Master's thesis

Volatility Curves of Incomplete Markets

The Trinomial Option Pricing Model

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Thesis for the Degree of Master Science

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Abstract

The graph of the implied volatility of call options as a function of the strike price is called volatility curve. If the options market were perfectly described by the Black-Scholes model, the implied volatility would be independent of the strike price and thus the volatility curve would be a flat horizontal line. However the volatility curve of real markets is often found to have recurrent convex shapes called volatility smile and volatility skew. The common approach to explain this phenomena is by assuming that the volatility of the underlying stock is a stochastic process (while in Black-Scholes it is assumed to be a deterministic constant). The main purpose of this project is to propose and explore the idea that the occurrence of non-flat volatility curves is the result of market incompleteness. A market is incomplete if it admits more than one risk-neutral probability. In other words, within an incomplete market, investors do not necessarily agree on the market price of risk. The hypothesis that volatility curves are linked to market incompleteness is, at least from a qualitative perspective, reasonable and justified, since the convex shape of volatility curves indicates that investors demand an extra premium for call options which are out of the money, that is to say, they assume that out-of-the money options are more risky than predicted by Black- Scholes. Mathematically this means that investors use a different risk-neutral probability to price call options with different strikes. This hypothesis will be tested quantitatively by using the trinomial model, which is the simplest example of one- dimensional incomplete market.

Keywords: Implied volatility, Incomplete markets, Trinomial option pricing model, Black-Scholes option pricing model, Risk-neutral probability.

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Contents

1

Introduction

Option pricing theory is one of the core topics in financial mathematics. It has a long and illustrated history with a revolutionary breakthrough made by Fisher Black and Myron Scholes [8] in 1973. They derived the first option pricing model with the closed formula to calculate the price of European options on a stock that pays no dividends. In the same year, Robert Merton extended their model in several important ways with a more general approach. This development has become known as Black–Scholes–Merton (or Black–Scholes) option pricing model. Up to this day, this model has a huge influence on the way traders price derivatives and is widely used [9].

The Black-Scholes model provides a unique theoretical price of an option in a complete market based on the fundamental principle of absence of arbitrage opportunities. By this pricing formula, the discounted price of the stock of a European derivative is a martingale under the corresponding risk-neutral measure. The Black-Scholes model also implies the constant volatility assumption. Nevertheless, it is an empirical fact that in the real world, prices do not correspond to the fixed value of volatility as theory requires [7]. The solution adopted by traders is to extract implied volatilities from market prices of options. Implied volatility can be seen as what the market believes the volatility should be and depends on the strike price and maturity of an option. A plot of the implied volatility as a function of the strike price is commonly referred as the implied volatility smile [5].

Another example of complete market and a very popular and useful technique for valuing options is the binomial option pricing model. It was firstly introduced by John Cox, Stephen Ross and Mark Rubinstein [9] in 1979. This approach has been widely used due to its simplicity, and the fact that it is more accurate than the Black-Scholes model, particularly for longer-dated options on securities with dividend payments. Subsequently, the trinomial model was formulated by Phelim Boyle [10] in 1986 as an extension of the binomial model. The trinomial asset pricing model introduces three possible price movements that an underlying asset can have in one time period. It incorporates that the risk-neutral probability measure is not unique and that the price of the European derivative is also not uniquely defined.

The trinomial asset pricing model is the simplest example of an incomplete market.

The purpose of this thesis is to investigate the possible connection between the implied volatility and incomplete markets. The common approach to resolve this shortcoming of the Black–Scholes model is by assuming that the volatility of the underlying stock is a stochastic process itself. While stochastic volatility models can reproduce the smile, they have a major disadvantage. For general stochastic volatility models, the pricing equations of European derivatives are not analytically solvable, and they are very hard to calibrate. Examples of stochastic volatility models are Heston, SABR and GARCH models [14]. This thesis approaches this problem in a different way, namely the trinomial option pricing model as an example of incomplete market is used to justify the volatility curves.

The thesis begins with an introduction to the risk-neutral probability measure and pricing under it including some basic notions about the option pricing of a standard European derivative. It also covers the theory behind implied volatility and incomplete markets. Chapter three is devoted to formulating the trinomial model and its interpretation in a probabilistic sense. The conditions of the existence of the risk-neutral (or martingale) probability measure are found for the general model and for important special case. Thereafter, the conditions under which the stock price converges to the geometric Brownian motion in the time continuous limit are investigated. Two cases will are treated separately, videlicet with and without the recombination condition. The chapter ends with a discussion dedicated to the trinomial option pricing theory and the fair price of a European derivative will be derived. The forth chapter is dedicated to the empirical analysis where Matlab is used for numerical computations. The purpose of this chapter is to numerically see the connection between the implied volatility and our incomplete model. Worth to mention, that only European call option is included in the analysis through the thesis.

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Background

The first part of this chapter gives a brief introduction to the risk-neutral measure and pricing under it. Further, two fundamental theorems of asset pricing are presented as a part of the discussion dedicated to the markets incompleteness. Thereafter, some basic notions about the option pricing of a standard European derivative including the binomial and Black-Scholes models are introduced. The purpose of the last section is to provide an introduction to the theory behind the implied volatility.

2.1 Risk-Neutral Probability Measure

Throughout this section, we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given and the filtration $\{\mathcal{F}_W(t)\}_{t\geqslant 0}$ is generated by a Brownian motion $\{W(t)\}_{t\geqslant 0}$. One can interpret $\mathbb{P}$ as the actual probability measure or the real-world probability.

Definition 2.1.1. A probability measure $\widetilde{\mathbb{P}}$ is called a risk-neutral (or martingale) probability measure if the following is true:

(i) $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, i.e., for every $A \in \mathcal{F}$, $\mathbb{P}(A) = 0$ if and only if $\widetilde{\mathbb{P}}(A) = 0$.

(ii) The discounted price of the stock $\{S^*(t)\}_{t\geqslant 0}$ is a $\widetilde{\mathbb{P}}$-martingale relative to filtration $\{\mathcal{F}_W(t)\}_{t\geqslant 0}$.

Recall that $S^*(t) = e^{-rt}S(t)$ is called the discounted price of a stock, where r is the risk-free rate of the money market. It means that at time $t = 0$ the amount $S^*(t)$ should be invested in the risk-free asset in the way that it replicates the value of the stock at time t. Denote $\widetilde{\mathbb{E}}$ the conditional expectation in the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. Then $S^*(t)$ is a martingale if and only if

$$\widetilde{\mathbb{E}}[S^*(t_2) \mid \mathcal{F}_\Omega(t_1)] = S^*(t_1), \quad t_2 \geqslant t_1$$

where $\{\mathcal{F}_S(t)\}_{t\geqslant 0}$ is the filtration generated by $\{S(t)\}_{t\geqslant 0}$. Moreover, since martingales have constant expectation $\widetilde{\mathbb{E}}[S(t)] = S_0 e^{rt}$ [3]. Put in another words, in the risk-neutral (or martingale) probability measure, the return of the stock is the same as of the risk-free asset.

Let $\{(h_S(t), h_B(t)\}_{t\geqslant 0}$ be a self-financing portfolio process consisting of $h_S(t)$ shares of stock and $h_B(t)$ shares of the risk-free asset at time t. Then its value $\{V(t)\}_{t\geqslant 0}$ is given by $V(t) = h_B(t)B(t) + h_S(t)S(t)$.

Theorem 2.1.1. *If $\{(h_S(t), h_B(t)\}_{t\geqslant 0}$ is a self-financing portfolio process with value $\{V(t)\}_{t\geqslant 0}$, then the discounted portfolio value $V^*(t) = e^{-rt}V(t)$ is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geqslant 0}$, that is*

$$V^*(t) = \widetilde{\mathbb{E}}[V^*(T) \mid \mathcal{F}_\Omega(t)], \quad t \leqslant T \tag{2.1}$$

Definition 2.1.2. An arbitrage is a portfolio value process whose value $\{V(t)\}_{t\geqslant 0}$ satisfies the following set of conditions:

$$V(0) = 0, \quad \mathbb{P}\{V(T) \geqslant 0\} = 1 \quad \mathbb{P}\{V(T) > 0\} > 0.$$

Thus, an arbitrage opportunity is a self-financing strategy where one starts with zero initial value and, at some future time T, is sure to have a non-negative final value and furthermore a positive probability of having positive final value, thereby making a profit [6].

2.2 Incomplete Markets

Theorem 2.2.1 (First fundamental theorem of asset pricing)**.** *The market is arbitrage-free if there exists a risk-neutral probability measure $\widetilde{\mathbb{P}}$.*

We will consider only the European-style derivative on the stock which means that the derivative can be exercised only at the maturity time T. Let Y be a $\mathcal{F}_W(T)$-measurable random variable with finite expectation. The fair price of the derivative at time $t < T$ we will denote as $\Pi_Y(t)$. By definition $\Pi_Y(t)$ equals the value of self financing-hedging portfolio $V(t)$. Having that the hedging condition is $V(T) = Y$ (that is $V(T, \omega) = Y(\omega)$ for all $\omega \in \Omega$) and due to the equation (2.1), we may write the risk-neutral pricing formula:

$$\Pi_Y(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[Y \mid \mathcal{F}_\Omega(t)] \tag{2.2}$$

The next result is the key for the further development.

Theorem 2.2.2 (Second fundamental theorem of asset pricing)**.** *An arbitrage-free market is complete if and only if the risk-neutral probability measure is unique.*

Therefore, if we assume that the market is free of arbitrage, meaning that there exists an equivalent martingale measure $\widetilde{\mathbb{P}}$, but it is not uniquely defined, the market will be considered incomplete [6]. Moreover, in this case, we have to choose one of this measures to price the derivative. Hence, when the market is incomplete, the risk-neutral price of the European derivative is not unique.

2.3 Option Pricing

We start this section with a small discussion dedicated to the option pricing theory. Recall that an option is a security that gives the right to buy or sell an asset at a fixed price within a specified period of time. The fixed price that is paid for the asset when the option is exercised is called the strike price and the given date on which the option may be exercised is called the expiration date. A call option gives the right to buy and a put option to sell. An American-type option is the one that can be exercise at any given time prior to maturity. A European option can be exercise only at maturity [8].

The option price is determined by the market as the fair value that the buyer is willing to pay and the seller is willing to get in order to create a binding contract. What is more, the fair price should be found in a way that it excludes arbitrage opportunities. In other words, neither of parties should benefit from the transaction, namely neither the buyer nor the seller is able to make the guaranteed risk-free profit from buying or selling the derivative [1].

The fair price of the derivative is the quantity that is needed to open a self-financing hedging portfolio. In order to understand this logic, assume that an option is sold for $\Pi_Y(t)$ at time t, and that the seller invests this amount in a $1 + 1$ dimensional market consisting of an underlying asset and a risk-free asset. Moreover, assume that the portfolio is self-financing, i.e., there is no movement of money in or out of the portfolio, and the value of the portfolio at maturity T is equal to the payoff Y of the option. From the above, we have that portfolio is hedging the option, and therefore one can conclude that the fair price $\Pi_Y(t)$ of the derivative should be equal to the value of a self-financing hedging portfolio [2]. Mathematically it means that the fair value of the option is equal to its discounted expected payoff at the expiration.

The binomial option pricing model is the simplest method to price an option. The option price under the Black-Scholes framework also has an analytical solution for the fair price, see equation (2.4) below. Both models provide the unique risk-neutral price of the derivative, and therefore the binomial and the Black-Scholes market are considered to be complete.

2.4 Binomial Option Pricing

One of examples of complete market is the binomial option pricing model. It is a simple time-discrete model for valuing options [13]. Let us denote by $S(t)$ the binomial stock price at time t. The model is interested in monitoring the evolution of the stock price on some finite time interval $[0, T]$ which can be viewed as the life of an option. The maturity day of the option $T > 0$ and the price of the stock at time $t = 0$, i.e., $S(0) = S_0$ are known.

The underlying assumption is that the stock price follows a random walk. Hence, the binomial stock price can only change at some given pre-defined time steps $0 = t_0 < t_1 < ... < t_N = T$. We assume that the time steps are equidistant, so for some $h < T$ the following is true $t_i - t_{i-1} = h > 0$ for all $i = 1, ..., N$.

By letting u, $d \in \mathbb{R}$, $u > d$ and $p \in (0, 1)$, the binomial stock price is determined by

$$S(t) = \begin{cases} S(t-1)e^u, & \text{with probability } p \\ S(t-1)e^d, & \text{with probability } 1-p \end{cases} \tag{2.3}$$

for all $t \in \mathcal{I} = \{1, ..., N\}$. If $S(t) = S(t)e^u$ the stock price goes up at time t and if $S(t) = S(t-1)e^d$ down respectively (in the application $u > 0$ and $d < 0$ are usually chosen). We may interpret p as the real-world or physical probability. The possible stock prices at time t belong to the set $\{S_0 e^{N_u(t)u+(t-N_u(t))d}, N_u(t) = 0, ..., t\}$, where $N_u(t)$ stands for the number of times the price can go up until the time t included. It follows that there exists 2^N possible paths of the stock price in a N-period model.

Given $p \in (0, 1)$, the probability space $\Omega_N = \{H, T\}^N$, $\mathbb{P}_p(\{\omega\}) = p^{N_H(\omega)}(1-p)^{N_T(\omega)}$ is called the N-coin probability space. Here $N_H(\omega)$ is the number of heads in the toss $\omega \in \Omega_N$ and $N_T(\omega)$ is the number of tails. The binomial stock price is a stochastic process defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$. Let us consider the following random variable:

$$X_t : \Omega_N \to \mathbb{R}, \quad X_t(\omega) = \begin{cases} u, & \text{if the } t^{th} \text{ toss } \omega \text{ is H} \\ d, & \text{if the } t^{th} \text{ toss } \omega \text{ is T} \end{cases}$$

We consider that the binomial stock price can be interpreted as a stochastic process defined on this probability space. The equation (2.3) then can be rewritten as

$$S(t) = S(t-1)\exp[(\frac{u+d}{2} + \frac{(u-d)X_t}{2})]$$

and by iteration

$$S(t) = S_0\exp[(\frac{u+d}{2} + \frac{(u-d)}{2})M_t], \quad M_t = X_1 + ... + X_t, t \in \mathcal{I}$$

Thus, $S(t) : \Omega_N \to \mathbb{R}$ is a random variable and $\{S(t)\}_{t\in\mathcal{I}}$ is a discrete stochastic process on the probability space $(\Omega_N, \mathbb{P}_p)$. Moreover, $S(1,\omega), ..., S(N,\omega))$ is a path for this stock price for each $\omega \in \Omega_N$.

