersubjective versus subjective is discussed in Section 3.2, and the distinction molecular versus holistic is discussed in Section 3.3. Further, in Section 3.4 the distinctions are combined in a frame, which is then used for analysing views on *concept* in texts.

3.1 The distinction mental versus non-mental

This first distinction addresses the fact that some views are concerned with concepts as non-mental objects, and others with concepts as mental representations. The discussion is divided into two parts. First, views considering concepts as non-mental will be the issue in Section 3.1.1. The diversity of examples claiming that concepts are non-mental can be regarded from different directions; Plato, Frege, Quine and Zalta have views that differ in many aspects, even though they all consider concepts as non-mental. Second, views considering concepts as mental will be the issue in Section 3.1.2. These involve different ideas about the complexity of the cognitive structure. In both these sections, the views are presented in historical order. Finally, in Section 3.1.3, the different views are compared and commented on.

3.1.1 Views where concepts are non-mental

As seen in Chapter 2, it may be claimed that concepts are non-mental objects from a Platonic realist position. Such a claim means that for example the concept *birthday cake* is not something that is developed in our minds, but in Plato (399 BCE/1997b) there is an idea of an ideal world, including concepts, that we learn about. When the fictive person Petter is talking with his daughter Alma about birthday cakes, Petter shares knowledge of the concept *birthday cake* that he in turn has developed from his parents, and Alma develops an understanding about the same concept. Further, when teaching about circles, there is a common meaning of the term 'circle' that can be described in an encyclopedia or by someone who is knowledgeable in mathematics. The teacher's understanding of the common concept allows her to talk about circles, so that her students can develop their own understanding of the concept.

The logicists from around 1900 are examples of philosophers having a Platonic approach to mathematics, where Frege may be taken as an example. He was one of those claiming that human thought cannot be understood just from studying mental processes:

> [The grasping of a thought] cannot be completely understood from a purely psychological standpoint. For in grasping [the thought] something comes into view whose nature is no longer mental in the proper sense, namely the thought; and this process is perhaps the most mysterious of all. (Frege, 1979, p. 145)

This point has, since Frege, been further developed during the 20th century, within the field of philosophy of language, by Quine (1951/1985; 1960), Katz (1972/1999), Peacocke (1991) and Zalta (2001), among others, who are all opposed to identifying concepts with mental representations. In this section, some different views are touched upon in order to exemplify different philosophical positions, all considering concepts as non-mental.

3.1.1.1 Concepts as senses

The view on *concept* in Frege (1892/1985; 1892/1951) is based on his distinction between *sense* (*sinn*), which is the linguistic meaning of a term, *reference* (*bedeutung*), which is the class of objects that the term refers to, and *image*, which is an internal image arising in the human mind. Even though those claiming that concepts are non-mental often assume that they are some kind of senses (Laurence & Margolis, 1999), Frege himself did not see concepts as senses, but rather as classes of objects, references. These are pointed out by senses in specific ways, through definitions or descriptions. This view is represented in Figure 3.2, where the oval represents the concept, the single lines represent connections between term and reference, between term and sense, and between sense and reference. The concept *birthday cake* is then the class of all birthday cakes together with a sense, which is a thought about what a birthday cake is. Another example is the mathematical concept *odd number*, which is the set {1, 3, 5, …} together with the sense, which could be *a number which may be written in the form 2n + 1 (where n is a natural number)*.

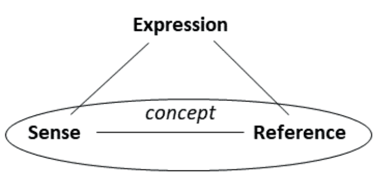

Figure 3.2 View on *concept* in Frege (1892/1985; 1892/1951)

Later, these ideas have been developed, and Peacocke (1989; 1991) is one of those who claim that concepts are senses in a Fregean meaning, roughly speaking. With this view we can modify Figure 3.2 to Figure 3.3.

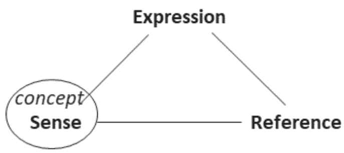

Figure 3.3 View on *concept* in Peacocke (1989; 1991)

3.1.1.2 Concepts as use in language

The view on *concept* described above is criticised by, among others, Quine (1960; 1951/1985) who has had a great influence in contemporary philosophy. He abandoned Frege's view on meaning and the idea that senses point out classes of objects in the world. Instead, for Quine (1960) a concept is the meaning of a term or expression, and we socialise concepts in communication, based on our mental representations. Consequently, concepts are non-mental and social objects. In this view, the reference no longer determines the concept. Hence, the meaning of 'birthday cake' is no longer determined by the class of all concrete birthday cakes and how members of this class are pointed out by the sense, but is instead determined by how the term 'birthday cake' is used. The idea that concepts relate to each other in many different ways, and that the meanings of terms overlap, is central. Quine (1960; 1969) considered that theoretical propositions get their meanings not as single expressions, but as larger blocks of theories, and claimed the holistic idea that an expression is meaningless except relative to a context. This indicates that it is impossible to tell if different terms refer to the same concept, and it cannot be judged whether the meanings of 'the square root of 16' and '√16' are the same, as Frege thought.

3.1.1.3 Concepts as modes of presentation

In Zalta's (2001) view, concrete objects such as roses exemplify properties and are discovered in an empirical way, while abstract objects such as numbers encode properties and are discovered with the help of logic. Both concrete objects and abstract objects can then be grasped by the human mind. In addition to the notion *sense*, Zalta (2001) uses *mode of presentation* as the different ways by which we can grasp an object, a property or a relation. While sense is the content of a linguistic expression, mode of presentation is the content of a mental representation[20]. Using the example of the birthday cake (Figure 3.4),

[20] Zalta uses 'mental token' or 'mental image'.

when Alma sees the cake she forms a mental representation of it, which is grasped through different modes of presentation. Some of them may represent concrete objects and substances, such as *strawberry* and *whipped cream*. Others may represent features of the objects, such as *round* and *red*. Still others may represent relations between the objects, such as *above* and *next to*. Further, not every mode of presentation can be expressed with a linguistic expression. Hence, the set of modes of presentations connected to linguistic expressions in this experience is a subclass of the class of modes of presentations, which is seen in Figure 3.4. (Zalta, 2001)

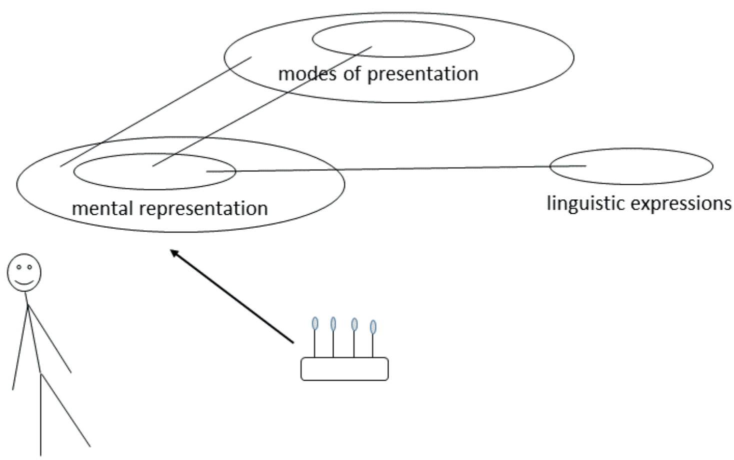

Figure 3.4 Mode of presentation

The difference between the view on *concept* in Peacocke (1989; 1991) and in Zalta (2001) is that in Peacocke (1989; 1991) concepts are senses in a Fregean way and in Zalta (2001) concepts are modes of presentations[21]. These concepts are non-mental objects encoding the content in the mental structure. What makes the view on *concept* non-mental is that, according to Zalta (2001), concepts do not exist in the individual mind even though they can be used to classify what is in the mind.

[21] In fact, Peacocke (1991) too uses *mode of presentation*, but in a way where *mode of presentation* is similar to *sense* in Frege's (1892/1985) view.

3.1.1.4 Summary

In this section, three examples of views regarding concepts as non-mental are examined:

1. The views in Frege (1892/1985; 1892/1951) and Peacocke (1989; 1991) are static objective views, where concepts are independent of culture.
2. The view in Quine (1960) is an intersubjective view, where concepts are meanings determined by how language is used.
3. In Zalta (2001), concepts are subjective contents of mental representations.

The example with the birthday cake can be used to show the different positions. In the first position, the concept *birthday cake* is immutable and does not change between different groups. In the second position, the concept *birthday cake* depends on cultural aspects. In the third position, the concept *birthday cake* is the content of an individual's experience of the cake. In this description, it sounds as if one important difference between the positions lies in the distinction between an objective, an intersubjective and a subjective view, which is something that will be further developed in Section 3.2.

Note that the views above are taken from contexts discussing philosophy of language. The position where concepts are non-mental has its origin in Platonism and has been adopted by the logicists, based on the work of Frege.

3.1.2 Views where concepts are mental

While concepts may be considered non-mental in philosophy of language, with Wittgenstein (1953/1992) as a possible exception, the representational theory of the mind (RTM) considers concepts as psychological entities, where thinking takes place in an internal system of representation (Margolis & Laurence, 2019). This position has a background in Aristotelian realism or conceptualism (see Chapter 2), and is held by classical empiricists, such as Hume (1888/1978). Today, contemporary empiricists, such as Jenkins (2008), psychologists, and cognitive scientists, such as Murphy (2004) and Carey (2009), claim that concepts are mental representations:

> If we have formed a concept (a mental representation) corresponding to that category (the class of objects in the world), then the concept will help us understand and respond appropriately to a new entity in that category. Concepts are a kind of mental glue, then, in that they tie our past experiences to our present interactions with the world, and because the concepts themselves are connected to our larger knowledge structures. (Murphy, 2004, p. 1)

When the fictive children Per and Nils, meet a dog, which happens to be a Labrador Retriever, in the street, they maybe hear his master call him Blue. Further, they see how Blue moves his ears, they can feel his soft fur and smell his scent. All these sensory inputs together create representations, individual concepts *Blue*. Later, they meet other dogs and recognise similarities and differences between these new dogs and Blue. With experiences of dogs of different kinds, in different situations, the concept *dog* is both formed and refined. Further, in a classroom students are supposed to develop mental representations of the mathematical object *circle*, which is abstract and hence not directly available to the senses. The access to the mathematical objects goes through external representations, such as pictures, graphs and written equations.

RTM has developed over the years and this section explores some aspects of this development. Here, views where concepts are considered as mental are used to exemplify different philosophical positions. While the first view, considering concepts as mental images, can be seen from a historical perspective, later discussions concern whether mental representations have language-like structure or not.

3.1.2.1 Representation as a mental image

In the Age of Enlightenment, views on *representation* only included representation of sensory impressions from the physical world. Early advocates of RTM called mental images of sensory impressions 'ideas', and Descartes, as one example, saw the relation between a physical object and the corresponding idea as a causality relation, where such ideas were considered faint copies of physical objects. British empiricists, such as Locke and Hume, were influenced by Descartes and began treating concepts as images:

> When I shut my eyes and think of my chamber, the ideas I form are exact representations of the impressions I felt (Hume 1888/1978, p. 3)

According to Hume (1888/1978), perceptions of the human mind could be divided into impressions and ideas, which are images of the impressions used in thinking and reasoning. In Figure 3.5, the arrows represent directions in the development of mental representations. With the example of the dog Blue, when Per sees Blue, he gets impressions from it, which are the sources for his ideas. (Margolis & Laurence, 2019; Furth 1969)

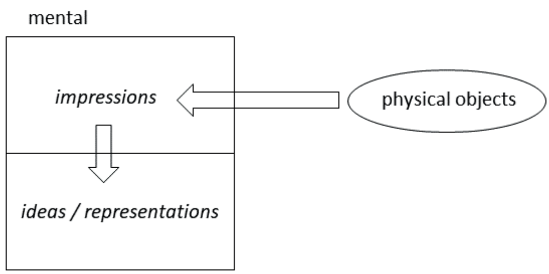

Figure 3.5 Representations as ideas

As another example of this view, when Kajsa is participating in a competition in sprint orienteering, she uses a mental map representing the layout of a city, based on the map that she has in her hands. There is an inherent 1-1-correspondence between her mental map and the concrete map, and in turn, between both and the city itself; the mental map implies corresponding information about the city and each house on this map represents a house on the concrete map.

It is easy to find counterexamples to this view, since there are abstract objects that do not represent concrete objects. For example, it is hard to explain how we can have ideas about infinity or acceleration in this view, since infinity and acceleration do not exist as concrete objects. Another mathematical example is variables, which do not represent mathematical objects in a 1-1-correspondence, but rather act as place holders or pronouns in ordinary language. Therefore, it seems as if some concepts, such as *infinity*, *acceleration* and *variable*, do not have equivalents in the real world. Furthermore, mental structures include ideas of properties and relations between different objects. This is why modern versions of RTM often take into account that much thought is not grounded in mental images (Margolis & Laurence, 2019).

3.1.2.2 Representation as a language-like structure

In a wider sense, a representation could be seen as a mental structure with a language-like syntax. Much thought is, in this position, grounded in concepts which are word-like mental representations. These are not always based on memories of impressions. In mathematics, some concepts, such as *infinity* and *variable*, are built from earlier concepts and are developed through an exploration of the axiomatic system. (Furth, 1969; Laurence & Margolis, 1999)

An example of such a view is found in the mathematical empiricism of Jenkins (2008), also described in Chapter 2, where concepts are sub-propositional mental representations which are related to propositional mental representations in almost the same way as words are related to sentences. Complex concepts are composed of more basic ones, and do not need to be direct images of particular physical objects, but rather the total conceptual structure represents the facts and mirrors the structure of the real world.

As an example, Faria's individual concept *five* may be developed from an experience of an amount of five things. Further, her individual concept *addition* may be developed from her experience of combining different classes of things together with the experience of thinking about what it means to combine, for example, three fingers with five fingers. A statement is then knowable through examination of our empirically grounded concepts.

3.1.2.3 Representation as a non-language-like structure

In Margolis and Laurence (2019), it is claimed that the so called language of thought hypothesis is not perfect and that the structure of thought lacks many of the properties which are associated with natural language. In the view of Carey (2009), a concept is a kind of mental representation, representing concrete or abstract objects, properties or events in the concrete or abstract world. The nature of these concepts can be explained relative to the mental structures in which they are embedded.[22] Here, there is a distinction between two types of representations, sensory representations and conceptual representations. When we come in contact with the world around us, impressions from different types of objects (1 in Figure 3.6) cause us to develop sensory representations (3 in Figure 3.6). These representations integrate with the conceptual representations (4 and 5 in Figure 3.6), which are embedded in conceptual structures. Further, conceptual representations may be divided into

[22] For a deeper discussion of the conceptual structure see Section 3.3.

theoretical conceptual knowledge (5 in Figure 3.6) and innate conceptual knowledge called 'core cognition' (4 in Figure 3.6)[23]. Theoretical conceptual knowledge is developed in communication and is affected by theoretical knowledge in our cultural surrounding (2 in Figure 3.6), sensory representations (3 in Figure 3.6) and core cognition (4 in Figure 3.6). As in Figure 3.5, the arrows in Figure 3.6 represent directions in the development of mental representations. (Carey, 2009)

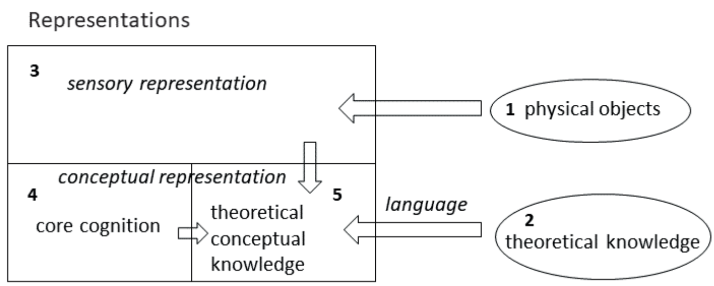

Figure 3.6 Different types of representations

Going back to the example with the dog Blue, the meeting with him causes the children to develop a range of sensory representations, including colours, movements, scents, and so on. First, Nils and Per may, from their concepts *number*, judge that there is one dog. Further, if Nils has theoretical conceptual knowledge about dogs, he can conclude that Blue is a Labrador Retriever and tell this to Per, and the younger boy may develop a similar knowledge. However, Blue's master can give even more information that develops both children's knowledge about dogs and Labrador Retrievers.

At the beginning of early childhood, the individual has access to sensory representations and core cognition. Later, this knowledge integrates with theoretical conceptual knowledge developed in a communicative surrounding. According to Carey, the difficulty of acquiring mathematics is the difficulty of acquiring the theoretical conceptual knowledge available in mathematics. Here,

[23] According to Carey (2009), there are studies indicating that animals without language can have mathematical concepts. In one such study, rats were trained to press a bar if presented with a sequence of two tones and another bar if presented with a sequence of eight tones. Further, Carey claims that infants have conceptual representations, before developing a language. As one of her examples, studies with 7 months old infants have shown that they are able to discriminate between 8 dots and 16 dots. Here I do not decide either for or against whether these examples actually show conceptual knowledge.

the ability to acquire natural numbers relies more on sensory representation and core cognition than other number concepts (*rational numbers*, *real numbers*, *imaginary numbers*, etcetera), which require that children transcend their innately given capacities and begin mastering culturally constructed ones. (Carey 2009)

3.1.2.4 Summary

Section 3.1.2 concerns different views on the nature of mental representations, in this section called concepts. Roughly speaking, three positions have been described:

1. A representation is a mental image representing physical objects in the world. Here, this position is exemplified by the views of Descartes and Hume (1888/1978).
2. A representation is a mental, language-like, structure, having elements which do not represent objects in the physical world. This position is exemplified by Jenkins (2008).
3. A representation is a structure having elements which cannot be described by language. This view is exemplified by Carey (2009), who claims that there exists innate conceptual knowledge.

Using the example of the concept *number*, if concepts are mental images of physical objects, then the concept *5* for a child, such as Faria, represent five objects and 1/2 may represent half an object or half of a number of objects. In line with that, the concept *number* excludes for example negative numbers, which cannot represent concrete objects. If concepts instead have language-like structure, then Faria develops the concept *number* over the years while communicating with her parents and others about the properties of different numbers, such as natural numbers and negative numbers, the relations between them and how different operations are used in different ways. In this view, negative numbers and complex numbers cannot refer to objects in the world, but are developed within a discussion in the community of mathematicians. In the third position, some of our mental representations of numbers are not dependent on language. In this view, Faria may have had conceptual representations before developing a language.

3.1.3 Comments on the distinction mental versus non-mental

In Section 3.1, different views on *concept*, exemplifying positions considering concepts as non-mental or as mental, are presented. There are both similarities and differences between these views. A similarity between the non-mental views is that concepts as senses and concepts as modes of presentations are abstract non-mental content of either linguistic expressions or mental representations. A difference concerns what these senses or modes of presentations are, which is an ontological issue, and whether they are objective, intersubjective or subjective, which is further explored in Section 3.2. A similarity between the mental views is that concepts are mental representations, while a difference concerns whether these representations are mental images, if they have language-like structure, or if they have elements that do not fit into a language-like structure. This latter issue concerns how mental representations are structured, which is further explored in Section 3.3.

In addition, there is a similarity between the view in Zalta (2001), claiming that there are concepts that are not connected to linguistic expressions, and the view in Carey (2009), claiming that children have conceptual representations before developing a language. In both these views, concepts do not depend on language. This indicates that the distinction between mental and non-mental views does not divide the discussion of concepts into two distinct domains.

The non-mental views seem to appear in some philosophies of language, but also in views relying on Platonism. The mental views seem to appear in concept research, but also in views relying on empiricism. Mathematics education shows influences from both mathematics and the educational field. Further, it deals with both communication and cognitive development. This makes the distinction between non-mental and mental views on *concept* relevant in the field.

3.2 The distinction intersubjective versus subjective

Historically, a concept has often been considered as something that exist independently of human beings or as something that are determined by its use, as something intersubjective. Later, in the 20th century, a concept has in some views been considered as something that each of us develop, as something subjective. Today, it is not unusual to see both an intersubjective and a subjective view on *concept* within the same context. In such a context, people may develop different subjective concepts. These can then be compared to an intersubjective concept. In this section, a distinction is made between concepts seen as intersubjective and concepts seen as objective, with the purpose of exemplifying different philosophical positions. Later in the text, the term 'intersubjective' is used in a wide sense, including both an objective view, independent of individuals, and an intersubjective view where concepts are constructed in a culture. This decision was made since the philosophical distinction between objective and intersubjective is not judged to be of importance for the purpose of separating different views on *concept* in mathematics education, even though this distinction may be important in philosophical discussions.

In this section, the discussion is divided into four parts, where the first section examines objective views, the second section examines intersubjective views and the third section examines subjective views on *concept*. Finally, Section 3.2.4 presents some comparisons and comments on the different positions.

The distinction between objective, intersubjective and subjective views on *concept* may appear in both a mental and a non-mental approach. Hence, the discussions below are divided based on whether they concern a non-mental or a mental view. One way of discriminating between when a non-mental or a mental view is discussed is by using two different examples in this section. In the non-mental views, an example considering Thomas who is a mathematics teacher at upper secondary school, is used. He is about to assess the students' knowledge about circles. Related to this task, it is relevant to ask whether concepts are given in a social context. In that case, concepts may be contained within a curriculum or in a formal mathematical context. Another alternative is to claim that students have their own concepts that are to be assessed by the teacher. In the discussion of the mental views, an example with the two fictive children Moine and Alex is used. They have both experiences of princesses, even though the experiences are different, and they have both developed subjective representations of *princess*. When they talk to each other, they together create an intersubjective representation, which is common to their small group of two people. When talking with other people, such as friends, parents and teachers, the representation is further developed. Consequently, in the different views considering concepts as mental, there is a distinction between subjective representations and intersubjective (in a wide sense) representations, sometimes denoted by the terms ‘individual concept’ and ‘concept’ (Carey, 2009).

3.2.1 Views where concepts are objective

The first section presents views seeing concepts as objective. Below, a non-mental view claiming that concepts are objective is followed by a mental view claiming a similar objective position.

3.2.1.1 An objective non-mental view on concept

An objective view on *concept* can be found in Frege (1892/1985; 1892/1951). His discussion concerns how expressions with different senses, but with the same reference, may differ in informational content. For instance, the expressions ‘2+2’ and ‘√16’ both refer to the number four, and the proposition 2+2 = √16 is an analytical truth. So how can we then explain that this proposition carries information? According to Frege, the answer to this is that ‘2+2’ and ‘√16’ refer to the number four in different ways. Here, each sense has a unique perspective on the reference, and 2+2 and √16 are different concepts, but with the same reference. Another example of two expressions with the same reference, but different senses is the following:

> A circle with radius 7 and centre at the point (3, -1)
>
> The graph of the equation $(x – 3)^2 + (y + 1)^2 = 49$

Here, there are two different expressions referring to the same object in different ways. In fact, two concepts are identical if and only if two objects or classes of objects, the references, are identical and are pointed out in the same way by the same sense. This may be the case for ‘the square root of 16’ and ‘√16’, which have the same meaning. A consequence of Frege’s view is that if the sense of an expression is changed, the expression has different meanings at different times, which point out different concepts. Thus, concepts are, in this view, static and objective.

Further, in this view there is a distinction between *sense* and the associated *idea*, which is an internal image arising from memories of impressions. Whilethe sense is objective and non-mental, the image is subjective and mental, and different people can have different images connected to the same sense. (Frege, 1892/1985)

3.2.1.2 An objective mental view on concept

An objective mental view may be found within classical empiricism. Here, there is a causality relation between the world and mental representations of this world. Concepts could then be said to refer to the concrete objects of which they are representations. This means that different people who are in contact with the same physical world develop the same mental representations, which are therefore independent of the individual:

> […] if our concepts are somehow sensitive to the way the independent world is, so that they successfully and accurately represent that world, then an examination of them may not merely be an examination of ourselves, but may rather amount to an examination of an accurate, on-board conceptual map of the independent world. (Jenkins, 2008, p. 126)

According to this view, both Moine and Alex should develop the immutable concept *princess*. This concept includes the representations of the Swedish Crown Princess Victoria and her daughter Princess Estelle, but also knowledge coming from different fairy tales. If Moine and Alex have different representations, then this means that they have not yet developed the concept, in this view.

3.2.2 Views where concepts are intersubjective

This second section concerns views seeing concepts as intersubjective. As with the objective views, these are divided into non-mental and mental.

3.2.2.1 An intersubjective non-mental view on concept

The view that concepts are intersubjective and non-mental can be exemplified by the ideas in Katz (1972/1999), using the distinction between concepts and cognitions, where concepts do not belong to the experience of a particular person:

> By a concept in this connection we do not mean images or mental ideas or particular thoughts. These, which we will refer to collectively as *cognitions*, form part of the conscious experience of some individual person, in the same way as do sensations, feelings, memories and hallucinations. Cognitions are individuated in part by the persons who have them. [...] Concepts, on the other hand, are abstract entities. They do not belong to the conscious experience of anyone. (Katz, 1972/1999, p. 133)

Further, people can share the same concept, and concepts are "capable of being the common property of several thinkers" (Katz, 1972/1999, p. 133). This intersubjective view is supported by Quine (1960), who claims that every person creates their own everyday mental representation[24]. This representation determines a reference, meaning that everyone has their own specific reference of a concept. These references are then compared through communication, where people socialise the concept. (Quine, 1960)

In relation to the example of *circle*, Karin can have a mental representation including the idea of a round figure, which has a constant radius of 7, and can be drawn with the help of a compass. Her teacher may instead have a mental representation which involves the idea that the same circle can be described with the help of the equation $(x - 3)^2 + (y + 1)^2 = 49$. When Karin and her teacher talk to each other, the different ideas about circles are compared and a social understanding appears. As a result of the communication, maybe combined with, for example, an understanding arising from study material, a social concept *circle* develops.

3.2.2.2 An intersubjective mental view on concept

In social representation theory, where Potters and Edwards (1999) can be seen as one example, representations are cognitive structures enabling people to perceive the world and also to construct meaning about the world. The collective nature and roles of these representations are to enable communication in cultures. Consequently, our internal representations have two roles: in part they are integrated into a collective understanding about the world, and in part they are our own representations, which are used for our own thoughts about the same world. At kindergarten, Moine's and Alex's different ideas about princesses meet in communication. In that process, their mental representations may change, becoming more akin to each other. The more they talk about

[24] Quine uses 'conceptual scheme'.

princesses with each other and with other people, the more their own representations will liken the common concept *princess*.

3.2.3 Views where concepts are subjective

The third section handles non-mental and mental views seeing concepts as subjective.

3.2.3.1 A subjective non-mental view on concept

According to Zalta (2001), there is not one single sense of a term, but rather the sense may vary from one person to another. While it is claimed in Quine (1960) that the sense of 'circle' in Euclidean geometry is different from the sense of 'circle' in analytical geometry, it is claimed in Zalta (2001) that the sense may be different for different people and at different times. In this view, an object o, which may be a circle, can be grasped in different ways, and three different people can have three different mental representations, r_1, r_2 and r_3, of the circle o. The same person can also have different representations at different times. These representations have different content, which can be described as three different modes of presentation, or concepts, representing the same object o. In the example of *circle*, Karin may have different concepts *circle* at different times:

> consider the property of being a circle, and let '**C**' denote the mental token for this property in x's brain, and let ξ_1 be an A-property that encodes one or more of the following properties of properties […]: being a shape that involves no straight lines, being a shape that is always uniformly curving, being a shape which can be inscribed with the help of a compass, being the geometric property exemplified by this particular figure on this particular printed page, *etc.* ξ_1 is clearly a mode of presentation for x of the property of being a circle. It is one of the concepts that x has for the property of being a circle. (Zalta, 2001, p. 344)

Here, ξ_1 is a concept associated with the mental representation $\boldsymbol{C}$. In the example where Karin talks with her teacher she may learn that a circle could be defined by the equation $(x - x_0)^2 + (y - y_0)^2 = r^2$. $\boldsymbol{C}$ develops to $\boldsymbol{C}'$ and a new concept ξ_2 is associated with the new $\boldsymbol{C}'$. Therefore, concepts are not just dependent on person, but also on time, and Karin switches between different concepts when learning mathematics.[25]

[25] A logic analysis of concept development in this spirit is presented in Bennet (2019).

3.2.3.2 A subjective mental view on concept

A subjective mental view can be found in Wittgenstein (1953/1992), where it is claimed that even if we use the same word for different phenomena, these phenomena must not have something in common. Further, it is not possible to delimit a reference, and a word thus has a family of meanings. To exemplify, Moine and Alex can have different individual concepts *princess*. Moine, on the one hand, may have a concept that is based on reading a lot of fairy tales about princesses wearing certain kinds of dresses, having princess crowns and living in castles. Alex, on the other hand, may have a concept that is based on watching the Swedish Crown Princess Victoria and her daughter Princess Estelle on television. It is not certain that Moine know about Princess Victoria. On the other hand, Alex may not have read any fairy tales about princesses. Their totally different experiences may lead to totally different individual concepts.

3.2.4 Comments on the distinction intersubjective versus subjective

Independently of whether concepts are seen as non-mental or as mental, there seems to be a distinction between objective, intersubjective and subjective views. We can summarise the discussion in Table 3.1.

The distinction between objective, intersubjective and subjective seems to be independent of the distinction between non-mental and mental. In fact, an intersubjective mental view and an intersubjective non-mental view seem to be similar in many aspects, as both views see concepts as constructed in a cultural context. Further, a subjective mental view and a subjective non-mental view have many similarities as well, as concepts seen as subjective mental representations and concepts seen as content of subjective mental representations may share features.

Table 3.1 Objective, intersubjective and subjective views on *concept*

	Non-mental	**Mental**
Objective	Concepts are static and independent of culture (Plato, 399 BCE/1997a; Frege, 1884/1950; 1892/1951)	Concepts represent the physical world and are independent of culture (Jenkins, 2008)
Intersubjective	Concepts are constructed within a cultural context and are something that we use in language (Quine, 1960; Katz, 1972/1999)	Concepts are social and are developed by the individual in a cultural context (Carey, 2009)
Subjective	Concepts depend both on time and individual (Zalta, 2001)	Concepts depend both on time and individual (Wittgenstein, 1953/1992)

Not all views fit within just one category. Carey (2009), for example, distinguishes between different representations, where some are based on theoretical knowledge in the community, and others are based on sensory impressions and core cognition. She uses the distinction between *concept* and *individual concept* (Carey, 2009), which I would say is similar to the distinction between *social* and *individual representation*. In addition, the distinction between *concept* and *individual concept* in Carey (2009) has a parallel in Frege's (1892/1985; 1892/1951) distinction between *concept* and *image*, and Katz's (1972/1999) distinction between *concept* and *cognition*. However, these are not precisely the same. Even though *individual concept* in Carey (2009) is similar to *image* in Frege (1892/1985) and *cognition* in Katz (1972/1999), since these are mental representations, concepts in the views of Frege (1892/1985; 1892/1951) and Katz (1972/1999) are non-mental while concepts in Carey (2009) are social mental representations.

There are arguments, such as the ones in Jackendoff (1989/1999), that it is possible to have a view that includes both intersubjective and subjective concepts[26]. In mathematics education, the students may develop subjective concepts for the intersubjective concepts inherent in mathematics. Here, clarity may be obtained by distinguishing between intersubjective and subjective concepts in mathematics education, and in fact in all educational research. This can be done by using terms such as 'concept' and 'conception', or 'concept' and 'individual concept'[27]. An objective view will from now on be seen as a special case of an intersubjective view, since this distinction seems to be less important for educational purposes and since it can be hard to distinguish between these two positions, when interpreting texts. As a result, when it is claimed that a view is intersubjective, it may or may not be objective.

Another issue in this chapter is a discussion about conceptual structures. This can be seen in Section 3.1.1 as a question of whether it is possible to delimit between references, or not. While Frege (1892/1985), as one example, claims that it is possible, Quine (1960; 1951/1985) considers it in principle impossible to draw sharp pictures of concepts. The discussion has a parallel in Section 3.1.2, where three different views on mental representation are presented. In a classical view, such as the one in Jenkins (2008), basic concepts are representations based on impressions from physical objects. These can then be combined into more complex concepts. On the other hand, there are also views where representations are regarded as more complex structures, where some representational knowledge is innate. The view in Carey (2009) can be taken as an example of that.

[26] Jackendoff (1989/1999) argues that it is possible to have a view that includes both externalised and internalised concepts but the argumentation holds for a view that includes both intersubjective and subjective concepts as well.
[27] This solution does not take into account the usage of the word 'concept' in a subjective sense.

3.3 The distinction molecular versus holistic

The third distinction on which my analysing tool is based concerns different views on how concepts are structured. The focus in this section is whether structures of concepts have molecular or holistic features. To exemplify, one may look at the concept *Retriever*[28]. In a molecular model[29], *Retriever* is a complex concept that can be composed of concepts such as *dog*, *retrieve*, *hunt* and *bird*. Each component specifies a necessary, but not sufficient, condition for being a Retriever and together these form the concept. Consequently, a definition of the concept may use terms such as 'dog', 'retrieve', 'hunt' and 'bird'. Together, the components in such a definition forms a condition that is both necessary and sufficient. In a holistic model, it is not possible to specify components in this way and the concept, which can be mental or non-mental, is rather thought to be included in a web-like structure. A difference between the two models is that in a molecular model, the concept *Retriever* is dependent on the concept *dog*. This is not the case in a holistic model. (Laurence & Margolis, 1999)

There are in fact two different structures of interest involved here. The first structure is non-mental, consisting of classes of objects that are named, and relations between these classes and between meanings of terms. The second structure is a mental representation, also referred to as the cognitive structure. In this section, non-mental and mental structures are discussed from two points of view. First, approaches[30] based on the two models are described, where Section 3.3.1 deals with approaches where concepts have molecular structure and Section 3.3.2 deals with approaches where concepts have holistic structure. Second, Section 3.3.3 concerns the question of whether structures are hierarchically ordered or not. Finally, Section 3.3.4 presents comparisons and conclusions.

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[28] In this paragraph, there are two interpretations of the Retriever example, one mental and one non-mental.

[29] Other names for these models are 'the containment model' and 'the inferential model'. The choice of vocabulary is made from the perspective that the models are used at a superficial level that suits an educational purpose. The term 'model' is used, since molecularism and holism may not be fully expressed in a certain text, but a view on *concept* may rely on the two models in different ways.

[30] The expression 'approach to a concept' is used to mean an idealised version of broad classes of views on *concept* that are grouped together due to certain similarities between the views.

The approaches used to exemplify the models are found in the area of concept research within cognitive science, where there have been philosophical discussions about the nature of concepts for the last 50 years or so.[31] These approaches are selected based on the criteria that they offer clear views on structures of concepts, that they are influential in concept research, and that they complement each other. Based on these criteria, the classical approach, the prototype approach and the knowledge approach have been chosen. While the classical approach can be seen within a long historical discussion in philosophy, the other two approaches have been developed more recently within concept research. Further, the classical approach can be used in both a non-mental and a mental position. That is why, when describing this approach, some paragraphs are about concepts in a mental view and other paragraphs are about concepts in a non-mental view. I have tried to make this explicit. In the prototype approach and the knowledge approach, on the other hand, concepts are considered as mental representations. In the descriptions of these approaches, concepts as non-mental objects will be dealt with, but in an indirect way using the terms 'word meaning', which is interpreted as synonymous with 'sense', and 'class', which is short for 'class of objects'. Even though concepts are considered mental representations, Murphy (2004) claims that many properties of concepts are also found in word meanings. This suggests that there are connections between word meanings and concepts, and that word meanings are mentally represented through the cognitive system. However, it is not obvious that a cognitive structure developed by the human mind has similar features to a non-mental structure seen in communication or formal mathematics.

3.3.1 Views where concepts are molecular

In a molecular model, a concept may be composed of other more basic concepts (Laurence & Margolis, 1999; Margolis & Laurence, 2019). In addition, the idea of compositionality entails that the meaning of a complex expression is determined by the meanings of its constituent terms and the rules combining them. The meaning of 'linear' and the meaning of 'function' thus determine the meaning of 'linear function'. A related issue is whether concepts are

[31] Cognitive science include different areas, such as psycholinguistics, reasoning, anthropology, neuropsychology, problem-solving and concept research (Murphy, 2004). It is the philosophical ideas within the area of concept research that are dealt with here.

hierarchically ordered or not. In a hierarchical structure, a concept can be reduced to basic ones, and sentences involving a complex concept can be transformed into sentences involving more basic concepts.

Here, a binary relation R is transitive if, for all A, B and C, ARB and BRC implies ARC. A typical example of a transitive relation is *subset*: if $A \subset B$ and $B \subset C$, then $A \subset C$. This can be concretised with the example of the dog Blue, which can be called a yellow Labrador Retriever, a Labrador Retriever, a Retriever, a dog, a mammal, a vertebrate and an animal. In this example, retrieved and modified from Murphy (2004, p. 27), the class of yellow Labrador Retrievers is included in the class of Labrador Retrievers, which in turn is included in the class of Retrievers, which is included in the class of dogs, and so on. As there are subclass relations between the different classes, the idea of transitivity entails that all yellow Labrador Retrievers must be dogs. Further, in this case, the definition of *Labrador Retriever* may use the term 'Retriever' and the content in the definition of the concept *Retriever*.

The idea of transitivity gives the molecular model hierarchical features. Consequently, the molecular model may be represented in two different ways, as a molecule or as a diagram showing hierarchy (see Figure 3.7). With the example of the dog *Blue*, the hierarchy can be represented as a tree diagram. However, with other examples, as with the concepts *rectangle* and *rhombus*, where the classes are not mutually exclusive and both include squares, the diagram may have a different appearance.

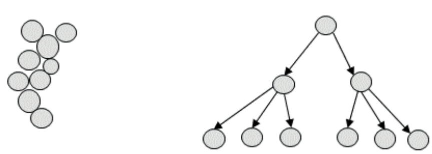

Figure 3.7 Two visualisations of concepts in a molecular model

Molecularism is a model to different extents underlying some approaches to concepts. While the classical approach[32] is totally based on molecularism, the prototype approach has abandoned some of these ideas. Below, these two different approaches, together with a dual approach combining the two, are described in order to exemplify how the molecular model can be expressed.

[32] Another name is 'the definitional approach'.

3.3.1.1 The classical approach

The classical approach is based on molecularism. The idea that definitions may be used for explicating concepts can be traced back to Aristotle and is prevalent in most works on concepts prior to 1970 (Murphy, 2004). By those who claim that concepts are mental representations, this approach is embraced by for example Locke and also the contemporary philosopher Jenkins (2008). They claim that complex concepts are built from primitive ones, which in turn are based on sensory data (for a more detailed description of Jenkins' view on *concept*, see Section 3.1.2.2). By those who claim that concepts are non-mental objects, this approach is held by some philosophers interested in language, such as Frege (1892/1985; 1892/1951) and Katz (1972/1999)[33]. In Frege (1892/1985) it is claimed that more complex expressions, such as 'natural number', get their senses through the senses of more basic terms, such as 'natural' and 'number', how they refer and how the more complex expression is built. This is the principle of compositionality, claiming that the structure of a complex expression and the meaning of its constituents determine the meaning of the expression (Szabó, 2017). In Katz (1972/1999), this idea is expressed by saying that semantic rules enable us to obtain the sense of an expression as a compositional function of the senses of its parts, ultimately of its component morphemes, and their properties and relations. The process begins with an assignment of a sense to each of the morphemes and ends with an assignment of a semantic interpretation of the whole expression (Katz, 1972/1999).[34]

In the classical approach, concepts can be represented by definitions, and one acquires a concept (in a mental view), or learn about a concept (in a non-mental view), by assembling its properties. Further, every object is either within or outside of a class of objects. Things that have the properties pointed out by the definition are members of the class, and things that do not have these properties are not members of the class. (Murphy, 2004)

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[33] The classical approach, unlike the prototype approach and the knowledge approach, can be held both by those holding that concepts are non-mental and those holding that concepts are mental.

[34] The philosophy of Frege (1892/1985) is also described in Section 3.1.1.1 and the view in Katz (1972/1999) is described in Section 3.2.1.2.

3.3.1.2 The prototype approach

One influential argument against the classical approach relies on psychological data, called 'typicality effects'. Findings have shown that people rank items in a class according to how typical they are as members of that class (Rosch & Mervis, 1975). If you ask people to say a number, some numbers come into their mind first. It could perhaps be natural numbers between 1 and 20. For most people, 19 is a more typical number than 19 249. Concepts are, in the prototype approach, complex mental representations with structures which are not governed by definitions, but by a list of properties that the objects in the class tend to have. The properties of the members overlap in a way that establishes a certain space of similarities. The properties of different dogs, such as Labrador Retrievers, Poodles and Beagles, establish a space of similarities of dogs. What makes something a dog is that it falls within the boundaries of this similarity space. As in the classical approach, one acquires a concept by assembling its properties, but there is no assumption that every member in the class shares all the properties. (Murphy, 2004; Laurence & Margolis, 1999)

Rosch (1978), drawing on Wittgenstein, claims that instead of using boundaries to distinguish between different classes, we can use clear cases. In a class, there is something called a prototype according to which the members in the class are assessed. The prototype is the most typical item in a class, defined operationally by people's judgements. It can be a specific knowledgeable member, or an average or ideal item that people extract from the knowledgeable members. It can for example be an idea of a typical dog, which is of average size, and has colours that many dogs have. Some dogs, such as Labrador Retrievers, may have many properties in common with the prototype dog, while other dogs, such as small Mexican hairless dogs, may have fewer properties in common. (Murphy, 2004)

3.3.1.3 A dual approach

The prototype approach explains typicality effects, which is one argument against the classical approach. However, a problem with the prototype approach is that it cannot explain the phenomenon of compositionality; it is difficult to see how the prototype of a more complex concept, such as *linear function*, can be explained as a function of the prototypes of more basic concepts, such as *linear* and *function*. Compositionality offers an explanation of how humans develop concepts, and an approach that does not explain this phenomenon is not complete. In addition, many concepts have definitions, and we can use them for developing our understanding about the world. Even though not all dogs have fur, it is still useful to use fur as a characteristic for identifying dogs. Consequently, categorisation cannot be understood within just one of these approaches and there may be a need for a dual approach to concepts that includes both a classical and a prototypical part. (Laurence & Margolis, 1999; Murphy, 2004)

To sum up this section, three approaches have been taken as examples of how a molecular model can be expressed. The classical approach is totally based on molecularism. The prototype approach has abandoned some of the ideas of molecularism, since the characteristics described in the definition are not considered necessary but are seen as properties that members in a class of objects tend to have. Further, a dual approach combines a classical and a prototype approach. As will be seen below, there are criticisms that these approaches cannot answer, which is an argument for a holistic model.

3.3.2 Views where concepts are holistic

The holistic model is from a mental position based on the idea of Wittgenstein (1953/1992) that concepts are related to each other in many different ways, and that objects that fall under a concept form a complicated network of similarities. As an example, the mental concept *Retriever* may have a structure including a relation to the concept *dog*, which in turn may have a structure including a relation to the concept *Retriever*. Furthermore, the holistic model is connected to a non-mental position based on Quine's (1960; 1951/1985) idea that all concepts are interconnected. It can be misleading to consider the content of a single term or expression; a term such as 'Labrador Retriever' is connected to 'dog', which in turn is connected to 'hot dog', and so on. In fact, every term can be related to every other term in the language, and the change of the meaning of one term affects the meanings of related terms. As a result, there are no basic concepts and one cannot define concepts from a delimited selection of such basic concepts (see also the discussion in Section 3.1.1.2). (Laurence & Margolis, 1999)

As a structure of concepts in a holistic model can be seen as a network, where generally speaking all concepts are related, this model can be represented as in Figure 3.8. However, it is not clear what a concept is in this picture: is it the structure or is it a part of the structure? Further, if we zoom in, focusing on one of the ovals, we can see new relations and subparts that are not seen at the general level. The structure of meaning is in a sense fractal.

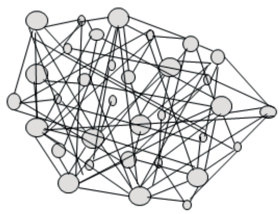

Figure 3.8 A visualisation of concepts in a holistic model

The holistic model is based on criticisms of the molecular model, which is summarised below. After that, the knowledge approach is presented as an example of an approach relying on holism.

3.3.2.1 Criticism of the molecular model

Laurence and Margolis (1999) claim that many of the discussions through history have concerned non-mental concepts which have definitions in dictionaries. For example, the concept *parallel straight line* has the following definition according to Euclid:

> *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction. Euclid (ca. 300 BCE/1956, Book 1: def 23)

However, some concepts are not as easy to define formally and even if a definition can be found, it could take quite a long time before it takes its final form. In mathematics, the idea of a function evolved in an explicit form at the beginning of the 18th century and the development since when can be described as an interplay between a geometric conception, an algebraic conception and a logical conception (Kleiner, 1989). In Lakatos (1978), it is claimed that definitions are not definitive, but develop through exploration, and that they are dependent on the mathematical context in which they are constructed (see Chapter 2).

There are philosophers who claim, from a non-mental position, that some concepts cannot be defined. As an example, in Euthyphro, Plato argues that there are concepts that cannot be defined without falling into a regress (Plato, 399 BCE/1997a). Murphy (2004) claims that with the dog example, the property of being canine can be proposed as a characteristic for dogs. However, the result of using that property in the definition would be a circular definition that does not explain anything, since *canine* is just a synonym for *dog*.