Recall from the previous chapter that a $1+1$ dimensional market is a market consisting of of a risky asset such that the price of this asset is given by the binomial model and a risk-free asset. The value of the risk-free asset at time t is given by a deterministic function of time $B(t) = B_0e^{rt}$ [1]. Here we assume that interest rate r of the money market is constant, and that the initial value of the risk-free asset $B_0 = B(0)$ is known.

Recall that the discounted price of the stock is defined as $S^*(t) = e^{-rt}S(t)$ and that by $\mathbb{E}_p$ we denote the conditional expectation in the corresponding probability space $(\Omega_N, \mathbb{P}_p)$.

Theorem 2.4.1. *If $r \neq (u,d)$, there exists a probability measure $\mathbb{P}_p$ on the sample space Ω_N such that the discounted stock price $\{S^*(t)\}_{t\in\mathcal{I}}$ is a martingale. Moreover, for $r \in (u,d)$, $\{S^*(t)\}_{t\in\mathcal{I}}$ is a martingale only in the special case when $p = q$, where*

$$q = \frac{e^r - e^d}{e^u - e^d}$$

Thus, due to the Theorem 2.4.1, the probability measure $\mathbb{P}_q$ is called the martingale probability measure. The focus of this thesis is the trinomial option pricing model which is an extension of the binomial model and which will be investigated more profoundly in the next chapter.

2.5 Black-Scholes Pricing

Another example of complete market is Black-Scholes. In this section we consider the Black-Scholes market consisting of a risky asset, i.e., a non-dividend-paying stock, a risk-free asset with a constant interest rate $r > 0$, and options on the stock.

In the Black-Scholes theory, it is assumed that the stock price $\{S(t)\}_{t\in[0,T]}$ is given by the geometric Brownian motion

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}$$

where r is the constant interest rate, $\widetilde{W}(t)$ is a Brownian motion in the risk-neutral probability measure and $\sigma > 0$ is the volatility of the stock price. The risk-free asset price $\{B(t)\}_{t\in[0,T]}$ is given by $B(t) = B(0)e^{rt}$.

Definition 2.5.1. Consider a simple derivative of a European type with a pay-off function $Y = g(S(T))$ at time of maturity $T > 0$. The Black-Scholes price of the derivative at time $t \in [0,T]$ is defined as

$$\Pi_Y(t) = v_g(t, S(t))$$

where

$$v_g(t,x) = e^{-r\tau}\widetilde{\mathbb{E}}[g(x)e^{(r-\frac{\sigma^2}{2})\tau+\sigma(\widetilde{W}(T)-\widetilde{W}(t))}) \mid F_w(t)]$$

where $\tau = T - t$ is the time left to the expiration of the derivative. Since the market parameters are constant, then $\mathcal{F}_W(t) = \mathcal{F}_{\widetilde{W}(t)}$, and the increment $\widetilde{W}(T) - \widetilde{W}(t)$ is independent of $\mathcal{F}_{\widetilde{W}(t)}$, the pricing function v_g above becomes

$$v_g(t,x) = e^{-r\tau}\widetilde{\mathbb{E}}[g(x)e^{(r-\frac{\sigma^2}{2})\tau+\sigma(\widetilde{W}(T)-\widetilde{W}(t))})] \tag{2.4}$$

$$= \frac{e^{r\tau}}{\sqrt{2\pi}}\int_{\mathbb{R}} g(x)e^{(r-\frac{\sigma^2}{2}\tau)+\sigma\sqrt{\tau}y})e^{-\frac{y^2}{2}}dy$$

The Black-Scholes model assumes that the price of an asset follows a geometric Brownian motion with constant drift and volatility [12]. However, it is an empirical fact that in the real world, prices do not correspond to the fixed value of volatility as theory requires. The difference between the Black-Scholes price and the market price is expressed in terms of the implied volatility of the derivative, which is discussed in more details in the section below.

2.6 Implied Volatility

In this section we will focus our discussion on the European call option. We thereby assume that the pay-off is given by

$$Y = (S(T) - K)_+, \quad i.e., \quad Y = g(S(T)), \quad g(z) = (z - K)_+$$

In the Black-Scholes model the price of the European call option with the strike price $K > 0$ and fixed maturity $T > 0$ on a stock with price $S(t)$ at time t is given by the following formula

$$C(t, S(t), K, T) = S(t)\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \tag{2.5}$$

where $r > 0$ is the constant interest rate of the money market, $\tau = T - t$ is the time left to the expiration of the call,

$$d_2 = \frac{ln\frac{S(t)}{K} + (r - \frac{\sigma^2}{2}\tau)}{\sigma\sqrt{\tau}}$$

$$d_1 = d_2 + \sigma\sqrt{\tau}$$

and Φ denoted the standard normal distribution

$$\Phi(z) = \int_{-\infty}^{z} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$$

Under the above setting, the interest rate and the volatility are model parameters which we want to estimate. Maturity T and strike price K differ for every option. From all the quantities listed above, the volatility is the one that is more difficult to estimate [2]. Therefore, we re-denote the price as

$$C(K, T, \sigma)$$

implying that we fix time t and price $S(t)$, and we disregard the dependence on r. Now assume that at some given fixed time t, we observe the real market price of a European call option, say $C_{obs}(K, T)$, with maturity time T and strike price K.

Definition 2.6.1. The implied volatility σ_{imp} of a European call option is a strictly positive solution of the equation

$$C_{obs}(K, T) = C(K, T, \sigma_{imp}) \tag{2.6}$$

In other words, the σ_{imp} is the volatility that, when substituted into the Black-Scholes formula, makes the theoretical call price agrees with the market price. Note that the implied volatility is a function of T, K.

Proposition 2.6.1. There can only be at most one solution to the Equation (2.6), and if

$$C_{obs}(K, T) \in ((S(t) - Ke^{-rt})_+, S(t)) := U \tag{2.7}$$

there exists exactly one strictly positive solution.

Proof. If the residual time τ is strictly positive

$$\frac{\partial C}{\partial \sigma} = S(t)\varphi(\frac{ln\frac{S(t)}{K} + (r + \frac{\sigma^2}{2}\tau)}{\sigma\sqrt{\tau}})\sqrt{\tau}$$

$$= S(t)\varphi(d_1)\sqrt{\tau} = S(t)e^{-\frac{d1^2}{2}}\sqrt{\frac{\tau}{2\pi}} > 0$$

It is easy to see that

$$\lim_{\sigma \longmapsto 0+} C(t, S(t), K, T, \sigma) = \lim_{\sigma \longmapsto 0+} e^{-r\tau} E[(S(t)e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y} - K)_+] = (S(t) - Ke^{-r\tau})_+$$

and

$$\lim_{\sigma \longmapsto \infty} C(t, S(t), K, T, \sigma) = \lim_{\sigma \longmapsto \infty} \{S(t)\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)\} = S(t)$$

□

Hence, $C(K, T, \cdot)$ is strictly increasing and takes values in $U \subset (0, \infty)$, by which the result follows.

Therefore, under the condition (2.7) there will always exist a unique solution σ_{imp} of the equation (2.6). If the Black-Scholes model was correct, then $\sigma_{imp}(K, T) = \sigma = constant$ for all K and T. However, empirical results indicate that in real world this is not true and that the implied volatility as a function of K is most often not a flat curve. Not only is the volatility surface not flat but it actually varies, often significantly, with time. This effect has been commonly referred as the implied volatility smile (a U-shaped curve) or skew (a downward sloping curve) or smirk (a downward sloping curve with increase for large strike price). Therefore, the implied volatility $\sigma_{imp}(K, T)$ can be seen as the quantitative measure of how real market deviates from the Black-Scholes option pricing theory. The empirical evidence thus shows that is does not seem plausible to price options with the constant volatility assumption and few possible reasons were proposed to explain this effect, one of which suggests that the market imperfection exists [11].

3

The Trinomial Model

3.1 Foundation of the Trinomial Model

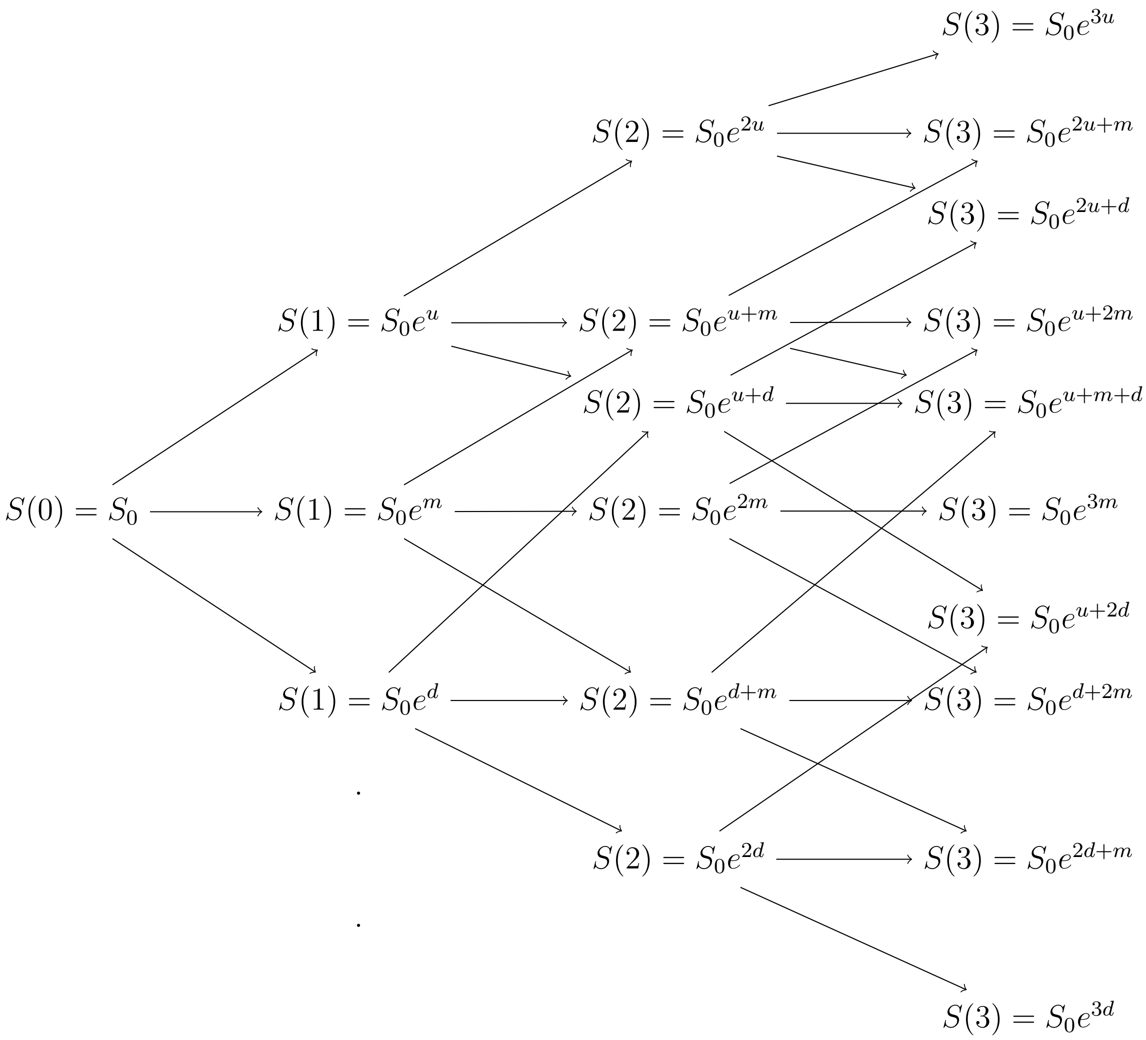
In this chapter, we will discuss the trinomial model as an example of incomplete market. As was mentioned previously, the trinomial model is an extension of the binomial model that incorporates three possible values that an underlying asset can have in one time period [4]. Within this chapter we formulate the basic concepts of the trinomial model and its probabilistic interpretation. Thereafter, the conditions under which this model converges to the geometric Brownian motion are investigated. The chapter ends with a discussion dedicated to the trinomial option pricing theory and the fair price of a European derivative will be derived.

We start by assuming that the asset price at the present time is known, i.e. $S(0) = S_0 > 0$, the possible values of $S(t)$ are:

$$S(t) = \begin{cases} S(t-1)e^{u}, \text{with probability } p_u \\ S(t-1)e^{m}, \text{with probability } p_m \\ S(t-1)e^{d}, \text{with probability } p_d, \end{cases}$$

where $u > m > d$, $p_u, p_m, p_d \in (0,1)$ and $p_u + p_m + p_d = 1$. We assume that the risky asset is a stock and that the risk-free asset has value $B(t) = B_0 e^{rt}$, $t \in \mathcal{I} = \{1, ..., N\}$, r is constant.

The figure below shows the general 3-period trinomial tree with 6 possible values of $S(N)$ when $N = 2$, and 10 possible values when $N = 3$ respectively.



Let us compute the number of possible prices at time t in the $N-$period binomial model and show that it is growing quadratically. The possible stock prices at time t belong to the set

$$\begin{aligned} A =& \{S_0e^{N_u(t)u+N_m(t)m+N_d(t)d}, N_u(t) = 0, ..., t; N_m(t) = 0, ..., t; N_d(t) = 0, ..., t; \\ & N_u + N_m + N_d = t\}, \end{aligned}$$

where N_u, N_m, N_d are the numbers of times that the stock price changes with rates u, m and d respectively.

It is clear that $N_u + N_m + N_d = t$, so the total number of elements of the set A is equal to the total number of integer positive solution of the following equation:

$$x_1 + x_2 + x_3 = t$$

When considering the general case, the equation $x_1 + ... + x_k = t$ has $\binom{t+k-1}{k-1}$ integer positive solutions. So in our case there are

$$\binom{t+3-1}{3-1} = \binom{t+2}{2} = \frac{(t+2)!}{2!t!} = \frac{(t+1)(t+2)}{2}$$

3.2 The Possible Stock Prices at time t

For simplification purposes, we shall later reduce the number of nodes by imposing the recombination condition of the following form

$$m = \frac{u+d}{2}$$

and thus restrict the trinomial stock price to

$$S(t) = \begin{cases} S(t-1)e^{u}, \text{with probability } p_u \\ S(t-1)e^{\frac{u+d}{2}}, \text{with probability } p_m \\ S(t-1)e^{d}, \text{with probability } p_d \end{cases} \tag{3.1}$$

with $u > d, \;\; t \in \mathcal{I}$.

Making use of the recombination condition, we obtain the 3-step trinomial tree with 7 nodes at $t = 3$.

We know show that in this case the number of possible stock prices at time t in the trinomial model is $2t + 1$. Recall that $N_u(t), N_m(t), N_d(t)$ are the numbers of times the stock changes in value with rates u, m and d respectively. For simplicity denote $N_u(t) = k, N_m(t) = l$ and $N_d(t) = t - (k + l)$. In our case, we have that $m = \frac{u+d}{2}$, which gives

$$\begin{aligned} ku + l\frac{u+d}{2} + (t-k-l)d = \\ = u(k+\frac{l}{2}) + (t-k-\frac{l}{2})d = \\ = uv + (t-v)d, \end{aligned}$$

where $v = k + \frac{l}{2}$.

So now the general form of the states of prices can be represented as $uv + (t - v)d$, where v takes its value from the set A, such that $A = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ..., t\}$. Therefore, the number of elements of the set A equals $2t + 1$.

Hence, we obtain that the trinomial model with the recombination condition has the linear rate of growth of number of nodes, as the binomial model. Moreover, as in the application of the binomial model, we shall later assume that $u = -d$.

3.3 Martingale Probability Measure

Let $\Omega = \{-1, 0, 1\}^N$ and $p = (p_u, p_m, p_d)$, we define probability $\mathbb{P}_p$ on the sample space Ω by letting

$$\mathbb{P}_p(\omega) = p_u^{N_+(\omega)} p_m^{N_0(\omega)} p_d^{N_-(\omega)}$$

where $p_u, p_m, p_d \in (0, 1)$, $p_u + p_m + p_d = 1$ and $N_\pm(\omega)$ is the number of ± 1 and $N_0(\omega) = N - N_+(\omega) - N_-(\omega)$ is the number of zeroes.

The trinomial stock price $S(t) : \Omega \to \mathbb{R}$ and $\{S(t)\}_{t \in \mathcal{I}}$ is a stochastic process in the probability space $(\Omega, \mathbb{P}_p)$.

Theorem 3.3.1. *There exists a probability measure $\mathbb{P}_p$ on the sample space Ω_N such that the discounted stock price $\{S(t)\}_{t \in \mathcal{I}}$ is a martingale if and only if $p = q = (q_u, q_m, q_d)$ where*

$$\begin{cases} e^u q_u + e^m q_m + e^d q_d = e^r \\ q_u + q_m + q_d = 1 \\ q_u, q_m, q_d > 0 \end{cases} \tag{3.2}$$

Proof. The process $S^*(t) = e^{-rt} S(t), t \in \mathcal{I}$ is a martingale for all $t \in \mathcal{I}$ if and only if

$$\mathbb{E}_p[e^{-rt} S(t) | S^*(1), ..., S^*(t-1)] = e^{-r(t-1)} S(t-1)$$

The expectation conditional to $S^*(1), ..., S^*(t-1)$ is the same as taking the expectation conditional to $S(1), ..., S(t-1)$, therefore

$$\begin{aligned} &\mathbb{E}_p[e^{-rt} S(t) | S^*(1), ..., S^*(t-1)] = \\ &= \mathbb{E}_p[e^{-rt} S(t) | S(1), ..., S(t-1)] = \\ &= e^{-rt} \mathbb{E}_p[S(t) | S(1), ..., S(t-1)] \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathbb{E}_p[S(t) | S(1), ..., S(t-1)] = \\ &= \mathbb{E}_p[\frac{S(t)}{S(t-1)} S(t-1) | S(1), ..., S(t-1)] = \\ &= S(t-1) \mathbb{E}_p[\frac{S(t)}{S(t-1)} | S(1), ..., S(t-1)] = \end{aligned}$$

where we used that $S(t-1)$ is measurable with respect to the conditional variables.

We know that the random variable

$$\frac{S(t)}{S(t-1)} = \begin{cases} e^u, \text{with probability } q_u \\ e^m, \text{with probability } q_m \\ e^d, \text{with probability } q_d \end{cases}$$

is independent of $S(t), ..., S(t-1)$, and therefore we have

$$\mathbb{E}_p[\frac{S(t)}{S(t-1)}|S(1), ..., S(t-1)] =$$
$$= \mathbb{E}_p[\frac{S(t)}{S(t-1)}] = e^u q_u + e^m q_m + e^d q_d$$

Then,

$$\mathbb{E}_p[e^{-rt}S(t)|S^*(1), ..., S^*(t-1)] =$$
$$= e^{-rt}S(t-1)(e^u q_u + e^m q_m + e^d q_d) =$$
$$= e^{-r(t-1)}S(t-1)$$

if and only if $e^u q_u + e^m q_m + e^d q_d = e^r$. So

$$\begin{cases} e^u q_u + e^m q_m + e^d q_d = e^r \\ q_u + q_m + q_d = 1 \\ q_u, q_m, q_d > 0 \end{cases}$$

are conditions for $S^*(t) = e^{-rt}S(t)$ to be a martingale. □

Let us now study condition of existence of martingale probability measure. We know that there exists infinitely many triples (q_u, q_m, q_d) that satisfy (3.2) when $m = \frac{u+d}{2}$. The solution of the equations in the system (3.2) can be written in the parametric form as

$$q_u = \frac{e^r - e^d}{e^u - e^d} - \omega \frac{e^{d/2}}{e^{u/2 + e^{d/2}}}, q_m = \omega, q_d = \frac{e^u - e^r}{e^u - e^d} - \omega \frac{e^{u/2}}{e^{u/2} + e^{d/2}} \tag{3.3}$$

where ω is a free parameter, and r, u, d are market parameters. It remains to show when the inequalities $q_u + q_m + q_d > 0$ are satisfied. We study first a special case.

Theorem 3.3.2. *Let $r > 0$, $u > 0$ and $u = -d$. q_u, q_d, q_m are probabilities that satisfy (3.3) if and only if*

$$u > r, \quad 0 < \omega = \frac{e^u - e^r}{e^u - 1} \tag{3.4}$$

where

$$q_u + q_d + q_m = 1, \quad q_u, q_d, q_m > 0 \tag{3.5}$$

Proof. The equality in 3.5 holds always. Having that $r > 0, u > 0$ and $u = -d$, then for q_d we have

$$q_d = \frac{e^{-d} - e^r}{e^{-d} - e^d} - \omega \frac{e^{d/2}}{e^{-d/2} + e^{d/2}} > 0$$

This inequality is equal to

$$\frac{e^{-d} - e^{r}}{e^{-d/2} - e^{d/2}} - \omega e^{-d/2} > 0$$

and therefore

$$\omega < \frac{e^{-d} - e^{r}}{e^{-d} - 1} = \frac{e^{u} - e^{r}}{e^{u} - 1} \tag{3.6}$$

Right-hand side in (3.6) is bigger than zero if and only if $-d > r$ or $u > r$. Thereby, q_d and q_m are > 0 if and only if

$$\begin{cases} u > r \\ 0 < \omega < \frac{e^{u}-e^{r}}{e^{u}-1} \end{cases} \tag{3.7}$$

Similarly we can show that

$$q_u = \frac{e^{-d/2}}{e^{-d/2} + e^{d/2}} \left(\frac{e^{r} - e^{d}}{1 - e^{d}} - \omega\right)$$

So $q_u > 0$ if $\frac{e^{r}-e^{d}}{1-e^{d}} > \omega$. But we know that $\omega < \frac{e^{u}-e^{r}}{e^{u}-1}$ from 3.7. Hence, for $\frac{e^{r}-e^{d}}{1-e^{d}} > \omega$ the following should hold:

$$\frac{e^{r} - e^{d}}{1 - e^{d}} > \frac{e^{u} - e^{r}}{e^{u} - 1} = \frac{e^{-d} - e^{r}}{e^{-d} - 1}$$

as $d = -u$ or

$$e^{r-d} - 1 > e^{-d} - e^{r}$$

The last inequality is equivalent to

$$e^{r} - e^{d} > 1 - e^{r+d}$$

This inequality is true if $0 < r < -d$. Indeed fot the function $f(r) = e^{r} - e^{d} - 1 + e^{r+d}$ we have that $\frac{d}{dr} f(r) = e^{r} + e^{r+d} > 0$ and hence the function $f(r)$ is increasing with respect to r. But $f(0) = 0$, so $f(r) > 0$ for $0 < r < -d$ or $0 < r < u$, which is equivalent to $q_u > 0$. Therefore, $q_u, q_d, q_m > 0$ if (3.7) holds. □

One can conclude that when $m = \frac{u+d}{2}$ and $u = -d$, the triples q_u, q_m, q_d given in (3.3) define a probability if and only if

$$u > r, 0 < \omega = \frac{e^{u} - e^{r}}{e^{u} - 1} \tag{3.8}$$

The first condition is rather intuitive. Recall that r determines the interest rate of the risk-free asset of a portfolio, i,e, the growth of value for the risk-free asset. Therefore, it would make little sense if the bond was growing more in value than the stock. The second condition is crucial for existence of the martingale probability measure from which it follows that the market is arbitrage-free according to the first fundamental theorem of asset pricing.

Generalisation: Let us also study the general case of the existence of martingal probability measure, i.e. when $u \neq -d$. Here q_u, q_d, q_m are probabilities that satisfy (3.3) if and only if

$$q_u, q_d, q_m > 0 \Leftrightarrow 0 < w < min(\frac{e^r - e^d}{e^m - e^d}, \frac{e^u - e^r}{e^u - e^m}) \tag{3.9}$$

where

$$q_u + q_d + q_m = 1, \quad q_u, q_d, q_m > 0 \tag{3.10}$$

Proof. The equality 3.10 holds as

$$q_u + q_d + q_m = \frac{e^r - e^d}{e^u - e^d} + \frac{e^u - e^r}{e^u - e^d} + w - (\frac{e^m - e^d}{e^u - e^d} + \frac{e^u - e^m}{e^u - e^d})w = 1 + w - w = 1$$

Now we consider

$$q_d = \frac{e^u - e^r}{e^u - e^d} - \omega\frac{e^u - e^m}{e^u - e^d} > 0$$

$$q_d = \frac{e^u - e^r}{e^u - e^d} > \omega\frac{e^u - e^m}{e^u - e^d} \tag{3.11}$$

Which can be rearranged as

$$q_d > 0 \Leftrightarrow w < \frac{e^u - e^r}{e^u - e^m}$$

Therefore, we must take $u > r$ for 3.11 to hold. In the same way,

$$q_u = \frac{e^r - e^d}{e^u - e^d} - \omega\frac{e^m - e^d}{e^u - e^d} > 0 \tag{3.12}$$

$$\Leftrightarrow \omega < \frac{e^r - e^d}{e^m - e^d}$$

Hence, in this case, we must have $r > d$ for 3.12 to hold. □

To sum up, it is rather intuitive that $d < r < u$ as risk-free rate cannot be bigger or smaller than the stock. The second condition imposed on the ω means that by choosing ω in such way, there will exist the martingale probability measure. Moreover, in the limit $\omega \to 0$, the trinomial model becomes binomial and the solution (3.9) converges to the martingale probability measure of the binomial model.

3.4 Convergence to the Geometric Brownian Motion

The purpose of this section is to show that the trinomial stock price converges to the geometric Brownian motion in the time-continuous limit. We treat separately the case with and without the recombination condition.

The Geometric Brownian Motion is a stochastic process $\{S(t)\}_{t\geqslant 0}$, where

$$\tilde{S}(t) = S(0)e^{\alpha t+\sigma W(t)}$$

where $\{W(t)\}_{t\geqslant 0}$ is a Brownian Motion, α is the instantaneous mean of log-return and σ is volatility of a stock with price $\tilde{S}(t)$. Let us consider a partition $0 = t_0 < t_1 < ... < t_N = t$ of the interval $[0, t]$ with uniform size $t_{i+1} - t_i = h$. Recall that our trinomial model with the recombination condition has the following structure:

$$S(t_i) = \begin{cases} S(t_{i-1})e^u, \text{with probability } p_u \\ S(t_{i-1})e^{\frac{u+d}{2}}, \text{with probability } p_m \\ S(t_{i-1})e^d, \text{with probability } p_d \end{cases} \tag{3.13}$$

We can rewrite 3.13 as

$$S(t_i) = S(t_{i-1})exp[N(\frac{u+d}{2}) + (\frac{u-d}{2})X_i],$$

where

$$X_i = \begin{cases} 1, \text{with probability } p_u \\ 0, \text{with probability } p_m \\ -1, \text{with probability } p_d \end{cases}$$

Iterating the previous identity, the trinomial stock price at time $t = t_N$ is

$$S(t) = S_0 \exp[N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N],$$

where $M_N = X_1 + ... + X_N$, $N = \frac{t}{h}$.

Theorem 3.4.1. *Let u and d be chosen in such way that*

$$u = \sigma(1 - p_u + p_d)\sqrt{\frac{h}{(p_u + p_d)(1 - p_u - p_d)}} + \alpha h \tag{3.14}$$

$$d = -\sigma\sqrt{\frac{h}{(p_u + p_d)(1 - p_u - p_d)}} + \alpha h \tag{3.15}$$

then $S(t) \to \tilde{S}(t)$ in distribution.

Proof. Firstly, we will obtain the expected values and variances of the logarithm of $\tilde{S}(t)$ and $S(t)$ divided by the initial stock price:

$$\mathbb{E}[\log \frac{\tilde{S}(t)}{S(0)}] = \mathbb{E}[\alpha t + \sigma W(t)] = \alpha t$$

$$\text{Var}[\log \frac{\tilde{S}(t)}{S(0)}] = \text{Var}[\alpha t + \sigma W(t)]\sigma^2 t$$

$$\mathbb{E}[\log \frac{S(t)}{S(0)}] = \mathbb{E}[N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N] =$$
$$= N(\frac{u+d}{2}) + (\frac{u-d}{2})\mathbb{E}[M_N] =$$

$$= N(\frac{u+d}{2}) + (\frac{u-d}{2})\mathbb{E}[\sum_{i=1}^{N} X_i] =$$
$$= N(\frac{u+d}{2}) + (\frac{u-d}{2})N\mathbb{E}[X_1] =$$
$$= N(\frac{u+d}{2}) + (\frac{u-d}{2})N(p_u - p_d)$$

In the same way,

$$\text{Var}[log \frac{S(t)}{S(0)}] = \text{Var}[N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N] =$$
$$= \text{Var}[(\frac{u-d}{2})M_N] = (\frac{u-d}{2})^2 \text{Var}[M_N] =$$
$$= (\frac{u-d}{2})^2 N\text{Var}[X_1] = (\frac{u-d}{2})^2 N(\mathbb{E}X_1^2 - (\mathbb{E}X_1)^2) =$$
$$= (\frac{u-d}{2})^2 N(p_u + p_d - (p_u + p_d)^2)$$

$S(t)$ and $\tilde{S}(t)$ must have the same expected value and the same variance:

$$\alpha t = N[(\frac{u+d}{2}) + (\frac{u-d}{2})(p_u - p_d)] =$$
$$= \frac{t}{h}[(\frac{u+d}{2}) + (\frac{u-d}{2})(p_u - p_d)] \Rightarrow$$
$$\alpha = \frac{1}{h}[(\frac{u+d}{2}) + (\frac{u-d}{2})(p_u - p_d)] \tag{3.16}$$
$$\sigma^2 t = \frac{t}{h}(\frac{u-d}{2})^2(p_u + p_d - (p_u + p_d)^2) \Rightarrow$$
$$\sigma^2 = \frac{1}{h}(\frac{u-d}{2})^2(p_u + p_d - (p_u + p_d)^2) \tag{3.17}$$

By solving the equations (3.16) and (3.17) in terms of u and d, one finds that the solution is given by (3.14) and (3.15).