Further, one can question the assumption that concepts can be defined at all. Wittgenstein (1953/1992) uses the concept *game* as an example of a concept that is hard to define, proposing that it is not possible to delimit a reference, and that a term must have a family of meanings. His argument consists of a series of definitions, followed by counterexamples. For example, the proposed definition of game as an activity that involves competition was rejected with the counterexample of a solitaire card game.

Further, one may question a hierarchical view on *concept* by criticising the idea of transitivity, that class inclusion is a logical basis for judgements about concepts. As seen above, this idea would suggest that if the class of Labrador Retrievers is included in the class of Retrievers and if the class of Retrievers is included in the class of dogs, then the class of Labrador Retrievers must be included in the class of dogs. Hampton (1982) offers counterexamples to this idea. For instance, even though the class of chairs is commonly considered to be included in the class of furniture, there are chairs, such as car seats, which are not judged to be furniture. (Murphy, 2004)

According to Murphy (2004), a classical view on *concept* may perhaps be applied within small, closed systems that do not permit exceptions and variations found in the natural world. The rules of chess can be taken as an example of such a system. Note that most examples in the criticism from the non-mental position are taken from non-mathematical domains, which strengthens the position where mathematical concepts are considered well determined. If it is true that mathematical concepts are well determined, then the classical approach can be used for describing mathematical concepts. (Laurence & Margolis, 1999)

3.3.2.2 The knowledge approach

The knowledge approach[35] seeks to explain so called knowledge effects, meaning that prior knowledge influences categorisation and learning of new objects and events. If you see a new animal that you have not seen before, you probably use earlier knowledge to understand what kind of animal it is and what kind of properties it has. If it looks similar to an animal that you have knowledge about, such as a dog, you may conclude that it has similar properties. If it has

[35] The knowledge approach has also been called 'the theory-theory' or 'the theory view'.

visible fur, four limbs, a tail, a symmetrical face with two eyes, a nose and two ears, then you may suspect that it is a mammal and use your knowledge about mammals to conclude that it has a four-chambered heart, gives birth to live young, and so on. Further, if the new animal has a property that is surprising such as a blue tail or three eyes, then it can change your knowledge about animals and biology in general. (Murphy, 2004)

In the knowledge approach, concepts may be identified as structured mental representations, where the structure involves relations to other representations, including facts and principles (Laurence & Margolis, 1999; Murphy, 2004). For example, the concept *function* is built from knowledge about functions and how functions could be used in various kinds of problem solving. Another way of describing concepts is as constituents of representations. Carey (2009), for example, explains concepts as constituents of thought embedded in conceptual structures. In this view, as we already have described in Section 3.1.2.3, impressions from concrete objects cause us to develop sensory representations, which integrate with conceptual representations embedded in cognitive structures (Carey, 2009).

While the prototype approach explains concept development based on typicality effects, the knowledge approach explains knowledge effects, which are not described within the classical approach or the prototype approach. However, one problem is that this approach allows people to have rather sketchy views, involving relatively little information, and the contexts in which the concepts *dog* and *wolf* are developed, for example, are almost the same. One could then ask how these concepts point out their respective references. Another issue is whether two different people can develop the same concept, when they probably have different ideas about a phenomenon.[36] To conclude, the mechanism for concept development seems to be poorly understood in the knowledge approach. (Laurence and Margolis, 1999)

It seems clear that the knowledge approach cannot explain concept development in full, and Murphy (2004) claims that in order to develop a complete view on concept development, you probably have to combine the advantages of the prototype approach and the knowledge approach. This can be seen as an argument for combining a molecular and a holistic model.

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[36] One answer to that may be that people do not need to develop the same concept, but similar concepts, where two concepts are similar when they have a sufficient number of properties in common. In that case, concepts are subjective with intersubjective features.

After presenting different approaches to concepts, the next section considers some results from concept research, concerning structures of concepts, independently of approach. Here, the issue of whether structures are hierarchical or not, and, if so, in what aspects, is of importance.

3.3.3 Hierarchical features of conceptual structures

Some classes form hierarchical structures. The classes determined by the terms 'Labrador Retriever', 'Retriever', 'dog', 'mammal' and 'animal', for example, form a hierarchy of progressively larger classes, where each class includes the previous ones as subclasses. A hierarchy is a special kind of network. It has nodes, which are the classes, connected by relations, which are part-whole relations. These are represented by lines in Figure 3.9 which is based on Figure 7.1 in Murphy (2004, p. 201). Here, the class of 'Retriever' is a subclass of the class of 'dog', which is a subclass of the class of 'mammal', which in turn is a subclass of the class of 'animal'. Further, the class of 'trout' is a subclass of the class of fish, which is also a subclass of the class of 'animal'. The hierarchy in Figure 3.9 forms a tree diagram. (Murphy, 2004)

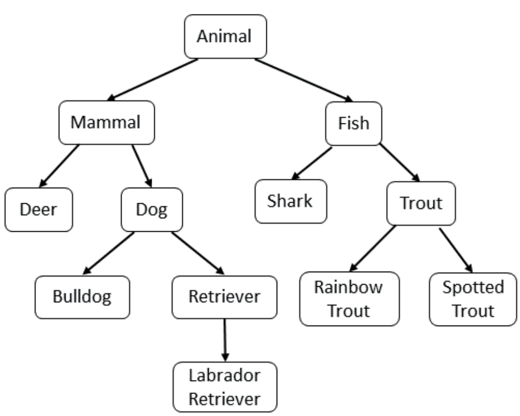


Figure 3.9 A hierarchy of classes

Classes do not always form clear hierarchies. The example in Section 3.3.2.1 with car seats and chairs shows that there is intransitivity in the conceptual structure; car seats do not share all features with chairs. Other classes do not form such a structure at all. Helge could be described as 'a man', 'a Norwegian', 'a cousin', 'a jogger' and 'a teacher', where none of the classes of these terms are included in any of the others. These examples show that classes are not always related in hierarchical ways. The requirement for the possibility of constructing a hierarchical structure is that some classes are subclasses of other classes (Murphy 2004). Further, the examples in Section 3.3.2 of typicality effects and knowledge effects show that people do not form strict hierarchical mental structures as in Figure 3.9.

In the case of mathematics, it can be discussed how hierarchical structures can be constructed. If we try to form a hierarchy for quadrilaterals as an example, it could be something like in Figure 3.10. One can see that the classes in this structure are not mutually exclusive; there are items belonging to different classes at the same level. As an example, squares are members both of the class of 'rectangle' and the class of 'rhombus': in fact squares are precisely those objects which are both rhombus and rectangle, with the consequence that there could be more than one link from higher-level classes to the same lower-level class.

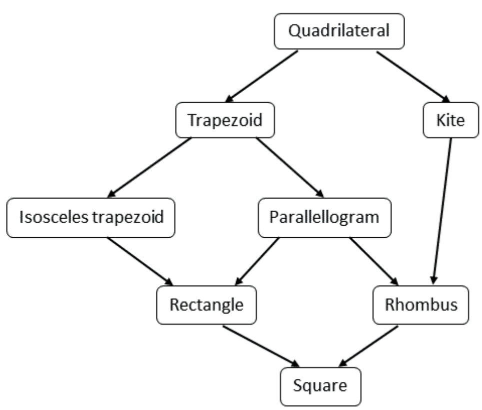


Figure 3.10 A hierarchical structure for quadrilaterals

A similar phenomenon can be seen if we take *number* as example and try to construct a hierarchical structure as in Figure 3.11.

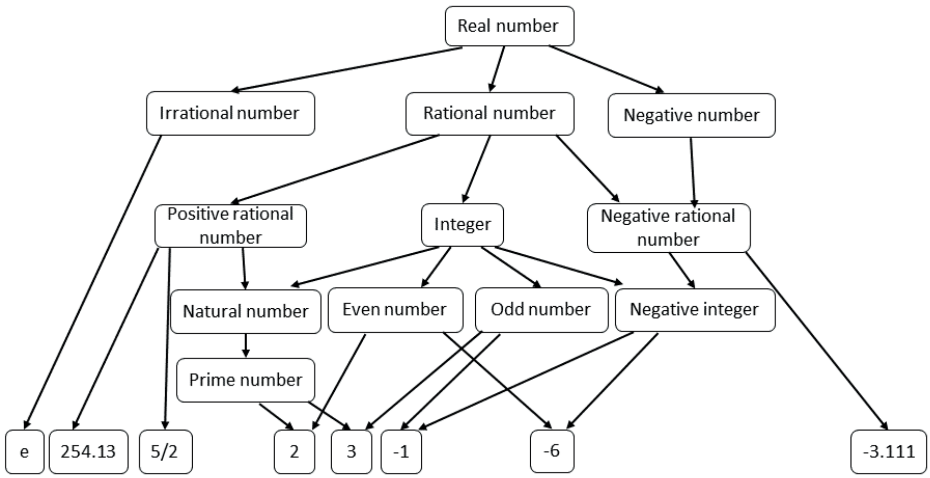


Figure 3.11 A hierarchical structure for numbers

There is an arbitrariness in the appearance of the structure; we could choose to consider terms where the corresponding classes form hierarchical structures, but this does not apply to every collection of terms and classes. In addition, by adding classes of terms such as 'transcendental number' to the diagram in Figure 3.11, we could get an even more complex structure, while some classes, such as the class of the term 'percent', do not fit into the structure at all.

There are also different types of links between classes. The part-whole links are the ones specifying a hierarchical structure. Other types of links may form connections between classes related in other ways. For the concept *triangle*, its relation to trigonometric functions, which do not fit into a hierarchical structure, may be as important as the relations to different types of triangles, which in turn do fit into a hierarchical structure. (Murphy, 2004)

It is not clear how hierarchies such as these are mentally represented. One possible alternative is the pre-stored view, where it is claimed that a person's mental concepts are structured in memory almost as in Figure 3.10 or 3.11. In such a view, concepts are connected in hierarchical networks, and the connections are used to make judgements about them. In this case, Figure 3.11 can illustrate concepts that a student actually has, where the lines indicate associations of the part-whole kind. Another alternative is the computed view, where it is claimed that the hierarchical structure of concepts is not directly stored in memory but comes from a reasoning process at every occurrence. Consider, for example, the fact that prime numbers are natural numbers and that natural numbers are rational numbers. If someone knows that all rational numbers can be written as a/b, then he or she may conclude from facts about class inclusion that all prime numbers can be written as a/b. There are indications that children store some hierarchical facts explicitly in memory. The fact that whales are mammals is an example of a fact that is probably stored, since whales do not look like other mammals. However, the arguments against the classical view, based on typicality effects, intransitivity etc., also hold against the pre-stored view. Such experimental results show that people do not organise information in an elegant hierarchical way. This is why Murphy (2004) argues for a model where the pre-stored view and the computed view are combined. (Murphy, 2004)

It is easier in some domains than in others to find examples of well determined concepts. This suggests that concepts in these domains are more likely to form hierarchical structures. If mathematical concepts are well determined, then there are hierarchical structures in the field of mathematics. However, there are also other types of links between mathematical concepts. There are connections between concepts from different mathematical domains, as well as to concepts in other domains. For example, when Per thinks about fractions he may relate them to geometrical figures, where eighths can be thought of from the perspective of octagons (Figure 3.12). Such relations cannot be represented in a hierarchical structure, since there is no part-whole relation between eighths and octagons.

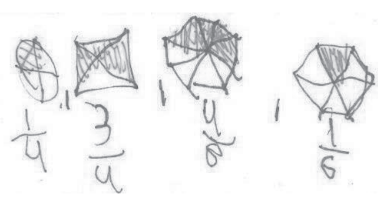


Figure 3.12 Representing fractions as geometrical figures

A conclusion from this section is that molecular and holistic features can show different aspects of the conceptual structure. It is possible to choose collections of classes that form hierarchical structures. When choosing classes that are well determined and mutually exclusive, and where classes include more basic ones as subclasses, the structure can be represented as a tree diagram. However, all collections of classes do not form hierarchical structures. Furthermore, since there are indications that structures of mental concepts have similar features to structures of non-mental ones, this could be seen as an argument for a combination of hierarchical and non-hierarchical approaches to mental concepts as well. This implies that there may not just be molecular and holistic views on *concept*, but also views that combine molecular and holistic features.

3.3.4 Comments on the distinction molecular versus holistic

Section 3.3 concerns conceptual structures and two different models for describing these structures. These two models, which are summarised in Table 3.2, underlie the discussions of how concepts are structured.

Table 3.2 Molecular and holistic models

The molecular model	• Concepts are composed of other, more basic, concepts • It is possible to define concepts from basic concepts • Relations between concepts are transitive • Concepts have hierarchical structure
The holistic model	• No concepts are more basic than others • It is not possible to define concepts • Relations between concepts are not transitive • Concepts have non-hierarchical structure

Returning to the example *linear function*, in a molecular model, the concept has at least two components, *function* and *linearity*, and a student develops *linear function* as a composition of these components. In a holistic model, the concept cannot be separated from a cluster of nearby concepts. In this case for example *graph*, *point-set*, *ordered pair* and *causality*, which in turn are correlated to the concept *linear function*. The individual needs the whole cluster in order to develop both the concept *linear function* and the concept *graph*.

Concept research within cognitive science uses different approaches to concepts, which are summarised in Table 3.3. In the classical approach, which is based on molecularism, concepts encode a list of necessary and sufficient properties. The prototype approach has abandoned some of the molecularism ideas, and the list contains properties that members in a class tend to have. Finally, in the knowledge approach structures of mental concepts are holistic as they contain various relations to other concepts and representations.

The different approaches are useful for describing concept development in different ways. The classical approach describes the effects of compositionality and how new concepts can be composed of already existing ones. This can be useful for describing judgement of class membership. The prototype approach and the knowledge approach explain results in psychological experiments concerning concept development (Murphy, 2004). While the prototype approach explains the identification procedure, based on typicality effects, the knowledge approach explains concept development based on background knowledge.

Table 3.3 Three approaches to concepts

Approach	**The nature of concepts**	**Usage**
The classical approach	Concepts encode a list of necessary and sufficient properties for members in a class	Useful for describing judgement of class membership, based on compositionality
The prototype approach	Concepts encode properties that members in a class tend to have	Useful for explaining the identification procedure, based on typicality effects
The knowledge approach	Concepts contain relations to other concepts and representations	Useful for explaining concept development based on knowledge effects

There are advocates of a dual approach, combining the classical approach and the prototype approach (Laurence & Margolis, 1999). Murphy (2004), in turn, proposes a combination of the prototype approach and the knowledge approach. This indicates that the classical approach, the prototype approach and the knowledge approach may all be combined, in order to develop a broader view on *concept*.

Whether mental or non-mental, concepts in some domains are more likely to form hierarchical structures than concepts in other domains. In mathematics, it is possible to find a collection of classes and part-whole relations forming a hierarchical structure. However, there are also collections of classes and relations that are not hierarchical. Relations between classes in different areas of mathematics, as an example, can hardly be put into hierarchical structures.

Through this section, one thread is an argumentation for a multiple approach to concepts, where a molecular model and a holistic model are combined. Consequently, molecularism and holism are not mutually exclusive and a view on *concept* can have both molecular and holistic features.

3.4 Putting the three distinctions together into a frame

The overall purpose of this chapter is to build a tool for analysing views on *concept* in mathematics education, such as they appear in texts. Three distinctions concerning views on *concept* in philosophy are combined. Section 3.1 concerns the distinction mental versus non-mental. Section 3.2 concerns the distinction intersubjective versus subjective. Section 3.3, finally, presents the molecular and holistic model for the conceptual structure. These distinctions can be put together into a three-dimensional matrix, as is seen in Figure 3.13[37].

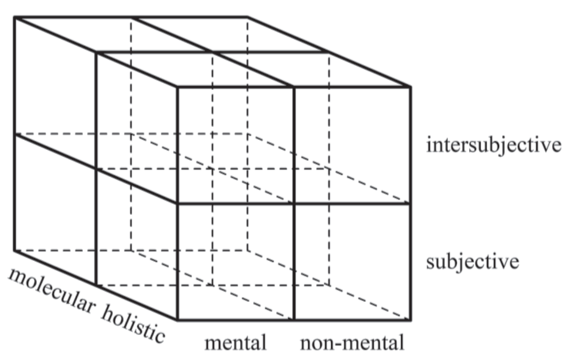


Figure 3.13 The three-dimensional matrix

[37] The matrices in the thesis are drawn with R (R Core Team (2016). R: A language and environment for statistical computing. *R Foundation for Statistical Computing*. Vienna: Austria).

As seen above, the philosophical discussions are complex and diverse. To highlight three distinctions in the way that is done here is a somewhat rough simplification. It is not always easy to tell which cell in the matrix a certain view belongs to, and a view could slip between positions. For instance, in Section 3.3, it is shown that there are arguments for a combined molecular-holistic view. Notably, the analysing tool is, so far, a philosophical construct, which is to be developed into a method for analysing texts in mathematics education. This is one of the issues in Chapter 4, where the method for my concept analysis is established. First however, in this section, the 3D matrix is used for categorising views on *concept* in philosophy. The purpose is two-fold: to show the usage of the frame and to sum up some of the views described in the chapter.

In the first group, some classical philosophers of language can be found, such as Frege (1892/1985; 1892/1951) and Katz (1972/1999), who consider concepts to be non-mental, intersubjective and molecular. While it is claimed in Frege (1892/1985; 1892/1951) that concepts are references, pointed out by senses, the view in Katz (1972/1999) is that concepts are just senses. One thing that is common to these views is that concepts are non-mental and intersubjective; they do not depend on the individual. Further, these views are built on compositionality, where the sense of a complex expression depends on the senses of its components. Consequently, concepts have molecular structure. Inserting the view on *concept* in the analytical frame results in the matrix in Figure 3.14. Further, in Frege (1892/1985; 1892/1951), there is a distinction between *concept* and *image*, which could be compared to Katz's (1972/1999) distinction between *concept* and *cognition*. Images and cognitions can be considered mental and subjective reflections of concepts.

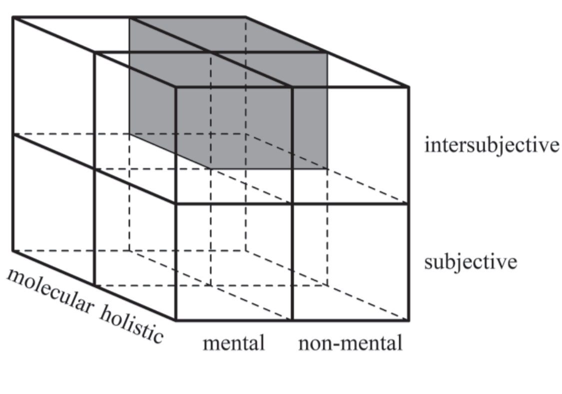


Figure 3.14 A non-mental, intersubjective and molecular view

In a second group, we can place the ideas of Zalta (2001). In this view, concepts are non-mental abstract contents of mental representations. These are subjective since they depend both on the individual and on time. Further, the idea that the mode of presentation encodes a collection of properties points at a molecular view. So, the view on *concept* here is non-mental, subjective and molecular (see Figure 3.15).

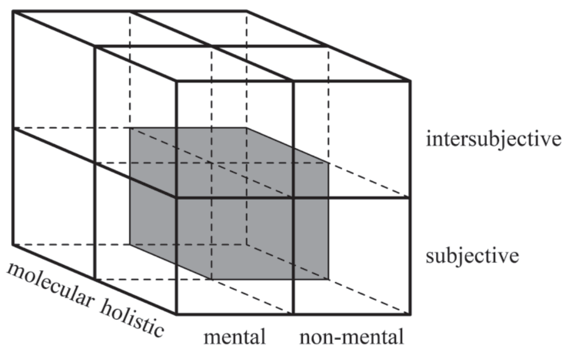


Figure 3.15 A non-mental, subjective and molecular view

A third group can be represented by Quine (1960; 1951/1985), where concepts are senses that are non-mental, intersubjective and holistic (see Figure 3.16). Senses are in this view culturally dependent constructions, that are time-dependent and determined by how terms are used in different languages. Further, Quine (1960) uses the notion *conceptual scheme* for the mental and subjective reflection of *concept*. As a consequence of the Quinean ideas, even the conceptual scheme has holistic structure, and the view on *conceptual scheme* would be similar to the view on *concept* in Figure 3.18, below.

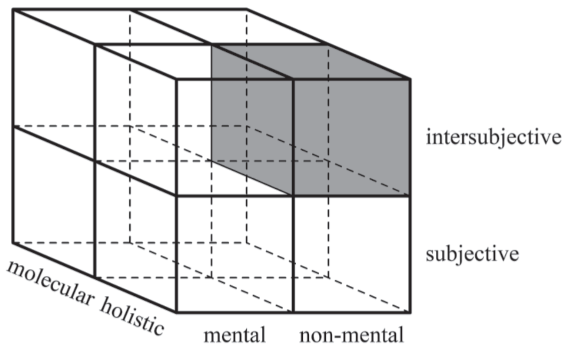


Figure 3.16 A non-mental, intersubjective and holistic view

In the fourth group, we find views of the classical empiricists and Jenkins (2008), where concepts are considered to be mental and intersubjective with molecular structure. In this view, which is inserted into the matrix in Figure 3.17, concepts are mental representations. Further, basic concepts are composed into more complex ones, and the total conceptual structure mirrors the structure of facts about the world.

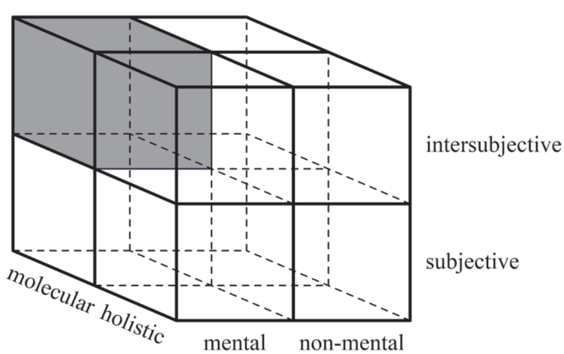


Figure 3.17 A mental, intersubjective and molecular view

The view on *concept* in Wittgenstein (1953/1992) can be placed in a fifth group, where concepts are mental, subjective and holistic (Figure 3.18). Here, concepts are defined as mental representations and different people can have different concepts. Further, the properties of the things that fall under a concept form a complicated network of similarities, where concepts are related in many different ways, which indicates a holistic view.

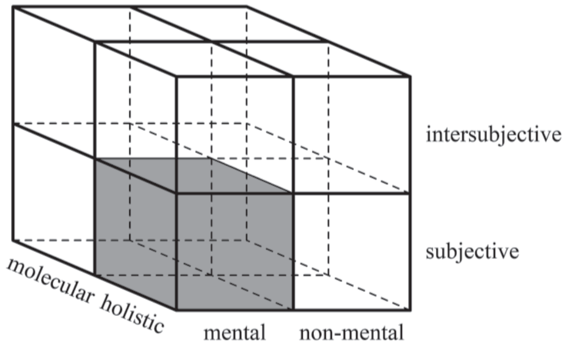


Figure 3.18 A mental, subjective and holistic view

Views of cognitive scientists using the knowledge approach, such as Carey (2009), and other views considering concepts as mental, intersubjective and holistic may be placed in the sixth group. Here, concepts are social mental representations, with holistic structures (Figure 3.19). Further, Carey (2009) uses the distinction between *concept* and *individual concept*, which is similar to the distinction between *social* and *individual representation*. The view on *individual concept* is mental, subjective and holistic, and hence similar to the view on *concept* presented in Figure 3.18.

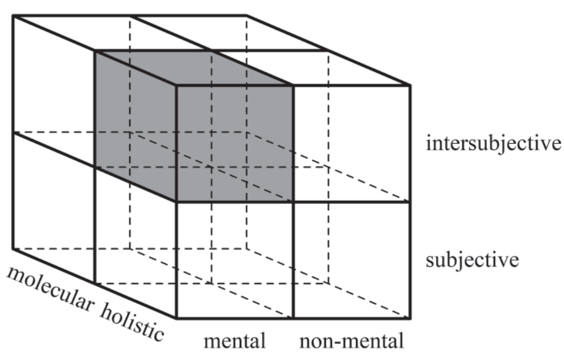


Figure 3.19 A mental, intersubjective and holistic view

Finally, a seventh group contains views using a combination of a molecular and a holistic model. In Murphy (2004), as one example, it is claimed that concepts are mental representations, and a combined molecular-holistic view is argued for (Figure 3.20). Here, the terms 'word meaning' and 'categories' are used for concepts seen as non-mental objects. If using these ideas for an intersubjective view, then the result is the view represented in Figure 3.20.

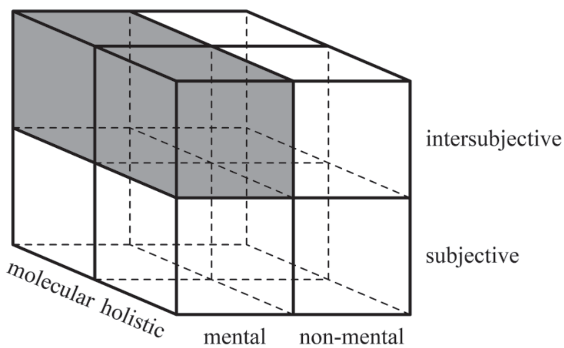


Figure 3.20 A mental, intersubjective and molecular-holistic view

Note that two cells in the 3D matrix are not represented by the philosophy presented in this chapter: a non-mental, subjective and holistic view, and a mental, subjective and molecular view. This does not mean that such views do not exist or are not possible, but that they are not found within the current philosophical literature review.

3.5 Local discussion

The purpose of this chapter is to clarify differences and similarities between views on *concept*, in a way that facilitates interpretation of views on *concept* in frameworks in mathematics education. Three distinctions, between mental and non-mental, between intersubjective and subjective, and between molecular and holistic, are highlighted and then combined in a 3D matrix (Figure 3.13).

While the distinctions mental versus non-mental and molecular versus holistic have been explicit in the texts (for example in Laurence and Margolis (1999)), the distinction intersubjective versus subjective has been more or less implicit. Note that the different distinctions are of different kinds. First, the distinction mental versus non-mental is an ontological distinction concerning the nature of concepts. In mathematics education, a mental view may describe how students develop concepts from a cognitive aspect, and a non-mental view may describe either how concepts develop in communication or how concepts develop in formal mathematics. However, as mentioned in Chapter 2, the distinction mental versus non-mental can also be seen as a question of view on mathematics, where a non-mental view is present in Platonic realism, and a mental view is seen in Aristotelian realism or in conceptualism.

Second, the distinction intersubjective versus subjective can be used to distinguish between a collective concept and individual's personal concepts. In mathematics education, the students' subjective concepts may be assessed and compared to an intersubjective concept that is common in a culture of teachers or mathematicians, and can be described in a curriculum. Third, the distinction molecular versus holistic seems to underlie the discussion about how concepts are structured.

I have chosen to use the terms 'molecularism' and 'holism' instead of 'containment model' and 'inferential model', which are the terms used in Laurence and Margolis (1999). This choice of vocabulary indicates that the models are used at a superficial level that suits an educational perspective. The ontological difference between intersubjective and objective views has also been ignored as I have judged that it is not relevant from an educational perspective, and that it may be difficult, or even impossible, to see the difference in a text in mathematics education.

Regarding demarcations, I want to comment on two views that have been omitted from the discussion. First, there are those considering concepts as abilities. In this position, concepts are neither mental representations, nor non-mental abstract objects. Instead, concepts are abilities connected to cognitive agents. In this view, the number-concept mirrors, or even is, the ability to distinguish numbers from non-numbers and, from that, draw conclusions about numbers. This idea is based on scepticism about the existence of mental representations, which appears in some frameworks[38]. This thesis, however, is written in a context where the three different arenas and the existence of representations are assumed. Thus, views based on scepticism about mental representations have not been considered.[39] (Margolis & Laurence, 2019)

[38] An example is the inferential framework (Bakker & Hußmann, 2017; Derry, 2017).

[39] According to Laurence and Margolis (1999, p. 6), the view claiming that concepts are abilities can be criticised using the same arguments that are used against behaviourism.

In addition, one assumption is that concepts have some kind of structure. There are philosophers, such as Fodor, Garrett, Walker and Parkes (1980/1999), who advocate conceptual atomism and deny conceptual structures. This atomism is opposed to the two models molecularism and holism. However, as will be seen later, a common opinion in the chosen frameworks is that concepts have some kind of structure, named with terms such as 'concept image', 'conception' and 'schema'. This is the reason why the frame includes only molecularism and holism.

Finally, it should be noted that the selection of philosophical views is not comprehensive. Naturally, the choices and simplifications, and the fact that philosophical views are adapted for the purpose of analysing educational texts, have affected the result. Here, the philosophical field and the educational field have different characteristics. It is not certain that the view in every educational framework fits into the matrix or contains all dimensions. How useful the philosophical frame is will be seen in Chapters 5 and 6.

The matrix in Figure 3.13 is developed for analysing views on *concept* in domains that are of interest for educational studies. In the next chapter, the method for the concept analysis of the notion *concept* in some frameworks within mathematics education is more thoroughly described, and indicators are developed for interpreting texts within this field.

4 Methodology

The purpose of conducting a concept analysis is usually to clarify the content of a concept, to find its relations to other concepts and its location within a conceptual framework, to reveal synonymy and equivalence between concepts, and, as a result, to create a basis for elaboration of explications and how terms are used (Nuopponen, 2010a). The overall approach is to start with a literature review and then carry out a text analysis, where different views on a concept are interpreted, compared and categorised (Nuopponen, 2010b). The study in Yoon (2006) can be taken as an example of a concept analysis within mathematics education. In her study, the notions *conceptual system*, *construct* and *model* are analysed within the Models and Modelling (M&M) literature. Yoon described her process of making a concept analysis as follows:

> I began by collecting a large number of excerpts from the M&M literature in which any of the three terms were used, which I then sorted into categories of distinct usages. This process gave me 11 distinct categories of term usage. I then created definitions to describe each of these categories (Yoon, 2006, p. 32)

The first step in the overall approach taken in my thesis was to conduct a literature review in mathematics education, searching for influential ideas about the notion *concept* in mathematics education, and for conceptual frameworks and texts to analyse. A second step was to interpret, analyse and compare views on *concept* in these texts, using the philosophy presented in Chapter 3. Nuopponen (2010b) claims that when systematic concept analysis constitutes the major research method in a study, it contains the following six phases:

- goal and delimitation
- acquisition of domain knowledge, creating a general idea of the field
- compilation of the material
- elaboration of a framework for the analysis
- systematic analysis of the material
- further analysis and conclusions

Here, the process included multiple iterations, going back and forth between the phases described above, using both deductive and inductive elements. The tool for analysing the texts was developed as a result of a literature review within another field, the common field of philosophy of language and concept research within cognitive science, described in Chapter 3. In order to use this tool for concrete analyses of texts, indicators were developed.

The first section below presents the method used for the literature review in mathematics education. A goal of this review was to find texts to analyse, and the selection of texts is presented in this section. The second section presents the initial analysis, using explications in the selected texts, focusing on explicated and non-explicated notions and how these are connected in the frameworks. As a result of this analysis, overview concept maps were developed. The third section describes how the analysing tool was used, and as an operationalisation of that, indicators used for interpreting the texts are presented in the fourth section. The fifth section contains delimitations of the object of study. The description of the methodology, which started in Chapter 1, is then completed and Chapter 4 ends with a compiled description of the study and some comments.

4.1 Method for the literature review

The first section in this chapter presents the grounds on which the literature review has been carried out, the criteria that have been employed for selecting literature and how the actual selection procedure has been conducted. Note that the concept analysis can be seen as a kind of literature review, where views on the notion *concept* are interpreted from texts using some frequently used frameworks for conceptual understanding in mathematics education.

According to Gough, Oliver and Thomas (2012), the aim of systematic research reviews within the social sciences can either be to aggregate, or add up, findings from multiple similar studies, or to configure, or arrange, findings of studies. A configurative approach, which is the approach taken in the current study, may have open questions about meaning, with few pre-defined concepts, that are answered with qualitative data and iterative methods. Often, the approach taken is inductive and does not use pre-specified methods. Further, in a configurative review, the syntheses can be seen as forming patterns, in which the findings together form a coherent whole. These involve interpretive concept analysis and are associated with reviews where the concepts are the data for the analysis. (Gough & Thomas, 2012; Gough, Thomas & Oliver, 2012)

4.1.1 How the review was conducted

Before describing in more detail how the review was conducted, the design is here presented at an overall level. First, the review took a stance on an overall question: Which views on *concept* can be found in mathematics education? A second step in the review was to map the field. Often, such a map is the object of the review. However, it can also be a useful stage in the review (Gough & Thomas, 2012), which is the case in the current study. A third step consisted of narrowing the topic, based on the insights from the second step. The result of this narrowing was a selection including a sub-group of the frameworks found in the second step. In the final step, texts exemplifying the selection were found, and later analysed. The whole process, including these steps, is seen in Figure 4.1.

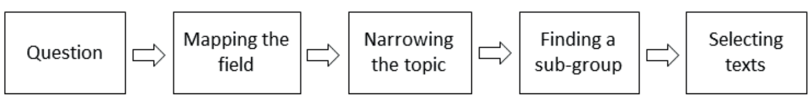


Figure 4.1 Design of the literature review

The search for literature had the aim of mapping the field and finding so called theoretical lines, describing how the conceptual frameworks had been developed over the years. For this purpose, it was important to understand the background of the frameworks and to include older texts, from 1968 onwards, that could give an understanding of later ideas. Further, I did not found it necessary to search for all articles about concepts, but the goal was to find influential conceptual frameworks and also to find alternative aspects of concepts. Since the concept *number* is central in the mathematical field, texts about arithmetical concepts were chosen to a higher extent compared to texts about geometrical concepts. There was also a focus on older students, at the expense of texts describing younger children. The reason for this was my interest in conceptual structures, and relations between concepts, which demands more conceptual knowledge. When it comes to, for example, arithmetic, this demands knowledge of different types of numbers. Generally, four different stages were used in multiple iterations for searching literature.

4.1.1.1 Stage 1

The review started with texts that I had found within teacher education and courses for PhD students, and that were recommended by researchers. Some of these texts (Tall & Vinner, 1981; Sfard, 1991; Duval, 2006; Vergnaud, 1998; 2009) were considered to be key texts in the field, based on how often they appeared in discussions of concepts.

4.1.1.2 Stage 2

The initial searches were made using electronic databases. First, more general notions, such as *concept* together with *mathematics* were used and then notions were picked from the key texts described above: *concept image*, *concept definition*, *conception*, *structural*, *operational*, *dual conception*, *representation* and *conceptual*. The searches were conducted with programmes such as SCOPUS, DIVA, and Google during spring 2014. A text found in these searches, which was later analysed, is Bingolbali and Monaghan (2008). In later stages, Google was used for searches aiming at getting further understanding of some of the notions required for the analysis. As an example, these searches led to the homepage of David Tall, which in turn led to some of his articles.

After these searches, I had a diverse list of texts without obvious connections between them; it was hard to see how the frameworks within the texts were related. At that time, I was inspired by Häggström (2008), who used a manual search method, browsing journals in mathematics education by reading abstracts, comparing the results with results using search engines on websites. In this study, more articles were generated by the manual search, which also allowed for more informed decisions concerning which articles to include in or exclude from the review. Based on that result, I abandoned the method of using search engines as the main method for the literature review.

4.1.1.3 Stage 3

After the initial searches, a manual survey was conducted, similar to the philosophical literature review described in the introduction of Chapter 3. In this survey, all articles from the journal *Educational Studies in Mathematics* were chosen from 1968 to August 2014. This journal contains theoretical discussions and three of the key texts from Stage 1 are published in this journal. Further, it is highly ranked within the field (Toerner & Arzarello, 2012). The reason for the chosen time period is that since some of the references in the key texts were from the 1970s and the journal started in 1968, there was no reason to delimit the literature review in time. First, all abstracts written in English or French from the articles in the journal were read, and ideas about how conceptual frameworks were used and how they related to each other were noted[40]. Even though I did not read all the articles, these notes gave an understanding of which frameworks were used, and how they had developed over time.

The articles that met certain criteria were selected. One such criterion was whether the abstracts used notions such as *concept*, *conception*, *concept image*, *number*, *arithmetic*, *representation*, *meaning* and *cognitive*. Another criterion was whether the content of the articles seemed to meet my interest, which now was focused on the notion *concept*, relations between concepts, and how concepts and cognitive structures are developed and relate to the world. The focus was not on ideas about arithmetic or numbers, but on concepts and number-concepts. Further, bibliographical articles were avoided. There has also been a greater focus on recent articles and articles from the 1970s were only put on the list when I had a clear reason to believe that they would be useful. After the manual search, the list contained 222 articles not including those found using the other approaches. Semadeni (2008) may be taken as an example of a text found in this process.

In the second step, parts or the whole of some of the chosen articles were read, with the purpose of obtaining essential ideas about concepts and how the notion *concept* and related notions were used. For each article, the introduction was read to see if the content in the text really met my interest, which is described above. If so, the theoretical parts and the discussion were read. If the conceptual framework in turn clearly described concepts, then the article was read in full. The questions guiding the reading were:

- Is the notion *concept* explicated?
- Is the notion *concept* used, and how is it used?
- Is the text referring to the literature presented in Chapter 3 above?

[40] The abstracts that were used for the study were those of articles, and not of book reviews, comments and editorial notes, which most often lacked abstracts. If a paper lacked an abstract, the beginning of the introduction was read instead. Smaller papers without abstracts, articles in other languages, and replies on other articles were avoided. Overall, more than 1000 abstracts of original papers have been read.

When I had read a number of texts, starting with the more recent ones, using this strategy, I chose some frameworks and began the analysis. The narrowing of the topic and the selection is described below. During this work, I started to look for references in the texts to find the origins of the ideas in the frameworks.

4.1.1.4 Stage 4

The fourth and final stage of the review included looking for references in the texts found in Stages 1, 2 and 3, and reading these texts based on the same strategies as in Stage 3. For example, Dubinsky (1991) and Asiala et al. (1996) were both found as references within Semadeni (2008).

4.1.2 Narrowing the topic

The four stages described above resulted in a diverse list of ideas and conceptual frameworks used for describing concepts in mathematics education. This list contained the concept image-concept definition (CICD) framework, various frameworks using the stage models of cognitive development, such as the operational-structural (OS) framework, the procept framework, APOS, and the van Hiele model. Further, it contained frameworks such as the framework of conceptual fields, the three worlds of mathematics, frameworks using semiotics, the commognition framework, variation theory, the inferential framework, frameworks using cognitive science of mathematics, and so on.

To carry out concept analyses of all these frameworks was judged to be an overly wide project, and I realised that the topic had to be narrowed. Among other issues, it was a question of how much background knowledge was needed, in order to understand the frameworks. As an example, I did not want to get into a situation where thorough interpretations of the perspectives of Piaget and Vygotsky, and how these were related, were needed. Further, frameworks depending on continental philosophy, such as the philosophy of Hegel, were deselected as it was less interesting and methodologically problematic to include such ideas in the analytical philosophical background and the analysing tool. The final subgroup includes some frameworks with a common philosophical background, that are related through referencing:

- The CICD framework
- The OS framework
- The procept framework
- The APOS framework

These different frameworks were classified by how they relate to each other, in a kind of historical order. Some texts that adhered to these frameworks, which were published earlier than the others or were frequently referred to, were considered basic texts. The views in these basic texts and, in some cases, how they have later been developed, were then analysed. In the next section, the frameworks and the analysed texts are presented and described.

4.1.3 Description of the analysed texts

Chapter 5 presents a concept analysis of two earlier texts using a basic version of the CICD framework and two later texts using developments of that framework. The notion *concept image* first appeared in Vinner and Hershkowitz (1980) in a study of students' understanding of geometric objects. Further, Tall and Vinner (1981) is, it seems, the earliest article that may naturally be placed within a modern discussion of concepts in mathematics education, and it is still used in current bibliographies. Consequently, the CICD framework may be considered to be the oldest framework that is still used for analysing conceptual understanding in mathematics education. In the later versions, the framework has been developed in various ways. As is seen in Chapter 5 below, in Semadeni (2008) the notions *concept image* and *concept definition* have been complemented with *deep intuition*, to describe how a developed concept image can be used instead of a concept definition. Further, in Bingolbali and Monaghan (2008), *concept image* is interpreted as a social idea, which depends on the context.

The frameworks using the stage models of concept development are based on interpretations of Piaget's perspective, and are still common within mathematics education. Chapter 6 contains analyses of several process to object (PO) frameworks, and differences and similarities between the views within them. The chosen frameworks describing the development of arithmetic concepts are the OS framework (Sfard, 1991), the procept framework (Gray and Tall, 1994), and the APOS framework (Dubinsky, 1991; Asiala et al., 1996).

In conclusion, the texts analysed are the ones seen in Table 4.1. As the reader may note, the structures of Chapters 5 and 6 differ from each other. In Chapter

5, a basic version of the framework and how it has been developed are analysed. In Chapter 6, basic versions of the different frameworks are analysed in the same chapter.

Table 4.1 Texts analysed in the study

Framework	**Texts**	**Abbreviations**
The basic version of the CICD framework (CICD^0)	Vinner and Hershkowitz (1980) Tall and Vinner (1981)	VIHE TV
A version of the CICD framework augmented with *deep intuition* (CICD$^+$)	Semadeni (2008)	SEM
A version of the CICD framework used for social studies (CICD*)	Bingolbali and Monaghan (2008)	BIMO
The OS framework	Sfard (1991)	SF
The procept framework	Gray and Tall (1994)	GT
The APOS framework	Dubinsky (1991) Asiala et al. (1996)	DUB ASI

In the analysis, the focus is on the content of the analysed texts, and not on the intentions of the authors. In addition, in Chapters 5 and 6, the different texts are given names, which are abbreviations of the names of the authors. To exemplify, the text written by Gray and Tall (1994) is called GT. The purpose of introducing abbreviations instead of repeating references is to facilitate reading, when the same text is continually discussed within longer sections.

Two methods are used for the analysis. In an initial analysis, explications of notions are used for classifying notions as explicated or non-explicated, and the connections between the notions are classified as hierarchical or non-hierarchical. Based on these classifications, superficial concept maps are constructed, representing overall conceptual structures of the frameworks in the different texts. The purpose of making these maps is to represent the roles of the notions, the relations between them, and how they interact in the frameworks. After that, views on *concept* and related notions, seen in how terms are used in the texts, are interpreted using the analysing tool developed in Chapter 3. In order to use the analysing tool for interpreting texts in mathematics education, indicators are developed (see Section 4.4).

4.2 Concept maps

It may be hard to analyse the view on *concept* in a text without also analysing how *concept* is presented as connecting to other notions. As an example, the notion *concept image* and the connection between *concept* and *concept image* are important for the interpretation of texts using the CICD framework. This is why, in the initial analysis, I constructed concept maps representing conceptual structures of the frameworks in the different texts.

In this thesis, I do not use a specific theory of concept maps or a method developed by others. Instead, I use concept maps as visual representations of the conceptual frameworks described in the texts analysed. They contain notions, which are enclosed in boxes, and connections between the notions are indicated by lines. Words on these lines specify the connections. I develop the concept maps from the content of explications of *concept* and notions described as connected to *concept*, in the theoretical backgrounds of the texts. Together, these notions describe conceptual structures. The purpose is to offer overall maps of the frameworks, and not to thoroughly analyse the meanings of the explications, or to penetrate the exact nature of the connections between the notions.

The origin of the notion *explication* can be found in Carnap (1947). According to him, the task of making a vague or not quite exact concept, used in everyday life or in an earlier stage of scientific development, more exact can be called explicating. An example from the analysis interpreted as an explication is shown below:

> The notion *mental picture* of a concept C is "the set of all pictures that have ever been associated with C, in P's [a person] mind" (Vinner & Hershkowitz, 1980, p. 177).

In the analysis, notions are divided into two groups, those which have explications in the texts and those which have not, but which appears in explications of other notions. The reason for this division is that it says something about the roles these notions play in the frameworks. Notions that lack explications may be basic notions used in explications of more complex notions, as seen in the philosophical perspective described in Section 3.3.1. However, a notion without explication may also be a complex one, where the explication is taken for granted. A prerequisite in the analysis for a description to count as an explication is that the notion is specified. It is not enough to give examples of what counts as falling under the notion. Neither does a discussion of the properties of a notion always count as an explication. Below, an example from the analysis which I have judged not to be an explication, is shown. This example describes what a student with a process conception can do, but the notion is not actually specified.

> An individual who has a process conception of a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps (Asiala et al., 1996, p. 7)

After dividing descriptions of notions into explications or not, I interpret how connections between the notions are described. These connections are divided into hierarchical or non-hierarchical. If a notion is included in another notion, then the connection between these two notions is judged to be a hierarchical relation. If, however, one of the notions is not included in another notion, then the described connection is judged to be non-hierarchical. To use the same examples as in Section 3.3.1, the relation between *dog* and *Labrador Retriever* is hierarchical as the class of Labrador Retrievers is included in the class of dogs. A connection between *eight* and *octagon*, however, indicating that octagons can be divided into eights, is not hierarchical, as neither the class of eights nor the class of octagons is included in the other. An example of a hierarchical relation found in the analysis is the part-whole relation between *cognitive structure* and *concept image*, where the latter is described as being part of the former (Vinner & Hershkowitz, 1980, p. 178). An example of a non-hierarchical connection, on the other hand, is when it is claimed that deep intuitions and concept definitions correspond to each other (Semadeni, 2008, p. 3). Naturally, non-hierarchical connections are of different types. However, I have chosen not to categorise different types of connections, but rather to use the distinction between hierarchical and non-hierarchical in a general sense.