Now we will show that if we choose our parameters so that (3.16) and (3.17) holds, then

$$N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N \to \alpha t + \sigma W(t) \tag{3.18}$$

in distribution.

Then (3.18) is equivalent to

$$N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N \to \alpha t + \sigma\sqrt{t}N(0,1)$$

or

$$\frac{N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N - \alpha t}{\sigma\sqrt{t}} \to N(0,1)$$

Then

$$\frac{N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N - \alpha t}{\sigma\sqrt{t}} =$$

$$= \frac{N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N - \frac{t}{h}[(\frac{u+d}{2}) + (\frac{u-d}{2})(p_u - p_d)]}{\sqrt{\frac{t}{h}(\frac{u-d}{2})^2(p_u + p_d - (p_u + p_d)^2)}} =$$

$$= \frac{N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N - N(\frac{u+d}{2}) - N(\frac{u-d}{2})(p_u - p_d)}{\sqrt{N(\frac{u-d}{2})^2(p_u + p_d - (p_u + p_d)^2)}} =$$

$$= \frac{(\frac{u-d}{2})M_N - N(\frac{u-d}{2})(p_u - p_d)}{(\frac{u-d}{2})\sqrt{N(p_u + p_d - (p_u + p_d)^2)}} =$$

$$= \frac{M_N - N(p_u - p_d)}{\sqrt{N(p_u + p_d - (p_u + p_d)^2)}} =$$

$$= \frac{\sum_{i=1}^N X_i - E(\sum_{i=1}^N X_i)}{\sqrt{Var[\sum_{i=1}^N X_i]}} \to N(0,1)$$

in distribution (according to the central limit theorem). □

Generalisation: Now we treat the general case without the recombination condition. The general trinomial model of the change of the price of an asset is the following

$$S(t_i) = \begin{cases} S(t_{i-1})e^u, \text{with probability } p_u \\ S(t_{i-1})e^m, \text{with probability } p_m \\ S(t_{i-1})e^d, \text{with probability } p_d \end{cases}$$

Or uniformly the latter can be replaced as

$$S(t_i) = S(t_i)e^{a+bx_i}$$

where x_i is a random variable with the distribution

$$X_i = \begin{cases} 1, \text{with probability } p_u \\ c, \text{with probability } p_m \\ -1, \text{with probability } p_d \end{cases}$$

Then we want

$$e^{a+bx_i} = \begin{cases} S(t_{i-1})e^u, \text{with probability } p_u \\ S(t_{i-1})e^m, \text{with probability } p_m \\ S(t_{i-1})e^d, \text{with probability } p_d \end{cases}$$

$$\begin{cases} a+b=u \\ a+cb=m \Rightarrow \\ a-b=d \end{cases}$$

$$\begin{cases} a=\frac{u+d}{2} \\ b=\frac{u-d}{2} \\ c=\frac{m-a}{b} \end{cases}$$

Therefore,

$$c = \frac{m-\frac{u+d}{2}}{\frac{u-d}{2}}$$

and $c=0$ when $m=\frac{u+d}{2}$.

If $c \neq 0$ then we can consider convergence $S(t) \to \tilde{S}(t) = S(0)e^{\alpha t+\sigma W(t)}$. Let us also consider a partition $0=t_o<t_1<...<t_N=t$ of the interval $[0,t]$ with uniform size $t_{i+1}-t_i=h$. We can write the same representation for $S(t_i)$ as in the previous case:

$$S(t_i) = S(t_{i-1})\exp[N(\frac{u+d}{2})+(\frac{u-d}{2})X_i],$$

But now

$$X_i = \begin{cases} 1, \text{with probability } p_u \\ c, \text{with probability } p_m \\ -1, \text{with probability } p_d \end{cases}$$

Iterating the previous identity we can write for $t=t_N$:

$$S(t) = S_0\exp[N(\frac{u+d}{2})+(\frac{u-d}{2})M_N],$$

where $M_N = X_1+...+X_N$, $N=\frac{t}{h}$. As in the previous case, let us obtain the expected values and variances of the logarithm of $\tilde{S}(t)$ and $S(t)$ divided by the initial stock price:

$$\mathbb{E}[\log \frac{\tilde{S}(t)}{S(0)}] = \mathbb{E}[\log \frac{S_o e^{\alpha t + \sigma W(t)}}{S_0}] =$$
$$\mathbb{E}[\log e^{\alpha t + \sigma W(t)}] = \mathbb{E}[\alpha t + \sigma W(t)] =$$
$$\alpha t + \mathbb{E}[\sigma W(t)] = \alpha t + \sigma \mathbb{E}[W(t)] = \alpha t$$

In the same way:

$$\mathrm{Var}[\log \frac{\tilde{S}(t)}{S(0)}] = \mathrm{Var}[\alpha t + \sigma W(t)] =$$
$$\mathrm{Var}[\alpha t] + \mathrm{Var}[\sigma W(t)] = \sigma^2 \mathrm{Var}[W(t)] = \sigma^2 t.$$

$\tilde{S}(t)$ and $S(t)$ must have the same expected value and the same variance, then:

$$\mathbb{E}[\log \frac{S(t)}{S(0)}] = N(\frac{u+d}{2}) + (\frac{u-d}{2})\mathbb{E}[\sum_{i=1}^{N} X_i] =$$
$$= N(\frac{u+d}{2}) + (\frac{u-d}{2})N\mathbb{E}[X_1] =$$
$$= N(\frac{u+d}{2}) + (\frac{u-d}{2})N(p_u + cp_m - p_d)$$

In the same way:

$$\mathrm{Var}[log \frac{S(t)}{S(0)}] = \mathrm{Var}[N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N] =$$
$$= \mathrm{Var}[(\frac{u-d}{2})M_N] = (\frac{u-d}{2})^2 \mathrm{Var}[\sum_{i=1}^{N} X_i] =$$
$$= (\frac{u-d}{2})^2 N \mathrm{Var}[X_1] = (\frac{u-d}{2})^2 N(p_u + c^2 p_m + p_d - (p_u + cp_m - p_d)^2)$$

Having that

$$\mathbb{E}[\log \frac{S(t)}{S(0)}] = \mathbb{E}[\log \frac{\tilde{S}(t)}{S(0)}]$$
$$\mathrm{Var}[\log \frac{S(t)}{S(0)}] = \mathrm{Var}[\log \frac{\tilde{S}(t)}{S(0)}]$$

we can write down the following equations:

$$\alpha t = N[(\frac{u+d}{2}) + (\frac{u-d}{2})(p_u + cp_m - p_d)] = \frac{t}{h}[(\frac{u+d}{2}) + (\frac{u-d}{2})(p_u + cp_m - p_d)] \tag{3.19}$$

and

$$\sigma^2 t = \frac{t}{h}(\frac{u-d}{2})^2(p_u + c^2 p_m + p_d - (p_u + cp_m + p_d)^2) \tag{3.20}$$

The exact form solution of (3.19) and (3.20) in terms of u and d can be found but is very complicated and not necessary here.

Now we will show that

$$N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N \to \alpha t + \sigma W(t) \tag{3.21}$$

Then (3.21) is equivalent to

$$N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N \to \alpha t + \sigma\sqrt{t}N(0,1)$$

or

$$\frac{N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N - \alpha t}{\sigma\sqrt{t}} \to N(0,1)$$

Substituting α and σ in (3.19) and (3.20), we will have:

$$\begin{aligned}
&\frac{N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N - \alpha t}{\sigma\sqrt{t}} = \\
&= \frac{N(\frac{u+d}{2}) + (\frac{u-d}{2})M_N - N[(\frac{u+d}{2}) + (\frac{u-d}{2})(p_u + cp_m - p_d)]}{\sqrt{(\frac{u-d}{2})^2(p_u + c^2 p_m + p_d - (p_u + cp_m - p_d)^2)}} = \\
&= \frac{(\frac{u-d}{2})M_N - N(\frac{u-d}{2})(p_u + cp_m - p_d)}{(\frac{u-d}{2})\sqrt{N(p_u + cp_m + p_d - (p_u + cp_m - p_d)^2)}} = \\
&= \frac{\sum_{i=1}^N X_i - E(\sum_{i=1}^N X_i)}{\sqrt{Var[\sum_{i=1}^N X_i]}} \to N(0,1)
\end{aligned}$$

according to the central limit theorem.

This result has significant implications for the relevance of the trinomial model. Not only does it mean that the model can be used to accurately approximate the Geometric Brownian Motion. It also follows from this that, under certain conditions, the trinomial model option price of European derivatives will converge to the Black-Scholes price.

3.5 Trinomial Option Pricing

As an incomplete model, the trinomial model exhibits infinitely many martingale measures (q_u, q_m, q_d) as we have proven in the chapter 3.3. Each of this martingale measures generates a different price of the derivative with pay-off Y and maturity $T = N$. Recall that at the moment of time $t < T$, the fair price of the option is given but the following formula:

$$\Pi_Y(t, w) = e^{-r(N-t)}\mathbb{E}_\omega[Y \mid S(1), ..., S(t)] \tag{3.22}$$

where $\mathbb{E}_\omega$ is the expectation in the probability measure 3.3.

As justified in the section before, the fair price of the European derivative should be equal to the corresponding value of a self-financing hedging portfolio. By setting $t = 0$ in (3.22), we obtain that the risk-neutral price of the derivative in the trinomial market with martingale probability (q_u, q_m, q_d) is given by

$$\Pi_Y(0, w) = e^{-rN} \sum_{x\in\{u,m,d\}^{N-t}} (q_u)^{N_u(x)}(q_m)^{N_m(x)}(q_d)^{N_d(x)}Y(x) \tag{3.23}$$

The following recurrence is crucial for the generalised valuation of the self-financing portfolio.

Theorem 3.5.1. *The fair price of European derivative at time $t \in t = \{0, ..., N-1\}$ satisfies the recurrence formula*

$$\Pi_Y(t, w) = e^{-r}[q_u\Pi^u_Y(t + 1, w) + q_m\Pi^m_Y(t + 1, w) + q_d\Pi^d_Y(t + 1, w)], \tag{3.24}$$

where $\Pi^u_Y(t+1, w), \Pi^d_Y(t+1, w), \Pi^m_Y(t+1, w)$ denote the one-step price of the option at time $t + 1$ if the stock price goes in the directions (u, d, w) respectively.

Proof. As $S(1), ..., S(t)$ are $S(t + 1)$-measurable then $\mathbb{E}[Y \mid S(1), ..., S(t)]$ also is $S(t + 1)$-measurable and

$$\mathbb{E}_\omega[\mathbb{E}_\omega[Y \mid S(1), ..., S(t)] \mid S(t + 1)]$$

$$= [\mathbb{E}_\omega[Y \mid S(1), ..., S(t)]$$

Then

$$\Pi_Y(t, w) = e^{-r(N-t)}\mathbb{E}_\omega[Y \mid S(1), ..., S(t)]$$

$$= e^{-r(N-t)}\mathbb{E}_\omega[\mathbb{E}_\omega[Y \mid S(1), ..., S(t)] \mid S(1), ..., S(t)]$$

$$= e^{-r}e^{-r(N-(t+1))}\mathbb{E}_\omega[Y \mid S(1), ..., S(t), \quad S(t + 1) = S(t)e^u]\times \tag{3.25}$$

$$\times P\{S(t + 1) = S(t)e^u\} + e^{-r}e^{-r(N-(t+1))}\mathbb{E}[Y \mid S(1), ..., S(t), \quad S(t + 1) = S(t)e^m]\times$$

$$\times P\{S(t + 1) = S(t)e^m\} + e^{-r}e^{-r(N-(t+1))}\mathbb{E}[Y \mid S(1), ..., S(t), \quad S(t + 1) = S(t)e^d]\times$$

$$\times P\{S(t + 1) = S(t)e^d\} =$$

Let us denote

$$\Pi_Y^u(t+1,\omega) = e^{-r(N-(t+1))}\mathbb{E}_\omega[Y \mid S(1),...,S(t), \quad S(t+1) = S(t)e^u]$$

$$\Pi_Y^m(t+1,\omega) = e^{-r(N-(t+1))}\mathbb{E}_\omega[Y \mid S(1),...,S(t), \quad S(t+1) = S(t)e^m]$$

$$\Pi_Y^d(t+1,\omega) = e^{-r(N-(t+1))}\mathbb{E}_\omega[Y \mid S(1),...,S(t), \quad S(t+1) = S(t)e^d]$$

Then 3.25 will be equal to

$$= e^{-r}\Pi_Y^u(t+1,\omega)P\{S(t+1) = S(t)e^u\}$$

$$+e^{-r}\Pi_Y^m(t+1,\omega)P\{S(t+1) = S(t)e^m\}$$

$$e^{-r}\Pi_Y^d(t+1,\omega)P\{S(t+1) = S(t)e^d\}$$

And eventually we will obtain

$$\Pi_Y(t,w) = e^{-r}[q_u\Pi_Y^u(t+1,w) + q_m\Pi_Y^m(t+1,w) + q_d\Pi_Y^d(t+1,w)]$$

and the initial value when $t = N$ will be

$$\Pi_Y(N,w) = \mathbb{E}_\omega[Y \mid S(1),...,S(N)] = Y$$

due to the properties of conditional expectation. Hence, pay-off Y can be interpret as measurable with regard to $S(1),...,S(N)$. □

In contrast to the binomial model, the fair price of the European derivative at time t is not uniquely defined due to the existence of infinitely many risk-neutral probabilities. For this reason, the trinomial market model is said to be incomplete.

4

Empirical Analysis

We start this chapter with the implementation of the special case of the trinomial model that has been used for the numerical results later on in this chapter. This chapter also covers the dependence of the trinomial price on the free-parameter of our model that is crucial for the empirical analysis. The second part of the chapter is dedicated to the results that were obtained using Matlab and real market data, and the corresponding discussion to it.

4.1 Implementation of the Trinomial Model

We recall that in applications to the real world, the trinomial model should be properly re-scaled in time. Indeed, as was previously proven in Chapter $3.5-3.6$, the trinomial stock price after being properly re-scaled, converges in distribution to the geometric Brownian motion in the time-continuous limit. Hence, let $T>0$ be the maturity of the European call option and consider the uniform partition of the interval $[0,T]$ with the size $h>0$ such that:

$$0=t_0<t_1<...<t_N=T, t_i-t_{i-1}=h$$

for all $i\in\mathcal{I}=\{1,...,N\}$.

We take a closer look at the trinomial model that will be studied numerically. Let us set up the model where $S(0)=S_0$ and it has the following structure:

$$S(t_i)=\begin{cases} S(t_{i-1})e^u, \text{with probability } p_u \\ S(t_{i-1})e^{\frac{u+d}{2}}, \text{with probability } 1-p_u-p_d \\ S(t_{i-1})e^d, \text{with probability } p_d \end{cases}$$

where $u>d$ and $p_u,\ \ p_d$ are defined in such way that they satisfy $p_u,p_d\in(0,1/2)$. The risk-free asset is defined as $B(t)=B_0e^{rt}$, t $\in\mathcal{I}$, and r is constant.

Then the trinomial stock price on the given partition is

$$S(t_i) = S(t_{i-1})exp[(\frac{u+d}{2}) + (\frac{u-d}{2})X_i], \quad i \in \mathcal{I} \tag{4.1}$$

where

$$X_i = \begin{cases} 1, \text{with probability } p_u \\ 0, \text{with probability } 1 - p_u - p_d \\ -1, \text{with probability } p_d \end{cases}$$

while $B(t_i) = B_0 e^{rhi}$.

The instantaneous mean of log-return and the instantaneous variance of trinomial stock price are defined as follows

$$\alpha = \frac{1}{h}\mathbb{E}_p[logS(t_i) - logS(t_{i-1})]$$

$$\sigma^2 = \frac{1}{h}Var_p[logS(t_i) - logS(t_{i-1})],$$

the parameter σ is called instantaneous volatility [13]. It is important to notice that these parameters are computed using the physical probability and not the risk-neutral probability. In our case α and σ are given by:

$$\alpha = \frac{1}{2h}[u + d + (p_u - p_d)(u - d)], \tag{4.2}$$

$$\sigma^2 = \frac{1}{4h}[p_u + p_d - (p_u + p_d)^2)(u - d)^2], \tag{4.3}$$

We assume for the simplification purposes that $p_u = p_d = p \in (0, 1/2)$, and let us invert the equations above in order to express u and d.

$$u = \alpha h + \sigma\sqrt{\frac{h}{2(1-2p)p}} \tag{4.4}$$

$$d = \alpha h - \sigma\sqrt{\frac{h}{2(1-2p)p}} \tag{4.5}$$

Note that most frequently in the application of the trinomial model, one sets the parameters α, σ and then computes u, d. In our case it can be done directly by using the equations (4.4)-(4.5).

Clearly, one can see that $u \neq -d$ here, unless $\alpha = 0$. Therefore, we go back to the second part of the Chapter 3.4, and recall that $\mathbb{P}_p(\omega)$ is the martingale probability measure if and only if $p = q = (q_u, q_m, q_d)$, where probabilities (q_u, q_m, q_d) satisfy the following conditions:

$$q_u = \frac{e^r - e^d}{e^u - e^d} - \omega \frac{e^{d/2}}{e^{u/2 + e^{d/2}}}, \quad q_m = \omega, \quad q_d = \frac{e^u - e^r}{e^u - e^d} - \omega \frac{e^{u/2}}{e^{u/2} + e^{d/2}} \tag{4.6}$$

whereas ω is conditioned as

$$0 < \omega < min(\frac{e^r - e^d}{e^m - e^d}, \frac{e^u - e^r}{e^u - e^m})$$

The latter can be interpret as that we choose the free parameter $q_m = \omega$ in such way that the discounted stock price $\{S^*(t)\}_{t \in \{0,...,N\}}$ is a $\mathbb{P}_\omega$-martingale.

It is necessary that $d < r < u$, i.e. that the risk-free rate is neither bigger nor smaller than the stock. Let us investigate the case when $m = \frac{u+d}{2}$, then $m = \frac{2\alpha h}{2} = \alpha h$. Having that $h > 0$ is the size of the given partition $0 = t_0 < t_1 < ... < t_N = T$ of the interval $[0, T]$, the condition imposed on w can be rewritten as

$$0 < \omega < min(\frac{e^{rh} - e^d}{e^{\alpha h} - e^d}, \frac{e^u - e^{rh}}{e^u - e^{\alpha h}}) := \omega_{max}(h) \tag{4.7}$$

For the simplification, let us denote

$$\hat{\sigma} = \frac{\sigma}{\sqrt{2(1 - 2p)p}}$$

Then u and d will be expressed as follows

$$u = \hat{\sigma}\sqrt{h} + \alpha h, \quad d = -\hat{\sigma}\sqrt{h} + \alpha h$$

This model is trustworthy only if $0 < \omega < \omega_{max}$. We now show that this holds provided h is small in comparison to T (i.e., $N >> 1$). If this condition is violated, N is considered too small. In fact, it is easy to see that when $N \to \infty$ and $h \to 0$ then

$$\omega_{max}(h) \to 1$$

Proof. First, we start with the first part in the minimum

$$\lim_{h \to 0} \frac{e^{rh} - e^{-\hat{\sigma}\sqrt{h} + \alpha h}}{e^{\alpha h} - e^{-\hat{\sigma}\sqrt{h} + \alpha h}}$$
$$= \lim_{h \to 0} \frac{e^{(r-\alpha)h} - e^{-\hat{\sigma}\sqrt{h}}}{1 - e^{-\hat{\sigma}\sqrt{h}}}$$

by L'Hospital's rule

$$= \lim_{h\to 0} \frac{(r-\alpha)e^{(r-\alpha)h} + \frac{\hat{\sigma}}{2\sqrt{h}}e^{-\hat{\sigma}\sqrt{h}}}{\frac{\hat{\sigma}}{2\sqrt{h}}e^{-\hat{\sigma}\sqrt{h}}}$$

$$= \lim_{h\to 0} \frac{(r-\alpha)2\sqrt{h}\times e^{(r-\alpha)h}}{\hat{\sigma}e^{-\hat{\sigma}\sqrt{h}}} = 1$$

And for the second part in the minimum, we prove in the same way

$$\lim_{h\to 0} \frac{e^{\hat{\sigma}\sqrt{h}+\alpha h} - e^{rh}}{e^{\hat{\sigma}\sqrt{h}+\alpha h} - e^{\alpha h}}$$

$$= \lim_{h\to 0} \frac{e^{\hat{\sigma}\sqrt{h}} - e^{(r-\alpha)h}}{e^{\hat{\sigma}\sqrt{h}} - 1}$$

$$= \lim_{h\to 0} \frac{\frac{\hat{\sigma}}{2\sqrt{h}}e^{\hat{\sigma}\sqrt{h}} - (r-\alpha)e^{(r-\alpha)h}}{\frac{\hat{\sigma}}{2\sqrt{h}}e^{-\hat{\sigma}\sqrt{h}}}$$

$$= \lim_{h\to 0} 1 - \frac{(r-\alpha)2\sqrt{h}\times e^{(r-\alpha)h}}{\hat{\sigma}e^{\hat{\sigma}\sqrt{h}}} = 1$$

Hence,

$$\omega_{max}(h) \to 1$$

as claimed. □

4.2 Dependence on the Free-Parameter

All the latter results are crucial for the empirical analysis that was conducted using Matlab. In this section, we will see how the free-parameter ω impacts the trinomial price of the derivative.

First, we start by calculating historical volatility. Historical volatility is a method with the help of which we can approximate the unknown σ. Historical volatility uses historical values of $S(t)$ in order to approximate volatility. Therefore, it can be computed as the standard deviation of log-returns of the asset based on historical data [16]. The asset that we will consider is a stock. The log-returns of the stock price in the interval t_{i-1}, t_i is given by

$$R_i = \log(\frac{S(t_i)}{S(t_{i-1})}), \quad \text{for} \quad i = 1, ..., n. \tag{4.8}$$

Let $T = t - t_0$, then the square root of the T-historical variance is the T-historical volatility defined as

$$\hat{\sigma_T}(t) = \frac{1}{\sqrt{h}}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2} \tag{4.9}$$

where $\bar{R}$ is sample mean of log-returns

$$\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$$

It is a known fact that the historical variance of a stock is an unbiased estimator for the instantaneous variance [1]. In order to numerically find the 20-days historical volatility of the stock, the real-world data was collected from Yahoo Finance. The observable data is the daily adjusted closing price of the S & P 500 stock market index between August, 22, 2019 and September, 20, 2019. Then by using Matlab, the following results of the stock were obtained:

$$\hat{\alpha}_{20} = 0.2945, \quad \hat{\sigma}_{20} = 0.1377$$

The market prices that are used are for the European call options on S & P 500 stock market index. These data were obtained from Yahoo Finance at close on September, 23, 2019 with $S_0 = 2991.78$. The conducted analysis includes 8 different maturity dates, i.e., $T = 1$ day, $T = 1$ week, $T = 1$ month, $T = 2$ months, $T = 3$ months, $T = 6$ months, $T = 9$ months and finally $T = 1$ year. Tables of data can be found in the Appendix B. To obtain the corresponding stock prices using the trinomial model, we have used the calculated above instantaneous mean of log-return $\hat{\alpha}$ and instantaneous variance $\hat{\sigma}$ of the stock. It can be shown that it is numerically most efficient to set $p_u = p_d = p = 0.25$. Thereafter, we find the parameters of the trinomial model u, d according to the discussion above and formulas (4.4) - (4.5). The Matlab code for the stock prices of a European call can be found in Appendix A.1.

As it was proven in the previous chapter, according to the recurrence formula, the fair price of the European derivative at time t satisfies the equation

$$\Pi_Y(t, w) = e^{-r}[q_u \Pi_Y^u(t+1, w) + q_m \Pi_Y^m(t+1, w) + q_d \Pi_Y^d(t+1, w)],$$

if and only if equations (4.6) - (4.7) hold. We compute the trinomial price of European call options with different strike prices and maturity times by using the Matlab function presented in Appendix A.2.

We have previously proven that there exists one parameter family of risk-neutral probabilities $\mathbb{P}_w$ such that $\{S^*(t)\}_{t\in\{0,\dots,N\}}$ is a martingale, thus the price of a derivative will depend on the free parameter $w \in (0, \omega_{max})$. While keeping volatility and other input parameters constant, we use the free-parameter ω to fit the data. In order to do so we also fix the maturity for options with different strike prices and calibrate the model in order to find the unique trinomial option price that corresponds to the market price. We estimate the value of ω in the way that it minimises the difference between the observed data and trinomial prices, i.e., by solving the problem

$$\min_{\omega}(C_{trn}(K,T;\omega) - C_{obs}(K,t))^2$$

In order to solve this problem numerically, the collected data from the real market, presented in the Appendix B, and the Matlab code in the Appendix A.3 were used. Having concluded that fixing $r = 0$ implies no significant errors, we have chosen the number of steps to be equal $N = 1000$ and $r = 0$. The remaining parameters $S(0)$, p, α and σ are kept constant and they are set to $S0 = 2991.78$, $p = 0.25$, $\alpha = 0.2945$ and $\sigma = 0.1377$ respectively. The obtained data for values of ω for European call options with different strike prices and maturities can also be found in the Appendix B. Worth to mentions that our free-parameter ω now depends on (K, T).

4.3 Results and Discussion

The purpose of this thesis is to investigate the possible connection between σ_{imp} and incompleteness of the market. The standard interpretation of σ_{imp} says that we are using a different value of implied volatility to price options with different T and K. Hence, σ_{imp} is a function of (K, T). The graph of the implied volatility of call options as a function of the strike price is called volatility curve. If the options market were perfectly described by the Black-Scholes model, the implied volatility would be a deterministic constant independent of the strike price and thus the volatility curve would be a flat horizontal line [5].

In this thesis, we approach this problem in a different way, namely by using the trinomial model as the simplest example of the incomplete markets. This is done by not changing the value of volatility as in our model now we have one extra parameter ω that can reproduce the curve. As was shown previously, ω depends on (K, T) which is a rigorous assumption due to the theory. Then by using the calibration method, we find such martingale measures that the trinomial prices of European call options correspond to the observed market prices for the options with the same input parameters K, T, and σ.

In order to investigate this connection, we first calculate numerically the implied volatility for different European call options with different strike and maturities. There is no closed formula to calculate the implied volatility, but by using numerical methods it is possible to get an approximation. Therefore, the implied volatility was determined by looking at the actual market prices and was computed with the inbuilt Matlab function *blsimpv*. All the data and input parameters were kept the same as in the trinomial model discussed in the previous section. The obtained data for values of σ_{imp} for European call options with different strike prices and maturities is also presented in the Appendix B. For the sake of comparison, we plotted ω and σ_{imp} across strike prices for options with the same time to expiration for eight different maturity times in Matlab .

Before we begin with the analysis dedicated to the obtained figures **??**-**??**, let us recall the concept of moneyness. Moneyness is the relative position of the current price of an underlying asset in relation to the strike price of a derivative. The derivative is said to be in the money if it would have positive intrinsic value if it was exercised today; if the current price and strike price are equal, it is said to be at the money; and out of the money if the current strike price is higher than the market price for this option [7]. One can interpret the options to the left from the vertical line in the figures **??**-**??** as in the money, and to the right, as out of the money.

Implied volatility can be seen as the proxy of the market risk. A higher implied volatility means that the stock price is predicted to move drastically before the expiration date. Since the payoff of the European call option $Y = max((S(t)-K), 0)$ has a lower limit zero but no upper limit, one is expected to gain from a high volatility than to lose from it. Therefore, it seems logic that the call option is an increasing function of the volatility. Usually, volatility smile implies that deep in the money and deep out of the money options are over-priced or under-priced in the real market compare to the Black-Scholes price. As a general rule, the lowest point of the volatility smile corresponds to the at the money options [15]. Nevertheless, not all data align with the volatility smile. In our example, we ended up with the volatility curves which fluctuate more for in the money options. This seems reasonable as implied volatility fluctuates the same way the prices do.