Based on the formulations of the explications, the explicated and non-explicated notions, and the hierarchical and non-hierarchical connections, are presented in concept maps representing the different frameworks. As seen in Figure 4.2, explicated and non-explicated notions are represented by boxes. Further, the hierarchical and non-hierarchical connections are represented by two different types of arrows. When it comes to the hierarchical relations, the arrows start with the higher-level notions and end with the lower-level notions, in view of the philosophy described in Section 3.3.3. When it comes to the non-hierarchical connections, on the other hand, the direction of the arrows is based on how these are formulated. As an example, when it is claimed that a concept image is associated with a concept (Tall & Vinner, 1981, p. 152), the arrow representing the connection *associated with* goes from *concept image* to *concept*.

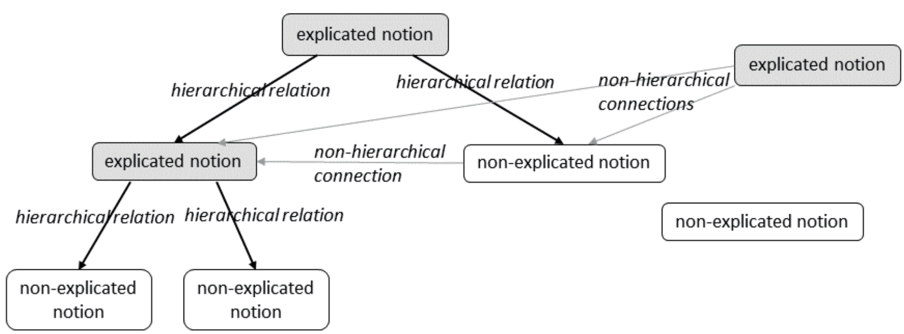


Figure 4.2 Example of a concept map

4.3 Using the analysing tool

After the initial analysis, the views on *concept* are interpreted by using the analysing tool described in Chapter 3. To repeat, this tool makes three distinctions, represented in the matrix in Figure 4.3, which is the same as Figure 3.13.

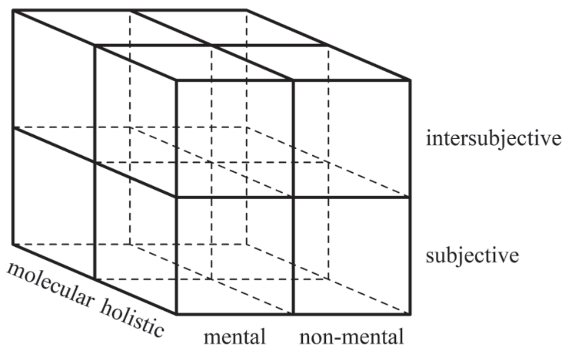


Figure 4.3 The three-dimensional matrix

As stated in Chapter 1, the tool is developed through an interplay between the philosophical literature review and the analysis of texts representing frameworks in mathematics education. For example, after the philosophical literature review, the preliminary analysing tool consisted only of the distinctions intersubjective versus subjective and molecular versus holistic. However, during the analysis of texts using the PO frameworks the distinction mental versus non-mental was added to the tool. Further, not all distinctions are seen in all texts. For example, when analysing Bingolbali and Monaghan (2008), and Gray and Tall (1994), only the distinctions mental versus non-mental and intersubjective versus subjective are used, since the distinction molecular versus holistic is not clearly visible in the texts, and the tool is then limited to a 2D matrix.

During the analysis, if explications of the notion *concept* are found in the texts, where it is clearly stated that concepts are mental or non-mental, intersubjective or subjective, and molecular or holistic, then the method consists of a straight forward interpretation of these explications. However, the most common method is to interpret how terms are used in more involved contexts.

Furthermore, even though there is a clear explication, one issue for the analysis is to see whether the view on *concept* shown in the way terms are used is coherent with the view shown in the explication.

When interpreting views on *concept* in the way terms are used, the method is to search for formulations, divided into what I call indicators, showing the different views. These indicators, described below, have been developed through an interplay between the characteristics of the analysing tool and the analysis of texts using the frameworks. In the initial reading, when the term 'concept' was found in the texts, I made a note in the margin and underscored the formulation where 'concept' was found. This formulation was then interpreted from the perspective of the three distinctions, to see whether it offered an argument for a certain view. The same procedure was used for related terms, such as 'conception' and 'concept image'. Afterwards, I sorted the formulations. As an example, if a formulation was interpreted as indicating a mental view, then it was sorted under the heading *mental*, and so on. The formulations under the respective headings were then categorised into the initial indicators.

For example, a view in a certain text could be interpreted as mental and intersubjective. Next, there could be uncertainties regarding whether the view is molecular or holistic, and the text was read again to see if there were arguments for a molecular or holistic view. These new formulations were then collected, together with the already existing ones, under the headings *Molecular* and *Holistic*. When the formulations had been categorised in this way, I looked at the overall picture emerging from the collected indicators to see whether they offered a complete description of the different views, from the perspective of the philosophy presented in Chapter 3. In some cases, the indicators were supplemented on the basis of this philosophical perspective. Next, the texts were read again to see whether these new indicators could be used for interpreting the texts as well, and so on. The process consisted of an interplay between reading, interpreting and writing, and the indicators were developed during the whole process.

The purpose of the indicators is to clarify how the analysis is made and to ensure that texts coming from different frameworks are interpreted in a similar way. The indicators should be seen as arguments for a text to be seen as belonging to a particular view. Clear arguments strengthen the position for that view, while an unclear argument diminishes the position.

An overall principle in the analysis is that if a text, in a description of concepts, refers to another text, then the references, in the shape of specific terms or formal references, indicate that the more recent text has a view on *concept* similar to the view in the text referred to. One example, referring to the perspective of Piaget, can be seen in the quote below. Here, the term 'schema' is naturally understood from a Piagetian perspective, and a concept is then a kind of schema. This makes the view on *concept* mental in this context.

> the duality between concept and schema is based on the same fundamental idea in which a named concept has rich internal links that reveal it to be a schema (Tall, 2013, p. 80)

To sum up, the procedure for finding views on *concept* in the educational texts consisted primarily of searching for explications and for references to texts that have already known views on *concept*, and using the indicators.

4.4 Indicators for interpreting views on *concept*

In this section, the indicators of the analysing tool are listed in three sections, based on the three different distinctions.

4.4.1 Concepts seen as mental or as non-mental

As mentioned in Chapter 3, mental views on *concept* are common in contemporary empiricism and concept research in cognitive science. Consequently, frameworks building on psychological ideas may have mental views. As was also mentioned in Chapter 3, non-mental views on *concept* are common in philosophies of language, where concepts may be senses in a Fregean view, which are independent of the mind, forming the meanings of words and other expressions. In texts in mathematics education, frameworks building on linguistic ideas may have non-mental views. Further, non-mental views can be seen in logicist approaches. Hence, frameworks using such ideas, taking the nature of mathematics into account, may have non-mental views as well.

4.4.1.1 Concepts seen as mental

In the analysis of texts in mathematics education, the mental position is concretised in formulations that can be separated by three indicators, at a general level.

[1] The first indicator for a mental view concerns formulations explicitly claiming that concepts are mental representations, e.g. 'concepts are part of mental structures' and 'a concept is an internal representation'. One example from the analysis can be found in Gray and Tall (1994), where it is stated that concepts are included in mental structures:

> The successful mathematical thinker uses a mental structure which is an amalgam of process and concept (Gray & Tall, 1994, p. 1)

This indicator is also used for formulations where concepts include mental elements. One example of a mental view on the notion *concept image* can be seen below, where concept images include mental attributes:

> all mental attributes associated with a concept, whether conscious or unconscious, should be included in the concept image (Tall & Vinner, 1981, p. 152)

Here, the indicator is used for analysing *concept image*, which is a notion used for the cognitive structure. That the indicators are used for other notions related to *concept* is discussed in Section 4.5 below.

[2] The second indicator for a mental view concerns formulations claiming that we have concepts or use concepts in our thinking, since the only way in which we can use something in our thinking is that this something is part of our thinking, and hence is mental. One example from the analysis can be found in the quote above, from Gray and Tall (1994), where it is stated that a mathematical thinker uses a structure that includes concepts. Consequently different indicators may be used to interpret the same quote. Further, in the quote below, concepts can be mentally manipulated, which is also an example of where this indicator can be used:

> the concept is given a symbol or name which enables it to be communicated and aids in its mental manipulation. (Tall & Vinner, 1981, p. 151)

[3] The third indicator for a mental view concerns formulations claiming that individuals build, develop, construct or acquire concepts. The terms 'build', 'develop' and 'construct' are not enough to judge views as mental, since a non-

mental concept may be developed in a cultural sense, or constructed theoretically. In the chosen texts, I have found that if it is claimed that an individual develops concepts, then there probably is a mental view present in the text. However, if it is claimed that several individuals together develop concepts, then the view can be either mental or non-mental. The following quotes are two examples from the analysis, where the view on *concept* is interpreted as mental:

> The concept can develop in the mind of an individual (Asiala et al., 1996, p. 4)

> How students actually might go about constructing the concept (Dubinsky, 1991, p. 96)

In the first quote, the formulation 'develop in the mind' is the reason for judging the view as mental. In the second quote, the term 'constructing' is interpreted based on the observation that the framework relies on Piaget's perspective (Asiala et al., 1996, p. 5; Dubinsky, 1991, pp. 95–107), and hence the view is judged as mental.

4.4.1.2 Concepts seen as non-mental

In the analysis, a non-mental view is concretised in formulations that can be separated by the following four indicators:

[4] The first indicator for a non-mental view concerns formulations claiming that concepts are building blocks in formal mathematics, since concepts then appear in mathematics outside of the individual mind. These can be formulations claiming that concepts are something that can be constructed or analysed theoretically. Two examples of formulations showing non-mental views, based on this indicator, are seen below:

> the word "concept" (sometimes replaced by "notion") will be mentioned whenever a mathematical idea is concerned in its "official" form - as a theoretical construct within "the formal universe of ideal knowledge" (Sfard, 1991, p. 3)

> conceptual understanding implies that the relationships inherent in all of the different components that form 3 are also available (Gray & Tall, 1994, p. 8)

In the second quote, the judgement of a non-mental view relies on the observation that in another part of the text (Gray & Tall, 1994, p. 118), a distinction is made between mathematics and cognition, where mathematics as a field should not be considered to be cognitive.

[5] The second indicator for a non-mental view concerns formulations claiming that concepts appear in the culture of mathematicians, or others, and may be seen in texts or other types of language. The following quote constitutes one example where the view is judged to be non-mental, based on this indicator, as the curriculum concerns concepts:

> The requirements of this curriculum distinguish between the skills or procedures that individuals need to have acquired in order to do things, and the concepts or basic facts, which they are expected to know, on which they operate with their skills. (Gray & Tall, 1994, p. 117)

Below, if propositions are non-mental objects, constructed in language, then a non-mental view on *schema* is indicated, as schemas include these propositions:

> According to this analysis, the objects in the propositional calculus schema are the propositions. (Dubinsky, 1991, p. 112)

[6] The third indicator for a non-mental view concerns formulations indicating that concepts are separate from the individual. One example can be seen in the expression "thoughts about a concept" (Dubinsky, 1991, p. 114), since in a mental view we use concepts in our thinking: we do not, at least not normally, think about them. Another example is seen in the formulation "[m]any concepts we meet in mathematics" (Tall & Vinner, 1981, p. 151), since something that you meet is not something that is included in existing mental representations. Further, formulations such as "a student is presented with concepts" (Dubinsky, 1991, p. 117) indicate a non-mental view as well.

4.4.2 Concepts seen as intersubjective or as subjective

An intersubjective view can be used to describe the idea that concepts are developed in a community of people, such as mathematicians. It may either be non-mental and something that several people share through the usage of language, or a mental representation partly integrated into a collective understanding. In both cases, our personal understanding can be evaluated against the intersubjective concept. Further, a subjective view on *concept* can be used to describe concepts developed by individuals.

4.4.2.1 Concepts seen as intersubjective

In the analysis, the intersubjective view is concretised in formulations that can be separated by the following four indicators:

[7] The first indicator for an intersubjective view concerns formulations indicating that concepts are developed within a particular culture. An example of such a formulation appears in the quote exemplifying Indicator [5] above, and in fact the formulations referring to concepts in the curriculum indicate both a non-mental and an intersubjective view. This is the reason why Indicator [5] and Indicator [7] are often connected. Another example of an intersubjective view, using the formulation 'standard academic presentation', is seen below:

> the standard academic presentation, based on set theory, of the concepts of 'ordered pair', 'relation', 'function', 'sequence' consists of the following six steps (Semadeni, 2008, p. 10)

[8] The second indicator for an intersubjective view concerns formulations showing that concepts are developed in mathematics, since mathematics as a field is often not seen as a subjective construction, but as something that is shared in a culture. An example is seen in the formulation "the mathematical concepts as formally defined" (Tall & Vinner, 1981, p. 151), indicating that concepts are something that are developed in the community of mathematicians.

Note that Indicator [8] often is connected to Indicator [4], and that Indicator [4], which is used for views where concepts are part of formal mathematics, implies that Indicator [8] can be used. However, as mathematics can be seen as a mental construction, Indicator [8] does not always imply Indicator [4].

[9] The third indicator for an intersubjective view concerns formulations indicating that there is a unique concept of a certain phenomenon, such as "the concept of an altitude" (Vinner & Hershkowitz, 1980, p. 182), and formulations indicating that concepts are culturally dependent, such as "a concept is presented in an 'official' form" (Sfard, 1991, p. 3). Further, formulations claiming that concepts are something that can be studied or formulated axiomatically, requiring that there is a unique concept of a phenomenon, indicate an intersubjective view. The same holds for formulations claiming that there is a necessity, when something necessarily leads to the development of a concept.

The idea that a cognitive structure may be incorrect and that it is possible to judge whether someone has reached a certain level of development points to an intersubjective view as well:

> It [the concept image] is a dynamic entity that develops, differentially over students, through a multitude of experiences. Some of these will, from a mathematical viewpoint, be incorrect, e.g. squaring a number could be defined as "multiplying a number by itself" and an associated property of squaring, grounded in students' experiences natural numbers, might be "squaring makes the number bigger". (Bingolbali & Monaghan, 2008, pp. 20–21)

In the quote above, there is something intersubjective, a concept or a culturally dependent concept image, that the subjective concept image can be judged against.

[10] The fourth indicator for an intersubjective view concerns formulations claiming that through communication, a conception or the meaning of a term obtains a shared reality and becomes an intersubjective concept. Further, an example of the usage of this indicator can be found when terms such as "reification" (Semadeni, 2008, p. 5) are used, referring to Piaget's perspective, where a schema is developed into an intersubjective concept through certain stages. Another example is seen below:

> At the stage of interiorization a learner gets acquainted with the processes which will eventually give rise to a new concept (like counting which leads to natural numbers, subtracting which yields negatives, or algebraic manipulations which turn into functions). (Sfard, 1991, p. 18)

4.4.2.2 Concepts seen as subjective

In the analysis, the subjective view is concretised in formulations that can be separated by the following two indicators:

[11] The first indicator for a subjective view concerns formulations indicating that people develop their own concepts. An example where this indicator can be used for the notion *concept image* is seen in the formulation "an individual's concept image" (Tall & Vinner, 1981, p. 153), where an individual has his or her own concept image. Further, a subjective view on *concept image* is seen in the same quote as is used in Indicator [9], where students develop different concept images, which can be incorrect. Here the students' individual concept images can be judged compared to an intersubjective view:

> [The concept image] is a dynamic entity that develops, differentially over students, through a multitude of experiences. Some of these will, from a mathematical viewpoint, be incorrect, e.g. squaring a number could be defined as "multiplying a number by itself" and an associated property of squaring, grounded in students' experiences natural numbers, might be "squaring makes the number bigger". (Bingolbali & Monaghan, 2008, pp. 20–21)

[12] The second indicator for a subjective view concerns formulations indicating that different students have different concepts. As an example, the formulation "revealing the concept images of our students" (Vinner & Hershkowitz, 1980, p. 180) points to a subjective view on *concept image*. Further, the idea that students' concepts or concept images change may point to a subjective view as well:

> it was designed to access students' concept image of the derivative over time (Bingolbali & Monaghan, 2008, p. 27)

Here, the concept image depends both on time and the individual.

4.4.3 Concepts seen as molecular or as holistic

In a molecular view, concepts are hierarchically structured. Here, some concepts are more basic than others and complex concepts could be defined from basic ones. In a holistic view, on the other hand, concepts interrelate with each other in a web-like structure, in which no concept is more basic than another.

4.4.3.1 Concepts seen as molecular

In the analysis, the molecular view on *concept* is concretised in formulations that can be separated by the following indicators:

[13] The first indicator for a molecular view concerns formulations claiming that concepts can be defined, including some formulations about composition, and, for example, formulations claiming the importance of realising in geometry that the class of squares is a subclass of the class of rectangles. This indicator can be used in relation to the following quote:

> If we take a scrutinizing look at any mathematical concept, more often than not we shall find that it can be defined - thus conceived - both structurally and operationally. (Sfard, 1991, p. 5)

An example where some concepts are more basic than others, thus indicating a molecular view, is seen below:

> Why mathematicians themselves needed several centuries to arrive at fully structural versions of the most basic concepts, such as number or function? (Sfard, 1991, p. 30)

[14] The second indicator for a molecular view concerns formulations claiming the importance of understanding definitions for conceptual understanding. It can also be formulations claiming that students develop concepts in a logical order, where some concepts are more basic than others. The following quote is an example, where this indicator may be used:

> Both psychologically and mathematically, multiplication is the addition of additions. (Dubinsky, 1991, p. 100)

The difference between Indicator [13] and Indicator [14] is that the first of these indicators is used for interpreting non-mental views on *concept* and that the second is used for interpreting mental views. In the quote above, both Indicator [13] and Indicator [14] may be used, as the formulation considers both mental and non-mental concepts.

[15] The third indicator for a molecular view concerns formulations about hierarchical structures, since hierarchy is one aspect of the molecular model. This indicator can be used both for mental and non-mental views. In the following, a molecular mental view is shown by the idea that a conception is developed partly by new layers being added to the structure:

> formation of a structural conception means reorganizing the cognitive schema by adding new layers - by turning sequential aggregates into hierarchical structures. (Sfard, 1991, p. 28)

Further, in the next quote, individuals' knowledge is hierarchical, since the knowledge of D may entail the knowledge of A, B and C:

> In such a group we will not find people who know D without knowing also A, B and C or people who know C without knowing also A and B and so on (Vinner & Hershkowitz, 1980, p. 182)

An example where this indicator is used in a non-mental context is when the historical development of mathematics is presented as a hierarchy in a figure in Sfard (1991, p. 13), where natural numbers are more basic than reals, which in turn are more basic than complex numbers.

4.4.3.2 Concepts seen as holistic

In the analysis, the holistic view on *concept* is concretised in formulations that can be separated by the following three indicators:

[16] The first indicator for a holistic view concerns formulations indicating that the cognitive structure uses relationships that are not hierarchical. An example can be seen below:

> Deep intuition may be thought of as a kind of conceptual synthesis or mental blend which includes the meaning of X, its basic properties derived from relationships with other objects, both mathematical and non-mathematical, in real life and in physics [...] (Semadeni, 2008, p. 9)

Here, relationships between mathematical objects and objects from real life and physics must be non-hierarchical, as the extension of a mathematical concept cannot be included in the extension of a concept describing real life and vice versa. In this quote, the view on the structure of deep intuition affects the view on the structure of concepts as well. Thus, the view on *deep intuition* as a kind of conceptual synthesis indicates a holistic view on *concept*. Note that this indicator is used for mental views on the cognitive structure. The fact that information other than definitions is used can be seen as a knowledge effect (see Section 3.3.2.2).

[17] The second indicator for a holistic view concerns formulations pointing to a complexity in a non-mental structure of concepts. As an example, it can be used for interpreting the following quote, where no schema is more basic than others:

> For example, there will be a proof schema, which can include a schema for proof by induction. This latter in turn could include a schema for proposition valued functions of the positive integers […]. Hence, there would be a relation with the schemas for number, for function, and for proposition. On the other hand, there is a sense in which a proof is an action applied to a proposition, so that the proof schema might be one of the processes in the proposition schema. (Dubinsky, 1991, p. 102)

[18] The third indicator for a holistic view concerns formulations criticising the idea that concepts have definitions, as in the following:

> For some of our concepts we also have concept definition in addition to the concept image. For many other concepts we do not. (Vinner & Hershkowitz, 1980, p. 177)

Here it may be noted that just the fact that every concept cannot be defined is not an argument for a holistic view. Even in a molecular view, there are basic concepts that are not defined, but used in definitions of more complex concepts. However, the fact that it is pointed out that there are concepts that cannot be defined, I would say, is a criticism against a definitional approach and an indication of a holistic view. Further, in Vinner and Hershkowitz (1980), examples of concepts without definitions are *house* and *orange*. Here, it would be possible to argue that these concepts actually have definitions. Hence, the quote above can be seen as an indication of holistic features.

4.5 The object of study

The main object of the analysis is the notion *concept*. However, there are other notions related to *concept*, which affect the analysis. It is in fact hard to interpret *concept* without describing or analysing these other notions as well, and the connections between the notions seem to be important for understanding views on *concept*. That is why an initial analysis of explications in the different texts focuses on which notions are explicated and which are not, and on connections between the notions. In that pre-analysis, whether a notion has an explication or not indicates the role the notion has in the framework, and whether the notion is considered a notion that needs to be theorised. Furthermore, a reason for not explicating a notion may be that the notion is taken for granted. In addition, some of the notions have features that enable an analysis from the perspective of the three distinctions, using the indicators. Which notions are analysed depends on the chosen framework. As an example, in the CICD framework the notion *concept image* is analysed, which is not the case in the OS framework. Further, some notions affecting the interpretation are discussed, but not analysed with the analysing tool. Table 4.2 offers a list of the notions, in addition to the notion *concept*, analysed in the different parts of the study.

Table 4.2 Notions analysed in the study

Framework	**Notions analysed with indicators**	**Notions discussed without indicators**
The CICD framework	*concept image*	*concept definition*
The OS framework	*conception*	
The procept framework		*procept*
The APOS framework	*schema*	

4.6 A summary of the study

The first four chapters in the thesis form the methodology of the study. In Chapter 1, the initial interest and the methodology are presented at a general level, the study is placed in a context within mathematics education, and aim and research question are presented. Chapter 2 contains a historical background to the study and a brief introduction to the basic philosophical terminology. Chapter 3 contains a philosophical literature review, aiming at finding some key references and at giving an overall picture of the contemporary discussion of concepts in philosophy of language and concept research within cognitive science. From this review the distinctions in the analysing tool are extracted. Finally, in the current chapter, methodological issues are presented, including delimitations of the object of study and methods used both for the literature review in mathematics education and for the analysis of the texts. This includes the development of the indicators. Before presenting the analyses in Chapters 5 and 6, I summarise below what is said about the study in the initial chapters. This includes different descriptions of goals and purposes, delimitations and underlying assumptions. Further, I make some comments on the methodology.

4.6.1 The nature of the study

The aim of the study is to carry out a concept analysis of the notion *concept*, in some frequently used frameworks for analysing conceptual understanding in mathematics education. The design is developed for answering the following research question:

> Which views on *concept* may be found in texts using the chosen frameworks, from the perspective of the distinctions mental versus non-mental, intersubjective versus subjective and molecular versus holistic?

The concept analysis involves a configurative literature review, searching for influential ideas in mathematics education from 1968 onwards. Among other things, there is a focus on the background of current views in order to further understand views on *concept* that are held today. A particular interest is taken in the concept *number* and texts about arithmetic concepts are chosen to a greater extent compared with texts on geometric concepts. Further, texts focusing on older students and adults are prioritised, as the focus is on conceptual structures. As described in Section 4.1.2, the final selection contains some frameworks having a common philosophical background, which are related through referencing.

The purpose is to influence mathematics education research in several ways. Theoretically, the goal is to clarify, categorise and compare views on *concept* in the studied frameworks, in order to increase the understanding of different views on *concept*, to find possible incoherencies, and to contribute to theory development in the field. From a wider perspective, the study can be seen as a first step in the development of a conceptual framework for analysing concepts. Methodologically, the goal is to contribute to the method of making concept analyses in the field. There are not many descriptions of how to conduct concept analyses, especially not in educational research, and the experiences from the current study may hopefully be of use for further development of such methods. The approach taken includes a tool for analysing views on *concept* in mathematics education. This tool includes three distinctions: mental versus non-mental, intersubjective versus subjective, and molecular versus holistic. There are reasons to believe that this method will both broaden and at the same time delimit what may be found in the analysis, since it may facilitate seeing philosophical aspects in the views and at the same time hinder the detection of aspects that are not in the analysing tool. In the analysis of the texts, explicated views are compared to the way terms are used, and one issue is whether the explications of the notion *concept* and the usage of the term 'concept' are coherent. Further, views on the notion *concept* are compared to views on *concept image*, *conception* and *schema*, listed in Table 4.2. Views on these notions are in turn analysed, and explicated views are compared to the way terms are used.

4.6.2 Philosophical assumptions

Underlying a study, there are always assumptions governing design and implementation. Below, assumptions expressed in other parts of the thesis are collected, and complemented with reflections that are unique to this section. These are divided into an overall ideal of research and assumptions underlying the design of the study.

The ideal of research involves the idea that it is important to seek coherence in a research field, in order to offer some kind of knowledge. One form of incoherence is disagreement about the meanings of terms, which may appear as vagueness or ambiguity. As a consequence of that, there is a need for theoretical research with the purpose of fighting such incoherence.

This study has not been governed by a predetermined template for how to conduct mathematics education research, or by methods developed by other researchers. Instead, the study has been problem-driven, where my own decisions have determined the process, even though inspired by ideas coming from other studies of various kinds. Further, no theoretical perspective coming from mathematics education has been assumed, but the study relies on an analytic philosophical approach. The study has been governed by philosophy in two different ways. First, the study takes an analytic philosophical approach as it assumes that a concept analysis may contribute new perspectives in the field of mathematics education. Second, it uses philosophy as a method, since the analysing tool is developed from philosophical views.

An overall assumption in the thesis is that it is possible to combine approaches and views. This can be concretised in the idea that it is possible to combine results from the field of concept research within cognitive science with results from the field of mathematics education. Further, it is possible to compare or combine different conceptual frameworks. As a third example, it is possible to combine different views on *concept*. Such combinations may require some theoretical work, where combining different approaches and perspectives demands an understanding of the notions in the original contexts, that the notions are clearly explicated in the new context, and a coherent usage of terms. However, the result of such work may be a framework that better suits the purpose of a particular study.

In addition to the overall research ideal, there are other assumptions governing the design. One such assumption is that distinctions made in the common field of philosophy and cognitive science can contribute by offering new perspectives in the field of mathematics education. Further, there are certain assumptions that accompany the analytic philosophical approach of the study, which are described below.

First, this approach brings with it the idea of reductionism in the spirit of Ockham's razor, claiming that simpler frameworks, including fewer concepts, are to be preferred. I have been governed by the principle of using few general text-independent terms, rather than terms that are used in the texts referred to. Further, I have tried to decrease the density of philosophical terms, in order to avoid certain connotations that these terms may bring and also to increase readability. A consequence of the idea of general text-independent terms is that different views on *concept* could and should be compared in order to broaden the understanding coming from a certain framework. Another consequence is the assumption that the meaning of the term 'concept' in different fields should basically be the same, but that there are different types of concepts, with different features. In this study, the view on *concept* is interpreted in texts describing mathematical concepts. Thus, the distinction between features of mathematical concepts and features of other types of concepts is relevant. This is seen in the development of the indicators, where Indicators [4; 8] are specifically used for mathematical concepts and the others are not. Further, even those that are not specifically used for mathematics may be formulated in other ways, when used in other fields.

Second, the analytic philosophical approach brings the assumption of the three different arenas: the concrete arena, the mental arena, and the non-mental arena. This does not mean that I take a particular ontological stance concerning the existence of these arenas, but rather that I claim that these can be used for interpreting views on *concept*. Other distinctions are the one between intersubjective and subjective, and the one between molecular and holistic. Further, from the perspective of classical philosophy of language, the distinction between syntax, which may be seen as the form of language, and semantics, which may be seen as the content of language, and the distinction between terms, objects and classes of objects (or situations) that the terms refer to, and the content of the terms, are important.

Third, the study includes only views assuming the existence of representations, where an object in one of the three arenas can point to an object in one of the others. Without this assumption, the mental view in the distinction mental versus non-mental becomes irrelevant. As above, this does not mean that I assume an existence of a particular type.

Even though I have tried to not take a stance when it comes to what a concept is, it may be asked what it indicates about my own view when I claim that it is necessary to analyse related notions, and how these are related, in order to analyse the view on *concept* in a text. This, I would say, presupposes a view where concepts have structure, which is also in line with the fact that an atomistic view on *concept* is not included in the analysing tool. Further, I divide connections into hierarchical and non-hierarchical ones, which presupposes a view where molecular and holistic features may be combined.

4.6.3 Comments on the methodology

Naturally, the study's theoretical nature has consequences for the methodology. For example, the configurative literature review has a different role compared with an aggregative literature review aiming at setting a theoretical background for an empirical study. The purpose of the current review is not just to map the field. Rather, the mapping is a step towards the goal, and the final selection of frameworks can be seen as a subgroup within this map, containing examples of common frameworks for conceptual understanding in mathematics education. As a result, the analyses offer examples, rather than a complete description, of views on *concept* in the field. That the used methods are not pre-determined, but developed during the work is a consequence of the configurative approach. The same can be said about the object and delimitations of the study, which are not set beforehand, but during the review.

The 3D matrix with its three distinctions has been developed through an interplay between the philosophical literature review and analyses of texts in mathematics education. The indicators in turn have been developed through an interplay between the 3D matrix (Figure 3.13 or 4.3) and the analyses. The purpose of the indicators is to clarify how the analyses are made and to ensure that texts from the different frameworks are analysed in similar ways. The indicators are used as arguments for specific views, where more and clearer arguments strengthen a claim that there is a certain view on *concept* in a text. Further, even though the indicators are developed for analysing views on *concept*, they can also, as will be seen in the following chapters, be used for analysing the notions presented in Table 4.2: *concept image*, *conception* and *schema*.

5 Concept analysis of texts using the concept image-concept definition framework

The concept image-concept definition (CICD) framework emerged in a tradition in mathematics education, in the seventies and early eighties, based on a belief that if teachers offered correct definitions, the students would more or less automatically assimilate the concepts. The framework, which was used for studies of the cognitive structures that students developed, offered an understanding of the observation that even though the teacher's definition could evoke the right associations for some students, others may develop unintended associations. (Bingolbali & Monaghan, 2008)

In the original version of Vinner and Hershkowitz (1980), the CICD framework was used for studies of students' understanding of concepts in geometry, such as the concepts *angle* and *triangle*. In this version, concept images were mental images of geometric figures. Instead, in Tall and Vinner (1981) the focus is on limits and continuity. In the 1980s and early 1990s, the framework was used, for example, for studies of how students understand infinitesimals, differentiation, integration and differential equations. Since then, the framework has been used by mathematics educators interested in advanced mathematical thinking (AMT), where the ideas grew in discussions in a group that met at PME conferences, as well as by those interested in early learners. Today, it is used for studies of the understanding of concepts in different areas of mathematics. (Bingolbali & Monaghan, 2008)

Below, I first describe and analyse what I call a basic version of the CICD framework, $CICD^0$, represented by Vinner and Hershkowitz (1980) and Tall and Vinner (1981), with the help of references to later texts (Tall, 2001; 2004; Vinner, 1997). Next, two examples showing how the framework has been developed into contemporary versions are analysed. Semadeni (2008), on the one hand, combines the original framework with constructivist ideas, in a version that I call $CICD^+$. Bingolbali and Monaghan (2008), on the other hand, uses the framework in a study focusing on social aspects, in a version that I call CICD*. In order to refer to the analysed texts in a way that facilitates reading, I use abbreviations. In the following, VIHE is short for Vinner and Hershkowitz (1980), TV is short for Tall and Vinner (1981), SEM is short for Semadeni (2008), and BIMO is short for Bingolbali and Monaghan (2008). In the analysis, the intent is not to find a possible or even a reasonable interpretation, but to interpret the texts from the philosophical perspective described in Chapters 1 to 4 above. Further, the analysis concentrates on the content of the texts and not primarily on the intentions of the authors.

As stated in Section 4.5, the notion *concept* is the main object of the analysis. However, it is hard to analyse how *concept* is used without also analysing how it is presented as relating to other notions. As an example, the notion *concept image* and the connections between *concept* and *concept image* are important for the interpretation. Further, the view on *concept definition* affects the view on *concept* as well. Which of these may be analysed with the help of the analysing tool is an issue that will become apparent further on.

5.1 The CICD^0 framework

As stated above, two texts using the CICD^0 framework, VIHE and TV, are analysed in this section. The analysis starts with examples of explicated and non-explicated notions, and of hierarchical and non-hierarchical connections, in the theoretical backgrounds of the texts. The connections described in Section 5.1.1 are the ones that the explications are based upon, not including connections that are used in other descriptions in the texts. Some of these other connections are commented on in later sections. Based on the formulations of the explications, the different notions and connections are presented in concept maps representing the different frameworks.

Next, the notion *concept* is analysed with the help of indicators from the perspectives mental versus non-mental and intersubjective versus subjective. After this, the notions *concept image* and *concept definition* are analysed, before looking at whether the structure of concepts has molecular or holistic features. As claimed in Section 3.3.2, there are several views on the structure of concepts, where concepts can either be seen as elements in a structure, or as structures in themselves. Consequently, the molecular and holistic features may either apply to the concepts, or to the relations between them. Here, it may be noted that concept images are seen as cognitive. Hence, one may ask how concept images are connected to concepts and what the difference is between concept images and structures in which mental concepts are embedded. In addition, in Section

3.3.3, different models for relations between mental and non-mental structures are described. Mental and non-mental structures do not necessarily have the same features, and the analysis distinguishes between whether mental structures have molecular or holistic features, and whether non-mental structures have these features. The following sections addresses these issues.

5.1.1 Concept maps of the $CICD^0$ framework

Below, notions with and without explications in the theoretical backgrounds of VIHE and TV, and the connections between these notions, are first described. Again, a notion without explication may be seen as a basic notion in the sense of the philosophical perspective described in Section 3.3.1. Similarly, a notion with explication may be seen as a complex notion. However, it may also be the case that a notion without explication is seen as a complex one, where the explication is taken for granted. Further, I distinguish between hierarchical and non-hierarchical connections. From the explicated and non-explicated notions, and the connections between them, concept maps are constructed representing the texts.

5.1.1.1 A concept map of notions in Vinner and Hershkowitz (1980)

The notion *concept* is not explicated in VIHE. However, as the notions having explications, *mental picture*, *concept image* and *concept definition*, directly or indirectly are connected to *concept*, one can see *concept* as an important notion in the text.

- The notion *mental picture* of a concept C is explicated as "the set of all pictures that have ever been associated with C, in P's [a person] mind" (VIHE, p. 177).
- *Concept image* is explicated as the mental picture together with the set of properties associated with the concept in a person's mind (VIHE, p. 177).
- *Concept definition* is explicated as a verbal definition that accurately explains the concept in a non-circular way (VIHE, p. 177).

In addition to the explicated notions, *cognitive structure*, *formal definition* and *informal definition* are examples of non-explicated notions in VIHE. Regarding the described connections, the relations between *concept image* and *mental picture* and between *concept image* and *set of properties*, where both the mental picture and the set of properties are included in the concept image (VIHE, p. 177), are hierarchical. Further, there are hierarchical relations between *cognitive structure* and *concept image* and between *cognitive structure* and *concept definition*, as concept images and concept definitions are cells in cognitive structures (VIHE, p. 178). The connection between *concept definition* and *concept*, on the other hand, where concept definitions explain concepts (VIHE, p. 177), may be seen as non-hierarchical. The fact that mental pictures and sets of properties are associated with concepts may be interpreted as non-hierarchical connections as well.

In Figure 5.1 the explicated and non-explicated notions, and the hierarchical and non-hierarchical connections, are represented in a concept map, as a way of describing the conceptual framework in the text.

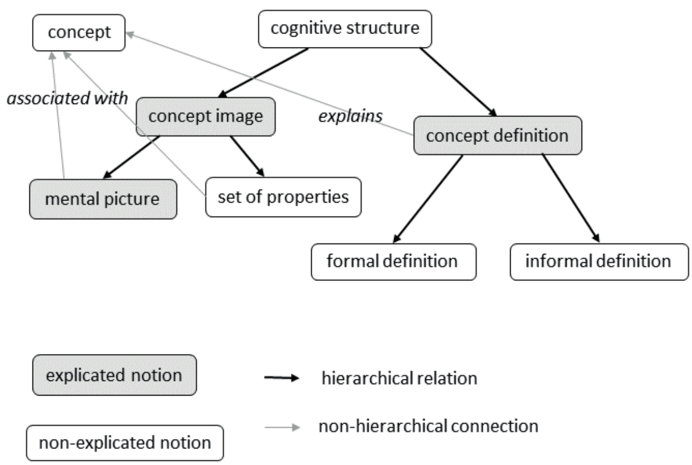


Figure 5.1 Concept map of Vinner and Hershkowitz (1980)

After analysing the explications of notions in VIHE, a similar analysis can be made of the notions in TV, offering a way of comparing the two texts.

5.1.1.2 A concept map of notions in Tall and Vinner (1981)

As in VIHE, the notion *concept* is not explicated in TV. However, here again, the notions having explications are, directly or indirectly, connected to *concept*.

- The notion *concept image* is explicated as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (TV, p. 152).
- The notion *evoked concept image* is explicated as "the portion of the concept image which is activated at a particular time" (TV, p. 152).

- The notion *concept definition* is explicated as "a form of words used to specify [a] concept" (TV, p. 152).

Other explicated notions are *personal concept definition*, *formal concept definition*, *concept definition image*, *potential conflict factor* and *cognitive conflict factor*. Examples of non-explicated notions are *mental picture*, *property* and *process*. When it comes to connections, the relations between *concept definition* and *formal concept definition* and between *concept definition* and *personal concept definition* are examples of hierarchical relations, as formal concept definitions and personal concept definitions obviously are different types of concept definitions (TV, p. 152). There are also hierarchical relations between *concept image* and *potential conflict factor* and between *concept definition* and *potential conflict factor*, as a potential conflict factor is a part of either a concept image or a concept definition (TV, p. 153). The connection between *concept definition* and *concept*, on the other hand, where concept definitions specify concepts (TV, p. 152) can be taken as an example of a non-hierarchical connection.

After interpreting the connections between the notions, this initial analysis results in a concept map that looks something like the one in Figure 5.2. In this map, the explicated and non-explicated notions, and the connections between them, in a sense represent the conceptual framework in TV.

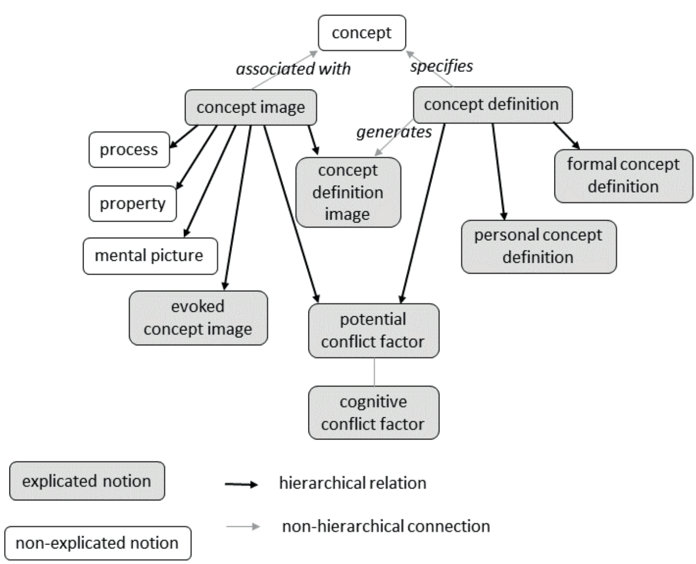


Figure 5.2 Concept map of Tall and Vinner (1981)

One difference between the concept maps in Figures 5.1 and 5.2 is that the notions in Figure 5.2 are not mutually exclusive, as a potential conflict factor may either be a part of the concept image or a part of the concept definition. Other comments on the analysis are given below.

5.1.1.3 Comments concerning the concept maps

From the above descriptions and concept maps, one can conclude that *concept image* and *concept definition* are the most connected notions, which justifies more thorough analyses of these notions. These are to be found below in later sections. Further, non-explicated notions can be basic notions that are used in explications of more complex notions. As an example, in VIHE the basic notion *set of properties* is used in the explication of *concept image*. Similarly, in TV the basic notions *mental picture*, *property* and *process* are used in the explication of *concept image*. Notably, a difference between the explications of *concept image* in the two texts is that processes are included in concept images in TV, which is not the case in VIHE.

It may be asked why the notion *concept* lacks explication in the texts. Here, there is a possibility that there was a clear view on *concept* in the field of mathematics education, at the time when the analysed texts were written. In that case, such a view was perhaps taken for granted, and thus was not explicated in the texts. However, today it is necessary to be able to read and interpret the texts without understanding the conceptual context in which the texts were actually written. That justifies a text analysis based on the actual content in the written texts.

5.1.2 Views on *concept* in the CICD^0 framework: mental vs non-mental and intersubjective vs subjective

> To understand how these processes occur, both successfully and erroneously, we must formulate a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived. (TV, p. 151)

In the above quote, a dualism is expressed between formally defined concepts and cognitive processes conceiving them, pointing to a dualism regarding a non-mental and a mental arena in the framework. This also implies a dualism concerning a non-mental and a mental view on *concept*, as will be seen below. As noted in Section 5.1.1, the notion *concept* is not explicated in neither VIHE nor TV. However, the way in which terms are used suggests that the main view is non-mental, which may also be seen from the number of formulations pointing to such a view. In VIHE, examples of such formulations are "learning of concepts" (VIHE, p. 178) and "the knowledge of […] concepts" (VIHE, p. 179), both pointing to a view where concepts are separated from the thoughts of an individual (Indicator [6][41]). In TV, the non-mental view is seen in formulations such as "[m]any concepts we meet in mathematics" (TV, p. 151), where concepts are separated from the thoughts of an individual (Indicator [6]), and in the following quote, where formal mathematics is based on concepts (Indicator [4]):

> mathematics is usually regarded as a subject of great precision in which concepts can be defined accurately to provide a firm foundation for the mathematical theory (TV, p. 151)

In addition to this non-mental view, there are also formulations indicating a mental view. In VIHE, this can be exemplified by the formulation "acquiring certain type [sic!] of concepts" (VIHE, p. 177) (Indicator [3]), where something that we acquire is mental. In TV, a mental view may be seen in the quote below, where concepts can be mentally manipulated (Indicator [2]):

> the concept is given a symbol or name which enables it to be communicated and aids in its mental manipulation. (TV, p. 151)

Also, in TV (pp. 151, 167), there are two formulations claiming that concepts can be evoked:

> a complex cognitive structure exists in the mind of every individual, yielding a variety of personal mental images when a concept is evoked. (TV, p. 151)

> for a small number [of students] there is an evoked concept of "a single formula" and for a minority there are other images (TV, p. 167)

As in other parts of the same text it is segments of the cognitive structure that are evoked (see the next section), both these formulations may point to a mental view on *concept*, where concepts are equated with some kind of cognitive structures. However, an alternative interpretation of these formulations is that there is a writing error here, and that 'concept' in these formulations should be

[41] Numbers of indicators refer to Section 4.4.

exchanged for 'concept image'. This may be a more reasonable interpretation, for which I will argue in Section 5.1.6.

Regardless of how to interpret these formulations, there are both a mental view and a non-mental view on *concept* in the framework. In both these views, concepts are mostly seen as intersubjective. In VIHE, this is seen in formulations such as "they are asked to identify the concepts" (VIHE, p. 179) and "the concept of an altitude" (VIHE, p. 182), both pointing to a view where there is a unique concept of a certain phenomenon (Indicator [9]). In TV, the intersubjective view is seen in formulations such as "the concept of complex number" (TV, p. 154), indicating that there is a unique such concept (Indicator [9]). It is also seen in the formulation "the mathematical concepts as formally defined" (TV, p. 151), indicating that concepts are something that are developed in the community of mathematics (Indicator [8]).

One of the quotes above may perhaps point to a subjective view on *concept* as well, since different students may have different evoked concepts. However, as noted above, there are two interpretations of this quote. Since there are several possibilities, I have judged this indication to be a too vague argument for a subjective view.

To conclude, there are hitherto two views on *concept* in the CICD^0 framework, a non-mental and intersubjective view, and a mental and intersubjective view. Later, these two views on *concept* will be analysed from the perspective molecular versus holistic, to see whether the non-mental and the mental structure of concepts have molecular or holistic features. First, however, since the analysis of *concept* depends on an analysis of the notions *concept image* and *concept definition*, these notions will be examined in the next sections.

5.1.3 Views on *concept image* in the CICD^0 framework: mental vs non-mental and intersubjective vs subjective

The notion *concept image* first appeared in VIHE, even though a similar notion had occurred earlier, in Vinner (1975), with the name 'mental image'. As is seen in Section 5.1.1.1, in this first version, *concept image* is defined as the set of all mental images that have ever been associated with a concept in the individual mind, together with the set of properties associated with the concept (VIHE, p. 177). This notion is related to *concept definition*, and the concept image and the concept definition are seen as two different cells in the cognitive structure, shown in a figure in VIHE (p. 178). Figure 5.3 depicts the ideas in that figure[42].