The hypothesis is that in our trinomial model we know without doubt that we are using different risk-neutral probabilities (martingale probabilities) in order to price an option. On the example of our experiment, we want to justify that investors when facing the volatility smile effect, are also using the different risk-neutral probabilities for each option. This should be logic for deeply in the money, and out of the money options as they appear to be more risky, so the different risk neutral measure must be used. The benefit of the risk-neutral pricing approach is that once the risk-neutral probabilities are calculated, they can be used to price every asset based on its discounted expected payoff.

It is worth pointing out that the trinomial price calculated with the $\omega = 0$ is identical to the price calculated with the binomial model. As ω decreases, q_u, q_d will increase since $q_u + \omega + q_d = 1$ and vice versa. The observed data in the Appendix B shows that when ω decreases, the call price seems to increase. One can see visually in the figures **??**-**??** that the curve for ω replicates the curve for the σ_{imp}. Therefore, we conclude that choosing the value of ω is equivalent to choosing the value of σ_{imp}.

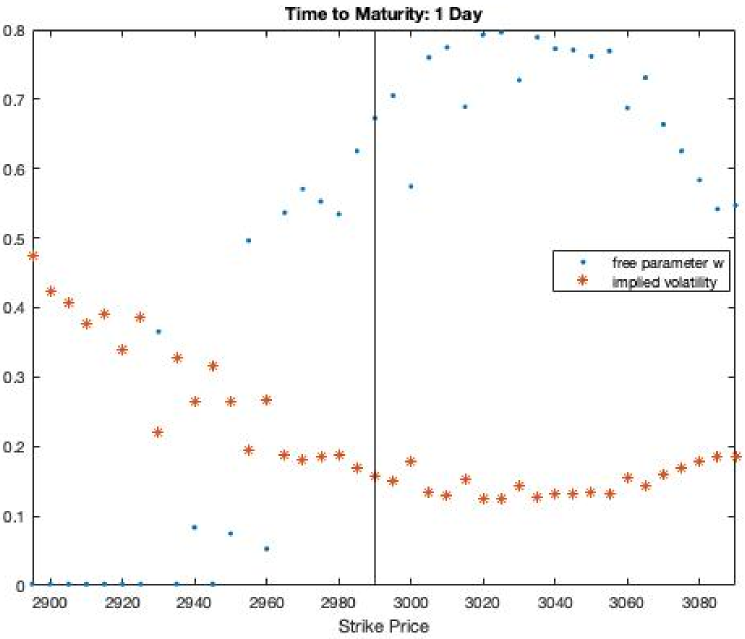


Figure 4.1: ω and σ_{imp} across strike prices for European call options with time to expiration 1 Day

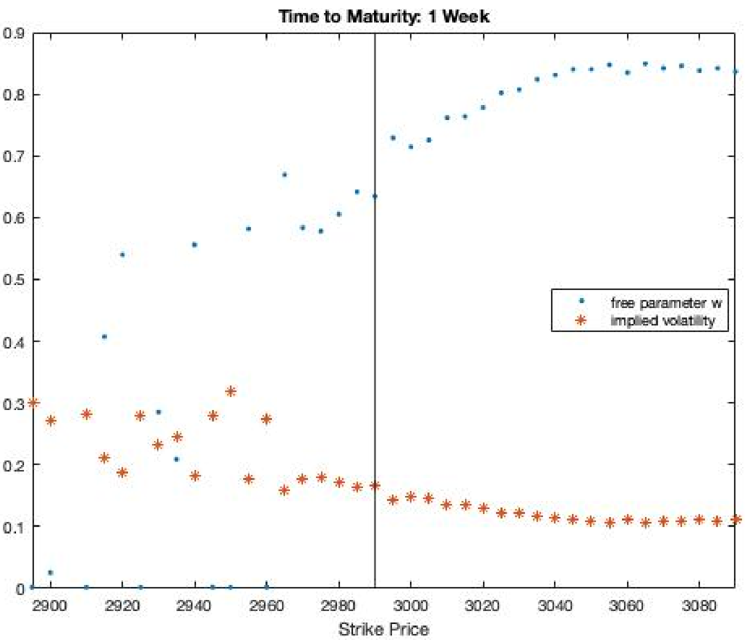


Figure 4.2: ω and σ_{imp} across strike prices for European call options with time to expiration 1 Week

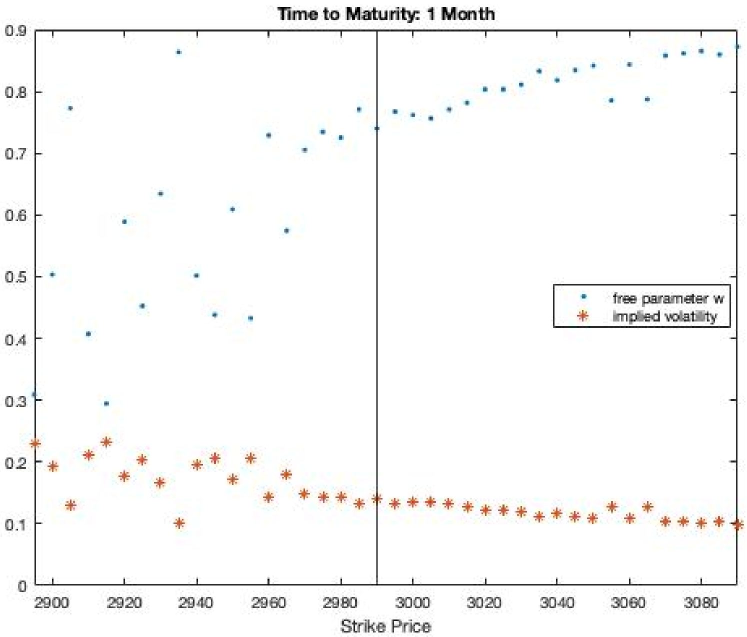


Figure 4.3: ω and σ_{imp} across strike prices for European call options with time to expiration 1 Month

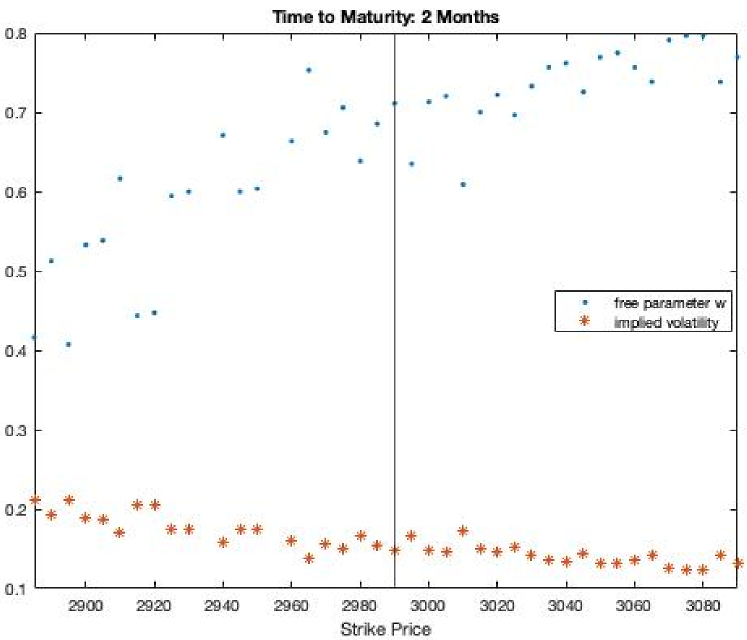


Figure 4.4: ω and σ_{imp} across strike prices for European call options with time to expiration 2 Months

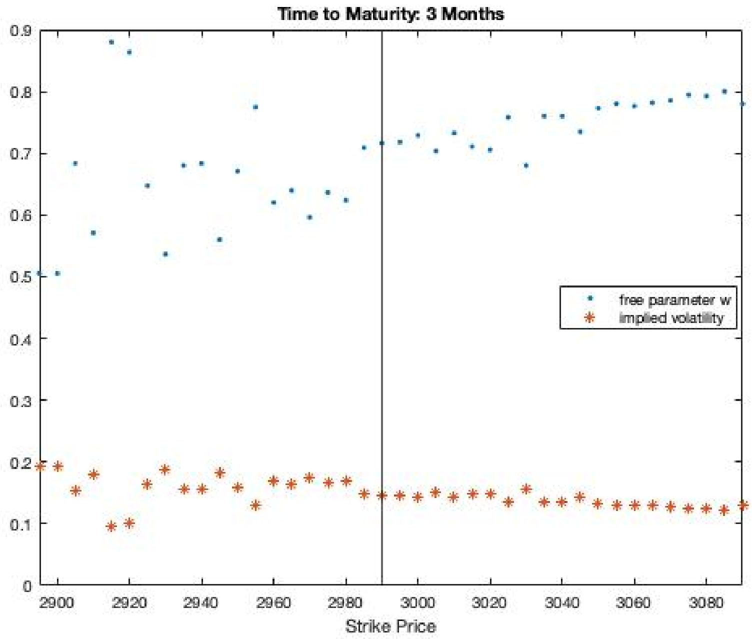


Figure 4.5: ω and σ_{imp} across strike prices for European call options with time to expiration 3 Months

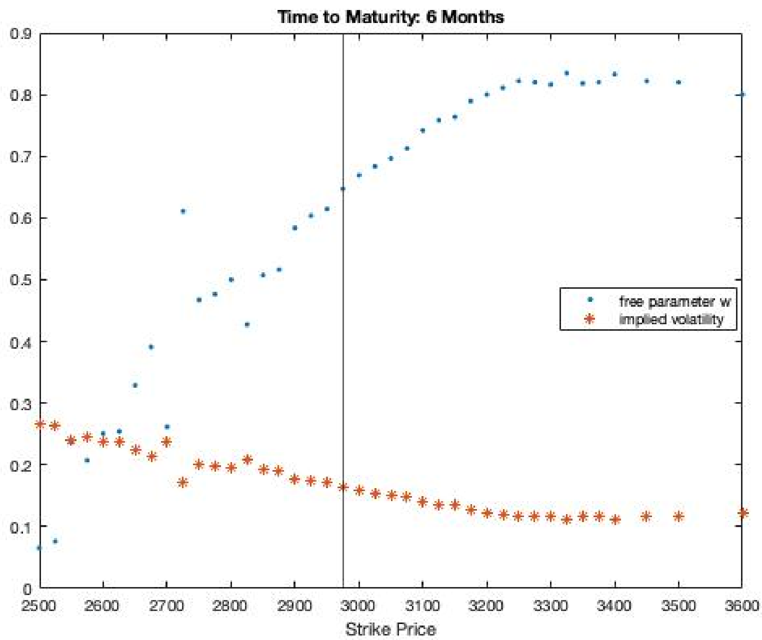


Figure 4.6: ω and σ_{imp} across strike prices for European call options with time to expiration 6 Months

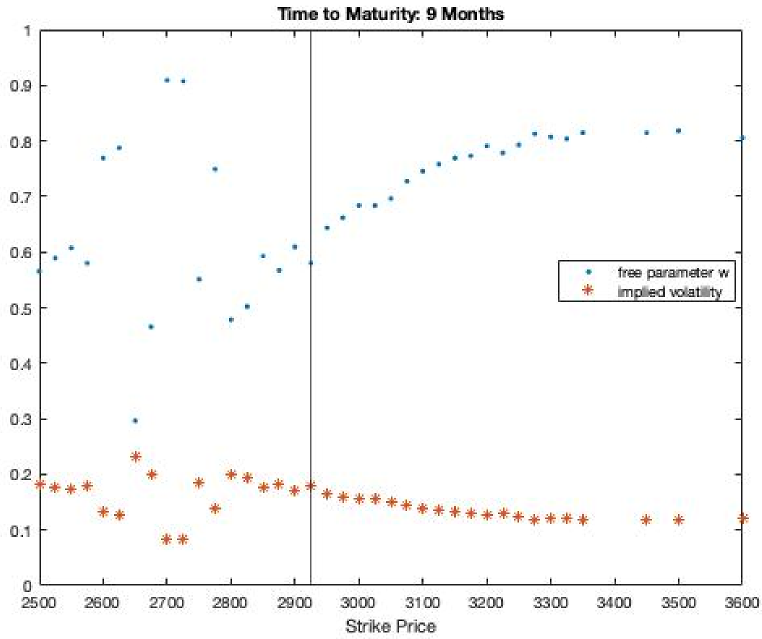


Figure 4.7: ω and σ_{imp} across strike prices for European call options with time to expiration 9 Months

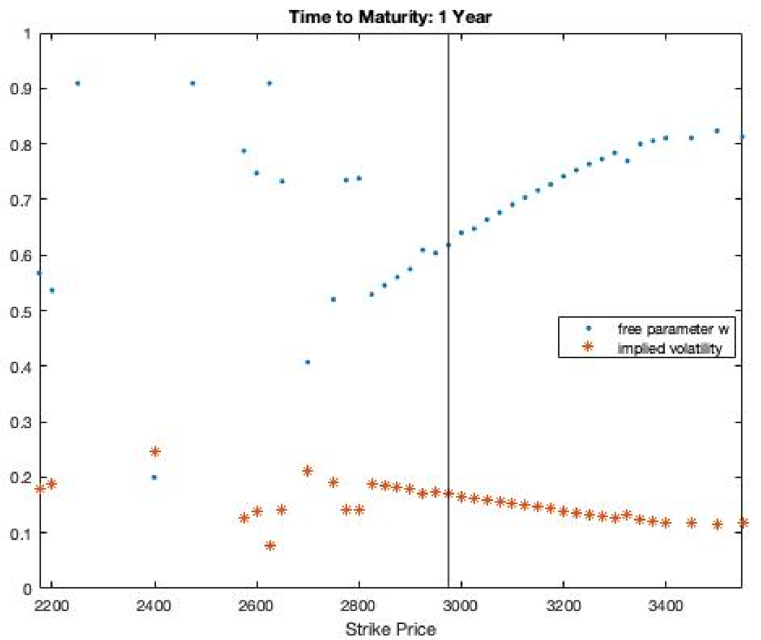


Figure 4.8: ω and σ_{imp} across strike prices for European call options with time to expiration 1 Year

4.4 Correlation

During this thesis, there was found an interesting result of some exact correlation between implied volatility and the free-parameter ω. Without a doubt, it seems reasonable as implied volatility is based on the probability p and is computed by using it. What is more, this relation appeared to be independent of maturity. By using Matlab, the figures of ω as a function of σ_{imp} for different maturities were created.