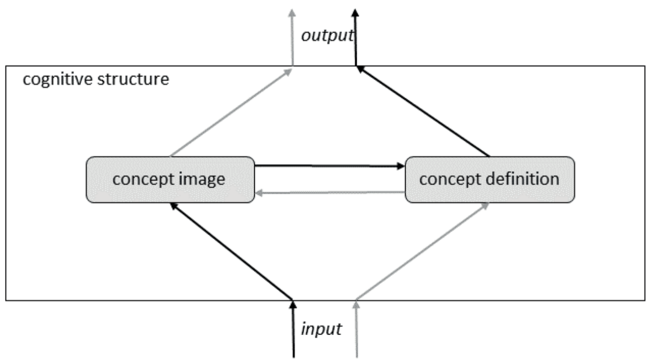


Figure 5.3 Cognitive structure in Vinner and Hershkowitz (1980)

In VIHE, the notion *concept image* is used in a study of basic geometrical figures, with mathematical concepts such as *right angle* and *altitude in a triangle*, where the pictures associated with them are essential for the study. In this original version of the framework, the notion *concept image* is used only for describing mental representations of geometric concepts, but is not explicitly restricted to geometry. (VIHE, pp. 177, 181–182)

In TV (p. 152), *concept image* is broadened and is defined as the total cognitive structure associated with a concept, including all mental pictures, and associated properties and processes. Here, the expression 'and processes' seems to shift the idea from a mental image to a more general mental representation, which could be used for studies of concepts such as *limit* and *continuity*. Note that the definition of *concept image* as the total cognitive structure associated with a concept is not in line with the usage of the terms 'cognitive structure' and 'concept image' in VIHE. There, the cognitive structure seems to include both the concept image and the concept definition.

In both VIHE and TV, *concept image* is explicated as a mental notion, which also is in line with how the term 'concept image' is used. In VIHE, an example of a mental view is seen in the formulation "76% of the students have a concept

[42] The model in Figure 5.3 does not really reflect practice. It is claimed in VIHE (pp. 178–179) that many definitions do not make sense for the students and the concept definition may be inactive or may not exist in the cognitive structure.

image" (VIHE, p. 182), where a concept image is something that is possessed by the individual (Indicator [2]). In TV, the following quotes indicate a mental view:

> all mental attributes associated with a concept, whether conscious or unconscious, should be included in the concept image (TV, p. 152)

> the concept definition image that it forms in his cognitive structure may be very weak (TV, p. 160)

In the first of these quotes, concept images contain mental attributes (Indicator [1]), and in the second the concept definition image, which is a part of the concept image, is formed in a cognitive structure (Indicator [3]).

Further, a subjective view on *concept image* appears in formulations such as "revealing the concept images of our students" (VIHE, 1980, p. 180), indicating that students may have different concept images (Indicator [12]), and "an individual's concept image" (TV, p. 153), where an individual has his or her own concept image (Indicator [11]). Another example is seen in the following quote, where students may have different concept images (Indicator [12]):

> What concept images do students have of continuity when they arrive at university? (TV, p. 165)

In order to further investigate the distinction intersubjective versus subjective concerning the notion *concept image*, one may look at the description in TV of how concept images develop. It is claimed that concept images are built up through experiences of different kinds, and that they change as individuals meet new situations. TV (p. 152) uses the example of subtraction and describes how a child first meets subtraction as a process involving natural numbers, in tasks such as 7 – 3. The child could, from these tasks, get the idea that subtraction always reduces the first term. This idea, which is included in the concept image, may later cause problems when subtraction is applied to negative numbers; 2 – (-3) is larger than the first term 2. Hopefully, the concept image will change when the child learns more about mathematics and uses subtraction for other sorts of numbers, and later on maybe even for objects such as vectors.

From this description, one can conclude that concept images have intersubjective features, based on the fact that students may develop similar images when participating in the same classroom, meeting the same situations. This implies a mental and subjective view on *concept image*, which also has intersubjective features. Further, there is an intersubjectivity in the fact that there are concept images that teachers encourage students to develop (Indicator [10]). This implies a second view on *concept image*, which is mental and intersubjective.

Note that a concept image does not need to be coherent and that sensory impressions may activate different parts of it. An illustration here could be that the two problems below may be solved using the same calculation, but it is likely that different parts of the concept image are evoked during the process of problem solving in each case.

1. Stina wants to buy a book which costs 123 SEK, but she only got 107 SEK in her wallet. She asks her brother Pentti if he could lend her some money. How much money does she need to borrow from him, in order to buy the book?
2. Solve the equation
   $107 + x = 123$

The different evoked concept images used in the two examples will affect how the student thinks about and solves the mathematical problems. Note that there are clear similarities between this view, including evoked concept images, and Zalta's (2001, p. 344) view where someone may have different mental representations at different times, which in turn could be described as different modes of presentation (see Section 3.2.3.1).

5.1.4 Views on *concept definition* in the CICD^0 framework

With the CICD framework, one may get the impression that concept images are more important than concept definitions. First, the term 'concept image' is more frequently used than the term 'concept definition'. Further, according to VIHE (p. 177), people need concept images in order to understand mathematics, but they do not need concept definitions. In the text, an example concerns children developing mental concepts of an orange, without a definition. In addition, even though we are introduced to concept definitions, we may forget them and the understandings of the definitions may be inactive in the concept images (VIHE, p. 177).

As seen in Section 5.1.1, the notion *concept definition* in VIHE (p. 177) is explicated as a verbal definition that accurately explains the concept in a non-circular way. One can compare this description with the explication in TV (p.

152), where a concept definition is a verbal expression[43], used to specify a concept. In the latter text, it is claimed that a concept definition may be a formal definition or a personal reconstruction of a formal definition. As the concept image varies, the personal definition varies in a similar way. From this description, it seems as if there are two types of concept definitions in TV, one intersubjective concept definition and one subjective concept definition. Both these views appear in the way terms are used in the same text. The intersubjective view is seen in formulations such as "the correct formal definition" (TV, p. 153), which can be compared with an incorrect or wrong formal definition (TV, p. 163). These formulations point to a view where the concept definition is developed in a community (by similar arguments to those given in Indicator [7]). A subjective view, on the other hand, is found in descriptions of how students' concept definitions are classified (TV, pp. 161–162), pointing to a view where students construct their own concept definitions (by similar arguments to those given in Indicator [11]).

The division into these two types of concept definitions is not seen in VIHE, neither in explications nor in how terms are used. Hence, one may see the notion *concept definition* in TV as a broadened notion compared with the one in VIHE, including personal definitions.

From the perspective that a concept definition is a verbal expression, the claim in VIHE (p. 178) that the concept definition can be seen as a cell in the cognitive structure, and the figure on the same page (see Figure 5.3), is problematic. In this description, the concept definition is mental and it may be asked how something can be an expression at the same time as it is a part of a mental structure. To further understand this, one may look at how terms are used in the texts. As the notion *concept definition* in these texts is explicated as an expression, the indicators for the distinction mental versus non-mental, presupposing that the analysed object is the meaning of terms or expressions, cannot be used. Therefore, I will interpret how terms are used in formulations, from the perspective of the distinction mental versus non-mental, without the indicators.

[43] I find the formulation 'a form of words' strange. I have interpreted that as 'a verbal expression', and will use that formulation instead.

In the usage of 'concept definition' in VIHE, there are both formulations pointing to a view where concept definitions are mental, and formulations pointing to a view where concept definitions are non-mental[44]. A mental view is seen in the formulation "[c]oncept definitions […] will remain inactive" (in the cognitive structure) (VIHE, p. 177), where concept definitions are part of the mental structure. A view where concept definitions are non-mental can be seen in the following quote, where something that the individual gives cannot be mental:

> the concept definition might become a "part of the game". Students are required to give definitions. (VIHE, p. 178)

As a result, there seems to be two views on *concept definition* in this text, one mental view and one view where concept definitions are non-mental. Here, there seems to be a category mistake, since concept definitions are described both as expressions and as contained in a cognitive structure that form the meanings of the expressions, at the same time.

With this in mind, it may be studied how terms are used in TV, to see if the same two views appear in that text. First, one can detect a mental view in the following quote, where the concept definition is part of the cognitive structure, in which it may be active or inactive:

> the concept definition is largely inactive in the cognitive structure (TV, p. 153)

Further, a view where concept definitions are non-mental appears in formulations where a concept definition is an expression, as in the following:

> the concept definition of a mathematical function might be taken to be "a relation between two sets A and B in which each element in A is related to precisely one element in B." (TV, p. 153)

Furthermore, a non-mental view may be seen in formulations indicating that a definition can be given (TV, p. 153), or that a definition is something that the students should have an understanding of (TV, p. 160), for the same reasons as in Indicator [6].

[44] Note that in a non-mental view on *concept*, concepts are often considered non-mental meanings of terms. However, the fact that concept definitions are seen as non-mental does not presuppose that they are considered meanings of terms. As an example, a view considering concept definitions as expressions is non-mental.

Consequently, the same two views on *concept definition* as are seen in VIHE appear in TV, one view where concept definitions are mental, and one view where concept definitions are non-mental. Notably, the category mistake where expressions and cognitive structures are not distinguished appears in this text as well, as *concept definition* is explicated as a verbal expression, but the notion is also used for parts of the cognitive structure.[45]

At this point it is appropriate to conclude the descriptions of the notions *concept*, *concept image* and *concept definition*, from the perspective of the two distinctions mental versus non-mental and intersubjective versus subjective, in the $CICD^0$ framework. First, there are two views on *concept* in the texts, one non-mental and intersubjective view and one mental and intersubjective view. Further, there are two views on *concept image* as well, one mental and subjective view, with intersubjective features, and one mental and intersubjective view. The latter is not very easy to distinguish in the text. Regarding the notion *concept definition*, there seems to be one intersubjective and one subjective view, connected to the formal concept definition and the personal concept definition respectively. The nature of these definitions is hard to understand from the perspective of the philosophy described in Chapters 2 and 3, since expressions and cognitive structures are not distinguished. *Concept definition* is explicated as an expression, but there are indications in the texts of a view where concept definitions are included in the cognitive structure.

5.1.5 Views on *concept* and *concept image* in the $CICD^0$ framework: molecular vs holistic

When it comes to the distinction molecular versus holistic, the analysis depends on the distinction between mental and non-mental views on structures of concepts. Regarding the mental view, there are both formulations pointing towards a molecular view and formulations pointing towards a holistic view. One indication of a molecular view in VIHE is seen in the formulation below, claiming that there are hierarchies in the cognitive structure (Indicator [15]):

[45] It may be noted that in the usage of the notion *concept definition image* it seems as if there is a difference between language and the cognitive structure, between concept definition and concept definition image. From that perspective, the formulations in the texts claiming that the concept definition is part of the cognitive structure, and consequently a part of the concept image as it is explicated as the total cognitive structure associated with a concept (TV, p. 152), do not make sense.

> In such a group [of people] we will not find people who know D without knowing also A, B and C or people who know C without knowing also A and B and so on (VIHE, p. 182)

In the next quote from TV, the fact that concept definitions may be important for developing concept images also points to molecular features (Indicator [14]):

> the concept image is intended [in The School Mathematics Project Advanced Level texts] to lead naturally to the concept definition, but in practice certain potential conflicts occur which can cause cognitive conflict for those who later study analysis (TV, p. 155)

Here, it may be asked what students can learn from working with concept definitions. A natural answer to that would be that definitions of complex concepts can be built on basic concepts. That is why such formulations are interpreted as indicating molecular features of concepts. However, even though formulations describing the cognitive structure are somewhat hard to interpret, it seems as if these structures have holistic features as well. The fact that potential conflicts occur is one indication of holistic features, as such conflicts must appear as a consequence of non-hierarchical connections. As one example of this, when it is claimed that a child could get the idea that subtraction reduces the first term (TV, p. 152), there is a non-hierarchical connection between *subtraction* and *term*.

Even the non-mental structure seems to have both molecular and holistic features. First, a molecular view is seen in formulations claiming that concepts can be defined, and in formulations claiming the importance of definitions in formal mathematics (Indicator [13]). Examples of this are:

> In formal learning the situation might be different. Here the concept definition might become a "part of the game". Students are required to give definitions. (VIHE, p. 178)

> Compared with other fields of human endeavour, mathematics is usually regarded as a subject of great precision in which concepts can be defined accurately to provide a firm foundation for the mathematical theory (TV, p. 151)

Additionally, a holistic view can be seen when it is claimed that not every concept has a definition (Indicator [18]), as in the following formulation:

> For some of our concepts we also have concept definition in addition to the concept image. For many other concepts we do not. (VIHE, p. 177)

It seems in the descriptions as if mathematical concepts can be accurately defined and provide a foundation for mathematical theory. The examples of concepts that do not have definitions in VIHE (p. 177) are *house* and *orange*, which are non-mathematical. Here, one has to distinguish between definability and molecularity. Even in a molecular view, not every concept has a definition, since there is a distinction between basic and complex concepts. However, in the quotes above it is claimed that one difference between mathematics and other subjects is that mathematical concepts can be defined. Thus, one can argue that the view on mathematical concepts in the $CICD^0$ framework is molecular, but that other types of concepts may have holistic features.

To sum up, there are two views on *concept* in the framework. The non-mental view is intersubjective and can be described with a combination of molecularism and holism, where it can be argued that the view on mathematical concepts involves the idea that the structure has molecular features. I have chosen to represent this using two different shades of grey in the left-hand matrix in Figure 5.4, where the molecular view has a darker shade than the holistic view. The mental view, on the other hand, is intersubjective and has a combined molecular-holistic structure. This is represented in the right-hand matrix in Figure 5.4.

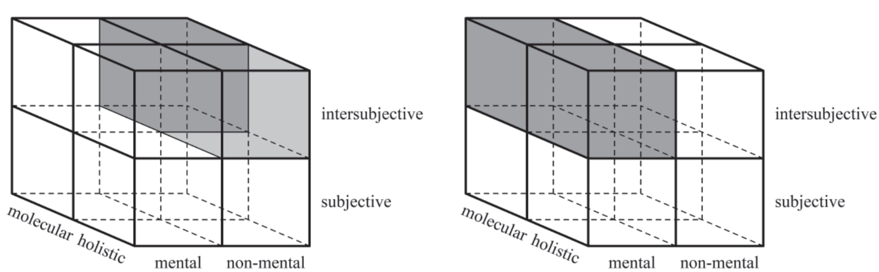


Figure 5.4 Views on *concept* in the $CICD^0$ framework

The relation between the structure of mental concepts and the concept image of a certain concept is not explicit in the text. It is natural to adopt the idea that the concept image of a certain concept contains other mental concepts, which have already been developed by the individual. In that case, structures of mental concepts and concept images must have the same features. Therefore, the two views on *concept image* both use a combination of molecularism and holism, like the view on the structure of mental concepts. Consequently, the first view is mental, subjective with intersubjective features, and the conceptual structure has both molecular and holistic features. The second view is mental, intersubjective, and the conceptual structure has both molecular and holistic features. These can be represented as the two matrices in Figure 5.5.

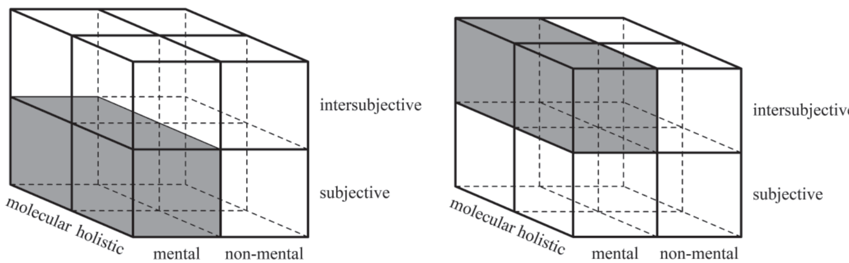


Figure 5.5 Views on *concept image* in the CICD^0 framework

Since the nature of *concept definition* is not established, the structure of the concept definition cannot be analysed. One can argue that if a concept definition is an expression, then it does not have conceptual structure. It may have syntactical structure, but that is another issue. If it instead is a part of the cognitive structure, then it may have a structure. However, I leave this discussion without making any conclusions, and I do not use the 3D matrix for representing the findings regarding the notion *concept definition*.

To sum up, in this section explicated and non-explicated notions in VIHE and TV, and connections between the notions, are first represented by concept maps, describing the conceptual frameworks. Next, views on *concept* and *concept image* in the CICD^0 framework are analysed with the help of indicators. Before analysing some lines of development of the framework, there are some comments to be made about the views in this basic version.

5.1.6 Comments on the analysis of the CICD^0 framework

To conclude the analysis, there are several notions connected to *concept* in the CICD^0 framework. The notion *concept* itself is not explicated and the analysis must therefore rely on the way terms are used. The two notions *concept image* and *concept definition* have most connections explicated in the analysed texts (see

Section 5.1.1), which justifies analyses of views on these notions as well. In these analyses, it is possible to use the indicators for analysing *concept image*, but not for *concept definition*.

As seen above, there are two views on *concept* in the texts, one where concepts are non-mental objects and one where they are mental representations. Later, Tall (2004, p. 32) declares that the term 'concept' has a dual usage; in some contexts it is used as an abstract, what I call non-mental, object and in some contexts it is used as a mental representation. This is in line with the views found in the analysis. Further, the fact that the non-mental view on the structure of mathematical concepts includes molecular features is later confirmed by Vinner (1997, p. 105), describing notions, what are here called concepts, as molecular, since concepts are either basic concepts or defined from basic concepts:

> Every notion is a primary notion or well-defined by means of other primary or well defined notions. The meaning of the primary notions is determined by the axioms. (Vinner, 1997, p. 105)

Notably, there are both subjective and intersubjective views on *concept image* visible in the framework. Further, it may be concluded that it is not clear what is meant by 'cognitive structure'. In Figure 5.3, representing the view in VIHE, the cognitive structure is something that contains both the concept image and the concept definition. In the descriptions in TV, on the other hand, the notion *concept image* has been broadened, and the concept image is the total cognitive structure. Here, it can be discussed whether the concept definition is included in this structure or not. David Tall describes the different meanings given to 'concept image' on his website:

> Shlomo's [Vinner] definition was philosophically based and was a thought experiment to analyse what happens when students focus in different ways on images and definitions. My perception was more humanly based, so that where Shlomo talked about 'the mind' and thought about it as separate from 'the brain' in a cartesian sense, I always thought of the mind as the way the brain works, so that it is an indivisible part of the structure of the brain. Shlomo has always written about 'concept image' and 'concept definition' as being 'two distinct cells' which enables him to make subtle analyses of different ways of employing the two distinct ideas. As the concept definition is a form of words that can be written or spoken, I regard this as part and parcel of the total concept image in the mind/brain. (Tall, 2003)

Consequently, there are two alternative views regarding the nature of and the connections between *concept image* and *concept definition*. In the first alternative, concept image and concept definition are separated. Such views are seen in the concept maps in Figures 5.1 and 5.2, based on the explications in VIHE and TV. Furthermore, such a view is seen in Figure 5.3, where the concept image and the concept definition are two distinct cells in the cognitive structure. In the second alternative, the concept definition is seen as a part of the concept image. In such a view, the concept image may include mental pictures, associated properties and processes, as well as strings of words and symbols, as stated in BIMO (p. 20). This implies a denial of the distinctions made in Chapter 2 between a mental and a non-mental arena, between expressions and the meanings of the expressions, and between internal and external representation. Here, it may be noted that in TV (p. 152), *concept image* is not explicated as a structure including strings of words and symbols, but the view where the concept definition is included in the concept image appears in the way terms are used.

Since *concept definition* is explicated as an expression, *concept image* in the second alternative, explicated as a cognitive structure, includes terms and also formal definitions of a non-mental nature according to the analysis. Since the view on *concept definition* is hard to understand, when denying a distinction between expressions and the meanings of the expressions, the view on *concept image* in this alternative becomes similarly hard to interpret.

The two alternatives support the developments of the framework in different ways. To exemplify, the first alternative is seen in a figure in Tall (2001, p. 203), representing the relation between concept image and concept definition in a map of the cognitive development. Figure 5.6 depicts some of the ideas in that figure:

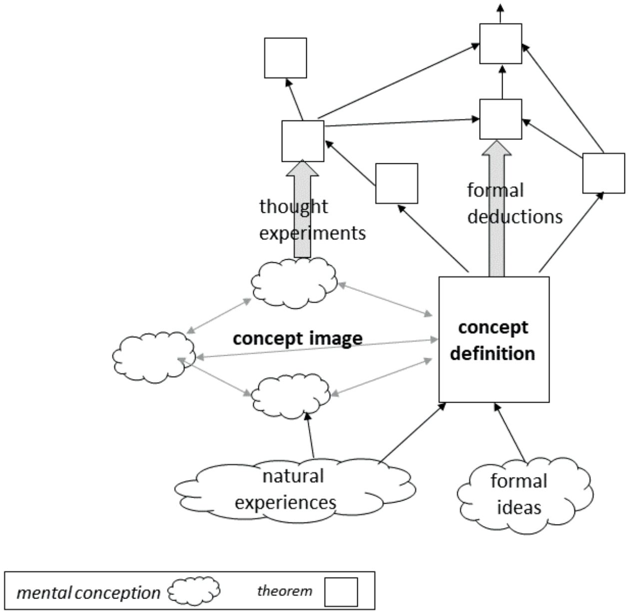


Figure 5.6 Concept image and concept definition in Tall (2001)

There are many questions that can be raised about this figure. Here, however, I am only interested in the fact that one can see a difference between the mental concept image and the formal concept definition, where the shape of a cloud represents a mental conception and the shape of a rectangle represents a theorem, which I interpret as non-mental.

The second alternative, where the concept definition is included in the concept image, is seen on the next page (Tall, 2001, p. 204) where it is claimed that the concept image contains a subpart called formal image, consisting of what is formally deduced from the axioms. Here, there is a distinction between formal and informal image, which, according to Tall (2001, p. 204), are both contained within the same biological brain. In this description, the three arenas (mental, non-mental and concrete) are combined.

To sum up, in the CICD framework there are two alternative views on the connections between *concept image* and *concept definition*, where in some articles (such as in VIHE) the concept image and the concept definition are separated, and in others (such as in Tall (2001)), the concept definition is described as included in the concept image. A related issue is whether the definition is an expression, or a part of the cognitive structure. In the descriptions, it seems as if the concept definition is both of these. But in that case, there is a category mistake in the texts, where expressions and the meanings of expressions are mixed up.

In addition, on some occasions, the most reasonable interpretation of the text seems to be that there is a writing error. As one such example, one may again discuss the following quotes:

> The identification of x + i0 = (x, 0) as the real number x is a potential conflict factor in the concept of complex number. (TV, p. 153)

> for a small number [of students] there is an evoked concept of "a single formula" and for a minority there are other images (TV, p. 167)

In both these quotes, the term 'concept' should probably be replaced with 'concept image', since concepts as they are described in other parts of the text do not involve conflict factors, and the expression 'evoked concept' does not make sense, regarding that it is parts of concept images that may be evoked.

Since the 1980s, when the framework appeared, the ideas have been further discussed and developed. One forum for the development has been the advanced mathematical thinking group (AMT) of PME, which in the late eighties began to extend cognitive theories to the construction of axiomatic systems. The framework then grew from discussions in this informal group (Tall, 2004; BIMO). Two examples of developments of the CICD framework are presented below.

5.2 The CICD^{+} framework

In SEM, it is claimed that mathematical thinking presupposes a certain stability of the concept images. However, it does not rely on concept definitions, and many students are able to understand proofs without knowing definitions of the concepts involved. According to the framework, this is because they have developed deep intuitions. The notion *deep intuition* is characterised by someone's concept image having reached a certain level of development, where the basic meanings are stable, context-independent and robust, even when the situations change. With the example of subtraction, the evoked concept image when working with tasks such as 7 – 3 may be different compared to when working with tasks such as 2 – (-3). The deep intuition, however, is evoked during any of these. (SEM, pp. 3, 8–9)

It is slightly difficult to analyse this framework as a mere development of the CICD framework, as the ideas in the CICD^0 framework are combined with, for example, the OS framework, the procept framework and the APOS framework, analysed in Chapter 6, together with the framework of three worlds of mathematics (Tall, 2004).[46] Even so, the analysis below concentrates on ideas that can be seen solely as a development of the CICD framework.

First, notions connected to *concept*, which are described as developed around the idea of concept image, are analysed depending on whether they are explicated or not, and how the notions connect to each other according to the explications. As in Section 5.1.1, the connections are divided into hierarchical and non-hierarchical ones. Based on these descriptions, the different notions and connections are inserted into a concept map. Second, the notion *concept* is analysed from the perspectives mental versus non-mental and intersubjective versus subjective. Third, the notion *concept image* is analysed from the same perspectives. As the notion *concept definition* is not discussed in SEM in a way that reveals new information, this notion will be dealt with during the analysis of *concept image*. Finally, the notions *concept* and *concept image* are discussed from the point of view of the distinction molecular versus holistic.

5.2.1 Concept map of the CICD$^+$ framework

As in the CICD^0 framework, the notion *concept* is not explicated in CICD$^+$. The notions that are mentioned below are all connected to the notion *concept image*, and directly or indirectly connected to *concept*. As stated above, the focus is on notions that may be seen as included in a developed version of the CICD framework. I have chosen to not include notions describing concept development, which are present in the analysed text, but have an origin in other frameworks.

- The notion *concept image* is explicated as "all the cognitive structure in the individual's mind that is associated with a given concept" (SEM, p. 4), with a reference to TV.
- The notion *evoked concept image* is explicated as "[t]he part of the concept image which is activated at a particular time" (SEM, p. 4).
- *Deep intuition* is explicated as "that part of the concept image of X [a mathematical concept] which is evoked during any reasoning concerning X" (SEM, p. 9). It is also claimed that a concept image has reached the

[46] As a consequence, the framework in SEM takes a constructivist perspective.

level of deep intuition if "the basic meaning of X is stable, context-independent and robust when the context or situation changes" (SEM, p. 9).

Other explicated notions are *concept definition* and *formal image*, and *mental picture*, *mental attribute* and *process* are examples of non-explicated notions. Regarding the connections, the relation between *evoked concept image* and *deep intuition* is hierarchical, as a deep intuition is a part of a concept image that is included in every evoked concept image (SEM, p. 9). Further, that a concept definition may be contained in a concept image (SEM, p. 4)[47] can be interpreted as a hierarchical relation between *concept image* and *concept definition*. An example of a non-hierarchical connection, on the other hand, is the one between *deep intuition* and *concept definition*, as it is claimed that deep intuitions and concept definitions correspond to each other in typical cases (SEM, p. 3).

After interpreting the connections between the explicated and non-explicated notions, this initial analysis results in the concept map in Figure 5.7.

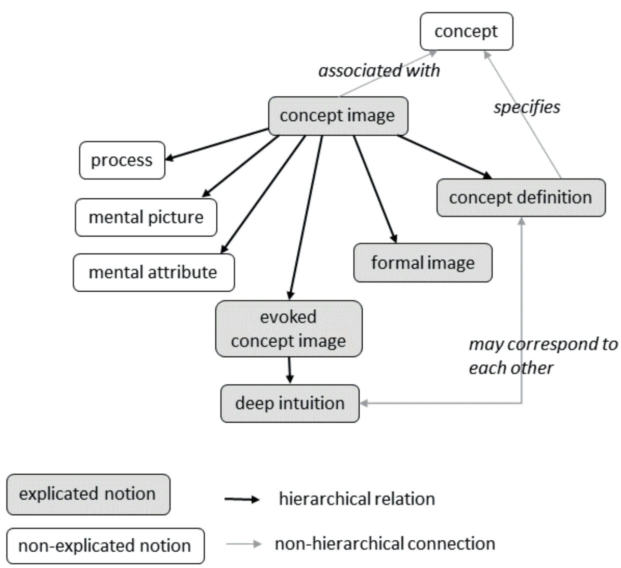


Figure 5.7 Concept map of Semadeni (2008)

[47] In the text, the second alternative regarding the relation between concept image and concept definition is used, where the concept definition is included in the concept image. In SEM (p. 4), it is expressed that this idea follows Tall, but not Vinner (1991).

Compared to the concept maps in Figures 5.1 and 5.2, the map in Figure 5.7 involves the idea that a concept image may contain a concept definition and also a formal image. Here, it may be noted that even though *concept definition* is explicated as an expression (literally 'a form of words'), the explication of *concept image* does not involve the idea that it may include words, and one may ask whether the concept definition should be seen as a mental picture, a mental attribute or an associated process.

5.2.2 Views on *concept* in the CICD$^+$ framework: mental vs non-mental and intersubjective vs subjective

As in the CICD^0 framework, there are both a non-mental and a mental view on *concept* in the text. A non-mental view is seen in formulations such as "concepts to be thought about" (SEM, p. 5) and "the concept was known to the students" (SEM, p. 13), pointing to a view where concepts are separated from individual thought (Indicator [6]). A mental view, on the other hand, is seen in formulations pointing to a view where concepts are developed or constructed by the individual (Indicator [3]):

> an abstract concept such as permutation was then constructed by an individual (SEM, p. 9)

> three stages of concept development: *interiorization*, *condensation* and *reification* (SEM, p. 5)

The last quote is from a context referring to the OS framework (Sfard, 1991), which is followed by references to the APOS framework (Asiala et al., 1996) and to Piaget's perspective. Further, there are references to the procept framework (Gray & Tall, 1994) on page 9 in SEM. In these frameworks, a mental concept development is described and the references point to a mental view on *concept* in SEM as well.

Throughout the text, there is an intersubjective view on *concept*, seen in formulations such as "the concept of addition of numbers" (SEM, p. 9), indicating that this concept is unique (Indicator [9]). Further, the following formulation points to a view where concepts are developed in a community (Indicator [7]):

> the standard academic presentation, based on set theory, of the concepts of 'ordered pair', 'relation', 'function', 'sequence' consists of the following six steps (SEM, p. 10)

Additionally, a suggested difference between the concept and the concept image is that the concept image varies, but the concept is context-independent, which may indicate a Platonic view on *concept*, such as the one in Frege (1892/1985; 1892/1951) (SEM, p. 3). As there are no formulations pointing to a subjective view, there are, from the perspectives mental versus non-mental and intersubjective versus subjective, two views on *concept* in the text, one non-mental and intersubjective view and one mental and intersubjective view.

5.2.3 Views on *concept image* in the CICD$^+$ framework: mental vs non-mental and intersubjective vs subjective

As seen in Section 5.2.1, the notion *concept image* is explicated in SEM (p. 4) as the cognitive structure associated with a given concept, with a reference to TV. Further, the concept image contains mental pictures, mental attributes and associated processes (SEM, p. 4). In that explication, it is clear that the view on *concept image* is mental (Indicator [1]), which is confirmed by the way terms are used. An example is seen in the fact that parts of the concept image can be evoked (SEM, pp. 4, 12), which, as seen in Section 5.1.2, may point to a mental view.

In addition, the fact that concept images are subjective is seen in formulations such as "her/his concept image" (SEM, p. 3), indicating that people have their own concept images (Indicator [12]). There are also intersubjective features in the view on *concept image*. This is seen in the following quote, where it is claimed that it is possible to describe when a concept image has reached the level of deep intuition:

> The concept image reaches the level of deep intuition when the individual can understand and follow simple Euclidean proofs (SEM, p. 12)

Consequently, what appears from the explication of *concept image* and the way terms are used is a view where concept images are mental and subjective, with intersubjective features. However, as noted in Section 5.2.1, it is claimed that the concept image includes what is called a formal image (introduced in Tall (2001, p. 204)), which is the part of the concept image that is formally deduced from axioms (SEM, p. 4). In that description, it seems as if the formal image is of a non-mental nature, and as a result, the mental concept image contains a non-mental formal subpart[48]. Further, the notion *concept definition* is explicated as words used to specify a concept, at the same time as it is claimed that the concept definition is included in the concept image (SEM, p. 4). Hence, the mental concept image contains words, which usually are seen as non-mental. Here, the nature of the concept definitions is somewhat hard to understand, since they are to be expressions and mental at the same time.

Consequently, concept images, often described as mental, contain non-mental elements, which makes it hard to understand the nature of the notions *concept image* and *concept definition*. It seems as if expressions and the meanings of expressions are mixed, and as if mental and non-mental arenas are not distinguished[49].

Further, the notion *conception* is used in a mental view, in parallel with the notion *concept image*. This notion is not explicated, and it is hard to see the relation between *concept image* and *conception*. However, *conception* is used both for describing mental representations of physical objects, and for describing more general ideas. As an example, in the following quote *conception* seems to be a cognitive structure, similar to *concept image*:

> an *embodied object* begins with the mental conception of physical objects (SEM, p. 5)

An example where the notion is more general, an idea of a topic, can be seen below:

> *Stability of meanings* is indispensable in mathematical reasoning and is a critical factor in the conceptions presented in this paper. (SEM, p. 8)

This more general usage of *conception* is not seen when it comes to *concept image*. Consequently, there are two views on *conception* in the text. In one of these, *conception* is similar to *concept image*. In the other view, *conception* is seen as a more general idea.

---

[48] Tall (2001) refers to the logicist ideas of Hilbert, Russell and Frege. Additionally, SEM refers to empiricism as well, claiming that "[f]or centuries the fact that mathematical knowledge stems from the senses and experience has been stressed by empirically minded philosophers" (SEM, p. 12). Consequently, from an epistemological point of view, the framework can be seen as trying to combine rationalism and empiricism.

[49] As in Section 5.1, I do not use the indicators for analysing *concept definition*.

5.2.4 Views on *concept* and *concept image* in the CICD+ framework: molecular vs holistic

When it comes to the distinction molecular versus holistic, the analysis is divided into a discussion about the non-mental structure and a discussion about the mental structure. Regarding the non-mental structure of concepts, a holistic view is seen in some critical examples as opposed to a definitional approach (SEM, pp. 10–12) (Indicator [18]). One argument used in the text is that in the twentieth century, there was no single formal definition of *triangle* accepted by the mathematical community (SEM, p. 12). Instead, it is claimed that there were different competing definitions. Another argument is that there are loops in certain collections of definitions (SEM, pp. 10–11). Together, these arguments form a criticism against a molecular view, and consequently indicate a holistic view.

Regarding the mental structure, the analysis relies on formulations about deep intuitions, which are included in concept images. The reason for this is that concepts and concept images are not discussed as much as deep intuitions. Further, from the fact that there are mental concepts and the explication of *concept image* as the total cognitive structure, one can conclude that mental concepts are included in concept images. Hence, the structure of the concept image is the structure in which the mental concepts are included.

First, one may argue that a holistic view is indicated just by the fact that deep intuitions are to be included in complex structures:

> Deep intuitions are not separable mental objects; they are part of various heterogeneous systems and form intricate webs. (SEM, p. 9)

These deep intuitions include properties derived from relationships with objects from mathematics, real life, and physics (SEM, p. 9). Such relationships are holistic in nature as a connection between a mathematical concept and real life cannot be a hierarchical relation (Indicator [16]). As it is claimed that deep intuitions are parts of concept images (SEM, p. 3), these formulations about deep intuitions affect the view on *concept image*, which in turn affects the view on *concept*. Therefore, the view on *concept image* and the mental view on *concept* are holistic as well.

To conclude, both the non-mental and the mental structures are holistic. However, there may also be molecular features in these structures. The fact that some concepts must be defined (SEM, p. 8) is an example of molecular features in a non-mental view. When it comes to a mental view, molecular features can be seen when it is claimed that some properties imply other properties:

> if certain properties of an embodied object of type X and related objects hold, then other properties follow (SEM, p. 12)

To sum up the analysis, there are two views on *concept* in the framework, one non-mental, intersubjective and holistic view, with molecular features, and one mental, intersubjective and holistic view, with molecular features (Figure 5.8).

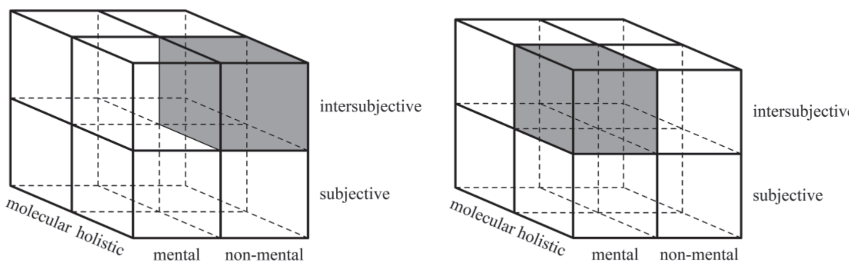


Figure 5.8 Views on *concept* in the CICD$^+$ framework

Regarding the notion *concept image* there is, generally speaking, one view where concept images are mental, subjective with intersubjective features, and holistic with molecular features (Figure 5.9). In order to judge this view as mental, one has to ignore the fact that it may contain a formal image of a non-mental nature and a concept definition defined as a form of words, which complicates the analysis.

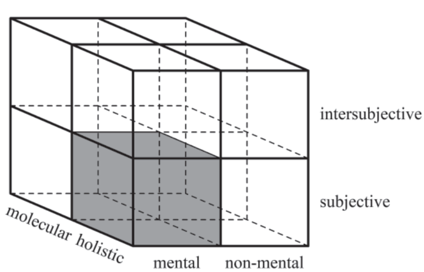


Figure 5.9 View on *concept image* in the CICD$^+$ framework

5.2.5 Comments on the analysis of the CICD$^+$ framework

In the preceding sections, the notions *concept* and *concept image* in the CICD$^+$ framework are analysed. This section contains some comments on the connections between these notions. From the above descriptions, one can conclude that there are two alternatives concerning the relation between *concept image* and *concept*. In the first, a concept is not included in a concept image. An argument for such an interpretation can be found in the fact that *concept image* is explicated as all cognitive structure that is associated with a given concept. From that, it can be concluded that the concept itself is not a part of that structure. However, one problem with this interpretation is that one can argue that concepts are part of formal mathematics. This can be seen, for example, in the formulation "concepts in formal structural theories" (SEM, p. 15). Following this, a second alternative would be that a concept is included in a formal image, which in turn is included in a concept image. There are, however, problems also with this interpretation. Since *concept image* is explicated as the cognitive structure associated with a given concept, one may ask how the concept image arises, if the given concept is included in the concept image.

Here one may end in a regress, where the concept generates a cognitive structure, which, according to the PO frameworks described in Chapter 6 (referred to in SEM), develops into a concept. This apparent regress, I would say, is a consequence of not distinguishing between a non-mental and a mental arena, and a non-mental and a mental view on *concept*, where traditionally the cognitive structure is seen as mental and formal mathematics is seen as non-mental.

Further, there is a lack of a distinction between terms and the meanings of terms, which is seen in the formulation "the basic meanings of the concepts should be stable" (SEM, p. 3). Usually, a concept is seen as a meaning of a term, and the distinction mental versus non-mental expresses that there are two interpretations of the nature of that meaning: the meaning is either mental or non-mental. The formulation 'the meanings of the concepts' can then be interpreted as 'the meanings of the meanings of the terms', which may seem a bit odd.

5.3 The CICD* framework

Initially, CICD studies of students' understanding have focused on the individual mind. However, in BIMO (p. 32) it is claimed that the development of the individual takes place in a context, where the concept image is influenced by the context and by the way that individuals position themselves in this context. In this version of the framework, *concept image* is used at a group level, and focus is on how different groups of students, mechanical engineering students and mathematics students, develop their concept images, based on teaching practices and department perspectives in their respective discipline.

Unlike the texts analysed in Sections 5.1 and 5.2, the only relevant notion that is frequently used in BIMO is *concept image*. The term 'concept' appears in some formulations, which have been used here for analysing the view on *concept*. However, the notion *concept definition* is considered less important, which is justified by the authors using the fact that they are not purely cognitivist theorists:

> We are less interested in concept definition than we are in concept image in this article because we are not purely cognitivist theorists, i.e. from our standpoint factors such as students' departmental affiliation enter into the mix that results in students' concept images. (BIMO, p. 20)

Further, how concepts are structured is not dealt with in the text. Consequently, the analysis concentrates on the notions *concept* and *concept image*, from the perspective of the distinctions mental versus non-mental and intersubjective versus subjective. However, the section starts with a description of explicated and non-explicated notions, and connections between them, with the purpose of constructing a concept map of the framework.

5.3.1 Concept map of the CICD* framework

As in the other versions of the framework, the notion *concept* is not explicated, but is used in the explications of other notions. Below, the notion *concept image* is taken as an example.

- *Concept image* is explicated as "the total cognitive structure associated with a concept in an individual's mind" (BIMO, p. 20). Further, a concept image "includes mental pictures, associated properties and processes as well as strings of words and symbols" (BIMO, p. 20).

Other explicated notions are *evoked concept image* and *concept definition*. *Mental picture*, *property*, *process*, *string of words and symbols*, and *mathematical concept* are examples of non-explicated notions. Regarding the connections, there are hierarchical relations between *concept image* and *mental picture*, *property*, *process*, and *string of words and symbols*. As the concept definition is a form of words/symbols (BIMO, p. 20), one may interpret the relation between *string of words and symbols* and *concept definition* as hierarchical as well. An example of a non-hierarchical connection is the one between *concept definition* and *mathematical concept*, where the concept definition is used for defining a mathematical concept (BIMO, p. 20).

As in Sections 5.1.1 and 5.2.1, the explicated and non-explicated notions, and the connections between them, can be represented in a concept map (Figure 5.10):

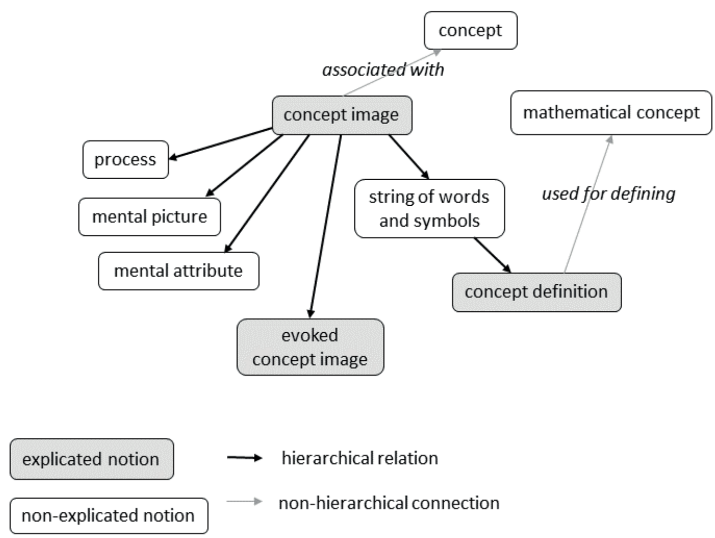


Figure 5.10 Concept map of Bingolbali and Monaghan (2008)

From the explications and the concept map, one can conclude that the view where the concept definition is seen as a part of the concept image has affected the explications. This is the first text analysed here, using a version of the CICD framework, where words and symbols are included in the explication of concept image. Notably, strings of words and symbols are seen as included in cognitive structures.

5.3.2 View on *concept* in the CICD* framework

As claimed above, the notion *concept* is not frequently used in BIMO, and the analysis becomes quite brief. What appears in the formulations is a non-mental and intersubjective view. A non-mental view is seen in the formulation "make sense of the concept" (BIMO, p. 34) and in the quote below, showing a view where concepts are separated from the thoughts of an individual (Indicator [6]):

> a questionnaire which sought to elicit student understanding of key mathematical concepts including the derivative (BIMO, p. 22)

In these formulations, an intersubjective view appears, where students can study concepts, indicating that there is a unique concept of a certain kind (Indicator [9]). This is also seen in the formulation "the derivative concept" (BIMO, p. 20) for a similar reason. Consequently, there is a non-mental and intersubjective view on *concept* in the framework, represented in Figure 5.11.

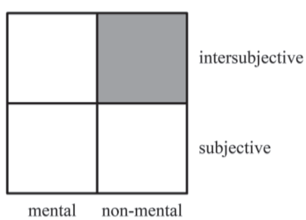


Figure 5.11 View on *concept* in the CICD* framework

5.3.3 Views on *concept image* in the CICD* framework

Again, the notion *concept image* is explicated in BIMO as the total cognitive structure associated with a concept, with a reference to TV. The reason for interpreting the following quote as an explication is that the framework is explicitly built on the perspective in TV:

> Tall and Vinner (1981) describe it as the total cognitive structure associated with a concept in an individual's mind. It includes mental pictures, associated properties and processes as well as strings of words and symbols. (BIMO, p. 20)

Here, it may be noted that the expression 'as well as strings of words and symbols' is not present in TV, but this is a formulation seemingly made to fit the view that concept definitions are included in concept images. Hence, concept images are mainly mental, but contain non-mental elements, such as words and symbols.

When it comes to the distinction intersubjective versus subjective, there are clear indications of both subjective and intersubjective features. First, a subjective view appears in formulations such as "it was designed to access students' concept image of the derivative over time" (BIMO, p. 27), indicating that different students have different concept images, and also different concept images at different times (Indicator [12]). Further, a subjective view is seen below where students develop different concept images, which can be incorrect (Indicator [11]):

> [The concept image] is a dynamic entity that develops, differentially over students, through a multitude of experiences. Some of these will, from a mathematical viewpoint, be incorrect, e.g. squaring a number could be defined as "multiplying a number by itself" and an associated property of squaring, grounded in students' experiences natural numbers, might be "squaring makes the number bigger". (BIMO, pp. 20–21)

Second, intersubjective features are shown when it is claimed that concept images depend on classroom practice (BIMO, p. 30) and departmental influences (BIMO, p. 32). The following quote is an example of such a view:

> students' developing concept images and the way they build relationships with its particular forms are closely related to teaching practices and departmental perspectives (BIMO, p. 32)

Further, an intersubjective view is found when it is claimed that different groups of students develop different kinds of concept images. The intersubjective features are used for studying relations between teaching or departmental affiliation and concept images that the students develop (BIMO, p. 20). In the following quote, the concept images of mechanical engineering (ME) students and of mathematics (M) students are different:

> ME students' concept images of the derivative developed in the direction of rate of change orientations and M students' concept images developed in the direction of tangent orientations. (BIMO, p. 30)

Consequently, both intersubjective and subjective features are shown in the framework. In BIMO (p. 21), it is claimed that the original view, where students develop their different concept images through experiences, is to be seen from a contextual viewpoint. This can be seen as a greater focus on the intersubjective features of the concept images, compared with other versions of the framework. Note that the students are considered at group level, and that the study in BIMO is aimed at identifying trends in the different groups of students, even though there is an awareness that individual concept images can vary in the groups:

> We speak of students in general terms in this article, noting trends in patterns of responses over time but are aware that some individuals do not fit with this trend and that the developmental paths of individuals providing similar responses will not be identical. (BIMO, p. 29)

To sum up, the main view on *concept image* is mental and has both subjective and intersubjective features. Further, there are signs of an intersubjective view, where concept images are described at group level. These two views are represented in Figure 5.12.

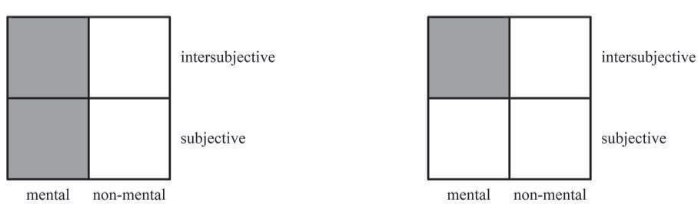


Figure 5.12 Views on *concept image* in the CICD* framework

Consequently, there is one non-mental and intersubjective view on *concept* and two mental views on *concept image* in the CICD* framework. Here it may be noted that there is a distinction between concepts, which are non-mental, and concept images, which are mental.

Since *concept image* is the notion used and discussed in the text, there is no point in commenting connections between different notions, except for what is seen in Section 5.3.1. Hence, the comments on the analysis of the CICD* framework are omitted, and the next section presents some conclusions and a local discussion of the chapter as a whole.

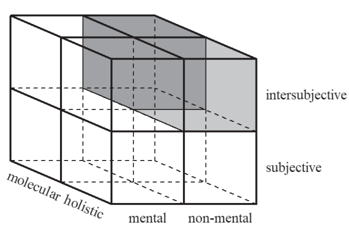
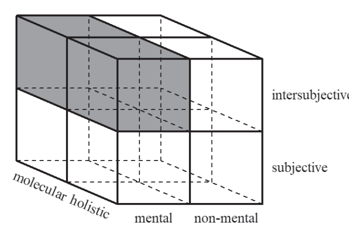
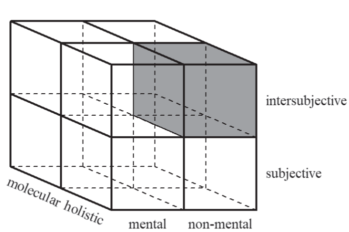
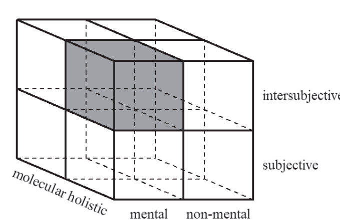
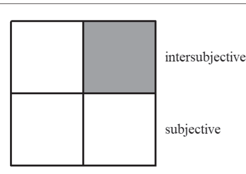
5.4 Comparisons, conclusions and local discussion

From 1980, when the CICD framework first appeared, it has developed into different versions. Below, I make some comparisons between the views on the notions in the analysed texts and comment on some of the findings.

5.4.1 Views on *concept* in the CICD framework

First, it may be noted that the notion *concept* is not explicated in the analysed texts, and the reason for this may be asked for. Is *concept* not considered as a notion that needs to be theorised, or is the meaning of 'concept' taken for granted? Furthermore, that there are several views on *concept* in the texts point to the necessity of clearly explicating *concept*. As a starting point for the comparisons between the views on *concept*, seen in how terms are used in the different versions of the framework, one may again consider the matrices in Figures 5.4, 5.8 and 5.11 (see Table 5.1).

Table 5.1 Views on *concept* in the CICD framework

CICD⁰	 
CICD⁺	 
CICD*	

Note that concepts are intersubjective in all these views. Further, in both the $CICD^0$ framework and the $CICD^+$ framework, there are two different views on *concept*, one non-mental and one mental view. In the CICD* framework, there is only a non-mental view on *concept*. Also, there seems to be a shift between the $CICD^0$ framework and the $CICD^+$ framework when it comes to whether the conceptual structure has molecular or holistic features. In VIHE and TV, non-mental mathematical concepts are considered molecular. In SEM, on the other hand, there are comments that are critical of a definitional approach, and a focus on holistic features.

5.4.2 Views on *concept image* in the CICD framework

Regarding the notion *concept image*, there are different explications in the analysed texts. These are seen in Table 5.2:

Table 5.2 Explications of *concept image* in the CICD framework

VIHE	The concept image is a set of properties together with the mental picture, which in turn is "the set of all pictures that have ever been associated with C [a concept], in P's [a person] mind" (VIHE, p. 177)
TV	The concept image is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." (TV, p. 152)
SEM	The concept image is "all the cognitive structure in the individual's mind that is associated with a given concept" (SEM, p. 4), with a reference to TV. Further, "[a]ll mental pictures (pictorial, symbolic, and others), all mental attributes (conscious or unconscious) and associated processes are included in the concept image" (SEM, p. 4).
BIMO	The concept image is "the total cognitive structure associated with a concept in an individual's mind" (BIMO, p. 20), with a reference to TV. Further, the concept image "includes mental pictures, associated properties and processes as well as strings of words and symbols" (BIMO, p. 20).

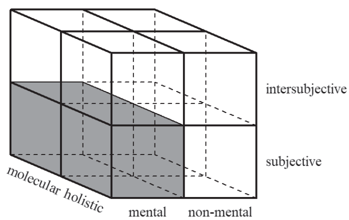
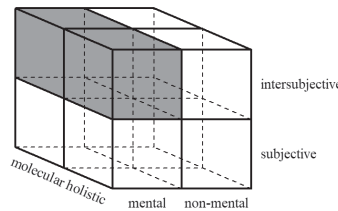
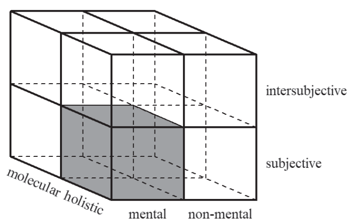
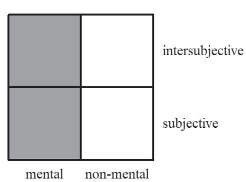
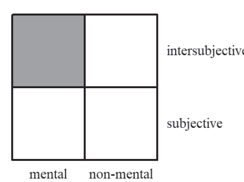
One difference between the explications in VIHE and TV is that *concept image* in the latter text is explicated as the total cognitive structure. This is not the case in VIHE, where the cognitive structure contains two cells, concept image and concept definition. Further, as suggested in Section 5.1.3, the expression 'and processes' seems to shift the idea from a mental image to a more general mental representation, including processes. The expression 'as well as strings of words and symbols' also appears in the explication in BIMO, seemingly to fit a view where concept images include concept definitions.

In addition, *concept image* and *concept definition* are separated in the concept maps in Figures 5.1 and 5.2. This may be compared with *concept definition* being included in *concept image* in Figures 5.7 and 5.10. However, that *concept definition* is included in *concept image* in SEM is not seen in the explication of *concept image*. Such an explication is found only in BIMO, where the concept image includes strings of words and symbols. Hence, the explication in this text seems more coherent, from this perspective, compared with the explication in SEM.

Regarding the views on *concept image* in the analysed texts, interpreted with indicators, one can compare them by considering the matrices in Figures 5.5, 5.9 and 5.12. This gives Table 5.3. Notably, concept images are mainly mental in all these views. This is in line with the explications of *concept image* as a cognitive structure. Further, the same shift as with the notion *concept* can be seen between the $CICD^0$ framework and the $CICD^+$ framework, regarding conceptual structures which are mainly holistic in SEM. It may also be noted that different versions of the framework focus more or less on subjective and intersubjective features. In the interpretations, I have found several aspects of intersubjectivity in the frameworks:

- The first aspect concerns the fact that culturally dependent concept images may be studied at group level, where students in different groups develop different structures.
- The second aspect concerns the fact that the teacher and the context affect individual concept images.
- The third aspect concerns the fact that concept images may be assessed, and that it is possible to decide when the concept image has reached different levels of development.

Table 5.3 Views on *concept image* in the CICD framework

CICD^0	 
CICD$^+$	
CICD*	 

5.4.3 Mental and non-mental arenas

When considering the view where concept images contain concept definitions, and also formal mathematics, the problem arises that mental structures contain elements that are non-mental. From the philosophical background described in Chapters 2 and 3, this is a category mistake that opposes the distinction between the three different arenas as well as the distinction between internal and external representations. It also opposes the distinction between expressions (syntax) and meanings of expressions (semantics).

It seems as though the alternative of *concept image* where concept image and concept definition are separated has been less advocated than the alternative where concept images include concept definitions. The CICD$^+$ framework and the CICD* framework both advocate the second alternative.[50]

[50] Note that later texts written by Shlomo Vinner have not been analysed.

One thing to note is that when *concept image* in BIMO (p. 20) is seen as including mental pictures, associated properties and processes as well as strings of words and symbols, a reference is made to TV, where there is another explication of *concept image*, not including words and symbols. SEM (p. 4) has another approach, first explicating *concept image* as in TV, without strings of words and symbols, and then assuming that the concept image may contain the concept definition, which is not in line with the explication. From this perspective, the explication in BIMO is more in line with the view on *concept image* in the text. However, neither BIMO nor SEM comment on the fact that the explication should be changed from the one in TV, if concept definitions are seen as parts of concept images.

6 Concept analysis of texts using the process to object frameworks

Stage models of concept development are common in a constructivist approach to learning. In the PO frameworks, acquiring knowledge consists of the construction of mental representations, which are used for making sense of mathematics. The OS framework (Sfard, 1991), the procept framework (Gray & Tall, 1994) and the APOS framework (Dubinsky, 1991; Asiala et al., 1996) are three examples, focusing on concepts in arithmetic, algebra and calculus, called computational concepts. The OS framework is known for the distinction between operational and structural conceptions. Since it appeared in Sfard (1991), this distinction has influenced mathematics education and is still referred to, often in combination with other frameworks analysed in this chapter. In Gray and Tall (1994), using what may be called the procept framework, the two fundamentally different ways of thinking usually described by the terms 'procedural' and 'conceptual' are instead described as 'procedural' and 'proceptual', where proceptual thinking includes using procedures as well (Gray & Tall, 1994, p. 125). Further, the APOS framework is built on an assumption that by analysing concepts theoretically, one can obtain knowledge of how individuals construct mental knowledge (Dubinsky, 1991, pp. 96).[51] The framework appeared during the 1990s and was used by an informal community of researchers, *Research of Undergraduate Mathematics Education Community* (Asiala et al., 1996, p. 2). The concept analysis in this section concentrates on Dubinsky (1991), presenting the initial development of the framework, and Asiala et al. (1996), describing the framework as a whole.

All frameworks analysed in Chapter 6 contain the idea of a cognitive structure that develops from including the idea of processes to including the idea of an object that can be used in other processes. Hence, these frameworks are called process to object frameworks or PO frameworks. As in Chapter 5, I use abbreviations when referring to the analysed texts. Thus, in the following, SF is short for Sfard (1991), GT is short for Gray and Tall (1994), DUB is short for Dubinsky (1991), and ASI is short for Asiala et al. (1996). In the first section,

[51] It takes a stance built on an interpretation of Piaget's perspective.

the texts are analysed from the perspective of which notions are explicated and which are not, and of how these relate to each other. Based on this first analysis, notions and connections between them are represented in concept maps. Next, in Section 6.2, views on *concept* in the frameworks are interpreted with the help of the indicators. In Section 6.3, the related notions *conception* and *schema* are interpreted similarly. Finally, comparisons between views on *concept* and views on *conception/schema* are made, and some conclusions from the concept analysis are discussed.

6.1 Concept maps of the PO frameworks

In this first section, concept maps are constructed for the OS framework in SF, the procept framework in GT, and the APOS framework in DUB and ASI. The descriptions below concentrate on notions describing cognitive structures, in the theoretical parts of the texts, together with connections between different notions. The connections are divided into hierarchical and non-hierarchical. The latter may be of different types, but are not further categorised.

6.1.1 Concept map of the OS framework

In the OS framework, the notion *concept* is explicated as "a theoretical construct within 'the formal universe of ideal knowledge'" (SF, p. 3). Further, the notion *conception* is explicated as "the whole cluster of internal representations and associations evoked by the concept – the concept's counterpart in the internal, subjective 'universe of human knowing'" (SF, p. 3). Other notions explicated in the text are *structural conception* and *operational conception*.

In addition, non-explicated notions such as *mathematics*, *mathematical concept*, *object*, *process*, *algorithm*, *action*, *definition* and *representation* appear in the framework in SF, described under the heading *The dual nature of mathematical conceptions*. This list is a selection of the notions used, where I have chosen the ones that seem to be the most important, in the sense that there are several connections to other notions shown in the text.

Regarding the connections described in SF, there are hierarchical relations between *mathematics* and *concept*, and between *mathematics* and *conception*, as concepts and conceptions are seen as the building blocks of mathematics (SF, p. 3). Further, there are hierarchical relations between *conception* and *structural conception*, and between *conception* and *operational conception*, as structural conceptions and operational conceptions obviously are types of conceptions

(SF, p. 4). As an example of a non-hierarchical connection, there is a connection between *concept* and *conception*, as a conception is evoked by a concept (SF, p. 3). After interpreting the connections between the notions, this initial analysis results in a concept map that looks something like the one in Figure 6.1. In this map, the explicated and non-explicated notions, and the connections between the notions, in a sense represent the conceptual framework in SF.

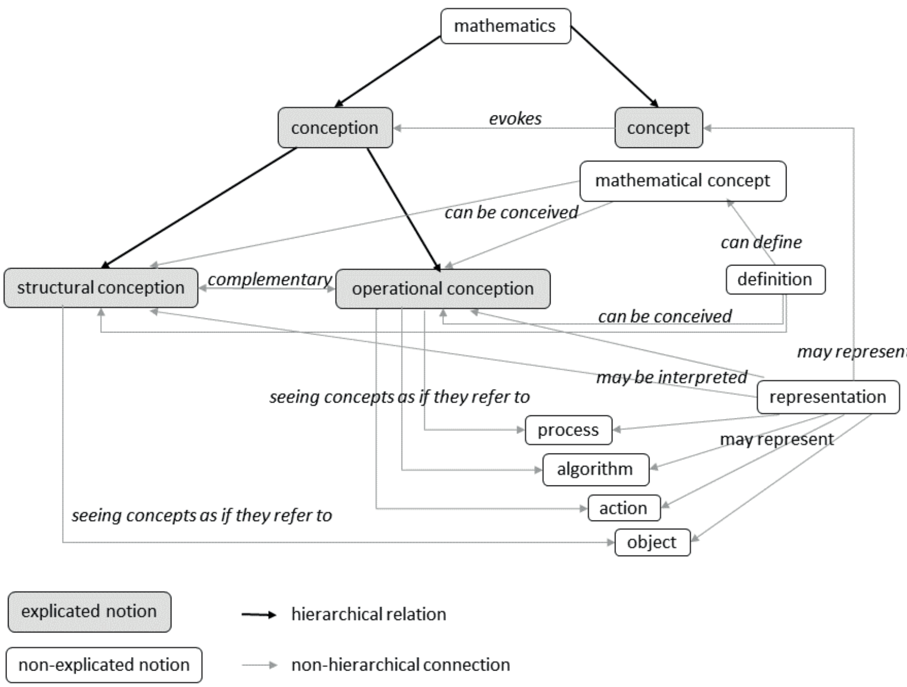


Figure 6.1 Concept map of Sfard (1991)

From comparing the concept map in Figure 6.1 to the concept maps in Figures 5.1, 5.2, 5.7 and 5.10 describing different versions of the CICD framework, it may be concluded that this is the first text, and actually the only text analysed in the study, which contains an explication of *concept*. In this explication, concepts are clearly placed apart from the cognitive structure, as concepts are theoretical constructs within formal and ideal knowledge (SF, p. 3). However, it may still be hard to understand the connections between, for example, *concept* and *mathematical concept*, on the one hand, and the notion *object*, on the other hand.

6.1.2 Concept map of the procept framework

In GT, the notion *concept* is not explicated. Rather, the focus is on the notion *procept*, which is explicated in two steps. First, *elementary procept* is explicated as "the amalgam of three components: a *process* that produces a mathematical *object*, and a *symbol*" (GT, p. 121). Second, *procept* is explicated as "a collection of elementary procepts that have the same object" (GT, p. 121). Other explicated notions in the text are *process*, *procedure*, *symbol*, and *proceptual thinking*. Non-explicated notions such as *object*, *thinking*, *procedural thinking*, and *conceptual thinking* appear in the framework as well.

Regarding the connections described in GT, there is, for example, a hierarchical relation between *procept* and *elementary procept*, as elementary procepts are elements in procepts (GT, p. 121). Further, there are hierarchical relations between *elementary procept* and *process*, *object*, and *symbol*, as processes, objects, and symbols are components of elementary procepts (GT, p. 121). The connections between *symbol* and *process*, *object*, and *concept*, however, where a symbol can represent a process, an object or a concept (GT, pp. 119, 121), can be taken as examples of non-hierarchical connections. After interpreting the connections between the notions in this manner, the explicated and non-explicated notions, and the connections between them, may be represented in a concept map similar to the one in Figure 6.2.

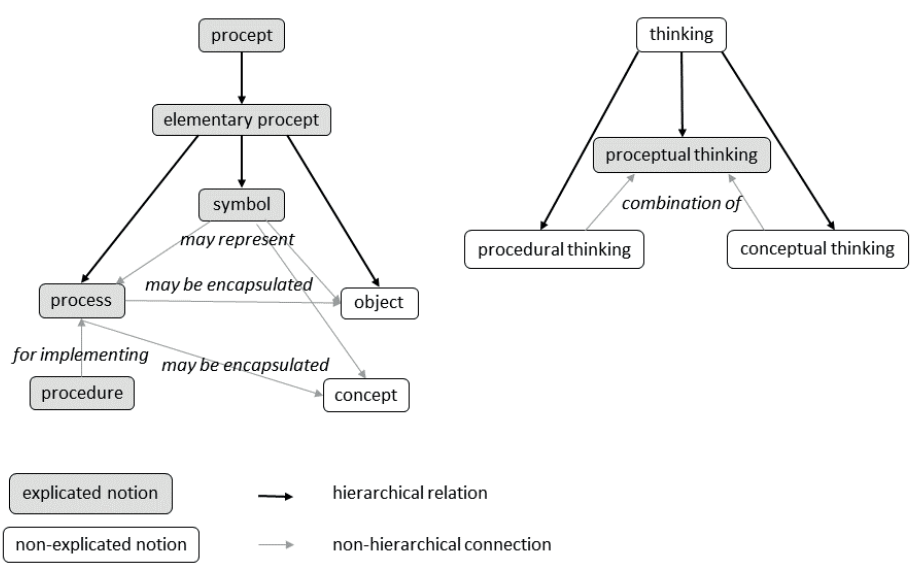


Figure 6.2 Concept map of Gray and Tall (1994)

Notably, on page 119 in GT, a symbol is described as representing either a process or an object, and on page 121, it is described as representing either a process or a concept. In addition, a process is described on page 118 as if it may be encapsulated into an object, and on page 119 as if it may be encapsulated into a concept. This raises the question of the difference between *concept* and *object*, and one can argue that these notions are used interchangeably.

It may also be noted that no clear notion used for the cognitive structure appears in the explications. When reading the whole text, there are several different notions used for the cognitive structure, such as *mental schema* (GT, p. 116), *mental structure* (GT, p. 116), *conceptual structure* (GT, p. 122) and *conception* (GT, p. 137). However, none of these notions appear more than occasionally.

6.1.3 Concept maps of the APOS framework

Below, notions with and without explications in the theoretical backgrounds of the two texts DUB and ASI, and the connections between these notions, are first described. Next, a concept map is constructed for each text.

6.1.3.1 Concept map of Dubinsky (1991)

In DUB, the notions *concept* and *mathematical concept* are not explicated. Rather, the notion *schema* is explicated as "a more or less coherent collection of objects and processes" (DUB, p. 101). Further, it is claimed that a schema is not static, but is a dynamic activity that may be described as a circular feedback system (DUB, p. 105). Other notions explicated under the heading *A theory of the development of concepts in advanced mathematical thinking* are *mathematical knowledge* and *process*. Non-explicated notions such as *action*, *object*, *mental object* and *physical object* appear in the text as well.

Regarding the connections described in DUB, there are, for example, hierarchical relations between *schema* and *object*, and between *schema* and *process*, as a schema is a collection of objects and processes (DUB, p. 101). An example of a non-hierarchical connection, on the other hand, may be the connection of construction between *schema* and *concept*, as concepts are constructed by coordinating schemas (DUB, p. 99). Also, an interpretation of the fact that schemas can be generalised to a wider collection of phenomena (DUB, p. 101), may be that there is a connection between the notion *schema* and itself.

The overall notion of *reflective abstraction*, involving the notions *interiorization*, *coordination*, *encapsulation*, *generalisation*, and *reversal*, deserves a note. In DUB, these are used for describing the development of cognitive structures, and for describing how different parts of a cognitive structure connect to each other. In the current section, I use these notions as connections between other notions. As will be seen further on when constructing a concept map of ASI, this approach fits with how notions are used in that text.

Interpreting DUB with a focus on connections between explicated and non-explicated notions results in a concept map similar to the one in Figure 6.3 below:

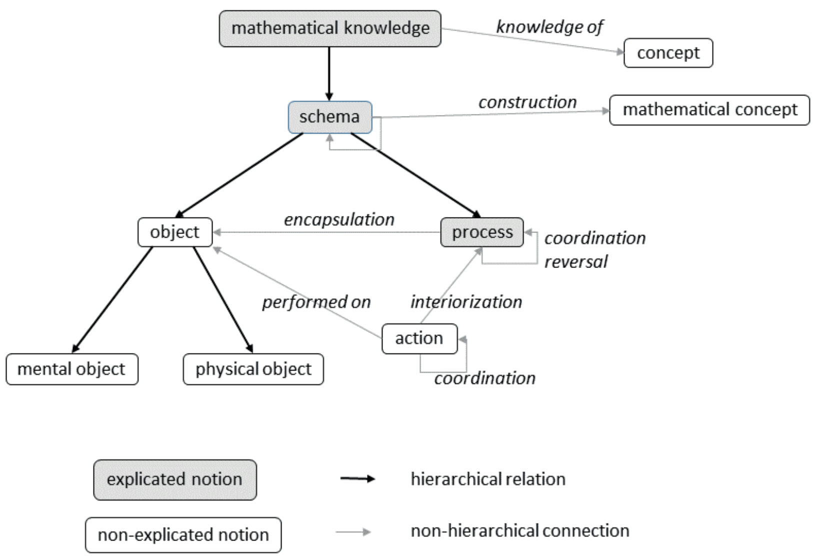


Figure 6.3 Concept map of Dubinsky (1991)

When comparing the concept map in Figure 6.3 with the concept maps describing the other frameworks, there is a difference regarding that in DUB there is no clear distinction between the view on how the cognitive structure is built and the view on concept development. Notably, in SF and GT, there is first a description of the cognitive structure, and next a description of concept development. However, when interpreting the connections between notions describing the cognitive structure in DUB, I would say that it is impossible not to use the notions describing concept development. In this framework, the view on concept development is partly integrated with the view on cognitive structures.

As is seen in Figure 6.3, objects may be mental or physical (DUB, p. 102). Similarly, it is claimed in DUB (pp. 99, 102) that an action may be mental or physical, where a mental action is a process. In addition, there seems to be a distinction between mental processes and other processes, as it is claimed that "the subject may respond by constructing, in her or his mind, a mental process relating to the function's process" (DUB, p. 103). It is unclear how the distinction between mental and physical objects, between mental and physical actions, and between mental and other processes should be interpreted. In DUB, such questions are avoided, which is seen in the formulation "the term *object* will refer to a mental or physical object (avoiding any discussion of the nature of the distinction)" (DUB, p. 102). Even so, the development explained in DUB is mental. To exemplify, this is seen in the formulation "reflective abstraction will be the construction of mental objects and of mental actions on these objects" (DUB, p. 101).

6.1.3.2 Concept map of Asiala et al. (1996)

The framework in ASI is a developed version, compared with the one in DUB, focusing more on instructional treatments and on gathering and analysing data (ASI, pp. 3–4). However, below the focus is on Section 3.1 in ASI, with the heading *Theoretical analysis*, where the notions can be compared to the notions above describing the framework in DUB.

First, it may be noted that the notions *concept* and *mathematical concept* lack explications. Further, the notion *schema* has the following explication: "[a]n individual's schema is the totality of knowledge which for her or him is connected (consciously or subconsciously) to a particular mathematical topic" (ASI, p. 9). Other explicated notions are *mathematical knowledge*, *action*, and *process*. Non-explicated notions are, among others, *object*, *mental object*, *physical object*, *conception*, *action conception*, *process conception*, *object conception*, and *action concept*.

Regarding the connections between the notions, examples of hierarchical relations, on the one hand, may be seen between *schema* and *process*, *object*, and *action*, as mathematical actions, processes, and objects are included in schemas (ASI, pp. 5, 8). Examples of non-hierarchical connections, on the other hand, may be seen between *process* and *object*, where a process may be encapsulated into an object, and an object may be de-encapsulated back to the process from which it was constructed (ASI, p. 8). Notably, in ASI, the different constructions describing reflective abstraction in DUB do not have explications and are not really focused. The concept map in Figure 6.4 represents the explicated and non-explicated notions, and the connections between them, as they are described in ASI.

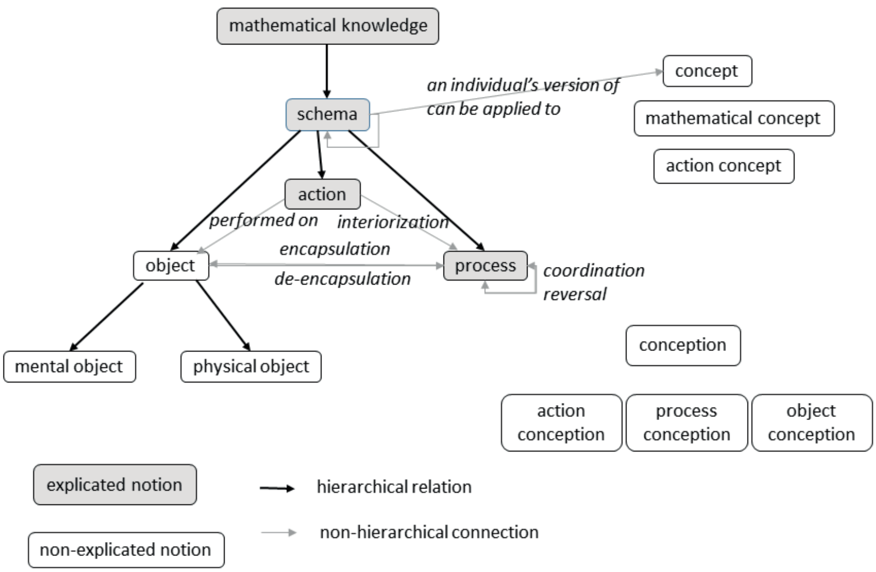


Figure 6.4 Concept map of Asiala et al. (1996)

From comparing the concept map in Figure 6.4, representing the framework in ASI, with the concept map in Figure 6.3, representing the framework in DUB, it can be concluded that they are quite similar. One difference is that in DUB, there is no hierarchical relation between *schema* and *action*. Further, the notion *conception* and different types of conceptions are used in ASI. As the notion *conception* is not explicated, it is hard to see how *schema* and *conception* are related. Naturally, one can interpret the connections between the notion *conception*, on the one hand, and the notions *action conception*, *process conception* and *object conception*, on the other hand, as hierarchical, even though these connections are not explicated. Further, there seem to be non-explicated connections between *action conception* and *action*, between *process conception* and *process*, and between *object conception* and *object*.

Finally, the notion *action concept* is used without explication, in addition to the notion *action conception*. It is unclear how the notion *action concept* fits into the framework. One alternative is that there are different types of concepts: action concepts, process concepts, and object concepts. Another possibility is that there is a writing error here, where 'action concept' should be replaced with 'action conception'.

6.1.4 Comments on the concept maps

As an overall observation from comparing the concept maps in Figures 6.1, 6.2, 6.3 and 6.4, it seems as though the frameworks aim at explaining different things. In SF, the concept map describes connections between the cognitive structure, mathematics and language. In GT, there is a static view on the cognitive structure, even if there is a description in the text of how this structure develops. In DUB and ASI, on the other hand, the view on the structure is intertwined with the view on the development of the structure.

Regarding the notion *concept* in the different maps, it seems as if the role of *concept* in GT is different, compared with those in SF, DUB and ASI. In GT, *concept* seems to be similar to a mental object, which is not true of the other texts. For example, in SF, which is the only text containing an explication of *concept*, concepts are theoretical constructs within formal and ideal knowledge (SF, p. 3).

Notably, *object* and *process* are frequently seen as mental in GT, DUB and ASI, which is not the case in SF. This makes it difficult to compare the notions *conception* in SF and *schema* in DUB and ASI, since these are explicated using different terms. However, one difference to be noted is that *conception* is a more static notion, and that *schema* is more dynamic.

Regarding the notion *procept*, the explication does not describe a procept as a cognitive structure, and it does not seem as if *procept* is similar to *conception* or *schema*. However, when comparing the concept map in Figure 6.2, describing the framework in GT, with the concept maps in Figures 6.3 and 6.4, describing the frameworks in DUB and ASI, it may be noted that one main difference between *procept* and *schema* is that a symbol is included in the procept. This may be related to the discussion in Chapter 5, concerning whether words and symbols are included in concept images or not. When comparing the role of *procept* in Figure 6.2, representing GT, with the role of *concept image* in Figure 5.10, representing BIMO, one can see that these seem to be similar. However, a main difference seems to be that the explication of *concept image* uses 'mental picture', and that the explication of procept uses 'object'. Consequently, the notion *procept* in GT appears to be similar to *concept image* in BIMO. In that case, a procept may be seen as a more developed concept image, where the cognitive structure presupposes that a symbol may refer to either a process or a concept. However, *procept* is not explicated as a cognitive structure, but as an amalgam of three components, which contradicts such an interpretation.

After this initial analysis of the PO frameworks, comparing explications in the analysed texts, the following sections present analyses of *concept* and of the notions *conception* and *schema*, using the indicators in Section 4.4.

6.2 Views on *concept* in the PO frameworks

This section presents analyses of views on *concept* in the PO frameworks. Below, views in the OS framework are followed by views in the procept framework and views in the APOS framework. After that, the different views are compared and commented on.

6.2.1 Views on *concept* in the OS framework

As is seen in Section 6.1.1, there is a distinction in SF (p. 3) between *concept* and the related notion *conception*, where a concept is presented as a theoretical idea and a conception is the cognitive structure evoked by the concept. Views on *concept* in the framework are intertwined with views on *conception*, especially when it comes to the distinction molecular versus holistic. Even so, in this section, *concept* is treated as a single notion, and we will return to views on *conception* in Section 6.3.1. Below, views on *concept* are first analysed from the perspective of the distinctions mental versus non-mental and intersubjective versus subjective, and next analysed from the perspective of the distinction molecular versus holistic.

6.2.1.1 The distinctions mental vs non-mental and intersubjective vs subjective

The notion *concept* is explicated as a mathematical idea that is presented as a theoretical construct within formal mathematics:

> the word "concept" (sometimes replaced by "notion") will be mentioned whenever a mathematical idea is concerned in its "official" form - as a theoretical construct within "the formal universe of ideal knowledge" (SF, p. 3)

This description, referring to formal mathematics, indicates a view considering concepts as non-mental (Indicator [4]). In the way terms are used, a non-mental view is confirmed by formulations such as "to think structurally about a concept" (SF, p. 18) and "a new concept is introduced" (SF, p. 23), where a concept is distinguished from the thoughts of an individual thinker (Indicator [6]). To continue, the view on *concept* is intersubjective since a concept is a mathematical idea presented in a, so to speak, official form (SF, p. 3) (Indicator [9]). The intersubjective view on *concept* is also seen in formulations such as "the concept of function" (SF, p. 5), indicating that there is a unique concept *function* (Indicator [9]). The following quote is another example. Here, the formulation 'we can take a look at any mathematical concept' indicates an intersubjective view, since a concept is something that we may study together and, hence, agree upon (Indicator [9]):

> If we take a scrutinizing look at any mathematical concept, more often than not we shall find that it can be defined - thus conceived - both structurally and operationally. (SF, p. 5)

Together, this points to a view where concepts are non-mental and intersubjective. However, the explication of *concept* and the usage of *concept* are not always coherent. In fact, in the article, 'concept' is occasionally used to refer to a mental representation. To exemplify, the text under the heading at page 16 forms a psychological context. Here, terms such as "concept acquisition" (SF, pp. 17, 21) seem to refer to a view where concepts are mainly seen as mental representations (Indicator [3]). Other examples of a mental view are seen in the following quotes:

> We shall call these three stages in concept development *interiorization*, *condensation* and *reification*, respectively. (SF, p. 18)

> At the stage of interiorization a learner gets acquainted with the processes which will eventually give rise to a new concept (like counting which leads to natural numbers, subtracting which yields negatives, or algebraic manipulations which turn into functions). (SF, p. 18)

In these quotes, the usage of the terms 'interiorization', 'condensation' and 'reification'[52], together with the fact that a learner may develop a concept, shows a mental view on *concept* (Indicator [3]). The necessity indicated in the quote above, where algebraic manipulations necessarily turn into functions, points to an intersubjective view (Indicator [10]). Further, the formulation "the concept of a circle" (SF p. 10), appearing in a context where concepts are seen as mental and where the individual develops concepts, indicates that there is a unique concept *circle* (Indicator [9]). Hence, the mental view is intersubjective.

---

[52] The cognitive process of forming a concept from a process has variously been called encapsulation, entification or reification. (Gray & Tall, 1994, p. 3)

Consequently, we have hitherto seen two views on *concept* in SF, a non-mental and intersubjective view and a mental and intersubjective view. These views may appear as a consequence of combining a philosophical and a psychological perspective. As an example, the following quote starts with the philosophical understanding of concepts, which indicates a non-mental view. Further, when the sentence forms a psychological context and it is suggested that concepts emerge from psychological processes, a mental view is indicated. Hence, in this quote, the non-mental and the mental views appear in the same sentence.

> It seems that the philosophical insight into the nature of mathematical concepts is what we need in order to understand in depth the psychological processes in which such concepts emerge. (SF, p. 2)

6.2.1.2 The distinction molecular vs holistic

When it comes to the distinction molecular versus holistic, the analysis is divided into two parts, considering first the non-mental and then the mental view. Regarding the non-mental view, the fact that concepts can be defined (SF, pp. 10–16), points to a molecular view on *concept* (Indicator [14]). Another indication of a molecular view is found when the historical development of mathematics is presented as a hierarchy, where natural numbers are more basic than reals, which in turn are placed at a lower level than complex numbers (SF, p. 13) (Indicator [15]).

In conclusion, the non-mental mathematical structure is molecular, and consequently the non-mental view on *concept* is intersubjective and molecular. This is the view represented into the left-hand matrix in Figure 6.6. Since the text refers to the works of Frege (SF, pp. 4, 15), this may be a static view on *concept* that is similar to the one in Frege (1892/1985; 1892/1951), described in Chapter 3.

Regarding the mental view, a molecular view is indicated in formulations claiming a hierarchical cognitive structure, such as in the following quotes (Indicator [15]):

> what is conceived purely operationally at one level should be conceived structurally at a higher level (SF, p. 16)

> formation of a structural conception means reorganizing the cognitive schema by adding new layers - by turning sequential aggregates into hierarchical structures. (SF, p. 28)[53]

Here, there is a connection between *conception* and *concept*, as conceptions include mental concepts and develop into concepts at a higher level. Consequently, the structure of the conception and the structure that concepts are embedded in cannot be distinguished. Figure 6.5, which is strongly inspired by Figure 7 in SF (p. 27)[54], shows a hierarchical view on conceptual structures, where the development of the structure consists of reorganising it and adding new layers, indicating molecular features (Indicator [15]). Here, the structure develops from unstructured to structured, with molecular features.

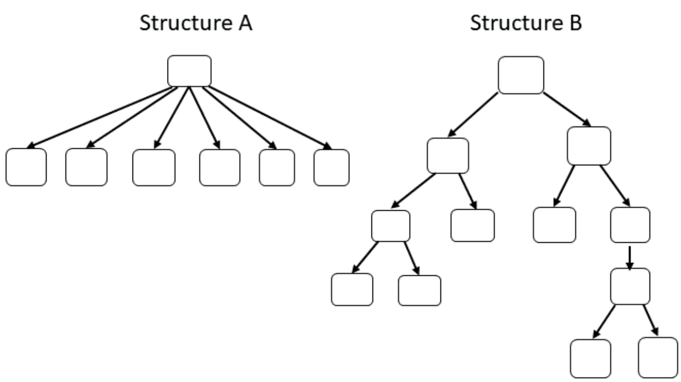


Figure 6.5 Organisations of cognitive structures

Consequently, concepts in a mental view have molecular features. Now, we can summarise the two different views on *concept*, the non-mental, intersubjective and molecular one, and the mental, intersubjective and molecular one, in Figure 6.6.[55]

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[53] In this quote, the term 'schema' is used.

[54] In this figure, the term 'schema' is used for the cognitive structure.

[55] It is worth mentioning that the word 'holistic' is used in the text (SF, p. 7), but not in a way that indicates a view where concepts have holistic structure. Rather, the formulation "[v]isual representation is holistic in its nature" (SF, p. 7) means that images preserve various aspects of a mathematical concept which may be grasped by sight.

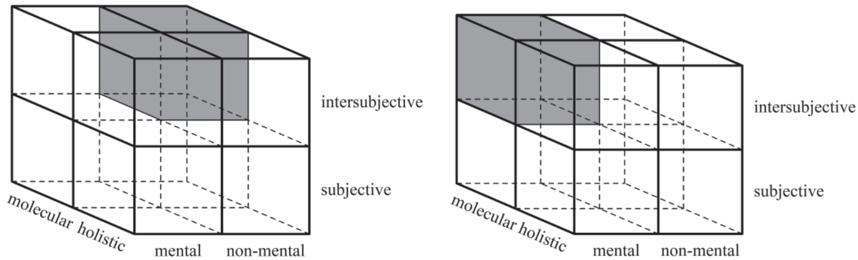

Figure 6.6 Views on *concept* in the OS framework

One can compare the views on *concept* resulting from analysing SF with the help of the indicators in this section with the concept map in Section 6.1.1. In this concept map, which emerges from the explications, it is neither seen that conceptions develop into concepts, nor that the conception includes mental concepts. This is due to *conception* being explicated as an internal structure and *concept* being explicated as a theoretical construct. The mental view on *concept* is not seen in these explications, but appears in the way terms are used later in the text.

6.2.2 Views on *concept* in the procept framework

My analysis of the procept framework is concentrated only on views on *concept*. The reason for not analysing the notion *procept* is that *procept* is not explicated as a cognitive structure. A discussion concerning this issue is found in Sections 6.1.4 and 6.3.3. Further, within GT, I have not found evidence for taking a stance regarding whether the structures of concepts have molecular or holistic features. Hence, only the distinctions mental versus non-mental and intersubjective versus subjective are used in the analysis.

6.2.2.1 The distinctions mental vs non-mental and intersubjective vs subjective

As is seen in Section 6.1.2, the notion *concept* is not explicated in GT[56]. In the usage of the term 'concept', two different views are shown. Occasionally 'concept' is used in a non-mental way. Otherwise it is used in a mental way. In the formulation "the concepts or basic facts, which they [the individuals] are

[56] The notion *conceptual entity* is explicated as "a cognitive object that can be manipulated as the input to a mental procedure" (GT, p. 118) (referring to Greeno (1983)). The relation between *concept* and *conceptual entity* is not commented on.

expected to know" (GT, p. 117) concepts seem to be something that individuals are expected to have knowledge about, indicating a non-mental view (Indicator [6]). Further, a non-mental view is visible when the text refers to concepts as a basis for mathematics:

> At the foundation of arithmetic is the concept of number. (GT, p. 118)

By the way in which GT (p. 118) distinguishes between cognition and mathematics, where mathematics as a field should be considered non-cognitive, this formulation indicates a non-mental view (Indicator [4]). Also, in this quote, concepts are intersubjective as there is a unique concept *number* (Indicator [9]). Hence, the view that appears is both non-mental and intersubjective.

In other parts of GT, a concept is seen as a mental structure:

> Symbolism that inherently represents the amalgam of process/concept ambiguity we call a "procept". We hypothesize that the successful mathematical thinker uses a mental structure that is manifest in the ability to think proceptually. (GT, p. 116)

In this quote, people use concepts, which are part of mental structures, in thinking, indicating a mental view (Indicators [1; 2]). Further, references to Piaget (GT, pp. 118–119) and the usage of the term 'encapsulation' (GT, pp. 118, 120, 123, 135–136) indicate a mental view as well (Indicator [3]). Through communication, the mental meaning of the symbol gets a shared reality and becomes intersubjective (Indicator [10]):

> The word *three* (and its accompanying symbol 3) can be spoken, it can be heard, it can be written, and it can be read. These forms of communication allow the symbol to be shared in such a way that it has, or seems to have, its own shared reality. (GT, p. 122)[57]

I have interpreted the fact that a symbol may be shared to indicate that a concept or a process may be shared as well, since symbols represent concepts or processes. Consequently, there are two separate views on *concept* in the text, a mental view and a (not very common) non-mental view. In both cases, concepts are intersubjective.

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57 Regarding this quote, it may be asked what it means for a symbol to be shared. Instead, from my perspective, it should be the concept that the symbol represents that is shared. This is an example where the text does not distinguish between symbols and the meanings of symbols.

When it comes to the conceptual structure, it is claimed that conceptual knowledge is a connected web that is rich in relationships (GT, p. 117). As an example, repeated addition becomes multiplication (GT, p. 135), which can be interpreted to mean that the concept *multiplication* depends on the concept *addition*. In this case, the conceptual structure has molecular features. However, the text also discusses how hierarchies in mathematics can hinder a student in their problem solving, and how more able students collapse the hierarchy into one single level of notions (GT, pp. 135–136). Therefore, I do not see this discussion as an argument for taking a position concerning the distinction molecular versus holistic. Hence, using only the distinctions mental versus non-mental, and intersubjective versus subjective, the views on *concept* may be represented in matrices as in Figure 6.7, where the distinction molecular versus holistic is omitted.

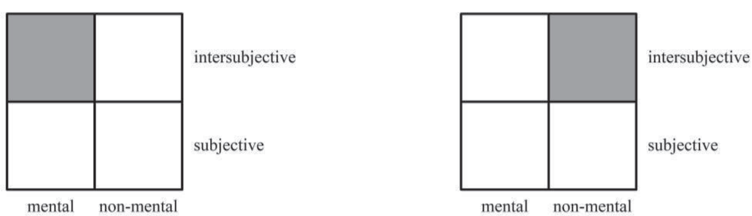

Figure 6.7 Views on *concept* in the procept framework

The views on *concept* resulting from analysing GT with the help of the indicators, in this section, may be compared to the concept map in Section 6.1.2. In the concept map, *concept* seems to be similar to *object* in a mental view. The non-mental view on *concept* is not seen in the concept map, but appears in the way terms are used. This may be compared with the analysis of SF, where the non-mental view is seen in the concept map, and the mental view is seen in the usage of terms.

6.2.3 Views on *concept* in the APOS framework

When it comes to the APOS framework, the distinction between *concept* and *schema* seems to be similar to the distinction between *concept* and *conception* in the OS framework. As stated in Section 6.1.4, a difference between the frameworks is that in SF, both *object* and *process* seem to be non-mental, but in DUB and ASI, *object* and *process* are mostly seen as mental. Hence, the notions *conception* in SF and *schema* in DUB and ASI are explicated using different terms.

In the description of the development of a schema, one can see that there are different levels of objects, where an object may be the input for an action or a process, which in turn can be encapsulated into an object at a higher level. An example taken from DUB (p. 100) is that multiplication is described as an addition of additions, and in order to multiply, it is necessary to first encapsulate the process of addition. Also, when objects and processes have been constructed, they can be organised into schemas. These schemas may in turn be treated as objects in higher-level actions and processes, and may be included in higher-level schemas. As an example, in ASI (p. 8) it is claimed that functions can be grouped into sets, operations can be introduced on these sets and properties of the operations, which are now seen as objects, can be explored. These can then be included in the construction of a schema for a function space. Consequently, there are actions, processes, objects, and schemas at different levels, as represented in Figure 6.8. (ASI, pp. 6–8; DUB, pp. 107, 116–117)

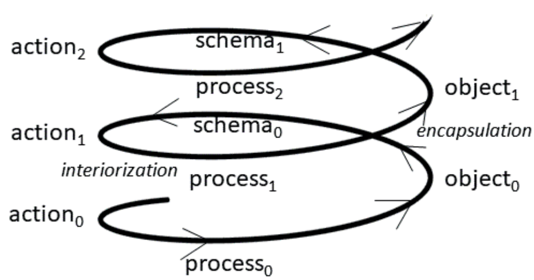

Figure 6.8 Development of cognitive structures

Below, the view on *concept* is first analysed from the perspective mental versus non-mental and intersubjective versus subjective, and next from the perspective molecular versus holistic.