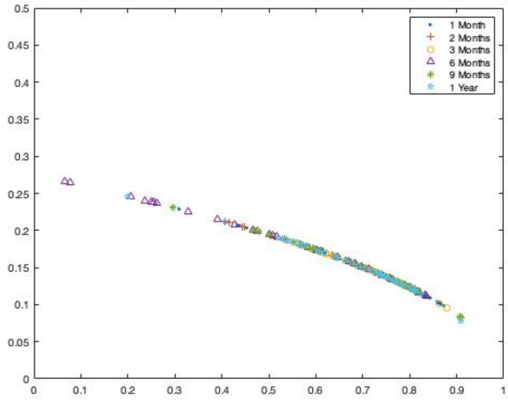


Figure 4.9: ω as a function of σ_{imp} for 1 Month, 2, 3, 6, 9 Months and 1 Year Maturities

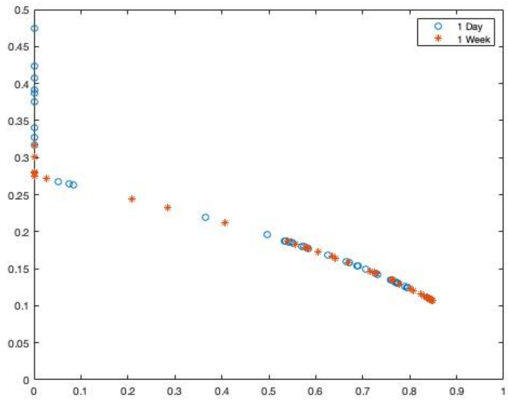


Figure 4.10: ω as a function of σ_{imp} for 1 Day and 1 Week Maturities

We treat the case for one day and one week maturity separately as we are very close to the expiration date, and as we know, the price of the call close to maturity tends to payoff. Therefore, in this case, the dependence on ω will be trivial and σ_{imp} is independent of ω. While computing this ω numerically in Matlab, the minimum value for which we can find ω is 0.0018 which will correspond to zero. It is a scale problem of our model for options that are very close to maturity and are very deeply in the money.

While looking at the figure 4.10, one can see that ω and σ_{imp} have a negative correlation. This means that when one variable increases the other decreases on average, and vice versa. The value of the correlation coefficients for different maturities where computed in Matlab and are presented in the table below.

Maturity	ρ
1 Day	-0.9430
1 Week	-0.9916
1 Month	-0.9944
2 Months	-0.9971
3 Months	-0.9949
6 Months	-0.9937
9 Months	-0.9909
1 Year	-0.9867

While looking at the correlation coefficients for different maturities, one can see that it is very close to -1 and is independent of the maturity essentially. By these results, we can conclude that σ_{imp} is a decreasing function of ω. Therefore, there is a unique correspondence between σ_{imp} and ω, which substantially means that if we have implied volatility curve, we can always find the corresponding curve for ω regardless of what the σ_{imp} curve is shaped as.

Taking everything into account, under the Black-Scholes assumption investors assumed to be indifferent to risk. In our interpretation of the trinomial model, the parameter $\Omega = 1 - \omega$ can be seen as the risk-aversion of the investors.

- The closer is Ω to zero, the more the stock price behaves as the risk-free asset $S(t) = S(0)e^{mt}$.
- The closer is Ω to one, the more the stock price behaves as the binomial model

Because the closer is ω to one, the less risk the investors assign to the option. If the investor prices the option with the martingale probability corresponding to Ω, then the closer is Ω to one (i.e., ω to zero), the more this investor considers the option as a risky investment. As we found a negative correlation between ω and σ_{imp}, then Ω is positively correlated with it. This seems reasonable because a large implied volatility is also considered an indication of the fact that investors are considering the option more risky and thus overpricing it in comparison to the Black-Scholes price.

5

Conclusion

The purpose of this thesis was to investigate the possible connection between implied volatility and market incompleteness. This hypothesis was tested quantitatively by using the trinomial model, which is the simplest example of one-dimensional incomplete market. First, the properties of the trinomial option pricing model were examined. In doing so, the conditions of the existence of the risk-neutral (or martingale) probability measure were found for the general case when $u \neq -d$ and for the special case when $u = -d$.

Another important topic of the investigation was a study of the conditions under which the stock prices converge to the geometric Brownian motion in the time continuous limit. We treated separately the case with and without the recombination condition. Thus we found the conditions under which the trinomial option price of the European derivatives converges to the Black-Scholes price. Finally, the trinomial fair price of a European derivative was derived.

Through empirical analysis, the implementation of the special case of the trinomial model when $u \neq -d$ but $p_u = p_d = p \in (0, 1/2)$ was studied. The conditions under which this model converges in distribution to the geometric Brownian motion in the way that the discounted stock price is a martingale with respect to the risk-neutral probability measure were found and the convergence proven. All these latter results were crucial for the empirical analysis of how the free-parameter ω impacts the trinomial price of the derivative. The dependence of ω on (K, T) was concluded, which is a rigorous assumption due to the theory. Then by using the calibration method, we found such values of ω that the trinomial prices of European call options correspond to the observed market prices for the options with the same K, T, and σ.

In order to investigate the connection between σ_{imp} and incompleteness of the market, we have calculated numerically the implied volatility for different European call options with different strike and maturities. There is no closed formula to calculate the implied volatility, but by using numerical methods it is possible to get an approximation. We plotted ω and σ_{imp} across strike prices for options with the same time to expiration for eight different maturity times, see figures **??**-**??**. In our example, we ended up with the volatility curves which fluctuate more for in the money options. This seems reasonable as implied volatility fluctuates the same way the prices do.

During this thesis, there was found an interesting result of some exact correlation (almost -1) of implied volatility and the free-parameter ω. Without a doubt, it seems reasonable as implied volatility is based on the probability p and is computed by using it. What is more, this relation appeared to be independent of maturity. By these results, we have concluded that σ_{imp} is a decreasing function of ω. Therefore, choosing the value of ω is equivalent to choosing the value of σ_{imp}, which substantially means that if we have implied volatility curve, we can always find the corresponding curve for ω regardless of what the σ_{imp} curve is shaped as. Overall, it may be said that the hypothesis that volatility curves are linked to market incompleteness is reasonable and justified from the qualitative perspective.

Nevertheless, the two parameters have a very different interpretation.

- An implied volatility that depends on (K, T) is inconsistent with Black-Scholes theory
- As the trinomial model is incomplete, there exists one more free-parameter which can be used to measure the investors risk-aversion. We can interpret volatility curves as a consequence of market incompleteness and investors assigning different risk to options far from being at the money.

Further research could focus on applicability of the trinomial model to different derivatives, such as European put options, American derivatives, Asian and lookback options, etc. The numerical studies of the convergence to the Black-Scholes model in comparison to the binomial model can be explored. Moreover, future research could also focus on investigating and testing the connection between implied volatility and market incompleteness for other incomplete multinomial models. Bonds valuation can be studied using this approach.

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Appendix A

Matlab code

A.1 Stock prices in the trinomial model

```
% TrinomialStock creates a (2N+1)x(N+1) matrix with the prices
% of a stock calculated by the trinomial model.  The initial
% price can be found in position (N+1,1). The input parameter a
% is the instantaneous mean of log returns, s is the
% instantaneous volatility, N is the number of steps,
% p is the probability, T is the maturity, and S0 is the initial
% stock price.



function [S,h,u,d] = TrinomialStock(S0,p,a,s,T,N)

% Compute the trinomial tree of the stock

S=zeros(2*N+1,N+1);
S(N+1,1)=S0;

h=T/N;
u=sqrt(h/(2*(1-2*p)*p))*s+h*a;
d=-sqrt(h/(2*(1-2*p)*p))*s+h*a;


for i=2:(N+1)
    S(:,i)=S(:,i-1)*exp((u+d)/2);
    S(N+2-i,i)=S(N+3-i,i-1)*exp(u);
    S(N+i,i)=S(N+i-1,i-1)*exp(d);

end
```

A.2 European call option prices in the trinomial model

```
% TrinomialPrice creates a matrix with the prices of call options
% calculated by the trinomial model. Each column corresponds
% to a time instant. S is the trinomial stock prices computed
% with the function TrinomialStock. K is the strike price of call
% options, r is the market risk-free rate, w is the
% free-parameter in trinomial model, and u and d correspond
% to the price change when the stock goes up or down respectively.



function P0 = TrinomialPrice(S,h,u,d,K,r,w)

% Compute the trinomial price of the call on the stock

qu=(exp(r*h)-exp(d))/(exp(u)-exp(d))-w*(exp((u+d)/2)-exp(d))/
  /(exp(u)-exp(d));
qd=(exp(u)-exp(r*h))/(exp(u)-exp(d))-w*(exp(u)-exp((u+d)/2))/
  /(exp(u)-exp(d));

% Check the conditions for the risk-neutral probabilities

if (qu<0 || qd<0)
    display('take smaller w');
    P0=0;
    return
end


M=size(S,1);
N=size(S,2);
P=zeros(M,N);
P(:,N)=max(S(:,N)-K,0); % Pay-off of the call option

% Recurrence formula to calculate the option prices

for j=N-1:-1:1
    for i=(N-j+1):(M-(N-j))
        P(i,j)=exp(-r*h)*(qu*P(i-1,j+1)+w*P(i,j+1)+qd*P(i+1,j+1));
         if P(i,1)>0
            P0 = P(i,1);
         end
    end
end
```

A.3 Dependence on the free-parameter

```
% Input parameters of the model

M=500;
x=zeros(M,1);
w=zeros(M,1);
wcal=zeros(40,1);
vol=zeros(40,1);
S0=2991.78;
p=0.25;
a=0.2945;
s=0.1377;
r=0;
T=1; % Changes with the data
N=1000;

[S,h,u,d] = TrinomialStock(S0,p,a,s,T,N);

% Compute the Black-Scholes implied volatility

for j=1:40
vol(j)=blsimpv(S0,dataK(j),r,T,dataPrice(j));

% Computes the free-parameter w of the trinomial model

for i=1:M
    w(i)=i/(M+M/10);
    x(i)=TrinomialPrice(S,h,u,d,dataK(j),r,w(i));
end

[y,I]=min((x-dataPrice(j)).^2);
wcal(j)=w(I);
end
```

Appendix B

Tables of Data and Results

The market prices that are presented in the following tables are for the European call options on S & P 500 stock market index. These data were obtained from Yahoo Finance at close on September, 23, 2019.

B.1 Time to Maturity: 1 Day

Maturity	*Strike*	*Price*	*w*	*ImpVol*
25/09/2019	2895	102.8	0.0018	0.4741
25/09/2019	2900	96.5	0.0018	0.4239
25/09/2019	2905	91.5	0.0018	0.4072
25/09/2019	2910	85.95	0.0018	0.3761
25/09/2019	2915	82.26	0.0018	0.3918
25/09/2019	2920	75.85	0.0018	0.3402
25/09/2019	2925	73.67	0.0018	0.3867
25/09/2019	2930	62.95	0.3600	0.2193
25/09/2019	2935	62.58	0.0018	0.3270
25/09/2019	2940	55.5	0.0836	0.2637
25/09/2019	2945	54.04	0.0018	0.3170
25/09/2019	2950	47.3	0.0745	0.2648
25/09/2019	2955	39.8	0.4964	0.1954
25/09/2019	2960	39.82	0.0527	0.2681
25/09/2019	2965	31.3	0.5364	0.1874
25/09/2019	2970	27.1	0.5709	0.1803
25/09/2019	2975	23.8	0.5527	0.1843
25/09/2019	2980	20.77	0.5345	0.1880
25/09/2019	2985	16.35	0.6255	0.1687
25/09/2019	2990	12.75	0.6727	0.1575
25/09/2019	2995	9.7	0.7055	0.1494
25/09/2019	3000	9.8	0.5745	0.1795
25/09/2019	3005	4.89	0.7600	0.1348
25/09/2019	3010	3.3	0.7745	0.1305
25/09/2019	3015	3.5	0.6891	0.1536
25/09/2019	3020	1.36	0.7927	0.1253
25/09/2019	3025	0.85	0.7964	0.1245
25/09/2019	3030	1	0.7273	0.1439
25/09/2019	3035	0.34	0.7891	0.1263
25/09/2019	3040	0.25	0.7727	0.1311
25/09/2019	3045	0.15	0.7709	0.1316
25/09/2019	3050	0.1	0.7618	0.1343
25/09/2019	3055	0.05	0.7691	0.1326
25/09/2019	3060	0.1	0.6873	0.1540
25/09/2019	3065	0.03	0.7309	0.1428
25/09/2019	3070	0.05	0.6636	0.1598
25/09/2019	3075	0.05	0.6255	0.1687
25/09/2019	3080	0.05	0.5836	0.1776
25/09/2019	3085	0.05	0.5418	0.1864
25/09/2019	3090	0.03	0.5473	0.1852

B.2 Time to Maturity: 1 Week

Maturity	*Strike*	*Price*	*w*	*ImpVol*
2/10/2019	2895	105	0.0018	0.3005
2/10/2019	2900	98.47	0.0255	0.2717
2/10/2019	2910	86.11	0.0018	0.2811
2/10/2019	2915	79.28	0.4073	0.2119
2/10/2019	2920	87.25	0.5400	0.1869
2/10/2019	2925	77.2	0.0018	0.2792
2/10/2019	2930	75.3	0.2855	0.2327
2/10/2019	2935	63.2	0.2091	0.2449
2/10/2019	2940	73.6	0.5564	0.1835
2/10/2019	2945	76.5	0.0018	0.2789
2/10/2019	2950	51.7	0.0018	0.3176
2/10/2019	2955	63.63	0.5818	0.1781
2/10/2019	2960	42	0.0018	0.2751
2/10/2019	2965	41.9	0.6691	0.1582
2/10/2019	2970	39.13	0.5836	0.1776
2/10/2019	2975	35.31	0.5782	0.1789
2/10/2019	2980	31.17	0.6055	0.1731
2/10/2019	2985	28.9	0.6418	0.1646
2/10/2019	2990	22.56	0.6345	0.1666
2/10/2019	2995	20.85	0.7291	0.1435
2/10/2019	3000	18.3	0.7145	0.1470
2/10/2019	3005	14.7	0.7255	0.1445
2/10/2019	3010	12.9	0.7618	0.1343
2/10/2019	3015	10.6	0.7636	0.1342
2/10/2019	3020	8.2	0.7782	0.1295
2/10/2019	3025	6.77	0.8018	0.1225
2/10/2019	3030	5.1	0.8073	0.1208
2/10/2019	3035	4	0.8236	0.1156
2/10/2019	3040	3	0.8309	0.1133
2/10/2019	3045	2.4	0.8400	0.1103
2/10/2019	3050	1.75	0.8400	0.1100
2/10/2019	3055	1.65	0.8473	0.1075
2/10/2019	3060	1.06	0.8345	0.1121
2/10/2019	3065	0.9	0.8491	0.1072
2/10/2019	3070	0.66	0.8418	0.1092
2/10/2019	3075	0.58	0.8455	0.1082
2/10/2019	3080	0.41	0.8382	0.1109
2/10/2019	3085	0.34	0.8418	0.1095
2/10/2019	3090	0.24	0.8364	0.1111