6.2.3.1 The distinctions mental vs non-mental and intersubjective vs subjective

In DUB and ASI, the notion *concept* is not explicated. Below, an analysis of how terms are used in DUB is followed by an analysis of how terms are used in ASI. In DUB, there is both a mental and a non-mental view. First, a mental view is seen in the usage of formulations such as "acquire concepts" (DUB, p. 95) (Indicator [3]). Another example of a mental view is seen in the formulation "the concept of number is constructed by coordinating the two schemas of classification" (DUB, p. 99), where the mental view on *schema* (see Section 6.3.2) affects the view on *concept* (Indicator [1]). Further, the fact that the mental view is intersubjective is seen in formulations such as "the concept of euclidean [sic!] ring" (DUB, p. 97), indicating that there is a unique such concept in a context where concepts are seen as mental (Indicator [9]). Further, it is seen in the following quote:

> One of our goals in elaborating the general theory is to isolate small portions of this complex structure and give explicit descriptions of possible relations between schemas. When this is done for a particular concept, we call it a genetic decomposition of the concept. We should also point out that although we only give, for each concept, a single *genetic decomposition*, we are not claiming that this is the genetic decomposition, valid for all students. Rather it represents one reasonable way that students might use to construct a concept. (DUB, p. 102)

Here, it is clear that a concept is intersubjective, since a concept is something that can be analysed, and the students are supposed to develop the same concept, even though there are several ways of developing it (Indicator [9]).[58]

In addition to the view where concepts are mental and intersubjective, there are, in DUB, formulations such as "thoughts about a concept" (DUB, p. 114) and "a student is presented with concepts" (DUB, p. 117), indicating a second non-mental view where a concept is separated from the thought of an individual (Indicator [6]). This non-mental view is seen in the following quote where a concept is being explained to the student:

> To say that the student does not understand could mean that the student has not and does not construct an appropriate schema for the concept being explained. (DUB, p. 117)

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[58] A genetic decomposition is a description of a mathematical topic and how an individual can create constructions for understanding this topic. (DUB, p. 96)

In this quote, there is a distinction between the non-mental concept being explained and the mental schema, which is the cognitive structure that the student constructs. Further, the non-mental view is intersubjective, since formulations such as 'thoughts about a concept' indicate that there is a unique concept of a certain phenomenon (Indicator [9]). Hence, there is a dual view on *concept* in the text, including both a mental and intersubjective view and a non-mental and intersubjective view. An example where the two views are combined can be found in the two sentences below, where the formulation 'description of any concept' points to a non-mental view (Indicator [6]) and the formulation about students 'constructing the concept' points to a mental view (Indicator [3]):

> Together these two are enough to obtain a description of any concept but the result would be far too *ex post facto* to expect it to have any relation to how students actually might go about constructing the concept. (DUB, p. 96)

The dual view on *concept* is clearly supported by the way terms are used in ASI. In this usage, a mental view is seen in formulations such as "the concept can develop in the mind of an individual" (ASI, p. 5) (Indicator [3]). A non-mental view, on the other hand, is seen in the following quote, where the concept first is something that may be analysed theoretically, indicating a non-mental view (Indicator [4]), and then the understanding of the concept can be constructed by the learner. Note that even this fact indicates a non-mental view, since in a mental view, it is the concept itself that is constructed and not the understanding of the concept.

> Research begins with a theoretical analysis modeling the epistemology of the concept in question: what it means to understand the concept and how that understanding can be constructed by a learner. (ASI, p. 4)

Hence, there are two different views on *concept* in both DUB and ASI, a mental and intersubjective view and a non-mental and intersubjective view. The non-mental concepts and the mental concepts are somehow connected in the framework, since one can study a non-mental concept theoretically in order to find indications for how to plan lessons that allow students to construct mental concepts. However, the non-mental and the mental concepts may not share the same features, as it is claimed that the theoretical analysis is not to be seen as a correct description of how the human mind actually constructs concepts (ASI, p. 20).

6.2.3.2 The distinction molecular vs holistic

Since there are several views on *concept* in the texts, the analysis from the perspective molecular versus holistic is divided into two parts[59]. First, the view on the structure of mental concepts is analysed. Here the mental view on *schema*, which is seen later in Section 6.3.2, affects the mental view on *concept*, since mental concepts are included in schemas, which in turn develop into concepts at a higher level. In the second part, the view on the structure of non-mental concepts is analysed.

Regarding the mental structure, it is claimed that objects and processes are interconnected in various ways. The formulation "through composition or in other ways" (ASI, 1996, p. 8) indicates a view combining a molecular and a holistic perspective. Here, a molecular view is seen when composition of processes and objects are described. The following quote can be taken as one example where such compositional features of both the mental and the non-mental structure are seen (Indicators [13; 14]):

> Both psychologically and mathematically, multiplication is the addition of additions. (DUB, p. 100)

The fact that it is claimed that there are processes and objects at different levels (DUB, p. 104) confirms a molecular view, where structures are hierarchical (Indicator [15]). On the other hand, a holistic view is seen when a mental schema is described as a circular feedback system. In the description below, none of the schemas mentioned are more basic than others, which points to a view with holistic features (Indicator [17]).

> For example, there will be a proof schema, which can include a schema for proof by induction. This latter in turn could include a schema for proposition valued functions of the positive integers […]. Hence, there would be a relation with the schemas for number, for function, and for proposition. On the other hand, there is a sense in which a proof is an action applied to a proposition, so that the proof schema might be one of the processes in the proposition schema. (DUB, p. 102)

Furthermore, it is claimed that "we cannot expect students to learn mathematics in the logical order in which it can be laid out" (ASI, p. 9). Hence, it can be concluded that the mental schema has holistic features. Consequently, the mental structure has both molecular and holistic features.

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[59] In this section, the views in DUB and ASI are not separated, but are analysed together.

Above, it was claimed that even though mathematics can be presented in a logical order, we do not learn mathematics in that way. This may indicate a view where the non-mental mathematical structure is seen as molecular. Other arguments for such a view are that multiplication is the addition of additions (DUB, p. 100) (Indicator [14]) and that there are levels in the mathematical structure (DUB, p. 101) (Indicator [15]), indicating a hierarchical and hence molecular view. Whether the non-mental structure is completely molecular or whether it could be described with a combination of molecular and holistic features is not clearly seen, since it is the mental structure that is discussed and not the structure of non-mental concepts. Here, the formulation "[c]ombining formal structures is a natural extension of the development of thought" (DUB, p. 99), and the suggestion that mental and non-mental development are intertwined, contradicts a view where the cognitive structure and the mathematical structure have different features. This, in turn, may also indicate a view where the non-mental structure has both molecular and holistic features. On the other hand, it is claimed that the theoretical analysis is not to be seen as a correct description of how the human mind actually constructs concepts (ASI, p. 20), which indicates that non-mental and mental concepts may not share the same features.

In summary, there are indications of both molecular and holistic features in the view on the mental structure, which affects the mental view on *concept*. When it comes to the non-mental structure, the picture is not clear. The fact that mathematics can be laid out in a logical order and that multiplication is the addition of additions indicate molecular features of the mathematical structure. To conclude, there are two different views on *concept* in the texts, a mental and intersubjective view, with both molecular and holistic features, and a non-mental and intersubjective view, with at least molecular features. These are represented in Figure 6.9.

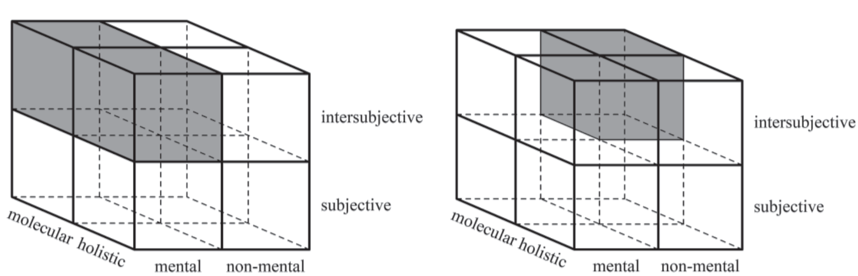

Figure 6.9 Views on *concept* in the APOS framework

This result may be compared to the concept maps in Figures 6.3 and 6.4, representing the frameworks in DUB and ASI. In such a comparison, it may be noted that in the concept map in Figure 6.3, there is both a connection of *knowledge of* between *mathematical knowledge* and *concept*, pointing to a non-mental view on *concept* (Indicator [6]), and a connection of *construction* between *schema* and *concept*, pointing to a mental view on *concept* (Indicator [3]).

6.2.4 Comparisons between views on *concept*

In this section, views on *concept* in texts using the PO frameworks are analysed. Below, the different views are compared to each other, and similarities and differences are discussed. To sum up, in all the frameworks analysed in this chapter there are two different views on *concept*, a mental and intersubjective view and a non-mental and intersubjective view, as represented in Table 6.1. A difference between the frameworks is how prominent the different views are in the texts. In SF, the non-mental view is the explicated one, and the mental view is shown by the way terms are used. In the explications in GT, however, as is commented on in Section 6.1.4, *concept* seems to be similar to *mental object*. In this text, the non-mental view is shown by the way terms are used. In DUB and ASI, both views appear in explications of other notions.

Also, the views differ when it comes to whether concepts are seen as having molecular or holistic features. The OS framework has views where both the non-mental and the mental structures have molecular features. In the APOS framework, the mental structure has holistic features as well, in addition to the molecular structure. In the procept framework, however, it is not clear whether the structures have molecular or holistic features.

Table 6.1 Views on *concept* in the PO frameworks

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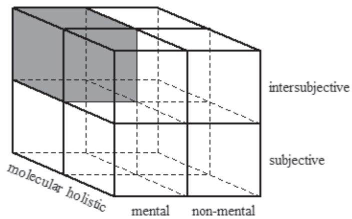

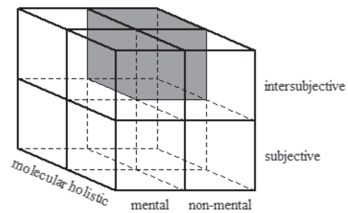

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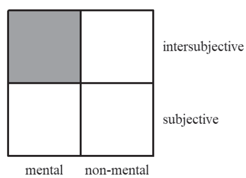

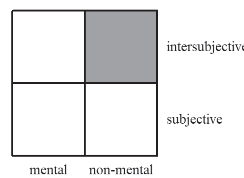

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<td>The APOS framework</td>
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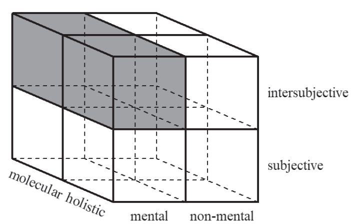

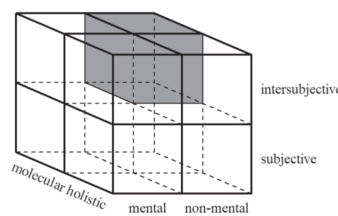

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6.3 Views on *conception*, *schema*, and *procept* in the PO frameworks

Three notions seem to be important for interpreting views on *concept* in the frameworks. First, the notion *conception* appears in the OS framework for describing conceptual structures. Second, the notion *schema* appears in the APOS framework and seems to be similar to *conception*. Third, the notion *procept* appears in the procept framework. Below, the two related notions *conception* and *schema* are first analysed, followed by a discussion of the notion *procept*.

6.3.1 Views on *conception*

As is seen in Section 6.1.1, *conception* in SF is explicated as "the total cluster of internal representations and associations evoked by the concept" (SF, p. 3). Further, there are two types of conceptions. If the conception consists of ideas of a concept as a process, an algorithm or an action, then it is called an operational conception. If instead it consists of ideas of a concept as an object, then it is called a structural conception. In the latter case, a conception is a structure that could be manipulated as a whole. For example, 3/4 could be either seen as operational (as 3 divided by 4) or structural (as 3 fourths). (SF, pp. 3–4)

6.3.1.1 The distinctions mental vs non-mental and intersubjective vs subjective

In the explication above, it is clear that the view on *conception* is mental, since it contains mental representations (Indicator [1]). The mental view is also seen in the way terms are used, as in the formulation "the operational conception is, for most people, the first step in the acquisition of new mathematical notions" (SF, p. 1), using the term 'acquisition' (Indicator [3]). The fact that the term 'structural conception' is sometimes replaced with "structural thinking" (SF, p. 4) indicates a mental view as well (Indicator [2]).

In addition, the formulation claiming that a conception is "the concept's counterpart in the internal, subjective 'universe of human knowing'" (SF, p. 3) shows a subjective view on *conception*. However, when it is claimed that a structural conception is static or "timeless" (SF, p. 4), with a reference to Frege (1952/1960), an intersubjective view appears. Here, the interpretation depends on the view on concept development, including the idea that an operational conception develops into a structural one. In the beginning, the conception is unstructured and subjective. As the conception develops towards a mental intersubjective concept, one can argue that the conception must have intersubjective features as well. Hence, it develops from a subjective unstructured representation towards an intersubjective structured representation.

In addition to the view on *conception* described above, there is also a view, seen in the way terms are used, where conceptions are seen as culturally dependent. This is the case in the section starting at page 10. The following quote can be taken as an example where intersubjective conceptions are described as developing from operational to structural:

> the history of numbers has been presented here as a long chain of transitions from operational to structural conceptions (SF, p. 14)

In definitions, culturally dependent operational conceptions can be seen as explicating ideas of computational processes. Similarly, culturally dependent structural conceptions can, again in definitions, be seen as explicating ideas of objects.[60] In the following, this is exemplified with the concepts *function* and *ratio*:

> Function can be defined not only as a set of ordered pairs, but also as a certain computational process (SF, p. 4)

> For instance, a ratio of two integers was initially regarded as a short description of a measuring process rather than as a number. (SF, p. 11)

Consequently, there are several views on *conception*, in the text. The first one is an individual, mental and subjective view, involving the idea that conceptions take on intersubjective features during development. The other one is a culturally dependent, intersubjective view. From the quotes above, it can be concluded that the culturally dependent conception is the meaning of a term, such as ‘function’ or ‘ratio’, as it is explicated in a definition. This meaning can change through history. Now, it may be asked whether this meaning is a culturally dependent mental understanding of mathematicians, or if it is a non-mental linguistic meaning connected to descriptions and definitions. As I interpret the text, a culturally dependent conception may be either mental or non-mental. It may be a collective mental understanding of mathematicians, but when a definition is presented it also acquires a non-mental meaning. Therefore, there are two culturally dependent, intersubjective, views on *conception*, a mental view and a non-mental view. Consequently, there are three views on *conception*, a mental and subjective view, a mental and intersubjective view, and a non-mental and intersubjective view.

6.3.1.2 The distinction molecular vs holistic

When it comes to the distinction molecular versus holistic, the structure of a conception is the structure that concepts are embedded in: a conception involves concepts, and develops into a concept at a higher level (SF, pp. 10–14). Further, to say that a view on *concept* has molecular features is to say that

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[60] Not only conceptions might be operational or structural. Also definitions and descriptions may be considered operational or structural (SF, pp. 5, 10).

the relations between concepts are hierarchical. And to say that a view on *conception* has molecular features is to say that the relations within a conception, between concepts, are hierarchical. Hence, a view on *concept* and the corresponding view on *conception* must take the same positions regarding the distinction molecular versus holistic. Consequently, if the non-mental structure has molecular or holistic features, this affects both a non-mental view on *concept* and a non-mental view on *conception*. The same holds for mental structures.

As is seen in Section 6.2.1.2, both the non-mental and the mental views on *concept* in SF have molecular features. From this it may be concluded that the three views on *conception* mentioned above have molecular features as well. A non-mental molecular view is indicated by the fact that culturally dependent conceptions are meanings of expressions used in definitions (SF, pp. 10–16) (Indicator [14]). When it comes to the mental view, a molecular view is seen in the fact that the development of a conception consists of organising it and adding new layers into a hierarchical structure (SF, p. 27) (Indicator [15]).

Consequently, the individual conception is mental, develops from subjective to intersubjective, and develops towards a molecular structure. Further, culturally dependent conceptions are intersubjective and develop towards molecular structures. These may be either mental or non-mental. In Figure 6.10, the initial states are represented with 2D matrices and the final states are represented with 3D matrices. That the distinction molecular versus holistic is omitted in the initial states indicates that the conceptions are unstructured. The arrows symbolise development between the initial and final states.

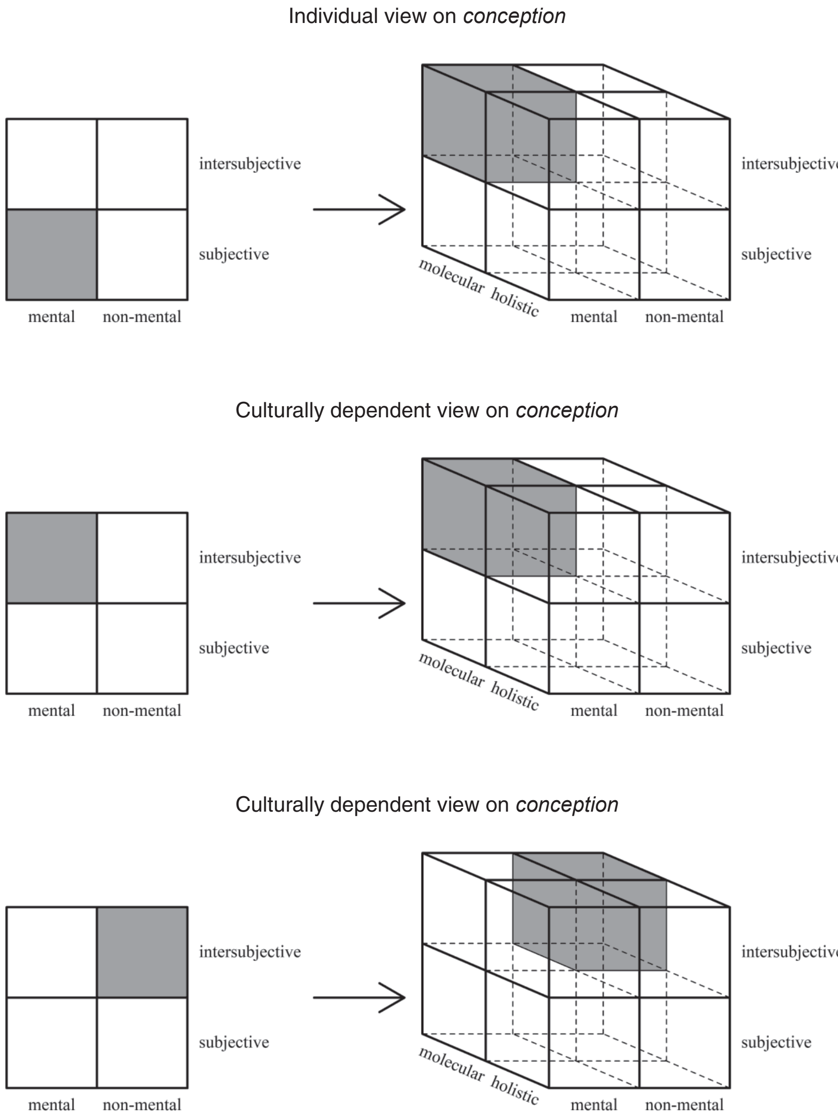


Figure 6.10 Views on *conception* in the OS framework

Here, it may be interesting to compare the results of the current section with the explication of *conception*, where a conception is seen as a cluster of internal representations and associations (SF, p. 3). Further, conceptions are explicated as subjective. Notably, both culturally dependent views, the non-mental one and the mental one, appear in the way terms are used.

Additionally, the notion *schema* is used in SF, besides the notion *conception*, as a consequence of that the framework uses cognitive schema theory (SF, p. 1). That is explicitly seen in Figure 7 in SF (p. 27) (Figure 6.5, this text). However, the difference between *conception* and *schema* is not commented on, which makes the connections somewhat hard to understand.

6.3.2 Views on *schema*

Instead of 'conception', the term 'schema' is used in the APOS framework. The reason for this may be that the framework takes an explicit stance in Piaget's perspective and that terms are chosen based on that background. Below, the analysis of views on *schema* in the two texts, DUB and ASI, is presented.

6.3.2.1 The perspectives mental vs non-mental and intersubjective vs subjective

In DUB, as is seen in Section 6.1.3.1, a schema is explicated as "a more or less coherent collection of objects and processes" (DUB, p. 101). A student constructs a schema for the non-mental concept being explained, and the tendency to invoke the schema for understanding or dealing with a problem situation is the student's knowledge of the mathematical concept (DUB, pp. 101, 119).

A mental view on *schema* is seen when it is claimed that a student constructs a schema (Indicator [3]). Further, formulations such as "the subject's schemas" (DUB, p. 106) indicate a subjective view, where the individual has their own schemas (Indicator [12]). However, the following quote indicates inter-subjective features as well, since it discusses what a schema should contain (Indicator [7]). Notably, just the fact that schemas are sometimes named indicates intersubjectivity.

> within a schema for a topological space there should be a schema for the concept of continuous function and within a vector space schema there should be a notion of linear function (DUB, p. 106)

In addition to this individual view, there is also a culturally dependent view on *schema*, where schemas are included in formal mathematics. In the following quotes, the objects in the schema of propositional calculus are claimed to be propositions, which in turn are formed using standard logical operations, which seems to point to a non-mental view.

> The predicate calculus schema appears to be obtained through a reconstruction of a schema resulting from coordinating a schema for first order propositional calculus with a function schema […] According to this analysis, the objects in the propositional calculus schema are the propositions. (DUB, p. 112)

> the formation of new propositions by the standard logical operations such as conjunction, disjunction, implication and negation (DUB, p. 112)

In DUB (p. 116), there is a distinction between formal mathematics and cognitive development, which is seen when it is claimed that the nature of mathematics and psychology are not similar. A non-mental view on mathematics can be seen in formulations such as "understanding mathematical ideas" (DUB, p. 119), where mathematics is separated from the thoughts of an individual. However, the fact that there is an explicit stance in radical constructivism in ASI is an argument against such an interpretation, even though this stance is not explicit in DUB. Further, when it is claimed that "the result becomes a proposition which is a mental object" (DUB, p. 114), a view where propositions are mental is indicated, which in turn may indicate that mathematics as a whole is mental. However, in that case it is difficult to understand the role of the non-mental concepts in the framework, and I would say that there is an implicit second view where schemas are non-mental. Consequently, there are two different views on *schema* in DUB: one individual, mental view with both subjective and intersubjective features, and one culturally dependent, non-mental and intersubjective view.

In ASI (p. 9), a schema is described as the totality of knowledge that is connected to a mathematical topic for a student. This is in line with the individual view in DUB. Further, it is claimed that the meaning of 'schema' is similar to the meaning of 'schemata' from Piaget's perspective, which in turn is claimed to be similar to the meaning of 'concept image' in Tall and Vinner (1981):

> Our use of the term in this paper is close to what Piaget calls *schemata* in [22][61] where his meaning appears to be in some ways similar to the *concept* image of Tall and Vinner [31][62] (ASI, p. 7)

[61] Reference to Piaget and Garcia (1989)
[62] Reference to Tall and Vinner (1981)

In these descriptions, *schema* seems to be similar to *concept image* and *conception*. Furthermore, there is an ambiguity where *schema* is partly seen as the cognitive structure that develops, which is seen in the quote above, and is partly seen as the fourth stage in the development, which is seen in Figure 6.8.

Since *schema* in ASI (p. 101) is a collection of objects and processes which are mental, the view on *schema* is mental. Further, it is claimed that:

> An individual's schema for a concept includes her or his version of the concept that is described by the genetic decomposition, as well as other concepts that are perceived to be linked to the concept in the context of problem situations. (ASI, p. 8)

Here, a schema has subjective features, since it is the student's own version of the concept (Indicator [7]). Hence, the view on *schema* is both mental and subjective. However, when the term 'schema' is referring to the last stage in the development, then the notion may have intersubjective features as well. The fact that schemas contain mental concepts, which are intersubjective, is one argument for schemas having intersubjective features. Also, an intersubjective view is seen in the following quote, where there is an indication of some mathematical schemas being included in others (Indicator [8]).

> The schema can then be included in higher level schemas of mathematical structures. For example, functions can be formed into sets, operations on these sets can be introduced, and properties of the operations can be checked. (ASI, p. 8)

In this quote, it is not really clear whether schemas are mental or non-mental. However, generally I would say that the view on *schema* in ASI is mental with both subjective and intersubjective features.

Comparing the views on *schema* in the two texts may result in the conclusion that there is a mental view with both subjective and intersubjective features within the framework. Further, in DUB there is a second view that is intersubjective and non-mental. In addition, in ASI there are two different interpretations, where a schema is either a cognitive structure that develops or the fourth stage in the development of a cognitive structure.

6.3.2.2 The perspective molecular vs holistic

As with the notion *conception*, a schema is a structure that contains concepts. This may be concluded from the suggestion that an individual's schema is the totality of knowledge connected to a topic (ASI, p. 9), and that there is a mental view on *concept*. If there are mental concepts, these must be included in the total knowledge of the individual in question, and hence be included in mental schemas. Further, in a non-mental view on *schema*, schemas include non-mental concepts. Consequently, the view on *concept* and the view on *schema* take the same position regarding whether the structure has molecular or holistic features. As is seen in Section 6.2.3.2, the mental structure has both molecular and holistic features and the non-mental structure has molecular features. A molecular view, on the one hand, is seen when composition is discussed (DUB, pp. 100, 103) and when it is claimed that there are processes, objects and schemas at different levels (DUB, p. 104), indicating a view where schemas are hierarchical (Indicator [15]). A holistic view, on the other hand, is seen when it is claimed that no mental schema is more basic than others (DUB, p. 102) (Indicator [17]). Further, both the mental and the non-mental schemas begin as unstructured, arising from the experience of actions (ASI, pp. 6–8). During the development, this experience is organised into structures. In the later stages of this development, the schemas become structures with features that are molecular or holistic.

Consequently, there are two different views on *schema* in the framework, as is seen in Figure 6.11. The first view is mental, with both subjective and intersubjective features. In this view, concepts develop from unstructured to structured, with both molecular and holistic features. The second view is intersubjective and non-mental. Here, concepts develop from unstructured to structured, with at least molecular features. This second view appears only in DUB.

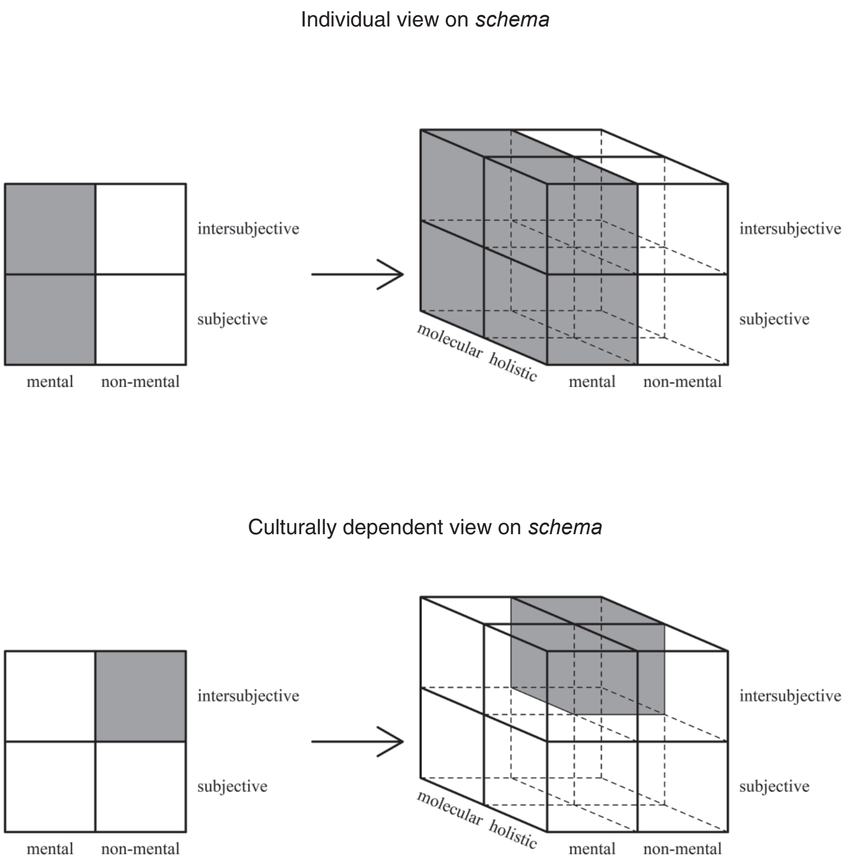


Figure 6.11 Views on *schema* in the APOS framework

In addition to *schema*, the notion *concept image* is used in DUB (p. 120) for the cognitive structure. Further, in ASI (pp. 6–8) the notion *conception* is used for the cognitive structure that contains ideas of actions, processes, objects or schemas, in different phases. In this usage, there seems to be an intension to distinguish between the cognitive structure that develops, the conception, and the fourth stage in the development, the schema.

6.3.3 Views on *procept*

The notion *procept* is introduced in GT as it is claimed that professional mathematicians often use the same symbol to represent both a process and a concept which is the product of that process (GT, pp. 119–120). As an example, the symbol '3/4' denotes both the process of dividing 3 by 4 and the rational number that is the product of that division. The role of the procept is to allow the symbol to evoke either the process of calculating a particular value, or the concept.

A *procept* is defined as a collection of elementary procepts having the same object, where an elementary procept is an amalgam of three components: a process, an object and a symbol (GT, p. 121). To exemplify, the procept 6 consists of a number of elementary procepts, all having the number 6 as the object. In this description, the procept contains a concept (or an object[63]) and can be seen as a concept (or an object). Further, the procept contains a process and can be seen as a process.

Here, there are two different interpretations of what a procept is. In the first interpretation, following the explication literally, a procept is just an amalgam of three components. In a second interpretation, a procept may be seen as a developed concept image. An argument for such a view can be seen in Section 6.1.4, where the concept map of BIMO, in Figure 5.10, is compared to the concept map of GT, in Figure 6.2, pointing out that there are similarities between the alternative of *concept image* containing symbols and words, and the view on *procept*. However, a main difference is that *concept image* is explicated as a cognitive structure, and *procept* is not. Below, I give some arguments for not analysing the notion *procept* with the help of my analysing tool.

The elements of a procept have different characteristics. For example, the notions *concept* and *symbol* are of different kinds. If *concept* has the property of having a structure, *symbol* may not share this property. Here, the concept *3* may have a structure including relations between different mathematical objects, such as *1 + 2 = 3* or *the number of angles in a triangle is 3*. The symbol '3' does not have such a structure. Therefore, it may be asked what it means when it is claimed that a procept, defined as an amalgam of a process, an object and a symbol, "starts out with a simple structure and grows in interiority" (GT, p.

[63] Comparing formulations including the term 'object' and formulations including the term 'concept' in the text has led to the conclusion that *object* and *concept* are similar. An argumentation for this conclusion is given in Section 6.1.2.

122). In this quote, no distinction is made between symbols and cognitive structures, which can be seen as meanings of symbols.

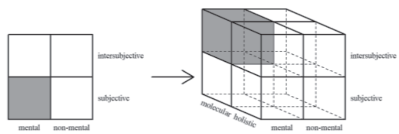
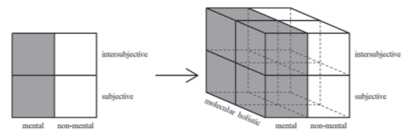
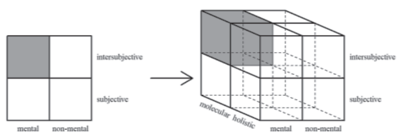
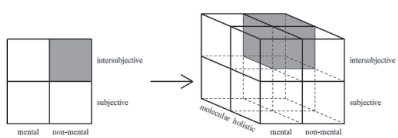
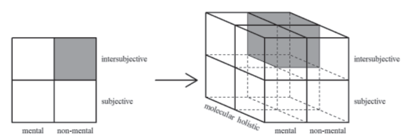
It is some kind of category mistake to combine *symbol* and *concept* into one common notion, and use this notion as though it inherits the nature of the different elements. Here, the notion *procept* does not inherit the nature of the notion *concept*, as a procept contains a symbol. As a symbol normally is not a mental or a non-mental meaning of a symbol, it is not possible to use the indicators for the distinction mental versus non-mental for analysing views on *procept*. In addition, as the symbol cannot have conceptual structure, it is not possible to use the distinction molecular versus holistic. Consequently, the indicators cannot be used for analysing views on *procept*.

Naturally, it is possible to reason in a similar way about the notion *concept image*, within the alternative where concept images include words and symbols. However, the difference is that *concept image* is explicated as a cognitive structure. Further, in the explications in the $CICD^0$ framework (Vinner & Hershkowitz, 1980; Tall & Vinner, 1981), concept images do not include words and symbols. This is why I analyse *concept image*, but not *procept*, with the help of the indicators.

6.3.4 Comparisons between views on *conception* and *schema*

Note that the frameworks analysed here use at least three different terms for the cognitive structure: 'conception', 'schema', and 'concept image'. In this thesis, I suppose that the meanings of these terms are similar at a general level. Further, as shown in Table 6.2, both 'conception' in SF and 'schema' in DUB are used with several meanings. A first meaning seems to indicate an individual view, where the cognitive structure is given subjective features in an initial state. In this view both conceptions and schemas are mental and develop from being unstructured to becoming more structured. A difference is that while the individual view on *conception* involves the idea that the cognitive structure develops from subjective to intersubjective, the individual view on *schema* has both subjective and intersubjective features, in both the initial and final state. Another difference is that while the more developed conception in the OS framework has molecular features, the more developed schema in the APOS framework has both molecular and holistic features. A second meaning seems to indicate a culturally dependent view. In this view, both conceptions and schemas are non-mental and intersubjective, and develop towards structures with molecular features. There is also a third view on *conception*, which is also culturally dependent. In this view, conceptions are mental and intersubjective, and develop towards structures with molecular features.

Table 6.2 Views on *conception* and *schema* in the PO frameworks

	Conception	**Schema**
Individual view		
Culturally dependent mental view		
Culturally dependent non-mental view		

In addition to these views on *conception*, there are both operational and structural conceptions in SF. Notably, what is claimed about procepts in GT may be compared to these two different types of conceptions. It seems as though an operational conception, according to SF (pp. 3–4), is similar to what is meant in GT (p. 7) when a symbol is described as evoking a process. Further a structural conception, according to SF (pp. 3–4), seems to be similar to what is meant when a symbol is described as evoking a concept (GT, p. 7). Finally, the two different views on *schema*, seen as the cognitive structure or as the fourth stage of development in the APOS framework, can be compared to *conception* and *structural conception*, in the OS framework.

6.4 Comparisons, conclusions and local discussion

In the previous sections, views on the notion *concept*, on the one hand, and on *conception* and *schema*, on the other, have been analysed, compared and commented on. In this section, the views on *concept* are compared to the views on *conception* and *schema*, in order to be able to draw some conclusions. What can the analyses of views on *conception* and *schema* tell about views on *concept*? This comparison is made from the perspective of the analysing tool. Next, the chapter ends with some comments on how the different arenas are seen in the frameworks.

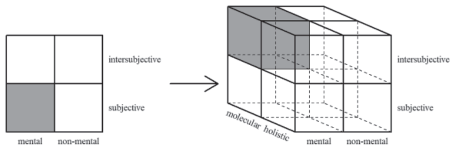
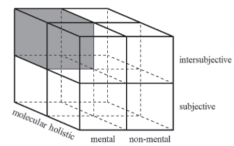
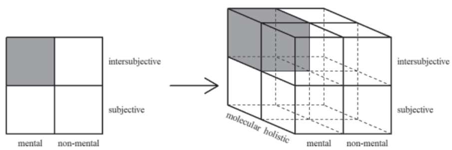
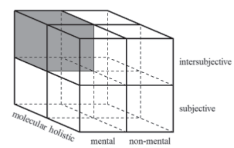
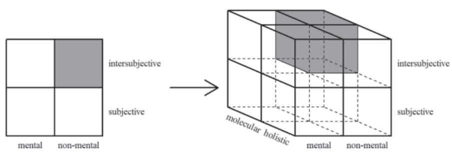
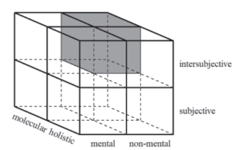
6.4.1 Comparisons between views on *concept*, and views on *conception* and *schema*

Below, the frameworks are first commented on one by one, regarding how views on *concept* and views on *conception* and *schema* relate to each other. Next, in Section 6.4.1.4, some general comments regarding the connections are made.

6.4.1.1 The OS framework

We begin the discussion with a comparison between the different views on *conception* and *concept* in SF (Figures 6.6 and 6.10), seen in the table below. First, it may be noted that the matrix representing the more developed individual conception and the matrix representing the mental view on *concept* are similar. Here, the difference between the structural conception and the mental concept is not evaluated in the text. Further, both the historical development of mathematics and the cognitive development of the individual are described as concept formation. From that perspective, it is natural to conclude that the historical development concerns formation of both mental and non-mental concepts. The culturally dependent mental conceptions of mathematicians develop in time into mental concepts, which are the concepts that we in turn want students to develop. Further, the non-mental meanings seen in the content of descriptions and definitions develop into non-mental concepts that form the basis of formal mathematics. The more developed non-mental conception may then be similar to a concept in a non-mental view. Note that the non-mental concept can either be a static timeless object or a non-static culturally dependent object that changes through history.[64]

Table 6.3 Views on *conception* and *concept* in the OS framework

	View on *conception*	**View on *concept***
Mental		
Mental		
Non-mental		

6.4.1.2 The procept framework

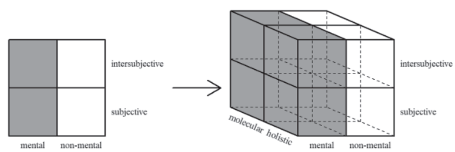
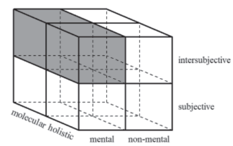
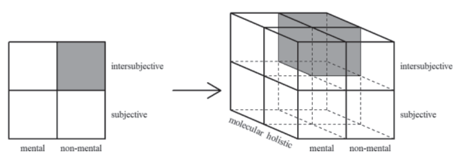
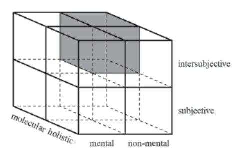
In GT, there are two different views on *concept*, one mental and intersubjective view, and one non-mental and intersubjective view. As the notion *procept* is not explicated as a cognitive structure, I do not compare the views on *concept* with views on *procept*. However, it may again be noted that there are two different interpretations of *procept*. In one of these, procepts are amalgams of three different elements: a process, a concept (or an object), and a symbol. In the other, a procept may be seen as a more developed concept image. In such an interpretation, the concept image develops from including the idea of a symbol as a process, to including the idea of a symbol as dually representing a process or a concept. However, this interpretation of a procept as a more developed concept image is not explicitly found in GT.

[64] It may be noted that there are references to both Frege (1952/1960) and Piaget (1952) within SF, which may be one cause for the two different views on *concept*.

6.4.1.3 The APOS framework

The comments on the APOS framework concern connections between views on *concept* and views on *schema* in the texts. In order to compare the different notions, the views from Figures 6.9 and 6.11 are inserted into a common table (Table 6.4).

Table 6.4 Views on *schema* and *concept* in the APOS framework

	View on *schema*	**View on *concept***
Mental		
Non-mental		

In the mental view, a schema is regarded as a cognitive structure that starts out as unstructured. The more developed schema can be compared to the mental view on *concept*, which is intersubjective. Here, the fourth stage is also called schema, which creates ambiguity as there are in fact two different views, one view where *schema* is the cognitive structure that develops and one view where *schema* is the fourth stage of development. In the first view, *schema* is similar to *conception* and *concept image*. In the second view, I would say that it is hard to see a difference between a schema and a mental concept. However, if comparing the matrices in Table 6.4, it seems as if the more developed mental schema has subjective features, but that the mental view on *concept* is intersubjective. The result of the analysis is a hierarchical view, seen in the helix in Figure 6.8, where there are objects, processes, schemas and concepts on different levels. In such a view, a concept may be included in a schema that develops into a concept at a higher level.

Regarding the non-mental views, the non-mental and intersubjective schema develops and obtains molecular features. Here, the view on the developed schema and the view on *concept* are similar. Again, the question arises regarding the difference between a more developed schema and a concept.

6.4.1.4 General comments

It is clear that there is a hierarchical view here, seen explicitly in SF, DUB and ASI, where a concept is included in a structure, which in turn develops into a higher-level concept. This is why concepts are included in a conception or a schema, which in turn develops into a new concept. This holds regardless whether the development described in the texts is mental or non-mental. A general comment can be made regarding the mental view on *concept*. According to the PO frameworks, mental concepts appear in the development of conceptions (or schemas) when a conception has reached a certain level. A similar line of reasoning holds regarding non-mental views on *conception* (or *schema*) and *concept*, where non-mental concepts appear in the development of non-mental conceptions. Here, one can ask whether a concept in fact is a more developed conception or if it is something else which appears in the development.

The analysis in this chapter shows that it is possible to use the analysing tool in the analysis of *conception* and *schema* as well, as these notions are mental or non-mental, intersubjective or subjective, and as they are structures that may have molecular or holistic features. Here, a difference between concepts and conceptions (or schemas) may be that a concept develops from an unstructured conception. That is why two matrices are used for the views on *conception* and *schema*, one for the initial state and one for the final state. When it comes to the notion *procept*, the analysing tool cannot be as easily used, since a procept contains an element, a symbol, which is not the meaning of a term and hence cannot have the property of being mental or non-mental. Further, the symbol does not have a structure in the sense of a structure of concepts.

6.4.2 Mental and non-mental arenas

Within the analyses, *concept* and *conception*, or *schema*, are described in a mental context where the individual develops concepts, or where the culturally dependent mental understanding develops historically, or else in a non-mental context where mathematics as a system develops historically. This may be what is meant in SF when it is claimed that two different contexts, a philosophical context and a psychological context, are combined:

> But up to now, not enough has been done in the direction of unified theory which would address philosophy and psychology of mathematics simultaneously, and would take an equal care of mathematical thinking and of mathematical thought (SF, p. 2)

The different contexts are clearly separated in the text, and transitions between them are seen in formulations such as:

> In the remainder of the present part I shall restrict myself to the preliminary problem: is the proposed model of concept formation in force also when individual learning is concerned? (SF, p. 16)

In this quote, there is a shift from a historical to a psychological context. Simultaneously, there is also a shift between the two views on *concept*. However, the view on *concept* is not coherent within either the philosophical or the psychological context. As an example, the formulation "to think structurally about a concept at hand" (SF, p. 18) points to a non-mental view, where concepts are separated from the thoughts of an individual (Indicator [7]) in a context where concepts are mostly seen as mental. Regarding the notion *conception*, there may be an awareness in SF of the different views, since it is claimed that *conception* has a combined ontological – psychological nature (SF, p. 8). Even so, the different views on *conception* are not explicated or commented on.

Within the APOS framework, the mental concepts and schemas are discussed within a mental arena, and the non-mental concepts and schemas are discussed within a non-mental arena. Here, the same notions are used for describing both the cognitive development and the intellectual development during history, and similar stages appear within these two developments. Sometimes, the different arenas make it hard to interpret the notions as they describe two types of structures, with different types of entities and relations. Consequently, the notions appear in different versions.

7 Summary and conclusions

The methodology of the study is presented in Chapters 1 – 4, and Chapters 5 and 6 present concept analyses of the chosen frameworks and texts. In this chapter, I summarise the results and draw some conclusions. There are three types of results involved. The first section concerns views on *concept*, the second section concerns views on *concept image*, *conception* and *schema*, and the third section concerns findings of how terms are used in the analysed texts. The chapter ends with a short summary of the study.

7.1 Views on *concept*

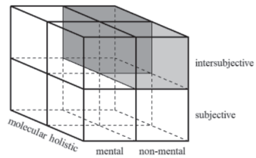
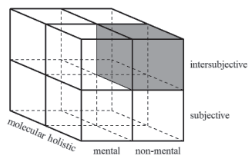
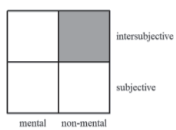
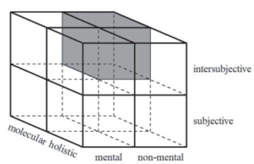
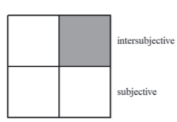
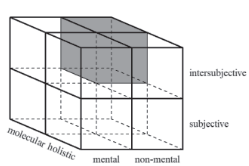
From the perspective of the analysing tool, it can be concluded that there are in fact, at an overall level, two common views on *concept* in the analysed frameworks, a non-mental and intersubjective view and a mental and intersubjective view. Both views are seen in the CICD^0 framework, the CICD$^+$ framework, the OS framework, the procept framework, and the APOS framework. In fact, the only framework involving only one view is the CICD* framework in Bingolbali and Monaghan (2008).

It is a simplification to say that all non-mental and intersubjective views, or all mental and intersubjective views, are similar. The non-mental views differ in a multitude of ways. One difference is whether conceptual structures have molecular or holistic features. Consequently, 'non-mental and intersubjective' and 'mental and intersubjective' may be seen as two categories including different views that are clustered based on the distinctions mental versus non-mental, and intersubjective versus subjective. Below, the views in these two categories are compared to each other.

7.1.1 Non-mental and intersubjective views

As a starting point for the comparison of the different non-mental and intersubjective views, one may regard the matrices describing the non-mental views in the different frameworks.

Table 7.1 Non-mental and intersubjective views on *concept*

CICD⁰	
CICD⁺	
CICD*	
The OS framework	
The procept framework	
APOS	

When comparing the matrices, it may be noted that they differ concerning whether non-mental structures have molecular or holistic features. In the procept framework and in the CICD* framework, there is no clear view on whether the structure is molecular or holistic. In the OS framework and the

APOS framework, there is a view including that the structure has only molecular features. In the CICD^0 framework, there is a view including that mathematical concepts have only molecular features, but other concepts may have holistic features as well. Finally, the view in the CICD$^+$ framework is unique in the aspect that mathematical concepts have holistic features.

The views above can be compared to the groups of philosophical non-mental views presented in Section 3.4 (see Figure 7.1). The first group includes views of some classical philosophers of language, such as Frege (1892/1985; 1892/1951) and Katz (1972/1999), who consider concepts to be non-mental, intersubjective and molecular. Further, the third group can be represented by the view in Quine (1960; 1951/1985), where concepts are non-mental, intersubjective and holistic. Notably, the non-mental views found in the analysis can be explained based on these two groups. No view considering concepts as non-mental and subjective has been found in the analysed texts.

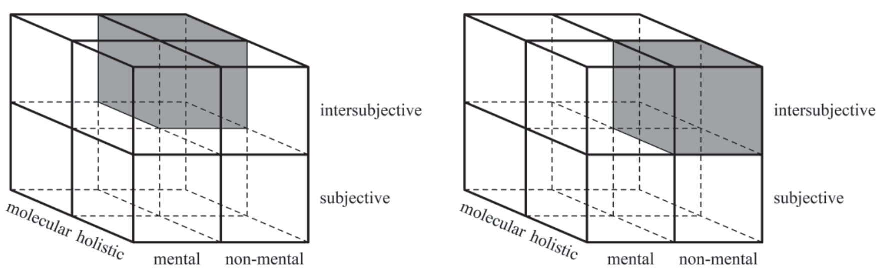


Figure 7.1 The 1st and 3rd groups of philosophical views

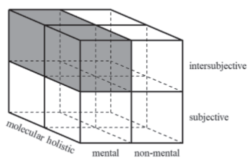
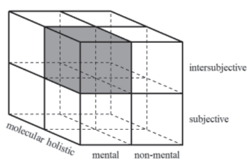
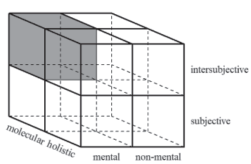
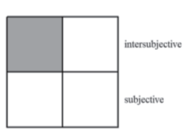
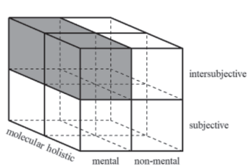
Further, the analyses have revealed two variants of a non-mental view in mathematics education. In the first variant, concepts are seen as context independent and may be included in formal mathematical structures. In the second variant, concepts are seen as culturally dependent, and may be seen through language, described in texts and other forms of communication. These two variants can be described as a static view and a non-static view. A static view is, for example, seen in Section 5.2.2 where concepts are seen as independent of context. Further, a non-static view is seen in Section 6.2.1 where a view including a historical development of concepts is seen.

Additionally, non-mental views on *concept* depend on a non-mental view on mathematics. If building a framework on an empiricist view on mathematics, then as a consequence concepts become mental. For example, from a radical constructivist position, such as the one in Asiala et al. (1996), individuals construct mathematics in their mind, and hence concepts are considered mental. Consequently, that there are two views in that text, where one of these is non-mental, may be seen as an incoherence.

7.1.2 Mental and intersubjective views

A comparison between the mental and intersubjective views may also start with the different matrices.

Table 7.2 Mental and intersubjective views on *concept*

CICD^0	
CICD$^+$	
CICD*	
The OS framework	
The procept framework	
APOS	

As with the non-mental views, the mental views take different positions regarding whether structures have molecular or holistic features. A framework that does not take a position is the procept framework. In the OS framework, there is a view where concepts have molecular features. In the $CICD^0$ framework and in the APOS framework, formulations indicate that concepts have both molecular and holistic features. Finally, in the $CICD^+$ framework there is an argumentation for a view where all types of concepts have holistic features.

Further, there are two variants of the mental views, which either describe concepts from the perspective of an individual development from an unstructured mental idea to a mental concept, or from the perspective of a development of culturally dependent understandings, shared by several people.

The views above can be compared to the groups of philosophical mental views presented in Section 3.4 (see Figure 7.2). In the fourth group, we find views of empiricists such as Jenkins (2008), considering concepts as mental, intersubjective and molecular. The sixth group includes views which consider concepts as mental, intersubjective and holistic, such as the one in Carey (2009). Also, as is seen in Section 3.4, there are combinations of different positions regarding mental views. For example, the seventh group contains a combination of molecularism and holism. Here, it may be noted that all these views are found in the analyses.

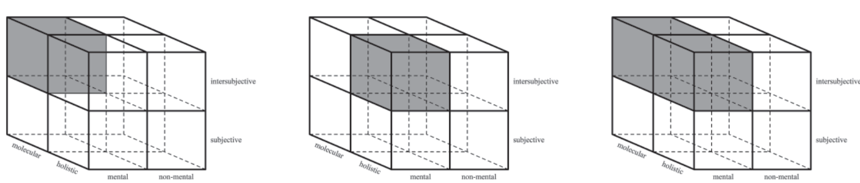


Figure 7.2 The 4th, 6th and 7th groups of philosophical views

7.1.3 Comparisons between views on *concept*

In the above descriptions, both non-mental and mental views are intersubjective, and no subjective view appears in the analysis. Consequently, the second and fifth group of philosophical views in Section 3.4 cannot be used for describing the views in the analysed frameworks. If also taking the distinction molecular versus holistic into account, there are different views on *concept* in different texts, both in texts using the same framework and in texts using different frameworks.

In some texts, there is a focus on molecular features, and in others, there is a focus on holistic features. For example, in Sfard (1991) both mental and non-mental views on *concept* are molecular, and in Semadeni (2008) both mental and non-mental views are holistic. In other texts, there is a position where mental structures and non-mental structures have different features. In some of the analysed frameworks, there seems to be a view where non-mental structures are molecular, but mental structures may have holistic features as well. One reason for this may be that non-mental structures in the analysed texts are often formal axiomatic systems, including concepts as building blocks. For example, in Dubinsky (1991) and Asiala et al. (1996), non-mental structures have only molecular features, and mental structures have both molecular and holistic features. As another example, in Vinner and Hershkowitz (1980) as well as in Tall and Vinner (1981) mental structures have both molecular and holistic features and there are different positions regarding non-mental structures. Here there is a difference between mathematical and non-mathematical concepts, where mathematical concepts seem to have solely molecular structure, while non-mathematical concepts have both molecular and holistic features. Based on this analysis, mathematical concepts are molecular, meaning that they have hierarchical structure and that it is possible to define them, but non-mathematical concepts may have holistic features and it is not always possible to define them. Furthermore, in Bingolbali and Monaghan (2008) as well as in Gray and Tall (1994) there are no clear views regarding whether concepts have molecular or holistic features. The conclusion that can be drawn from the perspective of the distinction between molecular and holistic views is that there is no clear position regarding whether conceptual structures have molecular or holistic features.

7.2 Views on *concept image*, *conception* and *schema*

During the concept analysis, it has been possible to use the analysing tool for some other notions than the notion *concept*. These notions can all be regarded as meanings of expressions. Further, they can be regarded as intersubjective or subjective, they are related in a structure and may be categorised as molecular or holistic. Clear cases of such notions are *concept image*, *conception* and *schema*. In the analysis of these notions, all distinctions have been used, considering that there is a developmental view on *conception* and *schema*. In an unstructured phase, these do not have molecular or holistic features, but these appear in later phases of the development.

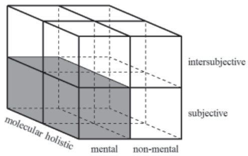
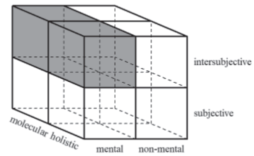
Other notions cannot be analysed with the help of my tool. One such example is the notion *procept* which includes a symbol that is not the meaning of a symbol, but the symbol itself. Hence, the distinction mental versus non-mental cannot be used. Further, a symbol does not have conceptual structure, so the distinction molecular versus holistic cannot be used either. The same holds for the notion *concept definition* which can be analysed with neither the distinction mental versus non-mental nor the distinction molecular versus holistic. So, if analysing *symbol*, *procept* and *concept definition*, only the distinction intersubjective versus subjective can be used. To conclude, the tool cannot be used for analysing symbols, terms, expressions and concept definitions, but just for analysing entities that are the meanings of these.

The notions *concept definition* and *procept* are discussed in Section 7.3 which presents category mistakes concerning syntax and semantics, between terms and meanings of terms. In the current section, however, a comparison is made between the notions *concept image*, *conception* and *schema*. On a superficial level, these seem to be used similarly. Below, the notions are first discussed one by one and next discussed together, in order to show similarities and differences between the notions, from the perspective of the analysing tool. The comparison intends to create a general understanding of views on the cognitive structure and to see how similar these notions are. Several of the frameworks have two different views on cognitive structures, one individual view and one culturally dependent view. Below, the individual views are compared, and so are the culturally dependent ones.

7.2.1 Views on *concept image*

As a starting point for the discussion of the notion *concept image*, the different views in Table 5.3, which is here called Table 7.3, may again be considered. First, it may be noted that there are two views in the $CICD^0$ framework and in the CICD* framework, and just one view in the $CICD^+$ framework.

Table 7.3 Views on *concept image* in the CICD framework

<table>
<tr><th></th><th>Individual views</th><th>Culturally dependent views</th></tr>
<tr><td>CICD⁰</td><td>
</td><td>
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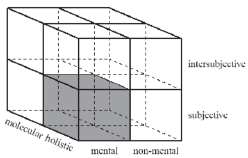
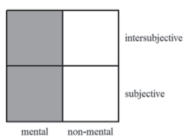
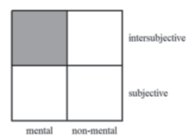
<tr><td>CICD⁺</td><td>
</td><td></td></tr>
<tr><td>CICD*</td><td>
</td><td>
</td></tr>
</table>

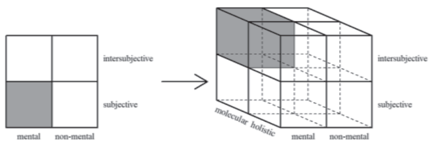
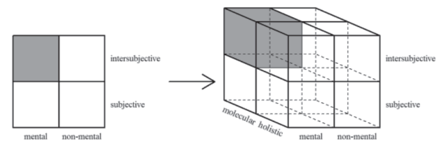
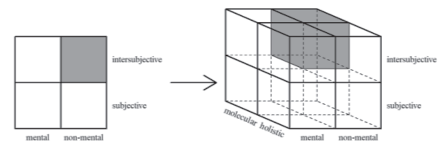
Notably, individual views are mental and have subjective features, but may have intersubjective features as well. Concept images are subjective in CICD^0 and in CICD$^+$, and in CICD* they have both subjective and intersubjective features. Culturally dependent views appear in CICD^0 and in CICD*. Common for these frameworks is that concept images are mental and intersubjective.

When it comes to the distinction molecular versus holistic, in CICD* there is not enough indications of either a molecular or a holistic view, in CICD^0 concept images have both molecular and holistic features, and in CICD$^+$ they have just holistic features. Additionally, there are two variants of *concept image*, due to the fact that concept definitions sometimes are considered included in the concept image. This is further commented on in Section 7.3.

7.2.2 Views on *conception*

When looking at the notion *conception*, it may be noted that it appears in the OS framework, where an unstructured conception develops into a structured one. Further, it is used in the CICD$^+$ framework, the procept framework and the APOS framework, in parallel with the notions *concept image* and *schema*. Below, the views in the OS framework are inserted into Table 7.4.

Table 7.4 Views on *conception* in the OS framework

<table>
<tr><th>Individual view</th><th>Culturally dependent views</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td></td><td>
</td></tr>
</table>

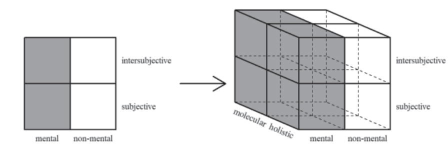
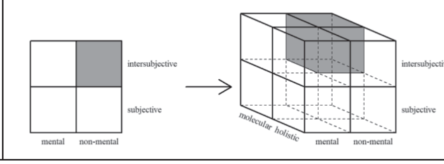
As is seen in this table, the individual conception is mental, develops from subjective to intersubjective and from unstructured to structured with molecular features. Further, there are two culturally dependent views. In the first view, the culturally dependent conception is intersubjective and mental, and develops from unstructured to structured with molecular features. In the second view, the culturally dependent conception is intersubjective and non-mental, and develops from unstructured to structured with molecular features.

The claim that conceptions in the beginning are unstructured is a simplification, since there are levels within the structure. In the development of conceptions, structural conceptions may be included in conceptions at a higher level. Consequently, operational conceptions of a higher-level concept may include structural conceptions of lower-level concepts.

7.2.3 Views on *schema*

The notion *schema* appears in the APOS framework, but also in the procept framework and the OS framework, in parallel to the notion *conception*. As is seen in Table 7.5, schemas develop from unstructured to structured in both the individual and the culturally dependent views found in APOS. In the individual view, schemas are mental, have both subjective and intersubjective features, and develop from unstructured to structured with both molecular and holistic features. In the culturally dependent view, schemas are non-mental and intersubjective, and develop from unstructured to structured with molecular features.

Table 7.5 Views on *schema* in the APOS framework

<table>
<tr><th>Individual view</th><th>Culturally dependent view</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

As with the notion *conception*, there are levels within schemas and it is a simplification to say that schemas are unstructured. Here, a schema is both seen as a structure that develops and as a stage in the development, where schemas may be included in higher-level structures. With the notion *conception*, the same duality does not exist since the term 'conception' is not used for one of the stages. It is only used for the structure that develops.

7.2.4 Comparisons between views on *concept image*, *conception*, and *schema*, and views on *concept*

After comparing different views on *concept* and different views on *concept image*, *conception*, and *schema*, it may be asked whether there are similarities between views on cognitive structures and views on *concept* in the different texts. In this section, I make some such comparisons of the different frameworks. I compare views on *concept* with views on *concept image* in the CICD framework, views on *concept* with views on *conception* in the OS framework, and views on *concept* with views on *schema* in the APOS framework.

When comparing views on *concept image*, *conception*, and *schema*, described in Section 7.2, with views on *concept*, described in Section 7.1, it may be noted that views on *concept image*, *conception* and *schema* are more often mental, and that they may have subjective features.

Within the CICD framework, on the one hand, views on *concept* do not have obvious connections to views on *concept image*. First, in the $CICD^0$ framework and the $CICD^+$ framework the subjective views on *concept image* are mental while the mental views on *concept* are intersubjective. As there is no description of how concept images develop into concepts, there is no reason for making connections of similarity between these views. The intersubjective view on *concept image* in the $CICD^0$ framework, however, is mental with a combined molecular-holistic structure and may be similar to the mental view on *concept*.

Secondly, in the CICD* framework, both views on *concept image* are mental, and the view on *concept* is non-mental. Hence, these are not similar.

Within the PO frameworks, on the other hand, views on more developed conceptions or schemas seem to be similar to views on *concept*. As seen in Section 6.4.1, the view on the more developed individual conception is similar to the mental view on *concept*. The view on the more developed culturally dependent mental conception is also similar to the mental view on *concept*. In addition, the view on the more developed culturally dependent non-mental conception is similar to the non-mental view on *concept*. An analogous result is found when comparing the view on the more developed mental schema with the mental view on *concept*. These seem to be similar, even though the more developed mental schema has subjective features and the mental view on *concept* involves the idea that concepts are intersubjective. Finally, the view on the more developed non-mental schema seems to be similar to the non-mental view on *concept*. To conclude, in the PO frameworks it is hard to see a difference between a more developed conception or schema and a concept. Here, I would say that the ideas of development within these frameworks have consequences, as concepts either are seen as more developed cognitive structures or as constructed in the development of cognitive structures.

7.2.5 Comparisons between views on *concept image*, *conception* and *schema*

From the analyses, it may be noted that the notions *concept image*, *conception*, and *schema* are often used in parallel[65]. It is relevant to ask in what aspects these notions are similar. Below, I make some comparisons.

When comparing views on *conception* seen in Table 7.4 with views on *concept image*, seen in Table 7.3, it may be noted that concept images are mainly mental in all views; there is no non-mental view on *concept image*. However, there is both an individual and a culturally dependent view, which may be compared to the two mental views on *conception*. In addition, in the CICD$^+$ framework concept images have holistic features, which is not the case for any view on *conception*.

When comparing views on *schema*, seen in Table 7.5, with views on *conception*, seen in Table 7.4, the individual view on *schema* may be compared to the individual view on *conception*. A difference is that the individual view on *schema*

[65] These notions may perhaps be similar to the notions *cognition*, *individual concept* and *image*, which are used in the philosophy presented in Chapter 3.

includes the idea that schemas have holistic features, besides the molecular ones. Another difference is that in the individual view on *schema*, both the initial and the final state of the schema includes both subjective and intersubjective features. In addition, the culturally dependent view on *schema*, which is non-mental, is similar to the non-mental culturally dependent view on *conception*.

In Asiala et al. (1996, p. 7), it is claimed that *schema* is similar to *concept image*. However, when comparing the views in Table 7.3 and Table 7.5, it can be concluded that a difference is that there is a developmental view on schema, where schemas develop from unstructured to structured. Further, in the concept maps in Section 6.1, a difference seems to be that the view on concept development is integrated with the views on *schema*, which is not the case with the views on *concept image* presented in Chapter 5, or with the views on *conception*.

If we generalise the findings, there are three different views on the cognitive structure:

1. The individual view is mental and subjective, but may have intersubjective features as well. In this view, the cognitive structure may be seen as an individual's understanding of a mathematical content.
2. The first culturally dependent view is mental and intersubjective. In this view, the cognitive structure may be seen as a culturally dependent mental understanding.
3. The second culturally dependent view is non-mental and intersubjective. In this view, the cognitive structure may be seen as a non-mental understanding, to be found in descriptions, definitions and formal mathematics.

However, a main difference between the CICD framework and the PO frameworks is that in the PO frameworks conceptions or schemas, which include mental concepts, develop into higher-level concepts. In this development, conceptions and schemas develop from unstructured to structured. Further, in the PO frameworks, there are non-mental views on the cognitive structure, which is not the case with the CICD framework. The analysed frameworks also have different ideas on whether cognitive structures have molecular or holistic features. Finally, as seen in Section 7.2.4, there is a clearer difference between cognitive structures and concepts in the CICD frameworks than in the PO frameworks. In the PO frameworks, it is hard to see a difference between a more developed conception or schema, and a mental concept.

The notions *concept image* and *schema* seem to be more theory-impregnated than *conception*. Further, from the perspective that both the OS framework and the APOS framework is based on cognitive schema theory, *conception* and *schema* seem to be more similar to each other than any of them to the notion *concept image*. However, the term 'schema' is clearly used in a Piagetian tradition and 'conception' is used in a more classical philosophical way, distinguishing between 'concept' and 'conception' with references to Frege (1952/1960). Here, a disadvantage with using *conception* is that this notion is used with many different meanings, and often in a vague everyday meaning, in mathematics education. For example, in Section 5.2.3 it is seen that *conception* is used both for describing mental representations of physical objects, and for describing more general ideas, in the CICD$^+$ framework. Additionally, I would say that the usage of the term 'concept image' may give the reader an idea of a view on mental representation seen as mental images (see Section 3.1.2.1), through the word 'image'. Initially, when concept images were introduced as mental images of basic geometrical concepts, such an idea would perhaps be appropriate. However, it is not in line with the view where processes are included in the concept image.

7.3 Usage of language

In Section 1.1, I claimed that it is important to seek coherence in the field of mathematics education. A type of incoherence is when a single term is used with different meanings or allowing for different interpretations. This appears when a term has different meanings in different texts, but also when a term has different meanings within a single text. In this section, I collect findings of how terms are used in the analysed frameworks, which eventually may hinder an understanding of the notions referred to.

An underlying assumption for the concept analysis is that there are three different arenas: a mental arena, a non-mental arena, and a concrete arena. The mental and the non-mental arenas are in the current study used for describing two types of concepts, which both may be seen as meanings of expressions. In Section 7.3.1, I discuss formulations where these different arenas are not distinguished, together with formulations where the distinction between syntax and semantics is not respected. These formulations may be seen as examples of category mistakes. That we use different notions for the same phenomenon may, on the one hand, not be a problem of incoherence, but may, on the other hand, blur the meanings of terms. This is discussed in Section 7.3.2. Section 7.3.3 presents findings of how different views are seen in the texts, if they are shown in explications or in the way terms are used. Finally, I describe how the indicators may be used for ensuring that there is but one single view on *concept* within a text, from the perspective of the distinctions in the analysing tool. Thereby, one can avoid mixing views.

7.3.1 Category mistakes

From the analysis, it may not only be concluded that there are several views present in the frameworks. The different views are also combined, which has consequences for how the reader may understand the notions involved. This appears when some formulations become incomprehensible by the fact that two different views on the same notion are found in the same paragraph, or even in a single sentence. This is for example seen when a formulation using 'thinking about a concept' is followed by a formulation claiming that a concept is a mental entity. Another example is seen in the quote below, discussed in Section 6.2.1, where concepts in a non-mental view is indicated from the perspective of a philosophical understanding of concepts, and concepts in a mental view is indicated from a perspective where concepts emerge from psychological processes:

> It seems that the philosophical insight into the nature of mathematical concepts is what we need in order to understand in depth the psychological processes in which such concepts emerge. (Sfard, 1991, p. 2)

Further, elements of different nature are sometimes combined. An example from the analysis is the construction of the notion *procept* which includes both a symbol and a concept. Regardless whether a concept is seen as mental or as non-mental, the symbol and the concept live on different arenas. Here, it is hard to point out properties of the notion *procept*. Procepts do not have conceptual structures, as symbols do not have structures. Further, the notion *procept* does not inherit properties from the notion *symbol* that are not shared by the notions *concept* and *process*. The two different variants of *concept image*, in the CICD framework, can be taken as another example. In one of these, concept definitions are included in cognitive structures, which gives definitions mental nature. When definitions are both expressions and mental at the same time the view becomes hard to understand.

The construction of the notion *procept* and the variant of *concept image* where concept definitions are included in cognitive structures may be interpreted as if the distinction between syntax and semantics is not respected. Another example is seen when it is claimed in Gray and Tall (1994, p. 122) that a symbol starts out with a simple structure that later links both procedural and conceptual aspects, when a symbol cannot have such a structure. A third example not distinguishing between expressions and meanings of expressions is the formulation "the basic meanings of the concepts should be stable" (Semadeni, 2008, p. 3). In this context, concepts are meanings of terms and 'meanings of concepts' becomes incomprehensible.

In addition, from the analyses it can be concluded that notions sometimes are mixed up. This may be seen when the difference between *concept* and *conception* is not respected, when *conception* is used instead of *concept*, and vice versa. An example of that may be when both 'an action concept' and 'an action conception' is used in Asiala et al. (1996, p. 7). Here, one may perhaps interpret 'an action concept' as a simple writing error, since at the action stage of the development the cognitive structure has not yet developed into a concept. Another example is seen when it is claimed that concepts contain conflict factors instead of that concept images contain conflict factors (Tall & Vinner, 1981, p. 153) and when the term 'evoked concept' is used instead of 'evoked concept image' (Tall & Vinner, 1981, p. 16). Here, the term 'evoked concept' should probably be replaced with 'evoked concept image', for which I have argued in Section 5.1.6.

A consequence of these category mistakes is that the different notions become hard to interpret and to understand. This affects the understanding of the frameworks, since it becomes unclear what the frameworks actually describe. Consequently, such mistakes appearing in the analysis is an essential part of the result.

7.3.2 Usage of different notions for the same phenomenon

As discussed above, it may be asked whether it is a problem with different notions for the same phenomenon. An example from the concept analysis is that the notions *conception*, *schema* and *concept image* are used in parallel and that similarities and differences between these notions are not described. As an example, the notion *conception* is used in the APOS framework in parallel with *schema*. However, while *schema* has an explication, *conception* has not, and connections between these notions are not described. Similarly, the notion *schema* is used in the OS framework, in parallel with the notion *conception*, as the framework presupposes cognitive schema theory. Here, the meanings of both 'conception' and 'schema' become unclear and the understanding of the frameworks is affected.

The notions *concept* and *object* in Gray and Tall (1994) can be taken as another example where it can be argued that different notions are used for the same phenomenon. As is seen in Section 6.1.2, arguments for the conclusion that *object* and *concept* are similar can be found in the text. Together with the fact that these notions are not explicated, it becomes unclear whether *concept* and *object* are similar or not. Again, the understanding of the notions and of the framework is affected.

7.3.3 Explications and the way terms are used

Within the analysed frameworks, there are different approaches to explications and the way terms are used. Table 7.6 shows which views on *concept* that are present, and how they are shown, in the different frameworks.

Table 7.6 How views on *concept* are shown

Framework	Mental view	Non-mental view
The $CICD^0$ framework	Used	Used
The $CICD^+$ framework	Used	Used
The CICD* framework		Used
The OS framework	Used	Explicated and used
The procept framework	Used	Used
The APOS framework	Used	Used

To me, there seems to be several ways in which a view on *concept* can be shown in a text. Some alternatives are:

- An explication and a consequent term usage
- No explication and a consequent term usage
- An explication and a non-consequent term usage
- No explication and a non-consequent term usage

It may be concluded that no text in the selection has an explication and a consequent term usage. In fact, only Sfard (1991) has an explication of *concept*. Even so, there are two different views on *concept* in Sfard (1991), shown in how terms are used. Hence, the term usage is not consequent. Further, in Bingolbali and Monaghan (2008), there is a consequent usage of the term 'concept' without an explication[66]. In Tall and Vinner (1981) there is no clear explication, even though the terminology in the beginning of the text indicates that concepts are seen as non-mental. However, later in the text formulations where concepts seem to be mental appear as well. Other frameworks without explication and a non-consequent term usage are the procept framework and the APOS framework.

After comparing how different views on *concept* are present in the different frameworks, a similar comparison can concern how different views on the notions *concept image*, *conception* and *schema* are shown. Table 7.7 shows which views on these notions that are present, and how they are shown, in the different frameworks.

Table 7.7 How views on *concept image*, *conception* and *schema* are shown

Framework	Mental view	Non-mental view
The CICD^0 framework	Explicated and used	
The CICD$^+$ framework	Explicated and used	
The CICD* framework	Explicated and used	
The OS framework	Explicated and used	Used
The APOS framework	Used	Used

From Table 7.7, it can be concluded that generally the notions are explicated as mental. For example, the notion *concept image* is explicated as mental, without any preference when it comes to the distinction subjective versus intersubjective. However, in the way terms are used two views are shown, a subjective view with intersubjective features and an intersubjective view. These views may be called an individual view and a culturally dependent view. There are two different mental views on *conception* as well. In Sfard (1991), *conception* is

[66] Here, it may be noted that the term 'concept' is not frequently used in this text, which may have affected the result.

explicated as mental and subjective, but in the way terms are used an intersubjective view appears. Here, the culturally dependent view may be seen as opposing the explication. Finally, in the APOS framework the explication of *schema* expresses the idea that schemas are subjective in Asiala et al. (1996), but not in Dubinsky (1991). As the culturally dependent view on *schema* appears in Dubinsky (1991), this view does not oppose an explication.

Furthermore, there is a tendency in the PO frameworks to use *conception* and *schema* in a non-mental view as well, describing a non-mental knowledge structure. This tendency connects to a view where the mental development of an individual is parallel to the development of mathematics throughout history, and the fact that the same notions are used for describing these two developments.

7.3.4 Usage of the indicators

The indicators have been developed in an interplay between the analysing tool and the analysis, and can be seen as part of the result. They can be used for interpreting texts, but also for ensuring that the way formulations are used is coherent with the explication of *concept*.

As an example, assume that we explicate *concept* as non-mental, intersubjective and molecular. Then, formulations claiming that concepts exist within mathematics, within a curriculum or in a community of mathematicians can be used (Indicators [4; 5]). Further, formulations such as 'thinking about concepts', 'knowing concepts' and 'describing concepts' can be used for describing students' knowledge of concepts (Indicator [6]), since such formulations are used for a non-mental view on *concept*. Many of the formulations indicating a non-mental view indicate an intersubjective view as well. This holds for the formulations claiming that concepts exist in mathematics or in a culture (Indicators [7; 8]). Formulations indicating that there is a unique concept of a certain kind, or that concepts are determined by its use, where concepts are intersubjective, can also be used (Indicator [9]). Additionally, a molecular non-mental view is described with formulations indicating that concepts can be defined, that there are concepts that are more basic than others, and that there are hierarchies within structures of concepts (Indicators [13; 15]).

On the other hand, with our example of a non-mental, intersubjective and molecular view, formulations appearing in Indicators [1; 2; 3; 10; 11; 12; 14; 16; 17; 18], indicate mental, subjective or holistic views and should be avoided in the text, since they oppose the explicated view. That formulations are used that oppose an intended view is a source for incoherence and may be an obstacle for readers to interpret the text.

7.4 A summary of the study

In Section 1.5, it is expressed that the study intends to make both a theoretical and a methodological contribution. Theoretically, the intention is to give clearer understanding of views on the notion *concept*, within the studied frameworks, examining the coherence of the texts. The intention is also to offer categories that can be used for describing concepts. These categories are the positions in the analysing tool. Methodologically, the design is meant to contribute to the method of making concept analyses in mathematics education.

As a result of the literature review in mathematics education, the selected texts were taken from a subgroup of frameworks for conceptual understanding. These had a common philosophical background and were related through referencing. Both the frameworks and the selected texts are described in Section 4.1. The purpose of making concept maps was to, in an initial analysis, represent the roles of the notions, the connections between them, and how they interact in the frameworks. After the construction of these maps, views on *concept* and related notions, seen in how terms are used in the texts, were interpreted using the analysing tool developed in Chapter 3. During the analysis, the object of study turned into also including views on the notions *concept image*, *conception* and *schema* in the analysed texts (see Section 4.5). When analysing the different texts, a need for indicators emerged in order to ensure that the texts were interpreted similarly. The indicators described in Section 4.4 may be seen as part of the method, but also, as claimed in Section 7.3.4, as a result, since they can be used for ensuring that a usage of formulations is coherent. This is one example where the methodology and the results are interacting with each other, and there is no strict border between methods and results.

Conclusions from the concept analysis are presented in this chapter. As a short answer to the research question, there are, as seen in Section 7.1, mainly two views on *concept*, one mental and intersubjective view, and one non-mental and intersubjective view. Most texts include both these views and they are not always distinguished within a single text. In fact, Bingolbali and Monaghan (2008) is the only text which contains one single view on *concept*, where concepts are non-mental and intersubjective.

If also taking the distinction molecular versus holistic into account, there are different views on *concept* in different texts, both in texts using the same framework and in texts using different frameworks. In some texts, there is a focus on molecular features, and in others, there is a focus on holistic features. Some frameworks take a position involving the idea that non-mental structures are molecular, but that mental structures have holistic features as well. From the analyses of these frameworks, one might get the idea that it is more common with holistic features in views on mental structures than in views on non-mental structures. However, in other frameworks non-mental and mental structures have similar features. From these frameworks, one might get the idea that the cognitive structure likens the formal structure of mathematics. Consequently, the conclusion that can be drawn from the perspective of the distinction between molecular and holistic views is that there is no clear position regarding whether conceptual structures have molecular or holistic features.

In Section 7.2, it is claimed that in order to use the tool for analysing views on different notions, it must be possible to regard these notions as meanings of signs, terms or expressions. Further, it must be possible to regard them as intersubjective or subjective, and they must have structures, or be included in structures, with relations that can be judged as molecular or holistic. As is seen in the same section, the notions *conception*, *concept image* and *schema* are all used for cognitive structures. There are generally three different views on such structures, one individual view and two culturally dependent views.

In the CICD framework, there is a clear distinction between concept images and concepts. However, within the PO frameworks, views on more developed cognitive structures seem to be similar to views on *concept*. The view on the more developed individual conception is similar to the mental view on *concept*, and the view on the more developed culturally dependent mental conception is also similar to the mental view on *concept*. Finally, the view on the more developed culturally dependent non-mental conception (or schema) is similar to the non-mental view on *concept*. Consequently, it is hard to see a difference between a more developed conception or schema, and a concept.

There are some differences between the notions *concept image*, *conception*, and *schema*. A main difference is that in the development described in the PO frameworks, conceptions and schemas develop into higher-level concepts, while an analogous description is not found in the CICD framework. In this development, conceptions and schemas develop from unstructured to structured. Here, the view on *schema* seems to be more theory impregnated, dependent on cognitive schema theory and the perspective of Piaget, than the view on *conception*. Another difference is that within the PO frameworks, there are non-mental views on the cognitive structure, which is not the case within the CICD framework.

In addition to the different views, one part of the result is the analysis of how terms are used in the frameworks. In Section 7.3 it is discussed how different views and different arenas sometimes are combined, and that there sometimes seems to be a confusion regarding terms, on the one hand, and the meanings of terms, on the other. Occasionally different notions also seem to be mixed up and there are examples where several notions are used for the same phenomenon.

The usage of formulations is not always coherent with how notions have been explicated in the texts. However, in Section 7.3.4 it is seen that different views interpreted from the way terms are used do not always oppose a view seen in an explication. Notably, the notion *concept* lacks explication in all texts except in Sfard (1991), where the mental view opposes the view in the explication of *concept* as a theoretical construct. In the other texts, the views found in the way terms are used cannot be compared to a view in an explication, as there is none.

Furthermore, in the OS framework the notion *conception* is explicated as mental and subjective, and the two culturally dependent views, the mental and intersubjective one and the non-mental and intersubjective one, oppose the explicated view. However, in the CICD framework the notion *concept image* has an explication that does not take a stance regarding whether concept images are subjective or intersubjective. In this case, the different mental views cannot be said to oppose the explication.

The philosophical assumptions that underlie the study are presented in Section 4.6.2. Besides the idea of the three arenas and the distinction between terms and meanings of terms, mentioned above, the study has been governed by the overall ideal that it is important to seek coherence within a field of research. An assumption that underlies the design is that the distinctions made in the common field of philosophy of language and concept research within cognitive science can contribute through offering new perspectives in the field of mathematics education. Additionally, an idea of reductionism claims that simpler frameworks, including fewer concepts, are to prefer. Consequently, different views on *concept* may be compared in order to broaden the understanding based on a certain framework, and the meaning of the term 'concept' in different fields and frameworks could basically be the same.

Finally, the study includes only views assuming the existence of representations, where an object in one of the three arenas can point to an object in one of the other. Without this assumption, the mental position, regarding concepts as mental representations, becomes irrelevant. As is claimed in Section 4.6.2, this does not mean that such an existence is assumed in the study, but that only texts based on this assumption are used and analysed.

8 Discussion

Considering the long history of philosophical discussions of concepts and the different views on *concept* in mathematics education that can be found through an initial reading of texts describing conceptual understanding, it is perhaps an ambitious project to carry out a concept analysis of the notion *concept*. In an early stage of the study, it became clear that an approach based on just reading texts, interpret views on *concept* and sort them into categories, would not give sufficient understanding of the different views. From that insight, the philosophical approach emerged. As claimed in Section 1.6, the study uses philosophy in two different ways. First, the study takes an analytic philosophical approach as it assumes that a concept analysis can contribute in general and in mathematics education in particular. Second, it uses philosophy as a method, since the analysing tool, with its three distinctions mental versus non-mental, intersubjective versus subjective, and molecular versus holistic, is developed from philosophical ideas.

From this, the aim of the study became to carry out a concept analysis of the notion *concept*, within some frequently used frameworks for analysing conceptual understanding in mathematics education. The purpose was to influence mathematics education research, both theoretically and methodologically. Theoretically, it was to give clearer understanding of views on the notion *concept*, within the studied frameworks, examining the coherence of the texts. The aim was also to offer categories that can be used for describing concepts. Methodologically, the design intended to contribute to the method of making concept analyses in mathematics education. The research question became:

> Which views on *concept* may be found in texts using the chosen frameworks, from the perspective of the distinctions mental versus non-mental, intersubjective versus subjective and molecular versus holistic?

In this final chapter, I discuss both the theoretical and the methodological contribution. It may be hard to distinguish between results and methodology in the study, since the analysing tool is developed in an interplay between the philosophical literature review, resulting in perspectives from which the texts could be analysed, and the analysed texts. Even so, the discussion is divided into a discussion of results and a discussion of methodology. After that, the philosophy used in the study is discussed from the perspective of mathematics education. The final section considers consequences of the results for future empirical research.

8.1 Discussion of results

The main result of the concept analysis is that different views on *concept* and different views on *concept image*, *conception*, and *schema* are found in the different texts. There are both different views within several of the texts, and different views in different texts. Additionally, an unexpected result turned out to be findings of how language is used within the texts. These findings are gathered in Section 7.3. These findings involve that the same term is used with different meanings or allowing for different interpretations, and that the distinction between different arenas or between terms and meanings of terms are not respected.

In the beginning of the study it was hard to understand the views in the different texts. From that background, it was surprising that there were so many similarities between them. Similar views appeared in the different frameworks. In fact, if ignoring the distinction molecular versus holistic, there are generally only two views on *concept*: a mental and intersubjective view, and a non-mental and intersubjective view. Further, there are similarities between the views on *concept image*, *conception*, and *schema* as well. This may be due to different views having a common origin in psychology, philosophy of language and mathematics, and that the frameworks are related through referencing. An alternative explanation is that the similarities arise as a consequence of the usage of the analysing tool. This may be due to a too rough simplification of aspects of views on *concept* in the developing of the analysing tool described in Chapter 3.

It may be asked whether, and if so why, there is need for two types of concepts in mathematics education, where one type is mental and another type is non-mental. Notably, these two types of concepts may be used for describing different aspects of learning. In the analysed frameworks, a mental view on *concept* is often used for describing individual development and a non-mental view is often used for describing the nature of mathematics. However, when mathematics develops through history, there is both a development of individuals' mental representations and a development of definitions and formal mathematics. For example, in Sfard (1991) there is first a description of a historical development of concepts. This historical description can be seen as a non-mental development, where conceptions are seen in definitions and other descriptions. It can also be seen as a historical development of how people develop mental conceptions. After this description, there is also a description of how people develop mental conceptions, from an individual perspective. Naturally, mental and non-mental views on *concept* can be connected to different views on learning, where some frameworks focus on mental concepts and other frameworks focus on non-mental concepts. If I allow myself to speculate, a non-mental view would perhaps be more common in texts using a socio-cultural perspective, while the texts analysed in this study have an interest in mental structures.

Mental and non-mental concepts have different nature, and it becomes confusing when the same term 'concept' is used both for the mental representation and for the non-mental object. When the same term is used for two different phenomena, there is an obvious risk that these different views are mixed up by the reader. There are in the literature ways of describing mental and non-mental arenas without using two different views on *concept*. As suggested in Section 3.3, the terms 'word meaning' and 'class' can be used for describing non-mental concepts, and the term 'concept' may in that case be used only for mental representations. Another alternative is to use a term such as 'conception' for mental concepts, and exclusively use 'concept' for non-mental objects. Such an alternative is advocated by philosophers such as Frege (1892/1951), and may be what is requested for example in the explication in Sfard (1991), referring to Frege (1952/1960). In spite of the explication, a mental view appears in the usage of terms in Sfard (1991), when results from psychology and Piaget's perspective are used. One can draw the conclusion that it is hard to use findings and ideas coming from another framework without adopting a view on *concept* that is present in this framework.

As another alternative, a terminology using both 'mental concept' and 'non-mental concept' may be used for distinguishing between different types of concepts that a framework may describe, and to keep track of these issues. Such a usage could reduce the risk for unclearness and the risk that different types of concepts are combined. If not distinguishing between them, however, there is a risk that different ontological views on the nature of concepts, and hence of the nature of knowledge, are not distinguished.

As seen in Section 4.6.2, an underlying assumption in the study is that it is beneficial to compare and combine knowledge from different frameworks, in order to obtain a deeper understanding. On the other hand, that there are similar notions for the mental representation, such as *concept image*, *conception* and *schema*, compared in Section 7.2, may be a consequence of combining ideas from different frameworks, still using the notions from the original texts. Furthermore, that there are similar notions for the mental representation may both affect the understanding of these notions, and of the framework at whole. This shows that when combining frameworks, a consciousness of different views, and translations between terms and notions, is needed. I would say that most often we do not need different terms for the same phenomenon in scientific contexts. However, if using similar notions within a text, I would say that clear explications, descriptions of the connections between the notions, and a term usage that is consistent with the explications is necessary. When the explications are presented, ideally one needs to be consistent and stick to the same meanings during a text, and avoid other similar notions. In a coherent text, different views are also distinguished by the way terms are used. Here, the indicators in Section 4.4 can be used to obtain a consistent term usage that fits the intended view on *concept*.

With the example of the OS framework in Sfard (1991), instead of saying that the individual develops concepts and that concepts emerge in psychological processes, a formulation claiming that conceptions develop and become structural is more in line with the explication. An alternative approach, using both ‘mental concept’ and ‘non-mental concept’, is to say that individuals develop mental concepts and that a non-mental concept is a theoretical construct that develops through history.

The distinctions non-mental versus mental, intersubjective versus subjective, and molecular versus holistic offer categories that can be used for describing views on *concept*. Above, I have discussed the distinction non-mental versus mental, and below I will make some comments on the other two.

From the perspective of the distinction intersubjective versus subjective, there is an agreement in the analysed frameworks that concepts are intersubjective. Hence, the subjective views on *concept* found in the philosophical review in Chapter 3 (see Section 3.2.3), are not represented in the analysed frameworks. However, subjective views on the notions *concept image*, *conception*, and *schema* are found in the analysis. Notably, when analysing individual views on these notions, intersubjective features appear as well. This might be a presumption for people to communicate with each other, as we need a common understanding in our communication.

The distinction molecular versus holistic has shown to be the hardest one to use, since it has not always been clear in the texts whether conceptual structures have molecular or holistic features. In some cases, I have chosen to omit this distinction from the 3D matrix. Further, even if there is a view mainly focusing on holistic features, the conceptual structure may have molecular features as well, and vice versa. The findings show that there is no agreement between the frameworks, but the texts have different views on whether structures have molecular or holistic features. To generalise the findings from the analysis of texts representing the $CICD^0$ framework and APOS, it seems as though mental structures have more holistic features than non-mental ones. From the analysis of the $CICD^0$ framework, it also seems as if mathematical concepts are molecular. This can be compared to the analysis of the $CICD^+$ framework, where mathematical concepts are holistic.

Additionally, within the construction of the concept maps, hierarchical and non-hierarchical connections may describe different features of the structure in a framework. Consequently, molecular and holistic features should perhaps not be held as mutually exclusive, but may be used in combination for describing conceptual structures. This is in line with the position in Murphy (2004), argued for in Section 3.3.

That different views on *concept* are combined may generate unclearness regarding what the framework actually describes. Consequently, it becomes hard to see how a framework can be used together with other frameworks and concept research within other fields. This may in turn generate more incoherencies in the future. Further, it may be asked whether the usage of several notions, such as *concept image*, *conception* and *schema*, depends on notions being underspecified in mathematics education, and that the meanings of the terms are so unclear that we do not want to use them. Instead, we invent new ones to specify a meaning in a certain framework. An alternative explanation may be that notions are understood as theory-dependent, bringing connotations that we want to avoid. For example, the notion *schema* may bring the idea of cognitive schema theory. With another view on concept development, the notion *schema* would perhaps give the reader wrong associations.

To conclude, clearer views may be requested in the field of mathematics education, and, as seen in Section 4.6.2, an assumption in the study is the idea of reductionism that few notions are to prefer. In order to get more coherent frameworks, explications of *concept* and other notions, are needed. Finally, as seen in Section 7.3, the distinction between syntax and semantics, between terms and meanings of terms, is not always respected. Here one can discuss which requirements are to be followed within the field of mathematics education, from a philosophy of language perspective. Can we always demand of the frameworks to be coherent? Here, I would say, that to not distinguish between terms and meanings of terms affects the understanding of the text. A questioning of how language is used, of unclear notions, and of usage of many notions, on the other hand, may be a driving force for theory development.

8.2 Methodological discussion

As claimed in Section 1.5, the study intends to make a methodological contribution to how concept analysis may be conducted in the field of mathematics education. The chosen design includes an analysing tool which is developed from philosophical views. Notably, two different literature reviews are used in this design. When it comes to the philosophical review, the used method is described in the introduction to Chapter 3. Any selection of philosophical texts is to some extent arbitrary. Even so, the delimited selection has offered enough content for creating an analysing tool, and consequently the purpose of the philosophical review has been fulfilled. When it comes to the review in mathematics education, this method is described in Section 4.1, and can be seen as a refined version of the method used in the philosophical review. The conceptual frameworks analysed in the study shall be seen as examples and, again, the purpose of the concept analysis is not to offer a complete picture of views on *concept* in mathematics education. Naturally, with the knowledge of which frameworks that actually was included in the study, it would be beneficial to redoing the literature review, searching for the chosen frameworks, aiming at finding texts that could complement the picture. However, as the analysed texts may be seen as essential in the field in the sense that they are frequently referred to, the concept analysis points to a fundamental problem.

In an initial analysis, concept maps were used as visual representations of the frameworks. The purpose was to offer overall pictures and not to go into detail about how notions are connected. This is the reason why connections are not categorised. Further, the connections used in the maps should be seen as examples, and I cannot claim that the chosen ones are the most important.

In an early phase, as is described in Section 4.5, I concluded that the connections between *concept* and *concept image*, *conception*, and *schema* were important for interpreting views on *concept*. Since cognitive structures in some frameworks are described as developing into concepts, and since cognitive structures include concepts, it became necessary to analyse views on these structures, in order to judge whether a view on *concept* is molecular or holistic.