B.3 Time to Maturity: 1 Month

$Maturity$	$Strike$	$Price$	w	$ImpVol$
25/10/2019	2895	137	0.3091	0.2289
25/10/2019	2900	123	0.5036	0.1941
25/10/2019	2905	101.5	0.7727	0.1311
25/10/2019	2910	121.7	0.4073	0.2119
25/10/2019	2915	124.65	0.2945	0.2313
25/10/2019	2920	103.9	0.5891	0.1764
25/10/2019	2925	109.35	0.4527	0.2037
25/10/2019	2930	94.1	0.6345	0.1664
25/10/2019	2935	71	0.8636	0.1021
25/10/2019	2940	96.95	0.5018	0.1944
25/10/2019	2945	98	0.4382	0.2064
25/10/2019	2950	83.5	0.6091	0.1723
25/10/2019	2955	92.5	0.4327	0.2073
25/10/2019	2960	67.7	0.7291	0.1432
25/10/2019	2965	77.3	0.5745	0.1795
25/10/2019	2970	64.12	0.7055	0.1495
25/10/2019	2975	58.77	0.7345	0.1420
25/10/2019	2980	56.83	0.7255	0.1441
25/10/2019	2985	49.82	0.7709	0.1316
25/10/2019	2990	50.42	0.7400	0.1405
25/10/2019	2995	45.29	0.7673	0.1329
25/10/2019	3000	43.51	0.7618	0.1345
25/10/2019	3005	41.65	0.7564	0.1357
25/10/2019	3010	38.08	0.7709	0.1318
25/10/2019	3015	34.83	0.7818	0.1285
25/10/2019	3020	30.52	0.8036	0.1218
25/10/2019	3025	28.6	0.8036	0.1218
25/10/2019	3030	26	0.8109	0.1195
25/10/2019	3035	21.97	0.8327	0.1124
25/10/2019	3040	21.96	0.8182	0.1173
25/10/2019	3045	18.8	0.8345	0.1121
25/10/2019	3050	16.52	0.8418	0.1092
25/10/2019	3055	20.6	0.7855	0.1273
25/10/2019	3060	13.94	0.8436	0.1091
25/10/2019	3065	17.88	0.7873	0.1272
25/10/2019	3070	10.4	0.8582	0.1039
25/10/2019	3075	9.1	0.8618	0.1024
25/10/2019	3080	7.9	0.8655	0.1009
25/10/2019	3085	7.5	0.8600	0.1026
25/10/2019	3090	5.92	0.8727	0.0983

B.4 Time to Maturity: 2 Months

Maturity	*Strike*	*Price*	*w*	*ImpVol*
29/11/2019	2885	165.1	0.4164	0.2104
29/11/2019	2890	153.7	0.5127	0.1922
29/11/2019	2895	159.5	0.4073	0.2121
29/11/2019	2900	145.5	0.5327	0.1883
29/11/2019	2905	141.8	0.5382	0.1872
29/11/2019	2910	131.02	0.6164	0.1705
29/11/2019	2915	144.1	0.4436	0.2054
29/11/2019	2920	140.7	0.4473	0.2046
29/11/2019	2925	123.94	0.5945	0.1754
29/11/2019	2930	120.26	0.6000	0.1740
29/11/2019	2940	106.75	0.6709	0.1581
29/11/2019	2945	111.5	0.6000	0.1740
29/11/2019	2950	108.4	0.6036	0.1735
29/11/2019	2960	96.19	0.6636	0.1599
29/11/2019	2965	82.2	0.7527	0.1369
29/11/2019	2970	89.32	0.6745	0.1569
29/11/2019	2975	83.06	0.7055	0.1496
29/11/2019	2980	88.45	0.6382	0.1656
29/11/2019	2985	80.45	0.6855	0.1546
29/11/2019	2990	74.8	0.7109	0.1483
29/11/2019	2995	81.38	0.6345	0.1663
29/11/2019	3000	69.59	0.7127	0.1474
29/11/2019	3005	66.47	0.7200	0.1459
29/11/2019	3010	77.3	0.6091	0.1722
29/11/2019	3015	64.37	0.7000	0.1506
29/11/2019	3020	59.55	0.7218	0.1453
29/11/2019	3025	60.7	0.6964	0.1519
29/11/2019	3030	53.9	0.7327	0.1423
29/11/2019	3035	48.8	0.7564	0.1359
29/11/2019	3040	46.2	0.7618	0.1345
29/11/2019	3045	49.05	0.7255	0.1443
29/11/2019	3050	41.5	0.7691	0.1323
29/11/2019	3055	39	0.7745	0.1307
29/11/2019	3060	39.8	0.7564	0.1360
29/11/2019	3065	40.5	0.7382	0.1411
29/11/2019	3070	31.89	0.7909	0.1257
29/11/2019	3075	29.82	0.7964	0.1243
29/11/2019	3080	28.3	0.7964	0.1241
29/11/2019	3085	34.3	0.7382	0.1411
29/11/2019	3090	29.1	0.7691	0.1324

B.5 Time to Maturity: 3 Months

$Maturity$	$Strike$	$Price$	w	$ImpVol$
20/12/2019	2895	171.1	0.5055	0.1937
20/12/2019	2900	168	0.5055	0.1936
20/12/2019	2905	142.9	0.6836	0.1548
20/12/2019	2910	154.4	0.5709	0.1804
20/12/2019	2915	103.7	0.8800	0.0951
20/12/2019	2920	103.8	0.8636	0.1017
20/12/2019	2925	135.63	0.6473	0.1634
20/12/2019	2930	146.78	0.5364	0.1874
20/12/2019	2935	125.3	0.6800	0.1557
20/12/2019	2940	122	0.6836	0.1550
20/12/2019	2945	135.56	0.5600	0.1826
20/12/2019	2950	118	0.6709	0.1578
20/12/2019	2955	99.04	0.7745	0.1306
20/12/2019	2960	119.8	0.6200	0.1698
20/12/2019	2965	114.4	0.6400	0.1652
20/12/2019	2970	117.6	0.5964	0.1748
20/12/2019	2975	109.6	0.6364	0.1658
20/12/2019	2980	109	0.6236	0.1690
20/12/2019	2985	94	0.7091	0.1485
20/12/2019	2990	90.4	0.7164	0.1466
20/12/2019	2995	87.75	0.7182	0.1462
20/12/2019	3000	83.6	0.7291	0.1433
20/12/2019	3005	85.2	0.7036	0.1498
20/12/2019	3010	78.2	0.7327	0.1420
20/12/2019	3015	79.54	0.7109	0.1479
20/12/2019	3020	78.18	0.7055	0.1493
20/12/2019	3025	67.54	0.7582	0.1353
20/12/2019	3030	77.93	0.6800	0.1559
20/12/2019	3035	63.1	0.7600	0.1348
20/12/2019	3040	61	0.7600	0.1346
20/12/2019	3045	63.4	0.7345	0.14199
20/12/2019	3050	55.15	0.7727	0.1313
20/12/2019	3055	51.9	0.7800	0.1289
20/12/2019	3060	50.7	0.7764	0.1299
20/12/2019	3065	48.2	0.7818	0.1286
20/12/2019	3070	45.8	0.7855	0.1274
20/12/2019	3075	42.6	0.7945	0.1246
20/12/2019	3080	41.4	0.7927	0.1253
20/12/2019	3085	38.7	0.8000	0.1232
20/12/2019	3090	40.4	0.7800	0.1291

B.6 Time to Maturity: 6 Months

Maturity	*Strike*	*Price*	*w*	*ImpVol*
20/03/2020	2500	538.4	0.0655	0.2661
20/03/2020	2525	517.7	0.0764	0.2647
20/03/2020	2550	485.53	0.2364	0.2405
20/03/2020	2575	468.15	0.2073	0.2452
20/03/2020	2600	444.95	0.2509	0.2383
20/03/2020	2625	425.45	0.2545	0.2376
20/03/2020	2650	399.52	0.3291	0.2256
20/03/2020	2675	374.3	0.3909	0.2147
20/03/2020	2700	370	0.2618	0.2364
20/03/2020	2725	310.69	0.6109	0.1717
20/03/2020	2750	311.2	0.4673	0.2009
20/03/2020	2775	292.65	0.4764	0.1990
20/03/2020	2800	272.65	0.5000	0.1944
20/03/2020	2825	266.97	0.4273	0.2084
20/03/2020	2850	239.83	0.5073	0.1931
20/03/2020	2875	223.5	0.5164	0.1915
20/03/2020	2900	197.91	0.5836	0.1776
20/03/2020	2925	180.45	0.6036	0.1734
20/03/2020	2950	164.9	0.6145	0.1708
20/03/2020	2975	146.02	0.6473	0.1634
20/03/2020	3000	129.7	0.6691	0.1584
20/03/2020	3025	115.2	0.6836	0.1546
20/03/2020	3050	102	0.6964	0.1515
20/03/2020	3075	88.7	0.7127	0.1473
20/03/2020	3100	73.7	0.7418	0.1399
20/03/2020	3125	61.8	0.7582	0.1351
20/03/2020	3150	53.58	0.7636	0.1339
20/03/2020	3175	41.64	0.7891	0.1263
20/03/2020	3200	33.88	0.8000	0.1230
20/03/2020	3225	27	0.8109	0.1196
20/03/2020	3250	21	0.8218	0.1161
20/03/2020	3275	17.7	0.8200	0.1164
20/03/2020	3300	15.19	0.8164	0.1174
20/03/2020	3325	10.5	0.8345	0.1118
20/03/2020	3350	10.4	0.8182	0.1172
20/03/2020	3375	8.26	0.8200	0.1163
20/03/2020	3400	5.75	0.8327	0.1124
20/03/2020	3450	4.39	0.8218	0.1160
20/03/2020	3500	2.91	0.8200	0.1165
20/03/2020	3600	1.69	0.8000	0.1228

B.7 Time to Maturity: 9 Months

Maturity	*Strike*	*Price*	*w*	*ImpVol*
19/06/2020	2500	518.66	0.5655	0.1812
19/06/2020	2525	494.93	0.5891	0.1764
19/06/2020	2550	471.76	0.6073	0.1724
19/06/2020	2575	454.36	0.5800	0.1782
19/06/2020	2600	408.62	0.7691	0.1319
19/06/2020	2625	384.25	0.7873	0.1266
19/06/2020	2650	435.5	0.2964	0.2309
19/06/2020	2675	376.6	0.4655	0.2011
19/06/2020	2700	253.75	0.9091	0.0825
19/06/2020	2725	232.9	0.9073	0.0836
19/06/2020	2750	295.57	0.5509	0.1842
19/06/2020	2775	237.45	0.7491	0.1378
19/06/2020	2800	278.99	0.4782	0.1986
19/06/2020	2825	260.62	0.5018	0.1944
19/06/2020	2850	228.2	0.5927	0.1756
19/06/2020	2875	220	0.5673	0.1809
19/06/2020	2900	198	0.6091	0.1719
19/06/2020	2925	192.02	0.5800	0.1782
19/06/2020	2950	165.9	0.6436	0.1644
19/06/2020	2975	150	0.6618	0.1599
19/06/2020	3000	133.8	0.6836	0.1546
19/06/2020	3025	123.6	0.6836	0.1545
19/06/2020	3050	111	0.6964	0.1515
19/06/2020	3075	94.37	0.7273	0.1437
19/06/2020	3100	81.4	0.7455	0.1388
19/06/2020	3125	70.4	0.7582	0.1351
19/06/2020	3150	60.77	0.7691	0.1321
19/06/2020	3175	53.6	0.7727	0.1309
19/06/2020	3200	43.47	0.7909	0.1254
19/06/2020	3225	41.56	0.7782	0.1293
19/06/2020	3250	33.75	0.7927	0.1250
19/06/2020	3275	25.58	0.8127	0.1186
19/06/2020	3300	23.3	0.8073	0.1204
19/06/2020	3325	21.1	0.8036	0.1218
19/06/2020	3350	16.4	0.8145	0.1180
19/06/2020	3450	12.3	0.8145	0.1183
19/06/2020	3500	8.6	0.8182	0.1170
19/06/2020	3600	5.35	0.8055	0.1210

B.8 Time to Maturity: 1 Year

$Maturity$	$Strike$	$Price$	w	$ImpVol$
18/09/2020	2175	823.95	0.5673	0.1807
18/09/2020	2200	801.85	0.5364	0.1872
18/09/2020	2400	658.3	0.2000	0.2462
18/09/2020	2575	437	0.7873	0.1265
18/09/2020	2600	423	0.7473	0.1383
18/09/2020	2625	370.85	0.9091	0.0774
18/09/2020	2650	385.4	0.7327	0.1421
18/09/2020	2700	413.5	0.4073	0.2119
18/09/2020	2750	359.6	0.5200	0.1905
18/09/2020	2775	293.5	0.7345	0.1416
18/09/2020	2800	276.05	0.7382	0.1408
18/09/2020	2825	312.11	0.5291	0.1888
18/09/2020	2850	294.06	0.5455	0.1855
18/09/2020	2875	276.69	0.5600	0.1825
18/09/2020	2900	259.51	0.5745	0.1793
18/09/2020	2925	237.66	0.6091	0.1718
18/09/2020	2950	226.46	0.6036	0.1731
18/09/2020	2975	210.34	0.6182	0.1698
18/09/2020	3000	193.1	0.6400	0.1652
18/09/2020	3025	179.36	0.6473	0.1631
18/09/2020	3050	164	0.6636	0.1592
18/09/2020	3075	150.3	0.6764	0.1563
18/09/2020	3100	136.5	0.6909	0.1529
18/09/2020	3125	123.25	0.7036	0.1495
18/09/2020	3150	110.8	0.7164	0.1462
18/09/2020	3175	99.33	0.7273	0.1432
18/09/2020	3200	87.9	0.7418	0.1397
18/09/2020	3225	77.44	0.7527	0.1365
18/09/2020	3250	68.06	0.7636	0.1337
18/09/2020	3275	59.3	0.7727	0.1308
18/09/2020	3300	50.9	0.7836	0.1276
18/09/2020	3325	50.09	0.7691	0.1321
18/09/2020	3350	37.5	0.8000	0.1228
18/09/2020	3375	32.1	0.8055	0.1208
18/09/2020	3400	27.6	0.8109	0.1194
18/09/2020	3450	21.6	0.8109	0.1191
18/09/2020	3500	14.8	0.8236	0.1151
18/09/2020	3550	12.85	0.8127	0.1185