The found views may not give an altogether fair picture of the views in the frameworks, considering that I have chosen a limited selection of texts and have made interpretations which highlight aspects of the notions from the perspective of the philosophical background. The less written and the more unclear content, the harder it became to interpret the texts and the higher risk for misinterpretation. For example, it may be asked whether the holistic view found in the CICD$^+$ framework (see Section 5.2.4) should instead be interpreted as a combined molecular-holistic view. Here, the judgement depended on that if there were molecular features in the text they were not clearly seen. The decision, however, became somewhat arbitrary, depending on how I as a reader interpreted the formulations. In order to make clear how the interpretations were made, and to ensure that the different texts were interpreted similarly, the indicators offered guidelines to how formulations could be interpreted.

As claimed before, in Section 1.2, the approach with the analysing tool both broadened and delimited what might be found in the analysis. On the one hand, it facilitated seeing philosophical aspects in the views on *concept*. On the other hand, it might have obstructed the discovery of aspects that were not included in the analysing tool.

I would like to end this section with some remarks about how the analysing tool can be used. Notably, the tool is developed for analysing the notion *concept*, and notions such as *concept image*, *conception* and *schema*. It is also developed from the perspective that it should be used in the field of mathematics education, and specifically for frameworks based on some assumptions regarding the distinction mental versus non-mental, and the existence of representations. Further, the tool is designed to be used for text analysis. In order to be used in other kinds of studies, there might be a number of aspects that have to be considered.

8.3 The analytic philosophical approach

The current study analyses the notion *concept* within mathematics education, from a philosophical perspective. In this section, I turn the perspectives around and discuss the analytic philosophical approach from the perspective of mathematics education. Notably, the concept analysis depends on the analysing tool, deciding what aspects of the notion *concept* are considered when interpreting the different texts. The three distinctions have been, so to speak, fetched from philosophy, based on that they should be useful for interpreting texts in mathematics education. Hence, it may be relevant to reflect on the roles of the distinctions within mathematics education, what they can tell about the field.

The distinction non-mental versus mental is ontological and concerns the nature of the relation between the human mind and the world. This distinction is related to the distinction between a Platonic and an Aristotelian view on *concept*, as presented in Section 2.1, and between rationalism and empiricism. Further, as is seen in Chapter 3, a non-mental view is frequently found in philosophy of language, and a mental view is frequently found in concept research within cognitive science. From the perspective of mathematics education, it may be asked whether a concept is developed mentally by a child, or if it exists within communication in a classroom or in a text, in a semantical meaning, or perhaps in an ideal world of mathematics. If considering a child learning mathematics, it is one thing to claim that the child develops concepts, and another thing to claim that the same child learns about concepts. Learning mathematics becomes different things in these alternatives.

Philosophically, a static view could be held against a non-static view. To exemplify, a static view is seen in the Platonic realist position in Frege (1892/1985; 1892/1951), where concepts are classes of objects, and in the empiricist view in Jenkins (2008), where there is a causality relation between concepts and the world. A non-static view, on the other hand, is seen in Katz (1972/1999), where a concept may be the common property of several individuals, in Quine (1960; 1951/1985), where concepts are meanings that are determined by how terms are used, and in Potter and Edwards (1999), where concepts may be integrated in a culturally dependent understanding of the world. The distinction between static and non-static can be expressed as a distinction between objective and intersubjective views. In the categorisations used in the concept analysis, however, these are not distinguished. The reason is that the views on *concept* in the analysed texts has not been expressed in a way enabling such an analysis.

It may not always be clear whether an intersubjective view is mental or non-mental. In formulations discussing the properties of the concept *number*, for example, the view may either be mental and intersubjective or non-mental and intersubjective. Further, if denying that concepts are intersubjective as is done in Wittgenstein (1953/1992) and Zalta (2001), then you end up in a subjective view, but such a subjective view on *concept* is not seen in the analysed texts. There are, however, subjective views on *concept image*, *conception* och *schema* in these texts, and, with these notions, perhaps there is no need for subjective views on *concept*. Notably, in the philosophical views it is possible to have an object including both intersubjective and subjective concepts. In Carey (2009), for example, there is a distinction between *concept* and *individual concept*. However, as seen above, when combining different types of concepts, explications and term usage become important issues.

The third distinction, between molecular and holistic views on *concept*, can be seen as a distinction between a classical view on *concept*, using compositionality, and criticism against such a view, based on the ideas in Quine (1960; 1951/1985) and Wittgenstein (1953/1992). This distinction is in a way empirical, as concept research can show whether a structure has molecular or holistic features. However, whether a structure has molecular or holistic features depends on how the structure of concepts is explicated. As shown in Section 3.3.3, if building the structure from part-whole relations, then it is possible to define the structure such that concepts get molecular features. The question then, from the perspective of mathematics education, concerns whether it is possible to define mathematical concepts, from basic ones. Another alternative, also expressed in Section 3.3.3, is to add other types of connections to the structure, between different types of mathematics, between mathematics and other subjects, and between mathematics and the everyday world. Naturally, if so the structure gets holistic features. This points to the importance of clearly explicating the type of structure, and what kinds of connections that are included. Here, explications of notions such as *conception* become important, as conceptions are in fact structures of concepts. For example, in the analysis in Section 6.1.1, it is seen that *conception* is explicated as "the total cluster of internal representations and associations evoked by the concept" (Sfard, 1991, p. 3). In this description, it seems as if conceptions have holistic features as all connections are included. However, in the analysis of the way terms are used in the same text (Sections 6.2.1 and 6.3.1) it is found that undeveloped conceptions are regarded unstructured and more developed conceptions are regarded structured with molecular features. Consequently, in this case the explication is not enough for judging the view as molecular or holistic.

Notably, the analysed texts do not always involve views focusing on structure from the perspective molecular versus holistic. For example, this is the case with the procept framework (Section 6.2.2). This raises some questions regarding similarities and differences between the frameworks. In Section 6.3.4, it is claimed that the description of an operational conception in the OS framework is similar to the description of a symbol evoking a process in the procept framework. Here, however, it may be noted that one difference between these two frameworks is that in the OS framework there is a focus on conceptual structures, and connections between concepts, which is not seen in the procept framework.

Furthermore, in the analysis of some frameworks, I have used a terminology including molecular or holistic views on *concept*, and in the analysis of other frameworks, such as the APOS framework, I have instead used a terminology including molecular and holistic features of the structure. It may be asked about the difference between these two ways of expressing structures, using the terms 'molecular' and 'holistic'. In the analysis, I have used 'molecular view' for a view that mainly have molecular features, which also may be described as hierarchical connections between notions. However, when there is a combined molecular-holistic view on *concept*, I have used expressions such as 'molecular and holistic features of the conceptual structure'.

Notably, the CICD^0 framework has a view where mathematical concepts are molecular, but non-mathematical concepts have holistic features (see Section 5.1.5). This can be compared to the position presented in Section 2.1 where mathematical concepts are distinguished from non-mathematical concepts by the fact that they are well determined and may have unique explications. For example, Priss (2017) suggests that mathematical concepts are well determined and Sjögren and Bennet (2014) argue that mathematical concepts have unique explications. Consequently, the view in the CICD$^+$ framework, where both mathematical and non-mathematical concepts are holistic, can be regarded as critic against such a philosophical position.

Finally, the claim in Chapter 3 that the positions as to the distinctions are not to be considered incommensurable can be commented on. In this chapter, it is suggested that it is possible to have a dual view, including a combination of a non-mental and a mental view, including a combination of an intersubjective and a subjective view, or including a combination of a molecular and a holistic view. This can be done by using different types of concepts and a consequent terminology, clearly distinguishing between the different types. However, in the concept analysis it is seen that there is a risk with combining different views, if these are not clearly distinguished in explications and term usage.

8.4 Consequences for future research

The importance of a theoretical foundation for empirical research was pointed out in Chapter 1. Again, it is of importance that theories of learning are based on a theoretical sound conceptual apparatus, without incoherence, which can describe and explain classroom practice. In order to know how concept formation can be studied, the object of study must first be clarified. For example, the object of study may be mental concepts that are developed by individuals, or it may be individuals' understanding of a mathematical concept that is described in a textbook. It may also be a culturally dependent concept which is developed through communication by a group of individuals. Different approaches for studying concept formation may be taken in the three alternatives.

In the concept analysis, it is shown that the analysed texts have often vague descriptions of cognitive structures and that there is no agreement between the texts concerning whether these structures are molecular or holistic. Consequently, there is a possibility for theory development taking this perspective into account.

Exactly how the results of this study may affect future empirical research is, of course, hard to tell. Some indications, however, may be seen in a future research project that I have planned, with the intention to study concept formation. Here, my overall project is to develop a framework for analysing conceptual understanding. The concept analysis presented in this thesis can be seen as the theoretical foundation for such a project. The next step is to interpret these theoretical findings in an empirical study about how students reason, using their individual concepts.

The planned study is based on a view which distinguishes between mental and non-mental structures, and between individual concepts and concepts. Further, as seen in Section 3.3, there are arguments from concept research within cognitive science for a view where mental concepts have both hierarchical and non-hierarchical features. In mathematics education, however, few existing studies about cognitive structures are looking at hierarchical and non-hierarchical features in combination.

Success in mathematics corresponds to an awareness of structure (Juter & Sriraman, 2011). While high achievers fluently can make connections between concepts, individual concepts of low achievers are often disconnected (Mulligan, 2011). Even so, there are few studies about how high achievers in mathematics make these connections (Szabo, 2017). The reason for focusing on high achievers is that if individual concepts of low achievers do not have a more developed structure, then it can be hard to see molecular and holistic features. This is also an experience from a smaller pre-study that I conducted in spring 2019, which used a questionnaire in a class in grade four. In addition, one of my assumptions is that if we can gain knowledge of how high achievers develop cognitive structures, then this knowledge can be used for developing models for how mathematics can be taught.

From that background, the aim of the planned study is to explore how molecular and holistic features of the cognitive structure can be detected in conversations with high achieving students. The study is hoped to increase knowledge of how high achievers make connections in mathematics, and aims at answering the following research question:

> How can high achievers' cognitive structures be described, from the perspective of the distinction between molecular and holistic features?

The mathematical content in the study is planned to be fractions. Naturally, there are connections between fractions and geometry, and research emphasises translation between different representations when teaching fractions (Cramer, 2003; Chahine, 2011). Further, in the pre-study, it was shown that questions about fractions could reveal both hierarchical and non-hierarchical connections.

As described above, the results presented in this thesis is a theoretical foundation for the planned study. Consequently, this is an example of how the theoretical results can be used in future empirical research. Simultaneously, findings of how molecular and holistic features of the cognitive structure can be detected in conversations may affect how cognitive structures can be described. Consequently, the empirical results of such a future study can also be used for theory development.

I end this thesis by commenting the trend in mathematics education, which is seen in Section 1.3, with a decrease of theoretical discussions and reflections. From that trend, it is natural to ask what theoretical research, such as the concept analysis in this study, can offer. From the above discussion, I would conclude that a meta-analysis of research in the field of mathematics education simplifies and affects interpretation of theories of learning, which in the extension may affect models of explanation for classroom practice.

Svensk sammanfattning

Ordet ’begrepp’ används med olika betydelser inom matematikdidaktik och det kan vara svårt att förstå olika synsätt på vad begrepp är, hur begreppsförståelse kan studeras och i förlängningen också hur lärare kan undervisa för att främja elevers begreppsbildning i matematik. Det finns idag ingen systematisk undersökning av hur begreppet *begrepp* används. Mitt avhandlingsarbete bottnar i ett intresse för begreppet *begrepp* (eller snarare det engelska begreppet *concept*) som det används i texter i matematikdidaktik och jag valde att göra en begreppsanalys. Denna analys förde mig till det filosofiska fältet och senare också till begreppsforskning inom kognitionsvetenskap. Genom att läsa filosofiska texter fick jag en djupare förståelse av olika synsätt på *begrepp* och denna förståelse bidrog senare i analysen av matematikdidaktiska texter. Här nedan görs först en beskrivning av min studie, som inte följer samma struktur som den som finns i avhandlingen. Denna beskrivning syftar till att förklara vad jag har gjort och varför, och också vad en sådan här typ av studie kan bidra med. Avslutningsvis följer ett kortare avsnitt där avhandlingens olika kapitel beskrivs.

Begreppet *begrepp* inom matematikdidaktik: En begreppsanalys

Sedan mitten av 1900-talet har begreppet *begrepp* varit vanligt förekommande inom det matematikdidaktiska fältet. Det används i kursplaner och också i studier om hur elever förstår olika matematiska fenomen. Det är dock otydligt vad begreppet *begrepp* innebär och det finns idag flera olika synsätt. I avsnitt 1.1 framgår att vissa matematikdidaktiker ser begrepp som beståndsdelar i formella matematiska system, medan andra ser begrepp som mentala representationer som konstrueras av individer. I ytterligare ett synsätt är begrepp symboler tillsammans med hur dessa symboler används. Även utan en djupare förståelse av de olika teorierna är det lätt att inse att de olika synsätten inte går att kombinera.

Att det finns olika synsätt på *begrepp* skulle jag säga beror på att det matematikdidaktiska fältet hämtat inspiration både från psykologi, där begrepp kan betraktas som mentala representationer, och från språkfilosofi och formell matematik, där begrepp vanligen betraktas som ickementala objekt som kan ingå i en språklig kontext eller en axiomatisk struktur. I kapitel 2 görs en historisk beskrivning av hur begrepp har diskuterats inom filosofin, som tar sin utgångspunkt i Platons och Aristoteles idéer. Historien beskrivs sedan via en distinktion mellan rationalism och empirism, en kort redogörelse för det logicistiska projektet, Willard Van Orman Quines och Ludwig Wittgensteins idéer och en förändrad syn på matematikens natur. Syftet med kapitlet är att skapa en filosofisk bakgrundskontext för studien, samtidigt som det introducerar en terminologi som senare används för att beskriva olika synsätt på *begrepp* inom matematikdidaktik.

När olika synsätt på *begrepp* kombineras mellan, eller till och med inom ett och samma, ramverk får det konsekvensen att det blir svårt att förstå vad som beskrivs eller studeras i olika undersökningar. I förlängningen kan det leda till att det även blir svårt att förstå vad lärare ska göra när de undervisar. Därför behöver användningen av ordet 'begrepp' i matematikdidaktiska sammanhang problematiseras. Mot den här bakgrunden blev mitt syfte att göra en begreppsanalys av begreppet *begrepp*, i några vanligt förekommande ramverk som används för att beskriva begreppslig förståelse i matematikdidaktik. Vidare undersöks koherensen i de analyserade texterna. Forskningsfrågan som ställs är:

> Vilka synsätt på *begrepp* finns i texter som använder de valda ramverken, utifrån distinktionerna mentalt relativt ickementalt, intersubjektivt relativt subjektivt och molekylärt relativt holistiskt?

Min avsikt är att bidra till forskningen inom matematikdidaktik, både teoretiskt och metodologiskt. Teoretiskt bidrar studien med en förståelse av olika synsätt på begreppet *begrepp* i de studerade ramverken. Metodologiskt bidrar studien med erfarenheter kring hur begreppsanalyser kan göras.

Före resultatet presenteras krävs några ord om vad de tre distinktionerna innebär. När det gäller mentalt relativt ickementalt, handlar det om i vad mån ett begrepp betraktas som en mental representation eller som ett ickementalt abstrakt innehåll i en representation eller i betydelsen hos ett ord. Här handlar det alltså om vilken typ av objekt begrepp är, något som knyter an till ontologiska frågor. När det gäller intersubjektivt relativt subjektivt handlar det istället om ifall begrepp är något som en grupp av individer har gemensamt eller om olika individer har olika begrepp. Slutligen handlar distinktionen molekylärt relativt holistiskt om olika synsätt på hur begrepp är strukturerade. Enligt den molekylära modellen har begrepp en hierarkisk struktur där komplexa begrepp definieras utifrån enklare begrepp. Enligt den holistiska modellen går det inte att avgränsa eller definiera begrepp och den begreppsliga strukturen är ickehierarkisk.

Analysen visar att det i huvudsak finns två olika synsätt inom fältet. Enligt det första är begrepp mentala representationer som enskilda personer utvecklar. Det finns en intersubjektiv aspekt av dessa begrepp. När jag har utvecklat ett begrepp *triangel* så liknar det i allt väsentligt andra personers begrepp *triangel*. Att olika personer utvecklar liknande begrepp skulle en empiristiskt sinnad person kunna förklara med att det finns ett kausalt samband mellan världen och våra begrepp, där begreppen bildar en mental karta som speglar världen. Enligt det andra synsättet är begrepp ickementala abstrakta objekt och utgör innebörden i symboler, ord och uttryck. Dessa objekt kan antingen ses utifrån en språklig kontext, där de används i kommunikation, eller utifrån att de är beståndsdelar i formell matematik. Inte heller i det här synsättet är begrepp beroende av individen, utan de utgör något som vi har gemensamt. Att begrepp är gemensamma gör att de kan beskrivas i till exempel en kursplan.

I avsnitt 4.1.3 beskrivs de texter som analyseras i studien:

- Vinner och Hershkowitz (1980): Concept image and common cognitive paths in the development of some simple geometrical concepts.
- Tall och Vinner (1981): Concept image and concept definition in mathematics with particular reference to limits and continuity.
- Semadeni (2008): Deep intuition as a level in the development of the concept image.
- Bingolbali och Monaghan (2008): Concept image revisited.
- Sfard (1991): On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin.
- Gray och Tall (1994): Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic.
- Dubinsky (1991): Reflective abstraction in advanced mathematical thinking.
- Asiala m.fl. (1996): A framework for research and curriculum development in undergraduate mathematics education.

De första fyra texterna, som analyseras i kapitel 5, använder det ramverk som utvecklats utifrån begreppen *begreppsbild* och *begreppsdefinition*. Analysen visar att i Bingolbali och Monaghan (2008) finns ett synsätt på vad begrepp är, där begrepp är ickementala och intersubjektiva. I Vinner och Hershkowitz (1980), Tall och Vinner (1981) och Semadeni (2008) finns istället två olika synsätt på *begrepp*, ett där begrepp är ickementala och intersubjektiva och ett där begrepp är mentala och intersubjektiva. De sista fyra texterna (Sfard, 1991; Gray & Tall, 1994; Dubinsky, 1991; Asiala m.fl., 1996), som analyseras i kapitel 6, använder tre olika ramverk. I vart och ett av dessa återfinns de två olika synsätten på *begrepp*, det ickementala och intersubjektiva och det mentala och intersubjektiva.

Vidare förekommer olika idéer om samband mellan begrepp. Som framgår i avsnitt 3.3 och som jag redan har beskrivit ovan innebär ett molekylärt synsätt att begrepp ingår i hierarkiska strukturer där komplexa begrepp är uppbyggda av enklare begrepp. Till exempel kan begreppet *linjär funktion* sägas vara uppbyggt av begreppen *funktion* och *linjäritet*. Dessutom finns det en hierarkisk relation mellan exempelvis kvadrater och rektanglar där en kvadrat är ett specialfall av en rektangel. Denna uppbyggnad kan uttryckas i definitioner. Enligt ett holistiskt synsätt är inga begrepp enklare eller mer grundläggande än andra och definitioner är svårare att uttrycka. Det går inte att tydligt avgränsa ett begrepp från ett annat, vilket gör det svårt att definiera begrepp utan att hamna i cirklar. Medan det molekylära synsättet bygger på ett klassiskfilosofiskt synsätt på *begrepp*, med representanter som Gottlob Frege och klassiska empirister, bygger det holistiska synsättet på idéer utvecklade av Quine och Wittgenstein. Distinktionen mellan ett molekylärt och ett holistiskt synsätt finns också i nutida begreppsforskning inom kognitionsvetenskap, där olika empiriska studier har gjorts för att undersöka hur individer bygger upp kognitiva strukturer.

En vanlig filosofisk ståndpunkt som beskrivs i kapitel 2 är att begreppsliga strukturer i matematik är molekylära och att matematikens begrepp är välbestämda och går att definiera. Huruvida begrepp är molekylära eller holistiska kan dock bero på vad som räknas som samband mellan begrepp, vilket diskuteras i avsnitt 3.3.3. Att en kvadrat är ett slags rektangel visar att det finns en hierarkisk relation mellan begreppen *kvadrat* och *rektangel*. Att ett barn som arbetar med bråk associerar det rationella talet 1/8 med en oktagon som tydligt kan delas in i åtta 'tårtbitar' kan istället ses som ett ickehierarkiskt samband som görs av barnet själv.

I de matematikdidaktiska artiklar som jag har analyserat finns inget entydigt synsätt på hur begrepp är strukturerade, vilket blir tydligt i kapitel 7. En del artiklar har ett större fokus på hierarkiska relationer och andra har ett större fokus på ickehierarkiska relationer. I Sfard (1991), till exempel, är både mentala och ickementala begrepp molekylära, vilket kan jämföras med att i Semadeni (2008) är både mentala och ickementala begrepp holistiska. Vidare finns i en del texter en position där mentala och ickementala begrepp är strukturerade på olika sätt. En sådan position kan ses i Dubinsky (1991) och i Asiala m.fl. (1996), där mentala begrepp har både molekylära och holistiska drag, medan ickementala begrepp är enbart molekylära. Ett annat exempel på att mentala och ickementala begrepp är strukturerade på olika sätt kan ses om Vinner och Hershkowitz (1980) och Tall och Vinner (1981) analyseras tillsammans. Även här har mentala begrepp både molekylära och holistiska drag. När det gäller de ickementala begreppen finns en distinktion mellan matematiska och ickematematiska begrepp, där matematiska begrepp verkar vara molekylära medan ickematematiska begrepp kan ha både molekylära och holistiska drag.

En slutsats som jag har dragit utifrån analysen av Vinner och Hershkowitz (1980) och Tall och Vinner (1981) är att matematiska begrepp i dessa texter har molekylär struktur, att de är hierarkiskt uppbyggda och att de går att definiera, men att andra typer av begrepp kan ha holistisk struktur och inte går att definiera. Utifrån vissa av de analyserade ramverken verkar det också finnas ett synsätt där ickementala strukturer är mer molekylära än mentala strukturer. Orsaken kan vara att de ickementala strukturerna i de matematikdidaktiska texterna ofta är formella axiomatiska system, med begrepp som beståndsdelar.

Det finns dock inte alltid ett tydligt synsätt när det gäller om begrepp är molekylära eller holistiska, vilket visar sig i analyserna av Bingolbali och Monaghan (2008) och Gray och Tall (1994). Då de analyserade texterna ger en otydlig och något spretig bild har jag dragit slutsatsen att det inte finns någon tydlig position utifrån distinktionen mellan molekylära och holistiska synsätt.

Förutom begreppet *begrepp* förekommer tre andra begrepp i de analyserade texterna: *begreppsbild* (på engelska *concept image*), *begreppsuppfattning* (på engelska *conception*) och *schema*. Dessa begrepp beskriver begreppsliga strukturer och är lika på många sätt. Till exempel finns både ett individuellt synsätt på begreppsliga strukturer, där begreppsbilder, begreppsuppfattningar och scheman är individers förståelse av ett fenomen, och ett kulturberoende synsätt, där samma begrepp beskriver den gemensamma förståelse som finns inom en kultur. Det individuella synsättet innebär att begreppsliga strukturer är mentala och subjektiva, men att de också kan ha intersubjektiva drag. Vidare finns två varianter av det kulturberoende synsättet. I den första varianten är begreppsliga strukturer mentala och intersubjektiva och är då integrerade i en gemensam förståelse av världen. I den andra varianten är strukturerna ickementala och intersubjektiva. Här är begreppsliga strukturer den förståelse som visar sig i beskrivningar, definitioner och formell matematik.

Det finns också vissa skillnader, både mellan de olika ramverken och mellan begreppen *begreppsbild*, *begreppsuppfattning* och *schema*. En skillnad är att i de texter som använder *begreppsuppfattning* (Sfard, 1991) och *schema* (Dubinsky, 1991; Asiala m.fl., 1996) beskrivs en utveckling där kognitiva strukturer innehåller mentala begrepp samtidigt som de utvecklas till begrepp på högre nivåer. I denna utveckling går begreppsuppfattningar och scheman från att vara ostrukturerade till att få en hierarkisk struktur. I Dubinsky (1991) och Asiala m.fl. (1996) finns en tydlig utgångspunkt i Piagets syn på lärande där begreppet *schema* används. Här är synen på begreppsutveckling integrerad med synen på *begrepp*. Detta gör att synsättet är mer dynamiskt jämfört med det som finns i Sfard (1991), trots att även denna text bygger på Piagets perspektiv. Dock verkar det som om ordet 'schema' är mer teoretiskt förankrat i Piagets synsätt än ordet 'begreppsuppfattning' som används i en mer klassiskfilosofisk betydelse, med en distinktion mellan *begreppsuppfattning* och *begrepp* och med referenser till Frege. I en jämförelse mellan de ramverk som använder *begreppsuppfattning* och *schema* och de ramverk som använder *begreppsbild* kan jag se att i de första ramverken finns ickementala synsätt på begreppsliga strukturer. Det finns dock inget ickementalt synsätt på begreppsbilder. En annan skillnad är att det inte står i texterna att begreppsbilder utvecklas till begrepp.

Förutom att det finns olika synsätt på vad begreppet *begrepp* är visar analysen att de olika synsätten ibland kombineras eller blandas ihop. Att ett synsätt där begrepp är mentala representationer kombineras med ett synsätt där begrepp är ickementala abstrakta objekt gör att texterna ibland blir svåra att läsa och förstå. Ett exempel där ett ickementalt synsätt följs av ett mentalt är när det först hävdas att det går att ha kunskap om ett begrepp (som då är något som till exempel kan beskrivas i en kursplan) och sedan hävdas att individen själv konstruerar ett begrepp (som då är mentala representationer). Här finns i avhandlingen ett antagande om tre olika arenor: en mental arena, en ickemental arena och en konkret arena. I en koherent text behöver dessa arenor hållas isär.

I avsnitt 7.3 presenteras resultat gällande hur språket används i de analyserade texterna. Här finns exempel på kategorimisstag. Ett sådant är när ord och betydelsen av ord blandas ihop. I analysen av begreppet *begreppsbild* i avsnitt 5.1 har jag till exempel sett att det finns två olika synsätt på relationen mellan begreppsbild och begreppsdefinition. Enligt det ena synsättet särskiljs begreppsbilder och begreppsdefinitioner. Enligt det andra ingår begreppsdefinitionen i begreppsbilden. En konsekvens av att begreppsbilder innehåller begreppsdefinitioner är att mentala representationer innehåller ord, som vanligen inte brukar ses som mentala. Utifrån ett filosofiskt perspektiv blir det då problematiskt att blanda en mental och en ickemental arena. Ett annat exempel där mentalt och ickementalt blandas är i konstruktionen av begreppet *procept* som presenteras i avsnitt 6.3.3. I definitionen beskrivs ett procept som en sammanslagning av tre olika element: en process, ett begrepp och en symbol. Här är det ett slags kategorimisstag att kombinera begreppet *symbol* och begreppet *begrepp* och använda begreppet *procept* som om det ärver egenskaper från de ingående elementen. Till exempel kan begreppet *procept* inte ha begreppslig struktur, då symboler inte har sådan struktur. Dessutom kan ett procept inte vara mentalt eller ickementalt då en symbol inte är en mental eller ickemental mening hos en symbol.

Vidare kan det bli svårt att förstå när ord som 'begreppsbild', 'begreppsuppfattning' och 'schema' används parallellt utan att det klargörs om dessa är synonymer eller vad som skiljer dem åt. Ibland används också ord i fel sammanhang, där till exempel ordet 'begrepp' används istället för 'begreppsuppfattning'. Denna ordanvändning är antagligen en felskrivning, men konsekvensen blir att det blir oklart vad ramverket beskriver.

Metodologiskt bidrar avhandlingen med en metod för att göra begreppsanalyser av matematikdidaktiska texter. I kapitel 4 skriver jag att begreppsanalysen utgår från en konfigurativ litteraturstudie med syftet att hitta inflytelserika idéer i det matematikdidaktiska fältet. I denna litteraturstudie finns ett fokus på bakgrunden till de synsätt på *begrepp* som används idag, för att bättre kunna förstå dem. Litteraturstudien har också fokuserat på aritmetiska begrepp och på äldre barn och vuxna. Orsaken till det senare är intresset för begreppsliga strukturer, som förutsätter att individer har utvecklat viss begreppslig förståelse. Det slutliga urvalet av texter har en gemensam filosofisk bakgrund och de är besläktade genom att de refererar till varandra.

När begreppsanalys används som den huvudsakliga metoden i en studie, kan följande sex faser användas: 1) sätta mål och göra avgränsningar, 2) skapa en allmän förståelse av fältet, 3) sammanställa materialet, 4) utarbeta ett ramverk för analysen, 5) systematiskt analysera materialet, och 6) göra ytterligare analyser och dra slutsatser. I min begreppsanalys ingår två olika litteraturstudier, där den ena användes för att hitta matematikdidaktiska texter att analysera och den andra gjordes inom språkfilosofi och begreppsforskning inom kognitionsvetenskap, med syftet att ta fram ett analysverktyg för begreppsanalysen.

När urvalet av texter hade gjorts påbörjades analysen. I en inledande analys klassificerades olika begrepp utifrån om de hade explikationer eller inte. I avhandlingen används ordet 'explikation', med utgångspunkt hos Rudolf Carnap, i betydelse av en beskrivning som syftar till att göra ett vagt begrepp mer exakt. Sedan klassificerades också samband mellan begrepp, utifrån hur de beskrevs i texterna, som antingen hierarkiska eller ickehierarkiska. Utifrån dessa klassifikationer konstruerades begreppskartor för att representera de olika ramverken på en övergripande nivå och de roller som olika begrepp spelar i ramverken. Begreppskartorna användes som en bakgrundskontext till den egentliga begreppsanalysen, som gjordes med hjälp av det analysverktyg som presenteras i kapitel 3.

I studien används filosofi på två olika sätt. För det första har den en analytiskfilosofisk utgångspunkt med antagandet att en klassisk begreppsanalys kan bidra med nya perspektiv inom det matematikdidaktiska fältet. För det andra utvecklas ett analysverktyg, baserat på en litteraturstudie i språkfilosofi och begreppsforskning inom kognitionsvetenskap. Som jag redan har nämnt bygger analysverktyget på tre distinktioner: mellan mentala och ickementala synsätt, mellan intersubjektiva och subjektiva synsätt, samt mellan molekylära och holistiska synsätt på *begrepp*. Här nedan följer en kort beskrivning av den filosofi som distinktionerna bygger på.

Distinktionen mellan ickementala och mentala synsätt presenteras i avsnitt 3.1. Där representeras den ickementala positionen av språkfilosofiska idéer. Medan ett begrepp hos Frege, Quine, Jerrold Katz och Christopher Peacocke är den abstrakta betydelsen hos ett ord eller uttryck, är ett begrepp enligt Edward Zalta det abstrakta innehållet i en mental representation. Idén att begrepp är ickementala objekt kan jämföras med idén att begrepp är mentala representationer, som representeras av Wittgenstein, empirister som David Hume, John Locke och Carrie Jenkins, samt kognitionsvetare som Gregory Murphy och Susan Carey. Bland tidiga förespråkare för denna position, som till exempel Hume, var representationer mentala bilder som uppstod från sinnesintryck och användes i tänkande och resonemang. Bland senare förespråkare kan mentala representationer istället antas ha en språklig struktur. I ett sådant synsätt, som till exempel finns hos Jenkins, är begrepp ord-lika mentala representationer som antingen baseras på minnen av sinnesintryck eller byggs upp utifrån redan utvecklade begrepp. Idag anser dock många att hypotesen att våra mentala representationer har en språkliknande struktur har en del brister, och att sådana representationer saknar många egenskaper som finns hos ett naturligt språk. Till exempel menar Carey att vi har medfödd begreppslig kunskap.

Distinktionen mellan intersubjektiva och subjektiva synsätt på *begrepp* presenteras i avsnitt 3.2. Här finns egentligen tre olika positioner: en där begrepp är objektiva, en där begrepp är intersubjektiva och en där begrepp är subjektiva. Den objektiva idén representeras av Frege som menar att två olika begrepp är identiska om och endast om de klasser av objekt som begreppen refererar till är identiska och pekas ut av samma betydelse. Vidare menar en del empirister att olika personer utvecklar samma begrepp när de kommer i kontakt med samma värld. I den meningen är begreppen oberoende av individen. Den intersubjektiva positionen representeras av Katz och Quine, som menar att begreppen utvecklas när vi jämför våra erfarenheter genom kommunikation. Det finns också de som anser att våra mentala representationer delvis innebär en gemensam förståelse av världen. Denna gemensamma förståelse är det som gör att vi kan kommunicera med varandra. Slutligen representeras den subjektiva positionen av Zalta som menar att begrepp, som i hans synsätt är det abstrakta innehållet i en mental representation, kan variera från person till person. Begrepp kan också variera över tid. Den subjektiva positionen finns också hos Wittgenstein, som menar att om olika personer använder samma ord för olika fenomen så behöver dessa fenomen inte ha något gemensamt. Tillsammans bildar dessa filosofiska idéer grunden för distinktionen mellan intersubjektiva och subjektiva synsätt på *begrepp*. Då det inte alltid är lätt att avgöra om ett synsätt på *begrepp* i matematikdidaktiska sammanhang är objektivt eller intersubjektivt har den objektiva positionen fått ingå i den intersubjektiva positionen i det analysverktyg som används i avhandlingen.

Den tredje distinktionen mellan molekylära och holistiska positioner presenteras i avsnitt 3.3. En molekylär position kan förstås utifrån ett klassiskt filosofiskt synsätt, där komplexa begrepp kan definieras utifrån enklare begrepp. I en holistisk position går det inte att avgränsa ett begrepp. Det finns samband mellan i stort sett alla begrepp och inga begrepp är mer enkla än andra. Därför blir det svårt att tydligt definiera begrepp utan att hamna i cirklar. Molekylarism och holism kan också ses som två modeller som används på olika sätt i nutida teorier om begrepp inom kognitionsvetenskap. Inom kognitionsvetenskapen förs en teoretisk diskussion utifrån forskning kring hur individer utvecklar begrepp. Förutom den klassiska positionen används prototypteorin och den inferentiella teorin för att exemplifiera idéer om hur begrepp är strukturerade. Utifrån den forskning som görs går det att sluta sig till att ingen av de teorier som används kan helt och hållet förklara hur individer bygger upp sina begreppsliga strukturer och det finns argument för att de olika teorierna behöver kombineras. Dessa argument kan även användas för en ståndpunkt där begreppsliga strukturer har både molekylära och holistiska drag.

Dessa tre distinktioner sattes sedan samman till en tredimensionell matris som kan användas för att analysera text (figur 3.13 i avhandlingen). Denna matris utvecklades i ett samspel mellan den filosofiska litteraturstudien och analysen av de texter som valdes ut i den matematikdidaktiska litteraturstudien. I ett inledande skede innehöll analysverktyget endast distinktionerna mellan intersubjektiva och subjektiva synsätt, och mellan molekylära och holistiska synsätt. Det visade sig dock i analysen av de texter som analyseras i kapitel 6 att distinktionen mellan mentala och ickementala synsätt var viktig. Alla distinktioner användes dock inte i alla analyser. I Bingolbali och Monaghan (2008) och Gray och Tall (1994) syns inga tydliga positioner när det gäller om begrepp har molekylära eller holistiska drag. Därför reducerades analysverktyget till en tvådimensionell matris, där distinktionen mellan molekylära och holistiska synsätt togs bort.

För att de olika texterna skulle tolkas på liknande sätt tog jag fram indikatorer. Dessa används här för att tolka texter, men de kan också användas vid textproduktion för att pröva om en text är koherent och håller sig till ett enda synsätt när det gäller vad begrepp är. Indikatorerna togs fram i ett samspel mellan analysverktyget och de analyserade texterna. I den första läsningen strök jag under formuleringar innehållande ordet 'begrepp' och gjorde noteringar i marginalen. Sedan tolkade jag formuleringarna utifrån om de kunde användas som argument för positionerna i de tre perspektiven. Samma tillvägagångssätt användes för ord som 'begreppsbild', 'begreppsuppfattning' och 'schema'. När den första tolkningen var gjord sorterades formuleringarna först utifrån de olika positionerna och sedan i underkategorier. Dessa kategorier bildade den första versionen av indikatorerna. Efter denna procedur kunde ett synsätt i en viss text till exempel kategoriseras som mentalt och intersubjektivt. Om det då fanns en osäkerhet kring om begrepp var molekylära eller holistiska läste jag texten igen för att se om det fanns textavsnitt som kunde bidra med ytterligare argument. När jag hade samlat ihop alla formuleringar under de olika indikatorerna tittade jag på dessa utifrån den filosofiska litteraturstudien. I något fall kompletterades indikatorerna med argument som kom från den filosofiska diskussionen. Processen blev ett samspel mellan läsning, tolkning och skrivande och indikatorerna utvecklades under hela denna process.

Avhandlingen bidrar med en fördjupad förståelse av synsätt på begreppet *begrepp* och hur olika termer används i det matematikdidaktiska forskningsfältet. I resultatdiskussionen ställs frågan om det behövs två olika typer av begreppsbegrepp i matematikdidaktik, ett där begrepp är mentala representationer och ett där begrepp är ickementala objekt. I så fall behövs också en terminologi som tydligt skiljer på ett mentalt och ett ickementalt synsätt. Vidare för jag en diskussion kring hur texter inom det matematikdidaktiska fältet kan bli mer koherenta. För att få en större koherens behövs explikationer, något som är extra viktigt då det finns olika synsätt på begreppet *begrepp* och begreppen *begreppsbild*, *begreppsuppfattning* och *schema* som alla beskriver kognitiva strukturer.

Den metodologiska diskussionen fokuserar på några aspekter i utformandet av studien. Till exempel förs en diskussion kring hur ett ytterligare steg i litteraturstudien hade kunnat bidra med ett mer systematiskt urval av texter, utifrån kunskap om vilka ramverk som faktiskt skulle analyseras. I ett sådant steg hade jag kunnat göra en mer systematisk sökning efter texter som använde de utvalda ramverken. På så vis hade jag kunnat få en bättre förståelse av synsätt inom dessa ramverk. Dessutom förs en diskussion kring hur användningen av analysverktyget påverkat analysen. Samtidigt som det har bidragit med en fördjupad förståelse av olika synsätt finns det en risk att det har begränsat vad som kunnat komma fram i analysen. Vidare har det ibland varit svårt att analysera texterna med hjälp av analysverktyget. Särskilt gäller detta tolkningen av olika positioner utifrån distinktionen mellan molekylära och holistiska synsätt.

Utifrån att filosofiska idéer används i matematikdidaktik är det intressant att tolka dessa idéer utifrån vilken roll de spelar i matematikdidaktik. Distinktionen mellan mentala och ickementala synsätt på *begrepp* verkar då handla om ifall begrepp utvecklas mentalt av individen eller om begrepp uppstår i en kommunikation i ett klassrum eller i en text, i en semantisk mening. Den ickementala positionen kan också innebära att begrepp finns i en matematisk idévärld. De olika positionerna hänger ihop med olika syner på vad lärande är, då det är en sak att hävda att barn konstruerar begrepp och en annan sak att hävda att de får kunskap om begrepp. När det gäller distinktionen mellan intersubjektiva och subjektiva synsätt på *begrepp* är det möjligt att kombinera dessa. Filosofiska teorier skiljer ibland mellan individuella begrepp och begrepp som vi har gemensamt. I de analyserade texterna används dock inte begrepp i ett sådant subjektivt synsätt, vilket kan bero på att begreppen *begreppsbild*, *begreppsuppfattning* och *schema* används istället. När det gäller frågan om ifall begrepp har molekylära eller holistiska drag verkar det delvis handla om hur den begreppsliga strukturen beskrivs. Om denna struktur definieras utifrån del-helhets-relationer, så får den begreppsliga strukturen molekylära drag. Om strukturen istället innehåller andra typer av samband, som till exempel samband mellan begrepp inom olika matematiska områden, mellan matematiska begrepp och begrepp inom andra vetenskaper, eller mellan matematiska begrepp och vardagliga begrepp, så får den begreppsliga strukturen holistiska drag.

Avslutningsvis är det rimligt att ställa frågan hur resultaten från denna teoretiska studie kan påverka empirisk forskning. För att svara på denna fråga beskrivs idéer kring ett framtida forskningsprojekt för att studera begreppsbildning. I den planerade studien är syftet att utforska hur molekylära och holistiska drag i den kognitiva strukturen kan analyseras via samtal med elever. Studien kommer att fokusera på högpresterande elever då det finns forskning som visar att en skillnad mellan högpresterande och lågpresterande elever är att högpresterande elever kan konstruera samband mellan begrepp, men att den mentala representationen hos lågpresterande ofta är ostrukturerad.

Avhandlingens olika delar

Det har varit omöjligt att skriva denna avhandling i en kronologisk ordning. Processen startade med att jag läste matematikdidaktiska texter. När jag insåg att jag hade svårt att förstå vad begrepp var började jag läsa filosofiska texter, något som sedan ledde mig till begreppsforskning inom det kognitionsvetenskapliga fältet. Syftet med att läsa filosofi var att försöka förstå synsätt på *begrepp* inom matematikdidaktik. Utifrån den filosofiska litteraturstudien utvecklades en första version av analysverktyget, och så var processen igång.

Kapitel 1 innehåller en beskrivning av forskningsintresset, vilket inbegriper en bakgrund med exempel på inkoherens när det gäller användandet av begreppet *begrepp*. Sedan görs en beskrivning av utformningen av studien på en övergripande nivå, som senare utvecklas i kapitlen 2, 3 och 4. Studien placeras också i en kontext utifrån det matematikdidaktiska fältet, där jag argumenterar för att det behövs mer teoretisk forskning. Vidare beskrivs den analytiskfilosofiska utgångspunkten utifrån några olika perspektiv. Slutligen presenteras syfte, forskningsfråga och avhandlingens struktur.

Eftersom studien använder sig av filosofi behövs en filosofisk bakgrund till de teorier som analyseras. Då avhandlingen görs inom matematikdidaktik kan också viss terminologi vara okänd för den tänkta läsaren. Därför innehåller kapitel 2 en förenklad historisk beskrivning av filosofiska diskussioner om begrepp.

I kapitel 3 beskrivs resultatet av den filosofiska litteraturstudien och det analysverktyg som senare används för att tolka texter. Detta analysverktyg utformas utifrån olika synsätt på *begrepp* inom språkfilosofi och begreppsforskning inom kognitionsvetenskap. Som jag redan har beskrivit så innehåller analysverktyget de tre distinktionerna mellan mentala och ickementala synsätt, mellan intersubjektiva och subjektiva synsätt, samt mellan molekylära och holistiska synsätt på *begrepp*.

Metodologin beskrivs sedan i kapitel 4, vilket bland annat inbegriper hur litteraturstudien genomfördes och urvalet av texter. Kapitlet innehåller också en beskrivning av den inledande analysen, hur analysverktyget har använts, de indikatorer som tagits fram för att tolka texterna och av studieobjektet. Sedan avslutas kapitel 4 med en sammanfattning av metodologin, med filosofiska antaganden och några kommentarer kring studiens utformning.

Kapitel 5 och 6 består av begreppsanalyser av de valda texterna. I kapitel 5 analyseras olika versioner av det ramverk som utvecklades av Vinner och Hershkowitz (1980) och av Tall och Vinner (1981) och som baseras på begreppet *begreppsbild*. Först analyseras det ursprungliga ramverket. Sedan analyseras en version som används i Semadeni (2008) och en version som används i Bingolbali och Monaghan (2008). Efter att ramverken presenterats via begreppskartor diskuteras begreppen *begrepp*, *begreppsbild* och *begreppsdefinition*. I kapitel 6 analyseras de ramverk som används i Sfard (1991), Gray och Tall (1994), Dubinsky (1991) och Asiala et al. (1996). Även i denna analys beskrivs ramverken först via begreppskartor innan begreppen *begrepp*, *begreppsuppfattning*, *procept* och *schema* undersöks.

Sedan sammanfattas resultaten av analyserna i kapitel 7, som också innehåller slutsatser. Först jämförs olika synsätt på begreppet *begrepp*. Sedan jämförs olika synsätt på begreppen *begreppsbild*, *begreppsuppfattning* och *schema*. När detta är gjort förs en diskussion kring hur terminologin används i de olika texterna. Till exempel presenteras kategorimisstag som görs när en mental, en icke-mental och en konkret arena blandas ihop, eller när ord och betydelsen av ord kombineras. Vidare beskrivs hur indikatorerna kan användas för att säkerställa att endast ett synsätt på *begrepp* används i en text, och hur en kombination av olika synsätt därmed kan undvikas. Kapitlet avslutas med en kort sammanfattning av studien.

Slutligen innehåller kapitel 8 en diskussion kring studiens resultat och metodologi. Vidare förs en diskussion kring hur den filosofi som används i studien kan tolkas utifrån ett matematikdidaktiskt perspektiv och hur de teoretiska resultaten kan användas i framtida empirisk forskning.

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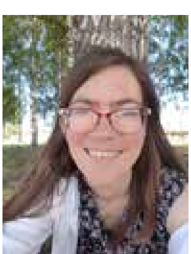
The concept *concept* in mathematics education:
A concept analysis

The notion *concept* is used in different ways within the field of mathematics education. Here, the aim is to carry out a concept analysis of the notion *concept*, within some frequently used frameworks describing conceptual understanding. The main findings of the study highlight two categories of views on *concept* within the texts: mental and intersubjective views, and non-mental and intersubjective views. One difference between these views is whether conceptual structures have molecular or holistic features. The thesis deepens the understanding of how terminology is used in mathematics education and discusses how terminology may be used coherently.



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